Motivation:

- We like to compose things w/ UWDS.
  - ↳ draw a UWD.
  - ↳ e.g. graphs, petri-nets, decapodes, dynamical systems, optimization problems.
- In these cases, boundary just designates which parts of system are "open" for future composition/nesting.
- But what if we want the boundary to be an event-horizon?
  - ↳ Answer: Decorated Corelations
- Mention the running example is "open" optimization problems

Outline

1) Brief review of decorated cospans.

2) Decorated corelation tutorial

   ↳ Done better/more clearly than Fong.

3) Relation to UWDs/cospan algebras.

   ↳ If time

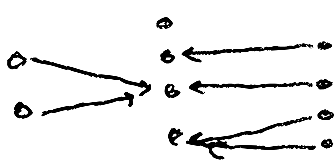
Decorated Cospans Intuition

E.g. "closed" minimization problem:

$f: \mathbb{R}^4 \to \overline{\mathbb{R}}$

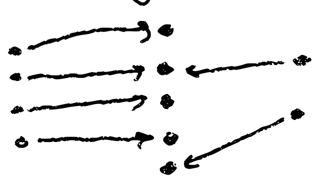
4 decision variables:

Idea: use a cospan to designate some variables as boundary vars.

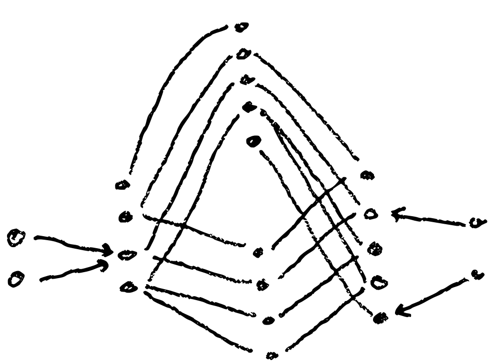


suppose we have $g: \mathbb{R}^5 \to \overline{\mathbb{R}}$ w/ boundaries:

Draw next to og



Draw pushout & leave on board!



composite function:

$(x_1, \ldots, x_5) \mapsto f(x_1, x_2, x_3, x_4) + g(x_2, x_3, x_4, x_4, x_5)$

How do we generalize to arbitrary systems?

Decorated Cospans Formally

- Need a way to "decorate" the apex of a cospan w/ a system defined on the apex.

  Solution: a functor $F: \mathrm{FinSet} \to \mathrm{Set}$

Def: An F-decorated cospan is a pair
$$(X \xrightarrow{\ell} N \xleftarrow{r} Y, 1 \xrightarrow{s} FN).$$

E.g. $\mathrm{Conv}: \mathrm{FinSet} \to \mathrm{Set}$
$$N \mapsto \{f: \mathbb{R}^N \to \overline{\mathbb{R}} \mid f \text{ convex}\}$$
$$(\phi: N \to M) \mapsto f \circ \phi^*$$ ← may be wondering why morph action needed?

- we compose cospans w/ pushout. How to compose decorated cospans?

  ↳ Answer: Need to upgrade F to a lax monoidal $(\mathrm{FinSet}, +) \to (\mathrm{Set}, \times)$.

  Define a laxator
  $$\varphi_{N,M}: FN \times FM \to F(N+M)$$

Composite of $(X \to N \leftarrow Y, 1 \xrightarrow{s} FN)$;
$(Y \to M \leftarrow Z, 1 \xrightarrow{t} FM)$ is

$$\begin{tikzcd}
 & & N+_Y M & & \\
 & & N+M \arrow[u, dotted, "p+q"] & & \\
X \arrow[r] & N \arrow[ur] \arrow[uur, bend left, "p"] & & M \arrow[ul] \arrow[uul, bend right, "q"'] & Z \arrow[l] \\
 & & Y \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

w/ decoration:

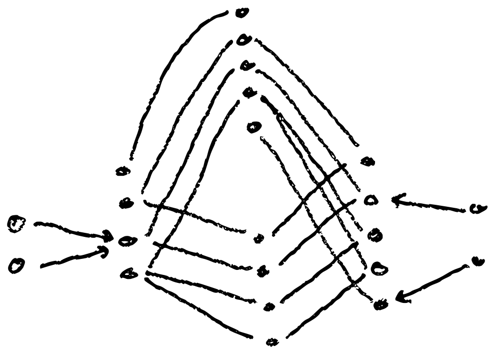
$$1 \cong 1 \times 1 \xrightarrow{s \times t} FN \times FM \xrightarrow{\varphi_{N,M}} F(N+M) \xrightarrow{F(p+q)} F(N+_Y M)$$ ✓

Hooray!

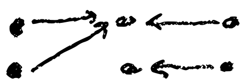
E.g. laxator for Conv is
$$\varphi_{N,M}(f,g) := f \circ \iota_1^* + g \circ \iota_2^* \qquad \iota_1: N \to N+M, \quad \iota_2: M \to N+M.$$

- Fong proves that F-decorated cospans form a hypergraph category.

Decorated corelations Intuition



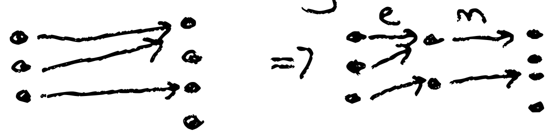
$\Downarrow$ restrict apex to part in image of feet.



E.g. we can "black box" a min problem $\mathbb{R}^5 \to \overline{\mathbb{R}}$ to a min problem $\mathbb{R}^2 \to \overline{\mathbb{R}}$ by taking the inf over non-exposed vars.

Decorated Corels Formally

- Formalize using $(\mathcal{E},\mathcal{M})$ factorization system.



Def: An $(\mathcal{E},\mathcal{M})$ corelation is a jointly $\mathcal{E}$-like cospan, i.e.
$$X+Y \xrightarrow{e} N.$$

Compose corels by

If anyone asks:
$F(X \xrightarrow{\mathrm{id}} X \xrightarrow{m} \overline{X})$

$$\begin{tikzcd}
& & N+_Y M & & \\
& & \overline{N+_Y M} \arrow[u, "m"] & & \\
& & X+Z \arrow[u, "e"] & & \\
& N \arrow[uuur, bend left] & & M \arrow[uuul, bend right] & \\
X \arrow[ur] \arrow[uurr, bend right, "\iota_1"'] & & Y \arrow[ul] \arrow[ur] & & Z \arrow[ul] \arrow[uull, bend left, "\iota_2"]
\end{tikzcd}$$

How to decorate corels?
↳ Need a way to restrict an apex system to one in $\mathcal{E}$-part of cospan along a mono.
↳ In other words, need a contra functor $\overline{F}: \mathrm{FinSet}^{op}_{mono} \to \mathrm{Set}$ w/
$$\overline{F}(N) = F(N) \quad \forall N \in \mathrm{FinSet}.$$

Def: An F-decorated corel is a pair
$(X \to N \leftarrow Y,\ s \in F(N))$
↑ jointly epic.

Compose decorations by
$$F(N+_Y M) = \overline{F}(N+_Y M) \xrightarrow{\overline{F}(m)} \overline{F}(\overline{N+_Y M}) = F(\overline{N+_Y M}).$$

<u>Thm.</u> Given a pushout square
$$\begin{tikzcd}
& N+_Y M & \\
N \arrow[ur, "p"] & & M \arrow[ul, "q"'] \\
& Y \arrow[ul, "m"] \arrow[ur, "\phi"'] &
\end{tikzcd}$$
in FinSet, if
$$\overline{F}q \circ Fp = F\phi \circ \overline{F}m$$
then F-decorated corels form a hypergraph cat.

E.g. $\overline{\mathrm{Conv}}: \mathrm{FinSet}^{op}_{mono} \to \mathrm{Set}$
$$N \mapsto \mathrm{Conv}(N)$$
$$(\phi: N \hookrightarrow M)(f: \mathbb{R}^M \to \overline{\mathbb{R}}) \mapsto$$
$$x \mapsto \inf_{\substack{y \in \mathbb{R}^M,\\ \phi^*(y) = x}} f(y)$$

Relation to Cospan Algs

A UWD is a cospan.

Φ $P_1 + \dots + P_n \xrightarrow{\phi} J \xleftarrow{m} O$

A cospan alg is a lax monoidal functor

$(\text{Cospan}, +) \to (\text{Set}, \times)$.

Given (F, φ) & $\overline{F}$, we can take

Φ to function $F(P_1) \times \dots \times F(P_n) \to F(O)$

by

$$F(P_1) \times \dots \times F(P_n) \xrightarrow{\varphi} F(P_1 + \dots + P_n) \xrightarrow{F(\phi)} F(J) \xrightarrow{\overline{F}(m)} F(O).$$

Backup Notes

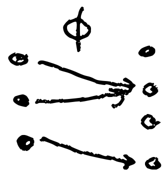
Contra FVS functor:

$(\cdot)^*: \mathrm{FinSet}^{op} \to \mathrm{Vect}$

$N \mapsto \mathbb{R}^N$

$(\phi: N \to M) \mapsto \phi^*: \mathbb{R}^M \to \mathbb{R}^N$ given by

$v \mapsto v \circ \phi$

e.g. 

$\phi^*(x_1, x_2, x_3, x_4) := (x_2, x_2, x_4)$