# Formalizing Intuition

A UWD is a cospan

$$P_1 + \dots + P_n \to J \leftarrow P'.$$

We compose UWDs by <u>nesting</u>

$$\begin{tikzcd}
 & & P' \arrow[d] \\
 & P_1 + \dots + P_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & J \arrow[d] \\
\begin{matrix} X_{11} + \dots + X_{1m_1} \\ \vdots \\ X_{n1} + \dots + X_{nm_n} \end{matrix} \arrow[r] & K_1 + \dots + K_n \arrow[r] & L
\end{tikzcd}$$

A <u>UWD algebra</u> is a way of assigning data to a UWD in a way that respects nesting.

So <u>UWD Alg</u> is a lax monoidal functor $F : (\mathrm{Cospan}(\mathrm{FinSet}), +) \to (\mathrm{Set}, \times)$

Given a UWD

$$\Phi := P_1 + \dots + P_n \xrightarrow{i} J \xleftarrow{o} P'$$

$$F(\Phi) : FP_1 \times \dots \times FP_n \longrightarrow FP'$$

Laxator gives us

$$FP_1 \times \dots \times FP_n \xrightarrow{\varphi} F(P_1 + \dots + P_n) \to FP'$$

Then use hom map

- Functoriality means we respect nesting.

For this talk, only care about

$$P_1 + \dots + P_n \xrightarrow{\phi} J = J$$

So just need

$$F : (\mathrm{FinSet}, +) \to (\mathrm{Set}, \times)$$

<u>Review</u>

$$\mathrm{Opt} : (\mathrm{FinSet}, +) \to (\mathrm{Set}, \times)$$

$$N \mapsto \{ f : \mathbb{R}^N \to \mathbb{R} \mid f \in C^1 \}$$

$$(\phi : N \to M)(f) \mapsto f \circ \phi^*$$

$$\varphi_{N,M} : \mathrm{Opt}(N) \times \mathrm{Opt}(M) \to \mathrm{Opt}(N+M)$$

$$(f,g) \mapsto f \circ \iota_N^* + g \circ \iota_M^*$$

$$\left( N \xrightarrow{\iota_N} N+M \xleftarrow{\iota_M} M \right)^* = \left( \mathbb{R}^N \underset{\iota_N^*}{\overset{\pi_N}{\longleftarrow}} \mathbb{R}^N \oplus \mathbb{R}^M \underset{\iota_M^*}{\overset{\pi_M}{\longrightarrow}} \mathbb{R}^M \right)$$

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$$\mathrm{Dynam} : (\mathrm{FinSet}, +) \to (\mathrm{Set}, \times)$$

$$N \mapsto \{ \mathbb{R}^N \to \mathbb{R}^N \}$$

$$(\phi : N \to M)(v) \mapsto \phi_* \circ v \circ \phi^*$$

$\varphi_{N,M}$:

$$(v, w) \mapsto \iota_{N*} \circ f \circ \iota_N^* + \iota_{M*} \circ g \circ \iota_M^*$$

$$\left( N \xrightarrow{\iota_N} N+M \xleftarrow{\iota_M} M \right)_* = \left( \mathbb{R}^N \xrightarrow{\iota_N} \mathbb{R}^N \oplus \mathbb{R}^M \xleftarrow{\iota_M} \mathbb{R}^M \right)$$

Gradient flow takes

$f \in Opt(N) \to -\nabla f \in Dynam(N)$

How do we know it respects all our UWD structure?

Answer: NTS a monoidal NT.

flow: $Opt \Rightarrow Dynam$

$flow_N(f) := -\nabla f$

Proof:

$$\begin{tikzcd}
Opt(N) \arrow[r, "Opt(\phi)"] \arrow[d, "flow_N"'] & Opt(M) \arrow[d, "flow_M"] \\
D(N) \arrow[r, "D(\phi)"'] & D(M)
\end{tikzcd}$$

Let $f \in Opt(N)$ & $P = \mathcal{M}(\phi^*)$. chain rule

$$\begin{tikzcd}
f \arrow[r, mapsto] \arrow[d, mapsto] & f \circ P \arrow[r, mapsto] & -\nabla(f \circ P) \arrow[r, phantom, "="] \arrow[d, phantom, "\|?"] & -\nabla P^T \circ \nabla f \circ P \arrow[d, phantom, "\| \text{ linearity}"] \\
-\nabla f \arrow[rr, mapsto] & & -P^T \circ \nabla f \circ P \arrow[r, phantom, "\overset{\checkmark}{=}"] & P^T \circ \nabla f \circ P
\end{tikzcd}$$

Monoidal:

$$\begin{tikzcd}
Opt(N) \times Opt(M) \arrow[r, "flow \times flow"] \arrow[d, "\varphi_{N,M}"'] \arrow[dr, phantom, "="] & D(N) \times D(M) \arrow[d, "\varphi_{N,M}"] \\
Opt(N+M) \arrow[r, "flow"'] & D(N+M)
\end{tikzcd}$$

Talk about how this enables message passing semantics.

- Draw a UWD. Illustrate

distribute → compute → collect
(in parallel)

- Naturality guarantees distr alg is correct!

# What about constraints & minimax?

- They are actually the same thing!

consider:

minimize $f(x)$ ← convex

subject to $Ax = b$ ← affine

$$L(x, \lambda) := f(x) + \lambda^T (Ax - b)$$

convex    concave

Uzawa's Alg:

$$x_{k+1} = x_k - \gamma \nabla_x L(x_k, \lambda_k)$$ ← gradient descent in cvx vars

$$= x_k - \gamma (\nabla f(x) + \lambda^T A)$$

$$\lambda_{k+1} = \lambda_k + \gamma \nabla_\lambda L(x_k, \lambda_k)$$ ← grd ascent in conc vars.

$$= Ax - b$$

integrate constraint violation

How to keep track of convex vs. concave vars?

Answer: Have you tried slice cats?

$$\mathrm{Saddle}: (\mathrm{FinSet}/_{\{\downarrow,\uparrow\} \cong 2}, +) \to (\mathrm{Set}, \times)$$

Review of slice cats:

Obs: $N \xrightarrow{\tau} 2$

Homs:
$$\begin{tikzcd} N \arrow[rr, "\phi"] \arrow[dr, "\tau_N"'] & {} \arrow[d, phantom, "="] & M \arrow[dl, "\tau_M"] \\ & 2 & \end{tikzcd}$$

For $\mathrm{Saddle}(N \xrightarrow{\tau} 2)$, let $N_1 := \tau^{-1}(\downarrow)$ &
$N_2 := \tau^{-1}(\uparrow)$.

$\tau \mapsto \{ f : \mathbb{R}^N \cong \mathbb{R}^{N_1} \oplus \mathbb{R}^{N_2} \to \mathbb{R} \mid f \in C^1 \wedge$
$f(\cdot, y)$ is convex $\forall y \in \mathbb{R}^{N_2} \wedge$
$f(x, \cdot)$ is concave $\forall x \in \mathbb{R}^{N_1} \}$.
& same as Opt.

$$\begin{tikzcd} \mathrm{FinSet}/2 \arrow[r, "\mathrm{Saddle}"] \arrow[d, "U"'] & \mathrm{Set} \arrow[dl, phantom, "\Downarrow"] \\ \mathrm{FinSet} \arrow[ur, "\mathrm{Dynam}"'] & \end{tikzcd}$$

flow: Saddle $\Rightarrow$ Dynam $\circ$ U.

$$\mathrm{flow}_{N \xrightarrow{\tau} 2}(L)(x) := i \mapsto \begin{cases} -\nabla L(x)(i) & \text{if } \tau(i) = 0 \\ \nabla L(x)(i) & \text{if } \tau(i) = 1 \end{cases}$$

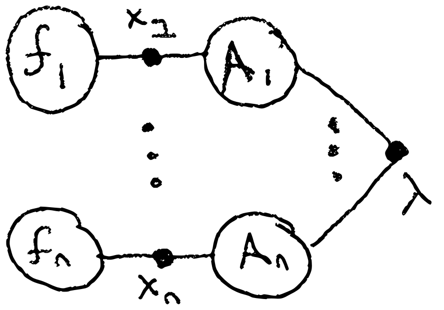
$\forall i \in N$.

$\mathrm{flow}(\tau) : \mathrm{Saddle}(\tau) \to \mathrm{Dynam}(N)$

Close w/ nice example:

Many probs of form

minimize $f_1(x_1) + \ldots + f_n(x_n)$
subject to $A_1 x_1 + \ldots + A_n x_n = 0$




Message Passing Uzawa's!

If time, talk abt current work.