Outline

0. Typed Petri nets

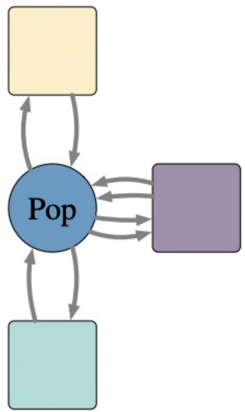
I. Motivation

II. Limits

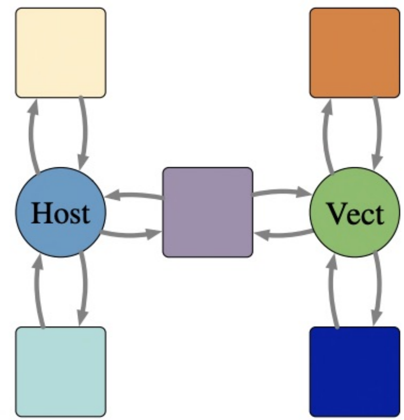
III. Model stratification

0. Typed Petri nets

A Petri net can represent a **domain-specific type system**



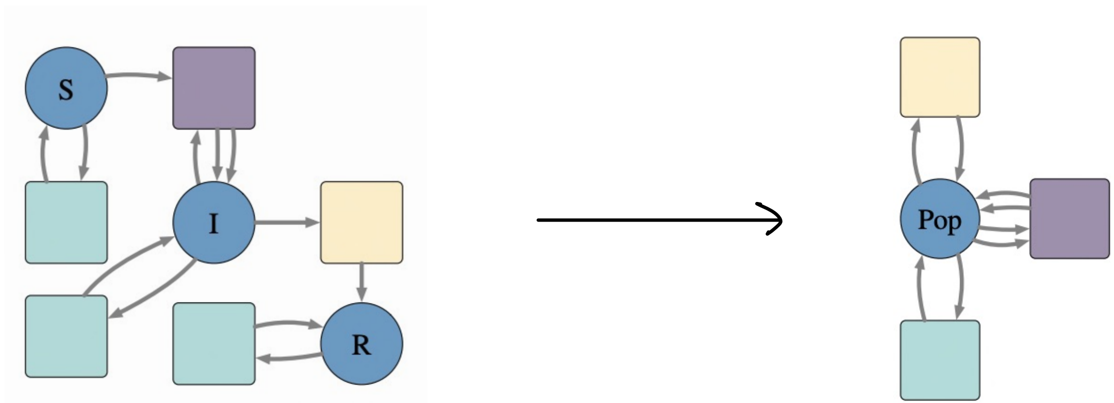
type system for
infectious diseases



type system for
vector-borne diseases

0. Typed Petri nets

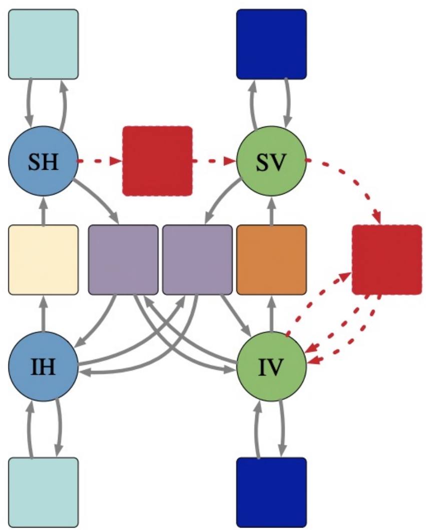
A **typed Petri net** is a $\mathcal{C}$-set homomorphism $P \to P_{type}$

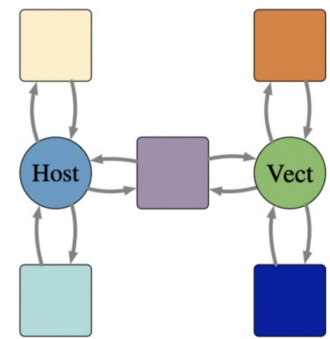


type system for infectious diseases

0. Typed Petri nets

A typed Petri net is a $\mathcal{C}$-set homomorphism $P \to P_{type}$





P_{type} = type system for vector-borne diseases

0. Typed Petri nets

Advantages:

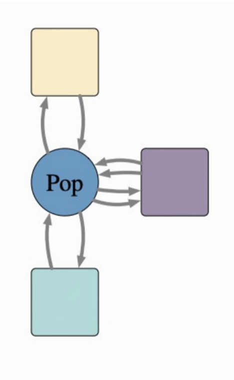
1. model checking

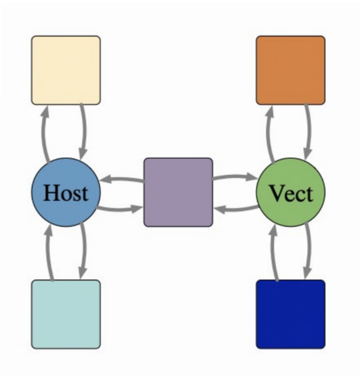
2. facilitate high-level critiques

3. features of type system imply features of model

4. guardrails for composition

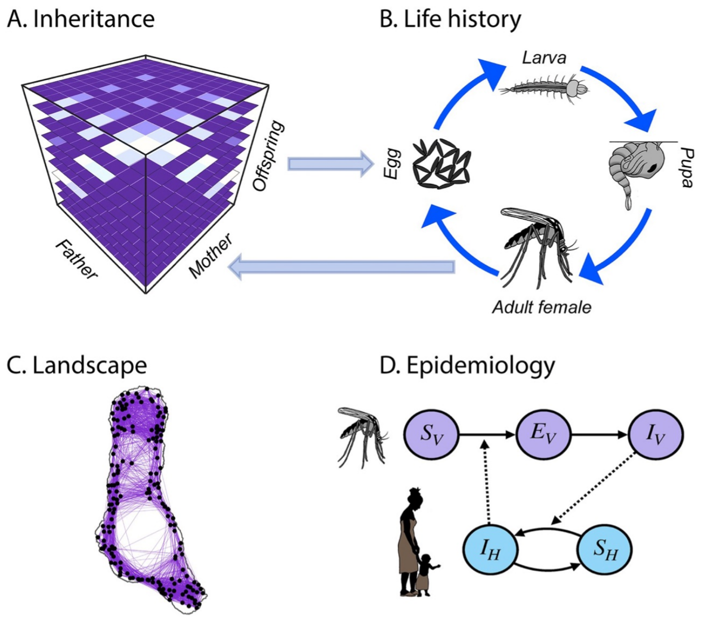
5. Models typed by P_{type} are organized in a slice category





I. Motivation

Stratified Models



Wu et al., 2021

I. Motivation

Comparing metapopulation dynamics of infectious diseases under different models of human movement

Daniel T. Citron[a,1], Carlos A. Guerra[b], Andrew J. Dolgert[a], Sean L. Wu[c], John M. Henry[a], Héctor M. Sánchez C.[c], and David L. Smith[a]

disease models

- SIR

$$\frac{dS}{dt} = -\beta\frac{SI}{N}$$
$$\frac{dI}{dt} = \beta\frac{SI}{N} - \gamma I.$$
$$\frac{dR}{dt} = \gamma I$$

movement models

- flux

$$\frac{dN_i}{dt} = -\sum_{j=1}^{K} f_{i,j}N_i + \sum_{j=1}^{K} f_{j,i}N_j,$$

$$\frac{dS_i}{dt} = -\beta_i\frac{S_iI_i}{N_i} - \sum_{j=1}^{K} f_{i,j}S_i + \sum_{j=1}^{K} f_{j,i}S_j$$

$$\frac{dI_i}{dt} = \beta_i\frac{S_iI_i}{N_i} - \gamma I_i - \sum_{j=1}^{K} f_{i,j}I_i + \sum_{j=1}^{K} f_{j,i}I_j.$$

$$\frac{dR_i}{dt} = \gamma I_i - \sum_{j=1}^{K} f_{i,j}R_i + \sum_{j=1}^{K} f_{j,i}R_j$$

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disease models

- SIR

$$\frac{dS}{dt} = -\beta \frac{SI}{N}$$
$$\frac{dI}{dt} = \beta \frac{SI}{N} - \gamma I.$$
$$\frac{dR}{dt} = \gamma I$$

$$\frac{dS_{i,i}}{dt} = -\beta_i \frac{S_{i,i} \sum_{k=1}^{K} I_{k,i}}{\sum_{k=1}^{K} N_{k,i}} - \sum_{k=1}^{K} \phi_{i,k} S_{i,i} + \sum_{k=1}^{K} \tau_{i,k} S_{i,k}$$
$$\frac{dS_{i,j}}{dt} = -\beta_j \frac{S_{i,j} \sum_{k=1}^{K} I_{k,j}}{\sum_{k=1}^{K} N_{k,j}} + \phi_{i,j} S_{i,i} - \tau_{i,j} S_{i,j}$$
$$\frac{dI_{i,i}}{dt} = \beta_i \frac{S_{i,i} \sum_{k=1}^{K} I_{k,i}}{\sum_{k=1}^{K} N_{k,i}} - \gamma I_{i,i} - \sum_{k=1}^{K} \phi_{i,k} I_{i,i} + \sum_{k=1}^{K} \tau_{i,k} I_{i,k}.$$
$$\frac{dI_{i,j}}{dt} = \beta_j \frac{S_{i,j} \sum_{k=1}^{K} I_{k,j}}{\sum_{k=1}^{K} N_{k,j}} - \gamma I_{i,j} + \phi_{i,j} I_{i,i} - \tau_{i,j} I_{i,j}$$
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$$\frac{dR_{i,j}}{dt} = \gamma I_{i,j} + \phi_{i,j} R_{i,i} - \tau_{i,j} I_{i,j}$$

movement models

- flux
- simple trip

$$\frac{dN_{i,i}}{dt} = -\sum_{j=1}^{K} \phi_{i,j} N_{i,i} + \sum_{j=1}^{K} \tau_{i,j} N_{i,j}.$$
$$\frac{dN_{i,j}}{dt} = -\tau_{i,j} N_{i,j} + \phi_{i,j} N_{i,i}$$

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disease models

- SIR
- SIS
- Ross-Macdonald

⋮

movement models

- flux
- simple trip

⋮

Products

- A type of limit
- Relatively simple, but important
- An abstraction of Cartesian products of sets

II. Limits

Products

Set

def A category $\mathcal{C}$ has products if we can choose

- for every pair of objects x and y, an object $x \times y$

  $\mathbb{N}$ $\quad$ $\mathbb{R}$ $\quad$ $(1,\pi) \in \mathbb{N} \times \mathbb{R}$

- such that morphisms into $x \times y$ are in natural bijection with pairs of morphisms into x and y.

$$\mathrm{Hom}_{\mathcal{C}}(z, x \times y) \simeq \mathrm{Hom}_{\mathcal{C}}(z, x) \times \mathrm{Hom}_{\mathcal{C}}(z, y) \qquad z = \mathrm{Ppl}$$

$$\mathrm{Ppl} \longrightarrow \mathbb{N} \times \mathbb{R} \quad \leftrightsquigarrow \quad \left(\text{age} : \mathrm{Ppl} \to \mathbb{N},\ \text{height} : \mathrm{Ppl} \to \mathbb{R}\right)$$

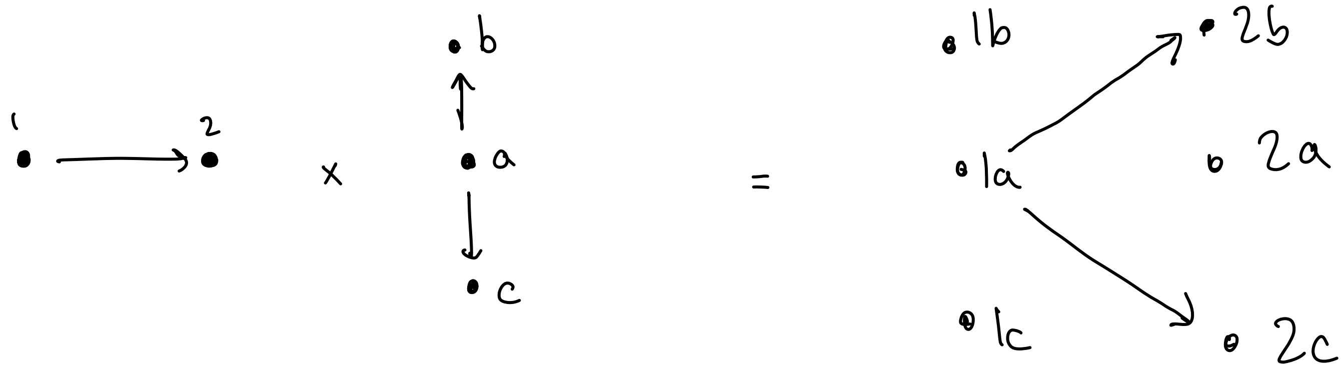
$$\longmapsto \left(\text{age}(\ ), \text{height}(\ )\right).$$

II. Limits

Products

In Graph:

Recall a graph is a $\mathcal{C}$-Set with schema $E \underset{tgt}{\overset{src}{\rightrightarrows}} V$



II. Limits

Products

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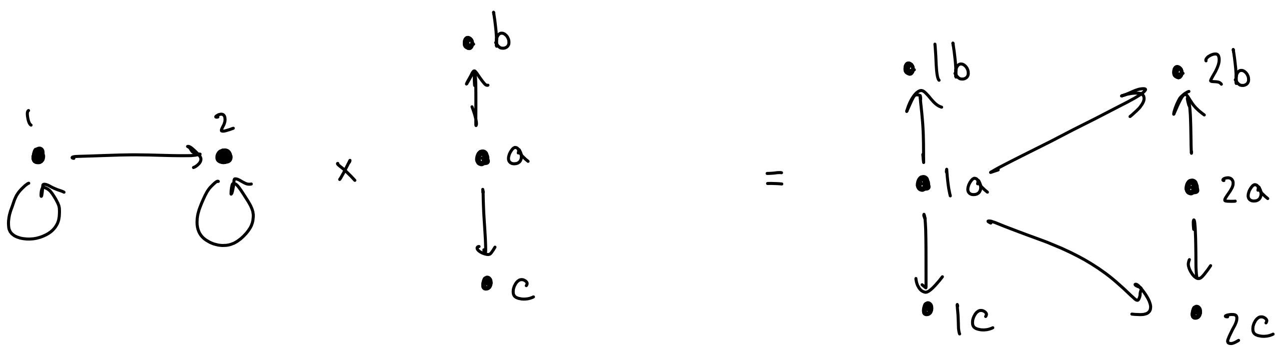


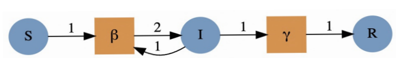
II. Limits

Products

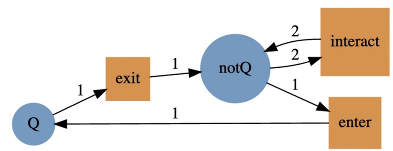
In Petri :

Recall a Petri net is a $\mathcal{C}$-Set with schema

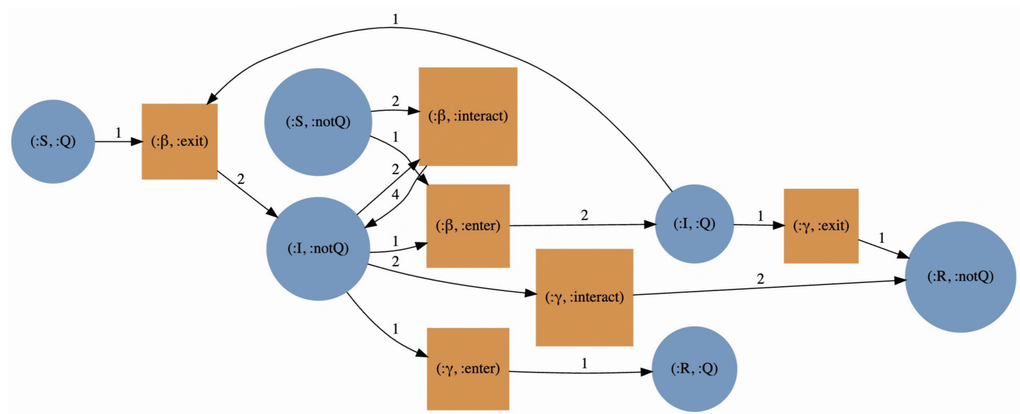
$$\begin{tikzcd} & S & \\ I \arrow[ur] \arrow[dr] & & O \arrow[ul] \arrow[dl] \\ & T & \end{tikzcd}$$



$\times$

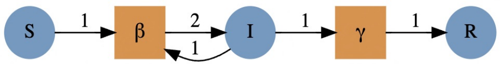


=



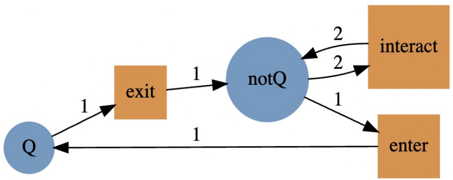
```
sir = LabelledPetriNet([:S, :I, :R],
    :β => ((:S, :I) => (:I, :I)),
    :γ => (:I => :R)
)

Graph(sir)
```



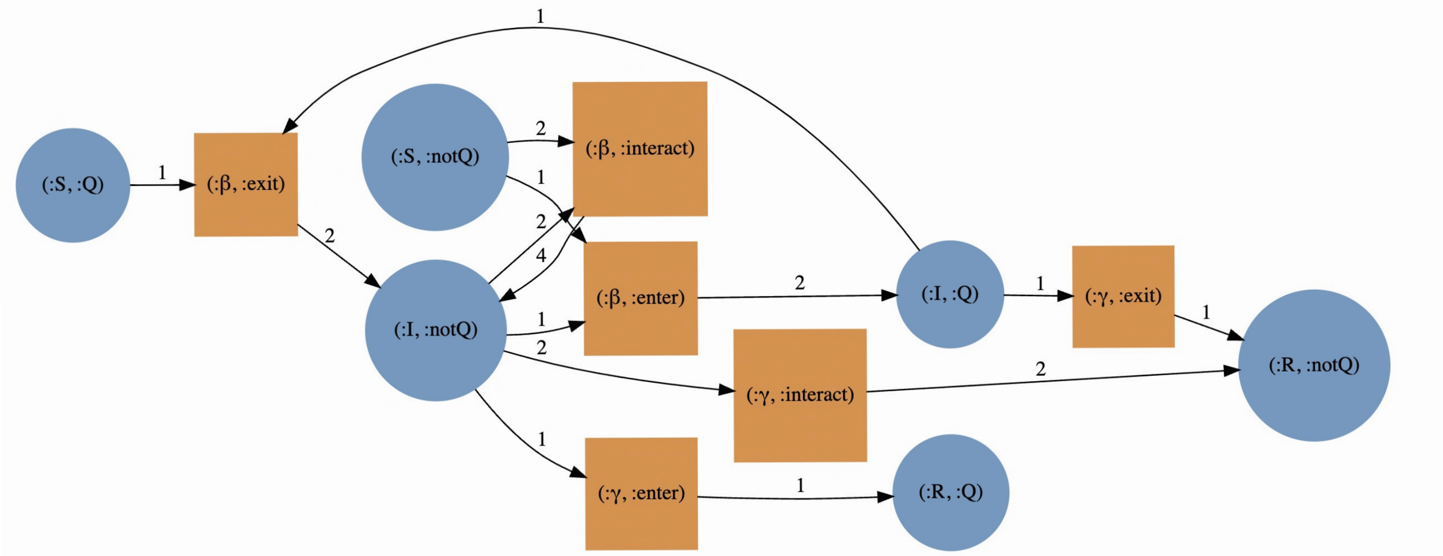

```
quarantine = LabelledPetriNet([:Q, :notQ],
    :enter => (:notQ => :Q),
    :exit => (:Q => :notQ),
    :interact => ((:notQ, :notQ) => (:notQ, :notQ))
)

Graph(quarantine)
```

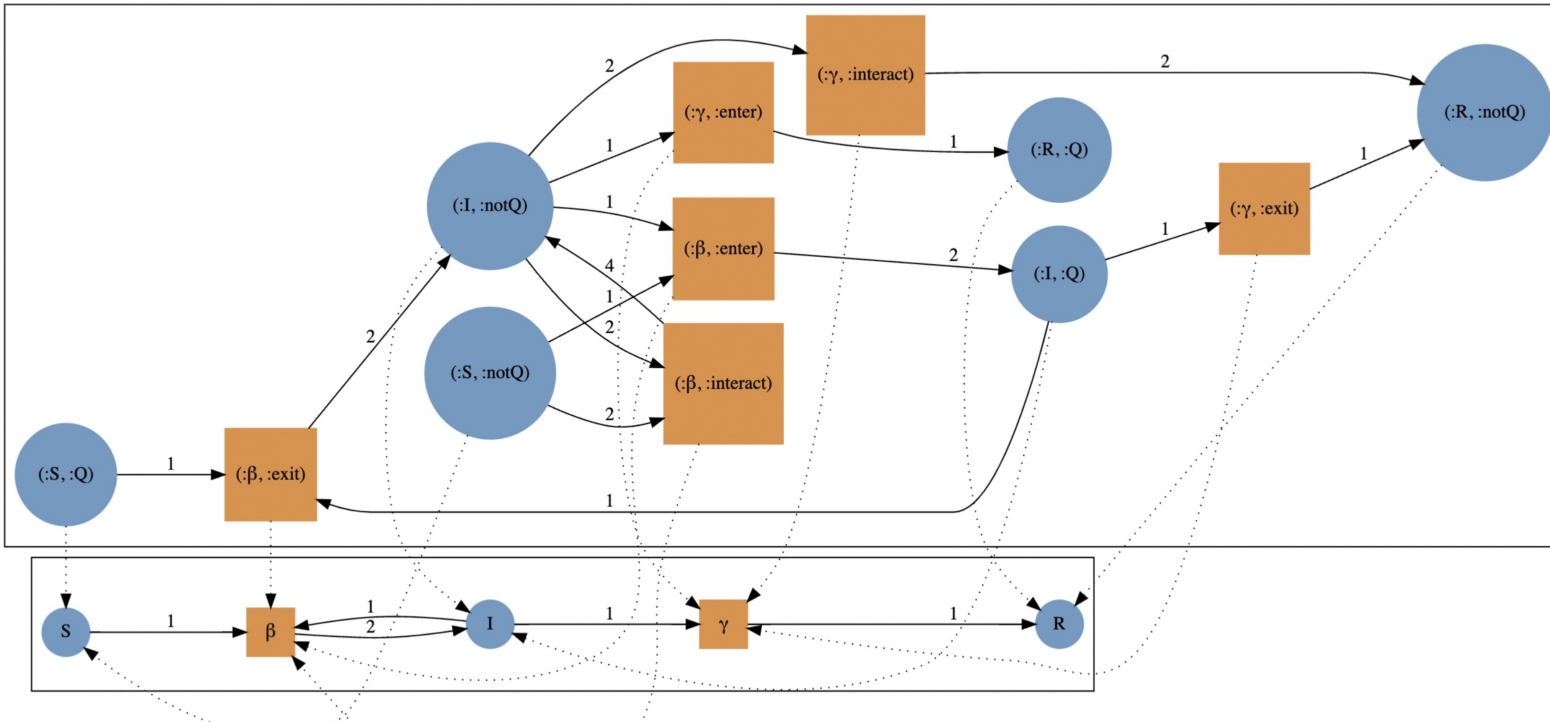


```
product_model = product(sir, quarantine)

Graph(apex(product_model))
```



```
Graph(legs(product_model)[1])
```



Pullbacks

- A type of limit
- A generalization of product

II. Limits

Pullbacks

In Set

$$\{(x,y) \mid f(x) = g(y)\}$$

$$\begin{tikzcd}
& \{(x,y) \mid f(x)=g(y)\} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\checkmark", very near start] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

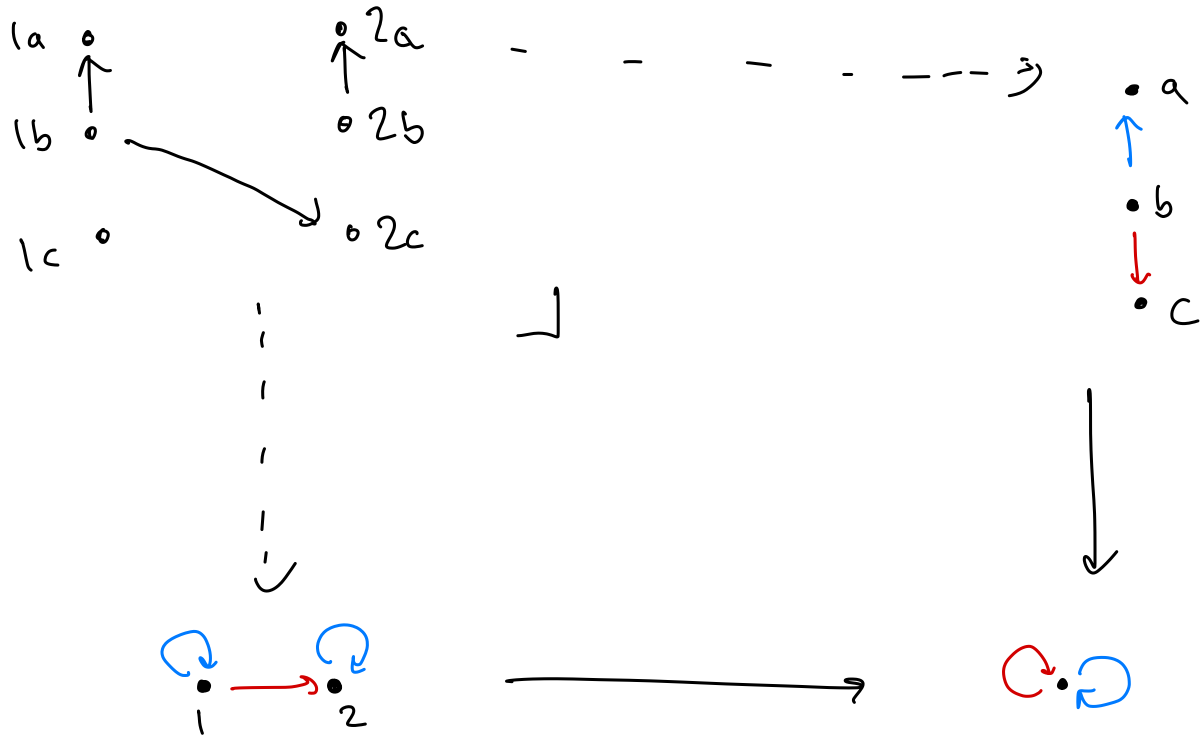
$$\{(🧍, 🐱) \mid \text{fav animal}(🧍) = \text{species}(🐱)\}$$

$$\begin{tikzcd}
& \bullet \arrow[dl, dashed] \arrow[dr, dotted] \arrow[dd, phantom, "\checkmark", very near start] & \\
\text{Ppl} \arrow[dr, "\text{fav animal}"'] & & \text{Pet} \arrow[dl, "\text{species}"] \\
& \text{Species} &
\end{tikzcd}$$

II. Limits

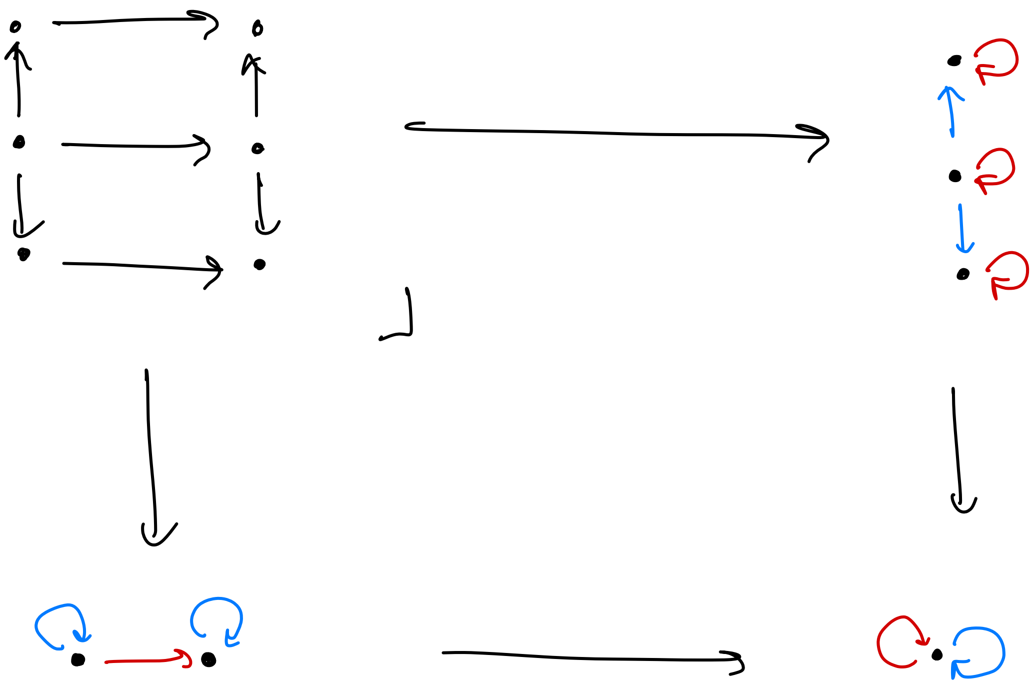
Pullbacks

In Graph:



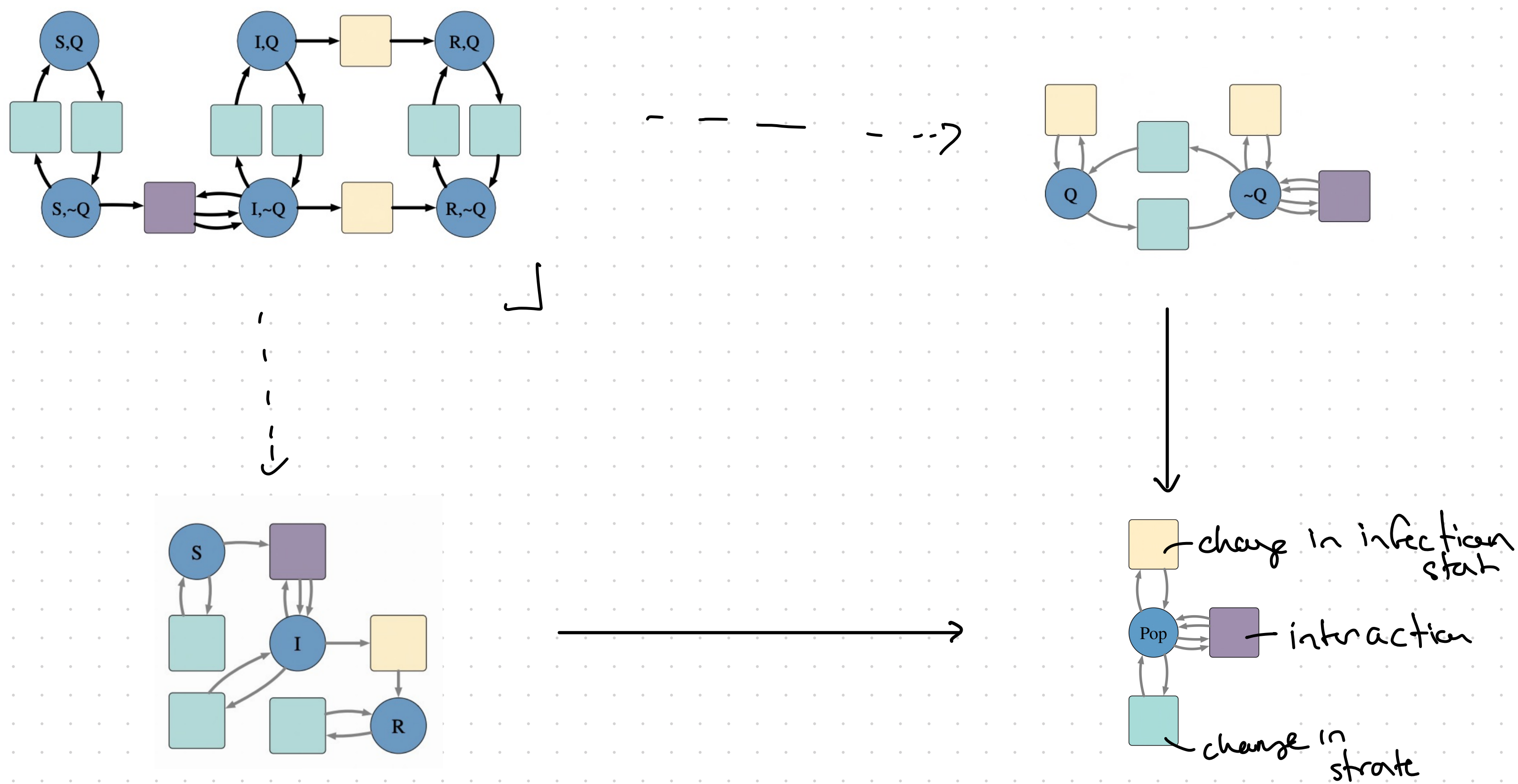
Pullbacks

In Graph:



II. Limits

Pullbacks in Petri:

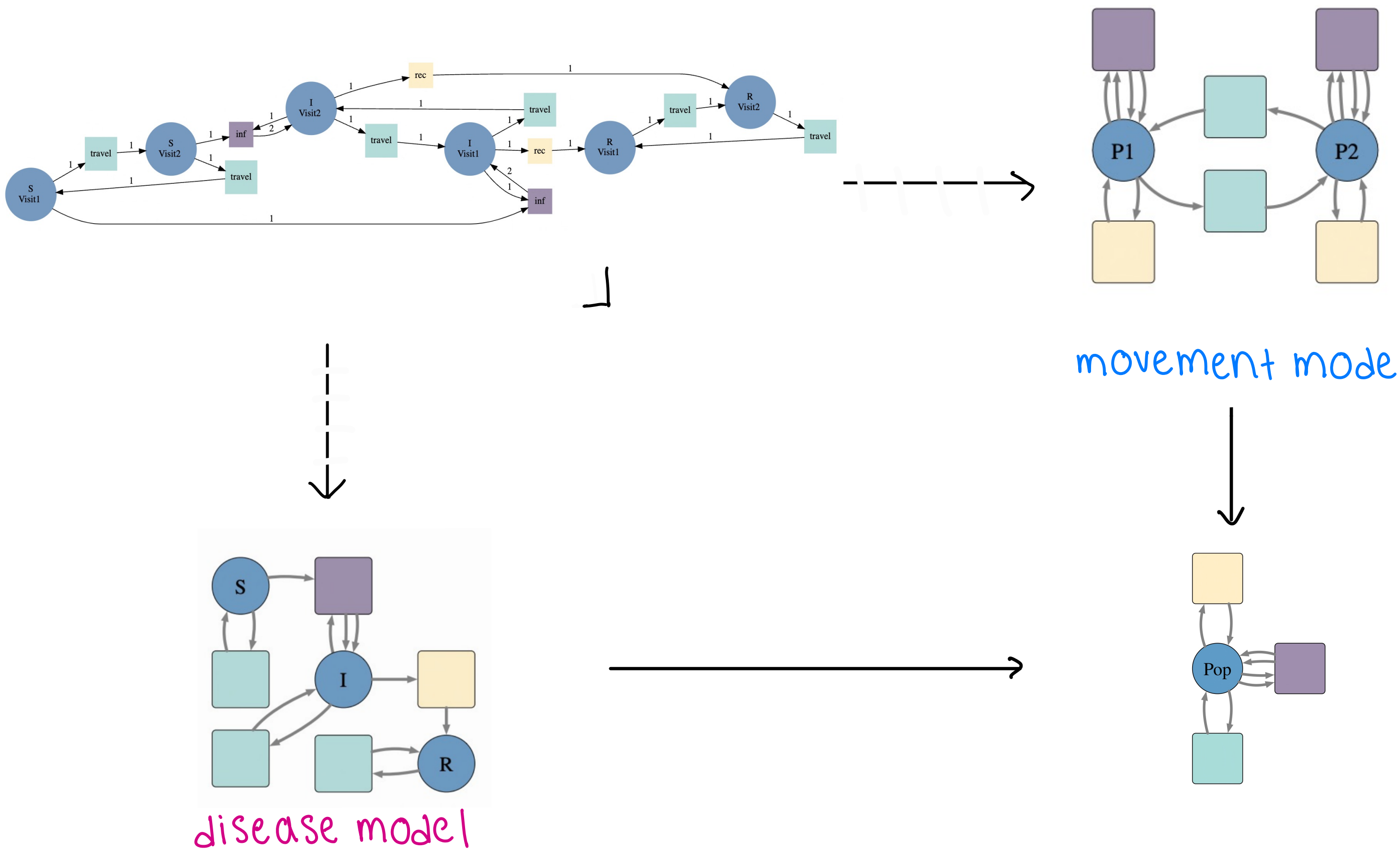


III. Stratified Models

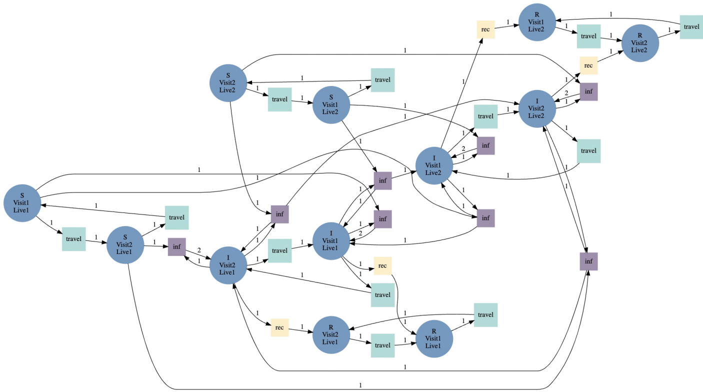
A product in $\mathsf{Petri}/P_{\mathsf{type}}$ is a stratified Petri net

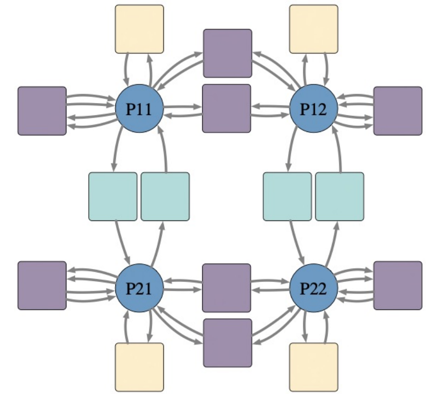
$$
\begin{tikzcd}
P_{\mathsf{stratified}} \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, very near start, "\lrcorner"] & P_{\mathsf{strata}} \arrow[d] \\
P_{\mathsf{disease}} \arrow[r] & P_{\mathsf{infectious}}
\end{tikzcd}
$$

III. Stratified Models

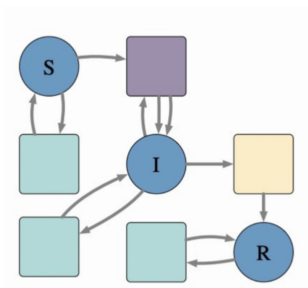


III. Stratified Models

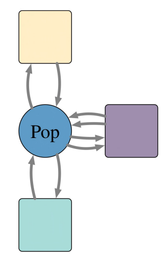




movement model



disease model



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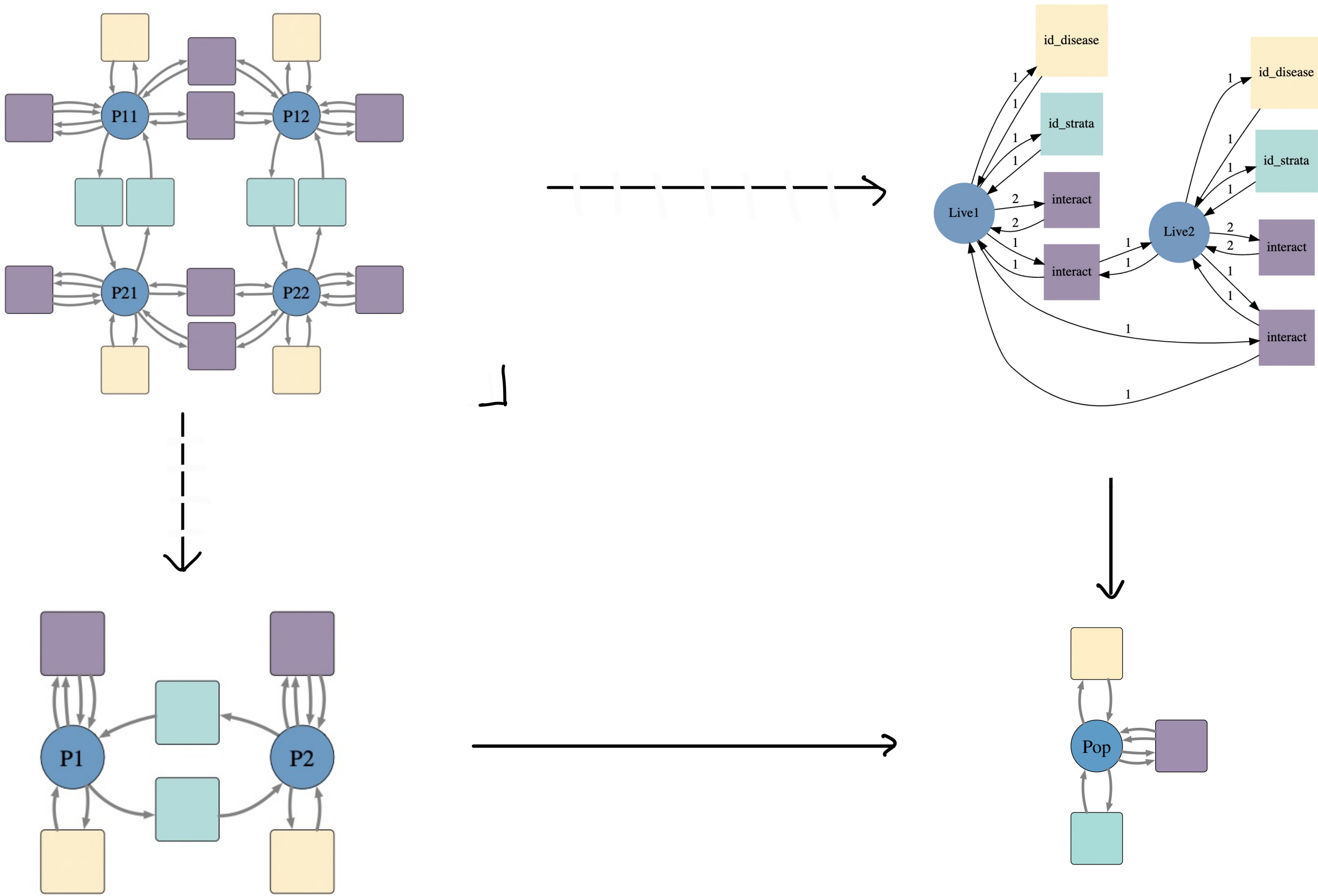

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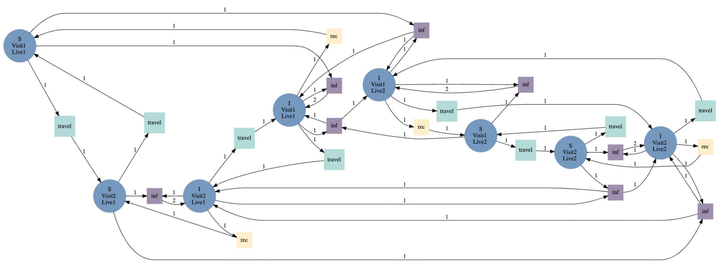
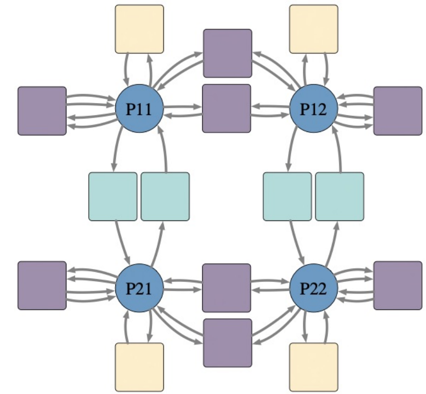
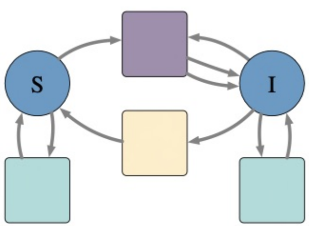




movement model



disease model