Discrete Exterior Calculus Tips & Tricks 3:

Ein neuer Zweig

GATAS Lab Seminar Series, Fall 2024

Luke Morris

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Discrete Exterior Calculus Tips & Tricks:

All you need is $\star\ d\ \wedge$

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Recap: Finite Difference Methods

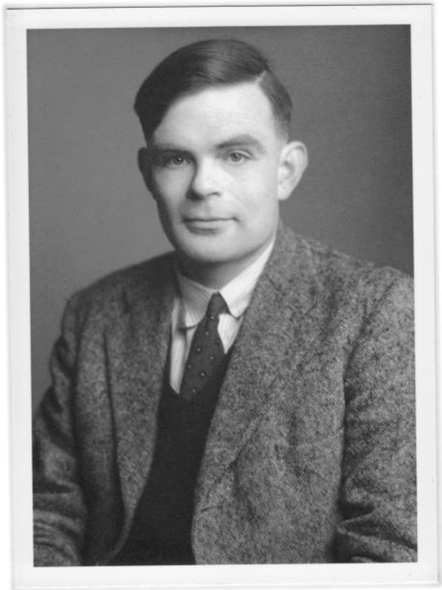
72 A. M. TURING ON THE CHEMICAL BASIS OF MORPHOGENESIS

from one pattern into another, rather than from homogeneity into a pattern. One would like to be able to follow this more general process mathematically also. The difficulties are, however, such that one cannot hope to have any very embracing *theory* of such processes, beyond the statement of the equations. It might be possible, however, to treat a few particular cases in detail with the aid of a digital computer. This method has the advantage that it is not so necessary to make simplifying assumptions as it is when doing a more theoretical type of analysis. It might even be possible to take the mechanical aspects of the problem into account as well as the chemical, when applying this type of method. The essential disadvantage of the method is that one only gets results for particular cases. But this disadvantage is probably of comparatively little importance. Even with the ring problem, considered in this paper, for which a reasonably complete mathematical analysis was possible, the computational treatment of a particular case was most illuminating. The

$$\begin{tikzcd}
\Omega_0 \arrow[r, "d"] \arrow[d, shift right, "\star"'] & \Omega_1 \arrow[r, "d"] \arrow[d, shift right, "\star"'] & \Omega_2 \arrow[r, "d"] \arrow[d, shift right, "\star"'] & \Omega_3 \arrow[d, shift right, "\star"'] \\
\tilde{\Omega}_n \arrow[u, shift right, "\star^{-1}"'] & \tilde{\Omega}_{n-1} \arrow[l, "\tilde{d}"] \arrow[u, shift right, "\star^{-1}"'] & \tilde{\Omega}_{n-2} \arrow[l, "\tilde{d}"] \arrow[u, shift right, "\star^{-1}"'] & \tilde{\Omega}_{n-3} \arrow[l, "\tilde{d}"] \arrow[u, shift right, "\star^{-1}"']
\end{tikzcd}$$

```julia
GrayScott = @decapode begin
  (U, V)::Form0{X}
  (f, k, rᵤ, rᵥ)::Constant{X}

  UV2 == (U .* (V .* V))
  ∂ₜ(U) == rᵤ * Δ(U) - UV2 + f * (1 .- U)
  ∂ₜ(V) == rᵥ * Δ(U) + UV2 - (f + k) .* V
end
```



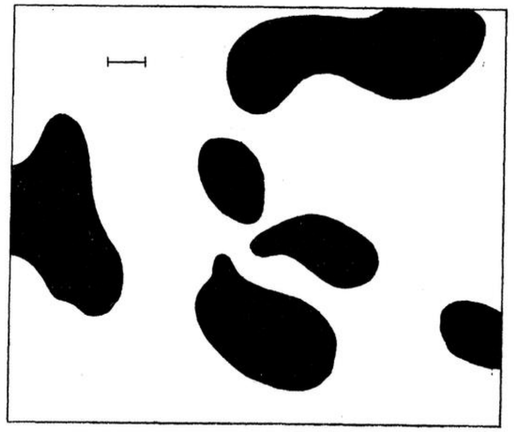
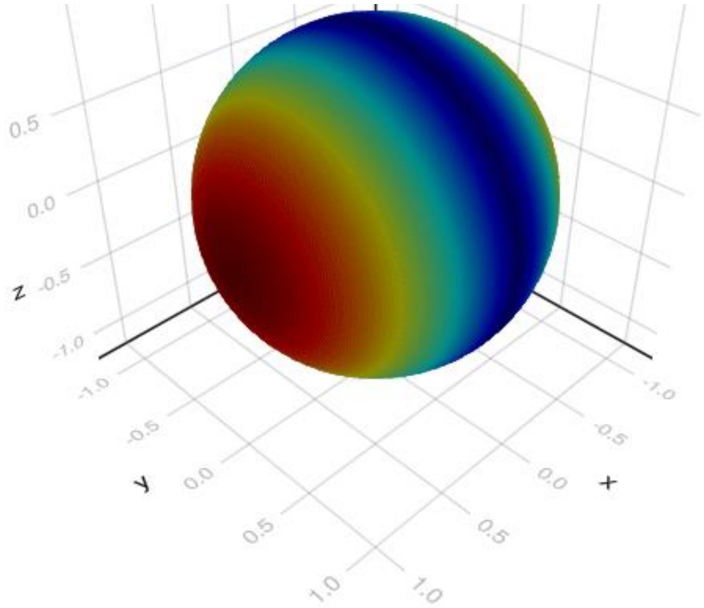


FIGURE 2. An example of a 'dappled' pattern as resulting from a type (*a*) morphogen system. A marker of unit length is shown. See text, §9, 11.



Discrete Exterior Calculus Tips & Tricks 2:

The Laplacian and Spectral Analysis

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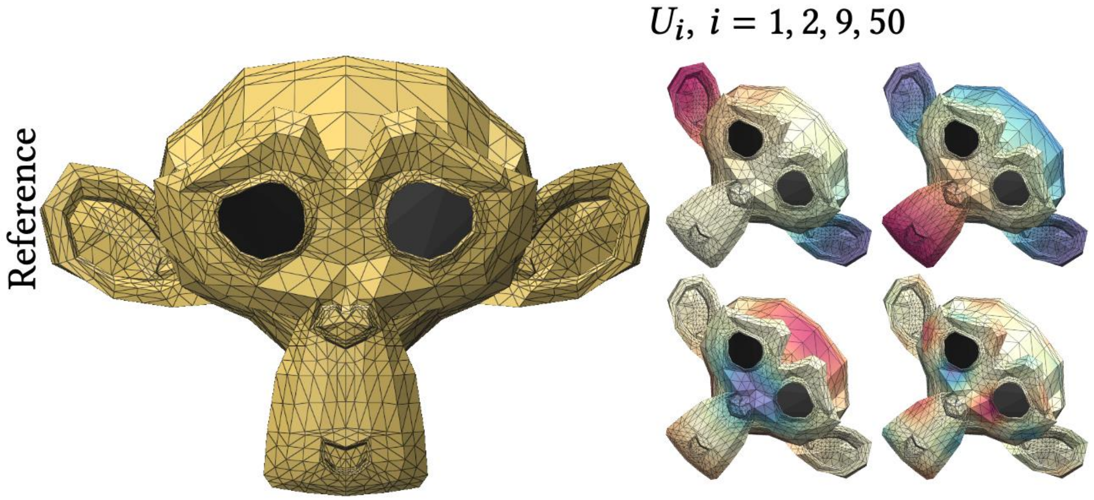
Recap: Eigenvectors of the Laplacian

- If the Laplacian is "**local deviation from average**"
- And if the Laplacian is "**imbued with geometry**"
- And if the eigenvectors are "**principal components**"…

- Then, we can study the principal ways in which we deviate from local position



Eigenvectors of the Laplacian on Blender's "Suzanne" mesh from Spectral Coarsening with Hodge Laplacians *from Keros & Subr*

Exterior Algebra: Grassmann

- Principles given by Hermann Grassmann in “Die Lineale Ausdehnungslehre”, 1844, reworked 1862
- Recovers “Geometry” from first principles

Exterior Algebra: Bases

- We pick “Einheiten”, quantities which serve to represent the range they represent. i.e. basis elements
- We form a notion of “extensive quantities”, linear combinations thereof
- An “Einheit” that is not a linear combination of others is “ursprünglich”. i.e. original

Exterior Algebra: Operations

- Extensive quantities are…
  - Added componentwise. This is associative and commutative.
  - Subtracted the same way
  - Scalar-multiplied by distributing
  - Scalar-divided the same way (if nonzero)

Exterior Algebra: Equivalence

- When are two extensive quantities equivalent?
  - When the geometric spaces they represent are the same
- When are to extensive quantities equal?
  - When their coefficients are the same

Exterior Algebra: Products

- The product must be another extensive quantity.
- But which? “[D]arüber sagt die Definition nichts aus”
- Algebraic:
  - [E1 E1], [E2 E2], and [E1 E2] = [E2 E1]
- Combinatoric:
  - [E1 E1] = [E2 E2] = 0 , and [E1 E2] = - [E2 E1]
- Inner:
  - [E1 E1] = [E2 E2] = 1 , and [E1 E2] = [E2 E1] = 0

Wedge Product, Example 1

- $(f_1\ E1 + f_2\ E2) \wedge (g_1\ E1 + g_2\ E2)$
- $f_1\ g_1\ E1 \wedge E1 + f_1\ g_2\ E1 \wedge E2 + f_2\ g_1\ E2 \wedge E1 + f_2\ g_2\ E2 \wedge E2$
- $f_1\ g_2\ E1 \wedge E2 + f_2\ g_1\ E2 \wedge E1$
- $(f_1\ g_2 - f_2\ g_1)\ E1 \wedge E2$

Wedge Product, Example 2

- $(f_1 \, E1 + f_2 \, E2) \wedge (g_2 \, E2)$
- $f_1 \, g_2 \, E1 \wedge E2 + f_2 \, g_2 \, E2 \wedge E2$
- $f_1 \, g_2 \, E1 \wedge E2$

Hodge Star

- What is an extensive quantity of class K “dual” to?
- Good answer: Something of class N-K
- $\star$[E1 E2] = 1
- $\star 1$ = [E1 E2]
- $\star$E1 = E2
- $\star$E2 = -E1

Hodge Star, Trick Question

- What is the difference between these two expressions?
- 1
- $\star$[E1 E2]

Discrete Exterior Calculus

- What is the natural duality to exploit in finite difference methods?

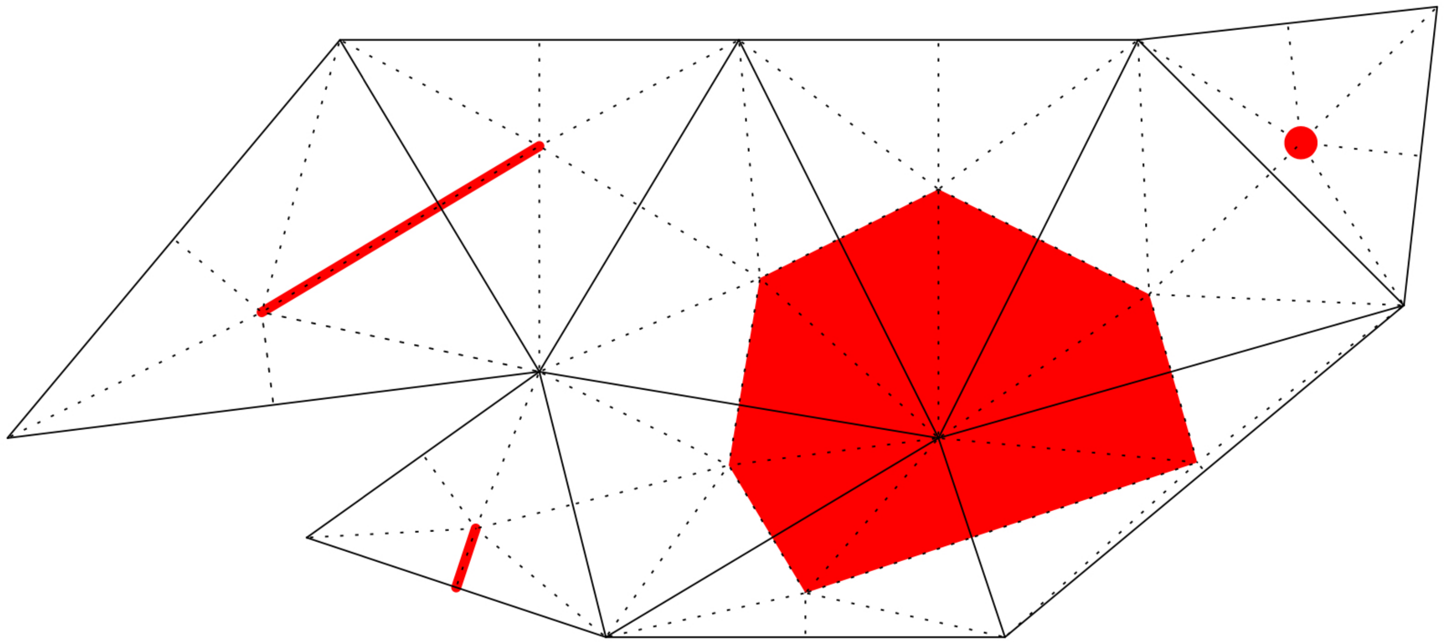


Figure 2.4: A simplicial complex K is subdivided into the simplicial complex $\mathrm{csd}\ K$ and some dual cells of dimension 0,1 and 2 are marked. See Example 2.4.4 and 2.4.7. The new edges introduced by the subdivision are shown dotted. The dual cells shown are colored red. Some elementary dual simplices and subdivision simplices appearing in this figure are pointed out in Example 2.4.7.

Annotated simplicial complex from Discrete Exterior Calculus *from Hirani*

Discrete Exterior Calculus

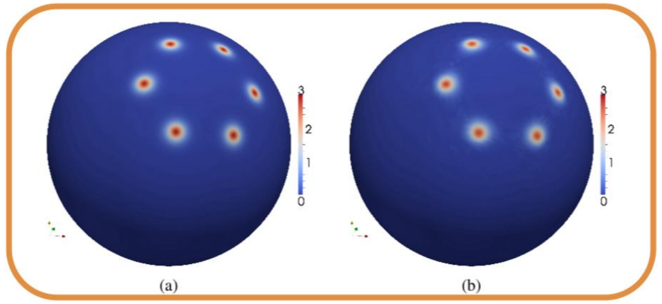
- Exploiting that natural duality, we can present quantities like so:

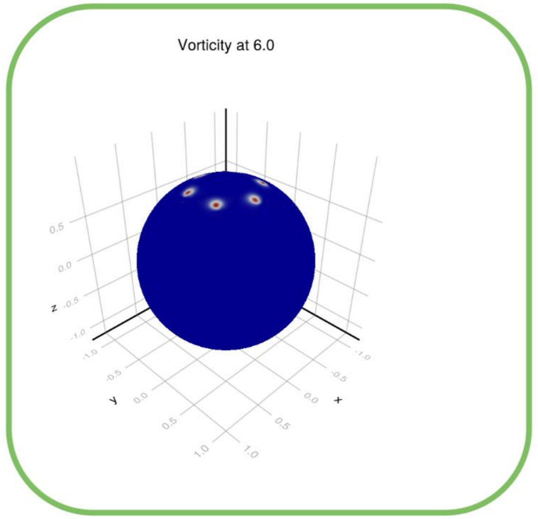
$$\begin{tikzcd} C^0(K) \arrow[r, "d_0"] \arrow[d, "*_0"] & C^1(K) \arrow[r, "d_1"] \arrow[d, "*_1"] & C^2(K) \arrow[d, "*_2"] \\ D^2(\star K) & D^1(\star K) \arrow[l, "-d_0^T"'] & D^0(\star K) \arrow[l, "d_1^T"'] \end{tikzcd}$$

A diagram representing the spaces of quantities stored on a primal and dual mesh from Discrete exterior calculus discretization of incompressible Navier-Stokes equations over surface simplicial meshes *from Mohamed et al.*

Discrete Exterior Calculus

- *"The appearance of dual complexes leads to a proliferation of the operators in the discrete theory."* - Hirani
- This leads to **nontrivial** in choices in specifying simulations





property dd = 0, the resulting governing equation is

$$\frac{\partial \mathbf{du}^{\flat}}{\partial t} + (-1)^{N+1}\mu \mathrm{d} * \mathrm{d} * \mathbf{du}^{\flat} + (-1)^{N} \mathrm{d} * (\mathbf{u}^{\flat} \wedge * \mathbf{du}^{\flat}) = 0. \qquad (11)$$

The DEC discretization of Navier-Stokes equations is carried out through the discretization of Eq. (11). The advantage of discretizing the vorticity form of the NS

```
eq11_inviscid_poisson = @decapode begin
  du::DualForm2
  u::DualForm1
  ψ::Form0

  ψ == Δ⁻¹(⋆(du))
  u == ⋆(d(ψ))

  ∂ₜ(du) ==  (-1) * ∘(♭♯, ⋆₁, d̃₁)(∧ᵈᵖ₁₀(u, ⋆(du)))
end
```

Thanks!

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About me:

I'm a 4th-year **Computer Science** PhD student in the Herbert Wertheim College of Engineering at the University of Florida.

I am employed as a graduate research assistant in the Mechanical & Aerospace Engineering Department

I graduated with my Bachelor's in Computer Science from the University of Kentucky in 2021, *summa cum laude*.

My advisor here in Gainesville is Dr. James Fairbanks of the GATAS Lab.

My Current Research Involves:

- Applied Category Theory
- Multiphysics Simulations
- Space Weather
- High Performance Computing
- Opinion Dynamics