

Advanced Techniques for Combinatorial Algorithms: Randomized Algorithms

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Karp-Rabin

Find all occurrences of Pattern in a Text

- hash function $H : \Sigma^{|P|} \mapsto \mathbb{N}$
- hash conflicts = false positives
- speedup: simple hash function
- $H(S) = \sum_{i=1}^{|S|} 2^{|S|-i} H(S[i]) \bmod p$, with p a large random prime
- $|P|$ -long sliding window on T
- $H(T[i+1 : i+|P|]) = ((H(T[i : i+|P|-1]) - T[i]) / 2 + 2^{|P|} T[i+|P|])$

Karp-Rabin: false positives

Kinds of error

- False positive (FP): reported false occurrence
- False negative (FN): occurrence not found
- No False negative in Karp-Rabin

Decreasing error probability

Choosing k random primes (independently, without repetitions).

Las Vegas vs. Monte Carlo

Classifying randomized algorithms

- Las Vegas:
 - Always correct
 - Sometimes not fast
 - Example: Quicksort with random pivot
- Monte Carlo:
 - Always fast
 - Sometimes not correct
 - Karp-Rabin

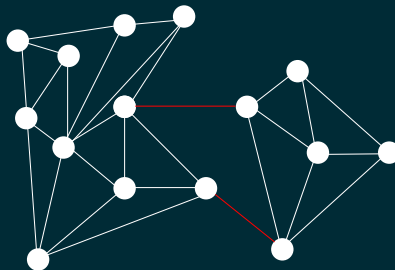
Min Cut

Instance

Undirected graph $G = \langle V, E \rangle$.

Question

Find a smallest subset $C \subset E$ of edges such that its removal from G results in a disconnected graph.



Min Cut: Karger's algorithm

Min Cut problem

If the probability that an object exists is > 0

The probabilistic method

Shows that an object exists

If the probability that an object exists is > 0

Proposition

Let K_n be the complete graph with n vertices. If $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$ then we can color the edges of K_n so that we have no monochromatic K_k subgraph.

The probabilistic method

Proposition

Let K_n be the complete graph with n vertices. If $\binom{n}{k} 2^{1-\binom{k}{2}} < 1$ then we can color the edges of K_n so that we have no monochromatic K_k subgraph.

Proof

- 1 There are $2^{\binom{n}{2}}$ colorings of K_n .
- 2 Color each edge at random
- 3 There are $\binom{n}{k}$ k -cliques
- 4 Let A_i be the event that k -clique i is monochromatic
- 5 $P[A_i] = 2^{1-\binom{k}{2}}$
- 6 $P[\bigcup A_i] \leq \sum P[A_i] = \binom{n}{k} 2^{1-\binom{k}{2}} < 1$
- 7 $P[\bigcap \bar{A}_i] = 1 - P[\bigcup A_i] > 0$

The expectation method

Shows that an object exists

Let $E[X]$ be the expected value of event X . Then there exists an event with value $\leq E[X]$ and an event with value $\geq E[X]$

Max Cut

- Let $G = (V, E)$ be an undirected graph, with $|V| = n$, $|E| = m$.
- The **cut** associated with the bipartition (V_1, V_2) of V is $E \cap V_1 \times V_2$.
- There exists a cut with at least $m/2$ edges.

The expectation method

Random cut

- For each vertex v , assign v to a side with $p = 1/2$

Proof

- $X_i = 1$ if edge i in the cut C , else 0
- Probability of each edge in the random cut C : $1/2$. $E[X_i] = 1/2$
- $E[\sum X_i] = \sum E[X_i] = m/2$

Question

The algorithm?

The expectation method

Random cut

$q = P[|C| \geq m/2]$, for random cut C . Then $q \geq \frac{1}{m/2+1}$

Proof

$$\begin{aligned}\frac{m}{2} = E[C] &= \sum_{i \leq m/2-1} i \cdot P[|C| = i] + \sum_{i \geq m/2} i \cdot P[|C| = i] \leq \\ &\leq (m/2 - 1)(1 - q) + qm \Rightarrow q \geq \frac{1}{m/2 + 1}\end{aligned}$$

Question

How many random cuts are needed?

The sample and modify method

Independent set

G : undirected graph. Average degree $d = 2m/n$

Algorithm 1: Random Independent Set

- 1 $S \leftarrow$ a random sample of V , each vertex is picked with $p = 1/d$;
 - 2 $I \leftarrow S$;
 - 3 **while** *there exists an edge e in $G|I$* **do**
 - 4 Remove an endpoint of e from I
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The sample and modify method

Bounding $|I|$

- $E(G) = \frac{nd}{2}$
- $E[|S|] = \frac{n}{d}$
- $E[E(G|S)] = \frac{nd}{2} \frac{1}{d} \frac{1}{d} = \frac{n}{2d}$
- At most $E(G|S)$ are removed in the second step, hence
 $E[|I|] = E[|S|] - E[E(G|S)] = \frac{n}{2d}$