Advanced Techniques for Combinatorial Algorithms: Parallel Algorithms

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- Advanced Techniques for Combinatorial Algorithms

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- Advanced Techniques for Combinatorial Algorithms
- `https://gitlab.com/dellavg/advanced-algorithms`

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- `https://algolab.eu/2018/01/07/advanced-techniques-for-combinatorial-algorithms-2018/`

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RAM model

- Random Access Memory

RAM model

- Random Access Memory
- One processor

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- Random Access Memory
- One processor
- sequential algorithms

RAM model

- Random Access Memory
- One processor
- sequential algorithms
- Flat memory

RAM model

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- One processor
- sequential algorithms
- Flat memory
- Infinite memory

PRAM model

- Parallel RAM

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- Parallel RAM
- p RAMs

PRAM model

- Parallel RAM
- p RAMs
- Shared memory

PRAM model

- Parallel RAM
- p RAMs
- Shared memory
- Synchronized

PRAM model

- Parallel computation is rapidly becoming a dominant theme in all areas of computer science and its applications. It is likely that, within a decade, virtually all developments in computer architecture, systems programming, computer applications and the design of algorithms will be taking place within the context of parallel computation.

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- Karp, R M. and Ramachandran, V. Chapter 17. Parallel Algorithms for Shared-Memory Machines. Handbook of Theoretical Computer Science: Algorithms and complexity, Volume 1. 1990.

PRAM model

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PRAM model

- It's a MODEL!

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- MIMD (Multiple Instruction Multiple Data)

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- Processor ID

PRAM model

- It's a MODEL!
- MIMD (Multiple Instruction Multiple Data)
- Processor ID
- No communication cost

PRAM model

	Read	Write
Exclusive	ER	EW
Concurrent	CR	CW

- Different accesses

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- CRCW is better than EREW.

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PRAM model

	Read	Write
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Concurrent	CR	CW

- Different accesses
- CRCW is better than EREW.
- But how much?
- Common CRCW: concurrent writes if same value from all processors
- Priority CRCW: highest priority processor wins

Efficient Algorithm

- $t(n)$ = polylogarithmic time

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- $p(n)$ = polynomial number of processors

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- Hardness $=$ P-complete problems

Optimal Algorithm

- work $w(n) \leq t(n)p(n)$

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- work $w(n) \leq t(n)p(n)$
- $t(n) =$ polylogarithmic time
- $w(n) = O(T(n))$, where $T(n) =$ time complexity of best known sequential algorithm

Simulations

- EREW PRAM can simulate CRCW PRAM

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- Time multiplied by $O(\log p(n))$

Algorithms

Prefix sum problem (PRAM)

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Prefix sum problem (PRAM)

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- Sequence $\langle x_1, \ldots, x_n \rangle$ of elements
- Associative operation $+$
- Output
- $S = \langle S_1 \ldots, S_n \rangle$, with $S_i = x_1 + \cdots + x_i$
- trivial sequential algorithm

Prefix sum

Algorithm 1: PrefixSum

for $i \leftarrow 1$ **to** $n/2$ **do**
    $y_i \leftarrow x_{2i-1} + x_{2i}$;
$S^* = \text{PrefixSum}([y_1, \ldots, y_{n/2}])$;
/* $S^*_j = x_1 + \cdots + x_{2j}$ */
for $i \leftarrow 1$ **to** n **do**
    if i *is even* **then**
        $S_i \leftarrow S^*_{i/2}$;
    else
        $S_i \leftarrow S^*_{i/2} + x_i$

Prefix sum

Algorithm 2: PrefixSum

for $i \leftarrow 1$ **to** $n/2$ *in parallel* **do**
    $y_i \leftarrow x_{2i-1} + x_{2i}$;
$S^* = \text{PrefixSum}([y_1, \ldots, y_{n/2}])$;
/* $S^*_j = x_1 + \cdots + x_{2j}$ */
for $i \leftarrow 1$ **to** n *in parallel* **do**
    if i *is even* **then**
        $S_i \leftarrow S^*_{i/2}$;
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Brent's scheduling principle

- w work on p processors in time t

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- w work on p processors in time t
- $p_1 < p$

Prefix sum

- EREW
- $O(\log n)$ time, $O(n)$ processors
- $w(n) = O(n)$
- $O(n/\log n)$ processors are enough

Brent's scheduling principle

- w work on p processors in time t
- $p_1 < p$
- time $\lfloor w/p_i \rfloor + t$, work w

Find Maximum

Instance

An array A of n integers

Find Maximum

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An array A of n integers

Question

Find the largest element in A.

Find Maximum

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An array A of n integers

Question

Find the largest element in A.

Goal

Fastest algorithm

Find Maximum

Algorithm 3: Find1. Find Maximum in an Array A

for $i \leftarrow 1$ **to** n *in parallel* **do**
    $B[i] \leftarrow$ true;
for $i \leftarrow 1$ **to** n *in parallel* **do**
    for $j \leftarrow 1$ **to** n *in parallel* **do**
        if $A[i] < A[j]$ *or* $A[i] = A[j]$ *and* $i < j$ **then**
            $B[i] \leftarrow$ false;
for $i \leftarrow 1$ **to** n *in parallel* **do**
    if $B[i]$ **then**
        Return $A[i]$

Time? Work?

Find Maximum

Algorithm 4: Find1. Find Maximum in an Array A

```
for i ← 1 to n in parallel do
    B[i] ← true;
for i ← 1 to n in parallel do
    for j ← 1 to n in parallel do
        if A[i] < A[j] or A[i] = A[j] and i < j then
            B[i] ← false;
for i ← 1 to n in parallel do
    if B[i] then
        Return A[i]
```

Time? Work? $T(n) = O(1)$, $W(n) = O(n^2)$

Find Maximum

Algorithm 5: Find2. Find Maximum in an Array A

if $n > 16$ **then**
    for $i \leftarrow 1$ **to** n *in parallel* **do**
        $B[i] \leftarrow \text{Find2}(A[1 + \lfloor (i-1)/\sqrt{n} \rfloor : \lfloor i/\sqrt{n} \rfloor]);$
    Find1(B);
else
    Find1(A);

$T(n) \leq T(\sqrt{n}) + c_1 \Rightarrow T(n) = O(\log \log n)$
$W(n) \leq \sqrt{n} W(\sqrt{n}) + c_2 n \Rightarrow W(n) = O(n \log \log n)$

Find Maximum

Algorithm 6: Find3. Find Maximum in an Array A

for $i \leftarrow 1$ **to** $n/\log\log n$ *in parallel* **do**
  $B[i] \leftarrow \min(A[1 + \lfloor (i-1)\log\log n \rfloor : \lfloor i/\log\log n \rfloor]);$

Find2(B);

$T(n) = O(\log\log n)$
$W(n) = O(n)$

List Ranking

- Problem: given a list L, find the position of each element in L

Algorithm 7: List Ranking via pointer jumping

```
foreach L[i] in parallel do
    if next(i) = NIL then
        rank[i] ← 0
    else
        rank[i] ← 1
    for k ← 1 to log_2 n do
        rank[i] ← rank[i] + rank[next[i]];
        next[i] ← next[next[i]]
```

List Ranking

Proof

List Ranking

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- At iteration k:

List Ranking

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- At iteration k:
- if $next[i] \neq NIL$ then $rank[i] = 2^k$

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- if $next[i] \neq NIL$ then $rank[i] = 2^k$
- if $next[i] = NIL$ then $rank[i]$ is the distance between $L[i]$ and the end of the list

List Ranking

Proof

- At iteration k:
- if $next[i] \neq NIL$ then $rank[i] = 2^k$
- if $next[i] = NIL$ then $rank[i]$ is the distance between $L[i]$ and the end of the list
- $next[i] = NIL$ for the last 2^k elements of L

Binary trees

- Problem: to determine depth of each node

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- parent, left child, right child

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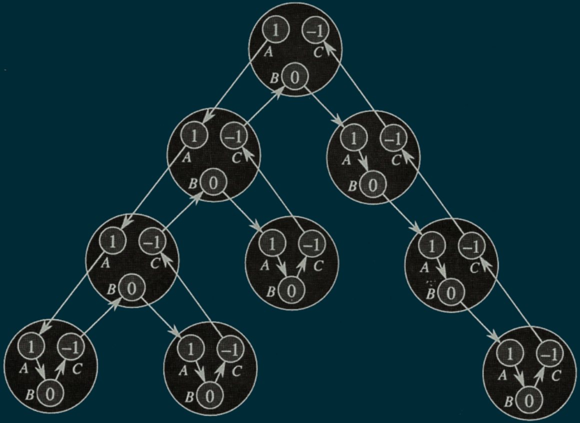
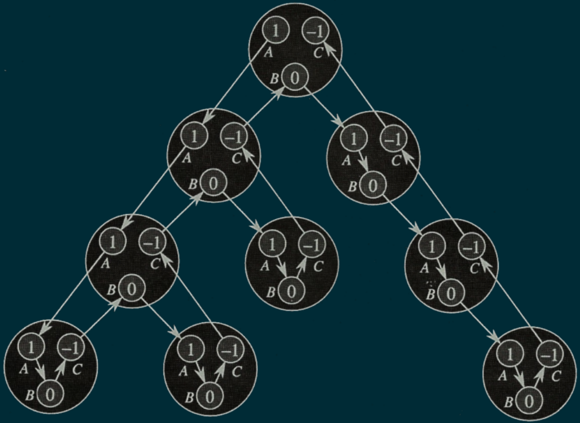
- parent, left child, right child
- 3 processors for each node
- $O(n)$-time sequential algorithm
- Algorithm 1: Level-wise visit, each node in parallel
- $t(n) =$height (not good)

Euler tour



Euler tour



Depth = prefix sum

Graph Algorithms

- depth-first visit

Graph Algorithms

- depth-first visit
- No NC algorithm

Graph Algorithms

- depth-first visit
- No NC algorithm
- breadth-first visit

Graph Algorithms

- depth-first visit
- No NC algorithm
- breadth-first visit
- $O(n^{2.37})$ processors

Graph Algorithms

- depth-first visit
- No NC algorithm
- breadth-first visit
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- Euler tour

Connected components

Instance

Undirected graph $G = (V, E)$

Connected components

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Data structure

Connected components

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Undirected graph $G = (V, E)$

Data structure

1. $M(v, w)$ adjacency matrix

Connected components

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Undirected graph $G = (V, E)$

Data structure

1. $M(v, w)$ adjacency matrix
2. $R(v) \leftarrow v$ representative. All vertices in the same connected components have the same representative.

Connected components

Instance

Undirected graph $G = (V, E)$

Data structure

1. $M(v, w)$ adjacency matrix
2. $R(v) \leftarrow v$ representative. All vertices in the same connected components have the same representative.
3. $C[v, w]$ connected components with representative v and w can be merged

Connected components

Algorithm 8: ConnectedComponents

```
for log₂ n times do  hookings
    foreach edge (v, w) such that R[v] ≠ R[w] do
        if R[v] < R[w] then
            C[R[v], R[w]] ← true;
    foreach vertex v such that R[v] = v do
        R[v] ← max w : C[R[v], R[w]] is true;
    for i ← 1 to log₂ n do  parallel pointer jumping
        foreach vertex v do
            R[v] ← R[R[v]];
```

Minimum Spanning Tree

Problem

Given an undirected edge-weighted connected graph $G = (V, E)$, find a minimum-weight subset $T \subseteq E$ such that T is a tree spanning V.

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Lemma

Let $G = (V, E)$ be an undirected graph, let (V_1, V_2) be a bipartition of V, let T be a minimum spanning tree of G, and let e be the lightest edge connecting V_1 and V_2. Then $e \in T$.

Additional Bibliography on PRAM

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