CHAPTER 2

Projectiles and Charged Particles

In this chapter, I present two topics: the motion of projectiles subject to the forces of gravity and air resistance, and the motion of charged particles in uniform magnetic fields. Both problems lend themselves to solution using Newton's laws in Cartesian coordinates, and both allow us to review and introduce some important mathematics. Above all, both are problems of great practical interest.

2.1 Air Resistance

Most introductory physics courses spend some time studying the motion of projectiles, but they almost always ignore air resistance. In many problems this is an excellent approximation; in others, air resistance is obviously important, and we need to know how to account for it. More generally, whether or not air resistance is significant, we need some way to estimate how important it really is.

Let us begin by surveying some of the basic properties of the resistive force, or **drag**, $\mathbf{f}$ of the air, or other medium, through which an object is moving. (I shall generally speak of "air resistance" since air is the medium through which most projectiles move, but the same considerations apply to other gases and often to liquids as well.) The most obvious fact about air resistance, well known to anyone who rides a bicycle, is that it depends on the speed, v, of the object concerned. In addition, for many objects, the direction of the force due to motion through the air is opposite to the velocity $\mathbf{v}$. For certain objects, such as a nonrotating sphere, this is exactly true, and for many it is a good approximation. You should, however, be aware that there are situations where it is certainly not true: The force of the air on an airplane wing has a large sideways component, called the **lift**, without which no airplanes could fly. Nevertheless, I shall assume that $\mathbf{f}$ and $\mathbf{v}$ point in opposite directions; that is, I shall consider only objects for which the sideways force is zero, or at least small enough

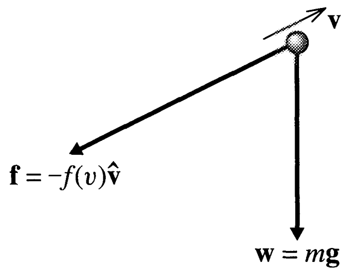

Figure 2.1 A projectile is subject to two forces, the force of gravity, $\mathbf{w} = m\mathbf{g}$, and the drag force of air resistance, $\mathbf{f} = -f(v)\hat{\mathbf{v}}$.

to be neglected. The situation is illustrated in Figure 2.1 and is summed up in the equation

$$\mathbf{f} = -f(v)\hat{\mathbf{v}}, \tag{2.1}$$

where $\hat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$ denotes the unit vector in the direction of $\mathbf{v}$, and $f(v)$ is the magnitude of $\mathbf{f}$.

The function $f(v)$ that gives the magnitude of the air resistance varies with v in a complicated way, especially as the object's speed approaches the speed of sound. However, at lower speeds it is often a good approximation to write[1]

$$f(v) = bv + cv^2 = f_{\text{lin}} + f_{\text{quad}} \tag{2.2}$$

where f_{lin} and f_{quad} stand for the linear and quadratic terms respectively,

$$f_{\text{lin}} = bv \quad \text{and} \quad f_{\text{quad}} = cv^2. \tag{2.3}$$

The physical origins of these two terms are quite different: The linear term, f_{lin}, arises from the viscous drag of the medium and is generally proportional to the viscosity of the medium and the linear size of the projectile (Problem 2.2). The quadratic term, f_{quad}, arises from the projectile's having to accelerate the mass of air with which it is continually colliding; f_{quad} is proportional to the density of the medium and the cross-sectional area of the projectile (Problem 2.4). In particular, for a spherical projectile (a cannonball, a baseball, or a drop of rain), the coefficients b and c in (2.2) have the form

$$b = \beta D \quad \text{and} \quad c = \gamma D^2 \tag{2.4}$$

where D denotes the diameter of the sphere and the coefficients β and γ depend on the nature of the medium. For a spherical projectile in air at STP, they have the approximate values

$$\beta = 1.6 \times 10^{-4} \text{ N}\cdot\text{s/m}^2 \tag{2.5}$$

[1] Mathematically, Equation (2.2) is, in a sense, obvious. Any reasonable function is expected to have a Taylor series expansion, $f = a + bv + cv^2 + \cdots$. For low enough v, the first three terms should give a good approximation, and, since $f = 0$ when $v = 0$ the constant term, a, has to be zero.

and

$$\gamma = 0.25 \text{ N}\cdot\text{s}^2/\text{m}^4. \tag{2.6}$$

(For calculation of these two constants, see Problems 2.2 and 2.4.) You need to remember that these values are valid only for a sphere moving through air at STP. Nevertheless, they give at least a rough idea of the importance of the drag force even for nonspherical bodies moving through different gases at any normal temperatures and pressures.

It often happens that we can neglect one of the terms in (2.2) compared to the other, and this simplifies the task of solving Newton's second law. To decide whether this does happen in a given problem, and which term to neglect, we need to compare the sizes of the two terms:

$$\frac{f_{\text{quad}}}{f_{\text{lin}}} = \frac{cv^2}{bv} = \frac{\gamma D}{\beta} v = \left(1.6 \times 10^3 \frac{\text{s}}{\text{m}^2}\right) Dv \tag{2.7}$$

if we use the values (2.5) and (2.6) for a sphere in air. In a given problem, we have only to substitute the values of D and v into this equation to find out if one of the terms can be neglected, as the following example illustrates.

EXAMPLE 2.1 A Baseball and Some Drops of Liquid

Assess the relative importance of the linear and quadratic drags on a baseball of diameter $D = 7$ cm, traveling at a modest $v = 5$ m/s. Do the same for a drop of rain ($D = 1$ mm and $v = 0.6$ m/s) and for a tiny droplet of oil used in the Millikan oildrop experiment ($D = 1.5\,\mu$m and $v = 5 \times 10^{-5}$ m/s).

When we substitute the numbers for the baseball into (2.7) (remembering to convert the diameter to meters), we get

$$\frac{f_{\text{quad}}}{f_{\text{lin}}} \approx 600 \qquad \text{[baseball]}. \tag{2.8}$$

For this baseball, the linear term is clearly negligible and we need consider only the quadratic drag. If the ball is traveling faster, the ratio $f_{\text{quad}}/f_{\text{lin}}$ is even greater. At slower speeds the ratio is less dramatic, but even at 1 m/s the ratio is 100. In fact if v is small enough that the linear term is comparable to the quadratic, both terms are so small as to be negligible. Thus, for baseballs and similar objects, it is almost always safe to neglect f_{lin} and take the drag force to be

$$\mathbf{f} = -cv^2\hat{\mathbf{v}}. \tag{2.9}$$

For the raindrop, the numbers give

$$\frac{f_{\text{quad}}}{f_{\text{lin}}} \approx 1 \qquad \text{[raindrop]}. \tag{2.10}$$

Thus for this raindrop the two terms are comparable and neither can be neglected — which makes solving for the motion more difficult. If the drop were a lot larger or were traveling much faster, then the linear term would be negligible; and if the drop were much smaller or were traveling much slower, then the quadratic term would be negligible. But in general, with raindrops and similar objects, we are going to have to take both f_{lin} and f_{quad} into account.

For the oildrop in the Millikan experiment the numbers give

$$\frac{f_{\text{quad}}}{f_{\text{lin}}} \approx 10^{-7} \qquad \text{[Millikan oildrop].} \tag{2.11}$$

In this case, the quadratic term is totally negligible, and we can take

$$\mathbf{f} = -bv\hat{\mathbf{v}} = -b\mathbf{v}, \tag{2.12}$$

where the second, very compact form follows because, of course, $v\hat{\mathbf{v}} = \mathbf{v}$.

The moral of this example is clear: First, there are objects for which the drag force is dominantly linear, and the quadratic force can be neglected — notably, very small liquid drops in air, but also slightly larger objects in a very viscous fluid, such as a ball bearing moving through molasses. On the other hand, for most projectiles, such as golf balls, cannonballs, and even a human in free fall, the dominant drag force is quadratic, and we can neglect the linear term. This situation is a little unlucky because the linear problem is much easier to solve than the quadratic. In the following two sections, I shall discuss the linear case, precisely because it is the easier one. Nevertheless, it *does* have practical applications, and the mathematics used to solve it is widely used in many fields. In Section 2.4, I shall take up the harder but more usual case of quadratic drag.

To conclude this introductory section, I should mention the Reynolds number, an important parameter that features prominently in more advanced treatments of motion in fluids. As already mentioned, the linear drag f_{lin} can be related to the viscosity of the fluid through which our projectile is moving, and the quadratic term f_{quad} is similarly related to the inertia (and hence density) of the fluid. Thus one can relate the ratio $f_{\text{quad}}/f_{\text{lin}}$ to the fundamental parameters η, the viscosity, and ϱ, the density, of the fluid (see Problem 2.3). The result is that the ratio $f_{\text{quad}}/f_{\text{lin}}$ is of roughly the same order of magnitude as the dimensionless number $R = Dv\varrho/\eta$, called the **Reynolds number**. Thus a compact and general way to summarize the foregoing discussion is to say that the quadratic drag f_{quad} is dominant when the Reynolds number R is large, whereas the linear drag dominates when R is small.

2.2 Linear Air Resistance

Let us consider first a projectile for which the quadratic drag force is negligible, so that the force of air resistance is given by (2.12). We shall see directly that, because the drag force is linear in $\mathbf{v}$, the equations of motion are very simple to solve. The two forces on the projectile are the weight $\mathbf{w} = m\mathbf{g}$ and the drag force $\mathbf{f} = -b\mathbf{v}$, as shown in Figure 2.2. Thus the second law, $m\ddot{\mathbf{r}} = \mathbf{F}$, reads

$$m\ddot{\mathbf{r}} = m\mathbf{g} - b\mathbf{v}. \tag{2.13}$$

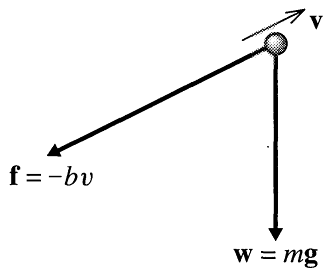

Figure 2.2 The two forces on a projectile for which the force of air resistance is linear in the velocity, $\mathbf{f} = -b\mathbf{v}$.

An interesting feature of this form is that, because neither of the forces depends on $\mathbf{r}$, the equation of motion does not involve $\mathbf{r}$ itself (only the first and second derivatives of $\mathbf{r}$). In fact, we can rewrite $\ddot{\mathbf{r}}$ as $\dot{\mathbf{v}}$, and (2.13) becomes

$$m\dot{\mathbf{v}} = m\mathbf{g} - b\mathbf{v}, \tag{2.14}$$

a first-order differential equation for $\mathbf{v}$. This simplification comes about because the forces depend only on $\mathbf{v}$ and not $\mathbf{r}$. It means we have to solve only a first-order differential equation for $\mathbf{v}$ and then integrate $\mathbf{v}$ to find $\mathbf{r}$.

Perhaps the most important simplifying feature of linear drag is that the equation of motion separates into components especially easily. For instance, with x measured to the right and y vertically downward, (2.14) resolves into

$$m\dot{v}_x = -bv_x \tag{2.15}$$

and

$$m\dot{v}_y = mg - bv_y. \tag{2.16}$$

That is, we have two separate equations, one for v_x and one for v_y; the equation for v_x does not involve v_y and vice versa. It is important to recognize that this happened only because the drag force was linear in $\mathbf{v}$. For instance, if the drag force were quadratic,

$$\mathbf{f} = -cv^2\hat{\mathbf{v}} = -cv\mathbf{v} = -c\sqrt{v_x^2 + v_y^2}\,\mathbf{v}, \tag{2.17}$$

then in (2.14) we would have to replace the term $-b\mathbf{v}$ with (2.17). In place of the two equations (2.15) and (2.16), we would have

$$\left.\begin{aligned} m\dot{v}_x &= -c\sqrt{v_x^2 + v_y^2}\,v_x \\ m\dot{v}_y &= mg - c\sqrt{v_x^2 + v_y^2}\,v_y. \end{aligned}\right\} \tag{2.18}$$

Here, each equation involves *both* of the variables v_x and v_y. These two *coupled* differential equations are much harder to solve than the uncoupled equations of the linear case.

Because they are uncoupled, we can solve each equation for linear drag separately and then put the two solutions together. Further, each equation defines a problem that is interesting in its own right. Equation (2.15) is the equation of motion for an object

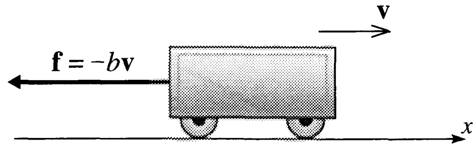

Figure 2.3 A cart moves on a horizontal frictionless track in a medium that produces a linear drag force.

(a cart with frictionless wheels, for instance) coasting horizontally in a medium that causes linear drag. Equation (2.16) describes an object (a tiny oil droplet for instance) that is falling vertically with linear air resistance. I shall solve these two separate problems in turn.

Horizontal Motion with Linear Drag

Consider an object such as the cart in Figure 2.3 coasting horizontally in a linearly resistive medium. I shall assume that at $t = 0$ the cart is at $x = 0$ with velocity $v_x = v_{xo}$. The only force on the cart is the drag $\mathbf{f} = -b\mathbf{v}$, thus the cart inevitably slows down. The rate of slowing is determined by (2.15), which has the general form

$$\dot{v}_x = -kv_x, \tag{2.19}$$

where k is my temporary abbreviation for $k = b/m$. This is a first-order differential equation for v_x, whose general solution must contain exactly one arbitrary constant. The equation states that the derivative of v_x is equal to $-k$ times v_x itself, and the only function with this property is the exponential function

$$v_x(t) = Ae^{-kt} \tag{2.20}$$

which satisfies (2.19) for any value of the constant A (Problems 1.24 and 1.25). Since this solution contains one arbitrary constant, it is the *general* solution of our first-order equation; that is, *any* solution must have this form. In our case, we know that $v_x(0) = v_{xo}$, so that $A = v_{xo}$, and we conclude that

$$v_x(t) = v_{xo}e^{-kt} = v_{xo}e^{-t/\tau}, \tag{2.21}$$

where I have introduced the convenient parameter

$$\tau = 1/k = m/b \qquad \text{[for linear drag]}. \tag{2.22}$$

We see that our cart slows down exponentially, as shown in Figure 2.4(a). The parameter τ has the dimensions of time (as you should check), and you can see from (2.21) that when $t = \tau$, the velocity is $1/e$ of its initial value; that is, τ is the "$1/e$" time for the exponentially decreasing velocity. As $t \to \infty$, the velocity approaches zero.

To find the position as a function of time, we have only to integrate the velocity (2.21). Integrations of this kind can be done using the definite or indefinite integral. The definite integral has the advantage that it automatically takes care of the constant

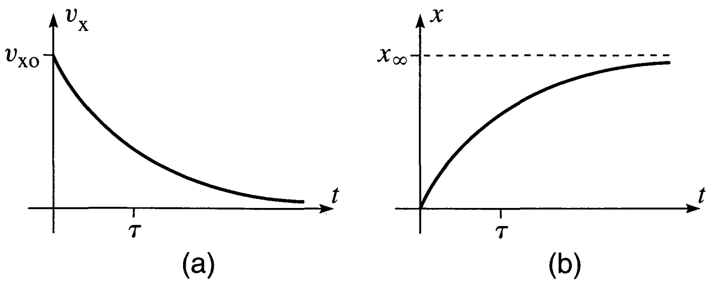

Figure 2.4 **(a)** The velocity v_x as a function of time, t, for a cart moving horizontally with a linear resistive force. As $t \to \infty$, v_x approaches zero exponentially. **(b)** The position x as a function of t for the same cart. As $t \to \infty$, $x \to x_\infty = v_{xo}\tau$.

of integration: Since $v_x = dx/dt$,

$$\int_0^t v_x(t')\,dt' = x(t) - x(0).$$

(Notice that I have named the "dummy" variable of integration t' to avoid confusion with the upper limit t.) Therefore

$$\begin{aligned} x(t) &= x(0) + \int_0^t v_{xo}e^{-t'/\tau}\,dt' \\ &= 0 + \left[-v_{xo}\tau e^{-t'/\tau}\right]_0^t \\ &= x_\infty\left(1 - e^{-t/\tau}\right). \end{aligned} \tag{2.23}$$

In the second line, I have used our assumption that $x = 0$ when $t = 0$. And in the last, I have introduced the parameter

$$x_\infty = v_{xo}\tau, \tag{2.24}$$

which is the limit of $x(t)$ as $t \to \infty$. We conclude that, as the cart slows down, its position approaches x_∞ asymptotically, as shown in Figure 2.4(b).

Vertical Motion with Linear Drag

Let us next consider a projectile that is subject to linear air resistance and is thrown vertically downward. The two forces on the projectile are gravity and air resistance, as shown in Figure 2.5. If we measure y vertically down, the only interesting component of the equation of motion is the y component, which reads

$$m\dot{v}_y = mg - bv_y. \tag{2.25}$$

With the velocity downward ($v_y > 0$), the retarding force is upward, while the force of gravity is downward. If v_y is small, the force of gravity is more important than the drag force, and the falling object accelerates in its downward motion. This will

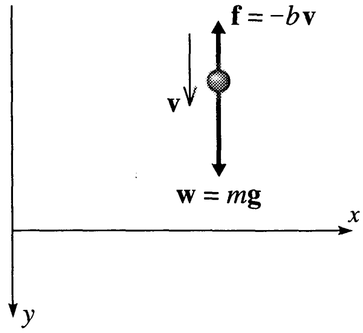

Figure 2.5 The forces on a projectile that is thrown vertically down, subject to linear air resistance.

continue until the drag force balances the weight. The speed at which this balance occurs is easily found by setting (2.25) equal to zero, to give $v_y = mg/b$ or

$$v_y = v_{\text{ter}}$$

where I have defined the **terminal speed**

$$v_{\text{ter}} = \frac{mg}{b} \qquad \text{[for linear drag].} \tag{2.26}$$

The terminal speed is the speed at which our projectile will eventually fall, if given the time to do so. Since it depends on m and b, it is different for different bodies. For example, if two objects have the same shape and size (b the same for both), the heavier object (m larger) will have the higher terminal speed, just as you would expect. Since v_{ter} is inversely proportional to the coefficient b of air resistance, we can view v_{ter} as an inverse measure of the importance of air resistance — the larger the air resistance, the smaller v_{ter}, again just as you would expect.

EXAMPLE 2.2 Terminal Speed of Small Liquid Drops

Find the terminal speed of a tiny oildrop in the Millikan oildrop experiment (diameter $D = 1.5\,\mu$m and density $\varrho = 840\,\text{kg/m}^3$). Do the same for a small drop of mist with diameter $D = 0.2$ mm.

From Example 2.1 we know that the linear drag is dominant for these objects, so the terminal speed is given by (2.26). According to (2.4), $b = \beta D$ where $\beta = 1.6 \times 10^{-4}$ (in SI units). The mass of the drop is $m = \varrho \pi D^3/6$. Thus (2.26) becomes

$$v_{\text{ter}} = \frac{\varrho \pi D^2 g}{6\beta} \qquad \text{[for linear drag].} \tag{2.27}$$

This interesting result shows that, for a given density, the terminal speed is proportional to D^2. This implies that, once air resistance has become important, a large sphere will fall faster than a small sphere of the same density.[2]

Putting in the numbers, we find for the oildrop

$$v_{\text{ter}} = \frac{(840) \times \pi \times (1.5 \times 10^{-6})^2 \times (9.8)}{6 \times (1.6 \times 10^{-4})} = 6.1 \times 10^{-5}\,\text{m/s} \qquad \text{[oildrop]}.$$

In the Millikan oildrop experiment, the oildrops fall exceedingly slowly, so their speed can be measured by simply watching them through a microscope.

Putting in the numbers for the drop of mist, we find similarly that

$$v_{\text{ter}} = 1.3\,\text{m/s} \qquad \text{[drop of mist]}. \tag{2.28}$$

This speed is representative for a fine drizzle. For a larger raindrop, the terminal speed would be appreciably larger, but with a larger (and hence also faster) drop, the quadratic drag would need to be included in the calculation to get a reliable value for v_{ter}.

So far, we have discussed the terminal speed of a projectile (moving vertically), but we must now discuss how the projectile approaches that speed. This is determined by the equation of motion (2.25) which we can rewrite as

$$m\dot{v}_y = -b(v_y - v_{\text{ter}}). \tag{2.29}$$

(Remember that $v_{\text{ter}} = mg/b$.) This differential equation can be solved in several ways. (For one alternative see Problem 2.9.) Perhaps the simplest is to note that it is almost the same as Equation (2.15) for the horizontal motion, except that on the right we now have $(v_y - v_{\text{ter}})$ instead of v_x. The solution for the horizontal case was the exponential function (2.20). The trick to solving our new vertical equation (2.29) is to introduce the new variable $u = (v_y - v_{\text{ter}})$, which satisfies $m\dot{u} = -bu$ (because v_{ter} is constant). Since this is *exactly* the same as Equation (2.15) for the horizontal motion, the solution for u is the same exponential, $u = Ae^{-t/\tau}$. [Remember that the constant k in (2.20) became $k = 1/\tau$.] Therefore,

$$v_y - v_{\text{ter}} = Ae^{-t/\tau}.$$

When $t = 0$, $v_y = v_{yo}$, so $A = v_{yo} - v_{\text{ter}}$ and our final solution for v_y as a function of t is

$$v_y(t) = v_{\text{ter}} + (v_{yo} - v_{\text{ter}})e^{-t/\tau} \tag{2.30}$$

$$= v_{yo}e^{-t/\tau} + v_{\text{ter}}\left(1 - e^{-t/\tau}\right). \tag{2.31}$$

[2] We are here assuming that the drag force is linear, but the same qualitative conclusion follows for a quadratic drag force. (Problem 2.24.)

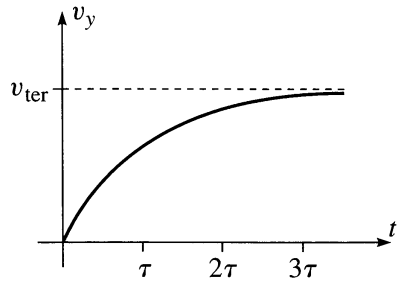

Figure 2.6 When an object is dropped in a medium with linear resistance, v_y approaches its terminal value v_{ter} as shown.

This second expression gives $v_y(t)$ as the sum of two terms: The first is equal to v_{yo} when $t = 0$, but fades away to zero as t increases; the second is equal to zero when $t = 0$, but approaches v_{ter} as $t \to \infty$. In particular, as $t \to \infty$,

$$v_y(t) \to v_{ter} \tag{2.32}$$

just as we anticipated.

Let us examine the result (2.31) in a little more detail for the case that $v_{yo} = 0$; that is, the projectile is dropped from rest. In this case (2.31) reads

$$v_y(t) = v_{ter}\left(1 - e^{-t/\tau}\right). \tag{2.33}$$

This result is plotted in Figure 2.6, where we see that v_y starts out from 0 and approaches the terminal speed, $v_y \to v_{ter}$, asymptotically as $t \to \infty$. The significance of the time τ for a falling body is easily read off from (2.33). When $t = \tau$, we see that

$$v_y = v_{ter}(1 - e^{-1}) = 0.63 v_{ter}.$$

That is, in a time τ, the object reaches 63% of the terminal speed. Similar calculations give the following results:

time t	percent of v_{ter}
0	0
τ	63%
2τ	86%
3τ	95%

Of course, the object's speed never actually reaches v_{ter}, but τ is a good measure of how fast the speed approaches v_{ter}. In particular, when $t = 3\tau$ the speed is 95% of v_{ter}, and for many purposes we can say that after a time 3τ the speed is essentially equal to v_{ter}.

EXAMPLE 2.3 Characteristic Time for Two Liquid Drops

Find the characteristic times, τ, for the oildrop and drop of mist in Example 2.2.

The characteristic time τ was defined in (2.22) as $\tau = m/b$, and v_{ter} was defined in (2.26) as $v_{\text{ter}} = mg/b$. Thus we have the useful relation

$$v_{\text{ter}} = g\tau. \tag{2.34}$$

Notice that this relation lets us interpret v_{ter} as the speed a falling object *would* acquire in a time τ, *if it had a constant acceleration equal to* g. Also note that, like v_{ter}, the time τ is an inverse indicator of the importance of air resistance: When the coefficient b of air resistance is small, both v_{ter} and τ are large; when b is large, both v_{ter} and τ are small.

For our present purposes, the importance of (2.34) is that, since we have already found the terminal velocities of the two drops, we can immediately find the values of τ. For the Millikan oildrop, we found that $v_{\text{ter}} = 6.1 \times 10^{-5}$ m/s, therefore

$$\tau = \frac{v_{\text{ter}}}{g} = \frac{6.1 \times 10^{-5}}{9.8} = 6.2 \times 10^{-6}\,\text{s} \qquad \text{[oildrop]}.$$

After falling for just 20 microseconds, this oildrop will have acquired 95% of its terminal speed. For almost every purpose, the oildrop *always* travels at its terminal speed.

For the drop of mist of Example 2.2, the terminal speed was $v_{\text{ter}} = 1.3$ m/s and so $\tau = v_{\text{ter}}/g \approx 0.13$ s. After about 0.4 s, the drop will have acquired 95% of its terminal speed.

Whether or not our falling object starts from rest, we can find its position y as a function of time by integrating the known form (2.30) of v_y,

$$v_y(t) = v_{\text{ter}} + (v_{yo} - v_{\text{ter}})e^{-t/\tau}.$$

Assuming that the projectile's initial position is $y = 0$, it immediately follows that

$$\begin{aligned} y(t) &= \int_0^t v_y(t')\,dt' \\ &= v_{\text{ter}}t + (v_{yo} - v_{\text{ter}})\tau\left(1 - e^{-t/\tau}\right). \end{aligned} \tag{2.35}$$

This equation for $y(t)$ can now be combined with Equation (2.23) for $x(t)$ to give us the orbit of any projectile, moving both horizontally and vertically, in a linear medium.

2.3 Trajectory and Range in a Linear Medium

We saw at the begining of the last section that the equation of motion for a projectile moving in any direction resolves into two separate equations, one for the horizontal and one for the vertical motion [Equations (2.15) and (2.16)]. We have solved each of these separate equations in (2.23) and (2.35), and we can now put these solutions together to give the trajectory of an arbitrary projectile moving in any direction. In this discussion it is marginally more convenient to measure y vertically *upward*, in which case we must reverse the sign of v_{ter}. (Make sure you understand this point.) Thus the two equations of the orbit become

$$\left.\begin{aligned} x(t) &= v_{xo}\tau\left(1 - e^{-t/\tau}\right) \\ y(t) &= (v_{yo} + v_{\text{ter}})\tau\left(1 - e^{-t/\tau}\right) - v_{\text{ter}}t. \end{aligned}\right\} \tag{2.36}$$

You can eliminate t from these two equations by solving the first for t and then substituting into the second. (See Problem 2.17.) The result is the equation for the trajectory:

$$y = \frac{v_{yo} + v_{\text{ter}}}{v_{xo}}x + v_{\text{ter}}\tau \ln\left(1 - \frac{x}{v_{xo}\tau}\right). \tag{2.37}$$

This equation is probably too complicated to be especially illuminating, but I have plotted it as the solid curve in Figure 2.7, with the help of which you can understand some of the features of (2.37). For example, if you look at the second term on the right of (2.37), you will see that as $x \to v_{xo}\tau$ the argument of the log function approaches zero; therefore, the log term and hence y both approach $-\infty$. That is, the trajectory has a vertical asymptote at $x = v_{xo}\tau$, as you can see in the picture. I leave it as an exercise (Problem 2.19) for you to check that if air resistance is switched off (v_{ter} and τ both approach infinity), the trajectory defined by (2.37) does indeed approach the dashed trajectory corresponding to zero air resistance.

Horizontal Range

A standard (and quite interesting) problem in elementary physics courses is to show that the horizontal range R of a projectile (subject to no air resistance of course) is

$$R_{\text{vac}} = \frac{2v_{xo}v_{yo}}{g} \qquad \text{[no air resistance]} \tag{2.38}$$

where R_{vac} stands for the range in a vacuum. Let us see how this result is modified by air resistance.

The range R is the value of x when y as given by (2.37) is zero. Thus R is the solution of the equation

$$\frac{v_{yo} + v_{\text{ter}}}{v_{xo}}R + v_{\text{ter}}\tau \ln\left(1 - \frac{R}{v_{xo}\tau}\right) = 0. \tag{2.39}$$

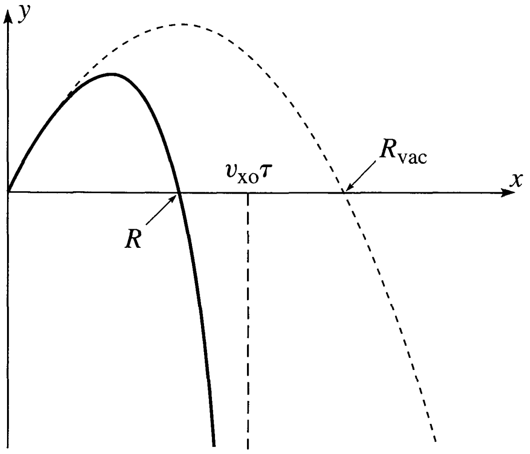

Figure 2.7 The trajectory of a projectile subject to a linear drag force (solid curve) and the corresponding trajectory in a vacuum (dashed curve). At first the two curves are very similar, but as t increases, air resistance slows the projectile and pulls its trajectory down, with a vertical asymptote at $x = v_{xo}\tau$. The horizontal range of the projectile is labeled R, and the corresponding range in vacuum R_{vac}.

This is a transcendental equation and cannot be solved analytically, that is, in terms of well known, elementary functions such as logs, or sines and cosines. For a given choice of parameters, it can be solved numerically with a computer (Problem 2.22), but this approach usually gives one little sense of how the solution depends on the parameters. Often a good alternative is to find some approximation that allows an *approximate* analytic solution. (Before the advent of computers, this was often the only way to find out what happens.) In the present case, it is often clear that the effects of air resistance should be *small*. This means that both v_{ter} and τ are large and the second term in the argument of the log function is small (since it has τ in its denominator). This suggests that we expand the log in a Taylor series (see Problem 2.18):

$$\ln(1-\epsilon) = -\left(\epsilon + \tfrac{1}{2}\epsilon^2 + \tfrac{1}{3}\epsilon^3 + \cdots\right). \tag{2.40}$$

We can use this expansion for the log term in (2.39), and, provided τ is large enough, we can surely neglect the terms beyond ϵ^3. This gives the equation

$$\left[\frac{v_{yo} + v_{ter}}{v_{xo}}\right] R - v_{ter}\tau \left[\frac{R}{v_{xo}\tau} + \frac{1}{2}\left(\frac{R}{v_{xo}\tau}\right)^2 + \frac{1}{3}\left(\frac{R}{v_{xo}\tau}\right)^3\right] = 0. \tag{2.41}$$

This equation can be quickly tidied up. First, the second term in the first bracket cancels the first term in the second. Next, every term contains a factor of R. This implies that one solution is $R = 0$, which is correct — the height y is zero when $x = 0$. Nevertheless, this is not the solution we are interested in, and we can divide out the common factor of R. A little rearrangement (and replacement of v_{ter}/τ by g) lets us rewite the equation as

$$R = \frac{2v_{xo}v_{yo}}{g} - \frac{2}{3v_{xo}\tau}R^2. \tag{2.42}$$

This may seem a perverse way to write a quadratic equation for R, but it leads us quickly to the desired approximate solution. The point is that the second term on the right is very small. (In the numerator R is certainly no more than R_{vac} and we are assuming that τ in the denominator is very large.) Therefore, as a first approximation we get

$$R \approx \frac{2v_{xo}v_{yo}}{g} = R_{\text{vac}}. \tag{2.43}$$

This is just what we expected: For low air resistance, the range is close to R_{vac}. But with the help of (2.42) we can now get a second, better approximation. The last term of (2.42) is the required correction to R_{vac}; because it is already small, we would certainly be satisfied with an approximate value for this correction. Thus, in evaluating the last term of (2.42), we can replace R with the approximate value $R \approx R_{\text{vac}}$, and we find as our second approximation [remember that the first term in (2.42) is just R_{vac}]

$$\begin{aligned} R &\approx R_{\text{vac}} - \frac{2}{3v_{xo}\tau}(R_{\text{vac}})^2 \\ &= R_{\text{vac}}\left(1 - \frac{4}{3}\frac{v_{yo}}{v_{\text{ter}}}\right). \end{aligned} \tag{2.44}$$

(To get the second line, I replaced the second R_{vac} in the previous line by $2v_{xo}v_{yo}/g$ and τg by v_{ter}.) Notice that the correction for air resistance always makes R smaller than R_{vac}, as one would expect. Notice also that the correction depends only on the ratio v_{yo}/v_{ter}. More generally, it is easy to see (Problem 2.32) that the importance of air resistance is indicated by the ratio v/v_{ter} of the projectile's speed to the terminal speed. If $v/v_{\text{ter}} \ll 1$ throughout the flight, the effect of air resistance is very small; if v/v_{ter} is around 1 or more, air resistance is almost certainly important [and the approximation (2.44) is certainly no good].

EXAMPLE 2.4 Range of Small Metal Pellets

I flick a tiny metal pellet with diameter $d = 0.2$ mm and $\mathbf{v} = 1$ m/s at 45°. Find its horizontal range assuming the pellet is gold (density $\varrho \approx 16\,\text{g/cm}^3$). What if it is aluminum (density $\varrho \approx 2.7\,\text{g/cm}^3$)?

In the absence of air resistance, both pellets would have the same range,

$$R_{\text{vac}} = \frac{2v_{xo}v_{yo}}{g} = 10.2\,\text{cm}.$$

For gold, Equation (2.27) gives (as you can check) $v_{\text{ter}} \approx 21$ m/s. Thus the correction term in (2.44) is

$$\frac{4}{3}\frac{v_{yo}}{v_{\text{ter}}} = \frac{4}{3} \times \frac{0.71}{21} \approx 0.05.$$

That is, air resistance reduces the range by 5% to about 9.7 cm. The density of aluminum is about 1/6 times that of gold. Therefore the terminal speed is one sixth as big, and the correction for aluminum is 6 times greater or about 30%, giving a range of about 7 cm. For the gold pellet the correction for air resistance is quite small and could perhaps be neglected; for the aluminum pellet, the correction is still small, but is certainly not negligible.

2.4 Quadratic Air Resistance

In the last two sections we have developed a rather complete theory of projectiles subject to a linear drag force, $\mathbf{f} = -b\mathbf{v}$. While we *can* find examples of projectiles for which the drag is linear (notably very small objects, such as the Millikan oildrop), for most of the more obvious examples of projectiles (baseballs, footballs, cannonballs, and the like) it is a far better approximation to say that the drag is pure quadratic, $\mathbf{f} = -cv^2\hat{\mathbf{v}}$. We must, therefore, develop a corresponding theory for a quadratic drag force. On the face of it, the two theories are not so very different. In either case we have to solve the differential equation

$$m\dot{\mathbf{v}} = m\mathbf{g} + \mathbf{f}, \tag{2.45}$$

and in both cases this is a first-order differential equation for the velocity $\mathbf{v}$, with $\mathbf{f}$ depending in a relatively simple way on $\mathbf{v}$. There is, however, an important difference. In the linear case ($\mathbf{f} = -b\mathbf{v}$), Equation (2.45) is a *linear* differential equation, inasmuch as the terms that involve $\mathbf{v}$ are all linear in $\mathbf{v}$ or its derivatives. In the quadratic case, Equation (2.45) is, of course, nonlinear. And it turns out that the mathematical theory of nonlinear differential equations is significantly more complicated than the linear theory. As a practical matter, we shall find that for the case of a general projectile, moving in both the x and y directions, Equation (2.45) cannot be solved in terms of elementary functions when the drag is quadratic. More generally, we shall see in Chapter 12 that for more complicated systems, nonlinearity can lead to the astonishing phenomenon of chaos, although this does not happen in the present case.

In this section, I shall start with the same two special cases discussed in Section 2.2, a body that is constrained to move horizontally, such as a railroad car on a horizontal track, and a body that moves vertically, such as a stone dropped from a window (both now with quadratic drag forces). We shall find that in these two especially simple cases the differential equation (2.45) *can* be solved by elementary means, and the solutions introduce some important techniques and interesting results. I shall then discuss briefly the general case (motion in both the horizontal and vertical directions), which can be solved only numerically.

Horizontal Motion with Quadratic Drag

Let us consider a body moving horizontally (in the positive x direction), subject to a quadratic drag and no other forces. For example, you could imagine a cycle racer, who has crossed the finishing line and is coasting to a stop under the influence of air resistance. To the extent that the cycle is well lubricated and tires well inflated, we can ignore ordinary friction,[3] and, except at very low speeds, air resistance is purely quadratic. The x component of the equation of motion is therefore (I'll abbreviate v_x to v)

$$m\frac{dv}{dt} = -cv^2. \tag{2.46}$$

If we divide by v^2 and multiply by dt, we get an equation in which only the variable v appears on the left and only t on the right:[4]

$$m\frac{dv}{v^2} = -c\,dt. \tag{2.47}$$

This trick — of rearranging a differential equation so that only one variable appears on the left and only the other on the right — is called **separation of variables**. When it is possible, separation of variables is often the simplest way to solve a first-order differential equation, since the solution can be found by simple integration of both sides.

Integrating Equation (2.47) we find

$$m\int_{v_o}^{v}\frac{dv'}{v'^2} = -c\int_0^t dt'$$

where v_o is the initial velocity at $t = 0$. Notice that I have written both sides as definite integrals, with the appropriate limits, so that I shan't have to worry about any constants of integration. I have also renamed the variables of integration as v' and t' to avoid

[3] As I shall discuss shortly, when the cyclist slows down to a stop, air resistance becomes smaller, and eventually friction becomes the dominant force. Nevertheless, at speeds around 10 mph or more, it is a fair approximation to ignore everything but the quadratic air resistance.

[4] In passing from (2.46) to (2.47), I have treated the derivative dv/dt as if it were the quotient of two separate numbers, dv and dt. As you are certainly aware this cavalier proceeding is not strictly correct. Nevertheless, it can be justified in two ways. First, in the theory of *differentials*, it is in fact true that dv and dt are defined as separate numbers (differentials), such that their quotient is the derivative dv/dt. Fortunately, it is quite unnecessary to know about this theory. As physicists we know that dv/dt is the limit of $\Delta v/\Delta t$, as both Δv and Δt become small, and I shall take the view that dv is just shorthand for Δv (and likewise dt for Δt), *with the understanding that it has been taken small enough that the quotient dv/dt is within my desired accuracy of the true derivative.* With this understanding, (2.47), with dv on one side and dt on the other, makes perfectly good sense.

confusion with the upper limits v and t. Both of these integrals are easily evaluated, and we find

$$m\left(\frac{1}{v_o}-\frac{1}{v}\right)=-ct \tag{2.48}$$

or, solving for v,

$$v(t)=\frac{v_o}{1+cv_ot/m}=\frac{v_o}{1+t/\tau} \tag{2.49}$$

where I have introduced the abbreviation τ for the combination of constants

$$\tau=\frac{m}{cv_o} \qquad \text{[for quadratic drag].} \tag{2.50}$$

As you can easily check, τ is a time, with the significance that when $t=\tau$ the velocity is $v=v_o/2$. Notice that this parameter τ is different from the τ introduced in (2.22) for motion subject to linear air resistance; nevertheless, both parameters have the same general significance as indicators of the time for air resistance to slow the motion appreciably.

To find the bicycle's position x, we have only to integrate v to give (as you should check)

$$\begin{aligned} x(t) &= x_o+\int_0^t v(t')\,dt' \\ &= v_o\tau\ln(1+t/\tau)\,, \end{aligned} \tag{2.51}$$

if we take the initial position x_o to be zero. Figure 2.8 shows our results for v and x as functions of t. It is interesting to compare these graphs with the corresponding graphs of Figure 2.4 for a body coasting horizontally but subject to a linear resistance. Superficially, the two graphs for the velocity look similar. In particular, both go to zero as $t\to\infty$. But in the linear case v goes to zero *exponentially*, whereas in the quadratic case it does so only very slowly, like $1/t$. This difference in the behavior of v manifests itself quite dramatically in the behavior of x. In the linear case, we

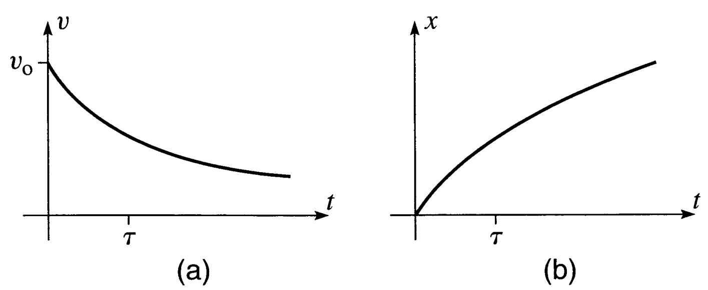

Figure 2.8 The motion of a body, such as a bicycle, coasting horizontally and subject to a quadratic air resistance. **(a)** The velocity is given by (2.49) and goes to zero like $1/t$ as $t\to\infty$. **(b)** The position is given by (2.51) and goes to infinity as $t\to\infty$.

saw that x approaches a finite limit as $t \to \infty$, but it is clear from (2.51) that in the quadratic case x increases without limit as $t \to \infty$.

The striking difference in the behavior of x for quadratic and linear drags is easy to understand qualitatively. In the quadratic case, the drag is proportional to v^2. Thus as v gets small, the drag gets *very* small — so small that it fails to bring the bicyle to rest at any finite value of x. This unexpected behavior serves to highlight that a drag force that is proportional to v^2 at *all* speeds is unrealistic. Although the linear drag and ordinary friction are very small, nevertheless as $v \to 0$ they must eventually become more important than the v^2 term and cannot be ignored. In particular, one or another of these two terms (friction in the case of a bicycle) ensures that no real body can coast on to infinity!

Vertical Motion with Quadratic Drag

The case that an object moves vertically with a quadratic drag force can be solved in much the same way as the horizontal case. Consider a baseball that is dropped from a window in a high tower. If we measure the coordinate y vertically down, the equation of motion is (I'll abbreviate v_y to v now)

$$m\dot{v} = mg - cv^2. \tag{2.52}$$

Before we solve this equation, let us consider the ball's terminal speed, the speed at which the two terms on the right of (2.52) just balance. Evidently this must satisfy $cv^2 = mg$, whose solution is

$$v_{\text{ter}} = \sqrt{\frac{mg}{c}}. \tag{2.53}$$

For any given object (given m, g, and c), this lets us calculate the terminal speed. For example, for a baseball it gives (as we shall see in a moment) $v_{\text{ter}} \approx 35$ m/s, or nearly 80 miles per hour.

We can tidy the equation of motion (2.52) a little by using (2.53) to replace c by mg/v_{ter}^2 and canceling the factors of m:

$$\dot{v} = g\left(1 - \frac{v^2}{v_{\text{ter}}^2}\right). \tag{2.54}$$

This can be solved by separation of variables, just as in the case of horizontal motion: First we can rewrite it as

$$\frac{dv}{1 - v^2/v_{\text{ter}}^2} = g\,dt. \tag{2.55}$$

This is the desired separated form (only v on the left and only t on the right) and we can simply integrate both sides.[5] Assuming the ball starts from rest, the limits of

[5] Notice that in fact any one-dimensional problem where the net force depends only on the velocity can be solved by separation of variables, since the equation $m\dot{v} = F(v)$ can always be

integration are 0 and v on the left and 0 and t on the right, and we find (as you should verify — Problem 2.35)

$$\frac{v_{\text{ter}}}{g} \operatorname{arctanh}\left(\frac{v}{v_{\text{ter}}}\right) = t \tag{2.56}$$

where "arctanh" denotes the inverse hyperbolic tangent. This particular integral can be evaluated alternatively in terms of the natural log function (Problem 2.37). However, the hyperbolic functions, sinh, cosh, and tanh, and their inverses arcsinh, arccosh, and arctanh, come up so often in all branches of physics that you really should learn to use them. If you have not had much exposure to them, you might want to look at Problems 2.33 and 2.34, and study graphs of these functions.

Equation (2.56) can be solved for v to give

$$v = v_{\text{ter}} \tanh\left(\frac{gt}{v_{\text{ter}}}\right). \tag{2.57}$$

To find the position y, we just integrate v to give

$$y = \frac{(v_{\text{ter}})^2}{g} \ln\left[\cosh\left(\frac{gt}{v_{\text{ter}}}\right)\right]. \tag{2.58}$$

While both of these two formulas can be cleaned up a little (see Problem 2.35), they are already sufficient to work the following example.

EXAMPLE 2.5 A Baseball Dropped from a High Tower

Find the terminal speed of a baseball (mass $m = 0.15$ kg and diameter $D = 7$ cm). Make plots of its velocity and position for the first six seconds after it is dropped from a tall tower.

The terminal speed is given by (2.53), with the coefficient of air resistance c given by (2.4) as $c = \gamma D^2$ where $\gamma = 0.25\ \text{N·s}^2/\text{m}^4$. Therefore

$$v_{\text{ter}} = \sqrt{\frac{mg}{\gamma D^2}} = \sqrt{\frac{(0.15\,\text{kg}) \times (9.8\,\text{m/s}^2)}{(0.25\,\text{N·s}^2/\text{m}^4) \times (0.07\,\text{m})^2}} = 35\ \text{m/s} \tag{2.59}$$

or nearly 80 miles per hour. It is interesting to note that fast baseball pitchers can pitch a ball considerably *faster* than v_{ter}. Under these conditions, the drag force is actually *greater* than the ball's weight!

The plots of v and y can be made by hand, but are, of course, much easier with the help of computer software such as Mathcad or Mathematica that can make the plots for you. Whatever method we choose, the results are as shown in Figure 2.9, where the solid curves show the actual velocity and position while the dashed curves are the corresponding values in a vacuum. The actual velocity levels out,

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written as $m\,dv/F(v) = dt$. Of course there is no assurance that this can be integrated analytically if $F(v)$ is too complicated, but it does guarantee a straightforward numerical solution at worst. See Problem 2.7.

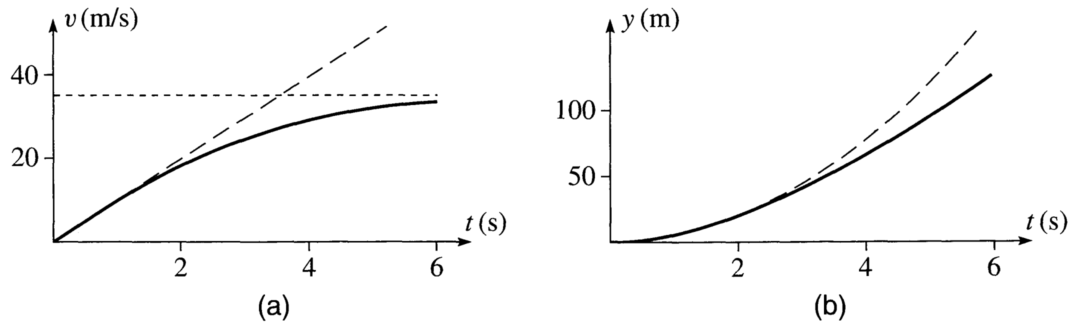

Figure 2.9 The motion of a baseball dropped from the top of a high tower (solid curves). The corresponding motion in a vacuum is shown with long dashes. **(a)** The actual velocity approaches the ball's terminal velocity $v_{\text{ter}} = 35$ m/s as $t \to \infty$. **(b)** The graph of position against time falls further and further behind the corresponding vacuum graph. When $t = 6$ s, the baseball has dropped about 130 meters; in a vacuum, it would have dropped about 180 meters.

approaching the terminal value $v_{\text{ter}} = 35$ m/s as $t \to \infty$, whereas the velocity in a vacuum would increase without limit. Initially, the position increases just as it would in a vacuum (that is, $y = \frac{1}{2}gt^2$), but falls behind as v increases and the air resistance becomes more important. Eventually, y approaches a straight line of the form $y = v_{\text{ter}}t +$ const. (See Problem 2.35.)

Quadratic Drag with Horizontal and Vertical Motion

The equation of motion for a projectile subject to quadratic drag,

$$\begin{aligned} m\ddot{\mathbf{r}} &= m\mathbf{g} - cv^2\hat{\mathbf{v}} \\ &= m\mathbf{g} - cv\mathbf{v}, \end{aligned} \tag{2.60}$$

resolves into its horizontal and vertical components (with y measured vertically upward) to give

$$\left.\begin{aligned} m\dot{v}_x &= -c\sqrt{v_x^2 + v_y^2}\,v_x \\ m\dot{v}_y &= -mg - c\sqrt{v_x^2 + v_y^2}\,v_y. \end{aligned}\right\} \tag{2.61}$$

These are two differential equations for the two unknown functions $v_x(t)$ and $v_y(t)$, but each equation involves *both* v_x and v_y. In particular, neither equation is the same as for an object that moves only in the x direction or only in the y direction. This means that we cannot solve these two equations by simply pasting together our two separate solutions for horizontal and vertical motion. Worse still, it turns out that the two equations (2.61) cannot be solved analytically at all. The only way to solve them is numerically, which we can only do for specified numerical initial conditions (that is, specified values of the initial position and velocity). This means that we cannot find the *general* solution; all we can do numerically is to find the particular solution corresponding to any chosen initial conditions. Before I discuss some general properties of the solutions of (2.61), let us work out one such numerical solution.

EXAMPLE 2.6 Trajectory of a Baseball

The baseball of Example 2.5 is now thrown with velocity 30 m/s (about 70 mi/h) at 50° above the horizontal from a high cliff. Find its trajectory for the first eight seconds of flight and compare with the corresponding trajectory in a vacuum. If the same baseball was thrown with the same initial velocity on horizontal ground how far would it travel before landing? That is, what is its horizontal range?

We have to solve the two coupled differential equations (2.61) with the initial conditions

$$v_{xo} = v_o \cos\theta = 19.3 \text{ m/s} \quad \text{and} \quad v_{yo} = v_o \sin\theta = 23.0 \text{ m/s}$$

and $x_o = y_o = 0$ (if we put the origin at the point from which the ball is thrown). This can be done with systems such as Mathematica, Matlab, or Maple, or with programming languages such as "C" or Fortran. Figure 2.10 shows the resulting trajectory, found using the function "NDSolve" in Mathematica.

Several features of Figure 2.10 deserve comment. Obviously the effect of air resistance is to lower the trajectory, as compared to the vacuum trajectory (shown dashed). For example, we see that in a vacuum the high point of the trajectory occurs at $t \approx 2.3$ s and is about 27 m above the starting point; with air resistance, the high point comes just before $t = 2.0$ s and is at about 21 m. In a vacuum, the ball would continue to move indefinitely in the x direction. The

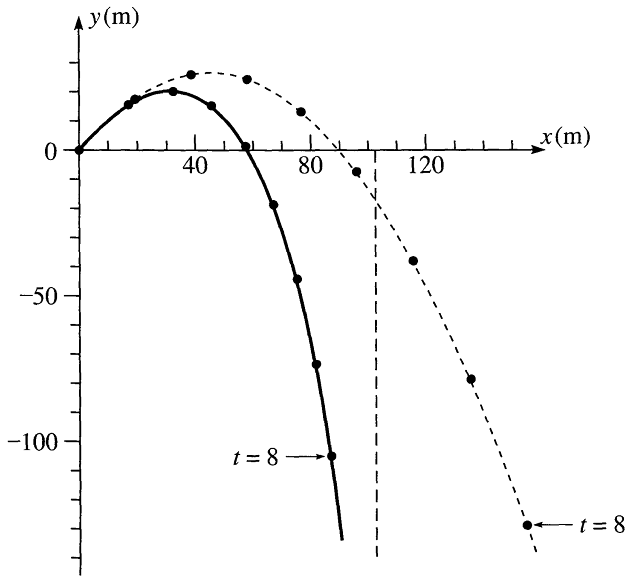

Figure 2.10 Trajectory of a baseball thrown off a cliff and subject to quadratic air resistance (solid curve). The initial velocity is 30 m/s at 50° above the horizontal; the terminal speed is 35 m/s. The dashed curve shows the corresponding trajectory in a vacuum. The dots show the ball's position at one-second intervals. Air resistance slows the horizontal motion, so that the ball approaches a vertical asymptote just beyond $x = 100$ meters.

effect of air resistance is to slow the horizontal motion so that x never moves to the right of a vertical asymptote near $x = 100$ m.

The horizontal range of the baseball is easily read off the figure as the value of x when y returns to zero. We see that $R \approx 59$ m, as opposed to the range in vacuum, $R_{vac} \approx 90$ m. The effect of air resistance is quite large in this example, as we might have anticipated: The ball was thrown with a speed only a little less than the terminal speed (30 vs 35 m/s), and this means that the force of air resistance is only a little less than that of gravity. This being the case, we should expect air resistance to change the trajectory appreciably.

This example illustrates several of the general features of projectile motion with a quadratic drag force. Although we cannot solve analytically the equations of motion (2.61) for this problem, we *can* use the equations to prove various general properties of the trajectory. For example, we noticed that the baseball reached a lower maximum height, and did so sooner, than it would have in a vacuum. It is easy to prove that this will always be the case: As long as the projectile is moving upward ($v_y > 0$), the force of air resistance has a *downward* y component. Thus the downward acceleration is greater than g (its value in vacuum). Therefore a graph of v_y against t slopes down from v_{yo} more quickly than it would in vacuum, as shown in Figure 2.11. This guarantees that v_y reaches zero sooner than it would in vacuum, and that the ball travels less distance (in the y direction) before reaching the high point. That is, the ball's high point occurs sooner, and is lower, than it would be in a vacuum.

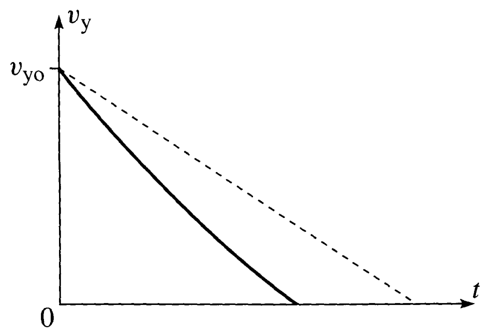

Figure 2.11 Graph of v_y against t for a projectile that is thrown upward ($v_{yo} > 0$) and is subject to a quadratic resistance (solid curve). The dashed line (slope $= -g$) is the corresponding graph when there is no air resistance. The projectile moves upward until it reaches its maximum height when $v_y = 0$. During this time, the drag force is downward and the downward acceleration is always greater than g. Therefore, the curve slopes more steeply than the dashed line, and the projectile reaches its high point sooner than it would in a vacuum. Since the area under the curve is less than that under the dashed line, the projectile's maximum height is less than it would be in a vacuum.

I claimed that the baseball of Example 2.6 approaches a vertical asymptote as $t \to \infty$, and we can now prove that this is always the case. First, it is easy to convince yourself that once the ball starts moving downward, it continues to accelerate downward, with v_y approaching $-v_{\text{ter}}$ as $t \to \infty$. At the same time v_x continues to decrease and approaches zero. Thus the square root in both of the equations (2.61) approaches v_{ter}. In particular, when t is large, the equation for v_x can be approximated by

$$\dot{v}_x \approx -\frac{cv_{\text{ter}}}{m} v_x = -kv_x$$

say. The solution of this equation is, of course, an exponential function, $v_x = Ae^{-kt}$, and we see that v_x approaches zero very rapidly (exponentially) as $t \to \infty$. This guarantees that x, which is the integral of v_x,

$$x(t) = \int_0^t v_x(t')\,dt',$$

approaches a finite limit as $t \to \infty$, and the trajectory has a finite vertical asymptote as claimed.

2.5 Motion of a Charge in a Uniform Magnetic Field

Another interesting application of Newton's laws, and (like projectile motion) an application that lets me introduce some important mathematical methods, is the motion of a charged particle in a magnetic field. I shall consider here a particle of charge q (which I shall usually take to be positive), moving in a uniform magnetic field $\mathbf{B}$ that points in the z direction as shown in Figure 2.12. The net force on the particle is just the magnetic force

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}, \tag{2.62}$$

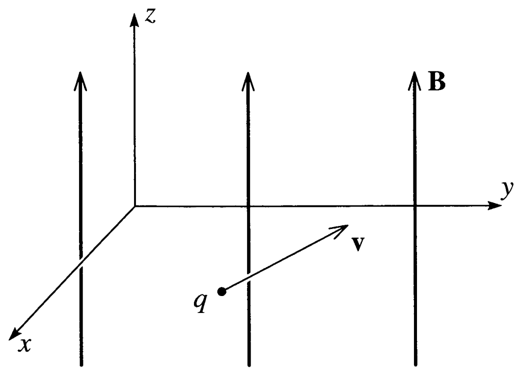

Figure 2.12 A charged particle moving in a uniform magnetic field that points in the z direction.