Dialectica Petri nets: A linear logic model of reaction Systems

Wilmer Leal
Gatas Lab
University of Floida
(joint work with Elena di Lavore and Valeria De Paiva)

November 9, 2023

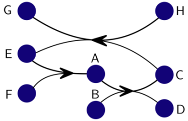
Outline

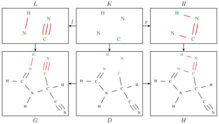
- Motivation
  - Why am I talking about this today?
  - Is this part of categorical chemistry?
  - Why linear logic?
- The paper in one slide!
- The structure of our category of Petri nets
- Questions for you

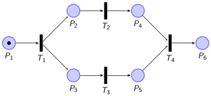
Motivation: Why am I talking about this today?

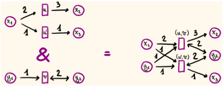
- We are revising and improving an arXived manuscript that want to submit to Fundamenta Informaticae
- Would appreciate your feedback
- I have specific questions for you

Motivation: Is this part of categorical chemistry?









- Composing reactions to reach a target substance using the network structure
- Looking at chemistry as a grammar of molecular copresheaves
- Dynamics on chemical reaction networks
- Composing small networks into bigger ones preserving algebraic properties of the components: random models of directed hypernetworks.
- Many more!

Categorical chemistry is the study of these structures and their interactions.

Motivation: Why linear logic?

- Chemistry's internal logic:
  - Duplication:
    $A \longrightarrow A + A$
  - Deletion:
    $A + A \longrightarrow A$

  are not allowed in chemical reactions.
- Linear logic's emphasis is on resource-boundedness, duality, and interaction.

$\&$	additive conjunction
$\oplus$	additive disjunction
$\otimes$	multiplicative conjunction
$⅋$	multiplicative disjunction
$?$	controlled access to contraction
$!$	controlled access to weakening

Binary connectives $\&$, $\oplus$, $\otimes$ and $⅋$ are associative and commutative. They have units: 1 for $\otimes$, 0 for $\oplus$, $\perp$ for $⅋$ and $\top$ for $\&$.

Construction of the category Net_L [1]

- We fix a lineale, that is, a poset version of a symmetric monoidal closed category: $(L, \leq, \circ, e, \dot{-})$
- We built a Dialectica-type category $M_L(Set)$ of *general relations:* Objects: (U, X, α), where U, X are sets and $\alpha\colon U \times X \to L$ is a function in Set.
- Define on $M_L(Set)$ the linear logic connectives (prove that it has binary products and coproducts, define the tensor product and its adjoint, the internal hom):
  &, $\oplus$, $\otimes$, and $[-,-]$.
- Define Net_L as the pullback of $M_L(Set)$ with itself in **Cat**:
  $A = ({}^{\blacktriangleright}\alpha, \alpha^{\blacktriangleright})$ of objects ${}^{\blacktriangleright}\alpha\colon U \times X \to L$ and $\alpha^{\blacktriangleright}\colon U \times X \to L$ in L.
- Show that Net_L inherits the linear logic structure from $M_L(Set)$:

Theorem (Linear logic structure of Net_L)

The category Net_L has products and coproducts and is a symmetric monoidal closed category.

[1] di Lavore E, Leal W and De Paiva V (2021). Dialectica Petri nets. ArXiv. (Submitted to *Log Methods Comput Sci*).

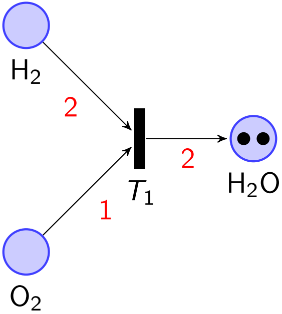
What can be modeled using Net_L

- Networks with diffrent kind of extensions

Basic extensions

- Weighted arcs:

$$2H_2 + 1O_2 \rightarrow 2H_2O$$



Basic extensions

- Inhibitor Arc.

$$2\text{X} \rightarrow 1\text{Y}$$

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$$2X \rightarrow 1Y$$

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Basic extensions

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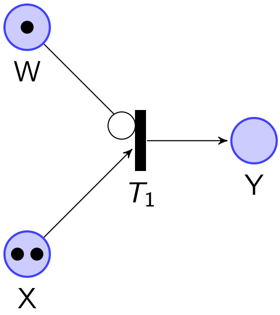

Basic extensions

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$$2X \rightarrow 1Y$$

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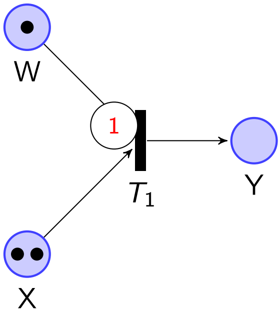

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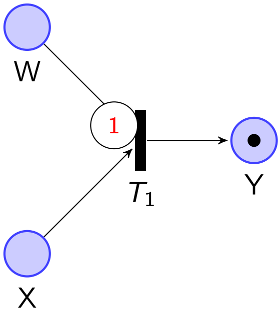
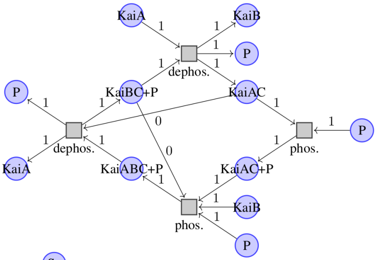
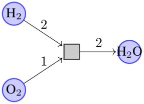
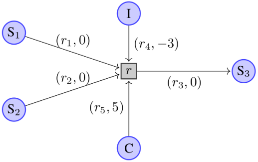
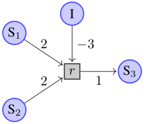


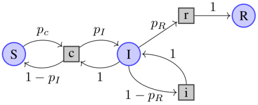
Net_L is a model for nets with several kinds of labels











What can be modeled using Net_L

- Networks with different kind of extensions
- Build larger networks from smaller ones using linear connectives

The Category MSets

Definition 3 *The category* MSets *consists of:*

- *Objects are triples* (U, X, α) *written as* $(U \xleftarrow{\alpha} X)$, *where* $U \times X \xrightarrow{\alpha} N$ *is a function in Sets, that is a multirelation.*
- *Morphisms in* MSets *from an object* $U \times X \xrightarrow{\alpha} N$ *or* $(U \xleftarrow{\alpha} X)$ *to an object* $V \times Y \xrightarrow{\beta} N$ *or* $(V \xleftarrow{\beta} Y)$ *are pairs of morphisms in Sets, (f,F) where f:* $U \to V$ *and F:* $Y \to X$ *are such that*

$$
\begin{tikzcd}
U \arrow[d, "f"'] & X \arrow[l, "\alpha"'] \\
V \arrow[ur, phantom, "\Downarrow"] & Y \arrow[l, "\beta"] \arrow[u, "F"']
\end{tikzcd}
$$

that means that $\forall u \in U$, $\forall y \in Y$ $\beta(fu, y) \geq \alpha(u, Fy)$

Examples of morphisms

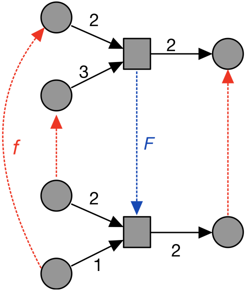


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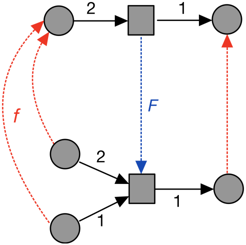
Figure 1:



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The Category MSets

Definition 4 *Given two objects* $U \times X \xrightarrow{\alpha} N$ *or* $(U \xleftarrow{\alpha} X)$ *and* $V \times Y \xrightarrow{\beta} N$ *or* $(V \xleftarrow{\beta} Y)$ *in* MSets *we define* $A \otimes B$ *their tensor product as the following object:*

$$A \otimes B = (U \times V \xleftarrow{\alpha \otimes \beta} X^V \times Y^U)$$

where the multirelation "$\alpha \otimes \beta$*" is given by*
$\alpha \otimes \beta(u, v, f, g) = \alpha(u, fv) + \beta(v, gu)$.

The Category MSets

Definition 5 *Given two objects* $U \times X \xrightarrow{\alpha} N$ *or* $(U \xleftarrow{\alpha} X)$ *and* $V \times Y \xrightarrow{\beta} N$ *or* $(V \xleftarrow{\beta} Y)$ *in* MSets *we define* $[A, B]$ *the internal-hom as the object,*

$$[A, B] = (V^U \times X^Y \xleftarrow{\alpha \dot{-} \beta} U \times Y)$$

the multirelation "$(\alpha \dot{-} \beta)$*" is given by* $(\alpha \dot{-} \beta)(f, F, u, y) = -\alpha(u, Fy) + \beta(fu, y)$*, where the dotted subtraction is truncated subtraction, that is* $-\alpha + \beta = \beta - \alpha$ *if* $\alpha \leq \beta$ *and* 0 *otherwise.*

The Category MSets

Proposition 3 *The construction above defines a bifunctor* $[-,-] : \mathsf{MSets}^{\mathsf{op}} \times \mathsf{MSets} \to \mathsf{Msets}$.

Theorem 1 *The category* MSets *is a symmetric monoidal closed category with respect to the tensor product* $\otimes$ *and the internal-hom* $[-,-]$ *defined above.*

Structure of $M_N\mathbf{C}$

Definition 6 *An ordered monoid* $(N, \leq, \circ, e)$ *is a poset* $(N, \leq)$ *with a given compatible symmetric monoidal structure* $(N, \circ, e)$*. The structures are compatible in the sense that, if* $a \leq b$*, we have* $a \circ c \leq b \circ c$*, for all* c *in* N*.*

Definition 7 *Suppose* $(N, \leq, \circ, e)$ *is an ordered monoid and* $a, b \in N$*. If there exists a largest* $x \in N$ *such that* $a \circ x \leq b$ *then this element is denoted as* $a \dot{-} b$ *and it is called the relative pseudocomplement of* a *wrt* b*. A closed posed is an ordered monoid* $(N, \leq, \circ, e)$ *such that* $a \dot{-} b$ *exist for all* a *and* b *in* N*.*

Structure of $\boldsymbol{M_N}\mathbf{C}$

Proposition 4 *A closed poset* $(N, \leq, \circ, e, \dot{-})$ *has the following properties:*

1. $a \circ b \leq c$ *iff* $a \leq b \dot{-} c$
2. *If* $a \leq b$*, then for any* c *in* N*,* $c \dot{-} a \leq c \dot{-} b$ *and* $b \dot{-} c \leq a \dot{-} c$*;*
3. *As 'e' is the identity for '∘'* $a \circ e = a \leq a$ *implies* $e \leq a \dot{-} a$ *for any* a *in* N*.*

Structure of $\boldsymbol{M_N}\mathbf{C}$

Definition 8 *Given two objects* $A = (U \xleftarrow{\alpha} X)$ *and* $B = (V \xleftarrow{\beta} Y)$ *in* $\boldsymbol{M_N C}$ *we define* $A \otimes_M B$ *their tensor product as follows:*

$$A \otimes_M B = (U \otimes V \xleftarrow{(\alpha \otimes \beta)_M} [V, X] \times [U, Y])$$

The morphism "$(\alpha \otimes \beta)_M$*" intuitively says* $(\alpha \otimes \beta)_M(u \otimes v, \langle f, g \rangle) = \alpha(u \otimes fv) \circ \beta(v \otimes gu)$, *where* $\circ$ *is the monoidal structure in* $(N, \leq, \circ, e, \dot{-})$.

Proposition 5 *The construction above induces a bifunctor,*

$$\otimes_M : \boldsymbol{M_N}\mathbf{C} \times \boldsymbol{M_N}\mathbf{C} \to \boldsymbol{M_N}\mathbf{C}$$

covariant in both coordinates, which is a tensor product. The identity I_M *is given by* $(I \xleftarrow{e} 1)$, *where the morphism* $I \otimes 1 \approx 1 \xrightarrow{e} N$, *just picks up the identity* $'e'$ *from the closed poset* $(N, \leq, \circ, e, \dot{-})$.

Structure of $\boldsymbol{M_N C}$

Definition 9 *Given two objects* $A = (U \xleftarrow{\alpha} X)$ *and* $B = (V \xleftarrow{\beta} Y)$ *in* $\boldsymbol{M_N C}$ *we define* $[A, B]_M$ *their internal hom as follows:*

$$[A, B]_M = ([U, V] \times [Y, X]) \xleftarrow{(\alpha \dot{-} \beta)_M} U \otimes Y)$$

The morphism "$(\alpha \dot{-} \beta)_M$*" intuitively says* $(\alpha \dot{-} \beta)_M(\langle f, F \rangle, u \otimes y) = \alpha(u \otimes Fy) \dot{-} \beta(fu \otimes y)$*, where* $\dot{-}$ *is the 'internal-hom' in* N*.*

Structure of $\boldsymbol{M_N C}$

Proposition 6 *The construction above induces a bifunctor $[-,-]_M$ contravariant in its first coordinate and covariant in its second coordinate.*

Since we have the adjunction

$$- \otimes A \dashv [-, A]$$

, we have the following

Theorem 2 *The category $\boldsymbol{M_N C}$ is a symmetric monoidal closed category.*

Structure of $\boldsymbol{M_N C}$

Definition 10 *Given two objects* $A = (U \xleftarrow{\alpha} X)$ *and* $B = (V \xleftarrow{\beta} Y)$ *in* $\boldsymbol{M_N C}$ *we define their categorical product as follows:*

$$A\&B = (U \times V \xleftarrow{\alpha\&\beta} X + Y)$$

The morphism "$\alpha\&\beta$*" is given intuitively by* $\alpha\&\beta(\langle u, v\rangle, \left(\begin{smallmatrix} x & 0 \\ y & 1 \end{smallmatrix}\right)) = \alpha(u, x) \cdot \beta(v, y)$

Defintion 11 *Given two objects* $A = (U \xleftarrow{\alpha} X)$ *and* $B = (V \xleftarrow{\beta} Y)$ *in* $\boldsymbol{M_N C}$ *we define their categorical coproduct*

$$A \oplus B = (U + V \xleftarrow{\alpha\oplus\beta} X \times Y)$$

The morphism "$\alpha \oplus \beta$*" is given by* $\alpha \oplus \beta(\left(\begin{smallmatrix} u & 0 \\ v & 1 \end{smallmatrix}\right), \langle x, y\rangle) = \alpha(u, x) \cdot \beta(v, y)$

Structure of $M_N C$

Proposition 8 *The category $M_N C$ has binary products and coproducts.*

Theorem 3 *The category $M_N C$ is a categorical model of Instuitionistic Linear Logic*

The !-modality in $\boldsymbol{M_N}\mathbf{C}$

In ILL, we have an operator (a "modality") called !. In a categorical model, this should be a comonad with certain properties. We will now construct this for $\boldsymbol{M_N}\mathbf{C}$, under some assumptions..

The !-modality in $\boldsymbol{M_N}\mathbf{C}$

Definition 12 *For each object U in a cartesian closed category* C *we have a monad $(()^U, \eta_1, \mu_1)$ in* C *given by the natural transformation below:*

$$X \xrightarrow{\eta_1} X^U \qquad\qquad X^{U \times U} \xrightarrow{\mu_1} X^U$$

Definition 13 *The endofunctor $T : \boldsymbol{M_N C} \to \boldsymbol{M_N C}$ takes an object $(U \xleftarrow{\alpha} X)$ of $\boldsymbol{M_N C}$ to the object $(U \xleftarrow{T\alpha} X^U)$, where intuitively the object $T\alpha$ is given by $T\alpha(u, f) = \alpha(u, fu)$.*

One can verify that this gives a *comonad* on $\boldsymbol{M_N}\mathbf{C}$
(We have assumed that C is Cartesian closed!)

The !-modality in $\boldsymbol{M_N C}$

For any symmetric monoidal category, we can define the category $\mathrm{Mon_c}(\mathrm{C})$ of commutative monoids in C, and there's a forgetful functor $U : \mathrm{Mon_c}(\mathrm{C}) \to \mathrm{C}$.
If this has a left adjoint, we call the monad of this adjunction the *free commutative monoids monad*, and denote it $(-)^*$.
Think of these as *unordered lists*
Definition 15 *The endofunctor* $S : \boldsymbol{M_N C} \to \boldsymbol{M_N C}$ *takes an object* $(U \xleftarrow{\alpha} X)$ *of* $\boldsymbol{M_N C}$ *to the object* $(U \xleftarrow{S\alpha} X^*)$*, where as intuitively* $\bar{x}$ *is* $\langle x_1, x_2, ..., x_n \rangle$*,* $S\alpha(u, \bar{x})$*, means* $\alpha(u, x_1)$ *and* $\alpha(u, x_2)$ *and ... and* $\alpha(u, x_n)$*.*
This is also a comonad.
(We've assumed that the left adjoint exists - i.e that C "has free commutative monoids").

The !-modality in $\boldsymbol{M_N}\mathbf{C}$

Definition 16 *The endofunctor* $! : \boldsymbol{M_N C} \to \boldsymbol{M_N C}$ *takes an object takes an object* $(U \xleftarrow{\alpha} X)$ *of* $\boldsymbol{M_N C}$ *to the object* $(U \xleftarrow{!\alpha} X^{*U})$*, where intuitively if* $\phi : U \to X^*$ *and* $\phi u = \langle x_1, x_2, \ldots, x_n \rangle$ *then* $!\alpha(u, \phi)$ *is given by* $\alpha(u, x_1)$ *and* $\alpha(u, x_2)$ *and ... and* $\alpha(u, x_n)$*.*
In other words, $! = T \circ S$.
To get the comonad structure, we need a natural transformation $! \to !!$, i.e $TS \to TSTS$. We can use the comonad structures on S and T to do $TS \to TTSS$, so we need a natural transformation $TS \to ST$. (A "distributive law").

The !-modality in $\boldsymbol{M_N C}$

There's a distributive law $\lambda : \left((-)^U\right)^* \to ((-)^*)^U$ in C. Intuitively, an element of $(X^U)^*$ is an unordered list of functions $U \to X$. Given such a list $f_1, f_2, \ldots,$ we can construction a function $U \to X^*$ which takes u to $f_1(u), f_2(u), \ldots$

Fact 4 *The distributive law λ in* C *induces a distributive law of comonads Λ in $\boldsymbol{M_N C}$, given by $\Lambda : TSA \to STA$:*

$$
\begin{tikzcd}
U \arrow[d, "1"'] & X^{*U} \arrow[l, "TS\alpha"'] \\
U & X^{U*} \arrow[l, "ST\alpha"] \arrow[u, "\lambda"']
\end{tikzcd}
$$

The !-modality in $\boldsymbol{M_N}\mathbf{C}$

Definition 19 *The endofunctor ! in $\boldsymbol{M_N C}$ acts on objects as*

$$!(U \xleftarrow{\alpha} X) = (U \xleftarrow{!\alpha} (X^*)^U)$$

where the morphism $!\alpha$ is given by composition

$$U \times (X^*)^U) \xrightarrow{\langle \pi, ev \rangle} U \times X^* \xleftarrow{S\alpha} N$$

if $(f, F) : A \to B$ is a morphism in $\boldsymbol{M_N C}$, then $!(f, F)$ is given by

$$\begin{tikzcd}
U \arrow[d, "f"'] & X^{*U} \arrow[l, "!\alpha"'] \\
V & Y^{*V} \arrow[l, "!\beta"] \arrow[u, "F^* \cdot (\ ) \cdot f"']
\end{tikzcd}$$

The !-modality in $\boldsymbol{M_N C}$

Proposition 9 *The comonad '!' in $\boldsymbol{M_N C}$ define above satisfies*

$$!(A\&B) \approx !A \otimes !B \quad \textit{and} \quad !1 \approx I$$

Theorem 4 *The comonad '!' in $\boldsymbol{M_N C}$ satisfies the rules for the modality '!' in linear logic.*

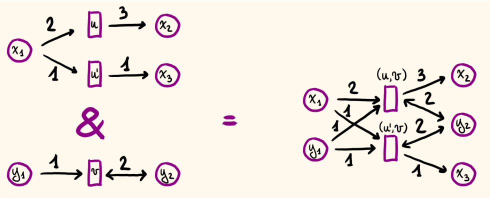
Let's take a look at the categorical product

Definition (Product in L)

Given two objects $A = ({}^{\blacktriangleright}\alpha, \alpha^{\blacktriangleright})$ and $B = ({}^{\blacktriangleright}\beta, \beta^{\blacktriangleright})$ in Net_L, we define their cartesian product $A \& B$ as the following object.

$$A \& B = ({}^{\blacktriangleright}\alpha \& {}^{\blacktriangleright}\beta, \alpha^{\blacktriangleright} \& \beta^{\blacktriangleright})$$

The function $\alpha \& \beta$ is $U \times V \times (X + Y) \xrightarrow{\alpha \times_V \beta \times_U} L$, where ${}_U$ is the function that discards U in .



Net_L is a model for nets with several kinds of labels