

Advanced Techniques for Combinatorial Algorithms: Fixed-Parameter Algorithms

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Fixed-Parameter

- An **NP**-hard problem does not go away
- Vertex cover
- Clique
- Independent set
- Dominating set
- Hamiltonian cycle

Vertex Cover

Instance

Undirected graph $G = \langle V, E \rangle$, integer k .

Question

Find a set $C \subset V$ such that for each edge $e \in E$ at least one endpoint of e belongs to C , and $|C| \leq k$



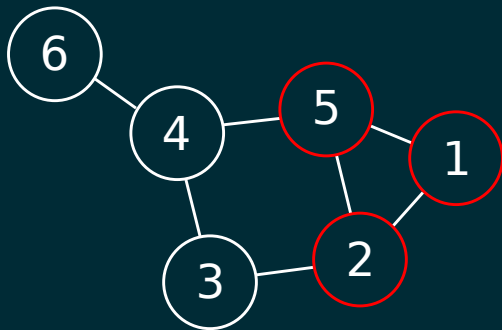
Clique

Instance

Undirected graph $G = \langle V, E \rangle$, integer k .

Question

Find a set $C \subset V$ such that all pairs of vertices in C are connected by an edge, and $|C| \geq k$



Independent Set

Instance

Undirected graph $G = \langle V, E \rangle$, integer k .

Question

Find a set $I \subset V$ such that no two vertices in I are connected by an edge, and $|I| \geq k$



Dominating Set

Instance

Undirected graph $G = \langle V, E \rangle$, integer k .

Question

Find a set $D \subset V$ such that for each vertex $v \notin D$, v is adjacent to some $d \in D$, and $|D| \leq k$



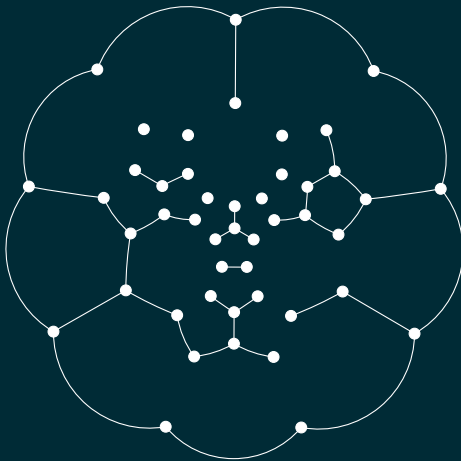
Hamiltonian cycle

Instance

Undirected graph $G = \langle V, E \rangle$.

Question

Find a cycle C that visits each vertex $v \in V$ exactly once.



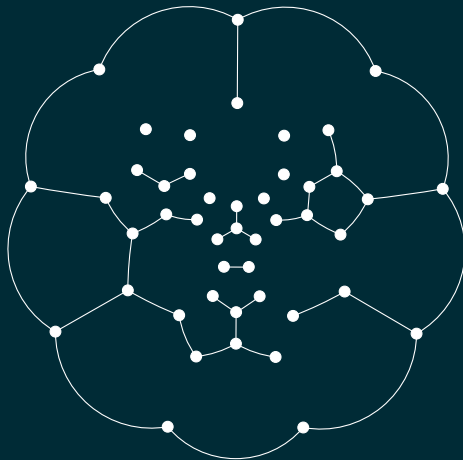
Longest Path

Instance

Undirected graph $G = \langle V, E \rangle$, integer k .

Question

Find a simple (no vertex is visited twice) path P of G , with P consisting of k vertices.



Fixed Parameter Tractable (FPT)

FPT

Parameterized problem: pair (I, k) . The problem is in FPT if there is an algorithm with time $f(k)n^\alpha$, with α a constant and f a function

Typical times

- $O(1.1^k)n$
- $O(2^k)n^3$
- $O(2^k + n^3)$
- $O(k^k)n^3$
- $O(2^{2^{2^{\dots^2}}})n^3$

Bounded Search Tree

The search tree has $O(f(k))n^\alpha$ nodes

Vertex Cover

Let $(u, v) \in E$. Then $u \in C$ or $v \in C$

- How is the search tree?
- $2^k n$ time

Smaller Search Tree

Proposition

Let v be a node, let $N(v)$ be its neighbors. Then $v \in C$ or $N(v) \subseteq C$

Proposition

No node has at least three neighbors. Then G consists of vertex-disjoint cycles or paths, and its smallest cover is trivial

Corollary

Consider only vertices v with $|N(v)| \geq 3$

Smaller Search Tree

Two cases

- 1 $v \in C$, then recurse on $G - v$ and height $k - 1$
 - 2 $v \notin C$, then recurse on $G - N(v)$ and height at most $k - 3$
-
- 1 Number of nodes $f(k) = f(k - 1) + f(k - 3) + 1$
 - 2 $f(k) \leq 5^{\frac{k}{4}} < 1.5^k$

Independent Set on Planar Graphs

Euler formula

Let G be a planar graph. Then G has at most $3n - 6$ edges.

Corollary

Let G be a planar graph. Then there exists a vertex v of G with at most 5 incident edges.

Independent Set on Planar Graphs

Algorithm 1: Finds an independent set of size k , if it exists

```
1  $v \leftarrow$  a minimum degree vertex of  $G$  foreach  $x \in \{v\} \cap N(v)$  do
2   Add  $x$  to  $I$ ;
3    $G_1 \leftarrow$  a copy of  $G$  where  $x$  and the edges incident on it are removed;
4   if  $G_1$  is not empty then
5     recurse on  $(G_1, k, I)$ ;
6   else
7     if  $|I| \geq k$  then
8       return  $I$ ;
```

Reduction to a Kernel

Main idea

- 1 Fix or exclude portions of the solution, so that the remaining part is small.
- 2 Then be naive on what remains.

Rule example

- 1 Let v be a vertex with exactly one neighbor w .
- 2 Remove v
- 3 Put w in the cover.

Problem

No such vertex $v \Rightarrow$ nothing is removed

Reduction to a Kernel

- 1 Find all vertices L with degree $\geq k$. If $|L| > k$, then no cover of size k . Let $k_1 = k - |L|$
- 2 $G_1 = G - L$. If G_1 has more than $k_1(k + 1)$ vertices, then no cover of size k .
- 3 Find k_1 -cover of G_1 , time $O(n + k^{2k})$
- 4 Improve to $O(n + 2^k k^2)$

Independent Set on Planar Graphs

4 color theorem

Let G be a planar graph. Then there is a coloring of the vertices of G that uses at most 4 colors and such that no two adjacent vertices are the same color.

Corollaries

- 1 The set of vertices that are the same color is an independent set
- 2 The largest such set has at least $1/4$ of all vertices
- 3 The set of all vertices is a kernel with $4k$ elements

Question

Is this kernel useful?

Independent Set on Planar Graphs

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Question

Is this kernel useful? No, $4k \geq n \Rightarrow 2^{[4k]} \geq 2^n$

Color Coding

Colorful Path

- 1 Color each vertex with k colors
- 2 Colorful path: each vertex has a distinct color

Random coloring

Probability that a path is colorful: $\frac{k!}{k^k} \geq e^{-k}$

Dynamic programming

- Given a coloring, find a colorful path (if it exists)
- Keep only the colors
- Time $(2^{O(k)}|E|)$

Color Coding

Dynamic programming

i colors from v to w : $i - 1$ colors from v to z and the color from z to w

Time complexity

- At time i , there are $\binom{k}{i}$ sets
- $O(\sum_{i=1}^k i \binom{k}{i}) = O(k2^k)$

Derandomize

- k -perfect family H of hash functions $h : [1 : n] \mapsto [1 : k]$ is such that for each $S \subseteq [1 : n]$ with $|S| = k$, there exists h that is 1-to-1 on S
- Compute in linear time a k -perfect family H of $2^{O(k)} \log n$ functions

Closest string

Input

s_1, \dots, s_m : strings of length n . Integer k

Problem

Find a string $t = t[1] \cdots t[n]$ such that t has Hamming distance at most k with each s_i

Hint

Let s_i and s_j be two input strings. Find an upper bound on their distance.

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Hint

Let s_i and s_j be two input strings. Find an upper bound on their distance. $2k$

Super-Sub sequence

Longest Subsequence

s_1, \dots, s_m : strings of length n . Does it exist $t = t[1] \cdots t[k]$ such that for each s_i , $s_i = w_0 t[1] w_1 \cdots t[k] w_k$, for some (possibly empty) strings w_j ?

Shortest Supersequence

s_1, \dots, s_m : strings of length n . Does it exist $t = t[1] \cdots t[k]$ such that for each s_i , $t_i = w_0 s_i[1] w_1 \cdots s_i[n] w_k$, for some (possibly empty) strings w_j ?

- Which problem is easier?

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