TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 315, Number 1, September 1989

MULTIRESOLUTION APPROXIMATIONS AND WAVELET ORTHONORMAL BASES OF $L^2(R)$

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ABSTRACT. A multiresolution approximation is a sequence of embedded vector spaces $(\mathbf{V}_j)_{j\in\mathbf{Z}}$ for approximating $\mathbf{L}^2(\mathbf{R})$ functions. We study the properties of a multiresolution approximation and prove that it is characterized by a 2π-periodic function which is further described. From any multiresolution approximation, we can derive a function $\psi(x)$ called a wavelet such that $(\sqrt{2^j}\psi(2^jx-k))_{(k,j)\in\mathbf{Z}^2}$ is an orthonormal basis of $\mathbf{L}^2(\mathbf{R})$. This provides a new approach for understanding and computing wavelet orthonormal bases. Finally, we characterize the asymptotic decay rate of multiresolution approximation errors for functions in a Sobolev space $\mathbf{H}^s$.

1. INTRODUCTION

In this article, we study the properties of the multiresolution approximations of $\mathbf{L}^2(\mathbf{R})$. We show how they relate to wavelet orthonormal bases of $\mathbf{L}^2(\mathbf{R})$. Wavelets have been introduced by A. Grossmann and J. Morlet [7] as functions whose translations and dilations could be used for expansions in $\mathbf{L}^2(\mathbf{R})$. J. Stromberg [16] and Y. Meyer [14] have proved independently that there exists some particular wavelets $\psi(x)$ such that $(\sqrt{2^j}\psi(2^jx-k))_{(j,k)\in\mathbf{Z}^2}$ is an orthonormal basis of $\mathbf{L}^2(\mathbf{R})$; these bases generalize the Haar basis. If $\psi(x)$ is regular enough, a remarkable property of these bases is to provide an unconditional basis of most classical functional spaces such as the Sobolev spaces, Hardy spaces, $\mathbf{L}^p(\mathbf{R})$ spaces and others [11]. Wavelet orthonormal bases have already found many applications in mathematics [14, 18], theoretical physics [6] and signal processing [9, 12].

Notation. $\mathbf{Z}$ and $\mathbf{R}$ respectively denote the set of integers and real numbers.

$\mathbf{L}^2(\mathbf{R})$ denotes the space of measurable, square-integrable functions $f(x)$.

The inner product of two functions $f(x)\in\mathbf{L}^2(\mathbf{R})$ and $g(x)\in\mathbf{L}^2(\mathbf{R})$ is written $\langle g(u), f(u)\rangle$.

The norm of $f(x)\in\mathbf{L}^2(\mathbf{R})$ is written $\|f\|$.

Received by the editors February 12, 1988.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 42C05, 41A30.

Key words and phrases. Approximation theory, orthonormal bases, wavelets.

This work is supported in part by NSF-CER/DCR82-19196 A02, NSF/DCR-8410771, Air Force/F49620-85-K-0018, ARMY DAAG-29-84-K-0061, and DARPA/ONR N0014-85-K-0807.

0002-9947/89 \$1.00 + \$.25 per page

The Fourier transform of any function $f(x) \in \mathbf{L}^2(\mathbf{R})$ is written $\hat{f}(\omega)$.
Id is the identity operator in $\mathbf{L}^2(\mathbf{R})$.
$\mathbf{l}^2(\mathbf{Z})$ is the vector space of square-summable sequences:

$$\mathbf{l}^2(\mathbf{Z}) = \left\{ (\alpha_i)_{i \in \mathbf{Z}} : \sum_{i=-\infty}^{+\infty} |\alpha_i|^2 < \infty \right\}.$$

Definition. A multiresolution approximation of $\mathbf{L}^2(\mathbf{R})$ is a sequence $(\mathbf{V}_j)_{j \in \mathbf{Z}}$ of closed subspaces of $\mathbf{L}^2(\mathbf{R})$ such that the following hold:

(1) $$\mathbf{V}_j \subset \mathbf{V}_{j+1} \quad \forall j \in \mathbf{Z},$$

(2) $$\bigcup_{j=-\infty}^{+\infty} \mathbf{V}_j \text{ is dense in } \mathbf{L}^2(\mathbf{R}) \quad \text{and} \quad \bigcap_{j=-\infty}^{+\infty} \mathbf{V}_j = \{0\},$$

(3) $$f(x) \in \mathbf{V}_j \Leftrightarrow f(2x) \in \mathbf{V}_{j+1} \quad \forall j \in \mathbf{Z},$$

(4) $$f(x) \in \mathbf{V}_j \Rightarrow f(x - 2^{-j}k) \in \mathbf{V}_j \quad \forall k \in \mathbf{Z},$$

(5) There exists an isomorphism $\mathbf{I}$ from $\mathbf{V}_0$ onto $\mathbf{l}^2(\mathbf{Z})$ which commutes with the action of $\mathbf{Z}$.

In property (5), the action of $\mathbf{Z}$ over $\mathbf{V}_0$ is the translation of functions by integers whereas the action of $\mathbf{Z}$ over $\mathbf{l}^2(\mathbf{Z})$ is the usual translation. The approximation of a function $f(x) \in \mathbf{L}^2(\mathbf{R})$ at a resolution 2^j is defined as the orthogonal projection of $f(x)$ on $\mathbf{V}_j$. To compute this orthogonal projection we show that there exists a unique function $\phi(x) \in \mathbf{L}^2(\mathbf{R})$ such that, for any $j \in \mathbf{Z}$, $(\sqrt{2^j}\phi(2^j x - k))_{k \in \mathbf{Z}}$ is an orthonormal basis of $\mathbf{V}_j$. The main theorem of this article proves that the Fourier transform of $\phi(x)$ is characterized by a 2π-periodic function $H(\omega)$. As an example we describe a multiresolution approximation based on cubic splines.

The additional information available in an approximation at a resolution 2^{j+1} as compared with the resolution 2^j, is given by an orthogonal projection on the orthogonal complement of $\mathbf{V}_j$ in $\mathbf{V}_{j+1}$. Let $\mathbf{O}_j$ be this orthogonal complement. We show that there exists a function $\psi(x)$ such that $(\sqrt{2^j}\psi(2^j x - k))_{k \in \mathbf{Z}}$ is an orthonormal basis of $\mathbf{O}_j$. The family of functions $(\sqrt{2^j}\psi(2^j x - k))_{(k,j) \in \mathbf{Z}^2}$ is a wavelet orthonormal basis of $\mathbf{L}^2(\mathbf{R})$.

An important problem in approximation theory [4] is to measure the decay of the approximation error when the resolution increases, given an a priori knowledge on the function smoothness. We estimate this decay for functions in Sobolev spaces $\mathbf{H}^s$. This result is a characterization of Sobolev spaces.

2. Orthonormal bases of multiresolution approximations

In this section, we prove that there exists a unique function $\phi(x) \in \mathbf{L}^2(\mathbf{R})$ such that for any $j \in \mathbf{Z}$, $(\sqrt{2^j}\phi(2^j x - k))_{k \in \mathbf{Z}}$ is a wavelet orthonormal basis of $\mathbf{V}_j$. This result is proved for $j = 0$. The extension for any $j \in \mathbf{Z}$ is a consequence of property (3).

Let us first detail property (5) of a multiresolution approximation. The operator $\mathbf{I}$ is an isomorphism from $\mathbf{V}_0$ onto $\mathbf{l}^2(\mathbf{Z})$. Hence, there exists a function $g(x)$ satisfying

$$\text{(6)} \qquad g(x) \in \mathbf{V}_0 \quad \text{and} \quad \mathbf{I}(g(x)) = \varepsilon(n), \quad \text{where } \varepsilon(n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

Since $\mathbf{I}$ commutes with translations of integers:

$$\mathbf{I}(g(x-k)) = \varepsilon(n-k).$$

The sequence $(\varepsilon(n-k))_{k\in\mathbf{Z}}$ is a basis of $\mathbf{l}^2(\mathbf{Z})$, hence $(g(x-k))_{k\in\mathbf{Z}}$ is a basis of $\mathbf{V}_0$. Let $f(x) \in \mathbf{V}_0$ and $\mathbf{I}(f(x)) = (\alpha_k)_{k\in\mathbf{Z}}$. Since $\mathbf{I}$ is an isomorphism, $\|f\|$ and $(\sum_{k=-\infty}^{+\infty} |\alpha_k|^2)^{1/2}$ are two equivalent norms on $\mathbf{V}_0$. Let us express the consequence of this equivalence on $g(x)$. The function $f(x)$ can be decomposed as:

$$\text{(7)} \qquad f(x) = \sum_{k=-\infty}^{+\infty} \alpha_k g(x-k).$$

The Fourier transform of this equation yields

$$\text{(8)} \qquad \hat{f}(\omega) = M(\omega)\hat{g}(\omega), \quad \text{where } M(\omega) = \sum_{k=-\infty}^{+\infty} \alpha_k e^{-ik\omega}.$$

The norm of $f(x)$ is given by

$$\|f\|^2 = \int_{-\infty}^{+\infty} |\hat{f}(\omega)|^2 \, d\omega = \int_0^{2\pi} |M(\omega)|^2 \sum_{k=-\infty}^{+\infty} |\hat{g}(\omega + 2k\pi)|^2 \, d\omega.$$

Since $\|f\|$ and $(\sum_{k=-\infty}^{+\infty} |\alpha_k|^2)^{1/2}$ are two equivalent norms on $\mathbf{V}_0$, it follows that

$$\text{(9)} \quad \exists C_1, C_2 > 0 \text{ such that } \forall \omega \in \mathbf{R}, \quad C_1 \leq \left(\sum_{k=-\infty}^{+\infty} |\hat{g}(\omega + 2k\pi)|^2 \right)^{1/2} \leq C_2.$$

We are looking for a function $\phi(x)$ such that $(\phi(x-k))_{k\in\mathbf{Z}}$ is an orthonormal basis of $\mathbf{V}_0$. To compute $\phi(x)$ we orthogonalize the basis $(g(x-k))_{k\in\mathbf{Z}}$. We can use two methods for this purpose, both useful.

The first method is based on the Fourier transform. Let $\hat{\phi}(\omega)$ be the Fourier transform of $\phi(x)$. With the Poisson formula, we can express the orthogonality of the family $(\phi(x-k))_{k\in\mathbf{Z}}$ as

$$\text{(10)} \qquad \sum_{k=-\infty}^{+\infty} |\hat{\phi}(\omega + 2k\pi)|^2 = 1.$$

Since $\phi(x) \in \mathbf{V}_0$, equation (8) shows that there exists a 2π-periodic function $M_\phi(\omega)$ such that

$$\hat{\phi}(\omega) = M_\phi(\omega)\hat{g}(\omega). \tag{11}$$

By inserting equation (11) into (10) we obtain

$$M_\phi(\omega) = \left(\sum_{k=-\infty}^{+\infty} |\hat{g}(\omega + 2k\pi)|^2\right)^{-1/2}. \tag{12}$$

Equation (9) proves that (12) defines a function $M_\phi(\omega) \in \mathbf{L}^2([0, 2\pi])$. If $\phi(x)$ is given by (11), one can also derive from (9) that $g(x)$ can be decomposed on the corresponding orthogonal family $(\phi(x-k))_{k\in\mathbf{Z}}$. This implies that $(\phi(x-k))_{k\in\mathbf{Z}}$ generates $\mathbf{V}_0$.

The second approach for building the function $\phi(x)$ is based on the general algorithm for orthogonalizing an unconditional basis $(e_\lambda)_{\lambda\in\Lambda}$ of a Hilbert space $\mathbf{H}$. This approach was suggested by Y. Meyer. Let us recall that a sequence $(e_\lambda)_{\lambda\in\Lambda}$ is a normalized unconditional basis if there exist two positive constants A and B such that for any sequence of numbers $(\alpha_\lambda)_{\lambda\in\Lambda}$,

$$A\left(\sum_{\lambda\in\Lambda} |\alpha_\lambda|^2\right)^{1/2} \leq \left\|\sum_{\lambda\in\Lambda} \alpha_\lambda e_\lambda\right\| \leq B\left(\sum_{\lambda\in\Lambda} |\alpha_\lambda|^2\right)^{1/2}. \tag{13}$$

We first compute the Gram matrix $\mathbf{G}$, indexed by $\Lambda \times \Lambda$, whose coefficients are $\langle e_{\lambda 1}, e_{\lambda 2}\rangle$. Equation (13) is equivalent to

$$A^2 \mathrm{Id} \leq \mathbf{G} \leq B^2 \mathrm{Id}. \tag{14}$$

This equation shows that we can calculate $\mathbf{G}^{-1/2}$, whose coefficients are written $\gamma(\lambda_1, \lambda_2)$. Let us define the vectors $f_\lambda = \sum_{\lambda\in\Lambda} \gamma(\lambda, \lambda')e_{\lambda'}$. It is well known that the family $(f_\lambda)_{\lambda\in\Lambda}$ is an orthonormal basis of $\mathbf{H}$. This algorithm has the advantage, with respect to the usual Gram-Schmidt procedure, of preserving any supplementary structure (invariance under the action of a group, symmetries) which might exist in the sequence $(e_\lambda)_{\lambda\in\Lambda}$. In our particular case we verify immediately that both methods lead to the same result. The second one is more general and can be used when the multiresolution approximation is defined on a Hilbert space where the Fourier transform does not exist [8].

In the following, we impose a regularity condition on the multiresolution approximations of $\mathbf{L}^2(\mathbf{R})$ that we study. We shall say that a function $f(x) \in \mathbf{L}^2(\mathbf{R})$ is regular if and only if it is continuously differentiable and satisfies:

$$\exists C > 0, \forall x \in \mathbf{R}, \quad |f(x)| \leq C(1+x^2)^{-1} \quad \text{and} \quad |f'(x)| \leq C(1+x^2)^{-1}. \tag{15}$$

A multiresolution approximation $(\mathbf{V}_j)_{j\in\mathbf{Z}}$ is said to be regular if and only if $\phi(x)$ is regular.

3. Properties of $\phi(x)$

In this section, we study the functions $\phi(x)$ such that for all $j \in \mathbf{Z}$, $(\sqrt{2^j}\phi(2^j x - n))_{n\in\mathbf{Z}}$ is an orthonormal family, and if $\mathbf{V}_j$ is the vector space generated by this family of functions, then $(\mathbf{V}_j)_{j\in\mathbf{Z}}$ is a regular multiresolution approximation of $\mathbf{L}^2(\mathbf{R})$. We show that the Fourier transform of $\phi(x)$ can be computed from a 2π-periodic function $H(\omega)$ whose properties are further described.

Property (2) of a multiresolution approximation implies that

$$\frac{1}{2}\phi\left(\frac{x}{2}\right) \in \mathbf{V}_{-1} \subset \mathbf{V}_0 .$$

The function $\frac{1}{2}\phi(\frac{x}{2})$ can thus be decomposed in the orthonormal basis $(\phi(x-k))_{k\in\mathbf{Z}}$ of $\mathbf{V}_0$:

$$(16) \quad \frac{1}{2}\phi\left(\frac{x}{2}\right) = \sum_{k=-\infty}^{\infty} h_k \phi(x+k), \quad \text{where } h_k = \frac{1}{2}\int_{-\infty}^{\infty} \phi\left(\frac{x}{2}\right)\overline{\phi}(x+k)\,dx .$$

Since the multiresolution approximation is regular, the asymptotic decay of h_k satisfies $|h_k| = O(1+k^2)^{-1}$. The Fourier transform of equation (16) yields

$$(17) \qquad \hat{\phi}(2\omega) = H(\omega)\hat{\phi}(\omega), \quad \text{where } H(\omega) = \sum_{k=-\infty}^{\infty} h_k e^{-ik\omega} .$$

The following theorem gives a necessary condition on $H(\omega)$.

Theorem 1. *The function $H(\omega)$ as defined above satisfies:*

$$(18) \qquad |H(\omega)|^2 + |H(\omega+\pi)|^2 = 1 ,$$

$$(19) \qquad |H(0)| = 1 .$$

Proof. We saw in equation (10) that the Fourier transform $\hat{\phi}(\omega)$ must satisfy

$$(20) \qquad \sum_{k=-\infty}^{+\infty} |\hat{\phi}(\omega + 2k\pi)|^2 = 1 ,$$

and therefore

$$(21) \qquad \sum_{k=-\infty}^{+\infty} |\hat{\phi}(2\omega + 2k\pi)|^2 = 1 .$$

Since $\hat{\phi}(2\omega) = H(\omega)\hat{\phi}(\omega)$, this summation can be rewritten

$$(22) \qquad \sum_{k=-\infty}^{+\infty} |H(\omega + k\pi)|^2 |\hat{\phi}(\omega + k\pi)|^2 = 1 .$$

The function $H(\omega)$ is 2π-periodic. Regrouping the terms for $k \in 2\mathbf{Z}$ and $k \in 2\mathbf{Z}+1$ and inserting equation (20) yields

$$|H(\omega)|^2 + |H(\omega+\pi)|^2 = 1.$$

In order to prove that $|H(0)| = 1$, we show that

$$|\hat{\phi}(0)| = 1. \tag{23}$$

Let us prove that this equation is a consequence of property (2) of a multiresolution approximation. Let $\mathbf{P}_{\mathbf{V}_j}$ be the orthogonal projection on $\mathbf{V}_j$. Since $(\sqrt{2^j}\phi(2^j x - n))_{n\in\mathbf{Z}}$ is an orthonormal basis of $\mathbf{V}_j$, the kernel of $\mathbf{P}_{\mathbf{V}_j}$ can be written:

$$2^j K(2^j x, 2^j y), \quad \text{where } K(x,y) = \sum_{k=-\infty}^{\infty} \phi(x-k)\overline{\phi}(y-k). \tag{24}$$

Property (2) implies that the sequence of operators $(\mathbf{P}_{\mathbf{V}_j})_{j\in\mathbf{Z}}$ tends to Id in the sense of strong convergence for operators. The next lemma shows that the kernel $K(x,y)$ must satisfy $\int_{-\infty}^{\infty} K(x,y)\,dy = 1$.

Lemma 1. *Let $g(x)$ be a regular function (satisfying (15)) and $A(x,y) = \sum_{k=-\infty}^{\infty} g(x-k)\overline{g}(y-k)$. Then the following two properties are equivalent:*

$$\int_{-\infty}^{+\infty} A(x,y)\,dy = 1 \quad \textit{for almost all } x. \tag{25}$$

(26) *The sequence of operators $(\mathbf{T}_j)_{j\in\mathbf{Z}}$ whose kernels are $2^j A(2^j x, 2^j y)$, tends to* Id *in the sense of strong convergence for operators.*

Proof. Let us first prove that (25) implies (26). Since $g(x)$ is regular, $\exists C > 0$ such that

$$|A(x,y)| \le C(1+|x-y|)^{-2}. \tag{27}$$

Hence, the sequence of operators $(\mathbf{T}_j)_{j\in\mathbf{Z}}$ is bounded over $\mathbf{L}^2(\mathbf{R})$. For proving that

$$\forall j \in \mathbf{Z}, \forall f \in \mathbf{L}^2(\mathbf{R}) \quad \lim_{j\to+\infty} \|f - \mathbf{T}_j(f)\| = 0, \tag{28}$$

we can thus restrict ourselves to indicator functions of intervals. Indeed, finite linear combinations of these indicator functions are dense in $\mathbf{L}^2(\mathbf{R})$. Let $f(x)$ be the indicator function of an interval $[a,b]$,

$$f(x) = \begin{cases} 1 & \text{if } a \le x \le b, \\ 0 & \text{otherwise.} \end{cases}$$

Let us first prove that $\mathbf{T}_j f(x)$ converges almost everywhere to $f(x)$:

$$\mathbf{T}_j(f)(x) = \int_a^b 2^j A(2^j x, 2^j y)\,dy. \tag{29}$$

Equation (27) implies that

$$(30) \qquad |\mathbf{T}_j(f)(x)| \le C2^j \int_a^b (1+2^j|x-y|)^{-2}\,dy \le \frac{C'}{1+2^j \mathrm{dist}(x,[a,b])^2}.$$

If x is not a member of $[a,b]$, this inequality implies that

$$\lim_{j\to+\infty} \mathbf{T}_j(f)(x) = 0.$$

Let us now suppose that $x \in]a,b[$,

$$(31) \qquad \mathbf{T}_j(f)(x) = \int_{2^j a}^{2^j b} A(2^j x, y)\,dy.$$

By applying property (25), we obtain

$$(32) \qquad \mathbf{T}_j(f)(x) = 1 - \int_{-\infty}^{2^j a} A(2^j x, y)\,dy - \int_{2^j b}^{+\infty} A(2^j x, y)\,dy.$$

Since $x \in]a,b[$, inserting (27) in the previous equation yields

$$(33) \qquad \lim_{j\to+\infty} \mathbf{T}_j(f)(x) = 1.$$

Equation (30) shows that for $j \ge 0$, there exists $C'' > 0$ such that

$$|\mathbf{T}_j f(x)| \le \frac{C''}{1+x^2}.$$

We can therefore apply the theorem of dominated convergence on the sequence of functions $(\mathbf{T}_j f(x))_{j\in\mathbf{Z}}$ and prove that it converges strongly to $f(x)$.

Conversely let us show that (26) implies (25). Let us define

$$(34) \qquad \alpha(x) = \int_{-\infty}^{\infty} A(x,y)\,dy.$$

The function $\alpha(x)$ is periodic of period 1 and equation (27) implies that $\alpha(x) \in \mathbf{L}^\infty(\mathbf{R})$. Let $f(x)$ be the indicator function of $[-1,1]$. Property (26) implies that $\mathbf{T}_j f(x)$ converges to $f(x)$ in $\mathbf{L}^2(\mathbf{R})$ norm. Let $1 > r > 0$ and $x \in [-r,r]$,

$$\mathbf{T}_j(f)(x) = \int_{-1}^{1} 2^j A(2^j x, 2^j y)\,dy.$$

Similarly to equation (32), we show that

$$(35) \qquad \mathbf{T}_j(f)(x) = \alpha(2^j x) + O(2^{-j}).$$

Since $\alpha(2^j x)$ is 2^{-j} periodic and converges strongly to 1 in $\mathbf{L}^2([-r,r])$, $\alpha(x)$ must therefore be equal to 1. This proves Lemma 1.

Since $(\mathbf{P}_{\mathbf{V}_j})_{j\in\mathbf{Z}}$ tends to Id in the sense of strong convergence for operators, this lemma shows that the kernel $K(x,y)$ must satisfy $\int_{-\infty}^{\infty} K(x,y)\,dy = 1$. Hence, we have

$$(36) \qquad \int_{-\infty}^{\infty} K(x,y)\,dy = \sum_{k=-\infty}^{\infty} \overline{\gamma}\phi(x-k) = 1,$$

with

$$\gamma = \int_{-\infty}^{\infty} \phi(y)\, dy = \hat{\phi}(0). \tag{37}$$

By integrating equation (36) in x on $[0,1]$, we obtain $|\hat{\phi}(0)|^2 = 1$. From equation (17), we can now conclude that $|H(0)| = 1$. This concludes the proof of Theorem 1.

The next theorem gives a sufficiency condition on $H(\omega)$ in order to compute the Fourier transform of a function $\phi(x)$ which generates a multiresolution approximation.

Theorem 2. *Let* $H(\omega) = \sum_{k=-\infty}^{\infty} h_k e^{-ik\omega}$ *be such that*

$$|h_k| = O(1+k^2)^{-1}, \tag{38}$$

$$|H(0)| = 1, \tag{39}$$

$$|H(\omega)|^2 + |H(\omega+\pi)|^2 = 1, \tag{40}$$

$$H(\omega) \neq 0 \quad on\ [-\pi/2, \pi/2]. \tag{41}$$

Let us define

$$\hat{\phi}(\omega) = \prod_{k=1}^{\infty} H(2^{-k}\omega). \tag{42}$$

The function $\hat{\phi}(\omega)$ *is the Fourier transform of a function* $\phi(x)$ *such that* $(\phi(x-k))_{k\in\mathbf{Z}}$ *is an orthonormal basis of a closed subspace* $\mathbf{V}_0$ *of* $\mathbf{L}^2(\mathbf{R})$. *If* $\phi(x)$ *is regular, then the sequence of vector spaces* $(\mathbf{V}_j)_{j\in\mathbf{Z}}$ *defined from* $\mathbf{V}_0$ *by* (3) *is a regular multiresolution approximation of* $\mathbf{L}^2(\mathbf{R})$.

Proof. Let us first prove that $\hat{\phi}(\omega) \in \mathbf{L}^2(\mathbf{R})$. To simplify notations we denote $M(\omega) = |H(\omega)|^2$ and denote by $M_k(\omega)$ $(k \geq 1)$ the continuous function defined by

$$M_k(\omega) = \begin{cases} 0 & \text{if } |\omega| > 2^k\pi, \\ M\left(\frac{\omega}{2}\right) M\left(\frac{\omega}{4}\right) \cdots M\left(\frac{\omega}{2^k}\right) & \text{if } |\omega| \leq 2^k\pi. \end{cases}$$

Lemma 2. *For all* $k \in \mathbf{N}$, $k \neq 0$,

$$I_k^n = \int_{-\infty}^{\infty} M_k(\omega) e^{i2n\pi\omega}\, d\omega = \begin{cases} 2\pi & if\ n = 0, \\ 0 & if\ n \neq 0. \end{cases} \tag{43}$$

Proof. Let us divide the integral I_k^n into two parts:

$$I_k^n = \int_{-2^k\pi}^{0} M_k(\omega) e^{i2n\pi\omega}\, d\omega + \int_{0}^{2^k\pi} M_k(\omega) e^{i2n\pi\omega}\, d\omega.$$

Since $M(2^{-j}\omega + 2^{k-j}\pi) = M(2^{-j}\omega)$ for $0 \leq j < k$ and

$$M(2^{-k}\omega) + M(2^{-k}\omega + \pi) = 1,$$

by changing variables $\omega' = \omega + 2^k\pi$ in the first integral, we obtain

$$I_k^n = \int_0^{2^k\pi} M\left(\frac{\omega}{2}\right)\cdots M\left(\frac{\omega}{2^{k-1}}\right) e^{i2n\pi\omega}\, d\omega .$$

Since $M(\omega)$ is 2π-periodic, this equation implies

$$I_k^n = \int_{-2^{k-1}\pi}^{2^{k-1}\pi} M\left(\frac{\omega}{2}\right)\cdots M\left(\frac{\omega}{2^{k-1}}\right) e^{i2n\pi\omega}\, d\omega = I_{k-1}^n .$$

Hence, we derive that

$$I_k^n = I_{k-1}^n = \cdots = I_1^n = \begin{cases} 2\pi & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

This proves Lemma 2.

Let us now consider the infinite product

$$M_\infty(\omega) = \lim_{k\to\infty} M_k(\omega) = \prod_{j=1}^{\infty} M(2^{-j}\omega) = |\hat{\phi}(\omega)|^2 . \tag{44}$$

Since $0 \leq M(\omega) \leq 1$, this product converges. From Fatou's lemma we derive that

$$\int_{-\infty}^{\infty} M_\infty(\omega)\, d\omega \leq \lim_{k\to\infty} \int_{-\infty}^{\infty} M_k(\omega)\, d\omega = 2\pi . \tag{45}$$

Equation (42) thus defines a function $\hat{\phi}(\omega)$ which is in $\mathbf{L}^2(\mathbf{R})$. Let $\phi(x)$ be its inverse Fourier transform. We must show that $(\phi(x-k))_{k\in\mathbf{Z}}$ is an orthonormal family. For this purpose, we want to use Lemma 2 and apply the theorem of dominated convergence on the sequence of functions $(M_k(\omega)e^{i2n\pi\omega})_{k\in\mathbf{Z}}$. The function $M_\infty(\omega)$ can be rewritten

$$M_\infty(\omega) = e^{-\sum_{j=1}^{\infty} \mathrm{Log}(M(2^{-j}\omega))} . \tag{46}$$

Since $H(\omega)$ satisfies both conditions (38) and (39), it follows that $\mathrm{Log}(M(\omega)) = O(\omega)$ in the neighborhood of 0, and therefore

$$\lim_{\omega\to 0} M_\infty(\omega) = M_\infty(0) = 1 . \tag{47}$$

As a consequence of (38), $H(\omega)$ is a continuous function. From property (41) together with (47), we derive that

$$\exists C > 0 \text{ such that } \forall\omega \in [-\pi, \pi] \quad M_\infty(\omega) \geq C . \tag{48}$$

For $|\omega| \leq 2^k\pi$, we have

$$M_\infty(\omega) = M_k(\omega) M_\infty(\omega/2^k) .$$

Hence, equation (48) yields

$$0 \leq M_k(\omega) \leq \tfrac{1}{C} M_\infty(\omega) . \tag{49}$$

Since $M_k(\omega) = 0$ for $|\omega| > 2^k\pi$, inequality (48) is satisfied for all $\omega \in \mathbf{R}$. We proved in (45) that $M_\infty(\omega) \in \mathbf{L}^1(\mathbf{R})$, so we can apply the dominated convergence theorem on the sequence of functions $(M_k(\omega)e^{i2n\pi\omega})_{k\in\mathbf{Z}}$. From Lemma 2, we obtain

$$\int_{-\infty}^{\infty} M_\infty(\omega)e^{i2n\pi\omega}\,d\omega = \begin{cases} 2\pi & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases} \tag{50}$$

With the Parseval theorem applied to the inner products $\langle\phi(x), \phi(x-k)\rangle$, we conclude from (50) that $(\phi(x-k))_{k\in\mathbf{Z}}$ is orthonormal.

Let us call $\mathbf{V}_0$ the vector space generated by this orthonormal family. We suppose now that the function $\phi(x)$ is regular. Let $(\mathbf{V}_j)_{j\in\mathbf{Z}}$ be the sequence of vector spaces derived from $\mathbf{V}_0$ with property (3), $(\sqrt{2^j}\phi(2^jx-k))_{k\in\mathbf{Z}}$ is an orthonormal basis of $\mathbf{V}_j$ for any $j \in \mathbf{Z}$. We must prove that $(\mathbf{V}_j)_{j\in\mathbf{Z}}$ is a multiresolution approximation of $\mathbf{L}^2(\mathbf{R})$. We only detail properties (1) and (2) since the other ones are straightforward.

To prove (1), it is sufficient to show that $\mathbf{V}_{-1} \subset \mathbf{V}_0$. The vector spaces $\mathbf{V}_0$ and $\mathbf{V}_{-1}$ are respectively the set of all the functions whose Fourier transform can be written $M(\omega)\hat{\phi}(\omega)$ and $M(2\omega)\hat{\phi}(2\omega)$, where $M(\omega)$ is any 2π-periodic function such that $M(\omega) \in \mathbf{L}^2([0, 2\pi])$. Since $\hat{\phi}(\omega)$ is defined by (42), it satisfies

$$\hat{\phi}(2\omega) = H(\omega)\hat{\phi}(\omega), \tag{51}$$

with $|H(\omega)| \leq 1$. The function $M(2\omega)H(\omega)$ is 2π-periodic and is a member of $\mathbf{L}^2([0, 2\pi])$. From equation (51), we can therefore derive that any function of $\mathbf{V}_{-1}$ is in $\mathbf{V}_0$.

Let $\mathbf{P}_{\mathbf{V}_j}$ be the orthogonal projection operator on $\mathbf{V}_j$. To prove (2), we must verify that

$$\lim_{j\to+\infty} \mathbf{P}_{\mathbf{V}_j} = \mathrm{Id} \quad \text{and} \quad \lim_{j\to-\infty} \mathbf{P}_{\mathbf{V}_j} = 0. \tag{52}$$

Since $(\sqrt{2^j}\phi(2^jx-k))_{k\in\mathbf{Z}}$ is an orthonormal basis of $\mathbf{V}_j$, the kernel of $\mathbf{P}_{\mathbf{V}_j}$ is given by

$$2^j \sum_{k=-\infty}^{\infty} \phi(2^jx-k)\bar{\phi}(2^jy-k) = 2^jK(2^jx, 2^jy). \tag{53}$$

Since $(\phi(x-k))_{k\in\mathbf{Z}}$ is an orthogonal family, we have

$$\sum_{-\infty}^{+\infty} |\hat{\phi}(\omega + 2k\pi)|^2 = 1.$$

We showed in (47) that $|\hat{\phi}(0)| = 1$, so for any $k \neq 0$, the previous equation implies that $\hat{\phi}(2k\pi) = 0$. The Poisson formula yields

$$\sum_{k=-\infty}^{\infty} \phi(x-k) = \int_{-\infty}^{+\infty} \phi(u)\,du = \hat{\phi}(0). \tag{54}$$

We can therefore derive that

$$\int_{-\infty}^{+\infty} K(x,y)\,dy = |\hat{\phi}(0)|^2 = 1 \quad \text{for almost all } x\,.$$

Lemma 1 enables us to conclude that $\lim_{j\to+\infty} \mathbf{P}_{\mathbf{V}_j} = \mathrm{Id}$. Since $\phi(x)$ is regular, similarly to (27), we have

$$|2^j K(2^j x, 2^j y)| \le \frac{C2^j}{(1+2^j|x-y|)^2}\,. \tag{55}$$

From this inequality, we easily derive that $\lim_{j\to-\infty} \mathbf{P}_{\mathbf{V}_j} = 0$. This concludes the proof of Theorem 2.

Remarks. 1. The necessary conditions on $H(\omega)$ stated in Theorem 1 are not sufficient to define a function $\phi(x)$ such that $(\phi(x-k))_{k\in\mathbf{Z}}$ is an orthonormal family. A counterexample is given by $H(\omega) = \cos(3\omega/2)$. The function $\phi(x)$ whose Fourier transform is defined by (42) is equal to $\frac{1}{3}$ in $[-\frac{3}{2}, \frac{3}{2}]$ and 0 elsewhere. It does not generate an orthogonal family. A. Cohen [2] showed that the sufficient condition (41) is too strong to be necessary. He gave a weaker condition which is necessary and sufficient.

2. It is possible to control the smoothness of $\hat{\phi}(\omega)$ from $H(\omega)$. One can show that if $H(\omega) \in \mathbf{C}^q$ then $\hat{\phi}(\omega) \in \mathbf{C}^q$ and

$$\frac{d^n H(0)}{d\omega^n} = 0 \text{ for } 1 \le n \le q \Leftrightarrow \frac{d^n \hat{\phi}(0)}{d\omega^n} = 0 \text{ for } 1 \le n \le q\,. \tag{56}$$

I. Daubechies [3] and P. Tchamitchian [17] showed that we can also obtain a lower bound for the decay rate of $\hat{\phi}(\omega)$ at infinity. As a consequence of (40),

$$\frac{d^n H(0)}{d\omega^n} = 0 \quad \text{for } 1 \le n \le q$$

implies that

$$\frac{d^n H((2k+1)\pi)}{d\omega^n} = 0 \quad \text{for } 0 \le n \le q-1 \text{ and } k \in \mathbf{Z}.$$

Hence, we can decompose $H(\omega)$ into

$$H(\omega) = (\cos(\omega/2))^q\, M_0(\omega)\,, \tag{57}$$

where $M_0(\omega)$ is a 2π-periodic function whose amplitude is bounded by $A > 0$. One can then show that

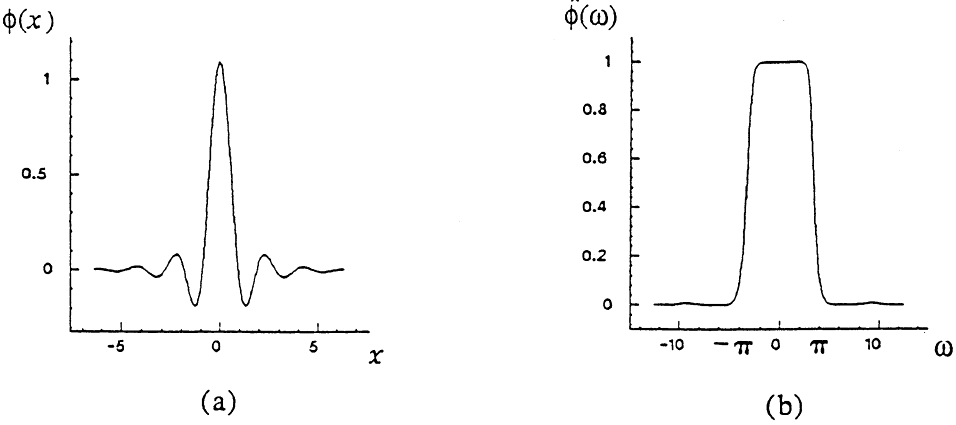
$$\prod_{j=-1}^{+\infty} |M_0(2^{-j}\omega)| = O(|\omega|^{\mathrm{Log}(A)/\mathrm{Log}(2)}) \tag{58}$$

at infinity. Since

$$\prod_{j=-1}^{+\infty} \cos\left(2^{-j}\frac{\omega}{2}\right) = \frac{\sin(\omega/2)}{\omega/2}\,,$$

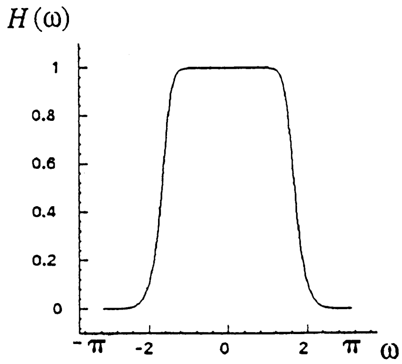
it follows that

$$|\hat{\phi}(\omega)| = O(|\omega|^{-q+\mathrm{Log}(A)/\mathrm{Log}(2)}) \quad \text{at infinity}. \tag{59}$$



(a) Graph of the function $\phi(x)$ derived from a cubic spline multiresolution approximation. It decreases exponentially.
(b) Graph of $\hat{\phi}(\omega)$. It decreases like $1/\omega^4$ at infinity.

FIGURE 1



Graph of the function $H(\omega)$ derived from a cubic spline multiresolution approximation.

FIGURE 2

Example. We describe briefly an example of multiresolution approximation from cubic splines found independently by P. Lemarie [10] and G. Battle [1].

The vector space $\mathbf{V}_0$ is the set of functions which are $\mathbf{C}^2$ and equal to a cubic polynomial on each interval $[k, k+1]$, $k \in \mathbf{Z}$. It is well known that there exists a unique cubic spline $g(x) \in \mathbf{V}_0$ such that

$$\forall k \in \mathbf{Z}, \quad g(k) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases}$$

The Fourier transform of $g(x)$ is given by

$$\hat{g}(\omega) = \left(\frac{\sin(\omega/2)}{\omega/2}\right)^4 \left(1 - \frac{2}{3}\sin^2\frac{\omega}{2}\right)^{-1}. \tag{60}$$

Any function $f(x) \in \mathbf{V}_0$ can thus be decomposed in a unique way,

$$f(x) = \sum_{k=-\infty}^{\infty} f(k) g(x-k).$$

Hence, for a cubic spline multiresolution approximation, the isomorphism $\mathbf{I}$ of property (5) can be defined as the restriction to $\mathbf{Z}$ of the functions $f(x) \in \mathbf{V}_0$. One can easily show that the sequence of vector spaces $(\mathbf{V}_j)_{j\in\mathbf{Z}}$ built with property (3) is a regular multiresolution approximation of $\mathbf{L}^2(\mathbf{R})$. Let us define

$$\Sigma_8(\omega) = \sum_{-\infty}^{+\infty} \frac{1}{(\omega + 2k\pi)^8}. \tag{61}$$

It follows from equations (60), (12) and (17) that

$$\hat{\phi}(\omega) = \frac{1}{\omega^4\sqrt{\Sigma_8(\omega)}} \quad \text{and} \quad H(\omega) = \sqrt{\frac{\Sigma_8(\omega)}{2^8\Sigma_8(2\omega)}}. \tag{62}$$

We calculate $\Sigma_8(\omega)$ by computing the 6th derivative of the formula

$$\Sigma_2(\omega) = \frac{1}{4\sin^2(\omega/2)}.$$

Figure 1 shows the graph of $\phi(x)$ and its Fourier transform. It is an exponentially decreasing function. Figure 2 shows $H(\omega)$ on $[-\pi, \pi]$.

4. The wavelet orthonormal basis

The approximation of a function at a resolution 2^j is equal to its orthogonal projection on $\mathbf{V}_j$. The additional precision of the approximation when the resolution increases from 2^j to 2^{j+1} is thus given by the orthogonal projection on the orthogonal complement of $\mathbf{V}_j$ in $\mathbf{V}_{j+1}$. Let us call this vector space $\mathbf{O}_j$. In this section, we describe an algorithm, which is now classic [13], in order to find a wavelet $\psi(x)$ such that $(\sqrt{2^j}\psi(2^jx-k))_{k\in\mathbf{Z}}$ is an orthonormal basis of $\mathbf{O}_j$.

We are looking for a function $\psi(x)$ such that $\psi(x/2) \in \mathbf{O}_{-1} \subset \mathbf{V}_0$. Its Fourier transform can thus be written

$$\hat{\psi}(2\omega) = G(\omega)\hat{\phi}(\omega), \tag{63}$$

where $G(\omega)$ is a 2π-periodic function in $\mathbf{L}^2([0, 2\pi])$. Since $\mathbf{V}_0 = \mathbf{V}_{-1} \oplus \mathbf{O}_{-1}$, the Fourier transform of any function $f(x) \in \mathbf{V}_0$ can be decomposed as

$$\hat{f}(\omega) = a(\omega)\hat{\phi}(\omega) = b(\omega)\hat{\phi}(2\omega) + c(\omega)\hat{\psi}(2\omega), \tag{64}$$

where $a(\omega)$ is 2π-periodic and a member of $\mathbf{L}^2([0,\pi])$, and $b(\omega), c(\omega)$ are both π-periodic and members of $\mathbf{L}^2([0,\pi])$. By inserting (17) and (63) in the previous equation, it follows that

$$a(\omega) = b(\omega)H(\omega) + c(\omega)G(\omega). \tag{65}$$

The orthogonality of the decomposition is equivalent to

$$\int_0^{2\pi} |a(\omega)|^2\, d\omega = \int_0^{\pi} |b(\omega)|^2\, d\omega + \int_0^{\pi} |c(\omega)|^2\, d\omega.$$

It is satisfied for any $a(\omega)$ if and only if

$$\begin{cases} |H(\omega)|^2 + |G(\omega)|^2 = 1, \\ H(\omega)\overline{G(\omega)} + H(\omega+\pi)\overline{G(\omega+\pi)} = 0. \end{cases} \tag{66}$$

These equations are necessary and sufficient conditions on $G(\omega)$ to build $\psi(x)$. The functions $b(\omega)$ and $c(\omega)$ are respectively given by

$$\begin{cases} b(\omega) = a(\omega)\overline{H(\omega)} + a(\omega+\pi)\overline{H(\omega+\pi)}, \\ c(\omega) = a(\omega)\overline{G(\omega)} + a(\omega+\pi)\overline{G(\omega+\pi)}. \end{cases} \tag{67}$$

Condition (66) together with (40) can also be expressed by writing that

$$\begin{bmatrix} H(\omega) & G(\omega) \\ H(\omega+\pi) & G(\omega+\pi) \end{bmatrix} \tag{68}$$

is a unitary matrix. A possible choice for $G(\omega)$ is

$$G(\omega) = e^{-i\omega}\overline{H(\omega+\pi)}. \tag{69}$$

Any vector spaces $\mathbf{V}_J$ can be decomposed as

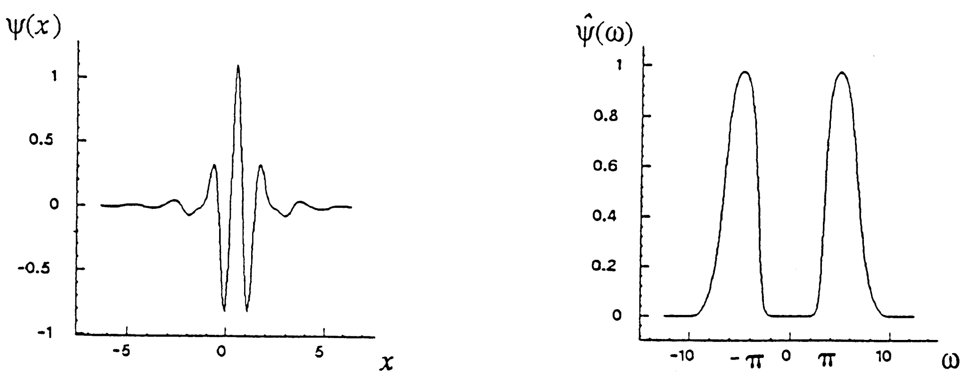
$$\mathbf{V}_J = \bigoplus_{j=-\infty}^{J-1} \mathbf{O}_j. \tag{70}$$

Since $\bigcup_{j=-\infty}^{+\infty} \mathbf{V}_j$ is dense in $\mathbf{L}^2(\mathbf{R})$, the direct sum $\bigoplus_{j=-\infty}^{+\infty} \mathbf{O}_j$ is also dense in $\mathbf{L}^2(\mathbf{R})$. The family of functions $(\sqrt{2^j}\,\psi(2^j x - k))_{(k,j)\in \mathbf{Z}^2}$ is therefore an orthonormal basis of $\mathbf{L}^2(\mathbf{R})$.

Multiresolution approximations provide a general approach to build wavelet orthonormal bases. We first define a function $H(\omega)$ which satisfies the hypothesis of Theorem 2 and compute the corresponding function $\phi(x)$ with equation (42). From equations (63) and (69), we can also derive the Fourier transform of a wavelet $\psi(x)$ which generates an orthonormal basis. Figure 3 is the graph of the wavelet derived from the cubic spline multiresolution approximation described in the previous section.

The Haar basis is a particular case of wavelet orthonormal basis with

$$\psi(x) = \begin{cases} 1 & \text{if } 0 \le x < 1/2, \\ -1 & \text{if } 1/2 \le x < 1, \\ 0 & \text{otherwise.} \end{cases}$$



(a) (b)

(a) Graph of the function $\psi(x)$ derivedfrom a cubic spline multiresolution approximation. It decreases exponentially.
(b) Graph of $|\hat{\psi}(\omega)|$. It decreases like $1/\omega^4$ at infinity.

FIGURE 3

The corresponding function $\phi(x)$ is the indicator function of $[0,1]$, so the Haar multiresolution approximation is not regular. It is characterized by the function $H(\omega) = e^{-i\omega/2}\cos(\omega/2)$. With some other choice of $H(\omega)$, we can build wavelets which are much more regular than the Haar wavelet.

The smoothness and the asymptotic decay rate of a wavelet $\psi(x)$ defined by (63) and (69) is controlleld by the behavior of $H(\omega)$. The asymptotic decay rate of $\psi(x)$ is estimated by observing that if $H(\omega) \in \mathbf{C}^q$, then $\hat{\psi}(\omega) \in \mathbf{C}^q$ and

$$(71) \qquad \frac{d^n H(0)}{d\omega^n} = 0 \text{ for } 1 \leq n \leq q \Leftrightarrow \begin{cases} \dfrac{d^n G(0)}{d\omega^n} = 0 \text{ for } 0 \leq n \leq q-1, \\ \dfrac{d^n \hat{\psi}(0)}{d\omega^n} = 0 \text{ for } 0 \leq n \leq q-1. \end{cases}$$

We can also obtain a lower bound of the asymptotic decay rate of $\hat{\psi}(\omega)$ from the lower bound (59) on the decay rate of $\hat{\phi}(\omega)$:

$$(72) \qquad |\hat{\psi}(\omega)| = O(|\omega|^{-q+\mathrm{Log}(A)/\mathrm{Log}(2)}) \text{ at infinity}.$$

Outside the Haar basis, the first classes of wavelet orthonormal bases were found independently by Y. Meyer [14] and J. Stromberg [16]. Y. Meyer's bases are given by the class of functions $H(\omega)$ satisfying the hypothesis of Theorem 2, equal to 1 on $[-\pi/3, \pi/3]$ and continuously differentiable at any order. The Fourier transform of Y. Meyer's wavelets are in $\mathbf{C}^\infty$, so $\psi(x)$ has a decay faster than any power. We can also easily derive that $\hat{\psi}(\omega)$ has a support contained in $[-8\pi/3, 8\pi/3]$ so $\psi(x)$ is in $\mathbf{C}^\infty$.

By using a multiresolution approach, I. Daubechies [3] has recently proved that for any $n \geq 1$, we could find some wavelets $\psi(x) \in \mathbf{C}^n$ having a compact support. Indeed, she showed that we can find trigonometrical polynomials

$$H(\omega) = \sum_{k=-N}^{N} h_k e^{-ik\omega}$$

such that the constant $q - \mathrm{Log}(A)/\mathrm{Log}(2)$ of equation (72) is as large as desired. The corresponding wavelet $\psi(x)$ has a support contained in $[-2N-1, 2N+1]$ and its differentiability is estimated from (72).

If $\psi(x)$ is regular enough, Y. Meyer and P. Lemarie showed that wavelet orthonormal bases provide unconditional bases for most usual functional spaces [11]. We can thus find whether a function $f(x)$ is inside $\mathbf{L}^p(\mathbf{R})$ $(1 < p < \infty)$, a Sobolev space or a Hardy space from its decomposition coefficients in the wavelet basis. As an example, one can prove that if a wavelet $\psi(x)$ satisfies

$$\exists C \geq 0, \quad |\psi(x)| < C(1+|x|)^{-1-q}, \tag{73}$$

$$\int_{-\infty}^{+\infty} x^n \psi(x)\, dx = 0 \quad \text{for } 1 \leq n \leq q, \tag{74}$$

$$\exists C' \geq 0, \quad \left|\frac{d^n \psi(x)}{dx^n}\right| < C'(1+|x|)^{1-q} \quad \text{for } n \leq q, \tag{75}$$

then for any $s \leq q$, the family of functions $(\sqrt{2^j}\psi(2^j x - k))_{(j,k)\in\mathbf{Z}^2}$ is an unconditional basis of the Sobolev space $\mathbf{H}^s$. As a consequence, for any $f(x) \in \mathbf{L}^2(\mathbf{R})$, if $\alpha(k,j) = \langle f(x), \sqrt{2^j}\psi(2^j x - k)\rangle$ then

$$f \in \mathbf{H}^s \Leftrightarrow \left(\sum_{j=-\infty}^{+\infty}\left\{\left(\sum_{k=-\infty}^{+\infty}|\alpha(k,j)|^2\right)^{1/2} 2^{js}\right\}^2\right)^{1/2} < +\infty. \tag{76}$$

Remarks. 1. The couples of functions $H(\omega)$ and $G(\omega)$ which satisfy (69) were first studied in signal processing for multiplexing and demultiplexing a signal on a transmission line [5, 15]. Let $A = (\alpha_n)_{n\in\mathbf{Z}}$ be a discrete time sequence and $a(\omega)$ the corresponding Fourier series. The goal is to decompose A in two sequences B and C each having half as many samples per time unit and such that B and C contain respectively the low and the high frequency components of A. Equations (67) enable us to achieve such a decomposition where $b(\omega)$ and $c(\omega)$ are respectively the Fourier series of B and C. In signal processing, $H(\omega)$ and $G(\omega)$ are interpreted respectively as the transfer functions of a discrete low-pass filter H and a discrete high-pass filter G. They are called *quadrature mirror* filters. The sequences B and C are respectively computed by convolving A with the filters H and G and keeping one element out of two of the resulting sequences.

2. If a function is characterized by N samples uniformly distributed, its decomposition in a wavelet orthonormal basis can be computed with an algorithm of complexity $O(N)$. This algorithm is based on discrete convolutions with the quadrature mirror filters H and G [12].

3. We proved that we can derive a wavelet orthonormal basis from any multiresolution approximation. It is however not true that we can build a multiresolution approximation from any wavelet orthonormal basis. The function $\psi(x)$ whose Fourier transform is given by

$$\hat{\psi}(\omega) = \begin{cases} 1 & \text{if } 4\pi/7 \leq |\omega| \leq \pi \text{ or } 4\pi \leq |\omega| \leq 4\pi + 4\pi/7, \\ 0 & \text{otherwise}, \end{cases} \tag{77}$$

is a counterexample due to Y. Meyer. The translates and dilates

$$(\sqrt{2^j}\psi(2^j x - k))_{(k,j)\in\mathbf{Z}^2}$$

of this function constitute an orthonormal basis of $\mathbf{L}^2(\mathbf{R})$. Let $\mathbf{V}_J$ be the vector space generated by the family of functions

$$(\sqrt{2^j}\psi(2^j x - k))_{k\in\mathbf{Z}, -\infty<j<J}.$$

One can verify that the sequence of vector spaces $(\mathbf{V}_J)_{J\in\mathbf{Z}}$ does not satisfy property (5) of a multiresolution approximation. Hence, this wavelet is not related to a multiresolution approximation. It might however be sufficient to impose a regularity condition on $\psi(x)$ in order to always generate a multiresolution approximation.

4. Multiresolution approximations have been extended by S. Jaffard and Y. Meyer to $\mathbf{L}^2(\Omega)$ where Ω is an open set of $\mathbf{R}^n$ [13]. This enables us to build wavelet orthonormal bases in $\mathbf{L}^2(\Omega)$.

5. Approximation error

When approximating a function at the resolution 2^j, the error is given by $\varepsilon_j = \|f - \mathbf{P}_{\mathbf{V}_j}(f)\|$. Property (2) of a multiresolution approximation implies that $\lim_{j\to+\infty} \varepsilon_j = 0$. A classical problem in approximation theory is to estimate the convergence rate of ε_j given an a priori knowledge on the smoothness of $f(x)$ or derive the smoothness of $f(x)$ from the convergence rate of ε_j [4].

Theorem 3. *Let $(\mathbf{V}_j)_{j\in\mathbf{Z}}$ be a multiresolution approximation such that the associated function $\phi(x)$ satisfies*

$$\exists C \geq 0, \quad |\phi(x)| < C(1+|x|)^{-3-q}, \tag{78}$$

$$\int_{-\infty}^{+\infty} x^n \phi(x)\, dx = 0 \quad \textit{for } 1 \leq n \leq q+1, \tag{79}$$

$$\exists C' \geq 0, \quad \left|\frac{d^n\phi(x)}{dx^n}\right| < C'(1+|x|)^{-1-q} \quad \textit{for } n \leq q. \tag{80}$$

Let $\varepsilon_j = \|f - \mathbf{P}_{\mathbf{V}_j}(f)\|$. For all $f \in \mathbf{L}^2(\mathbf{R})$, if $0 < s \leq q$ then

$$f(x) \in \mathbf{H}^s \Leftrightarrow \sum_{j=-\infty}^{+\infty} \varepsilon_j^2 2^{2sj} < +\infty. \tag{81}$$

Proof. Let $\mathbf{P}_{\mathbf{O}_j}$ denote the orthogonal projection on the vector space $\mathbf{O}_j$ and let $\psi(x)$ be the wavelet defined by (63) and (69). Let $f(x)$ be in $\mathbf{L}^2(\mathbf{R})$ and let $\alpha(k,j) = \langle f(x), \sqrt{2^j}\psi(2^j x - k)\rangle$. The approximation error is given by

$$(82)\quad \varepsilon_J = \|f - \mathbf{P}_{\mathbf{V}_J}(f)\| = \left(\sum_{j=J}^{\infty} \|\mathbf{P}_{\mathbf{O}_j}(f)\|^2\right)^{1/2} = \left(\sum_{j=J}^{+\infty}\sum_{k=-\infty}^{+\infty} |\alpha(k,j)|^2\right)^{1/2}.$$

In order to prove the theorem we show that if the function $\phi(x)$ satisfies conditions (78), (79) and (80) then $\psi(x)$ satisfies conditions (73), (74) and (75). We then apply property (76) to finish the proof.

It follows from (78) that the function $H(\omega) = \sum_{k=-\infty}^{+\infty} h_k e^{-ik\omega}$ satisfies

$$(83)\qquad h_k = O((1+|k|)^{-3-q}).$$

The function $G(\omega)$ defined in (69) can be written

$$G(\omega) = \sum_{k=-\infty}^{\infty} g_k e^{-ik\omega}, \quad \text{with } g_k = h_{-k-1}(-1)^{-k-1}.$$

The wavelet $\psi(x)$ is thus given by

$$\frac{1}{2}\psi\left(\frac{x}{2}\right) = \sum_{k=-\infty}^{+\infty} h_{-k-1}(-1)^{-k-1}\phi(x-k).$$

With the above expression and equations (78), (80) and (83) we can derive that

$$|\psi(x)| = O((1+|x|)^{-1-q}) \quad \text{and} \quad \left|\frac{d^n\psi(x)}{dx^n}\right| = O((1+|x|)^{-1-q}) \quad \text{for } n \le q.$$

Equations (78) and (79) imply that $\hat{\phi}(\omega) \in \mathbf{C}^{q+1}$ and

$$\frac{d^n\hat{\phi}(0)}{d\omega^n} = 0 \quad \text{for } 1 \le n \le q+1.$$

From (56) and (71) it thus follows that $d^n\hat{\psi}(0)/d\omega^n = 0$ for $0 \le n \le q$ and therefore

$$\int_{-\infty}^{+\infty} x^n \psi(x)\,dx = 0 \quad \text{for } 0 \le n \le q.$$

We can now finish the proof of this theorem by applying property (76). Let $\beta_j = \sum_{k=-\infty}^{+\infty} |\alpha(k,j)|^2$; equation (82) yields

$$\sum_{j=J}^{+\infty} \beta_j = \varepsilon_J^2.$$

This implies that $\beta_J = \varepsilon_J^2 - \varepsilon_{J+1}^2$. The right-hand side of property (76) is therefore given by

$$\sum_{j=-\infty}^{+\infty} \beta_j 2^{2js} = (1 - 2^{-2s}) \sum_{j=-\infty}^{+\infty} \varepsilon_J^2 2^{2sj}.$$

The right-hand side statement of property (76) is thus equivalent to the right-hand side statement of (81). This concludes the proof of Theorem 3.

Acknowledgments. I deeply thank Yves Meyer who helped me all along this research. I am also very grateful to Ruzena Bajcsy for her support and encouragement. I finally would like to thank Ingrid Daubechies for all her comments.

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