Introduction to Petri Nets

Nathaniel Osgood

Compositional Methods for Modeling Health & Infectious Disease 2022

Agenda

- Grammar
- Examples
- Petri Nets as C-Sets
- Composition
- Semantics

Petri Net Grammar

- Set of **Places** (Circles)
- Set of **Transitions** (Rectangles)
- For each **transition**
  - **Inputs**: For each place, a count of resources *required for* this transition
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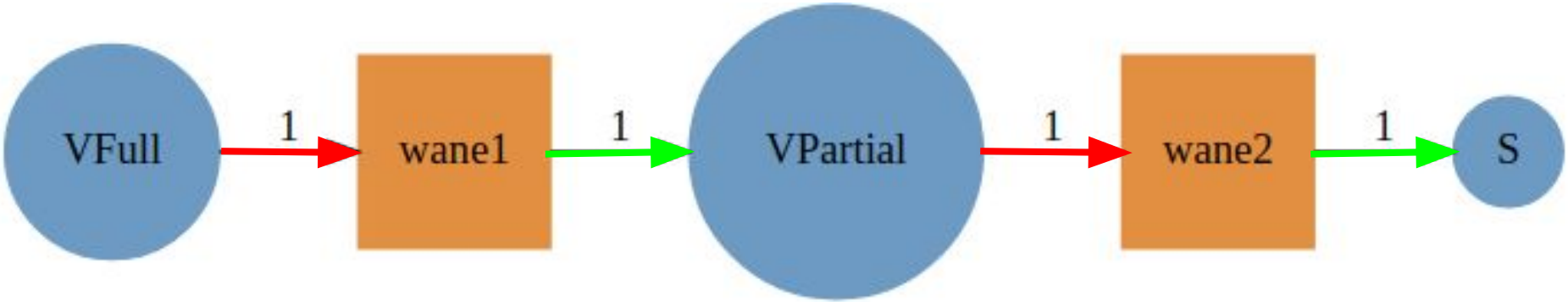

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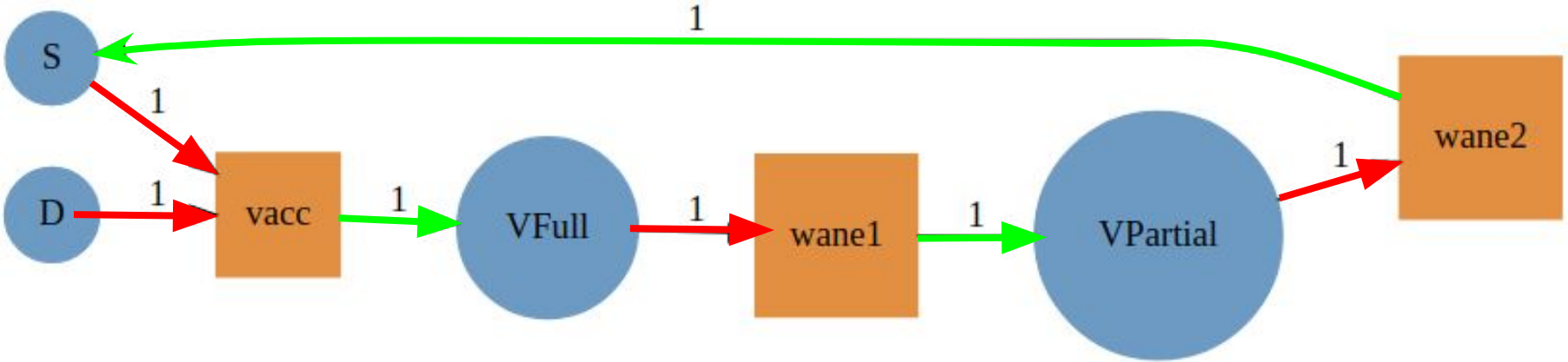

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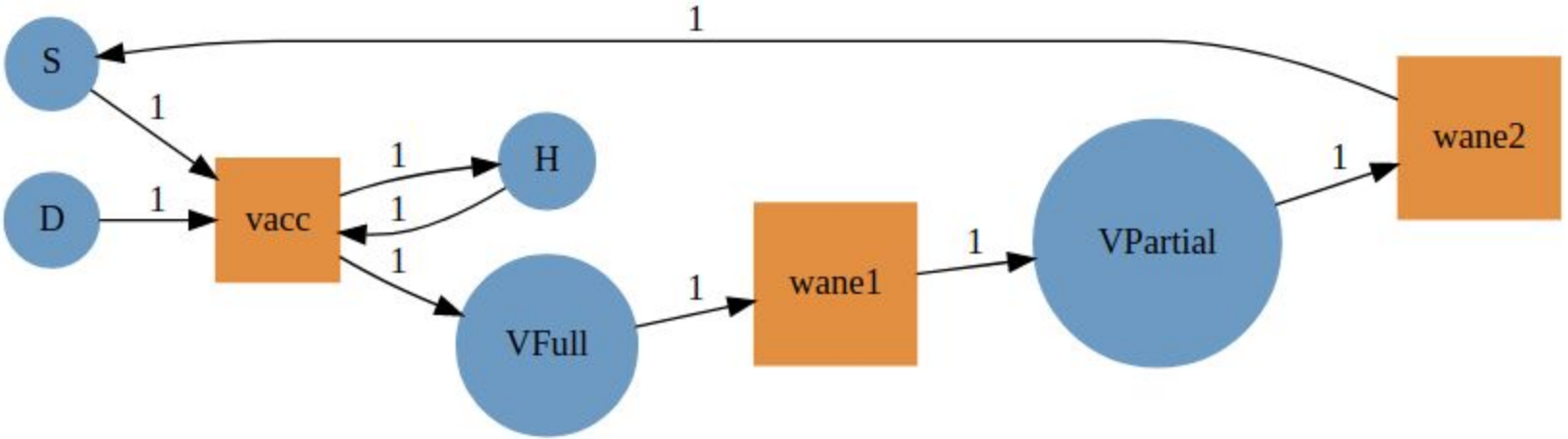

SIR Example

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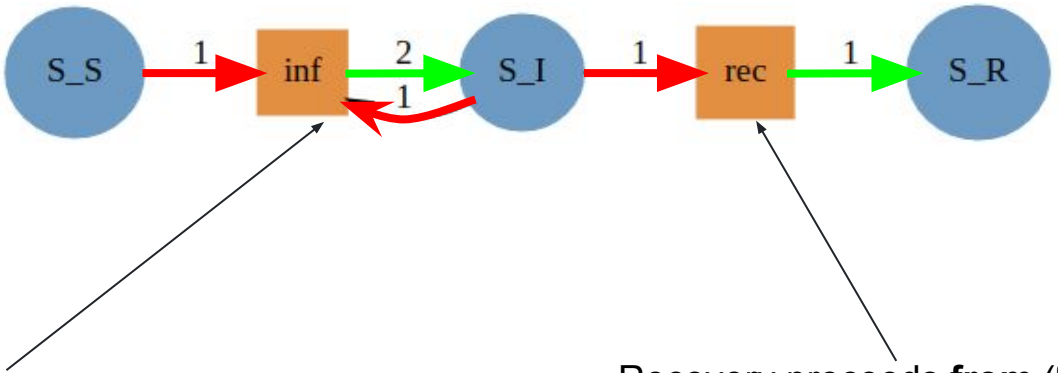

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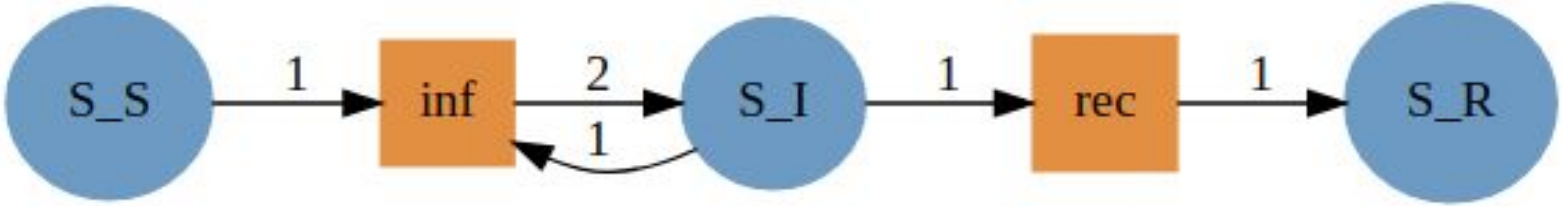
Elaborating our SIR for Waning of Immunity

How could we elaborate the diagram below to represent waning of immunity?



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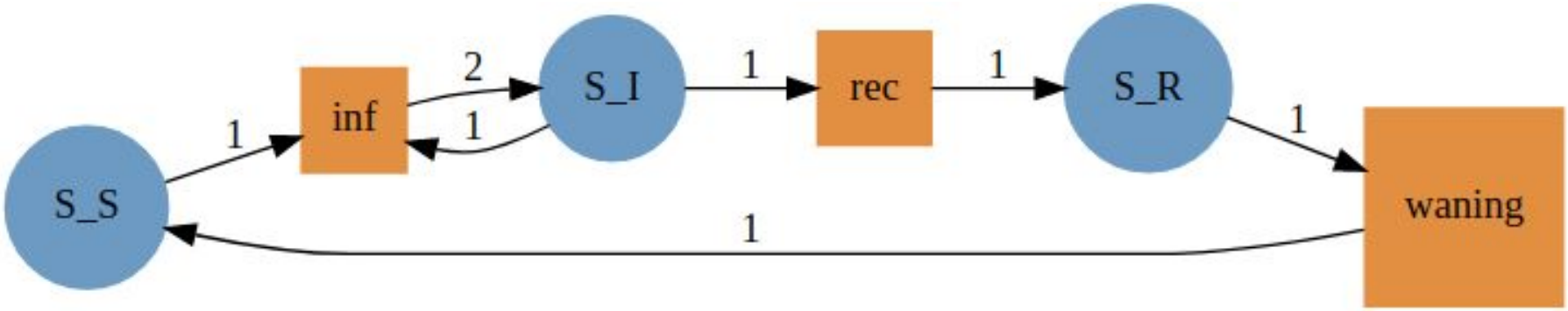

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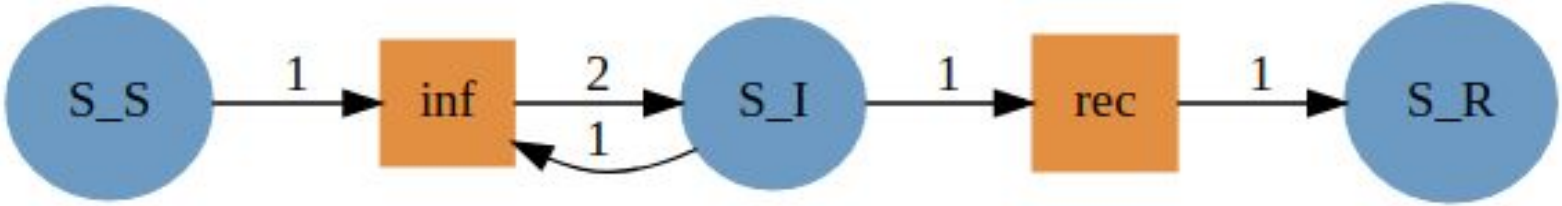

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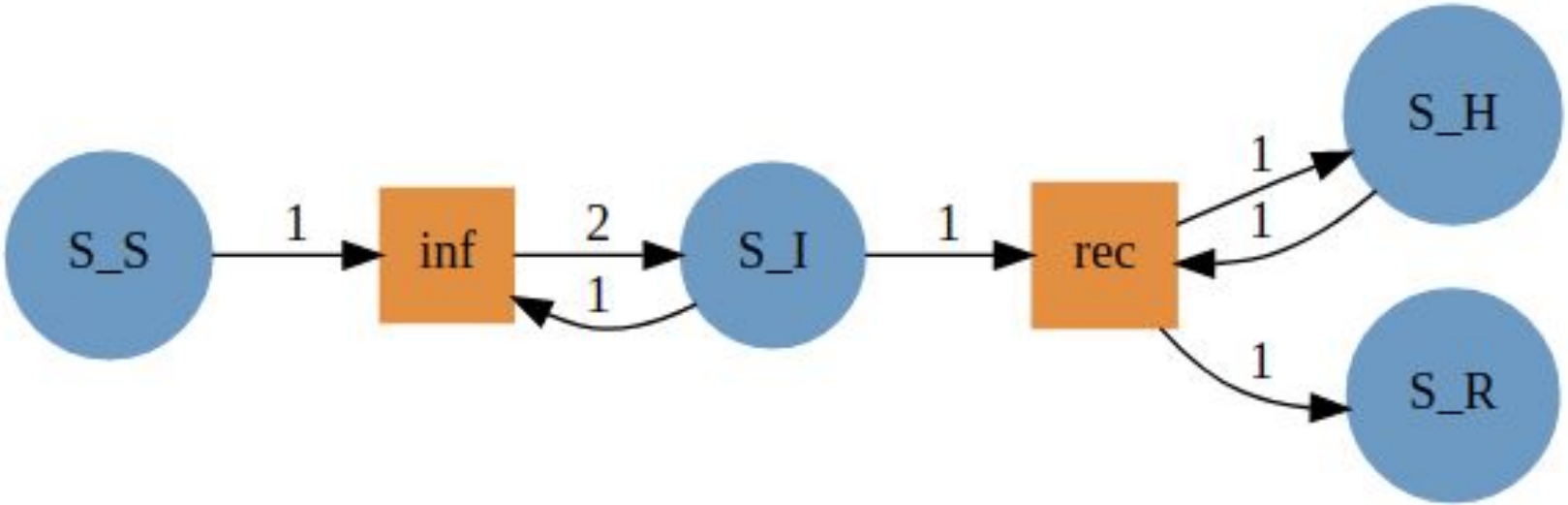

Elaborating our SIR for a Latent Stage of Infection

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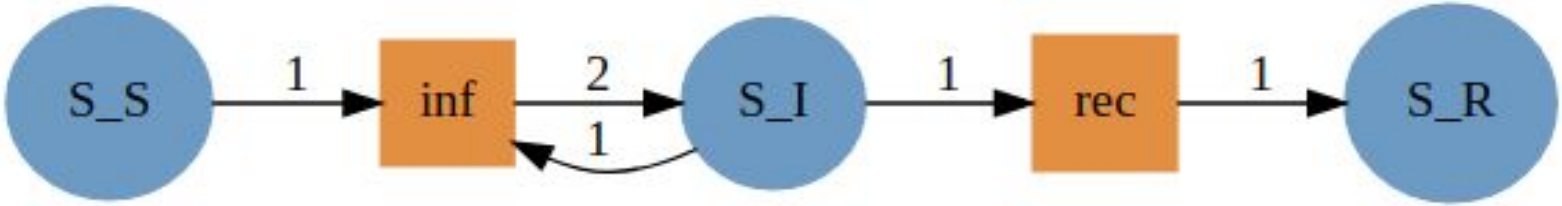

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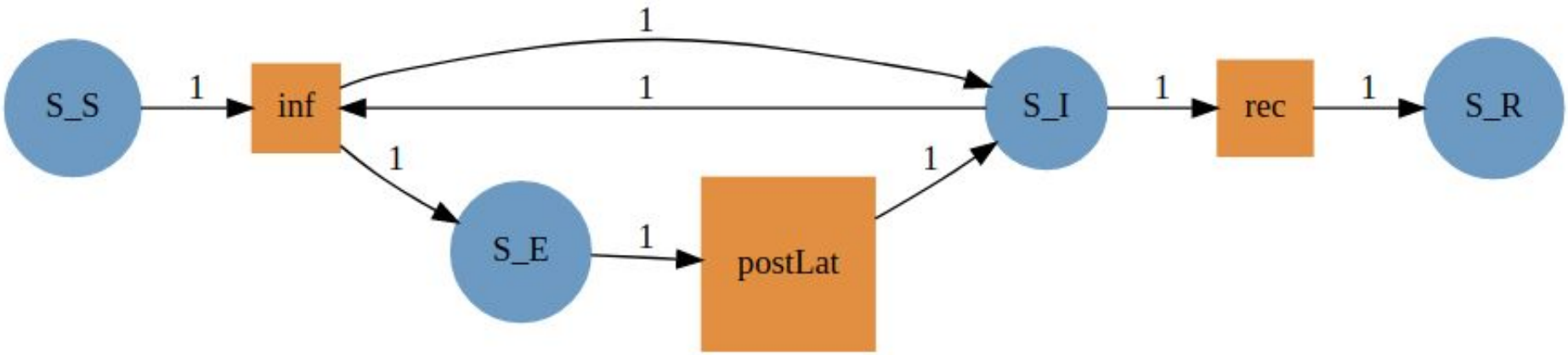

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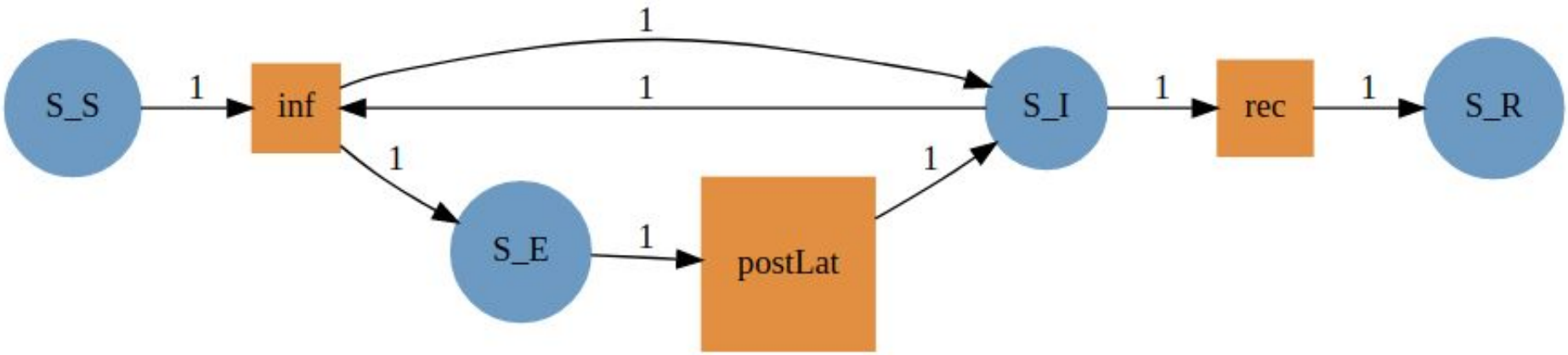

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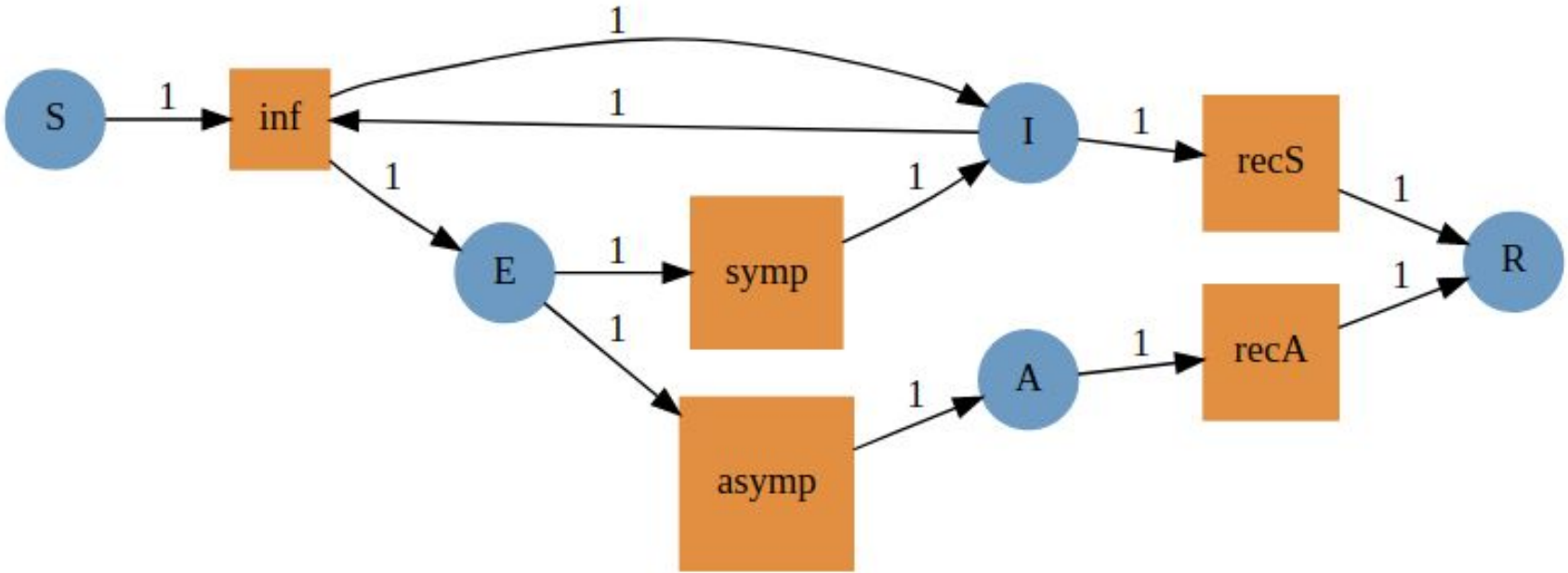


Models as Data: Schema for Petri Nets

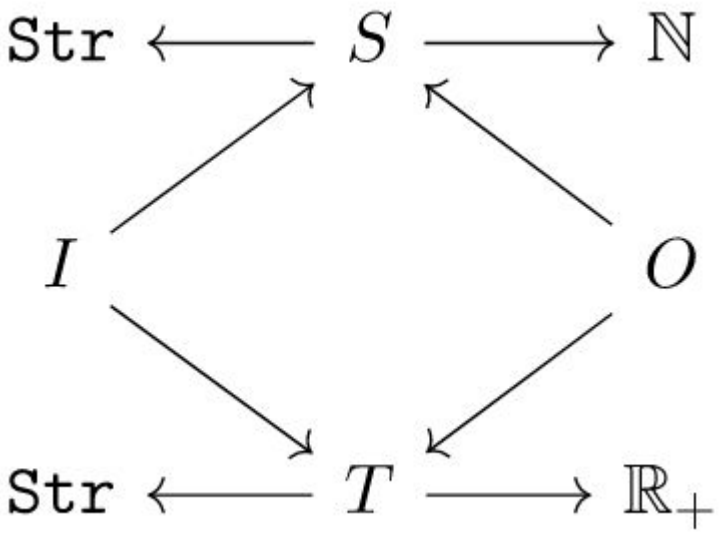
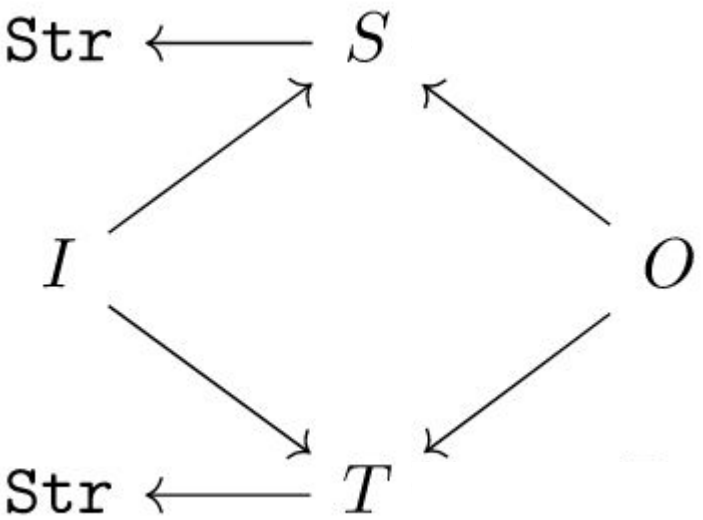


Image from: Halter, M. & Patterson E.,https://www.algebraicjulia.org/blog/post/2020/10/structured-cospans

Models as Data: Sub schema for the Moment



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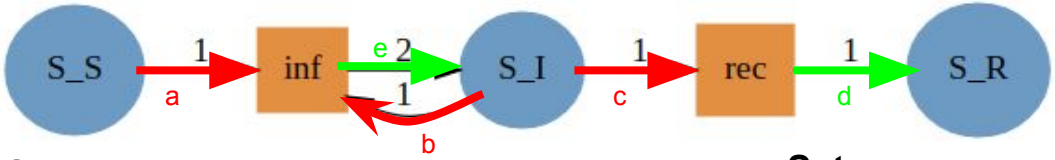
Models as Data Example: SIR

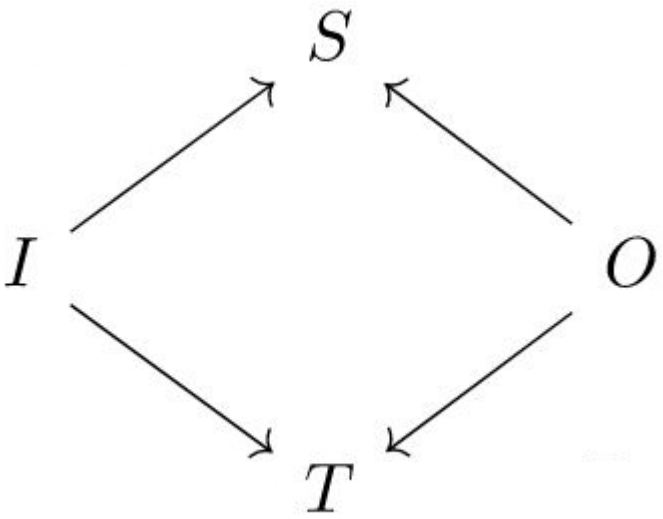


Petri Schema

Set

Inputs *to* a transition (from a state)

ansition (to a state)



{ S_S, S_I, S_R }

os(d)=rec, ot(d)=S_R
os(e)=inf, ot(e)=S_I

is

ot

{ a, b, c }

{ d, e }

os

it

is(a)=S_S, it(a)=inf
is(b)=S_I, it(b)=inf
is(c)=S_I, it(c)=rec

{ inf, rec }

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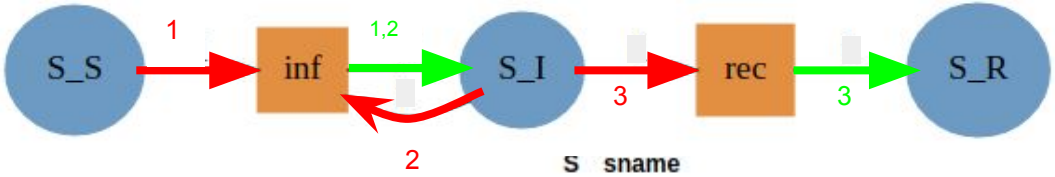

S	sname
1	S_S
2	S_I
3	S_R

I	it	is
1	1	1
2	1	2
3	2	2

O	ot	os
1	1	2
2	1	2
3	2	3

T	tname
1	inf
2	rec

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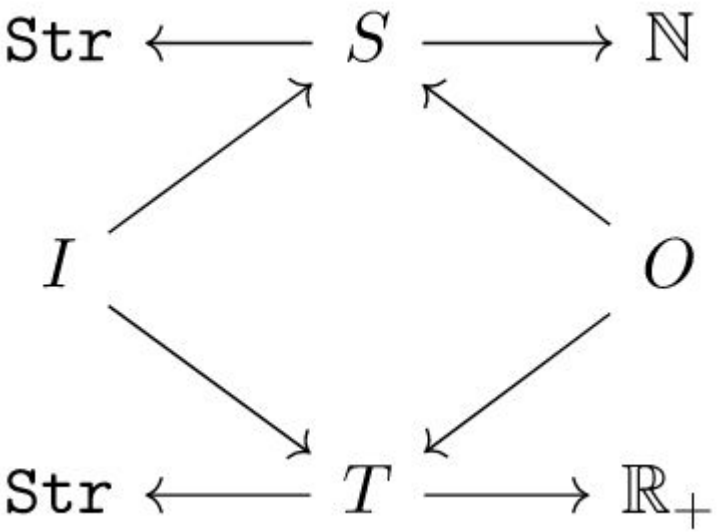
Schema for Petri Nets with Rates



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Recall: Original Elements of Petri Net

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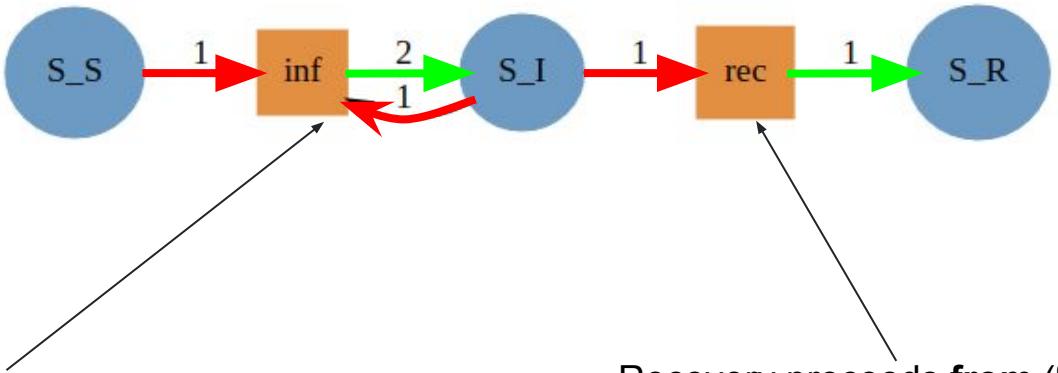

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Rate-Based Petri Net Grammar

- Set of **Places** (Circles), each with a name & initial quantity
- Set of **Transitions** (Rectangles)
- For each **transition**
  - A non-negative real number rate
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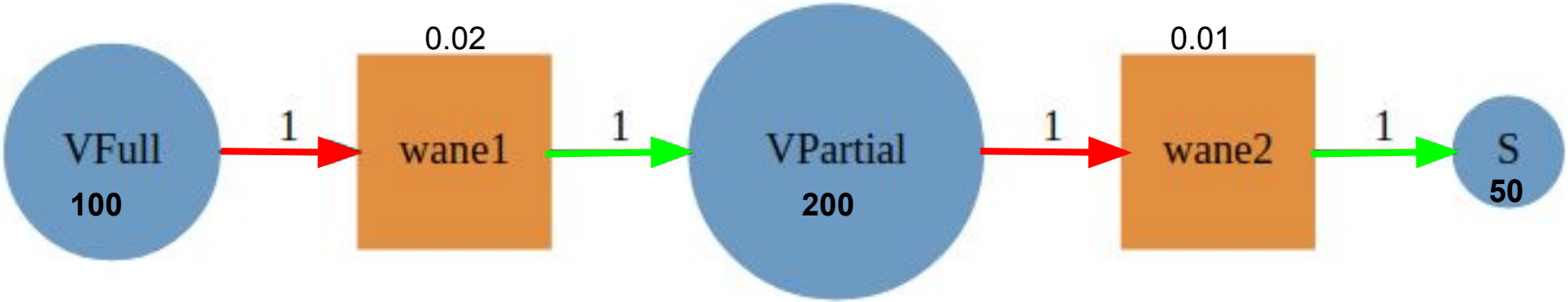

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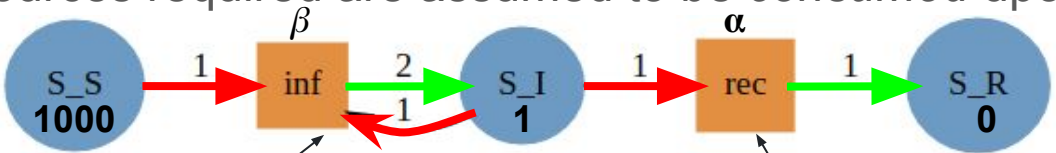

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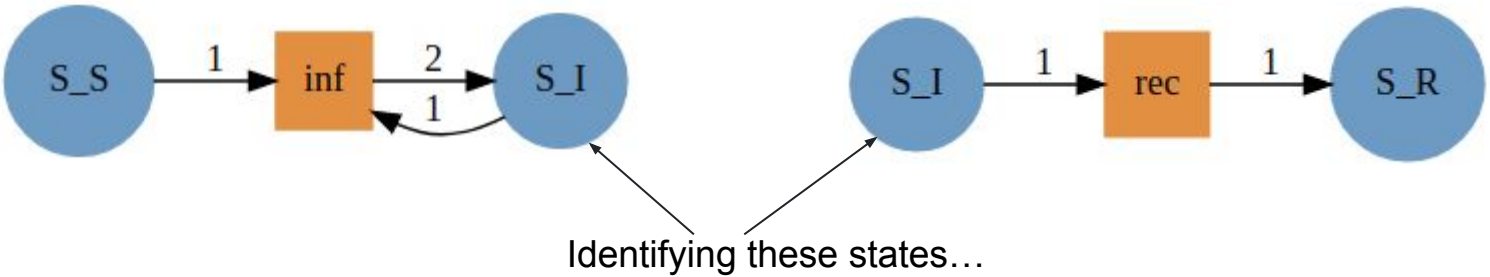
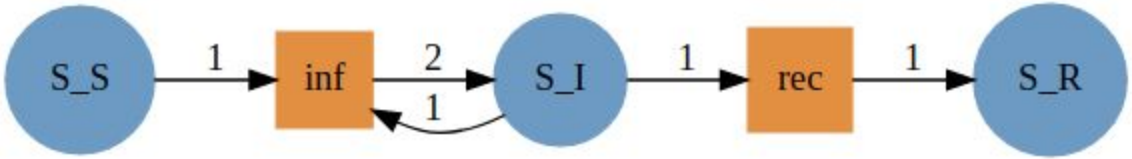
Composition: Gluing Petri Nets forms a Petri Net



yields the Petri Net below



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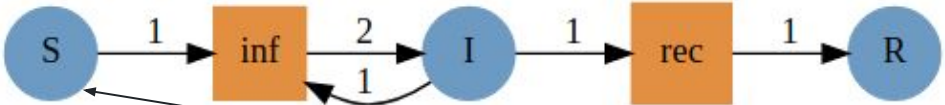
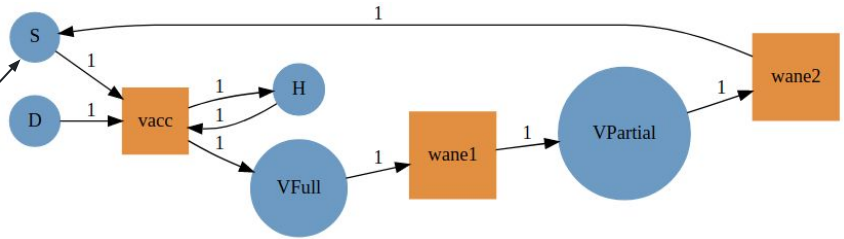
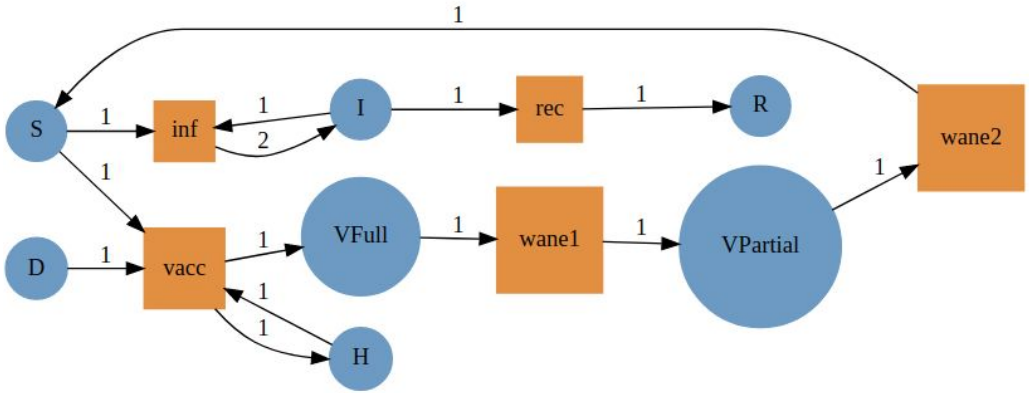
Infection Spread



Vaccination & Waning of Immunity



Identifying these states…
yields the Petri Net below



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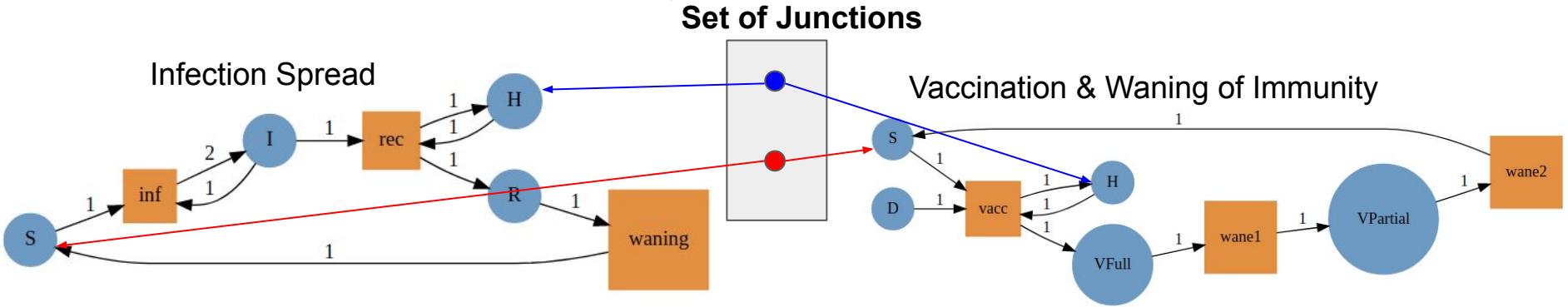
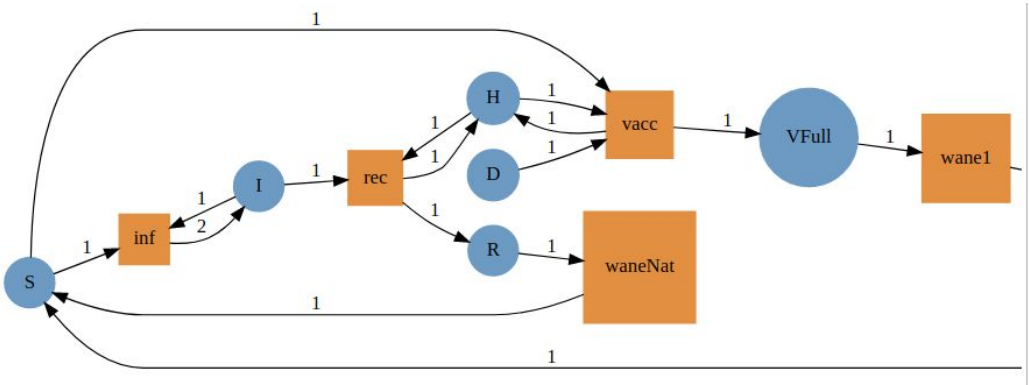


Identifying these states by connecting them to each other via the respective junction yields the Petri Net below



Semantics of Petri Nets

- Petri Nets offer multiple possible interpretations
- These different “meanings” of a Petri Net are characterized by semantics
- Some semantics relate to dynamic behaviour -- behaviour over time, e.g.,
  - Discrete time & state “marking game” semantics
  - Continuous time & state ordinary differential equation semantics
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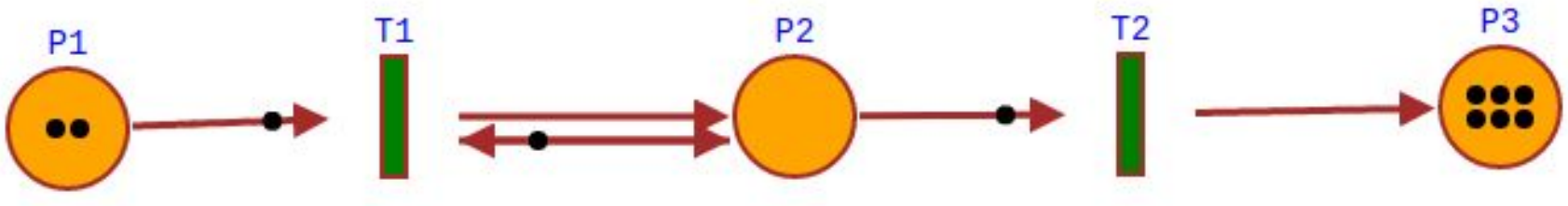
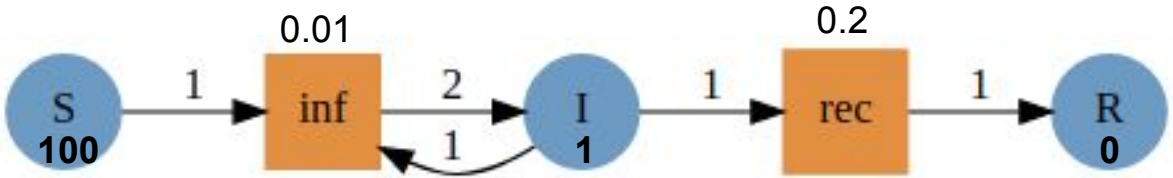
Classic “Token Game” Semantics

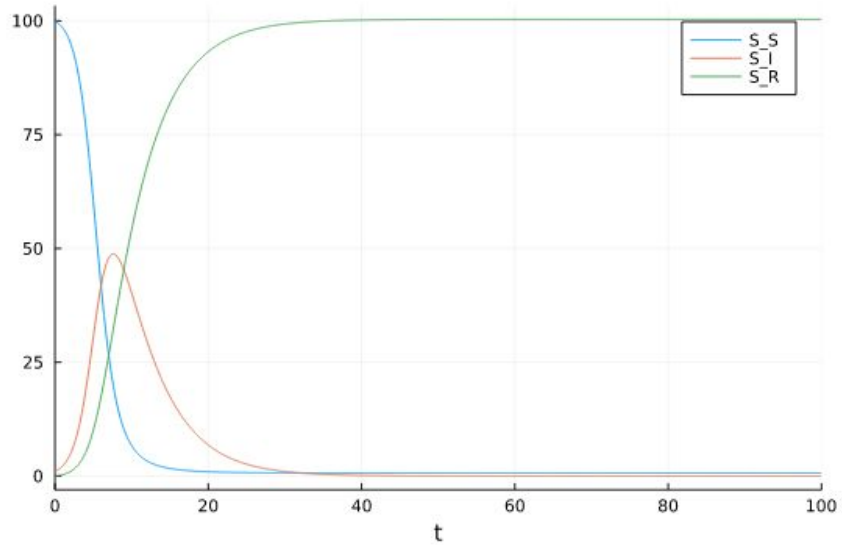


You can build very simple Petri Nets & run the token game at
https://apo.adrian-jagusch.de/
Alternative with different feature set: http://petri.hp102.ru/pnet.html

Continuous Time & State & Mass Action Semantics

- Interpreting the same Petri Net form in continuous time & continuous space assuming the law of mass action, we get ordinary differential equations





You can build both simple & sophisticated Petri Nets & run them with this semantics in AlgebraicPetri -- taught in this bootcamp

https://algebraicjulia.github.io/AlgebraicPetri.jl/dev/examples/covid/epidemiology/

https://www.algebraicjulia.org/blog/post/2020/10/structured-cospans/

Take Home Points

- Petri Nets are valuable for characterizing resource-constrained processes
- Petri Nets are characterized by a bipartite multigraph between places and transitions
- **Visual Models:** Petri Nets can be described visually
- **Model form as Data:** Petri Nets can be described as data -- and thus analyzed & transformed via programs
- **Composition**: We can combine smaller Petri Nets into a larger Petri Net
- **Separation of syntax & semantics:** In Petri Nets & other models, it is valuable to consider a variety of semantics for the same given model form