Advanced Techniques for Combinatorial Algorithms: Fixed-Parameter Algorithms

Gianluca Della Vedova

Univ. Milano–Bicocca
https://gianluca.dellavedova.org

April 28, 2020

Fixed-Parameter

- An **NP**-hard problem does not go away
- Vertex cover
- Clique
- Independent set
- Dominating set
- Hamiltonian cycle

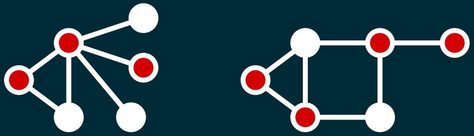
Vertex Cover

Instance

Undirected graph $G = \langle V, E \rangle$, integer k.

Question

Find a set $C \subset V$ such that for each edge $e \in E$ at least one endpoint of e belongs to C, and $|C| \leq k$



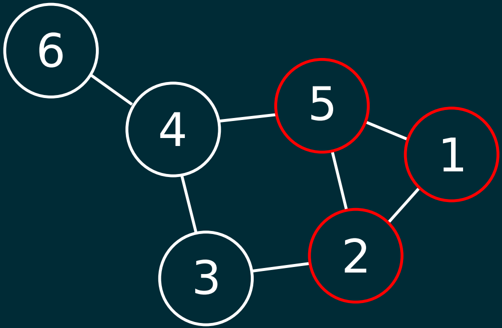
Clique

Instance

Undirected graph $G = \langle V, E \rangle$, integer k.

Question

Find a set $C \subset V$ such that all pairs of vertices in C are connected by an edge, and $|C| \geq k$



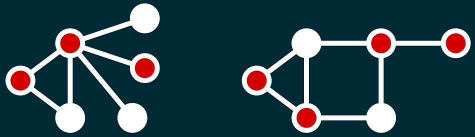
Independent Set

Instance

Undirected graph $G = \langle V, E \rangle$, integer k.

Question

Find a set $I \subset V$ such that no two vertices in K are connected by an edge, and $|K| \geq k$



Dominating Set

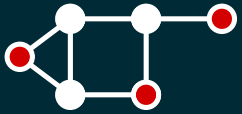
Instance

Undirected graph $G = \langle V, E \rangle$, integer k.

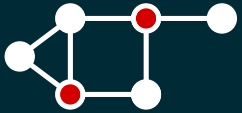
Question

Find a set $D \subset V$ such that for each vertex $v \notin D$, v is adjacent to some $d \in D$, and $|D| \leq k$

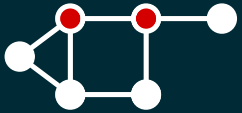
(a)



(b)



(c)



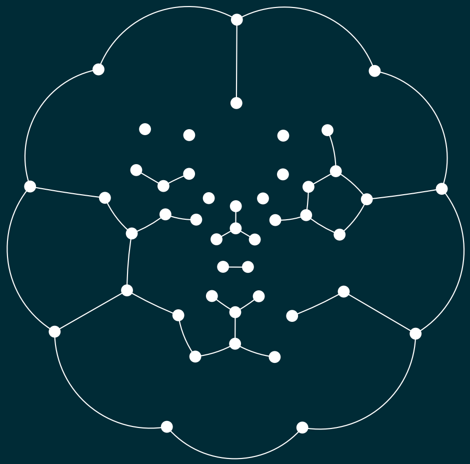
Hamiltonian cycle

Instance

Undirected graph $G = \langle V, E \rangle$.

Question

Find a cycle C that visits each vertex $v \in V$ exactly once.



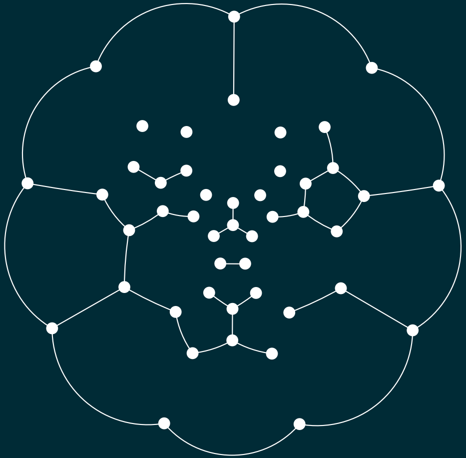
Longest Path

Instance

Undirected graph $G = \langle V, E \rangle$, integer k.

Question

Find a simple (no vertex is visited twice) path P of G, with P consisting of k vertices.



Fixed Parameter Tractable (FPT)

FPT

Parameterized problem: pair (I, k). The problem is in FPT if there is an algorithm with time $f(k)n^\alpha$, with α a constant and f a function

Typical times

- $O(1.1^k)n$
- $O(2^k)n^3$
- $O(2^k + n^3)$
- $O(k^k)n^3$
- $O(2^{2^{2^{\cdots^{2}}}})n^3$

Bounded Search Tree

The search tree has $O(f(k))n^\alpha$ nodes

Vertex Cover

Let $(u, v) \in E$. Then $u \in C$ or $v \in C$

- How is the search tree?
- $2^k n$ time

Smaller Search Tree

Proposition

Let v be a node, let $N(v)$ be its neighbors. Then $v \in C$ or $N(v) \subseteq C$

Proposition

No node has at least three neighbors. Then G consists of vertex-disjoint cycles or paths, and its smallest cover is trivial

Corollary

Consider only vertices v with $|N(v)| \geq 3$

Smaller Search Tree

Two cases

1. $v \in C$, then recurse on $G - v$ and height $k - 1$
2. $v \notin C$, then recurse on $G - N(v)$ and height at most $k - 3$

1. Number of nodes $f(k) = f(k-1) + f(k-3) + 1$
2. $f(k) \leq 5^{\frac{k}{4}} < 1.5^k$

Reduction to a Kernel

1. Find all vertices L with degree $\geq k$. If $|L| > k$, then no cover of size k. Let $k_1 = k - |L|$
2. $G_1 = G - L$. If G_1 has more than $k_1(k+1)$ vertices, then no cover of size k.
3. Find k_1-cover of G_1, time $O(n + k^{2k})$
4. Improve to $O(n + 2^k k^2)$

Color Coding

Colorful Path

1. Color each vertex with k colors
2. Colorful path: each vertex has a distinct color

Random coloring

Probability that a path is colorful: $\frac{k!}{k^k} \geq e^{-k}$

Dynamic programming

- Given a coloring, find a colorful path (if it exists)
- Keep only the colors
- Time $(2^{O(k)}|E|)$

Color Coding

Dynamic programming

i colors from v to w: $i-1$ colors from v to z and the color from z to w

Time complexity

- At time i, there are $\binom{k}{i}$ sets
- $O(\sum_{i=1}^{k} i\binom{k}{i}) = O(k2^k)$

Derandomize

- k-perfect family H of hash functions $h : [1:n] \mapsto [1:k]$ is such that for each $S \subseteq [1:n]$ with $|S| = k$, there exists h that is 1-to-1 on S
- Compute in linear time a k-perfect family H of $2^{O(k)} \log n$ functions

Closest string

Input

$s_1, \ldots, s_m$: strings of length n. Integer k

Problem

Find a string $t = t[1] \cdots t[k]$ such that t has Hamming distance at most k with each s_i

Super-Sub sequence

Longest Subsequence

$s_1, \ldots, s_m$: strings of length n. Does it exist $t = t[1] \cdots t[k]$ such that for each s_i, $s_i = w_0 t[1] w_1 \cdots t[k] w_k$, for some (possibly empty) strings w_j?

Shortest Supersequence

$s_1, \ldots, s_m$: strings of length n. Does it exist $t = t[1] \cdots t[k]$ such that for each s_i, $t_i = w_0 s_i[1] w_1 \cdots s_i[n] w_k$, for some (possibly empty) strings w_j?

- Which problem is easier?

Attribution

- Vertex Cover figure: By Miym - Own work, CC BY-SA 3.0, `https://commons.wikimedia.org/w/index.php?curid=6017739`
- Clique figure: Public Domain, `https://commons.wikimedia.org/w/index.php?curid=1072101`
- Dominating Set figure: By Miym - Own work, CC BY-SA 3.0, `https://commons.wikimedia.org/w/index.php?curid=6821042`