School of Economics and Business Administration

The pricing efficiency of the Heston – Nandi GARCH option valuation model: empirical evidence from the CAC40 index options market

MSc in Banking & Finance - Dissertation

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Abstract

The purpose of this thesis is to test the pricing performance of the Heston-Nandi (2000) GARCH option valuation model on the CAC 40 index options market. The GARCH process is estimated via maximum likelihood instead of non-linear least squares. As a benchmark the Ad Hoc Black-Scholes model of Dumas, Fleming and Whaley (1998) is used. Empirical results show that the Heston-Nandi GARCH model is greatly outperformed by the Ad Hoc model both in-sample and out-of-sample. The poor performance of the model is mainly due to the finite sample properties of the estimators employed.

Aknowledgments

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1.   Introduction

Following the work of Black and Scholes (1973) and Merton (1973) a voluminous literature has been developed regarding option pricing models. Black and Scholes derived their model (henceforth BS model) by assuming that the underlying asset price follows a geometric Brownian motion (GBM) with log-normal distribution and constant volatility. Many empirical studies have demonstrated that the BS model results in systematic biases. The underpricing of out-of-the-money options and options on low volatility securities (Gultekin et al. 1982) and the U-shaped implied volatility curve in relation to exercise price (Rubinstein 1985) – the so called "volatility smile" – are some of the well documented systematic biases linked with the model[1]. Furthermore, the assumption[2] that the asset price follows a GBM, or equivalently, that the logarithmic asset returns are normally distributed is in conflict with empirical research (e.g. Fama, 1965 and Mandelbrot, 1963) which has provided convincing evidence that returns distributions exhibit fat tails and excess peakdness at the mean (leptokurtosis).

In an effort to explain these biases researchers have focused on the development of option valuation models that incorporate stochastic volatility (i.e. the evolution of the asset volatility is determined by an exogenous process). There are two major categories of stochastic volatility option pricing models: continuous-time stochastic volatility models and discrete-time generalized autoregressive conditional heteroskedasticity (GARCH) models. Different specifications[3] for a continuous-time stochastic volatility process have been proposed by several authors including Hull and White (1987),

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[1]   More recent evidence against the constant volatility assumption have been presented by Rubinstein (1994) and Dumas, Fleming and Whaley (1998).

[2]   Another assumption of the original BS model that has been relaxed in order to extent the model is the assumption that the risk-free rate of interest is constant and the same for all maturities. In practice, the assumption of constant riskless interest rates is not realistic which has given rise to the development of models incorporating stochastic interest rates (for instance see Amin and Jarrow, 1991).

[3]   See Christoffersen and Jacobs (2004b) for a brief review of the GARCH option pricing models and Dotsis, Markellos and Mills (2009) for a thorough presentation of continuous-time stochastic volatility models.

Johnson and Shanno (1987), Scott (1987), Wiggins (1987), Stein and Stein (1991) and Heston[4] (1993). In the GARCH family of option pricing models Engle and Mustafa's (1992) model, Amin and Ng's (1993) model, Duan's (1995) non-linear asymmetric GARCH (NGARCH) model and Heston and Nandi's (2000) model have attracted the most attention.

The GARCH models, as Heston and Nandi (2000) point out, have the significant advantage that the volatility is readily observable in the historical prices of the underlying asset[5]. In contrast, the volatility process in continuous-time stochastic volatility models is unobservable and has to be estimated separately. In most GARCH option valuation models there is not a closed-form analytic solution available for the option price. The option price is available only through Monte Carlo simulation (for instance see Duan, 1995), a slow and computationally intensive procedure for empirical analysis. However, Heston and Nandi (2000) developed a closed-form[6] solution for European option values in a GARCH model (henceforth HN GARCH). The HN GARCH model describes option values as functions of the current spot price and the observed path of historical spot prices and captures both the stochastic nature of volatility and correlation between volatility and spot returns.

The purpose of this thesis is to test the pricing performance of the HN GARCH model on the CAC 40 index options market. As a benchmark the Ad Hoc Black-Scholes[7] (henceforth Ad Hoc BS) model of Dumas, Fleming and Whaley (1998) is used. This model is based on the BS pricing formula but uses a deterministic function of

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[4]   Significant contributions in the continuous-time option pricing literature also include jump diffusion models (Bates, 1996 and Bakshi, Cao and Chen, 1997) and models with jumps in both the asset price and volatility (Duffie et al., 2000 and Chernov et al., 2003).

[5]   One additional distinctive characteristic is that the GARCH option pricing models fail to satisfy the Markov property. Past volatility and returns are important pieces of information regarding the future of the process.

[6]   Even though the model provides a closed-form solution, the coefficients for the generating function have to be derived recursively by working backward from the time of maturity of the option.

[7]   In the literature this model is also referred to as the Practitioner Black-Scholes model (see Christoffersen and Jacobs - 2004a).

moneyness and maturity to fit the volatility to the parabolic shape of the volatility smile. The most important difference in the methodology used in this thesis compared to the study of Heston and Nandi (2000) is that the GARCH process is estimated via maximum likelihood (ML) instead of non-linear least squares (NLLS).

The rest of this study proceeds as follows: In section 2 the necessary theoretical framework is provided discussing the implied volatility curve, the Ad Hoc BS model, the standard GARCH process and the HN GARCH model. Section 3 reports the empirical results including in-sample and out-of-sample estimation. Finally, conclusions and recent developments in the GARCH option pricing literature are presented in sections 4 and 5, respectively.

2.   Theoretical Framework

2.1   The Implied Volatility Curve

The BS model is by far the most widely used formula for the valuation of European options even though the underlying assumptions are known to be violated. The success of the model arises mainly from its ability to parsimoniously describe market options prices. In the BS world of constant volatility the value at time t of a European call option on a non-dividend paying asset is simply given by

$$C(t) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2) \qquad [1]$$

where
$$d_1 = \frac{\log\left(\frac{S(t)}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \qquad [2]$$

$$d_2 = d_1 - \sigma\sqrt{T-t}. \qquad [3]$$

S(t) is the price of the underlying asset at time t, N(x) is the cumulative probability distribution function for a standardized normal distribution, K is the strike price, r is the continuously compounded risk-free interest rate, σ is the asset price volatility and T is the option's time to maturity.

The BS implied volatility of an option is the volatility obtained when equating the option's market value to its BS value, given the same strike price and time to maturity. Option prices are embedded with the expectation about the future evolution of the price of the underlying asset and therefore implied volatilities form a forward-looking estimate of the volatility of the underlying asset. Moreover, since the volatility cannot be extracted from the BS formula analytically this must be done numerically by using a

root-finding algorithm. A numerical procedure such as the method of bisections or Newton-Raphson can be used to derive [8] the implied volatility by the option price.

If σ_{iv} denotes the implied volatility, $C_{obs}(K,T)$ denotes the observed market call price with strike price K and time to maturity T and $C_{BS}(\sigma_{iv},K,T)$ the BS price of the call with same strike price and maturity, then σ_{iv} is the value of volatility in the BS formula which satisfies the relationship

$$C_{obs}(K,T) = C_{BS}(\sigma_{iv},K,T)$$ [4]

Equation [4] has to be expressed as the root of the objective function [9]

$$f(\sigma) = C_{BS}(\sigma,K,T,) - C_{obs}(K,T)$$ [5]

so that the objective function equals zero at the value of implied volatility, $f(\sigma_{iv}) = 0$. As Rouah and Vainberg (2007) point out, even though the Newton-Raphson method is quick to converge to the implied volatility it requires an initial guess for the volatility and therefore, given the fact that is quite sensitive to the location of the initial guess, this algorithm could deviate significantly from the root. On the contrary, the bisection method is particularly well suited for finding the implied volatility when the objective function is [5].

If the market option prices were consistent with the BS formula, all the BS implied volatilities corresponding to different options written on the same asset would match with the volatility σ of the underlying asset. In reality this is not the case and the BS implied volatilities form a smile (also known as smirk) when plotted against strike price (K) or moneyness (K/S). This plot is referred to as the implied volatility curve. Figure 1 illustrates the smile pattern in the CAC 40 BS implied volatilities.

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[8]   The approach presented here for the calculation of the implied volatilities is similar to that described by Rouah and Vainberg (2007). For a more analytical approach see Watsham and Parramore (2004).

[9]   The objective function can also be defined as $f(\sigma) = \left[C_{BS}(\sigma,K,T,) - C_{obs}(K,T) \right]^2$, which ensures that the Newton-Raphson minimization algorithm does not produce large negative values.

Figure 1: Volatility Smile
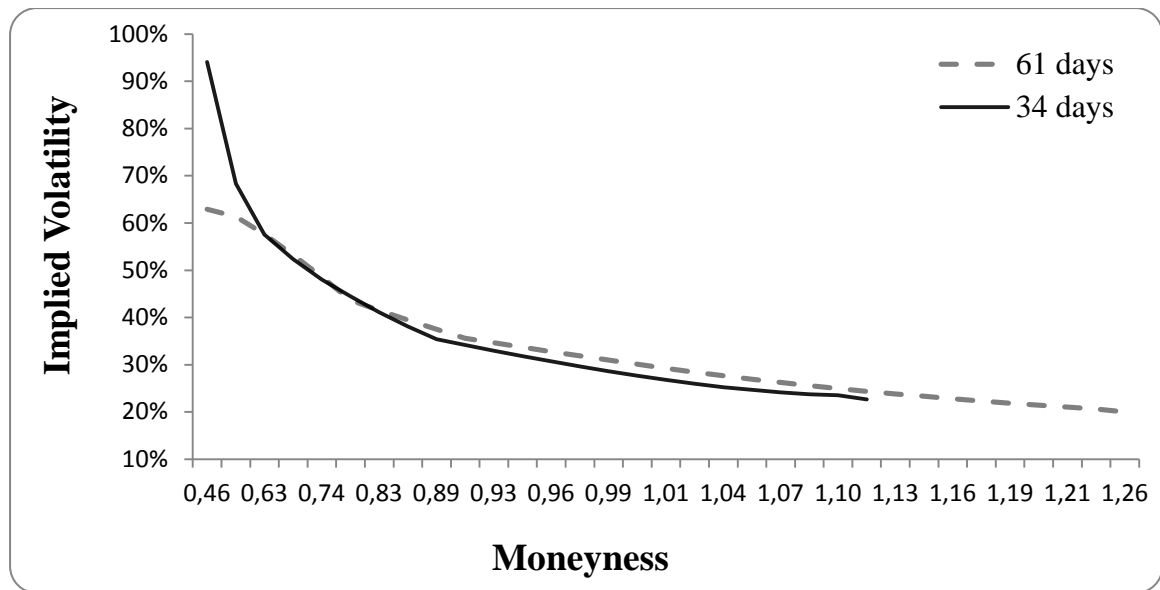
Note: BS implied volatilities on July 16, 2010. Implied volatilities are computed from CAC 40 index call option prices for the August and September 2010 option expirations. Moneyness is defined as K/S where S is the index level and K the option's exercise price.

From figure 1, it is evident that the amplitude of the smile increases when time to maturity decreases (the smile effect is more pronounced for options with short maturities).

2.2   The Ad Hoc Black-Scholes Model

The Ad Hoc BS model constitutes a simple way to price options based on implied volatilities and the BS pricing formula. The underlying assumption of constant volatility made in the BS model is dropped by using volatility that is no constant but depends on a deterministic function. Dumas, Fleming and Whaley (1998) describe this as the deterministic volatility function (DVF) approach to modeling implied volatility. They consider the following four different structural forms for the DVF

Model 0: $\sigma = \max(\,0.01,\, \alpha_0\,)$ [6]

Model 1: $\sigma = \max(\,0.01,\, \alpha_0 + \alpha_1 K + \alpha_2 K^2\,)$ [7]

Model 2: $\sigma = \max(\ 0.01,\ \alpha_0 + \alpha_1 K + \alpha_2 K^2 + \alpha_3 T + \alpha_5 KT\ )$ [8]

Model 3: $\sigma = \max(\ 0.01,\ \alpha_0 + \alpha_1 K + \alpha_2 K^2 + \alpha_3 T + \alpha_4 T^2 + \alpha_5 KT\ )$ [9]

where α_0, α_1, α_2, α_3, α_4, α_5 are the model parameters, K is the strike price and T time to maturity.

Each of these specifications generates a different form of the volatility function. Model 0 assumes constant volatility with no dependence on the strike price or the time to maturity and corresponds to the BS model. Model 1 describes volatility as a quadratic function of the strike price with no dependence on maturity. Model 2 adds a dependence on maturity and an interaction between the strike price and maturity. Lastly, model 3 allows for the relationship between volatility and maturity to be quadratic also. A minimum value of the local volatility rate is imposed to prevent negative values of fitted volatility.

Dumas, Fleming and Whaley (1998) show that the BS model leads to the largest valuation errors, a result which is consistent with the notion that the volatility is not constant across moneyness and maturity, while model 2 (in which the DVF includes K, K^2, T and KT) has the best performance.

The implementation of the model is straightforward and can be summarized into a four-step procedure. First, the BS implied volatilities from each option are abstracted. Second, the parameters for the DVF are estimated by ordinary least squares (OLS). Third, the model implied volatility for each option is obtained using the estimated parameters from the second step. Finally, the options prices are calculated with the BS formula.

A relevant class of volatility models is the implied binomial tree or the deterministic volatility models[10] of Derman and Kani (1994), Dupire (1994) and Rubinstein (1994) in which the asset return volatility is a flexible but deterministic function of the current asset price and time only. Rather than posting a structural form for the volatility function, these models search for a binomial or trinomial lattice that achieves an exact cross-sectional fit of reported option prices.

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[10] In the option pricing literature, the deterministic volatility models are often mentioned as local volatility models.

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2.3   The GARCH Process

The generalized autoregressive conditional heteroskedasticity (GARCH) process, which was developed independently by Bollerslev (1986) and Taylor[11] (1986), provides a useful extension to the ARCH model by Engle (1982). The standard GARCH (p, q) model is given by[12]

$$y_t = \mu_t + u_t, \text{ where } u_t = \sqrt{h_t}\,\varepsilon_t \qquad [10]$$

$$h_t = a_0 + \sum_{i=1}^{q} a_i u_{t-i}^2 + \sum_{i=1}^{p} \beta_i h_{t-i} \qquad [11]$$

In equation [10] $\mu_t = E(y_t | F_{t-1})$ is the conditional mean of the rate of the return y_t and $h_t = Var(u_t | F_{t-1})$ denotes the conditional variance of u_t where F_{t-1} is the information set (σ - field) available at time t-1. The term ε_t is a white noise process such that $\varepsilon_t \overset{iid}{\sim} N(0,1)$. The parameters α_0, α_i and β_i are restricted so that $h_t > 0$ for all t, which is ensured when $\alpha_0 > 0$, $\alpha_i \geq 0$ for $1 \leq i \leq q$ and $\beta_i \geq 0$ for $1 \leq i \leq p$. Note that for p = 0 an ARCH (q) process is obtained. Unlike ARCH models where the conditional variance is a linear function of past squared errors only, h_t in [11] is parameterized to depend upon q lags of the squared error and p lags of the conditional variance.

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[11] Taylor (1986) proposed the simple GARCH (1,1) specification. Under the GARCH (1,1) model , the current fitted variance is interpreted as a weighted function of a long term average value (dependent on α_0), information about volatility during the previous period ($\alpha_1 u_{t-1}^2$) and the fitted variance from the model during the previous period ($\beta_1 h_{t-1}$).

[12] The Gaussian assumption of ε_t is not always realistic. As noted by Markellos and Mills (2008) one can modify the model and allow heavy-tailed distributions. For instance, Bollerslev (1987) considers the case where the distribution is standardised-t with unknown degrees of freedom v that may be estimated from the data (for $v >2$ such a distribution is leptokurtic and hence has thicker tails than the normal). Other distributions that have been proposed include the normal Poisson mixture distribution (Jorion, 1988), the power exponential distribution (Baillie and Bollerslev, 1989), the normal–log-normal mixture (Hsieh, 1989) and the generalised exponential distribution (Nelson, 1991).

The conditional variance can also be written as $h_t = a_0 + a(L)u_t^2 + \beta(L)h_t$ where L is the lag (backward shift) operator. It can be shown that the GARCH (p, q) process is weakly stationary[13] if and only if the roots of $\alpha(L) + \beta(L)$ lie outside the unit circle, i.e. if $\alpha(1) + \beta(1) < 1$ or equivalently

$$\sum_{i=1}^{q} a_i + \sum_{i=1}^{p} \beta_i < 1.$$

If $\alpha(1) + \beta(1) = 1$ then $\alpha(L) + \beta(L)$ contains a unit root. Shocks to the conditional variance are permanent, in the sense that they remain important for all future forecasts, and the process is referred to as the integrated GARCH or IGARCH model (see Engle and Bollerslev, 1986).

One of the major restrictions of the standard GARCH model is that it enforces a symmetric response of volatility to positive and negative shocks. This arises since the conditional variance in equation [11] is a function of the magnitudes of the lagged errors and not their signs. Several extensions of the GARCH framework have been derived in order to increase the flexibility of the original model. Two popular asymmetric extensions of the GARCH model are presented below.

EGARCH

The exponential GARCH or EGARCH (p,q) model was proposed by Nelson (1991). Instead of [11] the conditional variance h_t is now specified as

$$\log(h_t) = a_0 + \sum_{i=1}^{q} a_i g(\varepsilon_{t-i}) + \sum_{i=1}^{p} \beta_i \log(h_{t-i}) \qquad [12]$$

where $u_t = \sqrt{h_t}\varepsilon_t$ and $g(\varepsilon_t) = \theta\varepsilon_t + \gamma[|\varepsilon_t| - E|\varepsilon_t|]$ are the weighted innovations that model asymmetric effects between positive and negative asset returns and θ, γ are

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[13] Conditions for weak stationarity are often more strict than those for strict stationarity. See Bougerol and Picard (1992) for stationarity conditions of the standard GARCH (p, q) process.

constants. Both ε_t and $|\varepsilon_t|-E(|\varepsilon_t|)$ are zero mean independent and identically distributed (iid) sequences and therefore $E[g(\varepsilon_t)]=0$. The function $g(\varepsilon_t)$ can be written as

$$g(\varepsilon_t) = \begin{cases} (\theta+\gamma)\varepsilon_t - \gamma E(|\varepsilon_t|) & if & \varepsilon_t \geq 0 \\ (\theta-\gamma)\varepsilon_t - \gamma E(|\varepsilon_t|) & if & \varepsilon_t < 0 \end{cases}$$

so that $\theta + \gamma$ and $\theta - \gamma$ reflect the asymmetry in response to positive and negative innovations. If $\theta < 0$ a positive return shock will increase volatility less than a negative one of the same magnitude (leverage effect). The use of the logarithm of the conditional variance relaxes the constraint of positive model coefficients since even in the case of negative parameters h_t will be positive.

GJR-GARCH

The GJR-GARCH model proposed by Glosten, Jagannathan and Runkle [1993] is a simple extension of the GARCH (p, q) model. In order to account for possible asymmetries one single term is added in the equation of the conditional variance. Thus, the conditional variance of the GJR-GARCH (p, q) model is given by

$$h_t = a_0 + \sum_{i=1}^{q} u_{t-i}^2 [a_i + \gamma_i I_{t-i}] + \sum_{i=1}^{p} \beta_i h_{t-i} \qquad [13]$$

where $u_t = \sqrt{h_t}\varepsilon_t$. For parameters α_0, α_i and β_i the same restrictions as in the standard GARCH model apply. I_{t-i} is an indicator function with $I_{t-i}=1$ if $u_{t-I} < 0$ and $I_{t-i}=0$ if if $u_{t-I} \geq 0$. The degree of asymmetry is governed by parameter γ.

2.4   The Heston-Nandi GARCH Option Valuation Model

Based on the local risk neutral valuation principle Duan (1995) developed a GARCH option pricing model with normal innovations. Following this direction, Heston and Nandi (2000) propose a conditionally-normal GARCH model for European option

values. The HN GARCH model has two basic assumptions. The first assumption is that the logarithmic spot price log(S(t)) of the underlying asset (including accumulated interest or dividends) follows the following GARCH(p,q) process over time steps of length Δ,

$$\log(S(t)) = \log(S(t-\Delta)) + r + \lambda h(t) + \sqrt{h(t)} z(t) \qquad [14]$$

$$h(t) = \omega + \sum_{i=1}^{p} \beta_i h(t-i\Delta) + \sum_{i=1}^{q} \alpha_i \left(z(t-i\Delta) - \gamma_i \sqrt{h(t-i\Delta)} \right)^2 \quad [15]$$

where r is the continuously compounded interest rate for the time interval Δ, z(t) is a standard normal disturbance and h(t) is the conditional variance[14] of the log-return between time t-Δ and t which is known from the information set at time t-Δ. The model uses the discrete-time random variable z(t) to drive both the spot asset and the variance and therefore is a discrete-time process.[15] The appearance of the conditional variance h(t) in the mean model can be interpreted as a return premium since it allows the average spot return to depend on the risk level. The conditional variance h(t) becomes constant in particular limiting cases: As the coefficients α_i and β_i governing the variance innovation approach zero the GARCH process reduces to the standard homoscedastic lognormal process in the BS model observed at discrete intervals. Furthermore, the term $\lambda h(t)$ implies that the expected spot return is assumed to exceed the risk-free rate by an amount proportional to the variance h(t).

On the single lag version (p=q=1) of the HN GARCH model one can directly observe the conditional variance h(t+Δ) as a function of the spot price at time t as follows:

$$h(t+\Delta) = \omega + \beta_1 h(t) + a_1 \frac{\left(\log S(t) - \log\left(S(t-\Delta) \right) - r - \lambda h(t) - \gamma_1 h(t) \right)^2}{h(t)} \qquad [16]$$

.

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[14] The conditional variance in equation [15] is similar to the NGARCH and VGARCH models of Engle and Ng (1993).

[15] On the contrary, the process that drives the asset price in the geometric brownian motion or the constant elasticity of variance model is a continuous-time Wiener process.

In equation [16] α_1 determines the kurtosis of the distribution and γ_1 the skewness of the distribution of the log returns. A zero value of the α_1 parameter, which is also known as the volatility of volatility, implies a deterministic time varying variance. When α_1 is not zero the kurtosis of the spot returns increases and as a result the distribution of returns exhibit fat-tails. This characteristic allows the model to be consistent with the stylized fact that the financial series have positive excess kurtosis and heavy-tail distributions. Also, if the γ_1 parameter (and λ) is zero the distribution is symmetric, while a value different than zero results in asymmetric influence of shocks where a large negative shock z(t) raises the variance more than a large positive shock. Moreover, the first-order process remains stationary with finite mean and variance only if $\beta_1 + \alpha_1\gamma_1^2 < 1$. If $\beta_1 + \alpha_1\gamma_1^2 = 1$ the variance process is integrated and shocks to volatility will be permanent.

The variance process h(t) and the spot return log(S(t)) are in general correlated as

$$Cov_{t-\Delta}\left[h(t+\Delta), \log(S(t))\right] = -2a_1\gamma_1 h(t) \qquad [17]$$

Given a positive value for the α_1 parameter and a positive value for γ_1, the correlation between spot returns and variance is negative. This asymmetric nature in stock returns and volatility is only partially consistent with the leverage effect documented by Christie (1982). As Dotsis and Markellos (2007) point out the quadratic specification of the GARCH process cannot generate a negative relationship between "good news" and volatility and therefore the model is not fully consistent with the leverage effect (see also, Yu - 2005).

Additionally, the HN GARCH(1,1) model includes Heston's (1993) stochastic volatility model as a continuous-time limit. It can be shown that as the observation interval shrinks the variance process h(t) converges weakly to a variance process v(t), which is the square-root process of Cox, Ingersoll and Ross (1985), and the corresponding option pricing model to that of Heston (1993):

$$d\log(S(t)) = \left(r + \lambda v(t)\right)dt + \sqrt{v(t)}d\xi(t) \qquad [18]$$

$$dv(t) = k\left(\theta - v(t)\right)dt + \sigma\sqrt{v(t)}d\xi(t) \qquad [19]$$

where $\xi(t)$ is a Wiener process. This result is considerably appealing since the model succeeds in reconciling the discrete-time GARCH approach to option pricing with the continuous-time stochastic volatility approach.

Heston and Nandi propose that under the risk neutral probability measure equations [14] and [15] can be written as

$$\log(S(t)) = \log\big(S(t-\Delta)\big) + r - \frac{1}{2}h(t) + \sqrt{h(t)}z^*(t) \qquad [20]$$

$$h(t) = \omega + \sum_{i=1}^{p} \beta_i h(t-i\Delta) + \alpha_1\left(z^*(t-\Delta) - \gamma_1^*\sqrt{h(t-\Delta)}\right)^2$$
$$+ \sum_{i=2}^{q} a_i\left(z(t-i\Delta) - \gamma_i\sqrt{h(t-i\Delta)}\right)^2 \qquad [21]$$
,

where

$$z^*(t) = z(t) + \left(\lambda + \frac{1}{2}\right)\sqrt{h(t)},$$

$$\gamma_1^* = \gamma_1 + \lambda + \frac{1}{2}.$$

The second assumption of the HN GARCH model is that the value of a call option with one period to expiration obeys the Black-Scholes-Rubinstein formula which is necessary in order for $z^*(t)$ to have a standard normal risk-neutral distribution.[16]

At time t a European call option with strike price K that expires at time T is worth

$$C(t) = e^{-r(T-t)}E_t^*\big[Max\big(S(T) - K,0\big)\big] = S(t)P_1 - Ke^{-r(T-t)}P_2, \quad [22]$$

where $E_t^*[\ ]$ denotes the expectation under the risk-neutral distribution and P_1 and P_2 are the risk-neutral probabilities. Under this model, P_1 is the delta of the call option (see appendix for the derivation of the Greeks from the HN GARCH(1,1) model) and P_2 is

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[16] The risk-neutral distribution of the asset price is lognormal with mean $S(t-\Delta)e^r$, if the BS formula holds for a single period. This implies that one can find a random variable $z^*(t)$, such that $z^*(t) \sim N(0,1)$ under the risk-neutral measure.

the probability of the asset price being greater than K at maturity, so that $P_2=Pr[S(T)>K]$. In order to obtain the call price, the two risk-neutral probabilities that appear in the pricing formula [22] have to be calculated. The first step is to solve for the generating function of the GARCH process of [14] and [15].

The generating function of the underlying asset price denoted as $f(\varphi) = E_t [S(T)^\varphi]$ is also the moment generating function of the logarithm of the spot price $S(T)$ since $E_t[S(T)^\varphi] = E_t [\varphi \log(S(T)]$. In the general case (i.e. for any p and q) the generating function takes the log-linear form

$$f(\varphi) = S(t)^\varphi \exp[A(t;T,\varphi) + \sum_{i=1}^{p} B_i(t;T,\varphi)h(t+2\Delta-i\Delta)$$
$$+\sum_{i=1}^{q-1} C_i(t;T,\varphi)\left(z(t+\Delta-i\Delta) - \gamma_i\sqrt{h(t+\Delta-i\Delta)} \right)^2]$$

[23]

where,

$$A(t;T,\varphi) = A(t+\Delta;T,\varphi) + \varphi r + B_1(t+\Delta;T,\varphi)\omega$$
$$-\frac{1}{2}\ln(1-2a_1 B_1(t+\Delta;T,\varphi) - 2C_1(t+\Delta;T,\varphi))$$

[24]

$$B_1(t;T,\varphi) = \varphi(\lambda+\gamma_1) - \frac{1}{2}\gamma_1^2 + \beta_1 B_1(t+\Delta;T,\varphi) + B_2(t+\Delta;T,\varphi)$$
$$+\frac{1/2\times(\varphi-\gamma_1)^2}{1-2a_1 B_1(t+\Delta;T,\varphi) - 2C_1(t+\Delta;T,\varphi)}$$

[25]

$$B_i(t;T,\varphi) = \beta_i B_1(t+\Delta;T,\varphi) + B_{i+1}(t+\Delta;T,\varphi) \quad \text{for } 1<i\leq p$$

[26]

$$C_i(t;T,\varphi) = \alpha_{i+1} B_1(t+\Delta;T,\varphi) + C_{i+1}(t+\Delta;T,\varphi) \quad \text{for } 1<i\leq q-1$$

[27]

and the coefficients for the generating function have to be calculated recursively from T to t by using the following conditions as starting values:

$$A(T;T,\varphi) = B_i(T;T,\varphi) = C_i(T;T,\varphi) = 0.$$

[28]

Next, since the generating function of the spot price is also the moment generating function of the logarithm of the spot price, $f(i\varphi)$ is the characteristic function of the

latter. Note that in order to use the characteristic function φ must be replaced by iφ everywhere in equations [24], [25], [26] and [27]. Furthermore, by inverting the characteristic function f*(iφ) the risk-neutral probabilities can be written as [17]

$$P_1 = \frac{1}{2} + \frac{e^{-r(T-t)}}{\pi S(t)} \int_0^\infty \text{Re}\left[\frac{K^{-i\varphi} f^*(i\varphi+1)}{i\varphi} \right] d\varphi, \qquad [29]$$

and

$$P_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re}\left[\frac{K^{-i\varphi} f^*(i\varphi)}{i\varphi} \right] d\varphi, \qquad [30]$$

where Re[ ] denotes the real part of the of complex numbers and f*(iφ) the characteristic function for the risk-neutral process described in [20] and [21]. Therefore, the value of a European call option[18] at time t is given by

$$C(t) = S(t)\left(\frac{1}{2} + \frac{e^{-r(T-t)}}{\pi S(t)} \int_0^\infty \text{Re}\left[\frac{K^{-i\varphi} f^*(i\varphi+1)}{i\varphi} \right] d\varphi \right)$$
$$-Ke^{-r(T-t)}\left(\frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re}\left[\frac{K^{-i\varphi} f^*(i\varphi)}{i\varphi} \right] d\varphi \right) \qquad [31]$$

.

The formula is a function of the current asset price S(t) and the conditional variance h(t+Δ) which appears in the characteristic functions f*(iφ+1) and f*(iφ). Since h(t+Δ) can be written as a function of the history of the asset prices , see equation [16] , the option formula is effectively a function of current and lagged asset prices.

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[17]   Feller (1971) and Kendall and Stuart (1977) show how to calculate probabilities by inverting the characteristic function.

[18]   European put option values can be calculated using the put – call parity.

3. Empirical Analysis

3.1 Description of Data

The CAC 40 index consists of the 40 stocks that are most representative of the various economic sectors quoted on the Euronext Paris market. The index is calculated continuously and disseminated every 15 seconds. The CAC 40 index options can be exercised only at maturity (European-style). Trading covers thirteen open maturities: 3 monthly, the following 7 quarterly (from the March, June, September and December cycle) and the rest 3 yearly (December cycle). The index options can be traded until their expiry date − the third Friday of the expiry month. The contract value is equal to the option price multiplied by 10 €.

The sample which consists of weekly closing call option prices for the CAC 40 index is obtained from Datastream for the period from January 6, 2010 to December 8, 2010 (the data set is sampled every Wednesday). The risk-free rate is computed using the France 90-day Treasury bill rates. Dividends are assumed to be zero and therefore there is no need to be subtracted from the current index level.

Three exclusionary criteria are used to filter the sample. First, call options with less than six or more than one hundred days to expiration are not included in the sample. Second, call option records in which moneyness, K/S, lies outside the region 0.9 − 1.1 are taken out of sample[19]. The third and final restriction is that the call option prices must satisfy the boundary condition $C(t,T) \geq \max(0, S(t) - Ke^{-r(T-t)})$. The inequality must be satisfied since it ensures that there is no arbitrage opportunity.

The data set consists of 2019 observations and 1003 observations are used for in-sample analysis (from 06/01/2010 to 09/06/2010) while the rest 1016 (from 16/06/2010

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[19] Short term options, DITM and DOTM options have relatively small time premiums and therefore the estimation of volatility is extremely sensitive to non-synchronous option prices and other possible measurement errors. Also, DITM and DOTM options are not actively traded and therefore price quotes are in general not supported by actual trades. For a further explanation of the exclusionary criteria see DFW (1998).

to 08/12/2010) are used for out-of-sample empirical analysis. The average option price in the sample is 133.67 €. In terms of moneyness, the data set is divided into five categories: deep in-the-money (DITM) call options with K/S<0.96, in-the-money (ITM) call options with 0.96≤K/S<0.98, at-the-money (ATM) call options with 0.98≤K/S<1.02, out-of-the-money (OTM) call options with 1.02≤K/S<1.05 and deep out-of-the-money (DOTM) call options with 1.05≤K/S<1.1. The time to maturity is classified into two categories: short term (less than 45 days) and long term options (between 45 and 100 days).

Table 1: Sample properties of CAC 40 options

Panel A		
Expiration Month	Number of Observations	Average Option Price
January	19	129,61
February	113	99,67
March	142	134,17
April	173	130,95
May	198	103,99
June	204	112,42
July	180	119,18
August	188	151,59
Septmeber	218	157,68
October	153	163,81
November	187	141,36
December	244	143,86
Total	**2019**	**133,67**

Panel B		
Moneyness (K/S)	Days to Expiration	
	< 45	≥ 45
DITM - [0,90 - 0,96)	311	241
ITM - [0,96 - 0,98)	135	95
ATM - [0,98 - 1,02)	282	196
OTM - [1,02 - 1,05)	191	136
DOTM - [1,05 - 1,10]	241	191
Total	**1160**	**859**

Note: Panel A reports the number of available option contracts, which are categorized based on their expiration month. The option data set starts on January 6, 2010 and ends on December 8, 2010. Panel B presents the number of available option contracts categorized by moneyness and days to expiration.

In addition, daily historical closing prices for the CAC 40 index are used to estimate the parameters in the GARCH model. The frequency distribution of CAC 40 log returns, during the period 02/01/2008 - 31/12/2009, is presented in figure 2. The distribution is highly peaked and fat tailed relative to the normal. This result provides a strong motive to model variance as a random variable, since fat tails and the high central peak are characteristics of mixtures of distributions with different variances (see Gatheral - 2006). Figure 3 depicts the log returns of the CAC 40 Index for the same period. The tendency for volatility to appear in bunches is apparent (large returns tend to be followed by large returns and small returns tend to be followed be small returns, irrespective of sign).

Finally, figure 4 illustrates the asymmetric relationship between CAC 40 index returns and volatility. This feature, which is common to many financial data, is known as leverage effect (i.e. volatility rises more following a large price fall than following a price rise of the same magnitude).

Figure 2: Frequency distribution of CAC 40 log returns
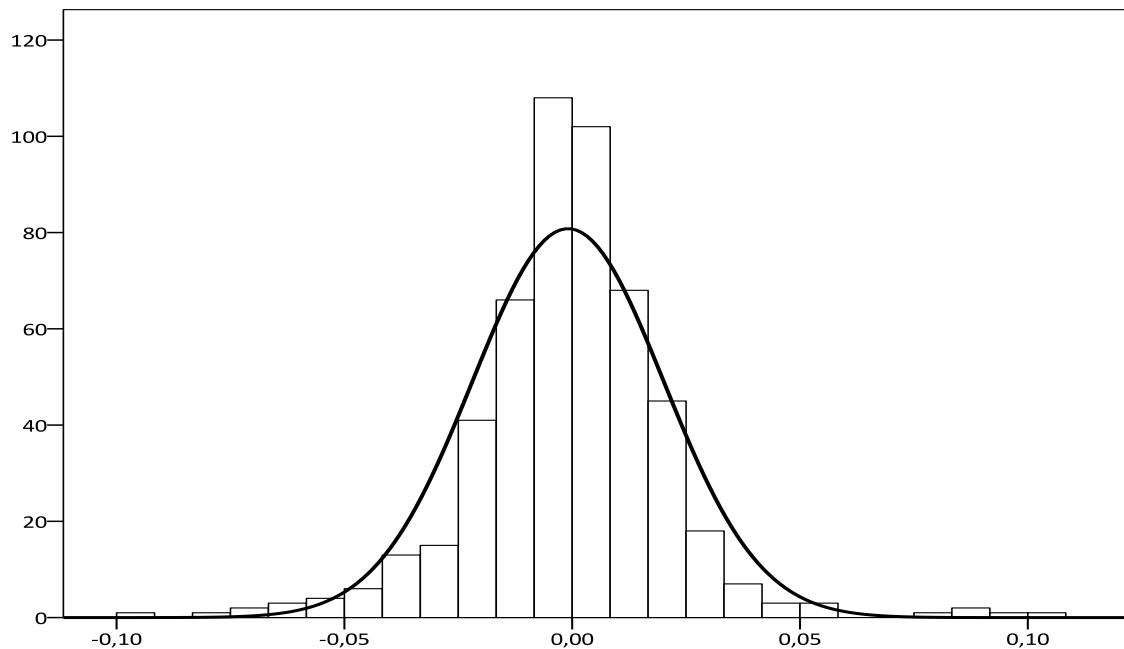
Note: Frequency distribution of CAC 40 daily log returns, during the period 02/01/2008 - 21/12/2009, compared with the normal distribution.

Figure 3: Volatility Clustering
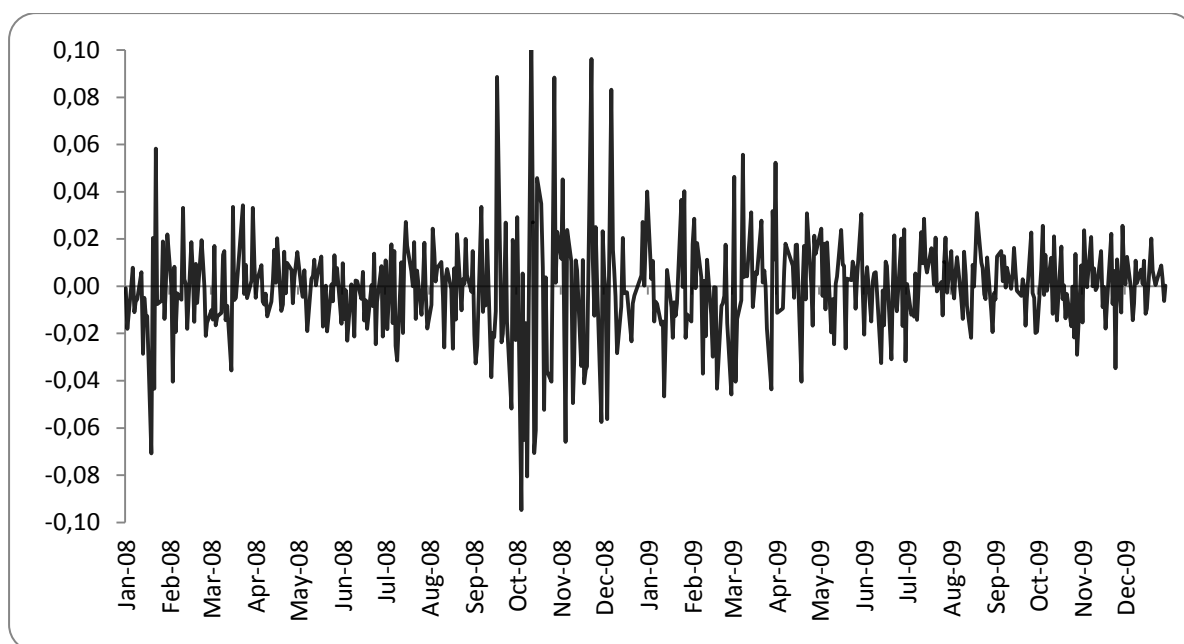
Note: CAC 40 daily log returns from January 2, 2008 to December 31, 2009.

Figure 4: Leverage Effect

Note: CAC 40 index level and BS implied volatility each Wednesday during the period 10/03/2010 - 8/12/2010.

3.2   Estimation

Heston and Nandi (2000) use non-linear least squares (NLLS) to estimate the GARCH model parameters. The following function is minimized over λ, ω, α_1, β_1 and γ_1:

$$\sum_{i=1}^{n}\left(C_i(S(t),T,K)-GARCH(\lambda,\omega,\alpha_1,\beta_1,\gamma_1;S(t),T,K,r)\right)^2 \qquad [32]$$
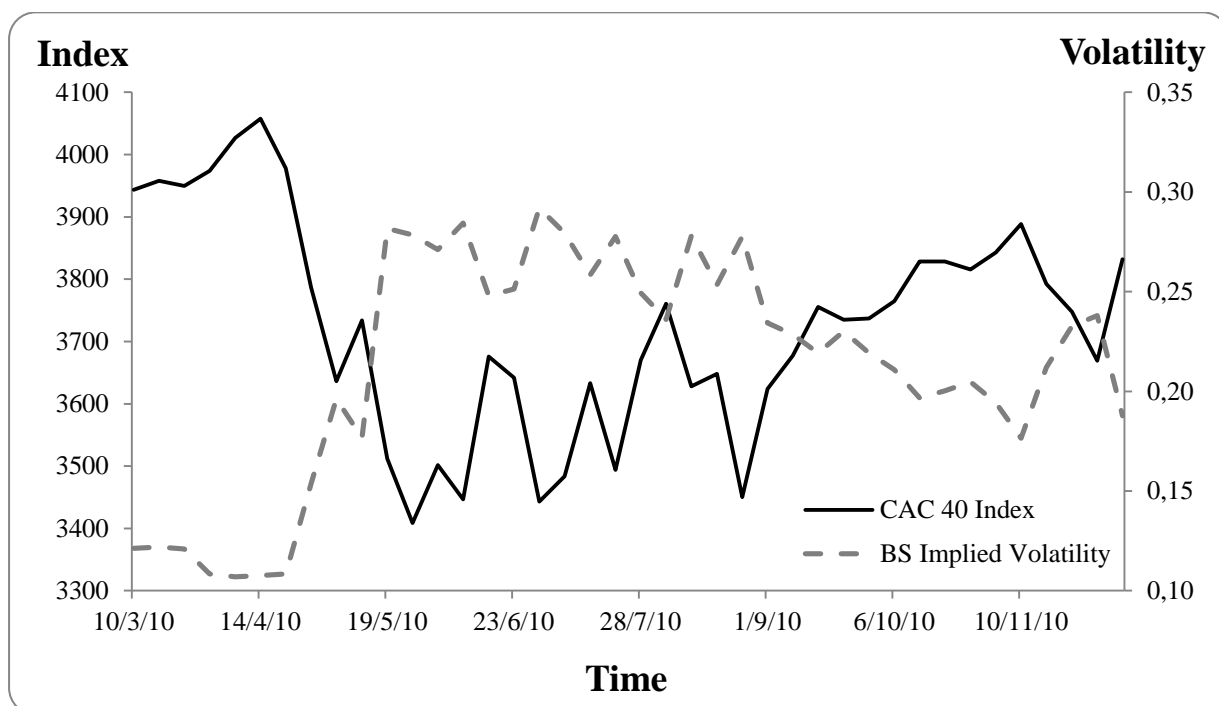
where n is the number of option prices in the sample, $C_i(S(t),T,K)$ is the observed price for the option with the underlying asset's price $S(t)$, time to maturity T and strike price K and GARCH( ) is the GARCH function with unknown parameters λ, ω, α_1, β_1 and γ_1. However, the optimization procedure is computationally quite demanding.

For the purpose of this thesis maximum likelihood (ML) estimation will be used. According to Dotsis and Markellos (2007) the ML estimation causes significant mispricings only in the case of short-term, out-of-the-money options. Furthermore, Dotsis and Markellos provided evidence that for adequate sample sizes the jackknife resampling method can be used effectively and with small computational cost to reduce estimation biases in the GARCH option prices.

3.2.1   Maximum Likelihood Estimation for the GARCH process

The ML estimation is performed on the first order case (p=q=1) of the HN GARCH model using daily ($\Delta=1$) index stock returns. By setting $\Delta=1$, the HN GARCH(1,1) specification can be written as

$$R(t) = r + \lambda h(t) + \sqrt{h(t)}z(t), \qquad\qquad [33]$$

$$h(t) = \omega + \beta_1 h(t-1) + a_1\left(z(t-1) - \gamma_1\sqrt{h(t-1)}\right)^2, \qquad [34]$$

where $R(t) = \log\left(\dfrac{S(t)}{S(t-1)}\right)$ and r is the continuously compounded interest rate over 1 trading day. From [33] and [34] we know that $R(t)\big| h(t-1) \sim N\big(r + \lambda h(t), h(t)\big)$.

Treating the first price observation and h_0 as fixed the likelihood function is given by

$$L(\lambda, \omega, \alpha_1, \beta_1, \gamma_1) = \prod_{t=1}^{n} \frac{1}{\sqrt{2\pi h(t)}} \exp\left(\frac{-\big(R(t) - (r + \lambda h(t))\big)^2}{2h(t)}\right) \qquad [35]$$

To maximize this function is the same as maximizing the logarithm of the function:

$$\log L(\lambda, \omega, \alpha_1, \beta_1, \gamma_1) = \log\left[\prod_{t=1}^{n} \frac{1}{\sqrt{2\pi h(t)}} \exp\left(\frac{-\big(R(t) - (r + \lambda h(t))\big)^2}{2h(t)}\right)\right]$$

$$= \sum_{t=1}^{n}\left[-\log\left(\sqrt{2\pi h(t)}\right) + \left(\frac{-\big(R(t) - (r + \lambda h(t))\big)^2}{2h(t)}\right)\right]$$

$$= \sum_{t=1}^{n}\left[-\frac{1}{2}\log 2\pi - \frac{1}{2}\log h(t) - \frac{1}{2}\underbrace{\left(\frac{R(t) - (r + \lambda h(t))}{\sqrt{h(t)}}\right)^2}_{z(t)}\right]$$

$$= -\frac{n}{2}\log 2\pi - \frac{1}{2}\sum_{t=1}^{n}\left[\log h(t) + z(t)^2\right]$$

If all the constants are ignored the function to be minimized is reduced to

$$-\frac{1}{2}\sum_{t=1}^{n}\left[\log h(t) + z(t)^2\right] \qquad [36]$$

The log-likelihood function [36] can be numerically maximized by choices of λ, ω, α_1, β_1 and γ_1.

The GARCH(1,1) process is estimated using the daily CAC 40 index returns during the period 02/01/2008 - 31/12/2009. In order to illustrate the importance of the skewness parameter γ_1 two specifications of the GARCH(1,1) model are estimated: an unrestricted model and a restricted model in which the parameter γ_1 equals zero (this model is equivalent to a symmetric GARCH). With a view to formally test the significance of the skewness parameter, a likelihood ratio (LR) test is made for each respective year and for the whole two-year period.

The LR test is a criterion for model selection among a class of parametric models with different number of parameters. The LR test statistic asymptotically follows a Chi-squared distribution and is given by LR = - 2(L_R - L_U) ~ χ^2(m) where m denotes the number of restrictions imposed and L_R and L_U are the maximized values of the log-likelihood functions for the restricted and the unrestricted case.

Table 2: Maximum Likelihood Estimation

	α_1	β_1	γ_1	ω	λ	θ	$\beta_1 + \alpha_1\gamma_1^2$	Log-Likelihood	LR test
2008									
GARCH	4,13E-05	0,86	30,55	2,75E-17	3,93	32,59%	0,900	852,74	9,81
GARCH, $\gamma_1 = 0$	5,03E-05	0,91	-	2,23E-15	0,15	37,97%	0,911	847,83	p-value 0,0017
2009									
GARCH	2,91E-06	0,73	293,20	7,09E-07	4,43	21,60%	0,980	937,14	20,91
GARCH, $\gamma_1 = 0$	7,66E-06	0,97	-	1,31E-15	0,71	24,80%	0,968	926,68	p-value 0,0000
2008 - 2009									
GARCH	2,62E-05	0,81	59,19	5,35E-06	3,72	29,12%	0,905	1.785,36	28,69
GARCH, $\gamma_1 = 0$	2,51E-05	0,94	-	8,87E-16	0,58	31,53%	0,935	1.771,02	p-value 0,0000

Note: Maximum likelihood estimates of the HN GARCH(1,1) model with $\Delta=1$ (day) using the CAC 40 stock index returns during the period 02/01/2008 – 31/12/2009 for the unrestricted ($\gamma_1 \neq 0$) and the restricted ($\gamma_1 = 0$) model. The log-likelihood function is $-0.5\sum_{t=1}^{n}\left(\log h(t) + z(t)^2\right)$ where n is the number of days in the sample. The likelihood ratio test statistic is computed as LR = - 2(L_R - L_U), where L_R and L_U are the maximized values of the log-likelihood functions for the restricted and the unrestricted model, respectively. Under the null hypothesis the LR test statistic follows a Chi-squared distribution with one degree of freedom. θ defined to be equal to $\sqrt{256(\omega+\alpha_1)/(1-\beta_1-\alpha_1\gamma_1^2)}$ is the annualized (assuming 256 trading days) long-run volatility implied by the parameter estimates. $\beta_1 + \alpha_1\gamma_1^2$ measures the degree of mean reversion, where $\beta_1 + \alpha_1\gamma_1^2 = 1$ implies that the variance process is integrated.

Figure 5a: Annualized daily volatility - unrestricted GARCH
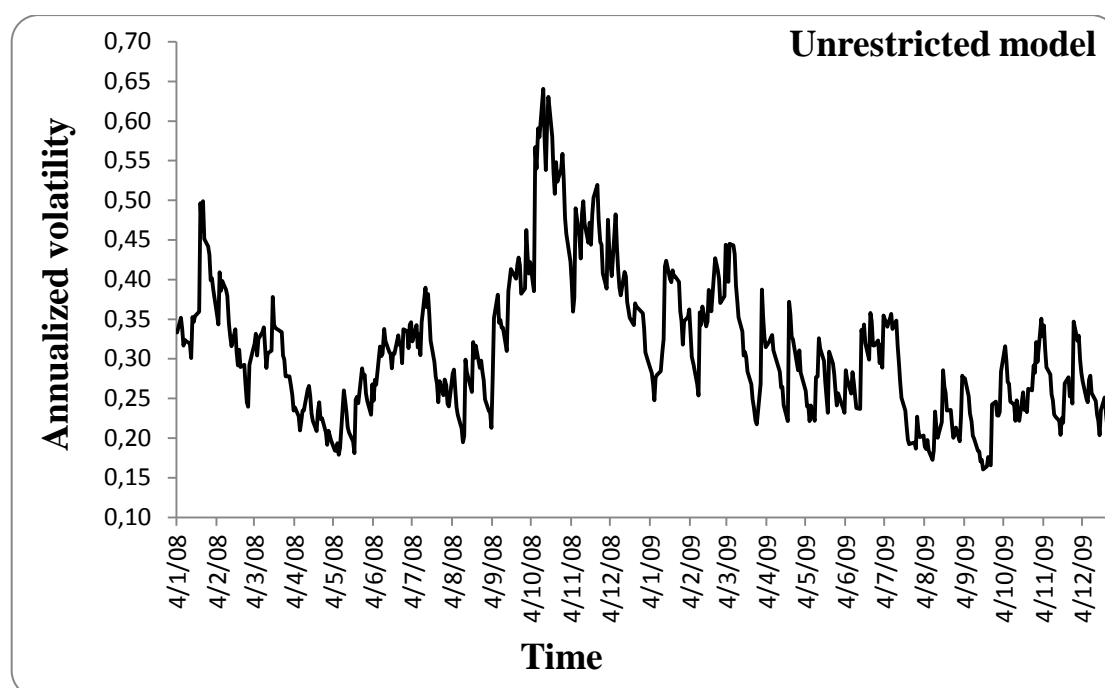
Note: Figure 5a illustrates the daily (annualized) volatility from the unrestricted (asymmetric) GARCH model, during the period 02/01/2008 - 31/12/2009 using daily CAC 40 index stock returns.

Figure 5b: Annualized daily volatility - restricted GARCH
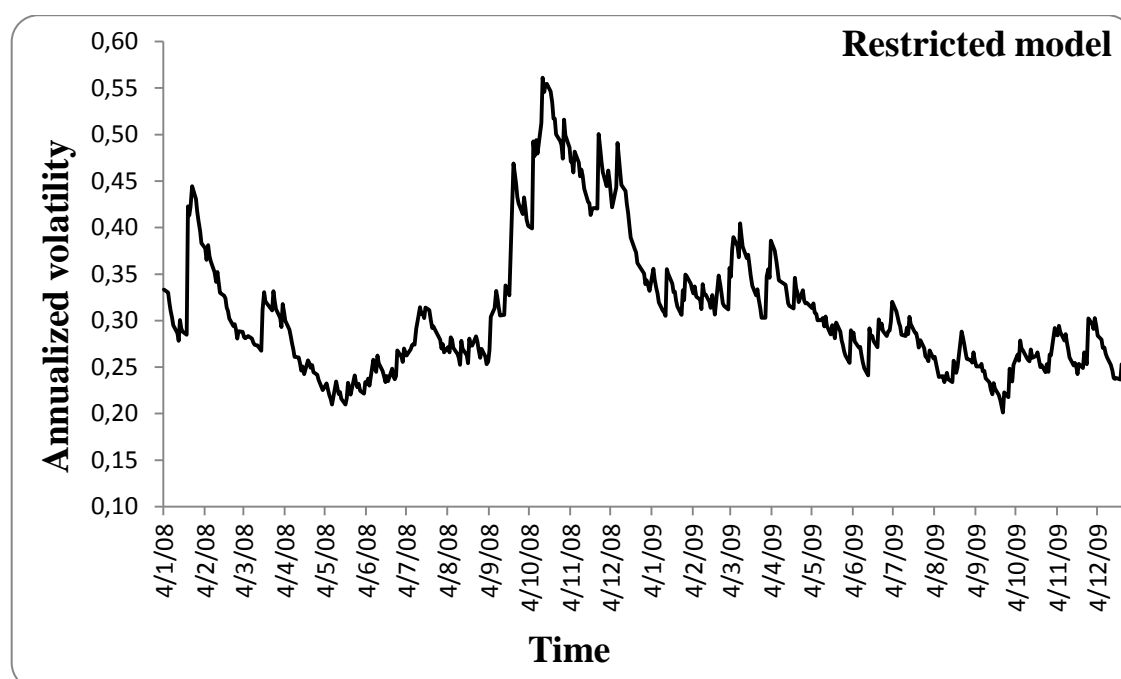
Note: Figure 5b illustrates the daily (annualized) volatility from the restricted (symmetric) GARCH model, during the period 02/01/2008 - 31/12/2009 using daily CAC 40 index stock returns.

Table 2 presents the maximum likelihood estimates of the GARCH model[20] from 02/01/2008 to 31/12/2009. The discussion here is focused on the parameter estimates for the whole sample period. The volatility of volatility, α_1, is 2,62E-05 in the unrestricted case of the model and 2,51E-05 in the restricted. The annualized long-run mean of volatility, defined to be equal to $\sqrt{256(\omega + \alpha_1)/(1 - \beta_1 - \alpha_1\gamma_1^2)}$ (256 trading days), is 29,12% for the full model and 31,53% for the restricted version of the model. The degree of mean reversion, given by $\beta_1 + \alpha_1\gamma_1^2$, is 0,905 for the unrestricted model and for the restricted is equal to 0,935 (given by β_1). The LR test strongly rejects the null hypothesis (H_0: $\gamma_1 = 0$) of a symmetric GARCH model for each respective year and for the whole two year period. This result implies that the negative correlation between returns and volatility is a significant feature of the CAC 40 index.

The annualized[21] daily volatility for the 2008 - 2009 period for the asymmetric and symmetric GARCH model is shown in figures 5a and 5b, respectively. It is evident that both annualized volatility series tend to oscillate around their long-run mean. This result is consistent with the previous findings that the variance process of the GARCH model is mean reverting. Moreover, by comparing figures 5a and 5b it can be deduced that including the skewness parameter makes the filtered variance more volatile and produces sudden drops in addition to sudden increases in volatility, which can explain why the risk premium parameter λ is higher in the unrestricted case.

Heston and Nandi (2000) argue that the pricing efficiency of the HN GARCH model in which the parameters are estimated by ML estimation is not accurate. This is mainly due to the fact that the information set of index levels is not the same with that of option prices. Option prices are embedded with the expectation about the future evolution of the asset price and therefore are forward looking while the information contained in asset prices is backward looking. In order to overcome any potential estimation biases in the GARCH option prices the jackknife procedure could be used.

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[20] The starting value for the conditional variance h(0) is set equal to the sample variance of the index returns (due to the strong mean reversion in volatility, the results are not sensitive to the starting value of h(0)). The starting values for the model parameters (α_1, β_1, γ_1, ω and λ) are similar to the estimates in table 9 of Chorro, Guegan and Ielpo (2010).

[21] For each day in the sample, the annualized level of volatility is $\sqrt{256h(t+1)}$.

3.2.2   The Jackknife Procedure

The jackknife procedure was originally proposed by Quenouille (1956) as a solution to finite sample bias in parametric estimation problems. Following the detailed description of the method by Phillips and Yu (2005), let T be the number of observations in the whole sample and let the sample be decomposed into m consecutive subsamples each with λ observations, so that T = m x λ. Then, the jackknife estimator of a certain parameter θ utilizes the subsample estimates to assist in the bias reduction method giving

$$\hat{\theta}_{jack} = \frac{m}{m-1}\hat{\theta}_T - \frac{\sum_{i=1}^{m}\hat{\theta}_{\lambda i}}{m^2 - m},\qquad [37]$$

where $\hat{\theta}_T$ and $\hat{\theta}_{\lambda i}$ are the estimates obtained by application of the ML estimation to the whole sample and the ith subsample, respectively. It can be shown that the bias in the jackknife estimate is of order $O(T^{-2})$ rather than $O(T^{-1})$.

Rather than jackknifing the parameter estimates on which the option prices depend and plugging the revised estimates into the options price formula, the jackknife is applied directly to the option prices as proposed by Dotsis and Markellos (2007). The reason is that the jackknife tends to increase the variance of the quantity being estimated and this additional variance adversely affects the performance of the procedure when the quantity is a non-linear function of its arguments, like the option price. Moreover, two subsamples (m=2) are used in implementing the jackknife. This configuration was used by Phillips and Yu (2005) since it is computationally easy and has satisfactory performance in bias reduction.

The implementation of the procedure can be summarized in the following specific steps: First, the GARCH parameters are estimated by ML using the entire sample. Second, the option prices are calculated based on the ML estimates obtained in step 1. Third, the GARCH parameters are estimated by ML for each sub-sample. Then, the option prices are calculated based on the ML estimates obtained in step 3 for each subsample. Finally, the jackknife estimators of the option prices are calculated using equation [37].

3.3   Model Comparisons

The pricing accuracy of the HN GARCH model is compared with the Ad Hoc BS model. As Heston and Nandi (2000) point out, even though the Ad Hoc BS model is theoretically inconsistent is a more challenging benchmark than the simple BS. Dumas, Fleming and Whaley (1998) show that the implied binomial tree or the deterministic volatility models of Derman and Kani (1994), Dupire (1994) and Rubinstein (1994) underperform the Ad Hoc BS model in out-of-sample options pricing errors in the S&P 500 index options market. Moreover, Christoffersen and Jacobs (2004a) suggest that when the same loss function is used for estimation and evaluation the Ad Hoc BS model outperforms the Heston (1993) model.

3.3.1 Option Model Evaluation

The performance of the different option pricing models is evaluated using the root mean squared error loss function (RMSE) and the relative root mean squared error loss function (%RMSE).

The root mean squared error loss function is the square root of the sample mean of the squared estimation errors

$$RMSE(\theta) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(C_i - C_i(\theta))^2} \; , \qquad\qquad [38]$$

where C_i and $C_i(\theta)$ are the market and the model options prices respectively and n is the number of option contracts used. The estimation errors, $e_i(\theta) = C_i - C_i(\theta)$, are squared before they are averaged meaning that higher weights are given to larger errors. The relative root mean squared error loss function is defined as

$$\%RMSE(\theta) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\big((C_i - C_i(\theta))/C_i\big)^2} \qquad\qquad [39]$$

Unlike the RMSE loss function which measures the raw difference between market prices and model prices, the %RMSE loss function measures the percentage (or relative) difference between these prices.

3.3.2   In-sample Model Comparison

The in-sample valuation performance of the two option pricing models is tested using call option prices from January 6, 2010 to June 9, 2010. In order to estimate the HN GARCH parameters the ML estimation is carried out[22]. Even though the options data are weekly, the conditional variance h(t+1) that is relevant for option values at time t is drawn from the daily evolution of index returns. The starting variance h(0) is kept fixed at the in-sample estimate of the variance[23]. Unlike the HN GRACH model where the parameters are held constant over the entire estimation period, the coefficients for the Ad Hoc model are re-estimated every week. The exact structural form of the implied volatility that is selected on a given day depends on the number of distinct option maturities in the sample on that day. The sum of squared errors between the BS implied volatilities across different strikes and maturities and the model's structural form of the implied volatility is minimized via OLS.

Table 3: In-sample model comparison (non-updated HN GARCH)

Model	Loss function RMSE	Average Option Price	Number of Observations
Ad Hoc BS	11,66	122,77	1003
HN GARCH (non-updated)	29,51		

Note: The parameters of the non-updated HN GARCH model are estimated via maximum likelihood and are held constant over the entire estimation period. The parameters of the Ad Hoc BS model are re-estimated every week. The options data range from 06/01/2010 to 09/06/2010.

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[22] It is crucial to note that the jackknife resampling method is not applied mainly due to significant time restrictions. Applying the jackknife procedure would require calculating an additional 4038 option values.

[23] The starting values for the model parameters (α_1, β_1, γ_1, ω and λ) are similar to the estimates in table 9 of Chorro, Guegan and Ielpo (2010).

The average option price in the sample is 122.77 €. The root mean squared error (RMSE) for the Ad Hoc model is 11.66 € and for the HN GARCH model is 29.51 €. The Ad Hoc BS model is designed to fit both the volatility smile in strike prices and the term structure of implied volatilities. In addition, it is updated every week and therefore outperforms the HN GARCH model.

Furthermore, an updated GARCH model is estimated. Although the model parameters change every week the variance h(t+1) is still drawn from the history of asset prices at time t. Since the main objective is to compare the model's out-of-sample valuation errors from June 16, 2010 to December 8, 2010, the updating of the model parameters is done only in that period. Table 4 reports the comparison between the updated HN GARCH model and the Ad Hoc BS.

Table 4: In-sample model comparison (updated HN GARCH)

Model	Loss function		Average Option Price	Number of Observations
	RMSE	% RMSE		
Ad Hoc BS	3,34	0,11	144,55	1016
HN GARCH (updated)	9,46	0,47		

Note: In-sample valuation errors (in €) from the weekly estimation using option prices from 16/06/2010 to 08/12/2010. Both the Ad Hoc BS and the HN GARCH model are estimated each week using ordinary least squares and maximum likelihood estimation respectively.

The average option price for this period is 144.55 € and the RMSE for the Ad Hoc model is 3.34 €. After updating the parameters of the GARCH process the RMSE of the HN GARCH model is 9.46 €. Even though there is a significant improvement of the pricing performance of the HN model, the mispricing errors are still large and the model is outperformed by the Ad Hoc. Table 5 describes the mean and standard deviation of the updated HN GARCH coefficients. The poor performance of the model can be explained by the fact that the estimated GARCH parameters[24] are unstable in the sample period. As Heston and Nandi (2000) point out[25]: *"... option*

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[24] The option value at time t is not only a function of the current level of variance but also of the parameters that drive the variance process namely α_1, β_1, γ_1, λ and ω.

[25] See Heston and Nandi (2000) page 605.

values are more sensitive to α_1 (that measures the volatility of volatility) and γ_1 (that controls the skewness of index returns) than they are to the other parameters. This stability is important for the GARCH model to fit the data reasonably well ..."

Table 5: Mean estimates from the updated HN GARCH model

Parameter	Mean	Standard Deviation
α_1	2,02E-05	1,14E-05
β_1	0,89	0,31
γ_1	101,56	59,61
ω	3,62E-06	6,06E-06

Note: This table reports the mean and standard deviation of the parameter estimates from the weekly estimation of HN GARCH model using maximum likelihood estimation. Variance h(t+1) is drawn from the daily history of CAC 40 levels.

3.3.3   Out-of-sample Model Comparison

The prediction performance of each option valuation model is tested based on call option prices from June 16, 2010 to December 8, 2010. The models are implemented using information at time t to value options at time t+1. In other words, the estimated parameters from the current week for the Ad Hoc BS model and the updated HN GARCH model are used to value options in the next week. The out-of-sample valuation errors for the two models are presented in table 6.

Table 6: Out-of-sample pricing errors

Model	Loss function		Average Option Price	Number of Observations
	RMSE	% RMSE		
Ad Hoc BS	8,45	0,24	144,55	1016
HN GARCH (updated)	14,61	0,50		

Note: Out-of-sample valuation errors for call options by the different models. The data range from 16/06/2010 to 08/12/2010.

The root mean squared valuation errors for the Ad Hoc and the HN GARCH model are 8.45 € and 14.61 € respectively.

4.   Conclusions

The pricing performance of the HN GARCH option pricing model on CAC 40 index options has been evaluated in this thesis. The GARCH model parameters have been estimated via maximum likelihood instead of non-linear least squares. The HN GARCH model is shown to suffer from significant mispricing errors as it is greatly outperformed by the Ad Hoc model both in-sample and out-of-sample. Even though these results contradict previous empirical studies, it is reasonably to argue that the poor performance of the model is not caused by a structural flaw but is mainly due to the finite sample properties of the estimators employed. As noted by Dotsis and Markellos (2007), option pricing models utilizing parameters estimated via asymptotic methods, such as maximum likelihood, entail substantial estimation biases. Applying the jackknife resampling method directly to the HN option prices as a solution for bias reduction is left for future research.

Another important methodological issue concerning the estimation of parameters for use in option valuation models was raised by Christoffersen and Jacobs (2004a). They suggest that in order to compare the pricing errors of competing models the same loss function should be used in parameter estimation and model evaluation otherwise unfair comparisons will be made. The importance of the loss function in option valuation was not taken into consideration in this thesis.

5.   Recent Developments

Finally, recent developments in the GARCH option pricing literature are presented in this section. Attention is drawn to the performance of the "new" option pricing models and a brief comparison with the HN GARCH model is provided.

Christoffersen, Heston and Jacobs (2006) propose an analytical discrete-time GARCH option pricing model with inverse Gaussian innovations in the variance process. The model allows for conditional skewness in addition to conditional

heteroskedasticity and a leverage effect and nests the HN GARCH model as a special case. The Inverse Gaussian GARCH (henceforth IG GARCH) process has two interesting continuous-time limits. One limit is the standard stochastic volatility model of Heston (1993) and the other a pure jump process with stochastic intensity. The performance of the model is examined on S&P 500 index options data and is compared to a number of benchmarks including the HN GARCH model. From the empirical results it can be deduced that the benefits of modeling conditional skewness are mixed. In-sample the IG GARCH model outperforms the HN GARCH model. Out-of-sample the performance of the IG GARCH model is superior only for the valuation of out-of-the-money put options.

Christoffersen, Jacobs, Ornthanalai and Wang (2008) present a new option valuation model, based on the work by Heston and Nandi (2000), in which the volatility of returns consists of two components. One of these components is a long-run component and it can be modeled as fully persistent. The other component is short-run and has a zero mean. The empirical performance of this new variance component model is significantly better than that of the HN GARCH model both in- and out-of-sample. As noted by Christoffersen et.al (2008) : "*an important aspect of the model's improved performance is that its richer parameterization allows for improved joint modeling of long-maturity and short-maturity options. The model captures the stylized fact that shocks to current conditional volatility impact on the conditional variance forecast up to a year in the future, which results in a very different implied volatility term structure for at-the-money options*". Also, the component model results in a different path for spot volatility compared to the HN GARCH model.

Chorro, Guegan and Ielpo (2010) develop an EGARCH option pricing model with generalized hyperbolic innovations (henceforth EGARCH-GH). The pricing performance of the EGARCH-GH model is examined on two data sets of options written on the CAC 40 index and the S&P 500 index respectively. In both cases the HN GARCH model is dominated by the EGARCH-GH.

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Appendix - Sensitivity Analysis

The value of a European call option is given by (*for simplicity the notation used here is slightly different from the one used in section 2.4*) $C = SP_1 - Ke^{-rT}P_2$, where the risk-neutral probabilities P_1 and P_2 are equal to

$$P_1 = \frac{1}{2} + \frac{e^{-rT}}{\pi S}\int_0^\infty \text{Re}\left[\frac{K^{-i\varphi}f^*(i\varphi+1)}{i\varphi}\right]d\varphi$$

$$P_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \text{Re}\left[\frac{K^{-i\varphi}f^*(i\varphi)}{i\varphi}\right]d\varphi$$

and the characteristic function f^* for the single version of the model is

$$f^*(\varphi) = S^\varphi \exp[A_t(\varphi) + B_t(\varphi)h(t+1)].$$

Delta (Δ)

Delta is defined as the rate of change of the option price with respect to the price of the underlying asset

$$\Delta_{call} = \frac{\partial C}{\partial S} = P_1.$$

Using the put-call parity, the delta of the put is

$$\Delta_{put} = P_1 - 1 .$$

Gamma (Γ)

Gamma is the rate of change off delta with respect to the price of the underlying asset. In order to obtain gamma the risk-neutral probability P_1 has to be written as

$$P_1 = \frac{1}{2} + \frac{e^{-rT}}{\pi}\int_0^\infty \text{Re}\left[\frac{K^{-i\varphi}S^{i\varphi}\exp(g(i\varphi+1))}{i\varphi}\right]d\varphi$$

where
$$g(\varphi) = A_t(\varphi) + B_t(\varphi)h(t+1).$$

Then

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial P_1}{\partial S} = \frac{e^{-rT}}{\pi} \int_0^\infty \mathrm{Re}\left[\frac{K^{-i\varphi} i\varphi S^{i\varphi-1} \exp(g(i\varphi+1))}{i\varphi} \right] d\varphi$$
$$= \frac{e^{-rT}}{\pi} \int_0^\infty \mathrm{Re}\left[\frac{K^{-i\varphi} f^*(i\varphi+1)}{S^2} \right] d\varphi$$
.

Vega (V)

Vega is defined as the rate of change of option price with respect to the volatility of the underlying asset

$$V = \frac{\partial C}{\partial \sigma(t+1)} = S\frac{\partial P_1}{\partial \sigma(t+1)} - Ke^{-rT}\frac{\partial P_2}{\partial \sigma(t+1)}$$
.

The two partial derivatives $\dfrac{\partial P_1}{\partial \sigma(t+1)}$ and $\dfrac{\partial P_2}{\partial \sigma(t+1)}$ have to be calculated

$$\frac{\partial P_1}{\partial \sigma(t+1)} = \frac{e^{-rT}}{\pi S} \int_0^\infty \mathrm{Re}\left[\frac{K^{-i\varphi}}{i\varphi} \times \frac{\partial f^*(z)}{\partial \sigma(t+1)}\bigg|_{z=i\varphi+1} \right] d\varphi$$

Since,

$$\frac{\partial f^*}{\partial \sigma(t+1)} = S^\varphi \exp(A_t + B_t h(t+1)) \times 2B_t \sigma(t+1) = 2B_t \sigma(t+1) f^*$$

the first partial derivative is equal to

$$\frac{\partial P_1}{\partial \sigma(t+1)} = \frac{2\sigma(t+1)e^{-rT}}{\pi S} \int_0^\infty \mathrm{Re}\left[\frac{K^{-i\varphi} f^*(i\varphi+1)B_t}{i\varphi} \right] d\varphi$$

and the second

$$\frac{\partial P_2}{\partial \sigma(t+1)} = \frac{2\sigma(t+1)}{\pi} \int_0^\infty \mathrm{Re}\left[\frac{K^{-i\varphi} f^*(i\varphi)B_t}{i\varphi} \right] d\varphi$$

Finally, vega can be calculated with the following formula

$$V = \frac{2\sigma(t+1)e^{-rT}}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{K^{-i\varphi}f^*(i\varphi+1)B_t}{i\varphi}\right]d\varphi - \frac{2\sigma(t+1)Ke^{-rT}}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{K^{-i\varphi}f^*(i\varphi)B_t}{i\varphi}\right]d\varphi .$$

Rho (ρ)

Rho is defined as the rate of change of the option price with respect to the interest rate

$$\rho_{call} = \frac{\partial C}{\partial r} = S\frac{\partial P_1}{\partial r} - K\left[-Te^{-rt}P_2 + e^{-rT}\frac{\partial P_2}{\partial r}\right] = S\frac{\partial P_1}{\partial r} + Ke^{-rT}\left[TP_2 - \frac{\partial P_2}{\partial r}\right].$$

In order to calculate $\dfrac{\partial P_1}{\partial r}$ and $\dfrac{\partial P_2}{\partial r}$ the partial derivative of the characteristic function $f^*(\varphi)$ with respect to r is needed

$$\frac{\partial f^*(\varphi)}{\partial r} = S^\varphi \exp\left(A_t + B_t\sigma(t+1)\right)\frac{\partial A_t}{\partial r} = f^*(\varphi)\frac{\partial A_t}{\partial r} = f^*(\varphi)T\varphi$$

since $\quad \dfrac{\partial A_t}{\partial r} = \varphi + \dfrac{\partial A_{t+1}}{\partial r} = 2\varphi + \dfrac{\partial A_{t+2}}{\partial r} = ... = T\varphi + \dfrac{\partial A_{t+T}}{\partial r} = T\varphi .$

Similarly, $\dfrac{\partial f^*(i\varphi)}{\partial r} = f^*(i\varphi)Ti\varphi \quad$ and $\quad \dfrac{\partial f^*(i\varphi+1)}{\partial r} = f^*(i\varphi+1)T(i\varphi+1) .$

Using the product rule, the first partial derivative is

$$\frac{\partial P_1}{\partial r} = T\underbrace{\frac{-e^{-rT}}{\pi S}\int_0^\infty \mathrm{Re}\left[\frac{K^{-i\varphi}f^*(i\varphi+1)}{i\varphi}\right]d\varphi}_{\frac{1}{2}-P_1} + \frac{e^{-rT}}{\pi S}\int_0^\infty \mathrm{Re}\left[\frac{K^{-i\varphi}}{i\varphi}\times\frac{\partial f^*(z)}{\partial r}\Big|_{z=i\varphi+1}\right]d\varphi$$

$$= T\left(\frac{1}{2}-P_1\right) + \frac{Te^{-rT}}{\pi S}\int_0^\infty \mathrm{Re}\left[\frac{K^{-i\varphi}f^*(i\varphi+1)\times(i\varphi+1)}{i\varphi}\right]d\varphi$$

Moreover,

$$\frac{\partial P_2}{\partial r} = \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left[\frac{K^{-i\varphi}}{i\varphi} \times \frac{\partial f^*(i\varphi)}{\partial r} \right] d\varphi = \frac{T}{\pi} \int_0^\infty \mathrm{Re}\left[K^{-i\varphi} f^*(i\varphi) \right] d\varphi$$

Finally, using the put-call parity rho for the put can be written as

$$\rho_{put} = \frac{\partial P}{\partial r} = \frac{\partial C}{\partial r} - TKe^{-rT} = \rho_{call} - TKe^{-rT}.$$

Theta (Θ)

Theta is the rate of change of the option price with respect to the passage of time. Obtaining theta in closed form is quite complicated since the terms $\{A_i\}_{i=t}^{T}$ and $\{B_i\}_{i=t}^{T}$ depend on the time to maturity. Theta can be estimated using the finite difference method.