Brief Introduction to Stock & Flow Diagrams

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Compositional Methods for Modeling Health & Infectious Disease 2022

Syntax

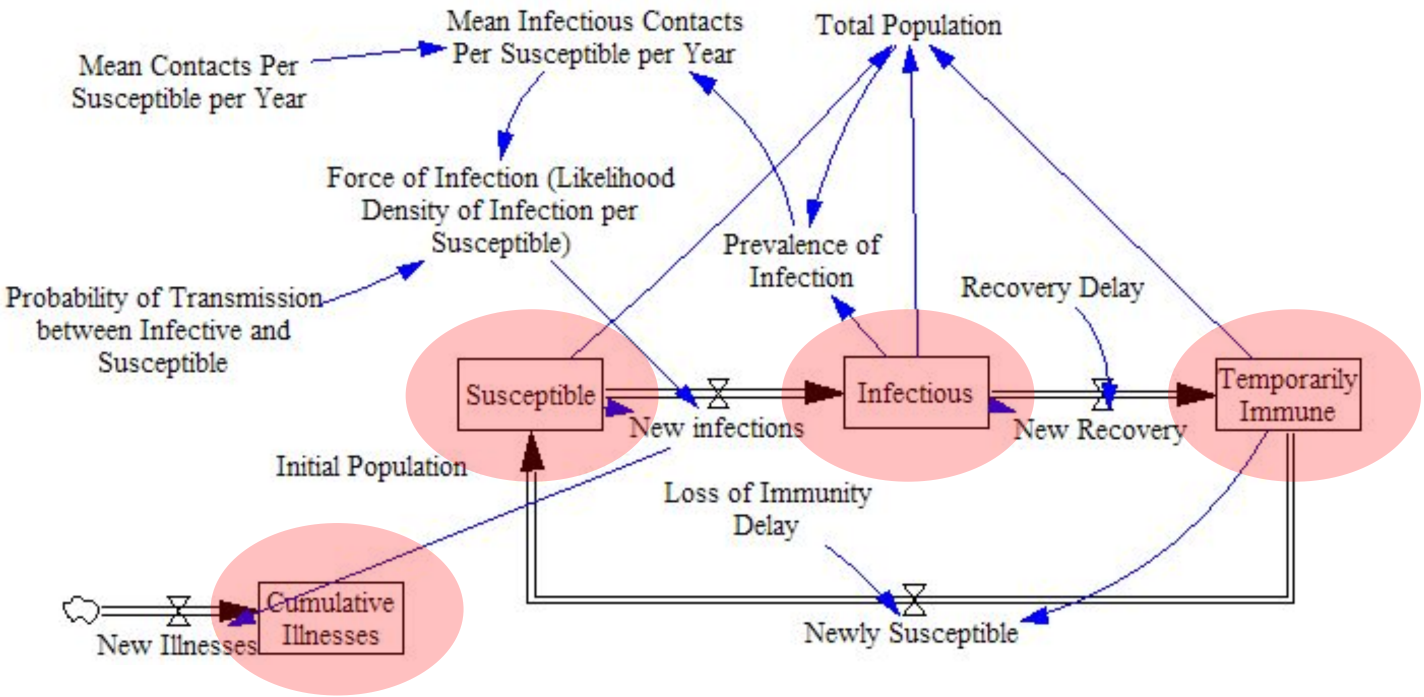
State of the System: Stocks
(“Levels”, “State Variables”, “Compartments”)

- State Variables represent accumulations
  - These capture the “state of the system”
- These can be measured at *one instant in time*
- State variables start with some initial value & are thereafter changed only by *flows* into & out of them
  - There are no inputs that immediately change stocks
- State variables are the source of delay in a system
- In a stock & flow diagram, shown as ***rectangles***
- ***Dimension:*** *Each state variable is associated with a dimension (e.g., Persons, Doses, $, Deaths)*

Examples of State Variables/Stocks

- Water in a bathtub (litres/minute)
- (Count of) {Susceptible, infective, immune} people
- (Count of) Healthcare workers
- (Count of) Cumulative { infections, deaths, $, tests administered}
- (Count of) Stockpiled doses of vaccine

Example Model: Stocks/State Variables



The Critical Role of Stocks in Dynamics

- Stocks determine current state of system
  - Stocks often provide the basis for making choices
- Stocks central to most disequilibria phenomena (buildup, decay)
- Lead to inertia
- Give rise to delays

State Changes: Flows

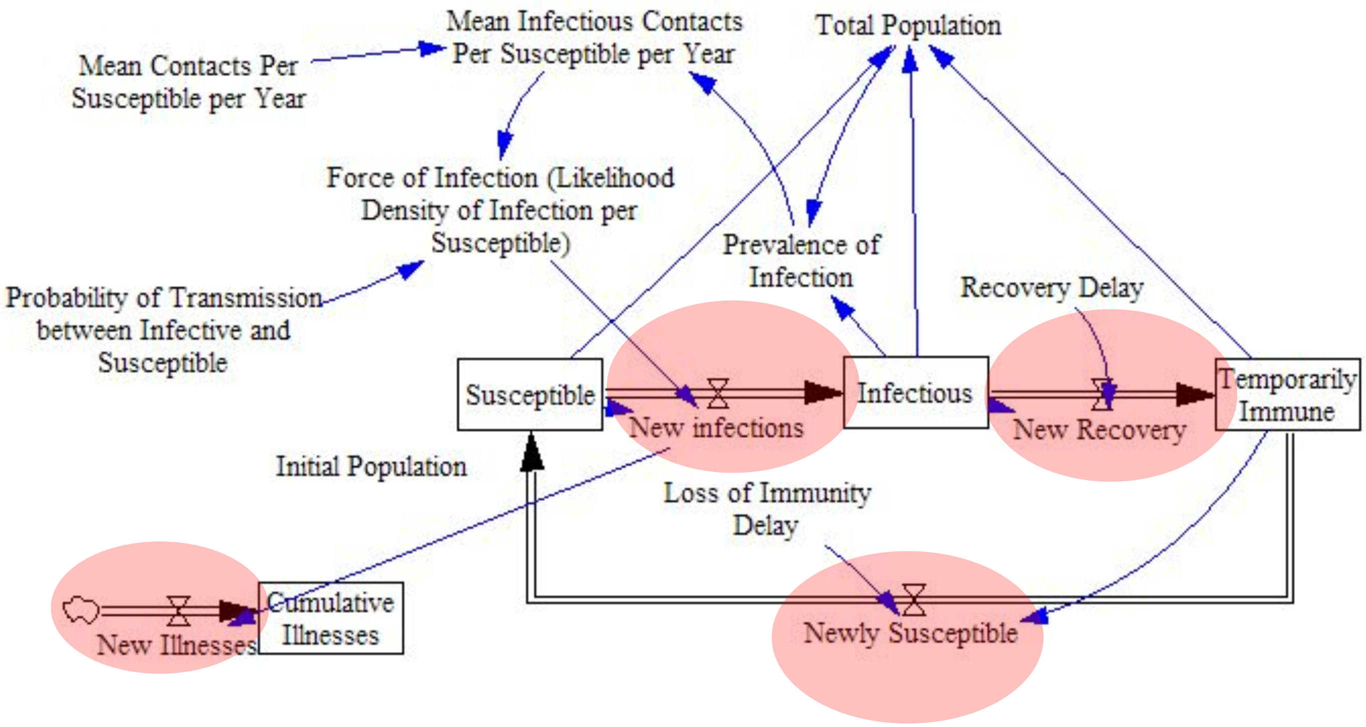
(“Fluxes”, “Transitions”, “Rates”, “Derivatives”, “Differentials”

- All changes to state variables/stocks occur via *flows*
- **Dimension:** Always expressed as **quantity per unit time**: If these flow into/out of a state variables/stock that keeps track of things of type *X (*e.g., persons), the rates are measured in *X*/(Time Unit) (e.g. persons/year, $/month, persons/day)
  - NB: Two state variables (stocks) connected by a given flow must have identical dimensions
- Typically measure over certain period of time (by considering accumulated quantity over a period of time) e.g.,
  - Yearly incidence is calculated by accumulating people over a year
  - Monthly death rates is calculated by accumulating deaths over a month
  - Can be estimated using such accumulations for any point in time

Examples of Flows

- Inflow or outflow of a bathtub (litres/minute)
- Incident cases (e.g., people/month)
- Recovery
- Vaccine administrations
- Mortality (e.g., people/year)
- Treatment (e.g., people/day)
- Rate of births (e.g., babies/year)
- Rate of caloric consumption (kcal/day)
- Costs
- Reactivation Rate (# of TB cases reactivating per unit time)

Example Model: Flows



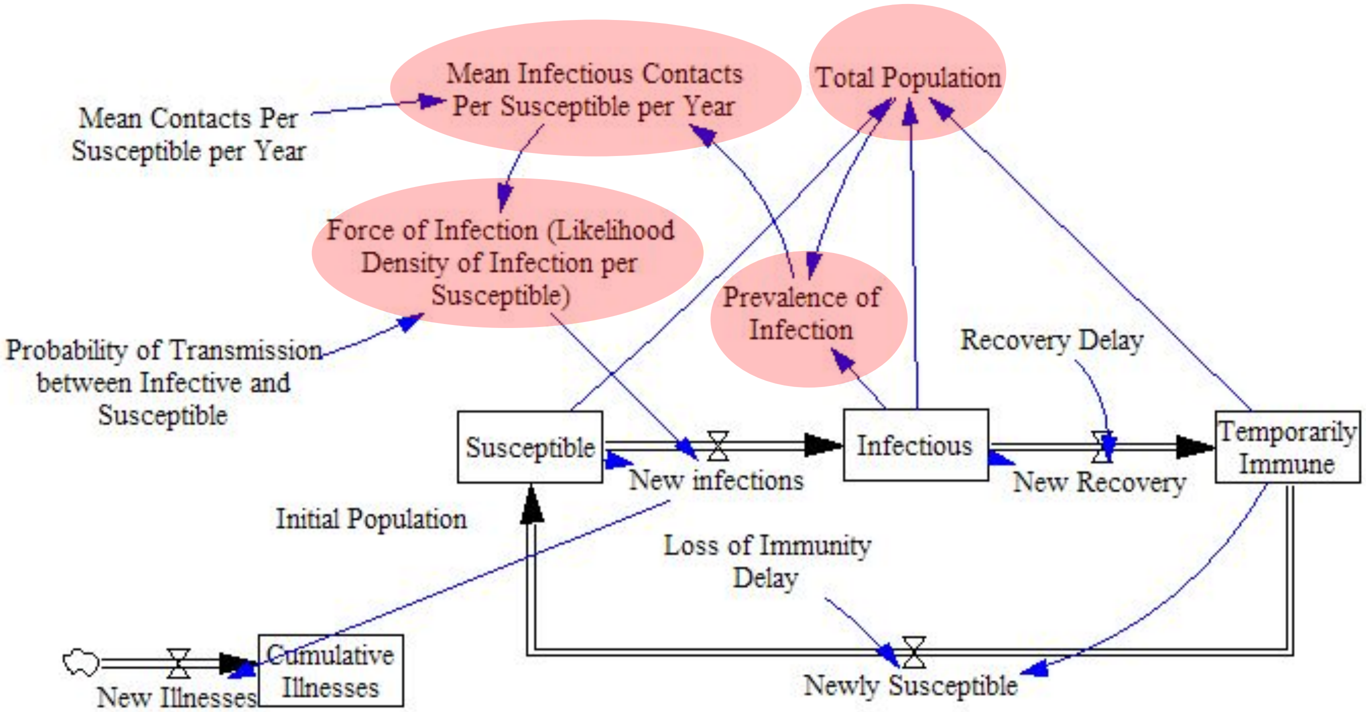
Formulas of Flows

- The value of each flow in a stock & flow model is associated with a formula
- At a mathematical level, the formula is given by a function that can depend on system state (values of stocks & [acyclically] other variables in the models)

Auxiliary (Dynamic) Variables

- Auxiliary (Dynamic) variables are convenience names we give to concepts that can be defined in terms of expressions involving stocks/flows at current time
  - Adding or eliminating an auxiliary variable does not change the mathematical structure of the system
- Elevate model transparency
  - Can be reused at many places
  - References to auxiliary variables prevents need for modeler to think about all of details of definition
- Enhance modifiability: Single place to define
- Convenient for reporting (graphing, tables) & analyzing model dynamics
- Like flows, auxiliary variables are associated with a *formula* depending on model state

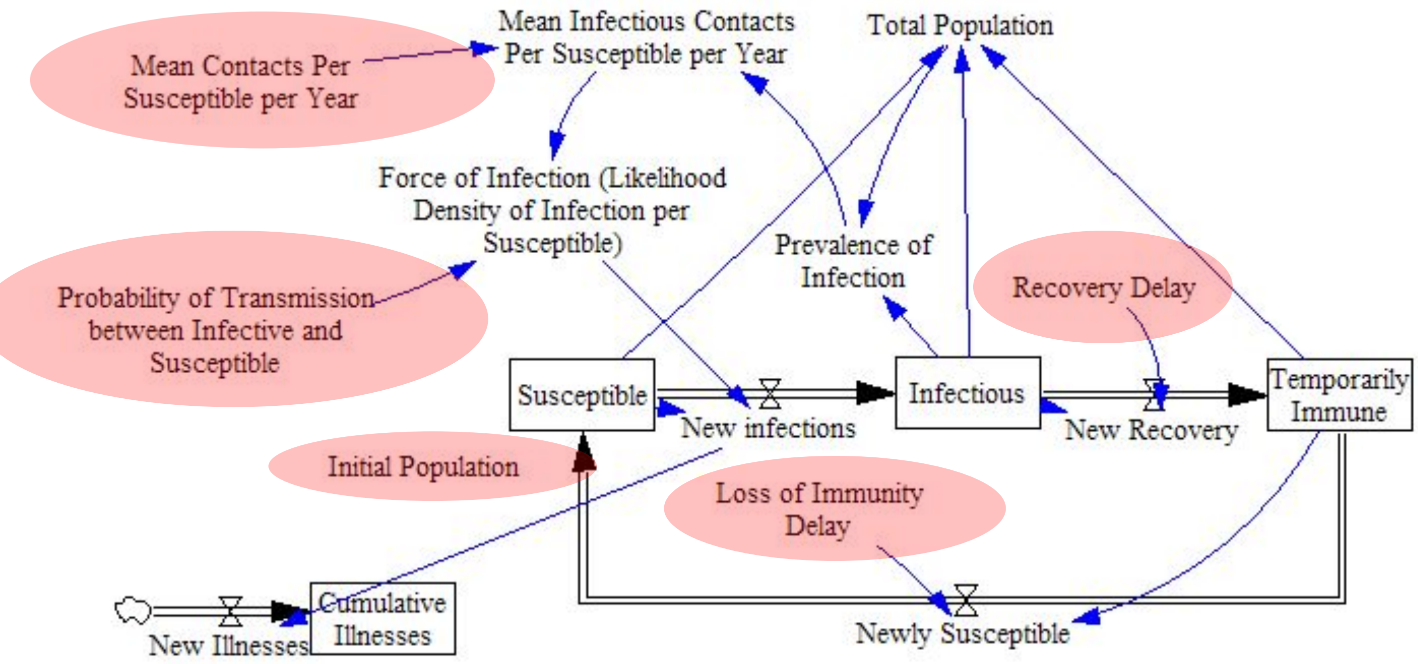
Example Model: Auxiliary (Dynamic) Variables



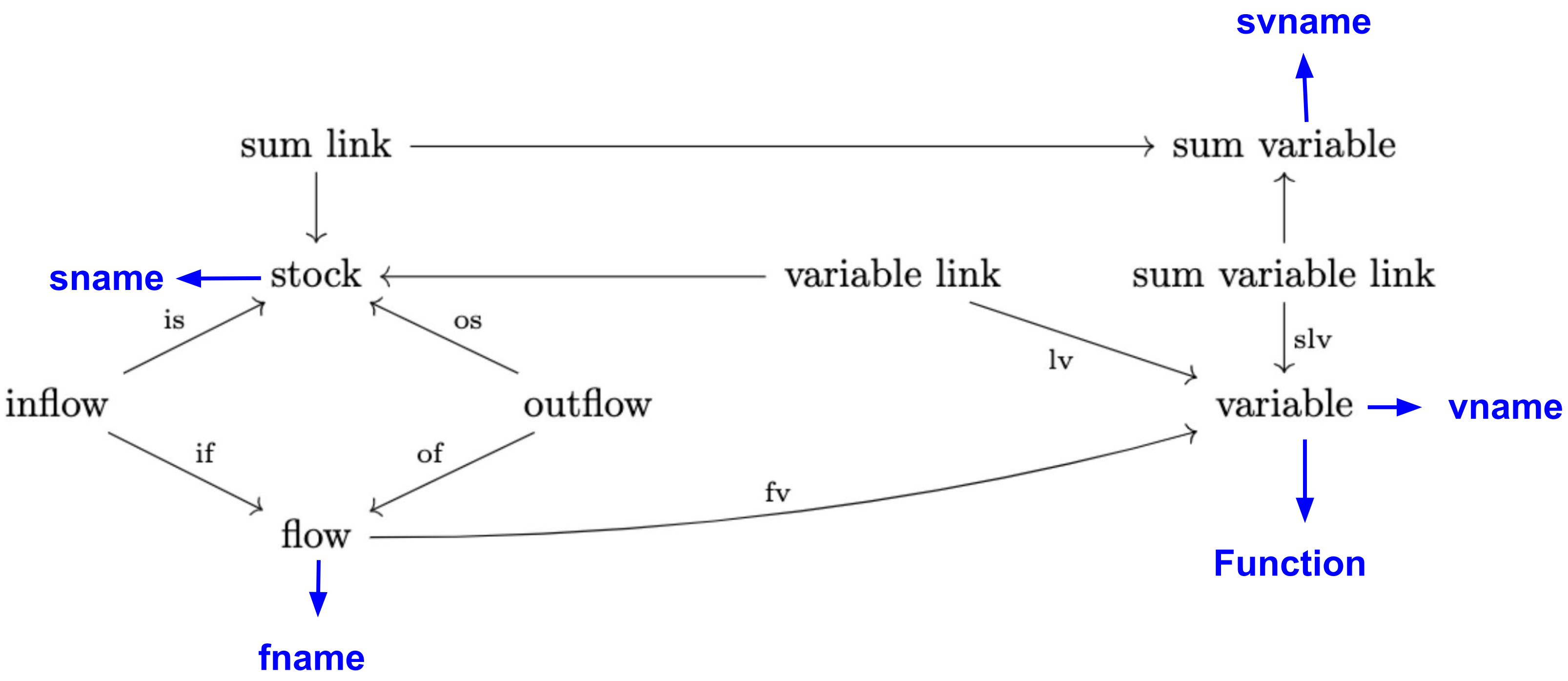
Constants & Time Series Parameters

- For similar reasons to auxiliary variables, we give names to
  - Model constants
  - Time series

Example Model: Parameters



Current Moderate Complexity Schema



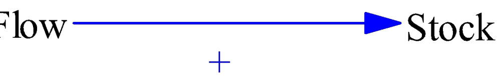
The Moderate Complexity Schema

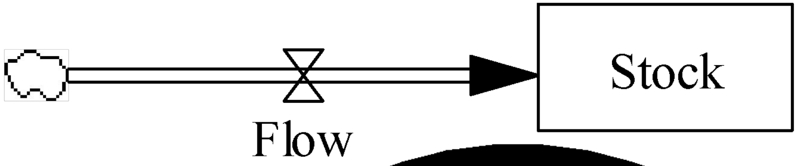
Sum Auxiliary (Dynamic) Variables

- There is a common need to sum up stocks -- and sometimes other variables within a Stock & Flow diagram
- Some of these variables have the meaning of being a sum of all stocks within the model
- To preserve the meaning of variables that seek to sum up all stocks, it can be useful to distinguish “sum” dynamic variables

Semantics

Invariants





If net flow into a stock is...
Positive: The value of the stock rises over time (arrivals exceed departures)
Zero: The value of the stock stays constant (arrivals = departures)
Negative: The value of the stock falls over time (departures exceed arrivals)

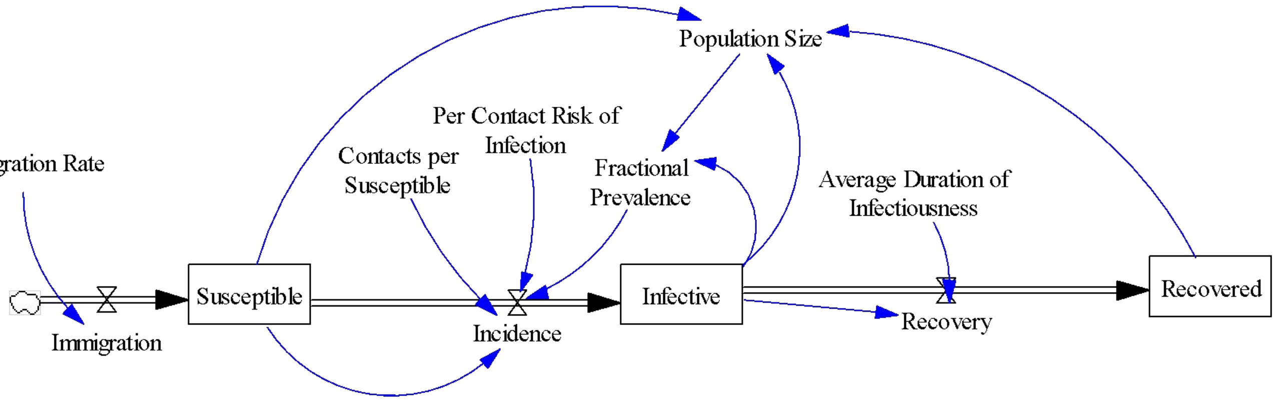
Example Semantics for Stocks & Flows

- Location and/or stability of equilibria (fixed points)
- Causal pathway & feedback loop identification & polarity
- Loop gains over time
- Eigenvalue elasticity over time
- Unit/dimensional inference/correctness
- Simulation semantics
  - Ordinary differential equations (currently gratuitously privileged by extant & past software)
  - Difference equations
  - Discrete state stochastic equations
  - Continuous-time, continuous state stochastic differential equations

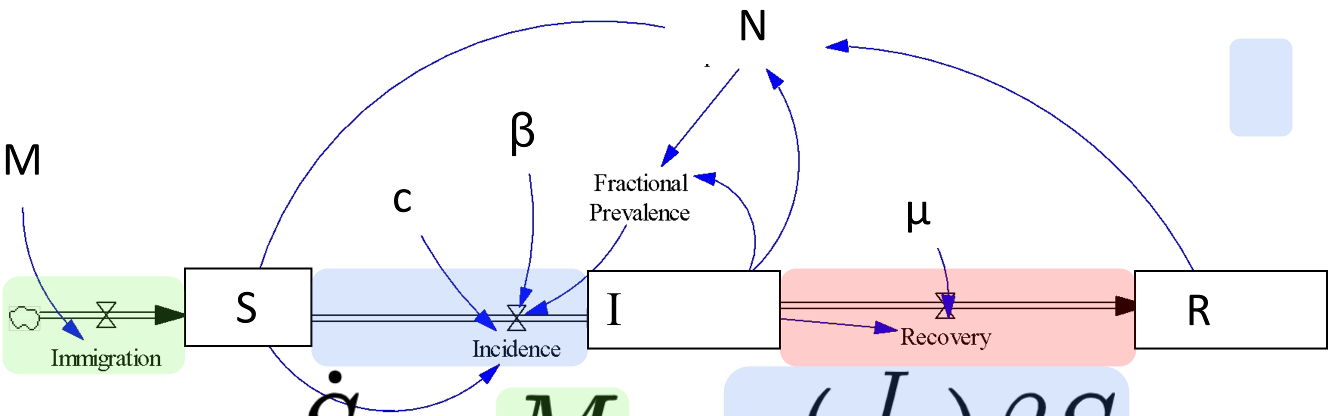
ODE Semantics of Stock (State Variable) & Incident Flows

- Each diagram is associated with a set of first order ordinary differential equations
- Each stock is associated with an element of that set (a first order ODE) having
  - Left hand side: The time derivative of the state variable (stock)
  - Right hand side: Sum of the formulae of each net flows
    - Flows in: Added to differential
    - Flows out: Subtracted from differential

Basic Model Structure



Basic Model Structure & Underlying Equations



$$\dot{S} = M - c\left(\frac{I}{N}\right)\beta S$$

$$\dot{I} = c\left(\frac{I}{N}\right)\beta S - \frac{I}{\mu}$$

$$\dot{R} = \frac{Ia}{\mu}$$