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Tutorial at ACT 2023

based on:

- papers by Marco Grandis and Robert Paré
- Double Fibrations - joint work with Geoff Cruttwell, Michael Lambert, and Martin Szyld

Double Categories

Intuitively:

A double category has:

- objects
- two types of arrows, each with its own composition structure, say $\nrightarrow$ and $\longrightarrow$
- double cells:

$$\begin{tikzcd}
\cdot \arrow[r, "\shortmid"{marking}] \arrow[d] \arrow[dr, phantom, "\alpha"] & \cdot \arrow[d] \\
\cdot \arrow[r, "\shortmid"{marking}] & \cdot
\end{tikzcd}$$

that can be composed in both directions and these compositions need to be compatible.

Definition For a 2-category $\mathcal{K}$, a **pseudo category** $\mathbb{C}$ in $\mathcal{K}$ consists of a diagram:

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "\otimes"] & C_1 \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & C_0 \arrow[l, "y"'] \end{tikzcd}$$

and iso. 2-cells:

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \times_{C_0} C_1 \arrow[r, "1\times\otimes"] \arrow[d, "\otimes\times 1"'] \arrow[dr, phantom, "\overset{\alpha}{\cong}"] & C_1 \times_{C_0} C_1 \arrow[d, "\otimes"] \\ C_1 \times_{C_0} C_1 \arrow[r, "\otimes"'] & C_1 \end{tikzcd}
\qquad
\begin{tikzcd} C_1 \arrow[r, "{(y,1)}"] \arrow[dr, bend right, "1_{C_1}"'] \arrow[dr, phantom, "\overset{l}{\cong}"] & C_1 \times_{C_0} C_1 \arrow[d, "\otimes"] & C_1 \arrow[l, "{(1,y)}"'] \arrow[dl, bend left, "1_{C_1}"] \arrow[dl, phantom, "\overset{r}{\cong}"] \\ & C_1 & \end{tikzcd}$$

normalized: l and r ids.

Definition A double category is a pseudo category object in $\underline{\mathrm{Cat}}$:

$$
\begin{tikzcd}
\underline{C}_1 \times_{\underline{C}_0} \underline{C}_1 \arrow[r, "\otimes"] & \underline{C}_1 \arrow[r, shift left=2, "s"] \arrow[r, "t"'] & \underline{C}_0 \arrow[l, bend left, "y"]
\end{tikzcd}
$$

$\otimes$ ps. assoc. and ps. unitary

$\underline{C}_0$, $\underline{C}_1$ categories.

Let's spell that out:

$$\begin{tikzcd}
 & \mathrm{Ar}(C)_1 \times_{\mathrm{Ob}(C)_1} \mathrm{Ar}(C)_1 \arrow[d] & \mathrm{Ar}(\underline{C}_0) \times_{\mathrm{Ob}(\underline{C}_0)} \mathrm{Ar}(\underline{C}_0) \arrow[d] \\
\mathrm{Ar}(C_1) \times_{\mathrm{Ar}(C_0)} \mathrm{Ar}(C_1) \arrow[r, "\otimes"] & \mathrm{Ar}(\underline{C}_1) \arrow[r, shift left=2] \arrow[r] \arrow[d, shift left=2] \arrow[d] & \mathrm{Ar}(\underline{C}_0) \arrow[l, shift left=2] \arrow[d, shift left=2] \arrow[d] \\
\mathrm{Ob}(\underline{C}_1) \times_{\mathrm{Ob}(\underline{C}_0)} \mathrm{Ob}(\underline{C}_1) \arrow[r, "\otimes"] & \mathrm{Ob}(\underline{C}_1) \arrow[r, shift left=2] \arrow[r] \arrow[u, shift right=4] & \mathrm{Ob}(\underline{C}_0) \arrow[l, shift left=2] \arrow[u, shift right=4]
\end{tikzcd}$$

$$
\begin{tikzcd}
\bullet \arrow[r, "/" marking] \arrow[d] \arrow[dr, phantom, "\alpha"] & \bullet \arrow[d] \\
\bullet \arrow[r, "/" marking] & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, "/" marking] \arrow[d] \arrow[dr, phantom, "\alpha"] & \bullet \arrow[d] \\
\bullet \arrow[r, "/" marking] \arrow[d] \arrow[dr, phantom, "\beta"] & \bullet \arrow[d] \\
\bullet \arrow[r, "/" marking] & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[d] \\
\bullet \arrow[d] \\
\bullet
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \mathrm{Ar}(\underline{C}_1) \times_{\mathrm{Ob}(\underline{C}_1)} \mathrm{Ar}(\underline{C}_1) \arrow[d] & \mathrm{Ar}(\underline{C}_0) \times_{\mathrm{Ob}(\underline{C}_0)} \mathrm{Ar}(\underline{C}_0) \arrow[d] \\
\mathrm{Ar}(\underline{C}_1) \times_{\mathrm{Ar}(\underline{C}_0)} \mathrm{Ar}(\underline{C}_1) \arrow[r, "\otimes"] & \mathrm{Ar}(\underline{C}_1) \arrow[r, shift left=2] \arrow[r] \arrow[d, shift right=2] \arrow[d] & \mathrm{Ar}(\underline{C}_0) \arrow[l, shift left=2] \arrow[d, shift right=2] \arrow[d] \\
\mathrm{Ob}(\underline{C}_1) \times_{\mathrm{Ob}(\underline{C}_0)} \mathrm{Ob}(\underline{C}_1) \arrow[r, "\otimes"] & \mathrm{Ob}(\underline{C}_1) \arrow[r, shift left=2] \arrow[r] \arrow[u, shift right=2] & \mathrm{Ob}(\underline{C}_0) \arrow[l, shift left=2] \arrow[u, shift right=2]
\end{tikzcd}
$$

squares of $\mathbb{C}$

pro-arrow comp^n

arrow comp^n

arrows of $\mathbb{C}$

$$
\begin{tikzcd}
\bullet \arrow[r, "/" marking] \arrow[d] \arrow[dr, phantom, "\alpha"] & \bullet \arrow[r, "/" marking] \arrow[d] \arrow[dr, phantom, "\alpha'"] & \bullet \arrow[d] \\
\bullet \arrow[r, "/" marking] & \bullet \arrow[r, "/" marking] & \bullet
\end{tikzcd}
$$

$$
\begin{tikzcd}
\bullet \arrow[r, "/" marking] & \bullet \arrow[r, "/" marking] & \bullet
\end{tikzcd}
$$

pro-arrows of $\mathbb{C}$

objects of $\mathbb{C}$.

$$
\begin{tikzcd}
\bullet \arrow[r, "/" marking] & \bullet
\end{tikzcd}
\qquad \bullet
$$

Proarrow composition is only required to be unitary and associative up to isomorphism:

these are double cells:

unitors:

$$
\begin{tikzcd}
A \arrow[r, "y_B h"] \arrow[d, "1_A"'] \arrow[dr, phantom, "\lambda_h"] & B \arrow[d, "1_B"] \\
A \arrow[r, "h"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "h y_A"] \arrow[d, "1_A"'] \arrow[dr, phantom, "\rho_h"] & B \arrow[d, "1_B"] \\
A \arrow[r, "h"'] & B
\end{tikzcd}
$$

associators:

$$
\begin{tikzcd}
A \arrow[r, "gf"] \arrow[d, "1_A"'] & C \arrow[r, "h"] \arrow[d, phantom, "\alpha_{hgf}"] & D \arrow[d, "1_D"] \\
A \arrow[r, "f"'] & B \arrow[r, "hg"'] & D
\end{tikzcd}
$$

- functorial w.r.t. dbl cells
- vertically invertible
- satisfying coherence.

1. **Rel** : objects are sets

proarrows are relations

arrows are functions

Squares :

$$\begin{tikzcd} S \arrow[r, "R", "/"{marking}] \arrow[d, "u"'] \arrow[dr, phantom, "\Downarrow"] & T \arrow[d, "v"] \\ S' \arrow[r, "R'"', "/"{marking}] & T' \end{tikzcd}$$

$R \subseteq S \times T$

$R' \subseteq S' \times T'$

such that

for $(s,t) \in R$, $(u(s), v(t)) \in R'$!

2. Span($\underline{\mathcal{C}}$), where $\underline{\mathcal{C}}$ is a category with pullbacks:

<u>obj</u>: are objects in $\mathcal{C}$

<u>arrows</u> are arrows in $\underline{\mathcal{C}}$.

<u>Proarrows</u> are spans

$$S \xleftarrow{s} A \xrightarrow{t} T$$

<u>squares</u> : commutative diagrams

$$\begin{tikzcd}
S \arrow[d, "u"'] & A \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "w"] & T \arrow[d, "v"] \\
S' & A' \arrow[l, "s'"'] \arrow[r, "t'"] & T'
\end{tikzcd}$$

3. For $\underline{C}_0 = \underline{1}$, a double category $\underline{C}_1 \times \underline{C}_1 \xrightarrow{\otimes} \underline{C}_1 \xleftarrow{y} \underline{1}$ is just a <u>monoidal category</u>.

4. A 2-category $\mathcal{A}$ gives rise to double categories

- $\mathbb{V}\mathcal{A}$ with cells
$$\begin{tikzcd} A \arrow[r, "1"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & A \arrow[d, "v"] \\ B \arrow[r, "1"'] & B \end{tikzcd}$$
when
$$\begin{tikzcd} A \arrow[d, bend right, "u"'] \arrow[d, bend left, "v"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \\ B \end{tikzcd}$$
in $\mathcal{A}$.

- $\mathbb{H}\mathcal{A}$ with cells
$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "1"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d] \\ A \arrow[r, "k"'] & B \end{tikzcd}$$
when
$$\begin{tikzcd} A \arrow[r, bend left, "h"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$$
in $\mathcal{A}$.

- $\mathbb{Q}\mathcal{A}$ with cells
$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "v"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$
when
$$\begin{tikzcd} A \arrow[r, bend left, "vh"] \arrow[r, bend right, "ku"'] \arrow[r, phantom, "\Downarrow\alpha"] & D \end{tikzcd}$$
in $\mathcal{A}$.

Note: if $\mathcal{H}$ and $\mathcal{V}$ are wide subcategories of $\mathcal{A}$, there is an obvious subdouble category of quintets with horizontal arrows in $\mathcal{H}$ and vertical arrows in $\mathcal{V}$.

For: double categories $\mathbb{C}$: $\underline{C}_1 \times_{\underline{C}_0} \underline{C}_1 \xrightarrow{\otimes} \underline{C}_1 \underset{y}{\overset{s,\,t}{\rightleftarrows}} \underline{C}_0$

and $\mathbb{D}$: $\underline{D}_1 \times_{\underline{D}_0} \underline{D}_1 \xrightarrow{\otimes} \underline{D}_1 \underset{y}{\overset{s,\,t}{\rightleftarrows}} \underline{D}_0$

strict in the arrow direction, pseudo in the proarrow direction

a pseudo double functor consists of

$F_0 : \underline{C}_0 \longrightarrow \underline{D}_0$ and $F_1 : \underline{C}_1 \longrightarrow \underline{D}_1$

with comparison cells:

$$\begin{tikzcd}
\underline{C}_1 \times_{\underline{C}_0} \underline{C}_1 \arrow[r, "\otimes"] \arrow[d, "F_1 \times_{F_0} F_1"'] \arrow[dr, phantom, "\overset{\varphi_1}{\underset{\cong}{\Rightarrow}}"] & \underline{C}_1 \arrow[d, "F_1"] \\
\underline{D}_1 \times_{\underline{D}_0} \underline{D}_1 \arrow[r, "\otimes"'] & \underline{D}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{C}_0 \arrow[r, "y"] \arrow[d, "F_0"'] \arrow[dr, phantom, "\overset{2}{\underset{\cong}{\Rightarrow}}"] & \underline{C}_1 \arrow[d, "F_1"] \\
\underline{D}_0 \arrow[r, "y"'] & \underline{D}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{C}_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[d, "F_1"'] \arrow[dr, phantom, "="] & \underline{C}_0 \arrow[d, "F_0"] \\
\underline{D}_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \underline{D}_0
\end{tikzcd}$$

Stricter!

subject to the usual coherence condns

These come in two flavours:

- with proarrow components — these are the internal transformations between internal functors:

$$\begin{tikzcd}
C_1 \arrow[r, shift left, "G_1"] \arrow[r, shift right, "F_1"'] \arrow[d, shift right=2, "s"'] \arrow[d, shift left=2, "t"] & D_1 \arrow[d, shift right=2] \arrow[d, shift left=2] \\
C_0 \arrow[u, "y"] \arrow[r, shift left, "G_0"] \arrow[r, shift right, "F_0"'] \arrow[ur, "\alpha"] & D_0 \arrow[u]
\end{tikzcd}$$

$\alpha : F \nRightarrow G$

$\alpha : C_0 \longrightarrow D_1$ is a functor

- for each object X in $\mathbb{C}$, $\alpha_X : F_0 X \nrightarrow G_0 X$ is a proarrow in $\mathbb{D}$
- for each arrow $v : X \longrightarrow Y$ in $\mathbb{C}$, there is a double cell

$$\begin{tikzcd}
F_0 X \arrow[r, "\alpha_X", "\shortmid" marking] \arrow[d, "F_0 v"'] \arrow[dr, phantom, "\alpha_v"] & G_0 X \arrow[d, "G_0 v"] \\
F_0 Y \arrow[r, "\alpha_Y", "\shortmid" marking] & G_0 Y
\end{tikzcd}$$

these need to satisfy:

- horizontal / proarrow naturality

$$\begin{tikzcd}
F_0 X \arrow[r, "F_1 h", "/"{marking}] \arrow[d, "F_0 v"'] \arrow[dr, phantom, "F_1 \theta"] & F_0 X' \arrow[r, "\alpha_{X'}", "/"{marking}] \arrow[d, "F_0 v'"] \arrow[dr, phantom, "\alpha_{v'}"] & G_0 X' \arrow[d, "G_0 v'"] \\
F_0 Y \arrow[r, "F_1 k"', "/"{marking}] & F_0 Y' \arrow[r, "\alpha_{Y'}"', "/"{marking}] & G_0 Y'
\end{tikzcd} \quad =$$

$$\begin{tikzcd}
F_0 X \arrow[r, "\alpha_X", "/"{marking}] \arrow[d, "F_0 v"'] \arrow[dr, phantom, "\alpha_v"] & G_0 X \arrow[r, "G_1 h", "/"{marking}] \arrow[d, "G_0 v"] \arrow[dr, phantom, "G_1 \theta"] & G_0 X' \arrow[d, "G_0 v'"] \\
F_0 Y \arrow[r, "\alpha_Y"', "/"{marking}] & G_0 Y \arrow[r, "G_1 k"', "/"{marking}] & G_0 Y'
\end{tikzcd}$$

- vertical / arrow functoriality: $\alpha_{wv} = \alpha_w \alpha_v$

- The transformations with arrow components are defined dually, with components

$$\begin{tikzcd}
F_0 X \arrow[r, "F_1 h", "\shortmid" marking] \arrow[d, "\beta_X"'] \arrow[dr, phantom, "\beta_h"] & F_0 X' \arrow[d, "\beta_{X'}"] \\
G_0 X \arrow[r, "G_1 h"', "\shortmid" marking] & G_0 X'
\end{tikzcd}$$

that need to be natural in the arrow direction:

$$\begin{tikzcd}
F_0 X \arrow[r, "F_1 h", "\shortmid" marking] \arrow[d, "\beta_X"'] \arrow[dr, phantom, "\beta_h"] & F_0 X' \arrow[d, "\beta_{X'}"] \\
G_0 X \arrow[r, "G_1 h"', "\shortmid" marking] \arrow[d, "G_0 v"'] \arrow[dr, phantom, "G_1 \theta"] & G_0 X' \arrow[d, "G_0 v'"] \\
G_0 Y \arrow[r, "G_1 k", "\shortmid" marking] & G_0 Y'
\end{tikzcd}
=
\begin{tikzcd}
F_0 X \arrow[r, "F_1 h", "\shortmid" marking] \arrow[d, "F_0 v"'] \arrow[dr, phantom, "F_1 \theta"] & F_0 X' \arrow[d, "F_0 v'"] \\
F_0 Y \arrow[r, "F_1 k", "\shortmid" marking] \arrow[d, "\beta_Y"'] \arrow[dr, phantom, "\beta_k"] & F_0 Y' \arrow[d, "\beta_{Y'}"] \\
G_0 Y \arrow[r, "G_1 k", "\shortmid" marking] & G_0 Y'
\end{tikzcd}$$

and functorial in the proarrow direction:

$$
\begin{tikzcd}
F_0X \arrow[rr, "F_1(gh)", "|" marking] \arrow[d, equal] & \arrow[d, phantom, "\varphi_{g,h}"] & F_0Z \arrow[d, equal] \\
F_0X \arrow[r, "F_1h", "|" marking] \arrow[d, "\beta_X"'] \arrow[dr, phantom, "\beta_h"] & F_0Y \arrow[r, "F_1g", "|" marking] \arrow[d, "\beta_Y"] \arrow[dr, phantom, "\beta_g"] & F_0Z \arrow[d, "\beta_Z"] \\
G_0X \arrow[r, "G_1h"', "|" marking] & G_0Y \arrow[r, "G_1g"', "|" marking] & G_0Z
\end{tikzcd}
=
\begin{tikzcd}
F_0X \arrow[rr, "F_1(gh)", "|" marking] \arrow[d, "\beta_X"'] & \arrow[d, phantom, "\beta_{gh}"] & F_0Z \arrow[d, "\beta_Z"'] \\
G_0X \arrow[rr, "G_1(gh)"', "|" marking] \arrow[d, equal] & \arrow[d, phantom, "\psi_{gh}"] & G_0Z \arrow[d, equal] \\
G_0X \arrow[r, "G_1h"', "|" marking] & G_0Y \arrow[r, "G_1g"', "|" marking] & G_0Z
\end{tikzcd}
$$

- objects: double categories $\mathbb{A}, \mathbb{B}, \mathbb{C}, \ldots$
- arrows: pseudo double functors
- transformations:
  - proarrow valued
  - arrow valued
- modifications

Given $F, G, H, K : \mathbb{C} \longrightarrow \mathbb{D}$

with

$$\begin{tikzcd} F \arrow[r, Rightarrow, "\alpha", "\shortmid"{marking}] \arrow[d, Rightarrow, "\beta"'] \arrow[dr, phantom, "\Omega"] & G \arrow[d, Rightarrow, "\gamma"] \\ H \arrow[r, Rightarrow, "\delta"', "\shortmid"{marking}] & K \end{tikzcd}$$

α, δ proarrow transformations

β, γ arrow transformations

a modification Ω is given by

a double cell

$$\begin{tikzcd} FX \arrow[r, "\alpha_X", "\shortmid"{marking}] \arrow[d, "\beta_X"'] \arrow[dr, phantom, "\Omega_X"] & GX \arrow[d, "\gamma_X"] \\ HX \arrow[r, "\delta_X"', "\shortmid"{marking}] & KX \end{tikzcd}$$

in $\mathbb{D}$ for each object X in $\mathbb{C}$.

natural in both directions

Results:

- $\mathrm{Hom}(\mathbb{C}, \mathbb{D})$ has the structure of a double category.
- By restricting to globular cells we can make this a category.
- DblCat is naturally enriched over itself; we can also make it a 2-category.
- There are ways to make it a double category.

The Grothendieck Construction

- many cool properties (see n-lab, listen to talks from this year's CT conference)

- today: I will mostly focus on two of them

- For an indexing pseudo functor $F : \mathcal{A}^{op} \longrightarrow \underline{Cat}$ with structure isomorphisms

$$\varphi_A : 1_{FA} \overset{\cong}{\Longrightarrow} F(1_A)$$

$$\varphi_{g,f} : Fg \circ Ff \overset{\cong}{\Longrightarrow} F(g \circ f)$$

the category of elements $\int_{\mathcal{A}} F \xrightarrow{\pi_F} \mathcal{A}$ has

<u>objects</u>: (A, x), $A \in Obj(\mathcal{A})$, $x \in Obj(F(A))$

<u>arrows</u>: $(g, \psi) : (A, x) \longrightarrow (B, y)$ with $g : A \to B$ and $\psi : x \longrightarrow F(g)(y)$ in $F(A)$.

identities:

$$\mathrm{id}_{(A,x)} = (\mathrm{id}_A, (\varphi_A)_x)$$

$$(\varphi_A)_x : x \to F(1_A)(x)$$

composition:

for $(A,x) \xrightarrow{(g_1,\psi_1)} (B,y) \xrightarrow{(g_2,\psi_2)} (C,z)$:

$$(g_2,\psi_2)\circ(g_1,\psi_1) = (g_2\circ g_1,\ \varphi_{g_1 g_2}\circ F(g_1)(\psi_2)\circ\psi_1)$$

$$x \xrightarrow{\psi_1} F(g_1)(y) \xrightarrow{F(g_1)(\psi_2)} F(g_1)F(g_2)(z) \xrightarrow{\varphi_{g_1,g_2}} F(g_1 g_2)(z)$$

- $\pi_F : \int_{\mathcal{A}} F \longrightarrow \mathcal{A}$ is defined by :

$$(A, x) \longmapsto A$$
$$(g, \varphi) \longmapsto g$$

- π_F is a <u>fibration</u>

Fibrations

$\exists \beta$

s.t. $\alpha = p\beta$

β is a lifting of α

$\forall \alpha$

path lifting property

Let $p : \underline{E} \longrightarrow \underline{B}$ be a functor. An arrow $\gamma : e' \to e$ in $\underline{E}$ is <u>cartesian</u> if for all

$$\begin{tikzcd} e'' \arrow[d, dashed, "\chi"', "!"] \arrow[dr, "\psi"] & \\ e' \arrow[r, "\gamma"'] \arrow[ur, phantom, "="] & e \end{tikzcd} \qquad \text{in } \underline{E}$$

$$\begin{tikzcd} pe'' \arrow[d, "p(\chi) = g"'] \arrow[dr, "p\psi"] & \\ pe' \arrow[r, "p\gamma"'] \arrow[ur, phantom, "="] & pe \end{tikzcd} \qquad \text{in } \underline{B}$$

there is a unique lifting χ.

① $p: \underline{E} \longrightarrow \underline{B}$ is a fibration if for any

$$e' \xrightarrow{\varphi} e \qquad \text{in } \underline{E}$$

$$b \xrightarrow[f]{} pe \qquad \text{in } \underline{B}$$

there is a Cartesian lifting φ (of f to e)

② A cleavage for p consists of a choice of a Cartesian lifting for each f and e.

$$\begin{tikzcd}
\int_{\mathcal{A}} F \arrow[d, "\pi_F"'] & (B, \quad) \arrow[r] & (A, x) \\
\mathcal{A} & B \arrow[r, "f"'] & A
\end{tikzcd}$$

$F : \mathcal{A}^{op} \to \underline{\mathbf{Cat}}$

$x \in FA$

$\in FB$

$$\begin{tikzcd} \int_{\mathcal{A}} F \arrow[d, "\pi_F"'] & (B, Ff(x)) \arrow[r, "{(f,\, 1_{Ff(x)})}"'] & (A, x) \\ \mathcal{A} & B \arrow[r, "f"'] & A \end{tikzcd}$$

Canonical Cartesian morphisms:

$$(f,\ 1_{Ff(x)})$$

they form a cleavage.

Result: for a fixed base category $\underline{B}$, the Grothendieck construction extends to a 2-equivalence:

$$\int_B : [\underline{B}^{op}, \underline{Cat}] \xrightarrow{\sim} \underline{Fib}(\underline{B})$$

$[\underline{B}^{op}, \underline{Cat}]$: pseudo-functors, pseudo natural transformations, modifications

$\underline{Fib}(\underline{B})$: fibrations over $\underline{B}$, cartesian functors, vertical natural transformations

$$\begin{tikzcd} \underline{E} \arrow[rr, bend left, "\Phi"] \arrow[rr, bend right, "\Psi"'] \arrow[rr, phantom, "\Downarrow\alpha"] \arrow[dr, "p"'] & & \underline{F} \arrow[dl, "q"] \\ & \underline{B} & \end{tikzcd}$$

$q(\alpha_e) = 1_{p(e)}$.

$\underline{\mathrm{Fib}}$ is the 2-category :

__objects__ : cloven fibrations $p : \underline{E} \to \underline{B}$ (arbitrary $\underline{B}$)

__arrows__ : $f : (p : \underline{E} \to \underline{B}) \longrightarrow (p' : \underline{E}' \to \underline{B}')$

is a commutative square :

$$
\begin{tikzcd}
\underline{E} \arrow[r, "f^{\top}"] \arrow[d, "p"'] & \underline{E}' \arrow[d, "p'"] \\
\underline{B} \arrow[r, "f^{\perp}"'] & \underline{B}'
\end{tikzcd}
$$

, $f^{\top}$ preserves Cartesian arrows.

__2-cells__ : $\alpha : f \Rightarrow g$:

$$
\begin{tikzcd}
\underline{E} \arrow[r, bend left, "f^{\top}"{name=A}] \arrow[r, bend right, "g^{\top}"'{name=B}] \arrow[d, "p"'] & \underline{E}' \arrow[d, "p'"] \\
\underline{B} \arrow[r, bend left, "f_{\perp}"{name=C}] \arrow[r, bend right, "g_{\perp}"'{name=D}] & \underline{B}'
\arrow[from=A, to=B, phantom, "\Downarrow \alpha^{\top}"]
\arrow[from=C, to=D, phantom, "\Downarrow \alpha_{\perp}"]
\end{tikzcd}
$$

Commutative cylinder.

$\underline{\mathrm{cFib}} \subseteq \underline{\mathrm{Fib}}$ is the locally full sub-2-category where we require the arrows $f^{\top}$ to preserve the cleavages.

We obtain two equivalences of 2-categories:

- $\underline{\mathrm{Fib}} \simeq \underline{\mathrm{ICat}}$

$$\begin{tikzcd}
\underline{\mathcal{B}} \arrow[dr] \arrow[d] \arrow[dr, phantom, shift right=3, "\Downarrow"] & \\
\underline{\mathcal{B}'} \arrow[r] & \underline{\mathrm{Cat}}
\end{tikzcd}$$

pseudo natural transf$^{\mathrm{ns}}$

- $\underline{\mathrm{cFib}} \simeq \underline{\mathrm{ICat}}_s$

<u>strict</u> ntl. transformations

- $\underline{\mathrm{Mod}} \longrightarrow \underline{\mathrm{Ring}}$ is a fibration (and an opfibration)

$(R, M) \longmapsto R$

M a left R-module

$(R_1, f^*M) \qquad (R_2, M)$

$R_1 \xrightarrow[f]{} R_2$

f^*M : restriction of scalars

- Codomain fibration over a category $\underline{C}$ with pullbacks:

$\mathrm{Arr}_s(\underline{C}) = \underline{C}^{\underline{2}}$ has obj. $\begin{array}{c} x \\ f \downarrow \\ y \end{array}$ in $\underline{C}$

and arrows:

$$\begin{tikzcd} x \arrow[r, "h"] \arrow[d, "f"'] & x' \arrow[d, "f'"] \\ y \arrow[r, "k"] & y' \end{tikzcd}$$

comm. in $\underline{C}$.

the codomain functor $\underline{C}^{\underline{2}} \to \underline{C}$ is a fibration and opfibration; cartesian arrows are pullback squares.

- for $\underline{C}$ any category, $\mathrm{Fam}(\underline{C})$ is the category of set-indexed families of obj.'s in $\underline{C}$:

$$(C_i)_{i \in I}$$

morphisms:

$$(C_i)_{i\in I} \xrightarrow[(f,(\phi_i)_{i\in I})]{} (D_j)_{j\in J}$$

$f : I \to J$ function

$\phi_i : C_i \to D_{f(i)}$

This is the Grothendieck construction for

$$\underline{\mathrm{Set}}^{op} \longrightarrow \underline{\mathrm{Cat}}$$

$$I \longmapsto \prod_{i\in I} \underline{C}$$

so we get that $\mathrm{Fam}(\mathcal{C}) \to \underline{\mathrm{Set}}$ is a fibration

The Category of Elements is also an oplax colimit for F as diagram in $\underline{\mathbf{Cat}}$:

- We have a universal cone:

$$\begin{tikzcd} A \arrow[d, "g"'] & & FA \arrow[rr, "\varepsilon_A"] \arrow[drr, phantom, "\Downarrow \varepsilon_g"] & & \int_{\mathcal{A}} F \\ B & & FB \arrow[u, "Fg"] \arrow[urr, "\varepsilon_B"'] & & \end{tikzcd}$$

$$\varepsilon_A : \quad x \longmapsto (A, x)$$

$$(\psi : x \to y) \longmapsto \left[(A,x) \xrightarrow[(1_A,\ \varphi_A \circ \psi)]{} (A,y) \right]$$

$$(\varepsilon_g)_y = (g, 1_{Fg(y)}) : (A, Fg(y)) \longrightarrow (B, y)$$

Other properties of the category of elements

- a functor $P : \mathcal{A}^{op} \longrightarrow \underline{\mathbf{Set}}$ is representable (i.e. $P \cong \mathrm{Hom}(-, X)$ for some X in $\mathcal{A}$) iff $\int_{\mathcal{A}} P$ has a terminal object.

- We can obtain $\int_{\mathcal{A}} P$ as a comma square (see nlab)

Tutorial

on Double Fibrations

Double Grothendieck

Double Colimits

Part II

- with a corresponding notion of double fibration
- extending the monoidal Grothendieck construction [Moeller, Vasilakopoulou, TAC 2020]
- Jaz-Myers' double Grothendieck constructions for open dynamical systems as special cases [EPTC 2021]
- structured and decorated cospans as special cases. [Baez-Courser-Vasilakopoulou, 2020, 2022] [Patterson, 2023]
- extending the discrete case [Lambert, TAC 2021]

Double Fibrations

Since it is not obvious what the codomain of an indexing functor

$$\mathbb{A} \longrightarrow \,??\qquad (\mathbb{A} \text{ dbl cat}^y)$$

should be, we start with double fibrations.

First Idea: a double category is a pseudo cat^y in $\underline{\mathrm{Cat}}$ $\rightsquigarrow$ a double fibration is a pseudo cat^y in $\underline{\mathrm{Fib}}$.

This would amount to:

$$
\begin{tikzcd}
\underline{E}_1 \times_{\underline{E}_0} \underline{E}_1 \arrow[r, "\otimes^{\top}"] \arrow[d, "P_1 \times_{P_0} P_1"] & \underline{E}_1 \arrow[r, shift left=2, "s^{\top}"] \arrow[r, shift right=2, "t^{\top}"'] \arrow[d, "P_1"] & \underline{E}_0 \arrow[l, "y^{\top}" description] \arrow[d, "P_0"] \\
\underline{B}_1 \times_{\underline{B}_0} \underline{B}_1 \arrow[r, "\otimes^{\perp}"] & \underline{B}_1 \arrow[r, shift left=2, "s^{\perp}"] \arrow[r, shift right=2, "t^{\perp}"'] & \underline{B}_0 \arrow[l, "y^{\perp}" description]
\end{tikzcd}
$$

$\rightsquigarrow$ strict dbl functor with extra properties?

Problem: $\underline{\mathrm{Fib}}$ doesn't have all 2-pullbacks required for this.

Observation:

- We require the same fibrational strictness for s and t that we require for y and $\otimes$.

Solution: We will require that s and t are in cFib.

(i.e., they preserve cleavages)

Definition: A <u>double fibration</u> is a (strict) double functor $P : \mathbb{E} \to \mathbb{B}$ between pseudo double categories:

$$
\begin{tikzcd}
E_1 \times_{E_0} E_1 \arrow[r, "\otimes_E"] \arrow[d, "P_1 \times_{P_0} P_1"'] & E_1 \arrow[r, shift left=2, "s_E"] \arrow[r, shift right=2, "t_E"'] \arrow[d, "P_1"'] & E_0 \arrow[l] \arrow[d, "P_0"] \\
B_1 \times_{B_0} B_1 \arrow[r, "\otimes_B"] & B_1 \arrow[r, shift left=2, "s_B"] \arrow[r, shift right=2, "t_B"'] & B_0 \arrow[l]
\end{tikzcd}
$$

s.t.

- P_0 and P_1 are fibrations with cleavages
- s_E and t_E are cleavage preserving
- y_E and $\otimes_E$ are cartesian-morphism preserving.

1. $\mathrm{Im} : \mathbb{S}\mathrm{pan}(\underline{\mathrm{Set}}) \longrightarrow \mathbb{R}\mathrm{el}$

$$\left(A \xleftarrow{s} S \xrightarrow{t} B\right) \longmapsto \left(\mathrm{Im}((s,t)) \subseteq A \times B\right)$$

is a double opfibration.

2. When $\underline{E}_0 = \underline{B}_0 = 1$, we get:

$$\begin{tikzcd}
\underline{E}_1 \times_{\underline{E}_0} \underline{E}_1 \arrow[r, "\otimes_E"] \arrow[d, "P_1 \times_{P_0} P_1"'] & \underline{E}_1 \arrow[d, "P_1"'] & 1 \arrow[l, "y"'] \arrow[d, "id"] \\
\underline{B}_1 \times_{\underline{B}_0} \underline{B}_1 \arrow[r, "\otimes_B"] & \underline{B}_1 & 1 \arrow[l, "y"']
\end{tikzcd}$$

a monoidal fibration!

3. The Grothendieck constructions given in David Jaz Myers' work are also double fibrations.

4. For any 2-functor $P : \underline{E} \to \underline{B}$, P is a 2-fibration as in Buckley's work if and only if $\mathbb{Q}(P) : \mathbb{Q}(E) \to \mathbb{Q}(\underline{B})$ is a double fibration.

5. if P_0 and P_1 are discrete fibrations, we recover discrete double fibrations.

6. For $\mathbb{D}$ a double cat^y, let $\mathbb{D}^2 = \begin{pmatrix} \mathbb{D}_1^2 \\ \downarrow\downarrow \\ \mathbb{D}_0^2 \end{pmatrix}$

$$\mathrm{dom} : \mathbb{D}^2 \longrightarrow \mathbb{D}$$

is a double fibration

7. The codomain fibration extends to a double codomain fibration $\mathrm{cod}: \mathbb{D}^2 \to \mathbb{D}$ if:

- $\mathbb{D}_1$ and $\mathbb{D}_0$ have chosen finite limits
- these limits are preserved on the nose by s and t
- and up to iso by y and $\otimes$.

8. for $\underline{\mathcal{C}}$ a small cat^y, $\mathrm{Fam}(\mathcal{C})$ has:

<u>obj</u>: $f : I \to \mathcal{C}$, or $(I, \{C_i\}_{i\in I})$

<u>arrows</u>: $$\frac{(h,\alpha) : f \longrightarrow g}{h : I \to J,\ \alpha : f \Rightarrow gh}$$

$$\begin{tikzcd} I \arrow[rr, "h"] \arrow[dr, "f"'] & \arrow[d, phantom, "\Rightarrow\alpha"] & J \arrow[dl, "g"] \\ & \mathcal{C} & \end{tikzcd}$$

<u>or</u>:

$$\begin{tikzcd} (I, \{C_i\}_{i\in I}) \arrow[d, "{(h : I \to J,\ \{\alpha_i : C_i \to C_{h(i)}\}_{i\in I})}"] \\ (J, \{C_j\}_{j\in J}) \end{tikzcd}$$

Proarrows : $\qquad f \xrightarrow[\quad + \quad]{(S,\, d_0,\, d_1,\, \theta)}$

for natural transformations

$$\begin{tikzcd} S \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\overset{\theta}{\Rightarrow}"] & K \arrow[d, "p"] \\ I \arrow[r, "f"'] & e \end{tikzcd}$$

for a span of functions $I \xleftarrow{d_0} S \xrightarrow{d_1} K$

Or :

$$(I, \{C_i\}_{i \in I}) \xrightarrow[((d_0, S, d_1), \theta_s)]{\quad + \quad} (K, \{C_k\})$$

for a span of functions $I \xleftarrow{d_0} S \xrightarrow{d_1} K$

+ a family of arrow

$$\theta_s : C_{d_0(s)} \longrightarrow C_{d_1(s)}$$

Cells: a cell from

$$\begin{tikzcd} S \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\overset{\theta}{\Rightarrow}"] & K \arrow[d, "p"] \\ I \arrow[r, "f"'] & \mathcal{e} \end{tikzcd}$$

to

$$\begin{tikzcd} T \arrow[r, "d_1'"] \arrow[d, "d_0'"'] \arrow[dr, phantom, "\overset{\delta}{\Rightarrow}"] & L \arrow[d, "q"] \\ J \arrow[r, "g"'] & \mathcal{e} \end{tikzcd}$$

is given by a morphism of spans:

$$\begin{tikzcd} I \arrow[d, "h"'] & S \arrow[l, "d_0"'] \arrow[r, "d_1"] \arrow[d, "m"] & K \arrow[d, "r"] \\ J & T \arrow[l, "d_0'"] \arrow[r, "d_1"'] & L \end{tikzcd}$$

with 2-cells:

$$\begin{tikzcd} & I \arrow[dl, "f"'] \arrow[dd, "h"] \\ \mathcal{e} \arrow[r, phantom, "\overset{\Rightarrow}{\alpha}"] & {} \\ & J \arrow[ul, "g"] \end{tikzcd} \qquad \begin{tikzcd} & K \arrow[dl, "p"'] \arrow[dd, "r"] \\ \mathcal{e} \arrow[r, phantom, "\overset{\Rightarrow}{\beta}"] & {} \\ & L \arrow[ul, "q"] \end{tikzcd}$$

st:

$$(\beta * d_1)\theta = (\delta * q)(\alpha * d_0)$$

<u>Or:</u> a family of cells:

$$\begin{tikzcd}
(I, \{C_i\}_{i\in I}) \arrow[r, "{((d_0,S,d_1),\theta)}", "/"{marking}] \arrow[d, "{(h,(\alpha_i))}"'] \arrow[dr, phantom, "m"] & (K, \{C_k\}_{k\in K}) \arrow[d, "{(r,(\beta_k))}"] \\
(J, \{C_j\}_{j\in J}) \arrow[r, "{((d_0',T,d_1'),\theta')}"', "/"{marking}] & (L, \{C_\ell\}_{\ell\in L})
\end{tikzcd}$$

where $m : S \to T$ fits in

$$\begin{tikzcd}
I \arrow[d, "h"'] \arrow[dr, phantom, "="] & S \arrow[l, "d_0"'] \arrow[r, "d_1"] \arrow[d, "m"] \arrow[dr, phantom, "\swarrow"] & K \arrow[d, "r"] \\
J & T \arrow[l, "d_0'"] \arrow[r, "d_1'"'] & L
\end{tikzcd}$$

and we require that for each $s \in S$:

$$\begin{tikzcd}
\mathcal{C}_{d_0(s)} \arrow[r, "\theta_s"] \arrow[d, "\alpha_{d_0(s)}"'] & \mathcal{C}_{d_1(s)} \arrow[d, "\beta_{d_1(s)}"] \\
\mathcal{C}_{h(d_0 s)} \arrow[r, "\theta'_{m(s)}"] & \mathcal{C}_{r(d_1 s)}
\end{tikzcd}$$

this square is well defined because

$$d_0'(m(s)) = h\, d_0(s)$$

and $\quad d_1'(m(s)) = r\, d_1(s)$

we only need to require that

$$\beta_{d_1(s)}\, \theta_s = \theta'_{m(s)} \circ \alpha_{d_0(s)},$$

because $\mathcal{C}$ has no 2-cells.

<u>Note</u>: this can be extended to $\mathcal{C}$ a dbl. cat^y

$$\Pi_0 : \mathbb{F}\mathrm{am}(\mathcal{C})_0 \longrightarrow \underline{\mathrm{Set}}$$

is a split fibration.

We extend this to

$$\pi : \mathbb{F}\mathrm{am}(\mathcal{C}) \longrightarrow \mathbb{S}\mathrm{pan}(\underline{\mathrm{Set}}).$$

(send proarrows to their underlying spans and cells to span morphisms)

<u>Claim:</u> this is a split double fibration.

Relation with

Street's Internal Fibrations

(a look-off trail we may not take)

- Internal fibrations in a 2-category were introduced by Street in 1974.
- We will use the following three 2-cats:

$$\underline{\mathrm{Dbl}}_s \subseteq \underline{\mathrm{Dbl}} \subseteq \underline{\mathrm{Dbl}}_\ell$$

strict dbl functors — pseudo dbl functors — lax dbl functors

say more about Street's definition?

- In all 3 cases we use <u>vertical transformations</u> $\alpha : F \Rightarrow G : \mathbb{D} \rightrightarrows \mathbb{E}$:
  - $*$ for each x in $\mathbb{D}$: an arrow $\alpha_x : Fx \to Gx$ in $\mathbb{E}$
  - $*$ for each pro arrow $x \overset{m}{\nrightarrow} y$ a dbl cell $\alpha_m : Fm \to Gm$ in $\mathbb{E}_1$. (natural and functorial)

Theorem (Cruttwell, Lambert, P, Szyld)

A strict double functor is an internal fibration in DblCat if and only if it is a double fibration.

In addition, a pseudo double functor P

- is an internal fibration in DblCat_ℓ iff P_0 and P_1 admit cleavages that are preserved by $s_{\mathbb{E}}$ and $t_{\mathbb{E}}$.
- is an internal fibration in DblCat iff, in addition, $y_{\mathbb{E}}$ and $\otimes_{\mathbb{E}}$ preserve cartesian morphisms.

Furthermore, a <u>strict</u> double functor P is an internal fibration in $\underline{\mathrm{DblCat}}_s$ if and only if P_0 and P_1 are fibrations that admit cleavages that are preserved by all of $s_{\mathbb{E}}$, $t_{\mathbb{E}}$, $y_{\mathbb{E}}$ and $\otimes_{\mathbb{E}}$

Double Indexing Functors

(Take 1)

Exercise !

Note: (1.) Categories are monoids in Span(Set)

$$\begin{tikzcd}
C_0 \arrow[d, equal] & C_0 \arrow[l, "id"'] \arrow[r, "id"] \arrow[d, "y"] & C_0 \arrow[d, equal] \\
C_0 & C_1 \arrow[l, "s"] \arrow[r, "t"'] & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
& & C_1 \times_{C_0} C_1 \arrow[dl] \arrow[dr] \arrow[dd, bend right, "\mu"] & & \\
C_0 \arrow[d, equal] & C_1 \arrow[l, "s"'] \arrow[r, "t"] & C_0 & C_1 \arrow[l, "s"'] \arrow[r, "t"] & C_0 \arrow[d, equal] \\
C_0 & & C_1 \arrow[ll, "s"] \arrow[rr, "t"'] & & C_0
\end{tikzcd}$$

Unit $\longleftrightarrow$ identities for the category

multiplication $\longleftrightarrow$ composition for the category.

(2) Moeller and Vasilakopoulou used:

$\underline{Fib} \simeq \underline{ICat} \implies \underline{Ps\,Mon}(\underline{Fib}, \times) \simeq \underline{Ps\,Mon}(\underline{ICat}, \times)$

To generalize this further we need:

* double 2-categories (pseudo category objects in $\underline{2\text{-Cat}}$)

* pseudo monoids in double 2-categories

$\underline{\text{Result}}$: pseudo categories in a 2-cat^y $\mathcal{C}$ correspond to pseudo monoids in $\mathcal{S}pan(\mathcal{C})$.

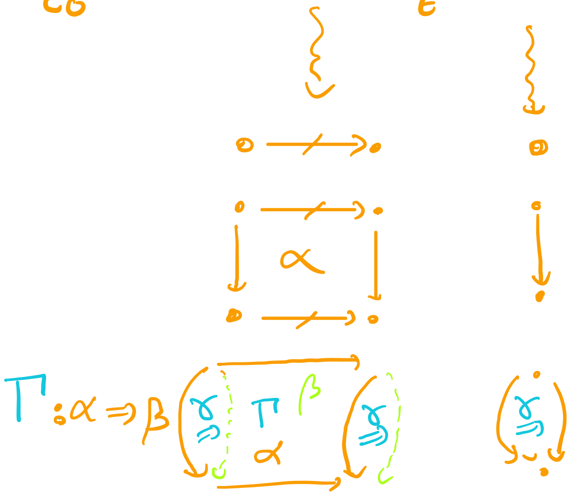
Recall that we want to take the source and target from a more restricted class of arrows, say Σ:

$\underline{\text{Result}}$: pseudo cat^s in $\mathcal{C}$ with s, t in Σ correspond to pseudo monoids in $\mathcal{S}pan_{\Sigma}(\mathcal{C})$.

A double 2-category is a pseudo cat^y in $\underline{2Cat}$

$$\mathcal{E}_1 \times_{\mathcal{E}_0} \mathcal{E}_1 \xrightarrow{\otimes} \mathcal{E}_1 \underset{t}{\overset{s}{\underset{y}{\rightleftarrows}}} \mathcal{E}_0$$

so we have



objects

arrows

2-cells

Objects categories $\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots$

Arrows functors

Proarrows spans $\mathcal{S} \xleftarrow{S} \mathcal{A} \xrightarrow{T} \mathcal{T}$

Dbl cells commutative diagrams

$$
\begin{tikzcd}
\mathcal{S} \arrow[d, "G"'] & \mathcal{A} \arrow[l, "S"'] \arrow[r, "T"] \arrow[d, "F"] & \mathcal{T} \arrow[d, "H"] \\
\mathcal{S}' & \mathcal{A}' \arrow[l, "S'"'] \arrow[r, "T'"] & \mathcal{T}'
\end{tikzcd}
$$

Dbl 3-cells

$$
\begin{tikzcd}
\mathcal{S} \arrow[d, "\mathcal{G}"'] & \mathcal{A} \arrow[l, "s"'] \arrow[r, "T"] \arrow[d, "F"] & \mathcal{J} \arrow[d, "H"] \\
\mathcal{X} & \mathcal{B} \arrow[l, "X"] \arrow[r, "Y"'] & \mathcal{Y}
\end{tikzcd}
\qquad (\gamma, \varphi, \theta) \qquad \Longrightarrow
$$

$$
\begin{tikzcd}
\mathcal{S} \arrow[d, "\mathcal{G}'"'] & \mathcal{A} \arrow[l, "s"'] \arrow[r, "T"] \arrow[d, "F'"] & \mathcal{J} \arrow[d, "H'"] \\
\mathcal{X} & \mathcal{B} \arrow[l, "X"] \arrow[r, "Y"'] & \mathcal{Y}
\end{tikzcd}
$$

Consists of

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathcal{S} \arrow[d, bend right, "\mathcal{G}"'] \arrow[d, bend left, dashed, "\mathcal{G}'"] \arrow[d, phantom, "\overset{\gamma}{\Rightarrow}"] & \mathcal{A} \arrow[l, "S"'] \arrow[r, "T"] \arrow[d, bend right, "F"'] \arrow[d, bend left, dashed, "F'"] \arrow[d, phantom, "\overset{\varphi}{\Rightarrow}"] & \mathcal{T} \arrow[d, bend right, "\mathcal{H}"'] \arrow[d, bend left, "\mathcal{H}'"] \arrow[d, phantom, "\overset{\theta}{\Rightarrow}"] \\
\mathcal{X} & \mathcal{B} \arrow[l, "X"] \arrow[r, "Y"'] & \mathcal{Y}
\end{tikzcd}$$

such that the two cylinders commute.

I $\mathbb{S}\mathrm{pan}(\underline{\mathrm{Cat}})$ is the slice category with

- <u>objects</u>: Contravariant lax double pseudo functors $F : \mathbb{A}^{op} \longrightarrow \mathbb{S}\mathrm{pan}(\underline{\mathrm{Cat}})$ ($\mathbb{A}$-indexed ps. dbl. categories)

- <u>morphisms</u>: $(H, \tau) : F \longrightarrow G$:

H lax dbl. pseudo functor

τ: lax dbl. ps. ntl. trafo

$$
\begin{tikzcd}
\mathbb{A}^{op} \arrow[rr, "H^{op}"] \arrow[dr, "F"'] & \arrow[d, phantom, "\overset{\Longrightarrow}{\tau}"] & \mathbb{B}^{op} \arrow[dl, "G"] \\
& \mathbb{S}\mathrm{pan}(\underline{\mathrm{Cat}}) &
\end{tikzcd}
$$

Definition A **lax double pseudo functor**

$$F : \mathbb{D} \longrightarrow \mathbb{E}$$

between dbl 2-categories $\mathbb{D}, \mathbb{E}$, consists of:

- Pseudo functors: $F_0 : \mathbb{D}_0 \longrightarrow \mathbb{E}_0$ (now 2-categories!)

  $F_1 : \mathbb{D}_1 \longrightarrow \mathbb{E}_1$ pseudo in the arrow direction!

- Comparison pseudo natural transformations:

$$\begin{tikzcd}
\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[r, "\otimes"] \arrow[d, "F_1 \times F_1"'] \arrow[dr, phantom, "\overset{\phi}{\Longrightarrow}"] & \mathbb{D}_1 \arrow[d, "F_1"] \\
\mathbb{E}_1 \times_{\mathbb{E}_0} \mathbb{E}_1 \arrow[r, "\otimes"'] & \mathbb{E}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}_0 \arrow[r, "y"] \arrow[d, "F_0"'] \arrow[dr, phantom, "\overset{\eta}{\Longrightarrow}"] & \mathbb{D}_1 \arrow[d, "F_1"] \\
\mathbb{E}_0 \arrow[r, "y"'] & \mathbb{E}_1
\end{tikzcd}$$

lax in the proarrow direction

- Invertible associativity and unitor modifications

$$
\begin{tikzcd}
 & \mathbb{D}_1^{(3)} \arrow[dl, "F_1^{(3)}"'] \arrow[dr, "\otimes\times 1"] \arrow[dd] & \\
\mathbb{E}_1^{(3)} \arrow[dd, "1\times\otimes"'] \arrow[r, phantom, "1\times\phi"] & & \mathbb{D}_1^{(2)} \arrow[dd, "\otimes"] \arrow[l, phantom, "\overset{\alpha}{\cong}"'] \\
 & \mathbb{D}_1^{(2)} \arrow[dl] \arrow[dr] & \\
\mathbb{E}^{(2)} \arrow[dr, "\otimes"'] \arrow[rr, phantom, "\overset{\phi}{\Rightarrow}"] & & \mathbb{D}_1 \arrow[dl, "F_1"] \\
 & \mathbb{E}_1 &
\end{tikzcd}
\quad \overset{\Phi}{\cong} \quad
\begin{tikzcd}
 & \mathbb{D}_1^{(3)} \arrow[dl, "F_1^{(3)}"'] \arrow[dr, "\otimes\times 1"] \arrow[dd, phantom, "\overset{\phi\times 1}{\Rightarrow}"] & \\
\mathbb{E}_1^{(3)} \arrow[dr] \arrow[dd, "1\times\otimes"'] & & \mathbb{D}_1^{(2)} \arrow[dl] \arrow[dd, "\otimes"] \\
 & \mathbb{E}_1^{(2)} \arrow[dd] & \\
\mathbb{E}_1^{(2)} \arrow[dr, "\otimes"'] \arrow[r, phantom, "\overset{\alpha}{\cong}"] & & \mathbb{D}_1 \arrow[dl, "F_1"] \arrow[l, phantom, "\overset{\phi}{\Rightarrow}"'] \\
 & \mathbb{E}_1 &
\end{tikzcd}
$$

etc.

satisfying well-definedness and coherence conditions.

A lax dbl ps. ntl transformation:

$$\tau : F \Rightarrow G : \mathbb{D} \rightrightarrows \mathbb{E}$$

consists of:

- pseudo ntl transformations

$$\tau_0 : F_0 \Rightarrow G_0, \quad \tau_1 : F_1 \Rightarrow G_1$$

(arrow-component transformations)

- modifications:

$$\begin{tikzcd}
\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[r, equal] \arrow[d, "F_1 \times_{F_0} F_1"'] & \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[r, "\otimes"] \arrow[d] \arrow[dl, phantom, "\overset{\tau_1 \times_{\tau_0} \tau_1}{\Rightarrow}"] & \mathbb{D}_1 \arrow[d, "G_1"] \\
\mathbb{E}_1 \times_{\mathbb{E}_0} \mathbb{E}_1 \arrow[r, equal] & \mathbb{E}_1 \times_{\mathbb{E}_0} \mathbb{E}_1 \arrow[r, "\otimes"] & \mathbb{E}_1
\end{tikzcd}
\quad \overset{\tau}{\Rrightarrow} \quad
\begin{tikzcd}
\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[r, "\otimes"] \arrow[d, "F_1 \times_{F_0} F_1"'] \arrow[dr, phantom, "\overset{\phi}{\Rightarrow}"] & \mathbb{D}_1 \arrow[r, equal] \arrow[d, "F_1"] \arrow[dr, phantom, "\overset{\tau_1}{\Rightarrow}"] & \mathbb{D}_1 \arrow[d, "G_1"] \\
\mathbb{E}_1 \times_{\mathbb{E}_0} \mathbb{E}_1 \arrow[r, "\otimes"'] & \mathbb{E}_1 \arrow[r, equal] & \mathbb{E}_1
\end{tikzcd}$$

and

$$
\begin{tikzcd}
\mathbb{D}_0 \arrow[r, equal] \arrow[d, "F_0"'] \arrow[dr, phantom, "\overset{\tau_0}{\Rightarrow}"] & \mathbb{D}_0 \arrow[r, "y"] \arrow[d] \arrow[dr, phantom, "\overset{2}{\Rightarrow}"] & \mathbb{D}_1 \arrow[d, "G_1"] \\
\mathbb{E}_0 \arrow[r, equal] & \mathbb{E}_0 \arrow[r, "y"'] & \mathbb{E}_1
\end{tikzcd}
\quad \overset{\mathrm{I}}{\Rrightarrow} \quad
\begin{tikzcd}
\mathbb{D}_0 \arrow[r, "y"] \arrow[d] \arrow[dr, phantom, "F_0 \overset{2}{\Rightarrow} F_1"] & \mathbb{D}_1 \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\overset{\tau_1}{\Rightarrow}"] & \mathbb{D}_1 \arrow[d, "G_1"] \\
\mathbb{E}_0 \arrow[r, "y"'] & \mathbb{E}_1 \arrow[r, equal] & \mathbb{E}_1
\end{tikzcd}
$$

satisfying multiplicativity and unitality conditions.

We write $\underline{\mathrm{Dbl2Cat}}(\mathbb{D}, \mathbb{E})$ for the cat^y of lax dbl pseudo functors and lax dbl. ps. ntt. transformations.

The Representation Theorem

Theorem (Cruttwell, Lambert, P., Szyld)

There is an equivalence of categories

$$\underline{\mathrm{DblFib}} \simeq \mathrm{I}\,\mathrm{Span}(\underline{\mathrm{Cat}})$$

Idea for the proof: use pseudo monoids in double 2-categories.

$$\underline{\mathrm{Fib}} \simeq \underline{\mathrm{ICat}} \rightsquigarrow \underline{\mathrm{cFib}} \simeq \underline{\mathrm{ICat}}_t$$

so: $\mathrm{Span}_c(\underline{\mathrm{Fib}}) \simeq \mathrm{Span}_t(\underline{\mathrm{ICat}})$

(Convince yourself!)

$\mathrm{Span}_c(\underline{\mathrm{Fib}})$ has objects:

$$\begin{tikzcd} \underline{E} \arrow[d, "p"] \\ \underline{B} \end{tikzcd}$$

fibrations with cleavage

Notation: p

$\mathrm{Span}_t(\underline{\mathrm{ICat}})$ has objects: $\underline{B}^{op} \xrightarrow{F} \underline{\mathrm{Cat}}$ ps. functor

F

arrows:

$$\begin{tikzcd} \underline{E} \arrow[r, "f^{\top}"] \arrow[d, "p"'] & \underline{E}' \arrow[d, "p'"] \\ \underline{B} \arrow[r, "f^{\perp}"'] & \underline{B}' \end{tikzcd}$$

cartesian arrow preserving

$$\begin{tikzcd} p \arrow[d, "f"] \\ p' \end{tikzcd}$$

arrows:

$$\begin{tikzcd} \underline{B}^{op} \arrow[rr, "H"] \arrow[dr, "F"'] & \arrow[d, phantom, "\overset{\theta}{\Rightarrow}"] & (\underline{B}')^{op} \arrow[dl, "F'"] \\ & \underline{\mathrm{Cat}} & \end{tikzcd}$$

θ pseudo trafo

$$\begin{tikzcd} F \arrow[d, "{(H,\theta)}"] \\ F' \end{tikzcd}$$

Proarrows: $P \xleftarrow{\ell} Q \xrightarrow{r} R$ $\quad \ell^T, r^T$ cleavage preserving

Proarrows: $F \xleftarrow{(L,\lambda)} \mathcal{G} \xrightarrow{(R,\rho)} K$ $\quad \lambda, \rho$ strict natural transf^{ns}

Cells:

$$\begin{tikzcd}
P \arrow[d, "f"'] & Q \arrow[l, "\ell"'] \arrow[r, "r"] \arrow[d, "g"] & R \arrow[d, "h"] \\
P' & Q' \arrow[l, "\ell'"'] \arrow[r, "r'"] & R'
\end{tikzcd}$$

$\ell^T, r^T, \ell'^T, r'^T$ cleavage preserving

f^T, g^T, h^T Cartesian arrow pres.

Cells:

$$\begin{tikzcd}
F \arrow[d, "{(H,\theta)}"'] & \mathcal{G} \arrow[l, "{(L,\lambda)}"'] \arrow[r, "{(R,\rho)}"] \arrow[d] & K \arrow[d] \\
F' & \mathcal{G}' \arrow[l, "{(L',\lambda')}"] \arrow[r, "{(R',\rho')}"'] & K'
\end{tikzcd}$$

Then lift:

$$\underline{\mathrm{DblFib}} := \underline{\mathrm{PsMon}}\left(\mathrm{Span}_c(\underline{\mathrm{Fib}})\right) \simeq$$

$$\simeq \underline{\mathrm{PsMon}}\left(\mathrm{Span}_t(\mathrm{I}\underline{\mathrm{Cat}})\right) \simeq \mathrm{ISpan}(\underline{\mathrm{Cat}}).$$

$$
\begin{tikzcd}
\text{Monoidal Fibrations} \arrow[dr, "\text{monoids are cats}"'] & & \text{2-Fibrations} \arrow[dl, "\text{quintet constr.}"] \\
& \text{Double Fibrations} & \\
\text{Discrete Dbl Fib.} \arrow[ur] & & \text{Double Gr. Constr.} \arrow[ul]
\end{tikzcd}
$$

All of these have an indexed notion and a Grothendieck category of elements.

In our case the composition of the functors used in the proof gives us also a category of elements construction.

The Double Category of Elements

Start with $\mathcal{F} : \mathbb{D}^{op} \longrightarrow \mathbb{S}\mathrm{pan}(\underline{\mathrm{Cat}})$:

$$\mathcal{F}_0 : \mathbb{D}_0^{op} \longrightarrow \mathbb{S}\mathrm{pan}(\underline{\mathrm{Cat}})_0 = \underline{\mathrm{Cat}} \qquad (1)$$

$$\mathcal{F}_1 : \mathbb{D}_1^{op} \longrightarrow \mathbb{S}\mathrm{pan}(\underline{\mathrm{Cat}})_1$$

and a further induced functor :

$$\mathbb{D}_1^{op} \xrightarrow{\mathcal{F}_1} \mathbb{S}\mathrm{pan}(\underline{\mathrm{Cat}})_1 \xrightarrow{\mathrm{apx}} \underline{\mathrm{Cat}} \qquad (2)$$

Apply the ordinary elements construction to (1) and (2):

$$\mathbb{E}\mathrm{l}(\mathcal{F})_0 \longrightarrow \mathbb{D}_0 \qquad \mathbb{E}\mathrm{l}(\mathcal{F})_1 \longrightarrow \mathbb{D}_1$$

cloven fibrations.

Some notation:

- for $m : A \xrightarrow{\mapstochar} B$ in $\mathbb{D}$, write Fm in $\mathrm{Span}(\underline{\mathrm{Cat}})$ as:

$$\begin{tikzcd}
& Fm \arrow[dl, "Lm"'] \arrow[dr, "Rm"] & \\
FA & & FB
\end{tikzcd}$$

- for $A \xrightarrow[\mapstochar]{m} B \xrightarrow[\mapstochar]{n} C$ in $\mathbb{D}$, we have the laxity comparison cell

$$\begin{tikzcd}
FA \arrow[d, equal] & Fm \times_{FB} Fn \arrow[l] \arrow[r] \arrow[d, "\phi_{m,n}"] & FC \arrow[d, equal] \\
FA & F(m \otimes n) \arrow[l, "L_{m\otimes n}"'] \arrow[r, "R_{m\otimes n}"] & FC
\end{tikzcd}$$

Now $\mathbb{E}l(F)$ is given by:

- objects: (C,x) $\quad C$ in $\mathbb{D}$, x in FC.

- arrows: $(f,\bar{f}) : (C,x) \longrightarrow (D,y)$

  with $f : C \to D$ in $\mathbb{D}$

  and $\bar{f} : x \longrightarrow f^* y \; (= F(f)(y))$ in FC.

- proarrows: $(m,\bar{m}) : (C,x) \xrightarrow{\;\;|\;\;} (D,y)$

  with $C \xrightarrow[\;|\;]{m} D$ in $\mathbb{D}$

  $\bar{m} \in Fm \quad \left(FC \xleftarrow{L_m} Fm \xrightarrow{R_m} FD \right)$

  s.t. $L_m(\bar{m}) = x$

  $R_m(\bar{m}) = y$

- double cells :

$$\begin{tikzcd}
(A,x) \arrow[r, "{(m,\bar{m})}", "|" marking] \arrow[d, "{(f,\bar{f})}"'] \arrow[dr, phantom, "{(\theta,\bar{\theta})}"] & (B,y) \arrow[d, "{(g,\bar{g})}"] \\
(C,z) \arrow[r, "{(n,\bar{n})}"', "|" marking] & (D,w)
\end{tikzcd}$$

with

$$\begin{tikzcd}
A \arrow[r, "m", "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\theta"] & B \arrow[d, "g"] \\
C \arrow[r, "n"', "|" marking] & D
\end{tikzcd}$$

in $\mathbb{D}$

$$\begin{tikzcd}
FA \arrow[d, "Ff"'] & Fm \arrow[l, "L_m"'] \arrow[r, "R_m"] \arrow[d, "F\theta"] & FB \arrow[d, "Fg"] \\
FC & Fn \arrow[l, "L_n"] \arrow[r, "R_n"'] & FD
\end{tikzcd}$$

and $\bar{m} \xrightarrow{\bar{\theta}} \theta^{*}\bar{n}$ an arrow in $\mathcal{F}m$ s.t.

$$L_m(\bar{\theta}) = \bar{f} \quad \text{and} \quad R_m(\bar{\theta}) = \bar{g}.$$

Composition in the "arrow direction" is as expected:

- for arrows $(A,x) \xrightarrow{(f,\bar{f})} (B,y) \xrightarrow{(g,\bar{g})} (C,z)$

  the composite is $\left(gf, \phi_{f,g}\, f^*(\bar{g})\,\bar{f}\right) : (A,x) \longrightarrow (C,z)$

- for cells $(m,\bar{m}) \xRightarrow{(\theta,\bar{\theta})} (n,\bar{n}) \xRightarrow{(\delta,\bar{\delta})} (p,\bar{p})$

  the composite is

  $$\left(\delta\theta, \phi_{\theta,\delta}\, \theta^*(\bar{\delta})\bar{\theta}\right) : (m,\bar{m}) \Longrightarrow (p,\bar{p}).$$

- Units: $\left(1_C, (\varphi_C)_x\right) : (C,x) \longrightarrow (C,x)$

  $\left(1_m, (\varphi_m)_{\bar{m}}\right) : (m,\bar{m}) \longrightarrow (m,\bar{m}).$

$\mathbb{E}l(F)_1 \overset{s}{\underset{t}{\rightrightarrows}} \mathbb{E}l(F)_0$ are defined by

$$s(\theta, \bar{\theta}) = (f, \bar{f})$$

$$t(\theta, \bar{\theta}) = (g, \bar{g})$$

- proarrow composition:

  for $(A, x) \xrightarrow{(m, \bar{m})} (B, y) \xrightarrow{(n, \bar{n})} (C, z)$

  the composite is:

  $$(m \otimes n, \phi_{m,n}(\bar{m}, \bar{n})) : (A, x) \longrightarrow (C, z)$$

- Composition for cells and proarrow units are given using appropriate components of the structure isos related to pseudo naturality.

- $\pi : \mathbb{E}l(F) \longrightarrow \mathbb{D}$ is a double fibration.
- This is the object part of an equivalence of categories

$$\underline{\mathrm{DblFib}} \simeq \mathbb{I}\,\mathrm{Span}(\underline{\mathrm{Cat}})$$

which specializes to

$$\underline{\mathrm{DblFib}}(\mathbb{B}) \simeq \underline{\mathrm{Dbl2Cat}}(\mathbb{B}^{op}, \mathrm{Span}(\underline{\mathrm{Cat}}))$$

for each dbl cat^{y} $\mathbb{B}$.

- Let $\underline{A}$ be a category with pushouts and $\mathbb{C}\mathrm{sp}(\underline{A})$ the double cat^y with cells:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "u"'] & Z \arrow[d, "w"] & Y \arrow[l] \arrow[d, "v"] \\ X' \arrow[r] & Z' & Y' \arrow[l] \end{tikzcd}$$

- For any lax double functor

$$F : \mathbb{C}\mathrm{sp}(\underline{A}) \to \mathbb{S}\mathrm{pan}(\underline{\mathrm{Cat}})$$

the dbl cat^y of elements $\mathbb{E}l(F)$ is F-$\mathbb{C}\mathrm{sp}$, the double category of F-decorated cospans. [Patterson, 2023].

This slightly generalizes the decorated cospans from Baez-Courser-Vasilakopoulou.