Discrete Exterior Calculus Tips & Tricks 4:

Qualities and Quantities

GATAS Lab Seminar Series, Fall 2024

Luke Morris

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Discrete Exterior Calculus Tips & Tricks:

All you need is $\star\, d\, \wedge$

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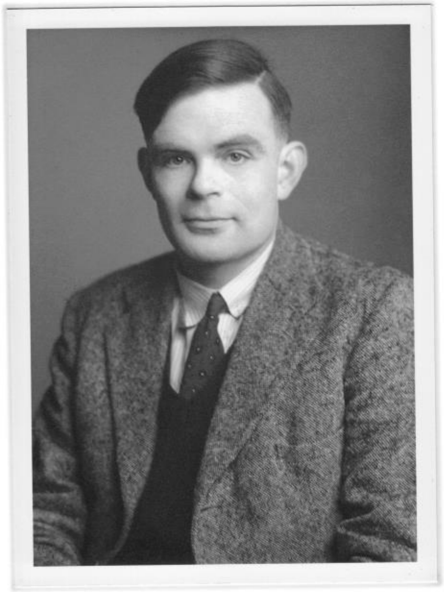
Recap: Finite Difference Methods

72 A. M. TURING ON THE CHEMICAL BASIS OF MORPHOGENESIS

from one pattern into another, rather than from homogeneity into a pattern. One would like to be able to follow this more general process mathematically also. The difficulties are, however, such that one cannot hope to have any very embracing *theory* of such processes, beyond the statement of the equations. It might be possible, however, to treat a few particular cases in detail with the aid of a digital computer. This method has the advantage that it is not so necessary to make simplifying assumptions as it is when doing a more theoretical type of analysis. It might even be possible to take the mechanical aspects of the problem into account as well as the chemical, when applying this type of method. The essential disadvantage of the method is that one only gets results for particular cases. But this disadvantage is probably of comparatively little importance. Even with the ring problem, considered in this paper, for which a reasonably complete mathematical analysis was possible, the computational treatment of a particular case was most illuminating. The



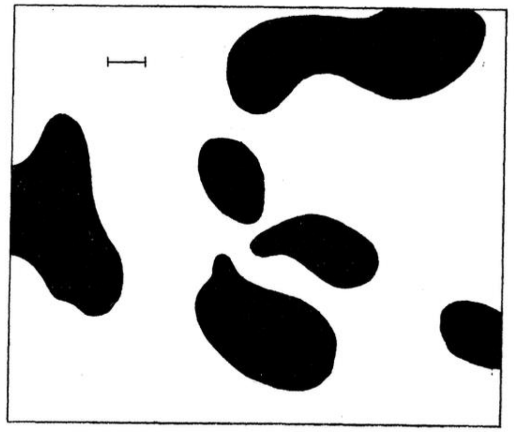
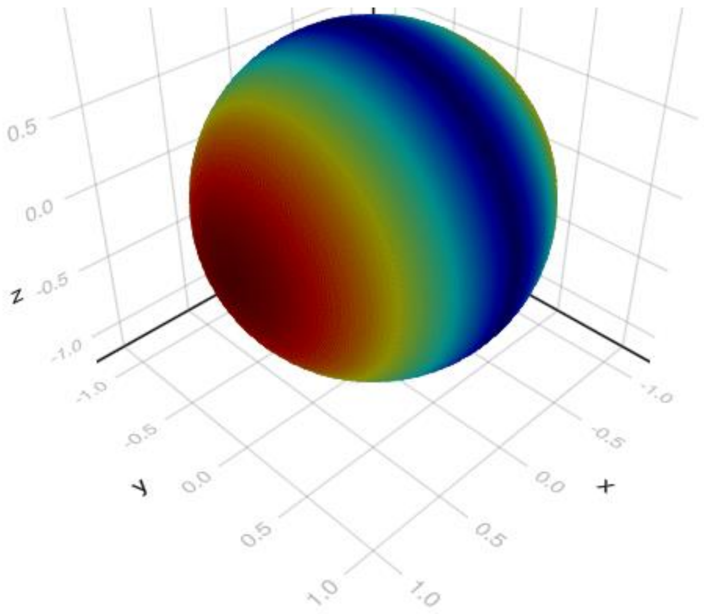


FIGURE 2. An example of a 'dappled' pattern as resulting from a type (*a*) morphogen system. A marker of unit length is shown. See text, §9, 11.

$$\begin{tikzcd}
\Omega_0 \arrow[r, "d"] \arrow[d, shift right, "\star"'] & \Omega_1 \arrow[r, "d"] \arrow[d, shift right, "\star"'] & \Omega_2 \arrow[r, "d"] \arrow[d, shift right, "\star"'] & \Omega_3 \arrow[d, shift right, "\star"'] \\
\tilde{\Omega}_n \arrow[u, shift right, "\star^{-1}"'] & \tilde{\Omega}_{n-1} \arrow[l, "\tilde{d}"] \arrow[u, shift right, "\star^{-1}"'] & \tilde{\Omega}_{n-2} \arrow[l, "\tilde{d}"] \arrow[u, shift right, "\star^{-1}"'] & \tilde{\Omega}_{n-3} \arrow[l, "\tilde{d}"] \arrow[u, shift right, "\star^{-1}"']
\end{tikzcd}$$

```
GrayScott = @decapode begin
  (U, V)::Form0{X}
  (f, k, rᵤ, rᵥ)::Constant{X}

  UV2 == (U .* (V .* V))
  ∂ₜ(U) == rᵤ * Δ(U) - UV2 + f * (1 .- U)
  ∂ₜ(V) == rᵥ * Δ(U) + UV2 - (f + k) .* V
end
```



Discrete Exterior Calculus Tips & Tricks 2:

The Laplacian and Spectral Analysis

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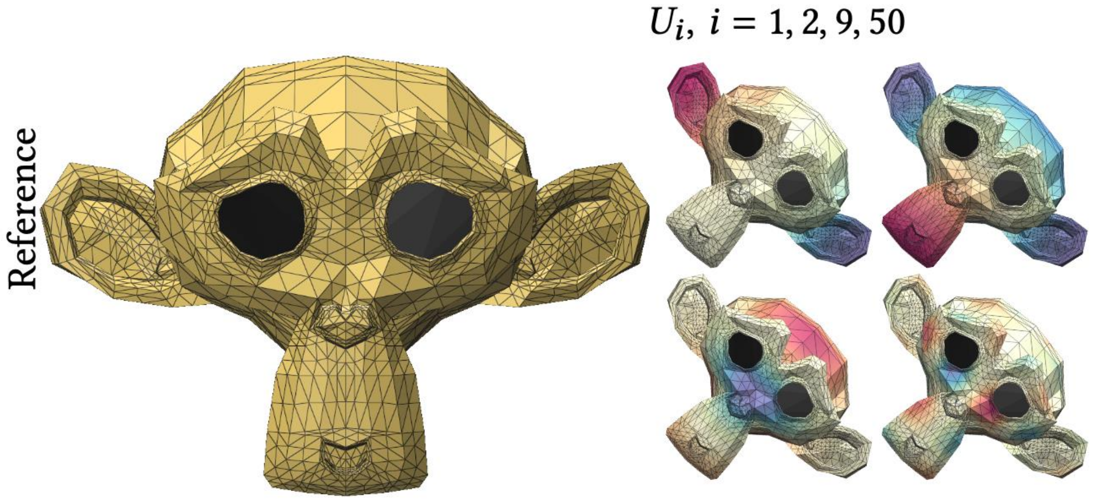
Recap: Eigenvectors of the Laplacian

- If the Laplacian is "**local deviation from average**"
- And if the Laplacian is "**imbued with geometry**"
- And if the eigenvectors are "**principal components**"…

- Then, we can study the principal ways in which we deviate from local position



Eigenvectors of the Laplacian on Blender's "Suzanne" mesh from Spectral Coarsening with Hodge Laplacians *from Keros & Subr*

Discrete Exterior Calculus Tips & Tricks 3:

Ein neuer Zweig

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Exterior Algebra: Products

- The product must be another extensive quantity.
- But which? “[D]arüber sagt die Definition nichts aus”
- Algebraic:
  - [E1 E1], [E2 E2], and [E1 E2] = [E2 E1]
- Combinatoric:
  - [E1 E1] = [E2 E2] = 0 , and [E1 E2] = - [E2 E1]
- Inner:
  - [E1 E1] = [E2 E2] = 1 , and [E1 E2] = [E2 E1] = 0

Outline of Today's Talk

GIT, EC

Grassmann
Whitney

DEC

Hirani
Eldred

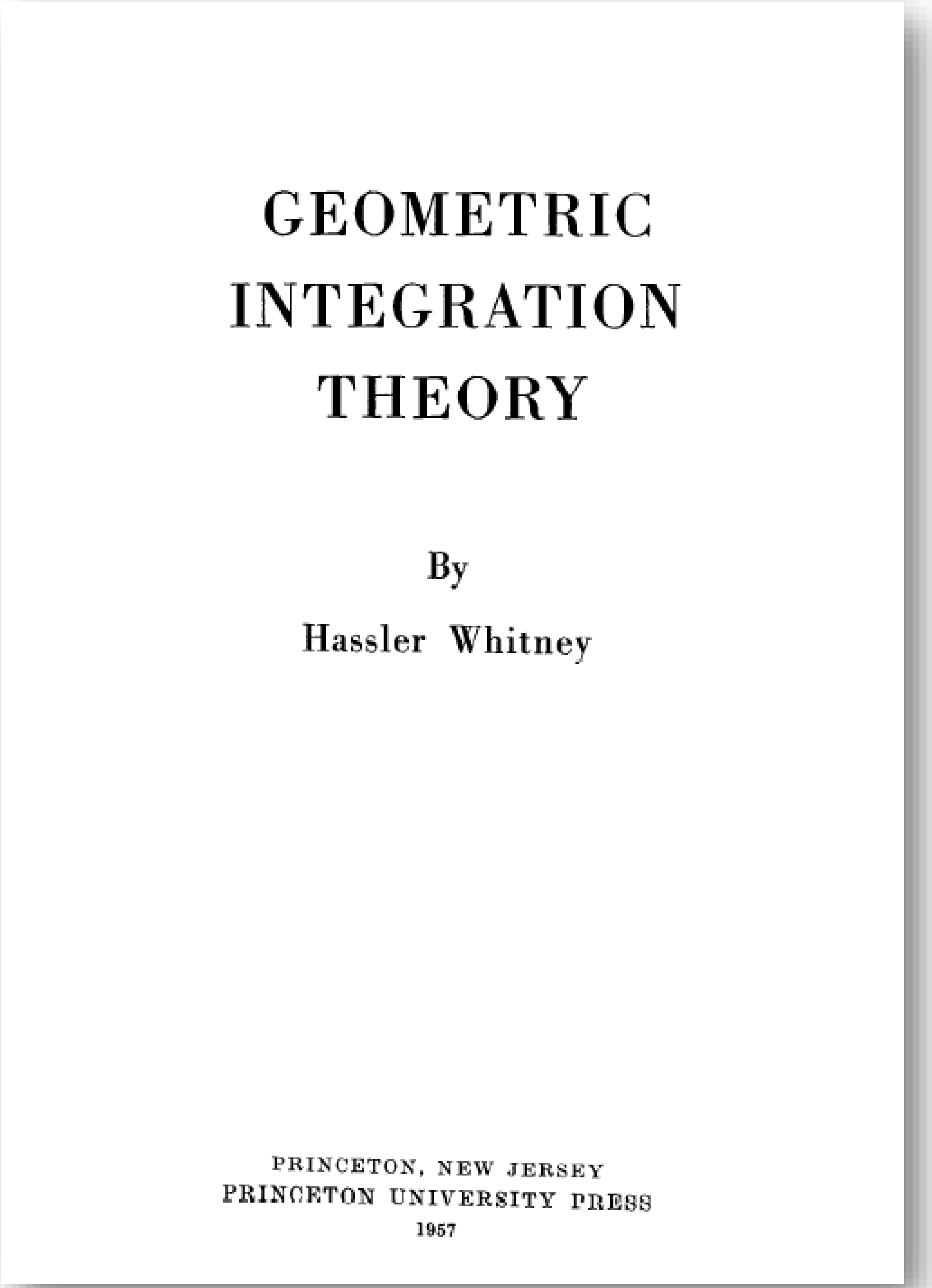
SEC

Rufat
Lessig
da Silva

FEEC

Arnold
Gates & Bittens

Whitney: Geometric Integration Theory (GIT)



GEOMETRIC
INTEGRATION
THEORY

By

Hassler Whitney

PRINCETON, NEW JERSEY
PRINCETON UNIVERSITY PRESS
1957

- Grassmann recovered “geometry” from first principles
- Whitney recovered “integration” from first principles
- Cells, chains, and co-chains

Whitney: Geometric Integration Theory (GIT)

- Those Principles:
- Cells have orientation. Switching switches sign of integral
- Subdivided cells yield same integral
- Chains are linear combinations of cells, respect 0 & 1 units
- Chains invariant under subdivision
- Take integrals piecewise, thus co-chains are integrands

Hirani: Discrete Exterior Calculus (DEC)

- "First" to encode Finite Difference Methods via exterior calculus
- Store operators as matrices
- Use "dual de Rham complex" (primal & dual complexes)

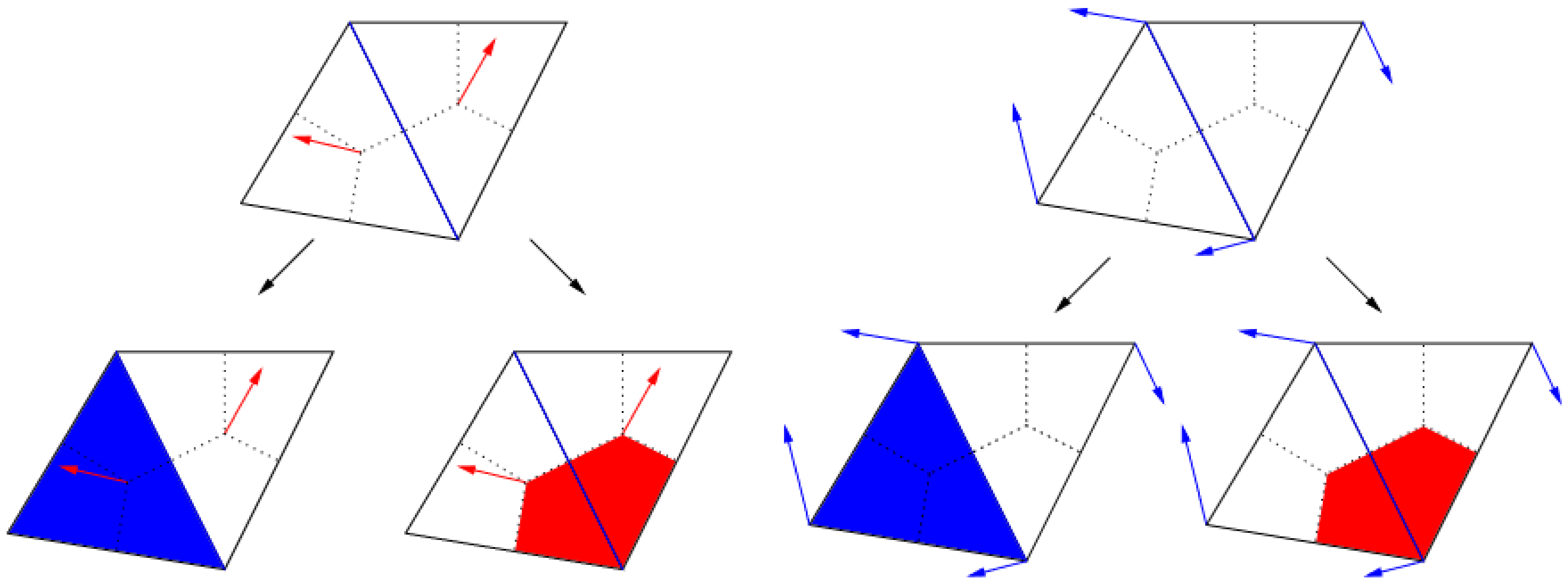
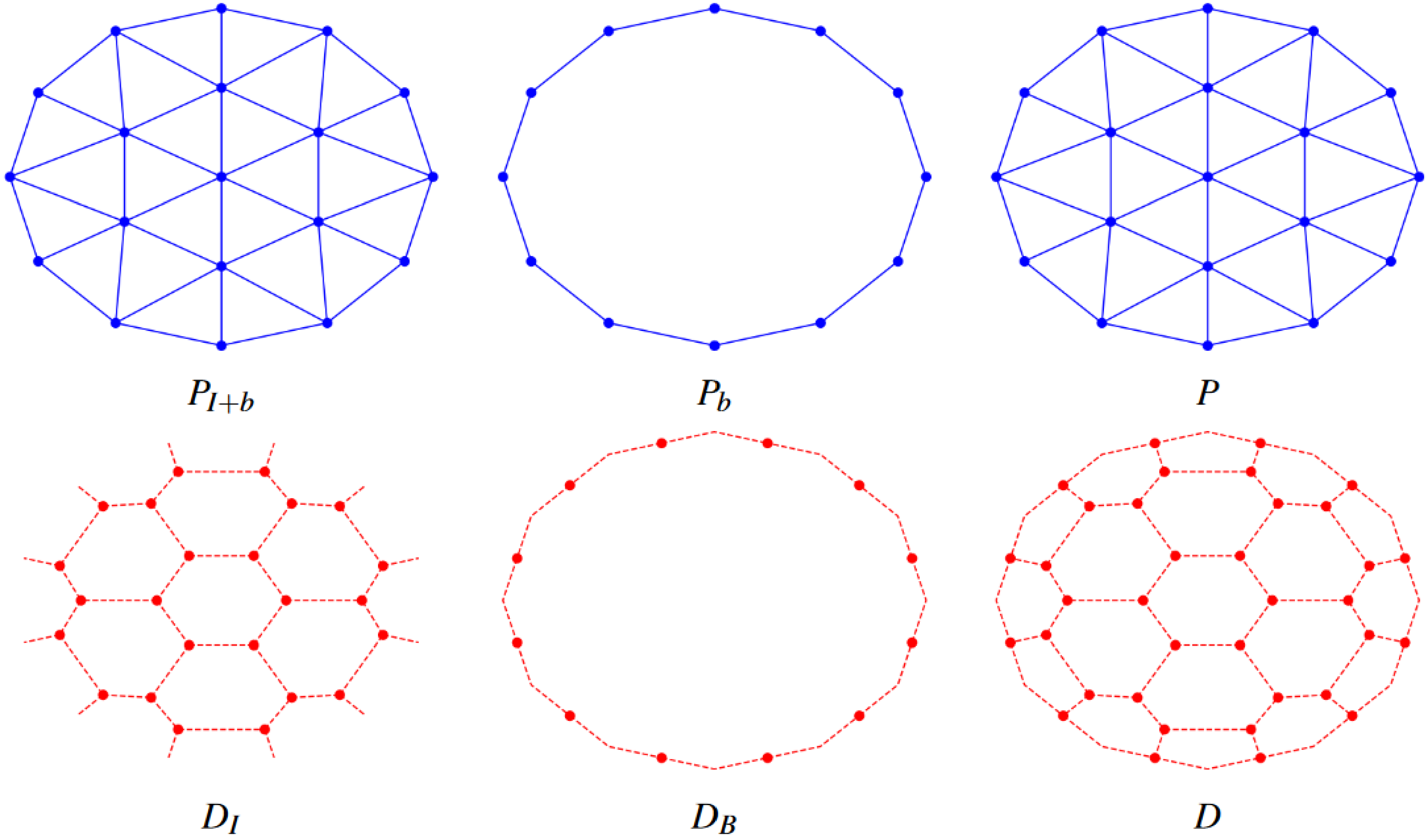


Figure 5.2: *Top row:* dual (left) and primal (right) vector fields X for which $X^\flat$ is desired on the shared edge ; *Bottom row:* (left to right) dual-primal, dual-dual, primal-primal and primal-dual interpolations. See Def. 2.7.1 for more on interpolation. The bottom row corresponds to the configuration for the discrete flats $\flat_{\mathrm{dpp}}$, $\flat_{\mathrm{ddp}}$, $\flat_{\mathrm{ppp}}$ and $\flat_{\mathrm{pdp}}$. A dual destination would yield 4 more flats for a total of 8 discrete flats.

Christopher Eldred: Bounded Domains

- Boundary conditions are “a rich topic”
- Use twisted/ straight forms, in/dependent of ambient orientation
- Boundary extension lets us define a discrete trace

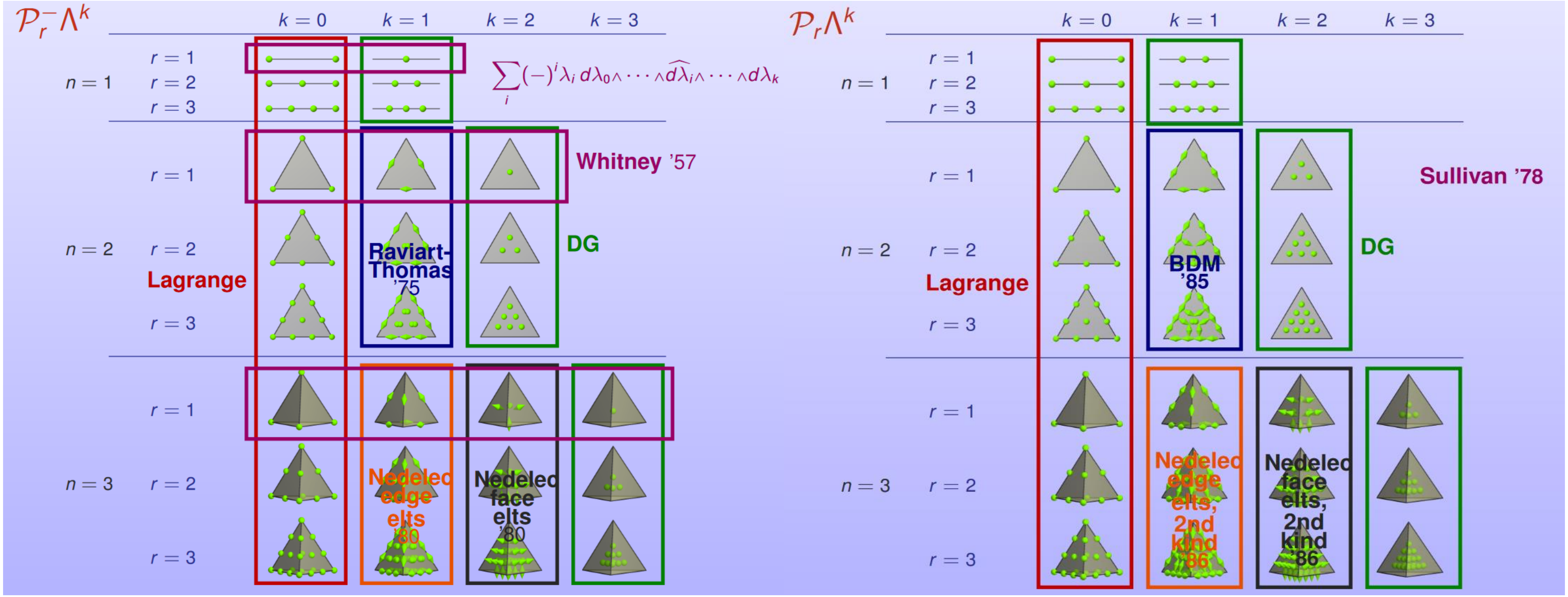
Christopher Eldred: Bounded Domains



Arnold et al.: Finite Element Exterior Calculus (FEEC)

- Encode Finite Element Methods via Exterior Calculus
- The real deal
- Enforce well-posedness: Stability via Compatibility
- A series of chain complexes
- Polynomial chain complex

Arnold et al.: Finite Element Exterior Calculus (FEEC)



https://www-users.cse.umn.edu/~arnold/femtable/

Gates & Bittens: *hp*-schemes for FEEC

- *h*-refinement: increase resolution
- *p*-refinement: increase degree of polynomial
- Generalize to higher dimensions as side effect
- “Reference element mapping”: Atlas to reference simplex

Gates & Bittens: *hp*-schemes for FEEC

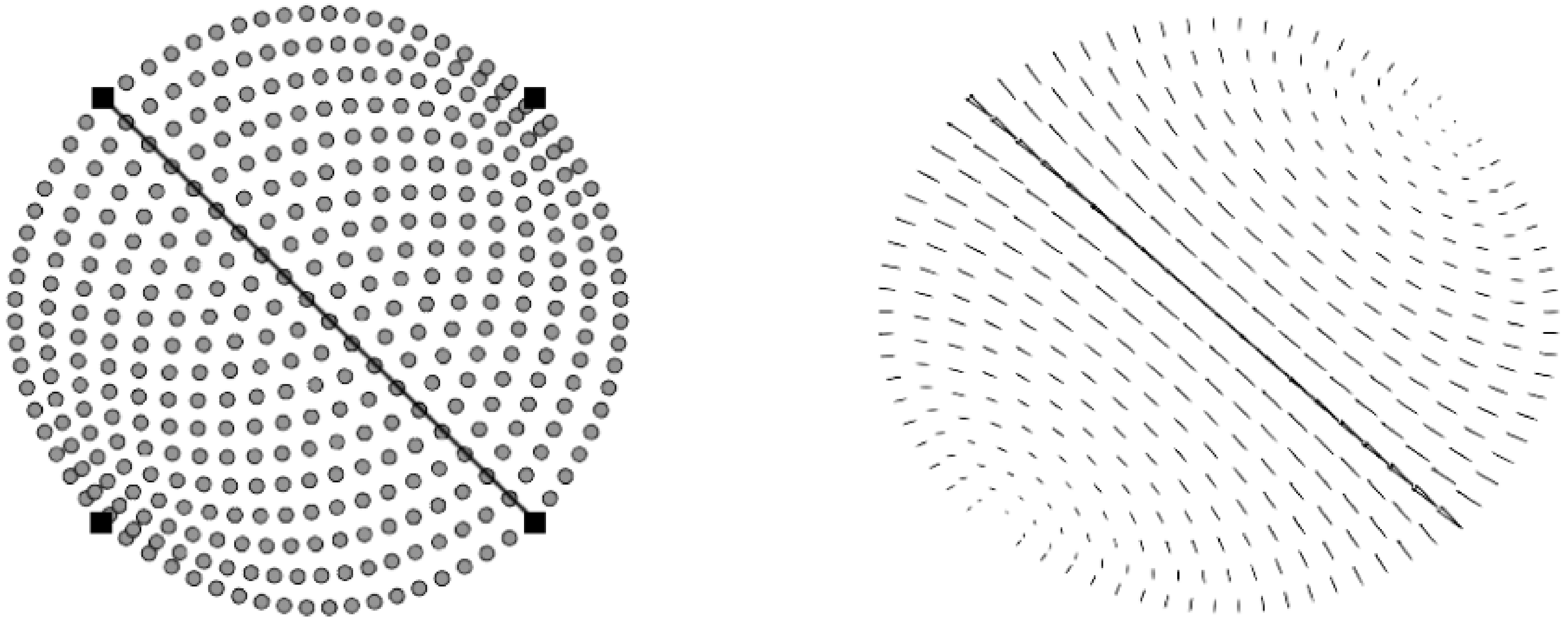


Figure 2: Cartesian grid of points on the reference element mapped onto two curvilinear triangles (vertices marked with squares, common edge marked as a black line) (left); Whitney 1-form basis function for the common edge mapped to the geometry (right)

Spectral Exterior Calculus

- Spectral Methods via Exterior Calculus
- Take Fourier expansion of both sides, match-up coefficients
- Not a "framework", but Vallet & Lévy:
  - Noisy Laplacian -> FFT -> Filter -> FFT -> De-noised Laplacian
- Not a "framework", but Berry & Giannakis:
  - Manifold learning shouldn't "just" learn 0-Laplacian, rather the entire exterior calculus on the manifold. Learn eigenforms of *all* Laplacians.

Spectral Exterior Calculus

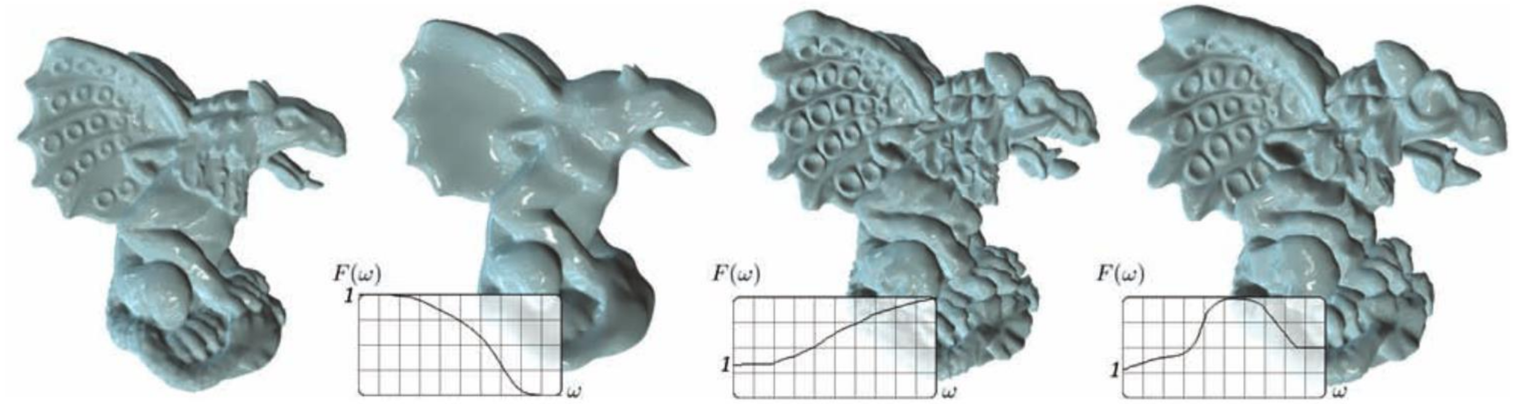


Figure 5: *Low-pass, enhancement and band-exaggeration filters. The filter can be changed by the user, the surface is updated interactively.*

Vallet & Lévy

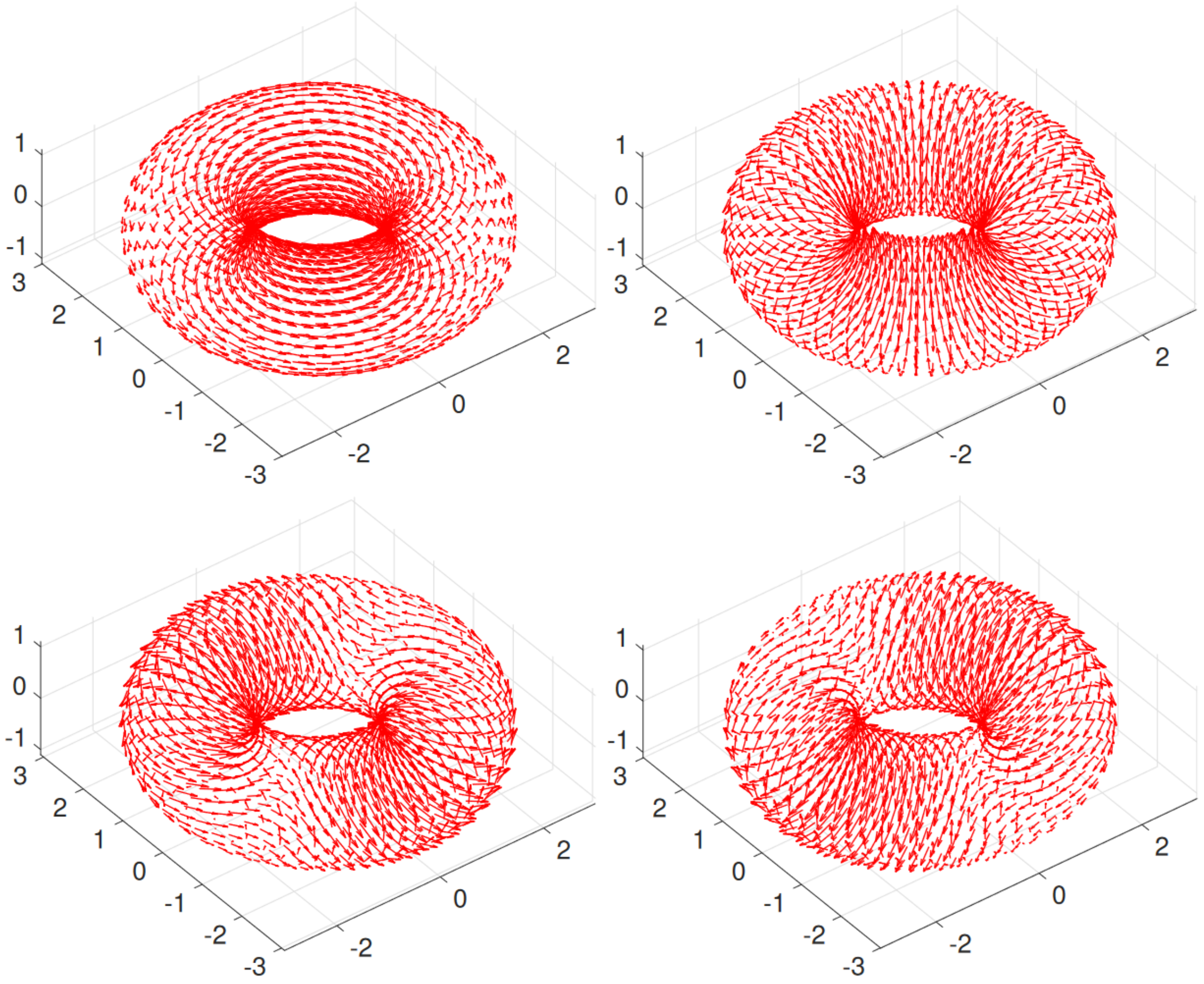


FIGURE 8.5. First four eigenforms on the standard embedding of the torus in $\mathbb{R}^3$. Notice that the first two represent the two 1-homology classes.

Berry & Giannakis

Rufat: Spex

- Collocation method:
  - Choose points where PDE will be enforced exactly; fit basis functions
- “Chain collocation method”
  - Choose *d*-cells where PDE will be enforced exactly
- Define on 1-manifold, then tensor to higher dimensions

Rufat: Spex

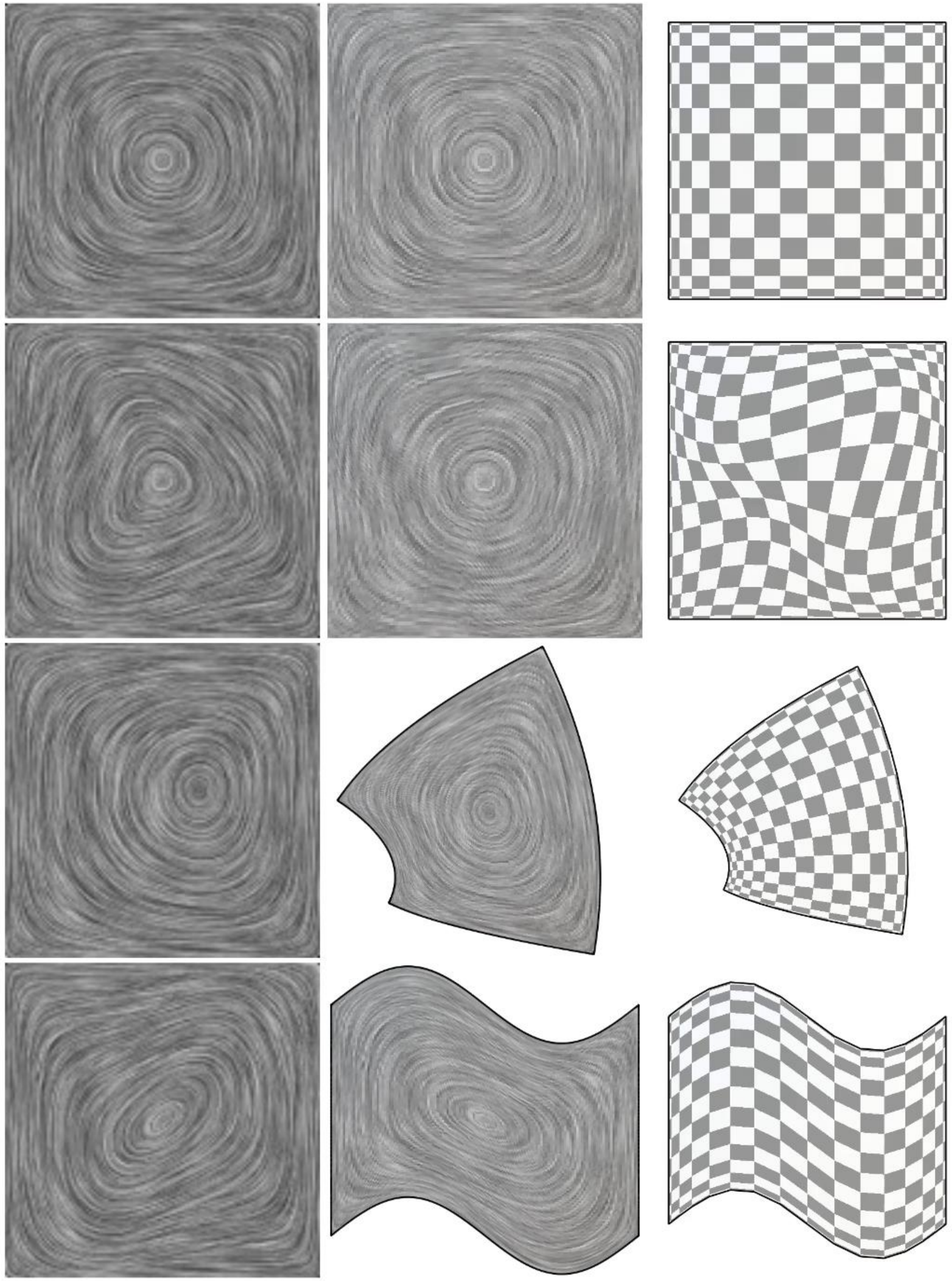


Figure 5.5: Solution to Laplace's equation for 1-forms on curved domains with boundary conditions $\star f|_{\mathcal{B}} = 0$ and $\star \mathbf{d} f|_{\mathcal{B}} = 1$. The rightmost column shows the grid, the middle column shows the solution, and the leftmost column shows the pull-back of the solution to the reference grid.

Lessig: ψec

- “[T]he Fourier transform of the differential form basis functions has a structure analogous to those of the Hodge dual”
- FT of a 0-form maps to volume-form
- Differential form wavelets
- FT of de Rham complex is a chain complex

Proposition 4. *The Fourier transform (or principal symbol) of the exterior derivative* $\mathrm{d} : \Omega^r(\mathbb{R}^n) \to \Omega^{r+1}(\mathbb{R}^n)$ *is the anti-derivation* $\widehat{\mathrm{d}} : \widehat{\Omega}^{n-r}(\widehat{\mathbb{R}}^n) \to \widehat{\Omega}^{n-r-1}(\widehat{\mathbb{R}}^n)$ *given by the interior product* $\mathrm{i}_{i\xi}\widehat{\alpha}$*, i.e.*

$$\mathfrak{F}\big(\mathrm{d}\alpha\big)(\xi) = \big(\widehat{\mathrm{d}}\,\widehat{\alpha}\big)(\xi) = \mathrm{i}_{i\xi}\widehat{\alpha}(\xi)$$

where ξ *is the coordinate vector* $\xi = \xi_1\,\mathrm{d}\xi^1 + \cdots + \xi_n\,\mathrm{d}\xi^n$*. The following diagram thus commutes*

$$\begin{array}{ccc}
\Omega^r(\mathbb{R}^n) & \xrightarrow{\quad\mathrm{d}\quad} & \Omega^{r+1}(\mathbb{R}^n) \\
\mathfrak{F}\Big\downarrow & & \Big\downarrow\mathfrak{F} \\
\widehat{\Omega}^{n-r}(\widehat{\mathbb{R}}^n) & \xrightarrow{\hat{\mathrm{d}} = \mathrm{i}_{i\xi}} & \widehat{\Omega}^{n-r-1}(\widehat{\mathbb{R}}^n)
\end{array}$$

and the Fourier transform of the de Rahm co-chain complex is a chain complex.

da Silva et al.: ψec(S^2)

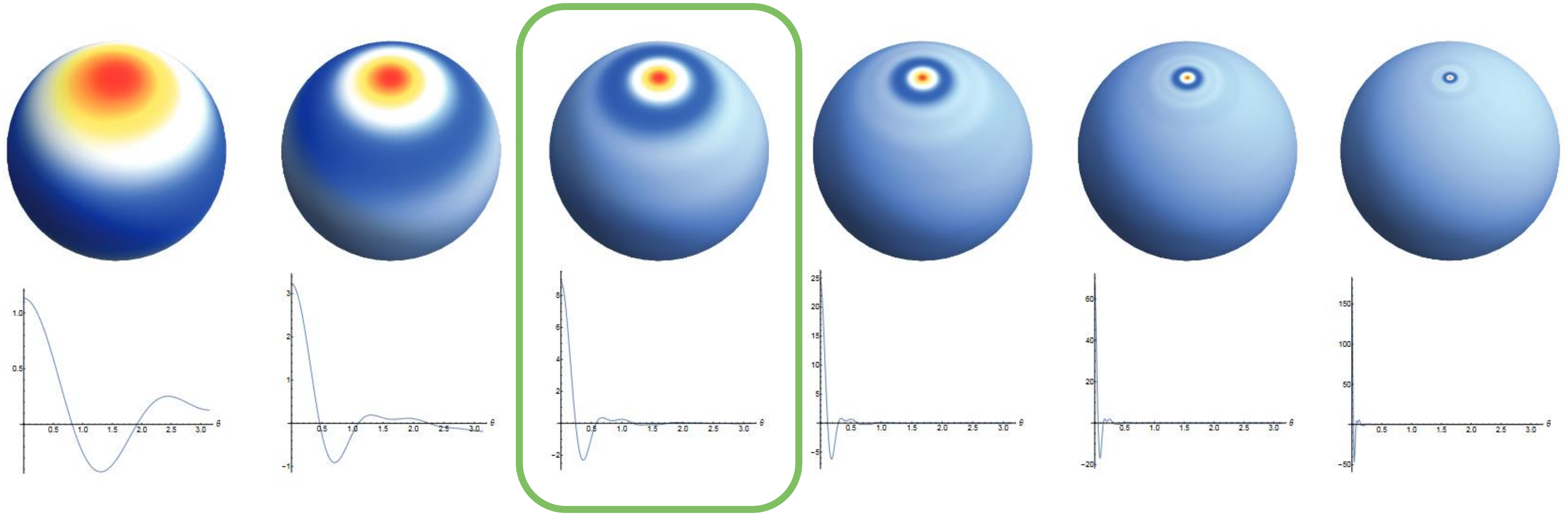


FIG. 6. *Wavelets $\psi_{j0}(\theta, \phi)$ and their profiles $\psi_{j0}(\theta, 0)$ for $j = 1$ to $j = 6$ based on the windows from [73]. The direct comparison to Fig. 2 shows the correspondence between the effective support of the wavelets and the density of the point set over which they are supported.*

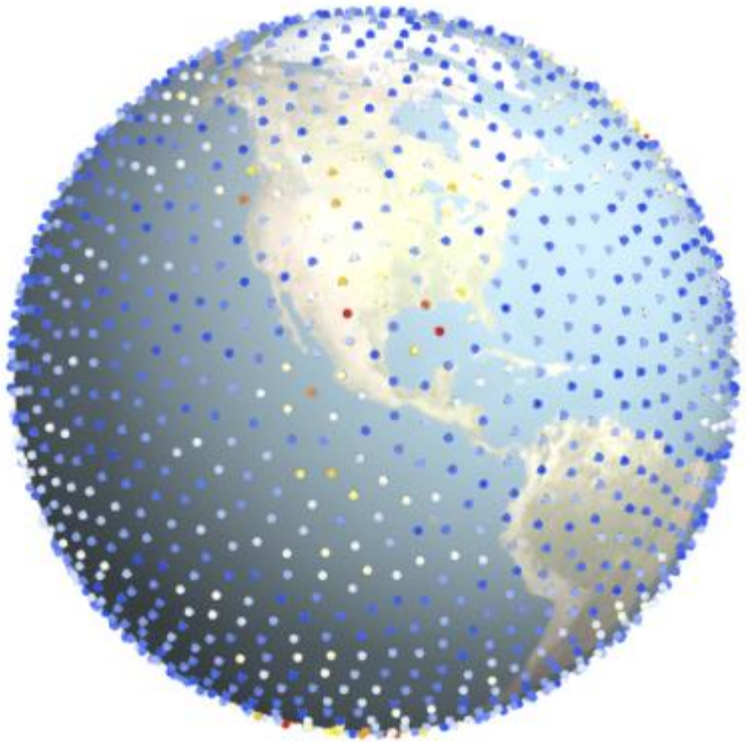
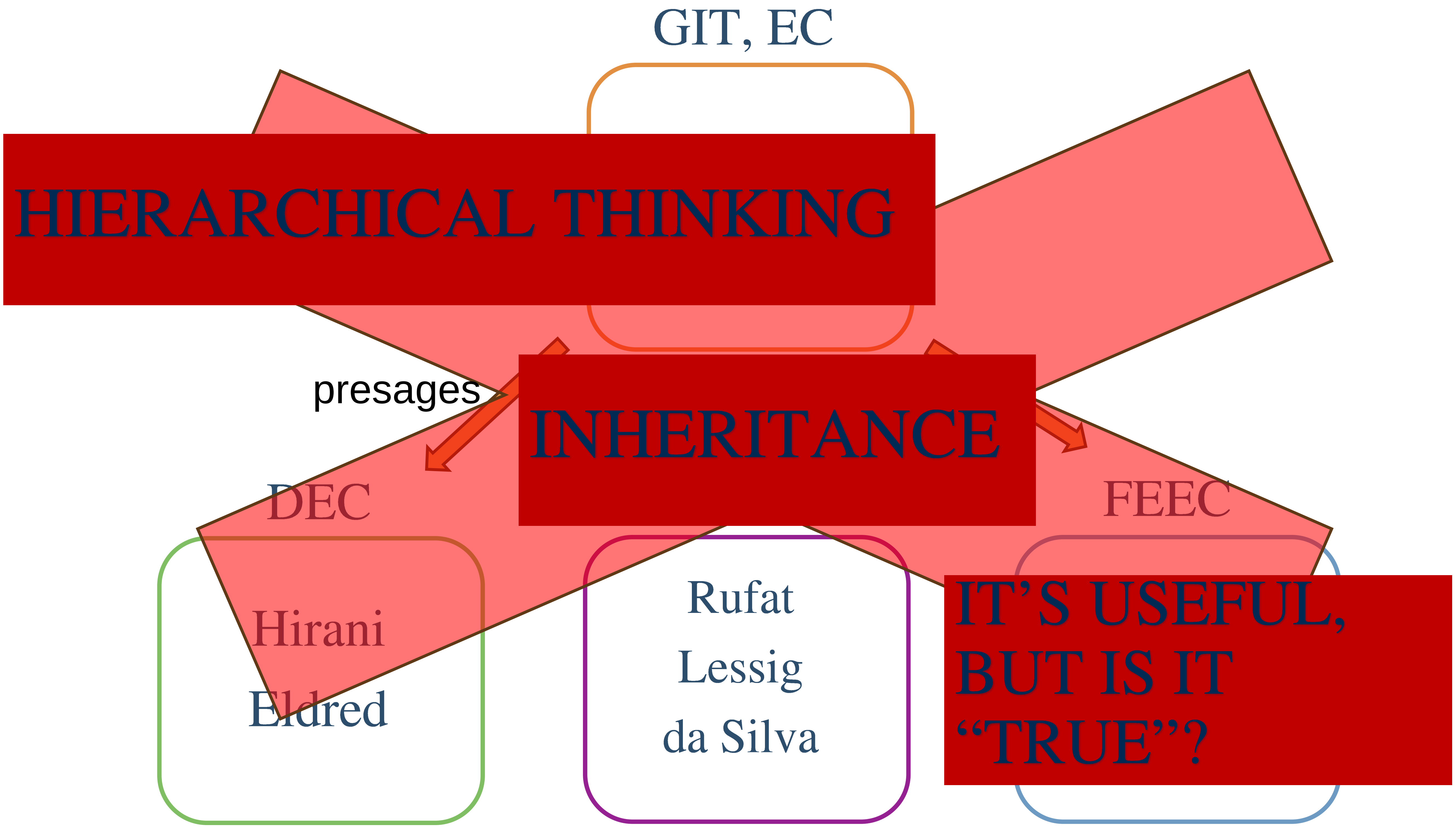


FIG. 8. *Visualization of wavelet coefficients for potential vorticity (PV) for 29/08/2005, the day hurricane Katrina made landfall at the gulf coast of the USA, see Fig. 7 for the spatial potential vorticity field. The coefficients are plotted at the locations of the basis functions with the magnitude encoded with a temperature map, with blue corresponding to low values and red to large ones.*

What I Presented



What Really Happened

THEORY		PRAXIS
Grassmann Whitney Hirani Eldred Arnold Gates & Bittens Rufat Lessig da Silva	contributes to → ← contributes to	Grassmann Whitney Hirani Eldred Arnold Gates & Bittens Rufat Lessig da Silva

Thanks!

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My LinkedIn My GitHub

About me:

I'm a 4th-year **Computer Science** PhD student in the Herbert Wertheim College of Engineering at the University of Florida.

I am employed as a graduate research assistant in the Mechanical & Aerospace Engineering Department.

I graduated with my Bachelor's in Computer Science from the University of Kentucky in 2021, *summa cum laude*.

My advisor here in Gainesville is Dr. James Fairbanks of the GATAS Lab.

My Current Research Involves:

- Applied Category Theory
- Multiphysics Simulations
- Space Weather
- High Performance Computing
- Opinion Dynamics