A Sheaf Theoretic Framework for Multi-Agent Model Predictive Control

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Background

MPC

- MPC is a broadly useful control method

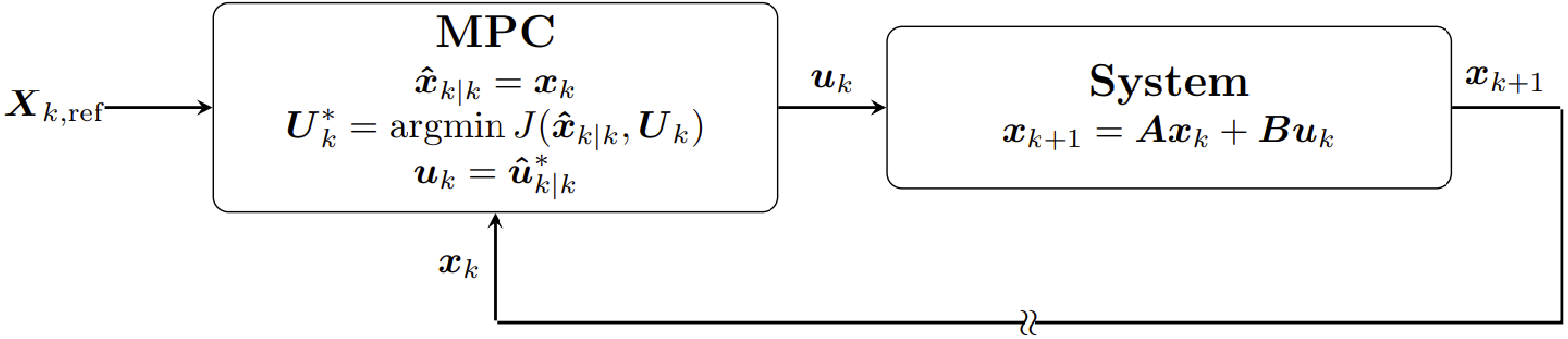
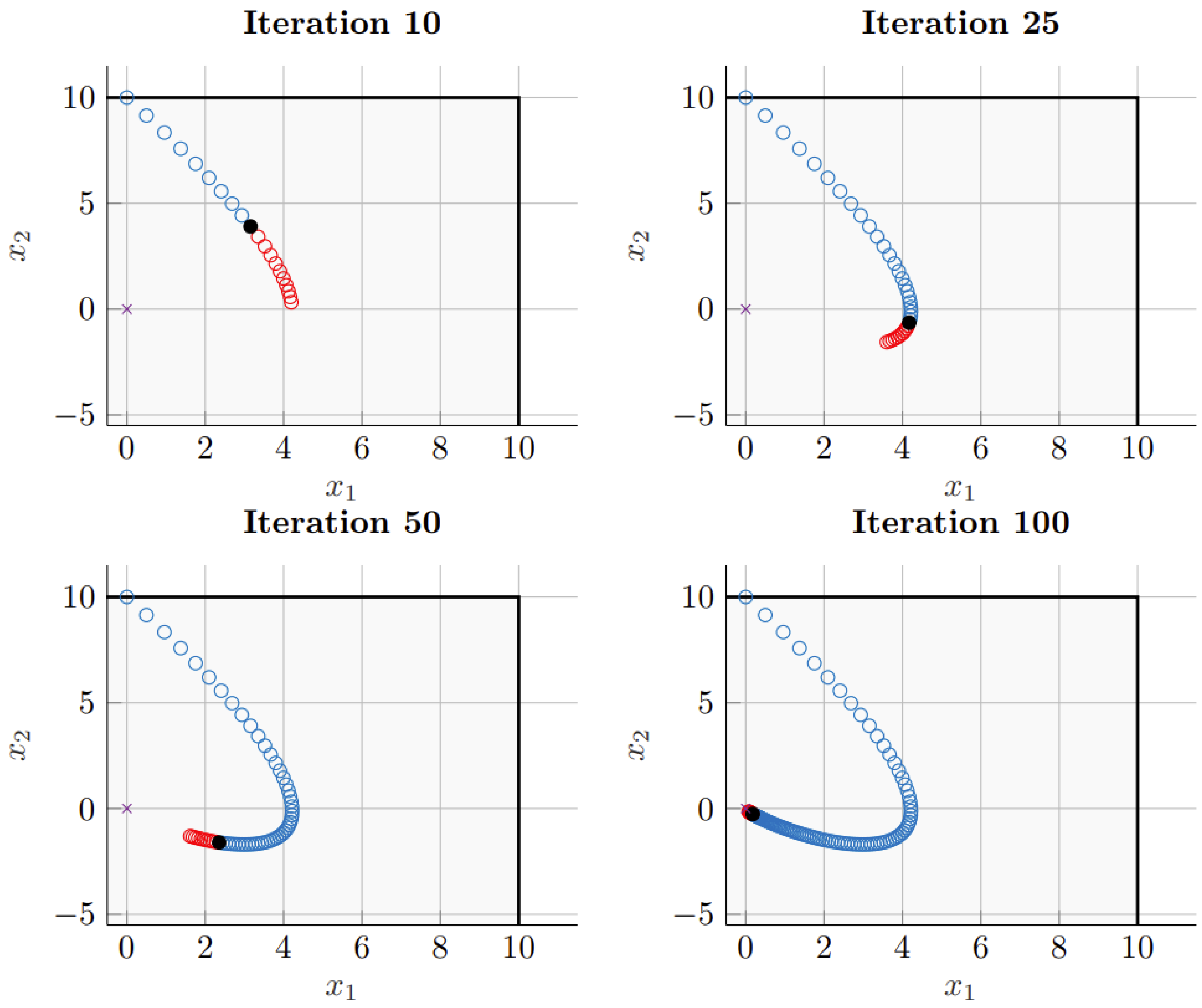


Figure 1: MPC scheme

MPC



Iteration 10
Iteration 25
Iteration 50
Iteration 100
x_1
x_2
10
5
0
−5
0
2
4
6
8
10

Multi-agent MPC

We consider a collection of $m \in \mathbb{N}$ agents with linear, discrete-time dynamics

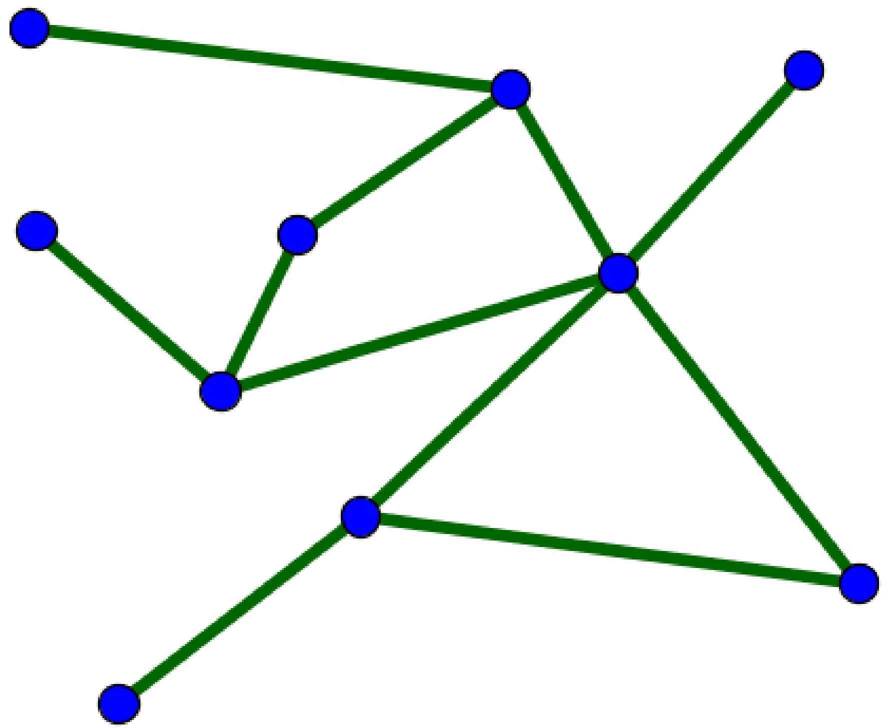
$$x^i(t+1) = A_i x^i(t) + B_i u^i(t), \quad x^i(0) = x_0^i, \tag{1a}$$

$$y^i(t) = C_i x^i(t), \tag{1b}$$

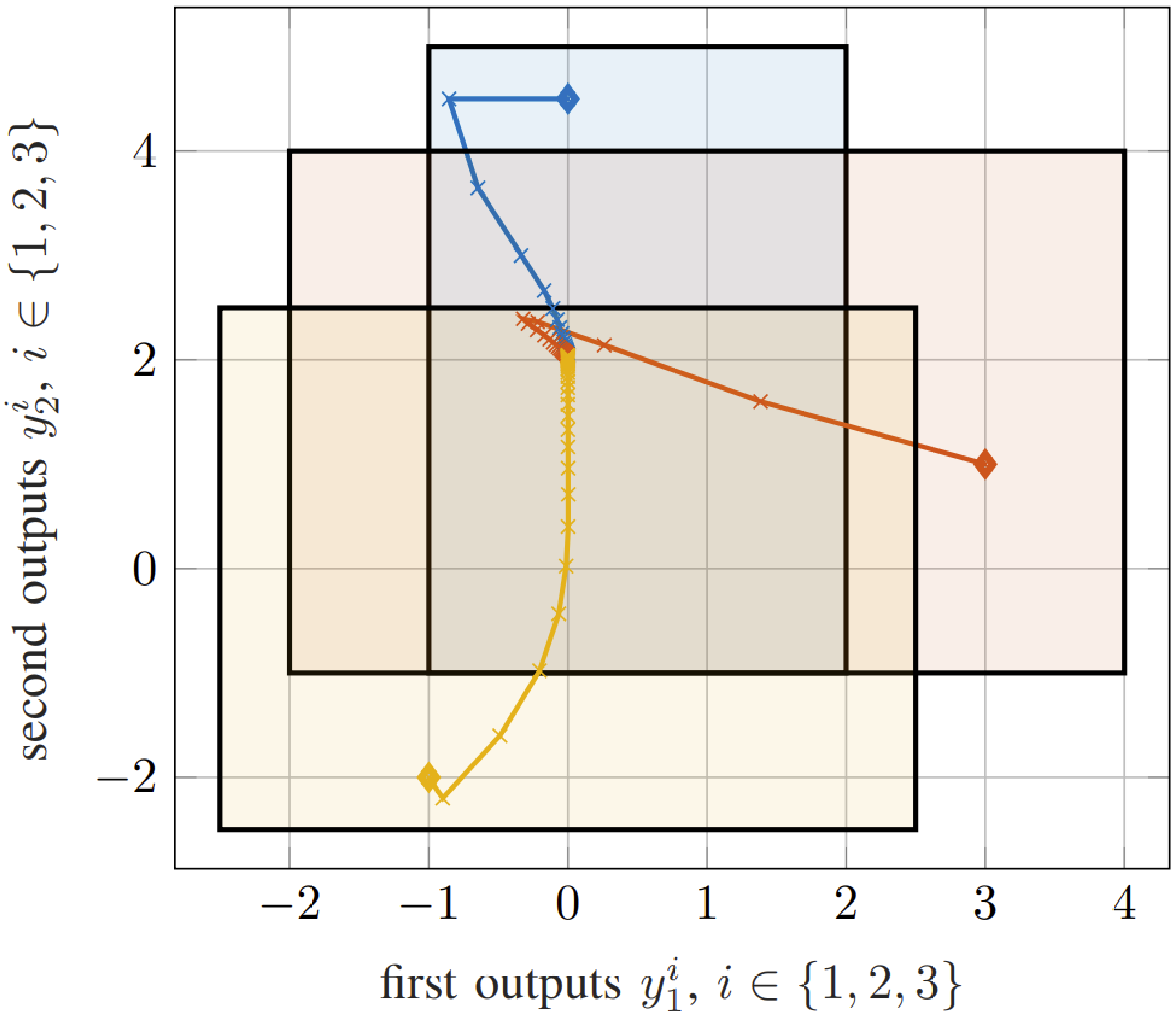
Multi-agent MPC

The communication topology of the multi-agent system is given by a connected, undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with the set of nodes, i.e. agents, $\mathcal{V} = \{1, \ldots, m\}$ and edges $\mathcal{E}$. Agents may bidirectionally share information with each other if they are connected by an edge. The induced set of neighbours of agent i is given as $\mathcal{N}_i = \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}$. The symmetric graph Laplacian is defined as $L = [l_{ij}] \in \mathbb{R}^{m \times m}$ where

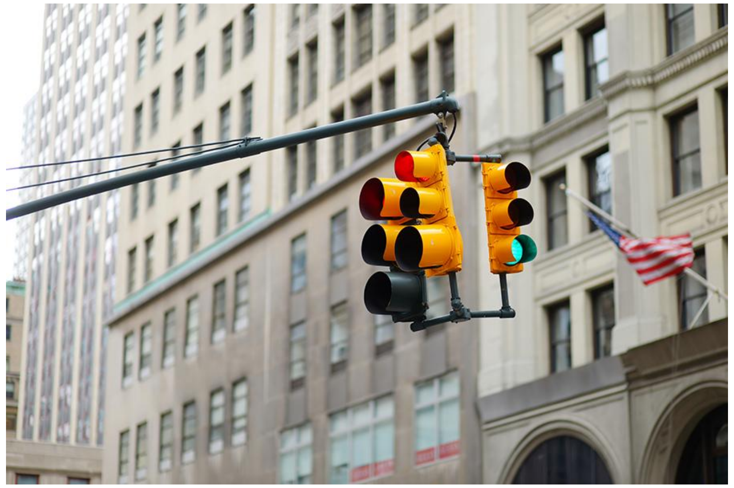
$$l_{ij} = \begin{cases} -1, & j \in \mathcal{N}_i, \\ |\mathcal{N}_i|, & j = i, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

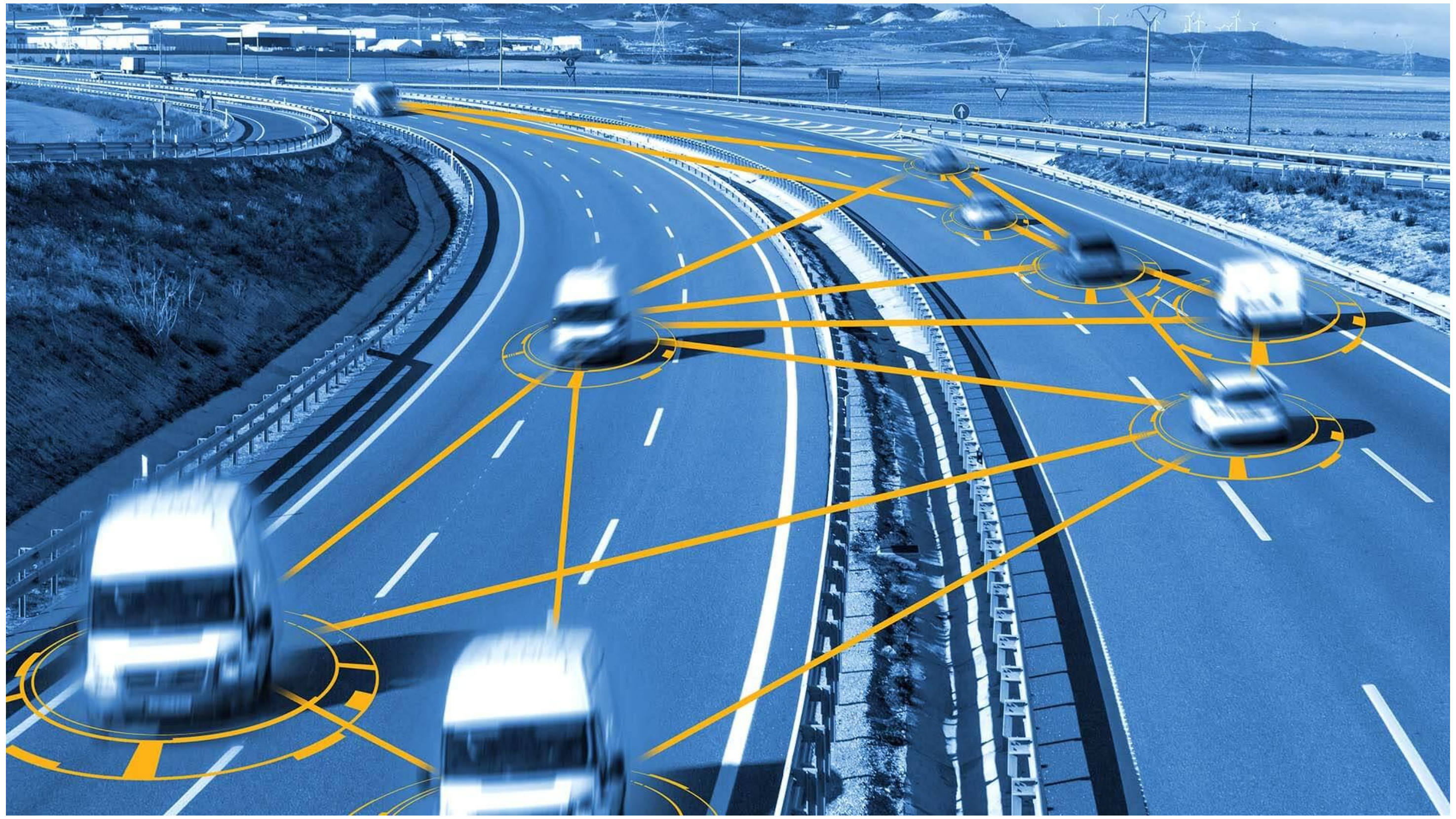


Multi-Agent MPC



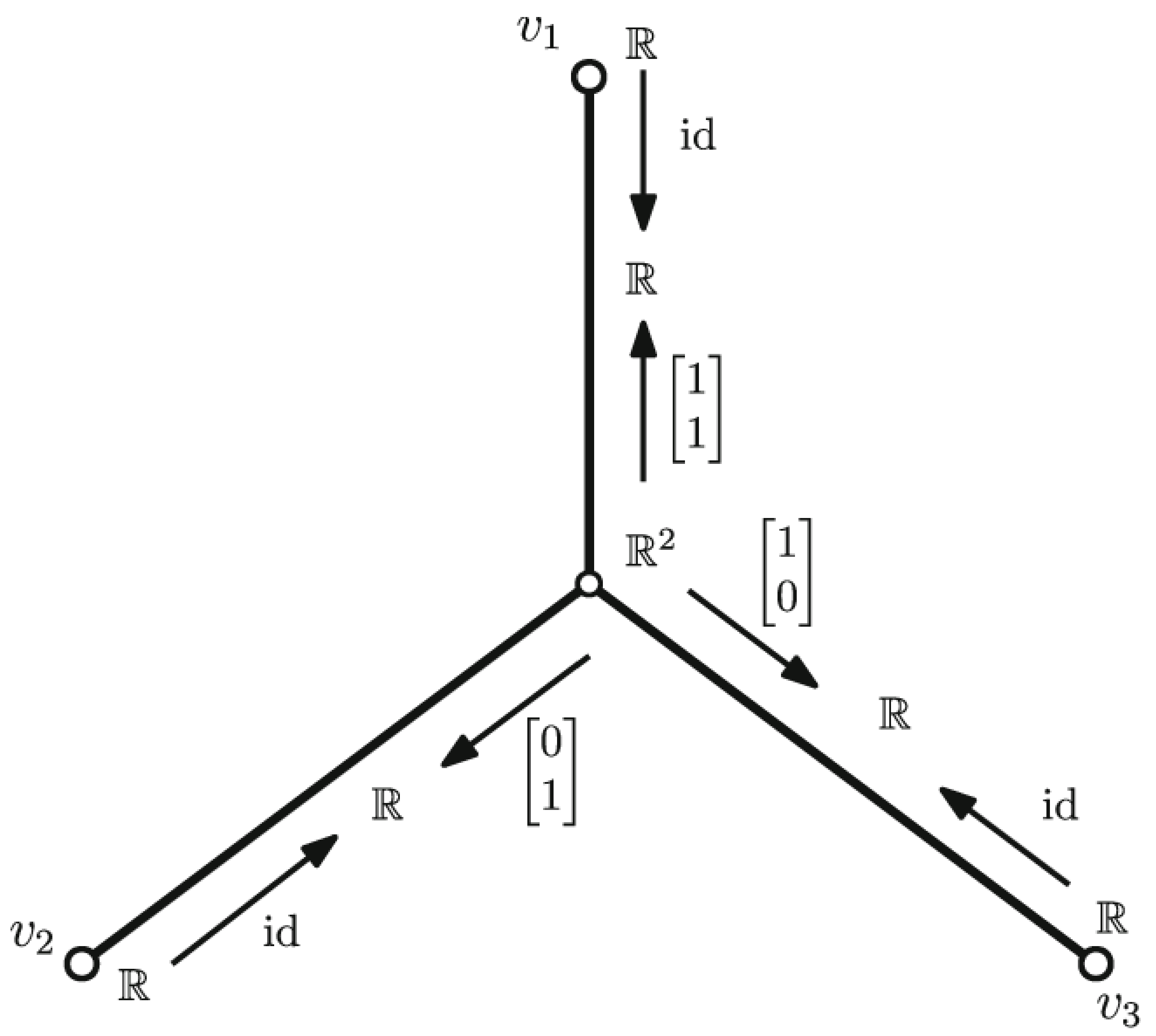
Multi-agent MPC





Cellular Sheaves

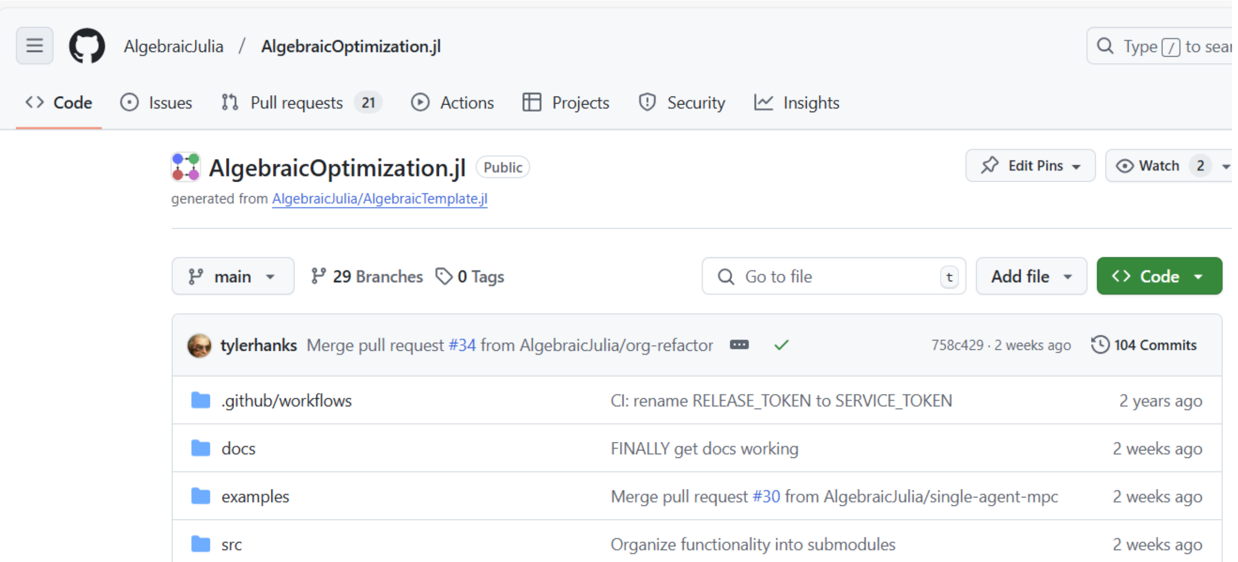
- Connected graph G = (V, E)
- Vertex and edge stalks
- Restriction maps
- Sheaf Laplacian

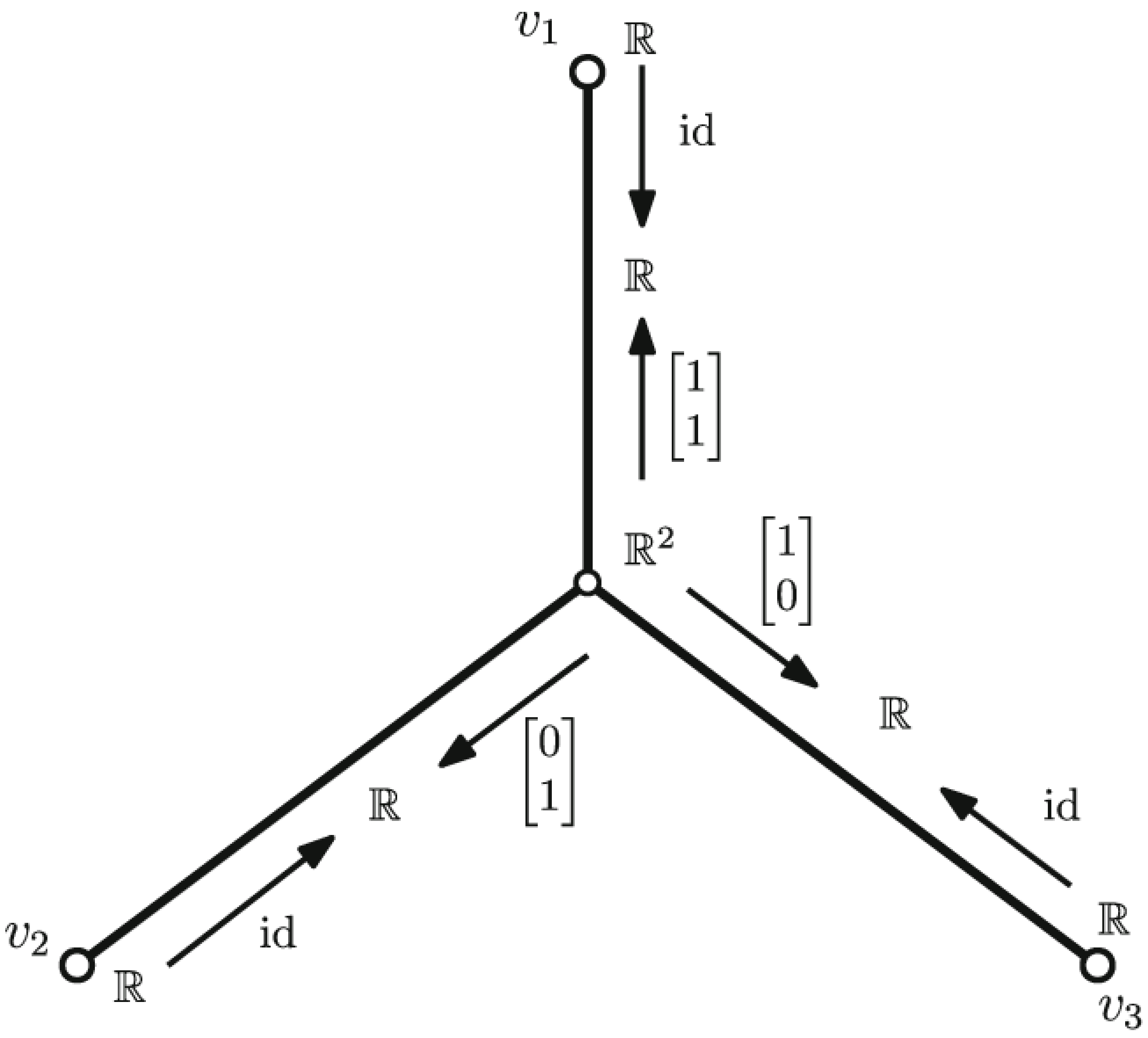


Cellular Sheaf Optimization

Minimize $\sum f(x)$

Subject to $Lx = 0$





A Sheaf Theoretic Framework for Multi-Agent Model Predictive Control

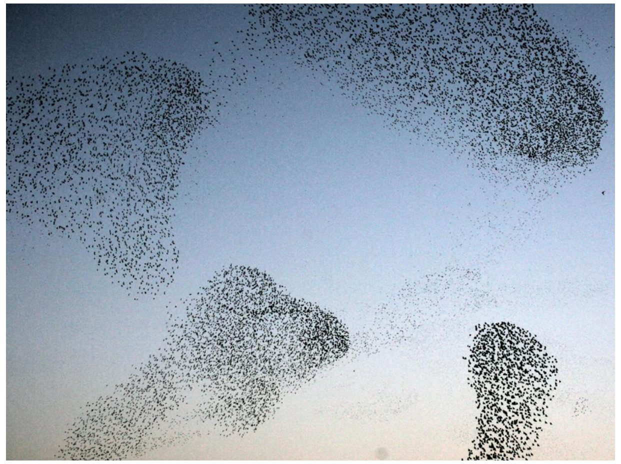
- We can do multi-agent MPC using cellular sheaves

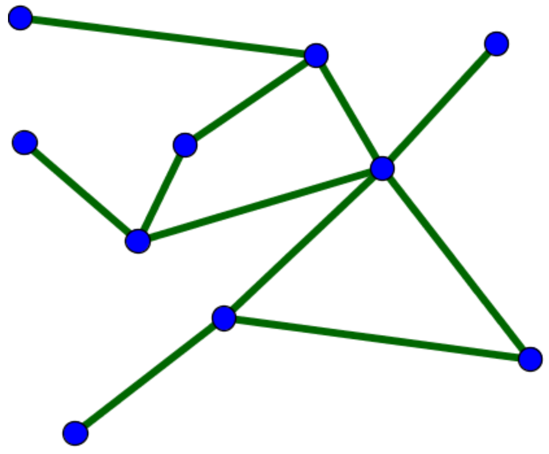
A. Flocking Problem

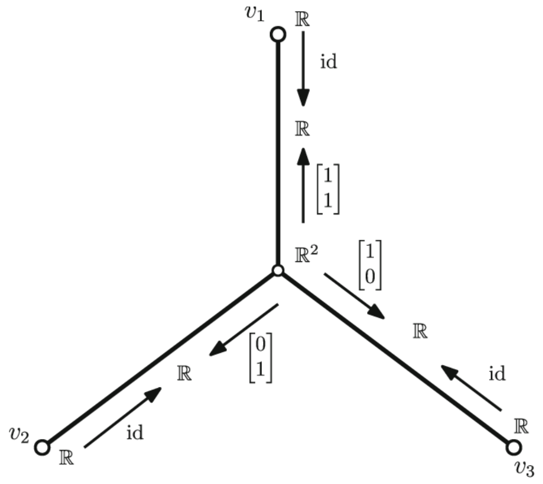
Develop a decentralized consensus protocol $u_i(k) = f_i(p_i(k), p_j(k), q_i(k), q_j(k)), j \in \mathcal{N}_i(k), i \in \mathcal{V}$, satisfying (2) and

$$\begin{aligned} \lim_{k\to\infty} \|q_i(k) - q_j(k)\| &= 0, \\ \lim_{k\to\infty} \|p_i(k) - p_j(k)\| &= 0, \quad \forall i, j \in \mathcal{V}. \end{aligned} \tag{4}$$

Note that "flocking" refers to a group of agents moving or migrating together with the same velocity.







Implementation

ADMM

- Iterative consensus algorithm

$$
\begin{aligned}
x_i^{k+1} &:= \operatorname{argmin}_{x_i} f_i(x_i) + (\rho/2)\|x_i - z_i^k + y_i^k\|_2^2 \\
\mathbf{z}^{k+1} &:= \Pi_{\mathcal{C}}(\mathbf{x}^{k+1} + \mathbf{y}^k) \\
y_i^{k+1} &:= y_i^k + x_i^{k+1} - z_i^{k+1}
\end{aligned}
$$

Implementation

1. Setup dynamical system $s(A, B, C)$ where $x' = Ax + Bu$
2. Setup agent objective function $x^T Q x + u^T R u$
3. Set time horizon and control bounds
4. Setup communication pattern, cellular sheaf c
5. Set initial state, MPC problem, and ADMM algorithm parameters
6. Run solver on these conditions
7. Plot resulting trajectories

JuMP

- Modeling language for mathematical optimization in Julia

```julia
function optimize_step(x_k, Q, R, s::DiscreteLinearSystem, x_target, ρ::Real)
    # Constants
    horizon = 10  # Prediction horizon

    # Define the optimization model using Ipopt solver
    model = Model(Ipopt.Optimizer)
    set_silent(model)  # Suppress solver output

    # Decision variables: state trajectory (x) and control inputs (u)
    @variable(model, x[1:2, 1:horizon])
    #@variable(model, u[1:2, 1:horizon])
    @variable(model, -20 <= u[1:2, 1:horizon] <= 20)  # Control limits

    # Initial state and control constraints
    @constraint(model, x[:, 1] .== x_k)
    #@constraint(model, u[:, 1] .== u_k)

    # System dynamics constraints: x[k+1] = Ax[k] + Bu[k]
    for k = 1:horizon-1
        @constraint(model, x[:, k+1] .== s.A * x[:, k] + s.B * u[:, k])
    end

    # Define the cost function (sum of squared states and inputs over the horizon)
    @objective(model, Min, sum((x[:, k]' * Q * x[:, k]) + (u[:, k]' * R * u[:, k]) for k = 1:horizon)) + ρ / 2 * ((x[:, horizon] - x_target)' * Q * (x[:, horizon] - x_target))

    # Solve the optimization problem
    optimize!(model)

    # Return the optimized control input for the next time step
    return value.(x[:, horizon]), value.(u[:, 1])
end
```

Global Sections

- Computes the projection of x onto the space of global sections of s

- Solve Laplacian: $Lx = b$

- b encodes the displacement

```
function nearest_section(s::CellularSheaf, x, b)
    d = coboundary_map(s)

    eL = LinearOperator(d) * LinearOperator(d')

    rhs = d * x - b

    y, stats = cg(eL, Array(rhs))

    return BlockArray(x - d' * y, s.vertex_stalks)

end
```

Documentation

[Moving Formation Example · AlgebraicOptimization.jl]

Paper submitted

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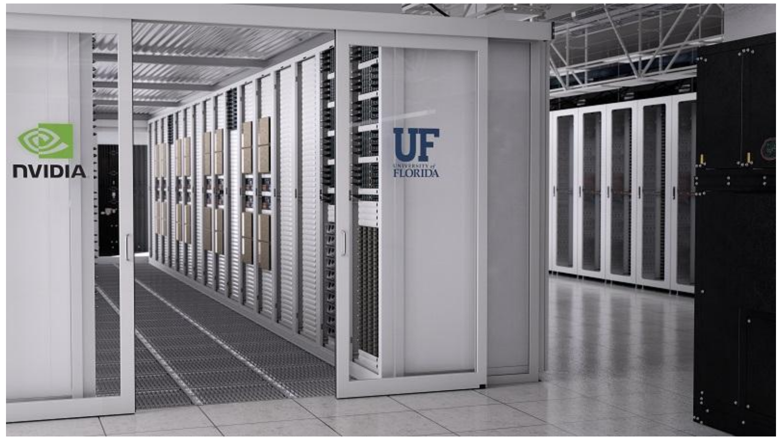
Up Next

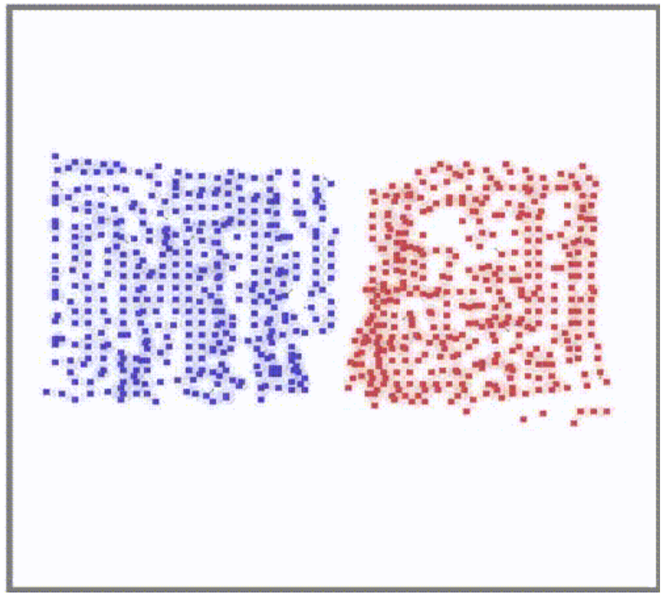
Paper extension

- Multithreaded implementation
- Distributed implementation
- Runtime analysis



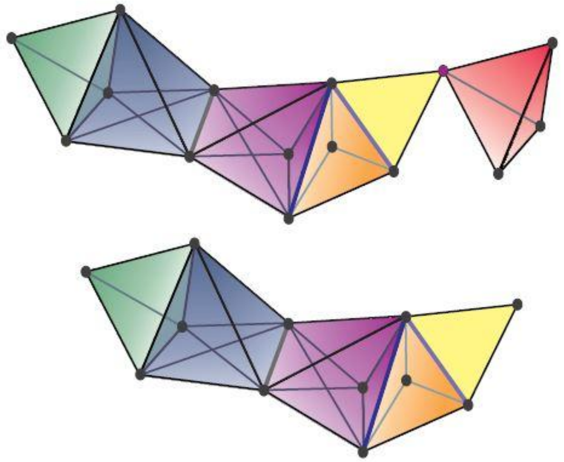
ML Applications

- Multitask learning
- Multiagent reinforcement learning



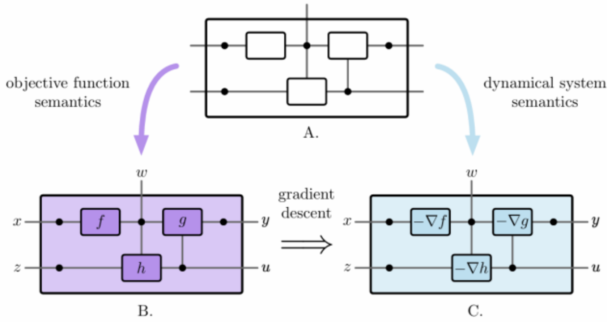
More sheaves

- Simplicial complexes
- Structures beyond graphs



More optimization

- Newton's Method
- Black boxing gradient descent
- Unified functorial gradient descent framework



Thanks for listening!