# Asymptotic asset pricing and bubbles

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**Abstract**

We define the concept of asymptotic superreplication, and prove an asset pricing theorem for sequences of financial markets (e.g., weakly converging financial markets and large financial markets) based on contiguous sequences of equivalent local martingale measures. This provides a pricing mechanism to calculate the fundamental value of a financial asset in the asymptotic market. We introduce the notion of asymptotic bubbles by showing that this fundamental value can be strictly lower than the current price of the asset. In the case of weakly converging markets, we show that this fundamental value is equal to an expectation of the terminal value of the asset in the weak-limit market. From a practical perspective, we relate the asymptotic superreplication price to a limit of quantile-hedging prices. This shows that even when a price process is a true martingale, it can have properties similar to a bubble, up to a set of small probability. For practical applications, we give examples of weakly converging discrete-time models and large financial models that present bubbles.

Asymptotic Arbitrage · Local Martingales · Contiguity of Probability Measures · Superreplication
JEL : G13 · G13

## 1 Introduction

In recent years, a mathematical theory of bubbles has been developed ([5], [14], [15], [24], to cite a few). The idea is simple but has far-reaching consequences. Under the No Free Lunch With Vanishing Risk (NFLVR) assumption, a nonnegative price process $S$ is a local martingale under an equivalent local martingale measure (ELMM) $\mathbf{Q}$. If $S$ is a strict local martingale (i.e., it is not a martingale) under $\mathbf{Q}$, it implies that its $\mathbf{Q}$-expectation is strictly less than its current value. In a complete market (i.e., the ELMM is unique), we can conclude from the fact that the $\mathbf{Q}$-expectation is equal to its replication price that one can replicate the cash flows associated to owning the asset for less than its current market value, giving birth to a "bubble". Note however that this does not imply the existence of an arbitrage since shorting the asset may entail taking on non-admissible risks. When the ELMM is not unique, one can interpret the expectation under a given ELMM as the fundamental value of the asset relative to that measure, which may also be lower than its current value. In particular, bubbles may exist under some ELMMs but not under others.

A well-known result from the theory of martingales provides an interesting perspective on the nature of the discrepancy between market and fundamental value: a positive continuous $\mathbf{Q}$-local martingale on $[0, T]$ satisfies

$$S_0 - \mathbf{E}_{\mathbf{Q}}(S_T) = \lim_{n\to\infty} \mathbf{E}_{\mathbf{Q}}\left[S_{\tau_n}\mathbf{1}_{\{\tau_n<T\}}\right] = \lim_{n\to\infty} n\mathbf{Q}(\sup_{t\le T} S_t > n) \tag{1}$$

with $\tau_n = \inf\{t \le T : S_t > n\}$. In the case of a strict local martingale $S$, the above equation shows that the market is assigning positive value to the event $\{\tau_n < T\}$ even though its probability of occurring converges

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to zero when $n \to \infty$. Shorting and hedging the asset would therefore be an arbitrage if this strategy were admissible.

For practical applications, the probability of the event $\{\tau_n < T\}$ only needs to be small relative to the value of $\mathbf{E_Q}\left[S_{\tau_n}\mathbf{1}_{\{\tau_n<T\}}\right]$, for some large $n$, in order to argue that, for all intended purposes, $S$ is a financial model that presents pricing anomalies that investors cannot easily arbitrage away, as in the theoretical setting of bubbles. In this paper, we extend the idea that events with small probabilities may be assigned positive market value by considering a pricing method consistent with the notion of asymptotic arbitrage, defined on a sequence of financial markets (a *large financial market*) as introduced by Kabanov and Kramkov [17]. We consider a sequence of financial market models $(\mathbb{M}^n)_{n\geq 1}$ each defined on a separate filtered probability space $(\Omega^n, \mathcal{F}^n, \mathbb{F}^n, \mathbf{P}_n)$ indexed by $n$. This setup applies to weakly converging financial markets ([8], [12], [26]), in which $n$ typically denotes the number of time steps in the $n$-th model, as well as large financial markets ([17], [18], [20], [22]), in which $n$ denotes the number of traded assets.[1] Note however that we do not assume a priori that the processes converge in any given sense. The key concept is the contiguity of sequences of ELMMs and we provide a mechanism to calculate the fundamental value of an asset with respect to a given sequence of ELMMs. By working with a sequence of financial markets, one can then use this pricing mechanism to make the theory of asset pricing and bubbles consistent with the premise that continuous processes can be approximated by discrete-time ones, so that pricing properties of the former class of processes should translate to the latter and vice-versa. We refer the reader to Section 11 of [27] which provides an insightful discussion on the role of discrete versus continuous-time models in finance.

The theory of bubbles has led to a number of extensions in the literature. Biagini et al. [4] model the birth of a bubble as a submartingale which then turns into a supermartingale by shifting the ELMM used for valuation. Bayraktar et al. [3], Kardaras et al. [19], and Pal and Protter [25] study the effect of the presence of bubbles in option pricing. More recently, Herdegen and Schweizer [11] specify the fundamental value of an asset as its superreplication price and define strong bubbles as the difference between this value and the current value of the price process. We refer the reader to [27] for an in-depth survey of the theory and other recent developments.

In Section 2, we define the notion of asymptotic superreplication in a way which is consistent with the definition of asymptotic arbitrage, and we relate it to a sequence of quantile-hedging problems. We prove an asymptotic asset pricing theorem by first considering the case of complete markets in Section 3, and incomplete markets in Section 4. In Section 5, we define the notions of asymptotic fundamental value and bubbles. For practical applications, we give examples of weakly converging discrete-time models and large financial models that present bubbles. We then provide a weak convergence result in Section 6 that shows that the asymptotic fundamental value of an asset is equal to an expectation of the terminal value in the limit market under a probability measure which is not necessarily the ELMM of the limit market. This gives a simple test to verify if an asset has an asymptotic bubble. As an application, we show that some sequences of GARCH models present asymptotic bubbles.

## 2 Asymptotic market models

For each $n \geq 1$, let $T_n \geq 0$, $\mathcal{T}_n \subset [0, T_n]$, and $(\Omega^n, \mathcal{F}^n, \mathbb{F}^n = (\mathcal{F}^n_t)_{t\in\mathcal{T}_n}, \mathbf{P}_n)$ a filtered probability space on which a (nonnegative) semimartingale $S^n$ is defined. We assume that $\{0, T_n\} \in \mathcal{T}_n$ and $\mathcal{F}^n_0$ is trivial for all $n \geq 1$. The dimension of the process $S^n$ is finite and denoted $d(n)$, as it may depend on $n$. However, $d(n)$ need not converge to infinity as $n$ grows, as most examples presented in this paper concern the case $d(n) = 1$. The vector $S^n_t$ represents the discounted prices of the $d(n)$ risky assets at time $t$. We assume the limit $\lim_n S^n_0$ exists and is finite. We also assume that the assets only give a lump payment of $S^n_{T_n}$ at time $T_n$. The extension to dividend-paying assets is straightforward. See [14], for instance. The $n$-th market is denoted $\mathbb{M}^n$. We refer to the sequence $(\mathbb{M}^n)_{n\geq 1}$ as the asymptotic (abbr. *asymp.*) market and the sequence

[1] Note that in the theory of large financial markets, the dimension of the process $S^n$ does not need, a priori, to converge to infinity.

$S = (S^n)_{n\geq 1}$ as the asymp. price processes. Each $S^n$ (resp. $\mathbb{M}^n$) is called an intermediate price process (resp. intermediate market). From an economic point of view, we consider the asymp. market and a given intermediary market $\mathbb{M}^n$ as approximations of each other when $n$ is large. From a mathematical point of view we do not assume a priori that the processes converge in any given sense.

**Definition 2.1** *A* payoff *in the asymp. market is a sequence $(h^n)_{n\geq 1}$ of nonnegative $\mathcal{F}^n_{T_n}$-random variables, each in $L^2(d\mathbf{P}_n)$ that satisfy*

$$\sup_n \mathbf{P}_n(h^n > M) \to 0 \text{ as } M \to \infty. \tag{2}$$

*In particular,*

$$\mathbf{P}_n(h^n > M_n) \to 0 \text{ as } n \to \infty,$$

*for any sequence $(M_n)_{n\geq 1}$ of positive real numbers converging to infinity.*

*The set of all payoffs in the asymp. market is denoted $\mathcal{H}$.*

Throughout the paper, if $X$ is a process, $X^*$ denotes its maximum process defined as $X^*_t = \sup_{s\in\mathcal{T}_n, s\leq t} X_s$. Furthermore, we write $X^n_T$ for $X^n_{T_n}$ and $\mathcal{F}^n_T$ for $\mathcal{F}^n_{T_n}$ in order to alleviate the notation.

## 2.1 Trading and arbitrage in intermediate markets

A *trading strategy* in intermediate market $\mathbb{M}^n$ is a $d(n)$-dimensional, predictable, $S^n$-integrable process $X$. The *wealth process* $W$ associated to the strategy $X$ and initial wealth $x^n$ is defined by the stochastic integral $x^n + \int X dS^n$. The strategy is said to be admissible if $\int X dS^n \geq -\alpha$ for some $\alpha \geq 0$.

Following [6], we denote by $K^n_0(x^n)$ the set of random variables of the form $x^n + \int_0^{T_n} X^n_s dS^n_s$ attainable with an admissible trading strategy $X^n$ and initial wealth $x^n$, and $C^n := (K^n_0(0) - L^0_+(d\mathbf{P}_n)) \cap L^\infty(d\mathbf{P}_n)$, the set of bounded random variables dominated by elements of $K^n_0(0)$.

**Definition 2.2** *An arbitrage opportunity in market $\mathbb{M}^n$ is an admissible wealth process $W^n$ such that $W^n_0 = 0$, $\mathbf{P}_n(W^n_T \geq 0) = 1$ and $\mathbf{P}_n(W^n_T > 0) > 0$.*

Each intermediate market $\mathbb{M}^n$ is assumed to satisfy the NFLVR assumption of [6]:

**Definition 2.3** *A free lunch with vanishing risk (FLVR) in market $\mathbb{M}^n$ is either an arbitrage according to Definition 2.2 or a sequence $(W^k)_{k\geq 1}$ of admissible wealth processes with $W^k_T \in K^n_0(0)$ for which there is a non-negative $\mathcal{F}^n_T$-measurable random variable $f_0$ such that $W^k_T \to f_0$ in probability as $k \to \infty$, and*

1. *$\mathbf{P}_n(f_0 > 0) > 0$ and*
2. *$W^k \geq -\epsilon_k$, for some sequence $\epsilon_k \to 0$, as $k \to \infty$.*

**Hypothesis 2.4** *For each $n \geq 1$, there is no free lunch with vanishing risk (NFLVR) in $\mathbb{M}^n$. Equivalently $\overline{C^n} \cap L^\infty_+(d\mathbf{P}_n) = \{0\}$, where $\overline{C^n}$ denotes the closure of $C^n$ with respect to the norm topology of $L^\infty(d\mathbf{P}_n)$. (See Proposition 3.6, [6].)*

The FLVR definition relaxes the economically relevant notion of making positive profits with positive probability without ever losing money. The NFLVR condition in market $\mathbb{M}^n$ implies that a trader cannot approximate an arbitrage strategy in probability by losing at most an infinitesimally small amount money. By Corollary 1.2 of [6], an equivalent condition for NFLVR in $\mathbb{M}^n$ is the existence of a probability measure, equivalent to $\mathbf{P}_n$ under which each component of the process $S^n$ is a local martingale. To this end, we denote by $\mathcal{M}^n_{loc}$ the set of equivalent local martingale measures for $S^n$ so that $\mathcal{M}^n_{loc} \neq \emptyset$ for all $n \geq 1$ under Hypothesis 2.4.

## 2.2 Trading and arbitrage in the asymptotic market

A priori, trading strategies are only defined for intermediate markets. When $n$ represents the number of time steps, a possible economic interpretation is that the frequency of observations and/or trading times is arbitrarily large. The asymptotic properties are then an approximation of trading in an intermediate market. On the other hand if a limit market can be well defined, $\mathbb{M}^n$ can then be thought as a numerical approximation of it, and the results below establish how fundamental prices can be consistently defined in both markets.

In order to prove the asset pricing theorem for asymp. markets, we first introduce the notion of asymp. arbitrage.

An admissible wealth process in the asymp. market is a sequence $(W^n)_{n\geq 1}$ of processes such that $W^n \in K_0^n(x^n)$, $n \geq 1$ for some sequence $(x^n)_{n\geq 1}$ of real numbers. The set of such wealth processes in the asymp. market is denoted $\mathcal{K}_0(x^n)$. We write $\mathcal{K}_0$ for admissible wealth processes that each start at 0.

In the definition of a free lunch with vanishing risk for an intermediary market, the parameter $n$ is fixed whereas the parameter $k$ goes to infinity. Another economically relevant definition of arbitrage is defined across markets: an admissible trading strategy in the asymp. market with zero cost in the limit $n \to \infty$ such that the associated wealth processes are nonnegative and the probability that the terminal value $W_T^n$ takes strictly positive values is strictly positive for $n$ large enough. The notion of asymp. arbitrage is due to Kabanov and Kramkov [17] (used in the context of large financial markets):

**Definition 2.5 (Kabanov and Kramkov [17])** *A sequence of wealth processes* $(W^n)$ *is an* asymptotic arbitrage of the first kind *(abbr. AA1$^0$) if*

*(a)* $W_t^n \geq 0$, *for all* $t \in \mathcal{T}_n$,

*(b)* $\lim_{n\to\infty} W_0^n = 0$, *and*

*(c)* $\lim_n \mathbf{P}_n(W_T^n \geq 1) > 0$.

*A sequence of wealth processes* $(W^n)$ *is an* asymptotic arbitrage of the second kind *(abbr. AA2$^0$) if*

*(a)* $W_t^n \leq 1$, *for all* $t \in \mathcal{T}_n$,

*(b)* $\lim_{n\to\infty} W_0^n > 0$, *and*

*(c)* $\lim_n \mathbf{P}_n(W_T^n \geq \epsilon) = 0$ *for all* $\epsilon > 0$.

In order to give it a more straightforward economic interpretation and motivate our definition of asymp. superreplication, we re-write this definition as follows:

**Definition 2.6** *A sequence of wealth processes* $(W^n)$ *satisfies AA1 if there exists* $\epsilon > 0$ *such that*

*(a)* $W_t^n \geq 0$, *for all* $t \in \mathcal{T}_n$, *and all* $n \geq 1$,

*(b)* $\lim_{n\to\infty} W_0^n = 0$, *and*

*(c)* $\lim_n \mathbf{P}_n(W_T^n \geq \epsilon) > \epsilon$.

*A sequence of wealth processes* $(W^n)$ *satisfies AA2 if there exists* $\epsilon, \alpha > 0$ *such that*

*(a)* $W_t^n \geq -\alpha$, *for all* $t \in \mathcal{T}_n$, *and all* $n \geq 1$,

*(b)* $\lim_{n\to\infty} W_0^n = 0$, *and*

*(c)* $\lim_n \mathbf{P}_n(W_T^n \geq \epsilon) = 1$.

In the above definitions, each intermediate wealth process need not be an arbitrage or an FLVR. The certainty of making positive arbitrage profits is only approached when $n$ is large. In Definition 2.6, we see that the terminal value of an arbitrage of the first kind is nonnegative and greater than a strictly positive constant with a positive probability limit, whereas the terminal value of an arbitrage of the second kind may be negative as long as the probability of such an event converges to 0.

**Proposition 2.7** *Definitions 2.5 and 2.6 are equivalent.*

*Proof:*

In the case of arbitrages of the first kind, it suffices to let $\epsilon = \lim \mathbf{P}_n(W_T^n \geq 1)$ and multiply $W^n$ by $\epsilon$ to go from AA$1^0$ to AA1. Conversely, dividing an arbitrage according to the second definition by the value of $\epsilon$ in $\lim \mathbf{P}_n(W_T^n \geq \epsilon)$ gives the converse.

On the other hand, assume $W^n$ satisfies AA$2^0$. Define $\tilde{W}^n = -W^n + \lim_{n\to\infty} W_0^n$. Then, $\tilde{W}^n \geq -(1 - \lim_{n\to\infty} W_0^n) \geq -\alpha$, when $\alpha = 1$. Furthermore, for $\epsilon_0 > 0$, $\lim_n \mathbf{P}_n(\tilde{W}_T^n \geq \epsilon_0) = \lim_n \mathbf{P}_n(W_T^n \leq -\epsilon_0 + \lim_n W_0^n)$. It suffices to take $0 < \epsilon_0 < \lim_n W_0^n$ to see that $\tilde{W}^n$ is an AA2. Conversely, if $W^n$ satisfies AA2 with $\lim_n \mathbf{P}_n(W_T^n > \epsilon_0) = 1$ for some $\epsilon_0 > 0$ and $W^n \geq -\alpha$ for some $\alpha > 0$, define $\tilde{W}^n = \frac{\epsilon_0 - W^n}{\epsilon_0 + \alpha}$. Then, $\tilde{W}^n \leq 1$ and $\lim_n \tilde{W}_0^n > 0$. Furthermore, for all $\epsilon > 0$,

$$\mathbf{P}_n(\tilde{W}_T^n \geq \epsilon) = \mathbf{P}_n(W_T^n \leq \epsilon_0 - (\epsilon_0 + \alpha)\epsilon) \leq \mathbf{P}_n(W_T^n \leq \epsilon_0) \to 0.$$

□

## 2.3 Asymptotic Superreplication

Consistent with the definition of asymp. arbitrage, we define the notion of superreplication in such a way that the certainty that a wealth process outperforms a payoff is approached asymptotically as $n \to \infty$.

**Definition 2.8** *An admissible non-negative wealth process $W = (W^n)$ in the asymp. market is said to* asymptotically strictly superreplicate *a payoff $H = (h^n) \in \mathcal{H}$ if there exists $\epsilon > 0$ such that one of the following two conditions holds:*

1. $\lim_{n\to\infty} \mathbf{P}_n \left(W_T^n \geq h^n + \epsilon\right) = 1$,
2. $W_T^n \geq h^n$ *for all* $n \geq 1$ *and* $\lim_{n\to\infty} \mathbf{P}_n \left(W_T^n \geq h^n + \epsilon\right) \geq \epsilon$.

*The wealth process $W$ is said to* asymptotically superreplicate *the payoff $H$ if one of the above two conditions holds for $\epsilon = 0$, in which case the second one is redundant (Condition 2 $\Rightarrow$ Condition 1).*

An admissible wealth process that satisfies AA1 or AA2 asymptotically strictly superreplicates the zero-payoff $h^n \equiv 0$.

Suppose a derivative claim, with asymp. payoff $(h^n)$ can be superreplicated by an admissible wealth process $(W^n)$. In order for this claim to have a no-asymp.-arbitrage price $h_0$, it should be the case that selling it and asymptotically superreplicating it is not an asymp. arbitrage. In other words, it must not be the case that

$$\lim_{n\to\infty} W_0^n - h_0 \leq 0.$$

The asymp. superreplication price of a payoff is therefore defined as the smallest amount of cash needed to asymptotically superreplicate it:

**Definition 2.9** *Let $H = (h^n) \in \mathcal{H}$. Its* asymptotic superreplication price *(abbrv. ASP) is defined as*

$$\pi(H) \quad = \quad \inf\{\limsup_n x^n : \mathcal{K}_0(x^n) \cap [H]_0 \neq \emptyset\},$$

*in which*

$$[H]_0 = \{(g_n) : g_n \in \mathcal{F}_T^n, g_n \geq 0, \text{ and } \lim_{n\to\infty} \mathbf{P}_n\left(g^n \geq h^n\right) = 1\}.$$

In the definition of asymp. superreplication, we allow to miss the target payoff on a set of small probability much in the spirit of quantile hedging, except that in this case the quantile probability (the probability of unsuccessful superreplication) converges to zero as $n$ goes to infinity. In the $n$-th market, the $\epsilon$-quantile hedging price of $h^n$ is defined as

$$\pi_\epsilon^n(h^n) = \inf\{x : \exists W \in K_0^n(x), W \geq 0, \text{ such that } \mathbf{P}_n(W_T^n \leq h^n) \leq \epsilon\}$$

for $\epsilon \geq 0$. (The quantity $\pi_0^n(h^n)$ is the superreplication price of $h^n$ in the $n$-th market.) The practical implication of Definition 4.2 is made clear in the following proposition, for which the proof can be found in the appendix.

**Proposition 2.10** *Let $H = (h^n) \in \mathcal{H}$. Assume that*

$$\limsup_n \pi_0^n(h^n) < \infty.$$

*Then,*

$$\pi(H) = \lim_{\epsilon\downarrow 0} \limsup_{n\to\infty} \pi_\epsilon^n(h^n).$$

From an economic point of view, one can find $\epsilon$ small enough and $n$ large enough so that $\epsilon$-quantile hedging price of $h^n$ and the ASP of $H$ are close quantities. We can then interpret $\pi_\epsilon^n(h^n)$ as an approximate cost of superreplicating $h^n$. From a pratical perspective, these considerations are relevant since the quantities $\limsup_n \pi_\epsilon^n(h^n)$, $\epsilon > 0$, are bounded away from $\limsup_n \pi_0^n(h^n)$ (the limit of the intermediate superreplication costs) in some cases, as we will see below.

We give throughout a number of examples and develop an asset pricing theory to support the economic significance of the definition of asymp. superreplication. However, we should first note that the ASP is a property of the sequence $(h^n)$ (and of its limit when it exists); it is not the price of every $h^n$. Indeed, the cost of superreplicating the payoff in every market is sometimes higher than the ASP. Furthermore, it is possible that a sequence of wealth processes $(W^n)$ does not superreplicate a payoff sequence $(h^n)$ in any of the intermediary markets in the usual sense but that it asymptotically superreplicates it. Consider for example the continuous price process of Section 2.2.1 of [5]:

$$S_0 = 1, \quad dS_t = \frac{S_t}{\sqrt{1-t}} dB_t, \quad t \leq 1$$

where $B$ is Brownian motion. The process $S$ is a true martingale on $[0, 1 - \frac{1}{n}]$ for any $n$, but it is a strict local martingale on $[0, 1]$ since it satisfies $S_1 = 0$, a.s. The replication price of $h^n := S_{1-1/n}$ for a fixed $n$ is 1, but the ASP of the sequence $(h^n)$ is 0 since $\lim_n \mathbf{P}\left(h^n \leq \epsilon\right) = 1$, for all $\epsilon > 0$. In particular, we conclude that almost all of the value of $S_{1-1/n}$ for fixed $n$ comes from a set of probability close to 0. Since $(h^n)$ is a proxy for $S_1$, the ASP equal to 0 is consistent with the fact that the replication price of $S_1$ is also 0 in the classical sense. From an economic point of view, one can replace $h^n$ by $S_1$ since they only differ by no more than $\epsilon$ on sets with probability converging to 0, as $n \to \infty$. If it were bounded by above, the sequence of processes $S^n := S$ on $[0, 1 - \frac{1}{n}]$ would therefore satisfy AA$2^0$, and the value 1 would not be an AA2-free price for $(S^n)$. Note however that this process is not bounded, therefore it is not an arbitrage: this process has an asymp. bubble since its ASP is strictly lower than its value at time 0. The notion of asymp. bubble is defined and further developed in Section 5.

Even when the NFLVR condition holds in each intermediary markets, there may exist an asymp. arbitrage. As shown in [17], the absence of asymp. arbitrage opportunities is related to the notion of contiguity of sequences of probability measures.

**Definition 2.11** *A sequence of measures $(\mathbf{Q}_n)_{n\geq 1}$ is contiguous to the sequence $(\mathbf{P}_n)_{n\geq 1}$, denoted $(\mathbf{Q}_n) \lhd (\mathbf{P}_n)$, if $\mathbf{P}_n(A_n) \to 0$ implies $\mathbf{Q}_n(A_n) \to 0$ as $n \to \infty$ for all sequences $A^n \in \mathcal{F}^n$. If both $(\mathbf{Q}_n) \lhd (\mathbf{P}_n)$ and $(\mathbf{P}_n) \lhd (\mathbf{Q}_n)$, then they are said to be mutually contiguous.*

Let $\mathbf{M}_{con}$ be the set of sequences $(\mathbf{Q}_n)_{n\geq 1}$, with $\mathbf{Q}_n \in \mathcal{M}^n_{loc}$, mutually contiguous with the sequence $(\mathbf{P}_n)$. Sequences in $\mathbf{M}_{con}$ play a major role in the asymp. asset pricing theorem.

**Hypothesis 2.12** $\mathbf{M}_{con}$ *is non-empty.*

Proposition 2 of [18] states that the existence of a sequence $\mathbf{Q}_n \in \mathcal{M}^n_{loc}$ to which $(\mathbf{P}_n)$ is contiguous is equivalent to the absence of AA1. On the other hand, Proposition 3 of [18] states that the existence of a sequence $\mathbf{Q}_n \in \mathcal{M}^n_{loc}$ contiguous to $(\mathbf{P}_n)$ implies there is no AA2. Note however here that the converse is not true in general as shown in [22], Example 2.3. The no-arbitrage condition that is equivalent to the existence of a sequence in $\mathbf{M}_{con}$ is the so-called no asymptotic free lunch (NAFL) and was provided by Klein [20] and [21]. The NAFL condition is presented in Section 4 when we present the incomplete market case.

The following example shows that the cost of asymptotically superreplicating the payoff can be strictly less than the limit of the superreplication prices in the case of incomplete intermediary markets.

**Example 2.13 (Multinomial Trees)** *Let $d(n) = 1$ and $T_n = 1$, $\mathcal{T}_n = \{i/n\}_{0\leq i\leq n}$ for all $n \geq 1$. Let $m \geq 1$ and $(R^n_{k/n})_{1\leq k\leq n}$ be a sequence of stock return processes defined by*

$$R^n_{k/n} = \frac{1}{\sqrt{n}}\sum_{i=1}^{k}(Y^n_i + \tfrac{1}{\sqrt{n}}\lambda), \quad R^n_0 = 0,$$

*in which $\lambda \in \mathbb{R}$, and for each $n \geq 1$, $(Y^n_i)_{i\leq n}$ are i.i.d. r.v. taking values in $\{y_1, y_2, ..., y_m\}$ such that $\mathbf{E}_n Y^n_1 = 0$, $\mathbf{E}_n (Y^n_1)^2 = 1$ and*

$$y_1 < y_2 < ... < y_m.$$

*We let $\mathbb{F}^n$ be the filtration generated by $R^n$ and assume a process $\sigma^n = (\sigma^n_{k/n})_{0\leq k\leq n}$ is adapted to this filtration. We then define the price process associated to $R^n$ as*

$$S^n_{i/n} = \prod_{1\leq k\leq i}\left(1 + \sigma^n_{(k-1)/n}\Delta R^n_{k/n}\right), \quad i \leq n, \tag{3}$$

*with $S^n_0 = 1$ and $\Delta R^n_{k/n} = R^n_{k/n} - R^n_{(k-1)/n}$ $(1 \leq k \leq n)$.*

*Let $p_i = \mathbf{P}_n(Y^n_1 = y_i)$ be independent of $n$, and assume $p_i > 0$ for all $i \leq m$. The price process $S^n$ satisfies NFLVR if and only if there exist $i, j \leq m$ such that*

$$y_i + \tfrac{1}{\sqrt{n}}\lambda < 0 < y_j + \tfrac{1}{\sqrt{n}}\lambda.$$

*Indeed, in that case there exist positive numbers $q^n_1, q^n_2, ..., q^n_m$ that add up to 1 and such that*

$$\sum_{i\leq m}(y_i + \tfrac{1}{\sqrt{n}}\lambda)q^n_i = 0, \tag{4}$$

*by the hyperplane separation theorem. An equivalent martingale measure $\mathbf{Q}_n$ is then given by the change of measure $Z^n_1 = \frac{d\mathbf{Q}_n}{d\mathbf{P}_n}$ with*

$$Z^n_{i/n} = \prod_{k=1}^{i}\sum_{i\leq m}\tfrac{q^n_i}{p_i}\mathbf{1}_{Y^n_k = y_i}.$$

*A specific solution of (4) for n large enough is* $q_i^n = (1 - \lambda \frac{1}{\sqrt{n}} y_i) p_i$. *In this case,*

$$Z_{i/n}^n = \prod_{k=1}^{i} (1 - \lambda \tfrac{1}{\sqrt{n}} Y_k^n),$$

*which converges weakly to the martingale* $Z = \mathcal{E}(-\lambda B)$ *(in which* $\mathcal{E}$ *denotes the Doléans-Dade exponential, and* $B$ *a Brownian motion). By Theorem 1 and Proposition 2 of [10],* $(\mathbf{Q}_n) \lhd \rhd (\mathbf{P}_n)$. *Hence,* $\mathbf{M}_{con}$ *is non-empty.*

*Assume now that* $\sigma^n$ *is a positive constant independent of* $n$, *and* $m \geq 3$. *Consider a call option with payoff equal to*

$$h^n = (S_1^n - K)_+.$$

*In this case, the ASP of the payoff* $(h^n)$ *is strictly less than the limit of its intermediary superreplication costs. Indeed, define* $\tilde{y}_i := y_i + \frac{1}{\sqrt{n}}\lambda$. *For fixed* $n$, *the fact that the payoff is a convex function of* $S_1^n$ *implies that the value* $\sup_{\mathbf{Q}_n \in \mathcal{M}_{loc}^n} \mathbf{E}_{\mathbf{Q}_n} h^n$ *is attained at the martingale measure* $\mathbf{Q}_n$ *defined by*

$$\mathbf{Q}_n(Y_1^n = y_i) = \begin{cases} \frac{\tilde{y}_m}{\tilde{y}_m - \tilde{y}_1}, & i = 1, \\ \frac{-\tilde{y}_1}{\tilde{y}_m - \tilde{y}_1}, & i = m; \\ 0, & \text{otherwise}, \end{cases}$$

*which is absolutely continuous w.r.t.* $\mathbf{P}_n$. *The superreplication cost of* $h^n$ *in market* $\mathbb{M}^n$ *is therefore*

$$\mathbf{E}_{\mathbf{Q}_n} \left( (1 + \sigma \tfrac{1}{\sqrt{n}} \tilde{y}_1)^I (1 + \sigma \tfrac{1}{\sqrt{n}} \tilde{y}_m)^{n-I} - K \right)_+ =: \breve{\pi}_n,$$

*in which* $I$ *is a binomial r.v. with parameter* $\frac{\tilde{y}_m}{\tilde{y}_m - \tilde{y}_1}$. *The quantity* $\breve{\pi}_n$ *converges to the Black-Scholes price with volatility parameter equal to* $\sigma\sqrt{-y_1 y_m}$, *which is the limit of the standard deviation of the r.v.* $\sigma Y_i$ *under* $\mathbf{Q}_n$.

*As shown in [8],* $S^n$ *(under* $\mathbf{P}_n$*) converges weakly to a geometric Brownian motion* $S$:

$$S_t = \exp\left( (\mu - \tfrac{1}{2}\sigma^2)t + \sigma B_t \right).$$

*Let* $\mathbf{Q}$ *be the unique EMM for this model. The processes* $S^n$ *satisfy the property*

$$(X^n, S^n) \Rightarrow (X, S) \text{ implies } (X^n, S^n, \int X_-^n dS^n) \Rightarrow (X, S, \int X_- dS),$$

*for any sequence of càdlàg* $S^n$*-integrable processes* $(X^n)$, *where* $\Rightarrow$ *denotes weak convergence. As a result, the trading strategy* $X_t^n = \Phi(d_1(S_{k/n}^n, T - kT/n))$, $t \in (kT/n, (k+1)T/n]$, *in which* $\Phi$ *is the normal cumulative distribution function and*

$$d_1(S, t) = \frac{\log(S/K) + 1/2\sigma^2 t}{\sigma\sqrt{t}},$$

*satisfies*

$$x_0 + \int_0^T X_t^n dS_t^n - h^n \Rightarrow 0$$

*where* $x_0 = \mathbf{E}_{\mathbf{Q}}(S_T - K)_+$, *the Black-Scholes price with volatility parameter equal to* $\sigma$. *In particular,* $\lim_{n\to\infty} \mathbf{P}_n(|x_0 + \int_0^T X_t^n dS_t^n - h^n| \leq \epsilon) = 1$ *for all* $\epsilon > 0$. *It is then evident that* $\pi(H) \leq x_0$. *Moreover, the quantity*

$$-y_1 y_m = y_1^2 \frac{y_m}{y_m - y_1} + y_m^2 \frac{-y_1}{y_m - y_1}$$

*is the maximum of* $\sum_i y_i^2 \alpha_i$ *over all* $(\alpha_i)$ *for which* $\sum_i \alpha_i = 1$ *and* $\sum_i y_i \alpha_i = 0$. *Since* $\sum_i y_i^2 p_i = 1$, *it is clear that* $-y_1 y_m$ *is strictly greater than 1 when* $m \geq 3$. *In particular,* $\pi(H) < \lim_n \breve{\pi}_n$ *in that case.*

# 3 Arbitrage Pricing Theorem : Complete Markets

The relation between local martingale measures and the superreplication price is well established in the classical theory (cf. [9], [23]). In this paper, we show that the ASP can be characterized in terms of mutually contiguous local martingale measures. We first consider the case of complete intermediary markets.

**Theorem 3.1** *Assume that $\mathcal{M}^n_{loc}$ consists of singletons for all $n \geq 1$ (i.e., $\mathbb{M}^n$ is complete), and let $\mathbf{Q}_n$ denote the unique equivalent local martingale measure for market $\mathbb{M}^n$. Let $H = (h^n) \in \mathcal{H}$. Then,*

$$\pi(H) = \inf_{(N_n)\to\infty} \limsup_{n\to\infty} \mathbf{E}_{\mathbf{Q}_n}\left(h^n \mathbf{1}_{\{h^n \leq N_n\}}\right), \tag{5}$$

*where the infimum is taken over all sequences $(N_n)$ of real numbers for which $\lim_{n\to\infty} N_n = \infty$.*
*Assume that $(W^n_t)_{t\in\mathcal{T}_n}$ is a $\mathbf{Q}_n$-local martingale for which $W^n_T = h^n$, for each $n$ and that satisfies*

$$\lim_{M\to\infty} \sup_n \mathbf{Q}_n\left(W^{n,*}_T > M\right) = 0, \tag{6}$$

$$\lim_{M\to\infty} \sup_n \mathbf{E}_{\mathbf{Q}_n}\left(\mathbf{1}_{\{\tau^M_n < T\}} |\Delta W^n_{\tau^M_n}|\right) = 0, \tag{7}$$

*in which $\tau^M_n := \inf\{t \in \mathcal{T}_n : W^n_t > M\}$. The ASP of $H$ can then also be represented as*

$$\pi(H) = \inf_{(N_n)\to\infty} \limsup_n \left(W^n_0 - N_n \mathbf{Q}_n\left(W^{n,*}_T \geq N_n\right)\right), \tag{8}$$

*where the infimum is taken over all sequences $N_n$ for which $\lim_{n\to\infty} N_n = \infty$.*

The proof of Theorem 3.1 is in Appendix A.

**Remark 3.2** *Condition 7 is based on a paper by Hulley and Ruf [13] who showed that the local martingale tail condition*

$$\lim_{M\to\infty} M\mathbf{Q}_n(W^{n,*}_T > M) > 0$$

*is also valid for strict local martingales $W^n$ that satisfy*

$$\lim_{M\to\infty} \mathbf{E}_{\mathbf{Q}_n}\left(\mathbf{1}_{\{\tau^M_n < T\}} |\Delta W^n_{\tau^M_n}|\right) = 0,$$

*for fixed $n$.*
*When the limit $\lim_n W^n_0$ exists, Equation 8 becomes*

$$\pi(H) = \lim_n W^n_0 - \sup_{(N_n)\to\infty} \liminf_n N_n \mathbf{Q}_n\left(W^{n,*}_T \geq N_n\right). \tag{9}$$

*Equation 9 is reminiscent of Equation 1 and states that the replication price is obtained by subtracting the risk-neutral value of extreme events given in terms of the maximum process associated to the local martingale $W^n$. If $\mathbb{M}^n$ is a complete market, an obvious $\mathbf{Q}_n$-local martingale for which $W^n_T = h^n$ is $W^n_t = \mathbf{E}_{\mathbf{Q}_n}\left(h^n|\mathcal{F}^n_t\right)$. In that case, the replication price of $h^n$ at time 0 in market $\mathbb{M}^n$ is $W^n_0 = \mathbf{E}_{\mathbf{Q}_n} h^n$. But even if $W^n$ is the market price process of $h^n$ and it is a true martingale in each market $\mathbb{M}^n$ (i.e., $W^n$ does not have a bubble), it is possible that the second term in (8) is non-zero so that the ASP is not the limit of $\mathbf{E}_{\mathbf{Q}_n} h^n$. Comparing this second term with (1), we see that the martingales $W^n$ may behave like strict local martingales when $n$ is large.*

**Example 3.3** *Let sequences $(x_n)$, $(y_n)$ and $(p_n)$ of positive real numbers satisfy*

$$0 < \lim_n x_n := x_0 < 1,$$
$$0 < p_n < 1, \ \ \textit{all } n \geq 1,$$
$$\lim_n y_n = \infty, \lim_n p_n = 0$$
$$x_n(1-p_n) + y_n p_n = 1.$$

*Fix $k \geq 1$, and for $i \leq k$ define*

$$S_i^n = S_{i-1}^n \left(x_n + (y_n - x_n)\delta_i^n\right)$$

*with $S_0^n = 1$ and $(\delta_i^n)_{i \leq k}$ a set of i.i.d. r.v. with Bernouilli distribution of parameter $p_n$. If $\mathbb{F}^n$ is the filtration generated by $S^n$, then $S^n$ is a martingale and the NFLVR and no asymp. arbitrage conditions are satisfied. In this case however, the payoff $(S_k^n)_{n \geq 1}$ can be asymptotically superreplicated for less than 1. By Theorem 3.1, the asymp. superreplication of $(S_k^n)_{n \geq 1}$ can be represented by a sequence $(M_n) \to \infty$ which minimizes*

$$\limsup_n \mathbf{E}_n(S_k^n \mathbf{1}_{\{S_k^n < M_n\}}).$$

*Indeed, $S_k^n = x_n^m y_n^{k-m}$ for some $0 \leq m \leq k$, so we only need to consider sequences of the form $M_n = x_n^m y_n^{k-m}$ for some $0 \leq m < k$. Therefore, a straightforward solution is $M_n = (x_n)^{k-1} y_n$ and*

$$\lim_n \mathbf{E}_n(S_k^n \mathbf{1}_{\{S_k^n < M_n\}}) = \lim_n \left[(x_n)^k \mathbf{P}_n \left(S_k^n = (x_n)^k\right)\right] = (x_0)^k < 1 = S_0^n$$

*because $\mathbf{P}_n \left(S_k^n = (x_n)^k\right) \to 1$.*

# 4 Pricing in incomplete market models

When an intermediate market is incomplete (i.e., the set $\mathcal{M}_{loc}^n$ consists of more than one measure), it is not possible to replicate all $\mathcal{F}_T^n$-measurable payoffs in $\mathbb{M}^n$. The superreplication price of $h^n$ is then given by $\sup_{\mathbf{Q}_n \in \mathcal{M}_{loc}^n} \mathbf{E}_{\mathbf{Q}_n} h^n$. (See [4] and [15]). This idea can be extended to asymp. markets by considering a pricing mechanism based on $\mathbf{M}_{con}$, as follows

$$V_0 = \sup_{(\mathbf{Q}_n) \in \mathbf{M}_{con}} \inf_{(g^n) \in [H]_0} \limsup_n \mathbf{E}_{\mathbf{Q}_n} (g^n).$$

In general, the set $\mathbf{M}_{con}$ does not contain all possible sequences $(\mathbf{Q}_n)$ with $\mathbf{Q}_n \in \mathcal{M}_{loc}^n$ (all $n \geq 1$). Furthermore NAA1 and NAA2 is not sufficient to imply that the set $\mathbf{M}_{con}$ is not empty as shown in Example 2.3 of [22].

In order to present the NAFL condition of Klein [21] which implies that $\mathbf{M}_{con}$ is not empty, we need the following definitions. We denote by $\mathcal{Y}$ the set of even convex functions $\phi : \mathbb{R} \to \mathbb{R}_+$ that satisfy $\phi(0) = \phi'(0) = 0$ and $\lim_{k \to \infty} \phi(k)/k = \infty$. Then, for $\phi \in \mathcal{Y}$, let

$$B^{\phi,n} = \{X \in L^1(d\mathbf{P}_n) : \mathbf{E}_n \phi(X) \leq 1\}.$$

**Definition 4.1** *The no asymptotic free lunch (NAFL) condition is violated if there exists $\epsilon > 0$ such that for all $\phi \in \mathcal{Y}$ there exists $n \geq 1$ for which*

$$C^n \cap (D^{\epsilon,n} - B^{\phi,n})) \neq \emptyset,$$

*in which*

$$D^{\epsilon,n} = \{X \in L^\infty(d\mathbf{P}_n) : X \geq 0, \mathbf{P}_n(X > \epsilon) > \epsilon\}.$$

Theorem 3.3 and Lemma 5.2 of [21] imply that NAFL is satisfied if and only if $\mathbf{M}_{con} \neq \emptyset$.

## 4.1 Economic relevance of NAFL

When the NAFL condition is violated there exists $\epsilon > 0$ such that for any $\phi \in \mathcal{Y}$ there is an $n$ and an admissible wealth process $W^n$ starting at zero that superreplicates some $(d^n - b^n) \in (D^{\epsilon,n} - B^{\phi,n})$ (we can assume $b^n \geq 0$). In other words, $W_T^n \geq d^n - b^n$ and

$$\mathbf{E}_n \phi(b^n) \leq 1. \tag{10}$$

Note that the lower bound of this admissible wealth process may depend on $n$, unlike in the case of AA1 or AA2. However, one can think of $b_n$ as the downside risk of this arbitrage strategy so that the condition (10) is interpreted as a bound on the downside risk with respect to $\phi \in \mathcal{Y}$. This downside risk condition is very stringent since the class of functions $\mathcal{Y}$ is very general and includes for instance $x \mapsto Cx^p$ for any $C, p > 0$. Furthermore, $m\mathcal{Y} \subset \mathcal{Y}$ for all $m > 0$ so the condition (10) can be replaced by

$$\mathbf{E}_n \phi(b^n) \leq \tfrac{1}{m}.$$

If $(W^n)$ is an AA1 or AA2, then NAFL is clealy violated. In fact in that case, there is $\epsilon > 0$ such that for any $n$ large enough, $W^n - W_0^n \in C^n$, $(W_T^n - W_0^n)\mathbf{1}_{\{W_T^n \geq \epsilon\}} \in D^{\epsilon,n}$ and

$$\mathbf{E}_n \phi(W_T^n \mathbf{1}_{\{W_T^n < \epsilon\}}) \leq \phi(\alpha) \mathbf{P}_n \left(W_T^n < \epsilon\right) \leq 1$$

for any $\phi \in \mathcal{Y}$ (where $\alpha$ is the constant that appears in the definition of AA2, or equals 0 for AA1). In other words, the process $W^n - W_0^n \in C^n$ does not depend on $\phi$.

One possible interpretation of the superreplication price is that it is the maximum price at which selling $H$ and superreplicating it does not lead to an asymp. arbitrage. In order to cover the incomplete market case, we extend our earlier definition of ASP to include bounds on the downside risk:

**Definition 4.2** *Let $H = (h^n) \in \mathcal{H}$. Its* asymptotic superreplication price with bounded downside risk *(ASP2) is defined as*

$$\pi_d(H) \quad = \quad \inf\{\limsup_n x^n : \forall \phi \in \mathcal{Y} \quad \mathcal{K}_0(x^n) \cap [H]_0^\phi \neq \emptyset\},$$

*in which*

$$[H]_0^\phi = \{(g_n) : \textit{ for all } n \geq 1 \ g_n \in \mathcal{F}_T^n, \ \mathbf{E}_n \phi((g_n)_-) \leq 1, \ \textit{and } \lim_{n\to\infty} \mathbf{P}_n \left(g^n \geq h^n\right) = 1\}.$$

The main theorem of this section is the following asset pricing theorem for asymp. markets.

**Theorem 4.3** *Let $H = (h^n) \in \mathcal{H}$. Then,*

$$\begin{aligned}
\pi_d(H) &= \sup_{(\mathbf{Q}_n) \in \mathbf{M}_{con}} \inf_{(g_n) \in [H]_0} \limsup_{n\to\infty} \mathbf{E}_{\mathbf{Q}_n}\left(g^n\right) \\
&= \sup_{(\mathbf{Q}_n) \in \mathbf{M}_{con}} \inf_{(N_n)\to\infty} \limsup_{n\to\infty} \mathbf{E}_{\mathbf{Q}_n}\left(h^n \mathbf{1}_{\{h^n \leq N_n\}}\right).
\end{aligned}$$

The notable fact of this result is that even though ASP2 is defined in terms of downwide risk functions $\phi \in \mathcal{Y}$, the value of $\pi_d$ can be calculated from $(g_n) \in [H]_0$ only. The proof of Theorem 4.3 can be found in Appendix A.

The following is an example of a large financial market based on the Black-Scholes-Merton model.

**Example 4.4 (A Large Financial Market)** *For $i \geq 1$, let $S_t^i$ denote the value of asset $i$ defined as*

$$S_t^i = S_0^i \exp\left((\mu^i - \tfrac{1}{2}(\sigma^i)^2 - \tfrac{1}{2}(\eta^i)^2)t + \eta^i B_t^0 + \sigma^i B_t^i\right),$$

*in which* $B^0, B^1, B^2, \ldots$ *are independent Brownian motions defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$, $\mu^i, \sigma^i, \eta^i, S_0^i > 0$, *for all* $i \geq 1$. *Define* $\theta^i = \frac{\mu^i}{\sigma^i}, i \geq 1$. *Consider the sub-filtrations* $\mathbb{F}^n$ *of* $\mathbb{F}$ *generated by* $B^0, B^1, ..., B^n$ *and* $\mathbf{P}_n$ *the measure* $\mathbf{P}$ *restricted to* $\mathcal{F}^n_{T_n}$. *A market index* $V^n$ *is defined by the weights* $(\alpha^i)_{i\geq n}$ *and constructed by investing a fixed proportion* $\alpha^i$ *of the value of the index in stock* $i$ $(i \leq n)$, *as follows:*

$$\frac{dV_t^n}{V_t^n} = \sum_{i\leq n} \alpha^i \frac{dS_t^i}{S_t^i}, \quad V_0^n = 1.$$

*Market* $\mathbb{M}^n$ *consists of assets* $S^1$ *to* $S^n$. *Market* $\mathbb{M}^n$ *is incomplete due to the presence of* $B^0$. *One equivalent martingale measure is given by the change of measure*

$$\frac{d\mathbf{Q}_n}{d\mathbf{P}_n} = \exp(-\tfrac{1}{2}\|\vec{\theta}^n\|^2 T_n - \langle \vec{\theta}^n, \vec{B}^n_{T_n}\rangle)$$

*in which* $\vec{\theta}^n$ *and* $\vec{B}^n_T$ *are the vectors* $(\theta^i)_{1\leq i\leq n}$ *and* $(B^i_T)_{1\leq i\leq n}$. *Under* $\mathbf{Q}_n$, $Z^i := B^i + \int \theta^i dt$ *is a Brownian motion,* $i \leq n$. *The market index can then be written*

$$V_t^n = \exp\left(-\tfrac{1}{2}\|\Lambda^n\|^2 t - \tfrac{1}{2}(\Upsilon^n)^2 t + \langle \Lambda^n, \vec{Z}_t^n\rangle + \Upsilon^n B_t^0\right),$$

*where* $\Lambda^n = (\alpha^i\sigma^i)_{i\leq n}$ *and* $\Upsilon^n = \sum_{i\leq n} \alpha^i \eta^i$.

*Assume* $\lim_n \|\vec{\theta}^n\|^2$ *exits. By adapting the proof of Proposition 8 of [18], one can show that* $(\mathbf{P}_n) \triangleleft (\mathbf{Q}_n)$. *Since*

$$\frac{d\mathbf{P}_n}{d\mathbf{Q}_n} = \exp(-\tfrac{1}{2}\|\vec{\theta}^n\|^2 T_n + \langle \vec{\theta}^n, \vec{Z}^n_{T_n}\rangle),$$

*a similar argument will give the relation* $(\mathbf{Q}_n) \triangleleft (\mathbf{P}_n)$.

*Let now* $(\mathbf{Q}_n)$ *be an arbitrary sequence in* $\mathbf{M}_{con}$ *and consider the probability measures* $\mathbf{R}_n$ *defined in terms of* $V^n_{T_n}$ *by*

$$\mathbf{R}_n(A) := \mathbf{E}_{\mathbf{Q}_n} V^n_{T_n} \mathbf{1}_A, \quad A \in \mathcal{F}^n_{T_n}.$$

*(Recall that* $V_0^n = 1$.*) Under* $\mathbf{R}_n$, $V^n_{T_n}$ *has a log-normal distribution and*

$$\mathbf{R}_n(V^n_{T_n} > M_n) = \Phi\left(\frac{-\log(M_n) + \frac{1}{2}\xi_n^2}{\xi_n}\right),$$

*in which* $\xi_n^2 = (\|\Lambda^n\|^2 + (\Upsilon^n)^2)T_n$. *If* $\xi_n \to \infty$, *we find that* $\limsup \mathbf{R}_n(V^n_{T_n} > M_n) = 1$ *by choosing the sequence* $M_n = e^{\frac{1}{4}\xi_n^2}$. *In this case,* $\mathbf{Q}_n\left(V^n_{T_n} > M_n\right) = \Phi\left(\frac{-\log(M_n) - \frac{1}{2}\xi_n^2}{\xi_n}\right) \to 0$, *hence*

$$\pi_d(V^n_{T_n}) = \sup_{(\mathbf{Q}_n)\in\mathbf{M}_{con}} \inf_{(N_n)\to\infty} \limsup_{n\to\infty} \mathbf{R}_n(V^n_{T_n} \leq M_n) = 0.$$

*On the other hand, if* $\lim_n \xi_n$ *exists and is finite, the limit of* $\Phi\left(\frac{-\log(M_n)+\frac{1}{2}\xi_n^2}{\xi_n}\right)$ *is 0 for all sequences* $M_n \to \infty$. *Hence,* $\pi_d(V^n_{T_n}) = 1$.

*It is also possible to calculate the ASP2 more directly by using the fact that the distribution of the maximum process* $V^{n,*}_{T_n}$ *is*

$$\mathbf{Q}_n(V^{n,*}_{T_n} > N_n) = \Phi\left(\frac{-\log(N_n) - \frac{1}{2}\xi_n^2}{\xi_n}\right) + \frac{1}{N_n}\Phi\left(\frac{-\log(N_n) + \frac{1}{2}\xi_n^2}{\xi_n}\right).$$

*If the limit of* $\xi_n$ *exists and is finite, then* $N_n\mathbf{Q}_n(V^{n,*}_{T_n} > N_n)$ *converges to 0 for all sequences* $N_n$ *converging to* $\infty$. *On the other hand, if* $\xi_n \to \infty$, *then* $N_n\mathbf{Q}_n(V^{n,*}_{T_n} > N_n)$ *converges to 1 for* $N_n := e^{1/4\xi_n^2}$.

## 5 Bubbles in asset prices

For a given payoff $h^n$, Jarrow et al. [15] interpret the quantity $\mathbf{E}_{\mathbf{Q}_n} h^n$ as the $\mathbf{Q}_n$-fundamental value of $h^n$ when $\mathbf{Q}_n$ is an equivalent local martingale measure. The interpretation is that among all possible arbitrage-free prices in the interval

$$\left( \inf_{\mathbf{Q}_n \in \mathcal{M}^n_{loc}} \mathbf{E}_{\mathbf{Q}_n} h^n, \sup_{\mathbf{Q}_n \in \mathcal{M}^n_{loc}} \mathbf{E}_{\mathbf{Q}_n} h^n \right)$$

the market has "selected" $\mathbf{Q}_n$ to value $h^n$ and all other financial claims. As demonstrated in Biagini et al. [4], the benefit of this approach is that the pricing measure can change over time, giving rise to new bubbles.

Theorem 4.3 gives us a possible interpretation of the quantity $\inf_{(g^n) \in [H]_0} \limsup_n \mathbf{E}_{\mathbf{Q}_n}(g^n)$ as the asympt. fundamental value of $H$ as long as $(\mathbf{Q}_n)$ is a contiguous sequence, since otherwise this value may be higher than the ASP2, which is an upper bound of the asymp. fundamental value. The definition of asymp. fundamental value is therefore as follows:

**Definition 5.1** *For a sequence $(\mathbf{Q}_n) \in \mathbf{M}_{con}$, we define the $(\mathbf{Q}_n)$-fundamental value of a payoff $H = (h^n)$ as*

$$\pi_{(\mathbf{Q}_n)}(H) = \inf_{(g^n) \in [H]_0} \limsup_n \mathbf{E}_{\mathbf{Q}_n} g^n.$$

Consider a traded asset with market price processes denoted $H^n$, for which $\lim_n H^n_0$ exists and is finite and positive. The asset is said to be a $\mathbf{Q}_n$-bubble in market $\mathbb{M}^n$ when its $\mathbf{Q}_n$-fundamental value, i.e. $\mathbf{E}_{\mathbf{Q}_n}(H^n_T)$, is strictly lower than its market price $H^n_0$. The asset has a $\mathbf{Q}_n$-bubble in a given market $n$ if $H^n$ is a strict local martingale under $\mathbf{Q}_n$.

On the other hand, even if $H^n_0 = \mathbf{E}_{\mathbf{Q}_n}(H^n_T)$ for all $n \geq 1$, the $(\mathbf{Q}_n)$-fundamental value of the payoff $(H^n_T)_n$ can be strictly less than its asymp. market price $\lim_n H^n_0$. In this case, the asset is said to have an asymp. $(\mathbf{Q}_n)$-bubble:

**Definition 5.2** *For a given sequence $(\mathbf{Q}_n) \in \mathbf{M}_{con}$, the asset with market price process $H^n$ in $\mathbb{M}^n$ $(n \geq 1)$ has an asymp. $(\mathbf{Q}_n)$-bubble if*

$$\lim_n H^n_0 > \pi_{(\mathbf{Q}_n)}(H^n_T).$$

By Equation 9, the size of a bubble, i.e. the difference between the market price and its fundamental value, is given by

$$\sup_{(N_n) \to \infty} \liminf_{n \to \infty} \mathbf{E}_{\mathbf{Q}_n} \left( H^n_T \mathbf{1}_{\{H^n_T > N_n\}} \right).$$

The processes $S^n$ of Example 3.3 have an asymp. bubble, whereas the processes $V^n$ of Example 4.4 have an asymp. bubble when $\xi_n \to \infty$.

A bubble may exist for some martingale measures sequences in $\mathbf{M}_{con}$ but not for others, as in this next example:

**Example 5.3** *Extending Example 3.3, consider sequences $(x_n)$, $(y_n)$, $(z_n)$ and $(p_n)$ of positive real numbers satisfying*

$$\begin{aligned} 0 < x_n < 1 < z_n, \ \textit{all } n \geq 1, \\ 0 < \lim_n x_n := x_0 < 1 < \lim_n z_n := z_0, \\ 0 < p_n < 1, \ \textit{all } n \geq 1, \\ \frac{x_0 + z_0}{2} < 1, \\ \lim_n y_n = \infty, \lim_n p_n = 0 \\ x_n(1 - p_n)/2 + z_n(1 - p_n)/2 + y_n p_n = 1. \end{aligned}$$

*Fix $k \geq 1$, and for $n \geq 1$ define*

$$S_i^n = S_{i-1}^n R_i^n, i \leq k$$

*with $S_0^n = 1$ and $(R_i^n)_{i \leq k}$ a set of i.i.d. r.v. with distribution given by*

$$\mathbf{P}_n(R_i^n = \omega) = \begin{cases} p_n, & \text{if } \omega = y_n; \\ \frac{1-p_n}{2}, & \text{if } \omega = x_n; \\ \frac{1-p_n}{2}, & \text{if } \omega = z_n. \end{cases}$$

*If $\mathbb{F}^n$ is the filtration generated by $S^n$, then $S^n$ is a $\mathbf{P}_n$-martingale. As before, we can show that its asympt. $(\mathbf{P}_n)$-fundamental value given by*

$$\inf_{(g^n) \in [S_k^n]_0} \lim_n \mathbf{E}_{\mathbf{P}_n} g^n$$

*is equal to $\frac{x_0+z_0}{2}$ by considering $g^n = S_k^n \mathbf{1}_{\{S_k^n < y_n\}}$. On the other hand, consider the martingale measure $\mathbf{Q}_n$ defined as*

$$\mathbf{Q}_n(R_i^n = \omega) = \begin{cases} (1/y_n)^2, & \text{if } \omega = y_n; \\ \frac{z_n(1-(1/y_n)^2)-(1-1/y_n)}{z_n - x_n}, & \text{if } \omega = x_n; \\ \frac{1-1/y_n - x_n(1-(1/y_n)^2)}{z_n-x_n}, & \text{if } \omega = z_n. \end{cases}$$

*We clearly have that $(\mathbf{P}_n) \lhd \rhd (\mathbf{Q}_n)$. The difference between these two sequences of measures however is that* $\lim_n y_n \mathbf{Q}_n(R_i^n = y_n) = 0$ *whereas* $\lim_n y_n \mathbf{P}_n(R_i^n = y_n) > 0$. *Therefore,*

$$\inf_{(g^n) \in [S_k^n]_0} \lim_n \mathbf{E}_{\mathbf{Q}_n} g^n = S_0^n = 1.$$

For $(\mathbf{Q}_n) \in \mathbf{M}_{con}$ and a price process $H^n$ which is a $\mathbf{Q}_n$-martingale, we define the measure $\rho[\mathbf{Q}_n, H^n]$ by setting

$$\rho[\mathbf{Q}_n, H^n](A_n) = \frac{1}{H_0^n} \mathbf{E}_{\mathbf{Q}_n} \left( H_T^n \mathbf{1}_{A_n} \right),$$

for $A_n \in \mathcal{F}_T^n$.

The following result highlights the relation between contiguity and asymp. bubbles.

**Proposition 5.4** *A sequence of price processes $(H^n)$ does not have an asymp. $(\mathbf{Q}_n)$-bubble if and only if $(\rho[\mathbf{Q}_n, H^n]) \lhd (\mathbf{Q}_n)$. In particular, $(H^n)$ does not have an asymp. $(\mathbf{Q}_n)$-bubble if and only if*

$$\lim_{M \to \infty} \sup_n \mathbf{E}_{\mathbf{Q}_n} \left( H_T^n \mathbf{1}_{\{H_T^n > M\}} \right) = 0.$$

*Proof:* The first result readily follows from the definition of $\rho[\mathbf{Q}_n, H^n]$ and the fact that

$$\pi_{(\mathbf{Q}_n)} = \inf_{(A_n): \mathbf{Q}_n(A_n) \to 1} \limsup_n H_0^n \rho[\mathbf{Q}_n, H^n](A_n).$$

The proof of this relation is similiar to the proof of Lemma A.3.

Proposition 1 of [10] states that $(\rho[\mathbf{Q}_n, H^n]) \lhd (\mathbf{Q}_n)$ if and only if $\left( \frac{d\rho[\mathbf{Q}_n, H^n]}{d\mathbf{Q}_n} \right)_n$ is uniformly integrable with respect to $\mathbf{Q}_n$. In other words,

$$\lim_{M \to \infty} \sup_n \tfrac{1}{H_0^n} \mathbf{E}_{\mathbf{Q}_n} \left( H_T^n \mathbf{1}_{\{H_T^n > M H_0^n\}} \right) = 0$$

since

$$\frac{H_T^n}{H_0^n} = \frac{d\rho[\mathbf{Q}_n, H^n]}{d\mathbf{Q}_n}.$$

The second statement readily follows from the fact that $\lim_n H_0^n$ exists and is positive. $\square$

The following gives a simple condition to rule out asymp. bubbles.

**Proposition 5.5** *A sequence of price processes $(H^n)$ does not have an asymp. $(\mathbf{Q}_n)$-bubble if* $\sup_n \mathbf{E}_{\mathbf{Q}_n}(|H_T^n|^p) < \infty$ *for some $p > 1$.*

*Proof:* Suppose $\sup_n \mathbf{E}_{\mathbf{Q}_n}(|H_T^n|^p) = C^p < \infty$. Let $A_n \in \mathcal{F}_T^n$, $n \geq 1$, such that $\mathbf{Q}_n(A_n) \to 0$. Then,

$$\rho[\mathbf{Q}_n, H^n](A_n) \quad = \quad \frac{1}{H_0^n}\mathbf{E}_{\mathbf{Q}_n}\left(H_T^n \mathbf{1}_{A_n}\right) \leq \frac{1}{H_0^n}\left(\mathbf{E}_{\mathbf{Q}_n}\left(|H_T^n|^p\right)\right)^{1/p}\left(\mathbf{Q}_n\left(A_n\right)\right)^{1/q} \leq \frac{C}{H_0^n}\left(\mathbf{Q}_n\left(A_n\right)\right)^{1/q} \to 0$$

as $n \to \infty$, where $1/p + 1/q = 1$. Hence, $(\rho[\mathbf{Q}_n, H^n]) \lhd (\mathbf{Q}_n)$, since $\lim_n H_0^n > 0$. The result follows from Proposition 5.4. □

The processes $S^n$ of Example 2.13 do not have an asymp. bubble when $\sigma$ is a constant since $\sup_n \mathbf{E}_{\mathbf{Q}_n}\left((S_T^n)^2\right) < \infty$ in this case.

**Example 5.6** *The odd-even model of Example 4.3 of [12] sets $S_t^n = S_0 \exp\left(\sum_{k=1}^{[nt]} \xi_k^n\right)$, in which*

$$\begin{aligned}
\mathbf{P}_n\left(\xi_{2k-1}^n = \sigma/\sqrt{n}\right) &= p^n \\
\mathbf{P}_n\left(\xi_{2k-1}^n = -\sigma/\sqrt{n}\right) &= 1 - p^n \\
\mathbf{P}_n\left(\xi_{2k}^n = \log(n)\right) &= q^n \\
\mathbf{P}_n\left(\xi_{2k}^n = -a/n\right) &= 1 - q^n
\end{aligned}$$

*for $k \geq 1$, $0 < p^n, q^n < 1$, $\sigma, a > 0$. The probabilities $p^n$ and $q^n$ are chosen so that $S^n$ is a $\mathbf{P}_n$-martingale. Consequently, $\mathbf{P}_n$ is the unique martingale measure for $S^n$.*

*The processes $S^n$ have an asymp. bubble. Indeed, the payoff*

$$h^n := S_0 \exp\left(-a/2T + \sum_{k=1}^{[n/2]} \xi_{2k-1}^n\right)$$

*has a replication price of $S_0 e^{-a/2T}$ in every intermediary market. Furthermore, the probability that $S_T^n$ exceeds $h^n$ is less than $\frac{n}{2}q^n$ which converges to 0 as $n \to \infty$ since*

$$\begin{aligned}
\tfrac{n}{2}q^n &= \frac{n - ne^{-a/n}}{2n - 2e^{-a/n}} \\
&= \frac{1 - e^{-a/n}}{2 - \frac{2}{n}e^{-a/n}} = O(1/n).
\end{aligned}$$

*From this, we conclude that $\pi(S_T^n) \leq S_0 \exp(-a/2T)$.*

*To show that $\pi(h^n) = S_0 \exp(-a/2T)$, we use Proposition 5.5 and the following bound on the second moment of $h^n$:*

$$\begin{aligned}
\mathbf{E}_n\left(S_0^2 \exp\left(-aT + 2\sum_{k=1}^{[n/2]} \xi_{2k-1}^n\right)\right) &= S_0^2 \exp(-aT)\left(\mathbf{E}_n e^{2\xi_1^n}\right)^{[n/2]} \\
&= S_0^2 \exp(-aT)\left(e^{2\sigma/\sqrt{n}} p_n + e^{-2\sigma/\sqrt{n}}(1 - p_n)\right)^{[n/2]} \\
&\to S_0^2 \exp(-aT + \sigma^2) \text{ as } n \to \infty.
\end{aligned}$$

*Since $h^n \leq S_T^n$, we obviously have $\pi(h^n) \leq \pi(S_T^n)$, hence it follows that $\pi(S_T^n) = S_0 \exp(-a/2T)$.*

# 6 Weak Convergence

Consider a sequence of weakly converging price processes $(S^n)_n$. When the limit defines a financial market with an ELMM, a natural question is whether an asympt. bubble for the sequence implies a bubble in the limit market. To answer this question, we begin with the following proposition which shows that our definition of asympt. fundamental value is consistent with pricing in a weakly converging market.

**Proposition 6.1** *Let $(\mathbf{Q}_n) \in \mathbf{M}_{con}, H = (h^n) \in \mathcal{H}$. Let $g$ be a r.v. defined on a probability space $(\Omega, \mathcal{F}, \mathbf{Q})$ for which $h^n \Rightarrow g$, and $g \in L^1(d\mathbf{Q})$. Then,*

$$\pi_{(\mathbf{Q}_n)}(H) = \inf_{(g^n)\in[H]_0} \limsup_n \mathbf{E}_{\mathbf{Q}_n} g^n = \mathbf{E}_{\mathbf{Q}} g.$$

*Proof:*

Let $(N_k)_{k\geq 1}$ be a sequence of positive reals diverging to infinity for which all $N_k$ are continuity points of the distribution of $g$ under $\mathbf{Q}$. For $n \geq 1$, $M \in \mathbb{R}_+$, define $p_{n,M} = \mathbf{E}_{\mathbf{Q}_n}\left(h^n 1_{\{h^n \leq M\}}\right)$, $p_M = \mathbf{E}_{\mathbf{Q}}\left(g 1_{\{g\leq M\}}\right)$ and $p = \mathbf{E}_{\mathbf{Q}} g$. By Lemma A.3, $\pi_{(\mathbf{Q}_n)}(H) \leq \limsup_n p_{n,N_{k_n}}$, for any sequence of integers $(k_n)$ diverging to infinity.

For all $k \geq 1$, $p_{n,N_k} \to p_{N_k}$, by weak convergence. Furthermore, $p_{N_k} \to p$ as $k \to \infty$ by dominated convergence. We can therefore apply Lemma A.1, and find $(k_n) \to \infty$ for which $p_{n,N_{k_n}} \to p$ as $n \to \infty$. From this we conclude that $\pi_{(\mathbf{Q}_n)}(H) \leq \mathbf{E}_{\mathbf{Q}} g$.

To prove the inverse inequality, let $\epsilon > 0$ and $(N_n)$ a sequence diverging to infinity (not necessarily continuity points). Take $M_\epsilon$ a continuity point of the distribution of $g$, large enough that $p_{M_\epsilon} \geq p - \epsilon$. For $n$ large enough, $\mathbf{E}_{\mathbf{Q}_n}\left(h^n \mathbf{1}_{\{h^n\leq M_\epsilon\}}\right) \geq \mathbf{E}_{\mathbf{Q}}\left(g \mathbf{1}_{\{g\leq M_\epsilon\}}\right) - \epsilon$ and

$$\mathbf{E}_{\mathbf{Q}_n}\left(h^n \mathbf{1}_{\{h^n\leq N_n\}}\right) \geq \mathbf{E}_{\mathbf{Q}_n}\left(h^n \mathbf{1}_{\{h^n\leq M_\epsilon\}}\right).$$

It follows from these inequalities that

$$p_{n,N_n} \quad \geq \quad p - 2\epsilon$$

for $n$ large enough. Hence, $\inf_{(N_n)\to\infty} \lim_n p_{n,N_n} \geq p$. □

When a sequence of price processes $(H^n)_n$ under $(\mathbf{Q}_n)$ converges weakly to a limit price process $G$ under some measure $\mathbf{Q}$), the size of the asymp. $(\mathbf{Q}_n)$-bubble of $(H^n)$ is simply

$$\lim_n H_0^n - \pi_{(\mathbf{Q}_n)}(H) = G_0 - \mathbf{E}_{\mathbf{Q}} G_T,$$

by the preceding proposition. A corollary of the preceding proposition is therefore that the asymp. $(\mathbf{Q}_n)$-fundamental value of an asset and the size of its asymp. $(\mathbf{Q}_n)$-bubble is equal to its $\mathbf{Q}$-fundamental value and $\mathbf{Q}$-bubble in the weak-limit market, when $\mathbf{Q}$ is an equivalent local martingale measure of the limit market. The definition is thus stable under weak limits, and gives an alternative to the classical definition of fundamental value, i.e. $\mathbf{E}_{\mathbf{Q}_n}(h^n)$, which does not always converge to the expectation of the weak limit, $\mathbf{E}_{\mathbf{Q}}(g)$. Note however that $\mathbf{Q}$ is not always the ELMM of the limit market, as in Example 6.2 below, so that an asset may have an asymp. bubble (even in the case of intermediary complete markets) but no bubble in the limit market.

**Example 6.2** *As noted in [12], the processes $S^n$ under $\mathbf{P}_n$ $(n \geq 1)$ of Example 5.6 converge weakly to a process $S$ of the form $S_t = S_0 \exp\left(-\frac{1}{4}\sigma^2 t - \frac{1}{2}at + \sqrt{\frac{\sigma^2}{2}} B_t\right)$, with $B$ a $\mathbf{P}$-Brownian motion. Therefore, $\pi(S^n) = \mathbf{E}_{\mathbf{P}} S_T = S_0 \exp\left(-\frac{1}{2}aT\right)$, so that the limiting measure $\mathbf{P}$ is not "useless for the purpose of option pricing" as stated in [12], even though it is not the EMM of $S$. The measure $\mathbf{P}$ can be used to calculate*

*ASPs. For instance, if an option $H$ has a payoff function $h(S^n_T)$ at time $T$ in market $\mathbb{M}^n$, in which $h$ is a continuous function, then its ASP is $\pi(H) = \mathbf{E}_{\mathbf{P}} h(S_T)$ (since $h^n$ under $\mathbf{P}_n$ converges weakly to $h(S_T)$ under $\mathbf{P}$), as opposed to $\mathbf{E}_{\mathbf{Q}} h(S_T)$ where $\mathbf{Q}$ is the EMM of $S$ defined by*

$$\frac{d\mathbf{Q}}{d\mathbf{P}} = \mathcal{E}\left(\tfrac{a}{\sqrt{2}\sigma} B\right).$$

The previous example highlights the fact that the ASP and the weak limit of a contiguous sequence $(\mathbf{Q}_n)$ are properties of the sequence $\mathbb{M}^n$ but not necessarily that of the limit itself. For example, one can easily construct another sequence of complete markets with no asymp. bubble for which the sequence of EMMs $(\mathbf{Q}_n)$ converges to the EMM $\mathbf{Q}$ of the limit process

$$S_t = S_0 \exp\left(-\tfrac{1}{4}\sigma^2 t - \tfrac{1}{2} a t + \sqrt{\tfrac{\sigma^2}{2}} B_t\right)$$

of the previous example.

**Example 6.3** *In Example 2.13, $(R^n|\mathbf{Q}_n) \Rightarrow (B|\mathbf{Q})$ for any $(\mathbf{Q}_n) \in \mathbf{M}_{con}$, with $B$ a Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{Q})$. Assume that there exists an $\mathbb{F}$-adapted càdlàg process $\breve{\sigma}$ such that $(\sigma^n, R^n|\mathbf{Q}_n) \Rightarrow (\breve{\sigma}, B|\mathbf{Q})$. Then, $S^n \Rightarrow S = 1 + \int \breve{\sigma} S dB = \mathcal{E}(\int \breve{\sigma} dB)$, by Theorem 4.4 of [8]. Then $\pi_{(\mathbf{Q}_n)}(S^n_1) = \mathbf{E}_{\mathbf{Q}} S_1$, if $\mathbf{E}_{\mathbf{Q}} S_1 < \infty$. For instance, the processes $S^n$ defined as*

$$S^n_{i/n} = \prod_{k=1}^{i} \left(1 + \tilde{\sigma}_0 S^n_{(k-1)/n} Y^n_k\right), \quad i \leq n, S^n_0 = 1,$$

*with $\tilde{\sigma}_0$ a positive constant therefore have an asymp. $(\mathbf{Q}_n)$-bubble since the solution of $dS_t = \tilde{\sigma}_0 S_t^2 dB_t$ satisfies $\mathbf{E}_{\mathbf{Q}} S_1 < 1$.*

**Example 6.4 (GARCH Models)** *For each fixed $n$, let $\lambda, \alpha^n_0, \alpha^n_1, c^n, \beta^n_1 \in \mathbb{R}$ and consider the following NGARCH model under $\mathbf{P}_n$:*

$$\begin{aligned} R^n_{k/n} &= \frac{1}{n}\lambda\sqrt{v^n_{k/n}} - \frac{1}{2n} v^n_{k/n} + \sqrt{\tfrac{1}{n} v^n_{k/n}}\,\epsilon_k \\ v^n_{k/n} &= \alpha^n_0 + \alpha^n_1 v^n_{(k-1)/n} \left(\epsilon_{k-1} - c^n\right)^2 + \beta^n_1 v^n_{(k-1)/n}, \end{aligned} \tag{11}$$

*with $(\epsilon_k)_{k\geq 1}$ independent and normally distributed with zero mean and unit variance. We let $S^n_0 = 1$ and define inductively $S^n_{k/n} = S^n_{(k-1)/n} \exp(R^n_{k/n})$, $k \leq n$. One can consider the equivalent martingale measure $\mathbf{Q}_n$ defined by*

$$Z^n := \frac{d\mathbf{Q}_n}{d\mathbf{P}_n} = \exp\left(-\frac{1}{2}\lambda^2 - \sum_{k\leq n} \lambda \sqrt{\tfrac{1}{n}}\epsilon_k\right).$$

*Under $\mathbf{Q}_n$, $\varepsilon_k := \epsilon_k + \lambda\sqrt{\frac{1}{n}}$ is normally distributed with mean 0 and unit variance. Furthermore, $(Z^n|\mathbf{P}_n)$ converges weakly to $(\mathcal{E}(-\lambda B)|\mathbf{P})$, and $(\frac{1}{Z^n}|\mathbf{Q}_n)$ converges weakly to $(\mathcal{E}(\lambda B)|\mathbf{P})$, where $B$ is a Brownian motion under $\mathbf{P}$. Therefore, $\mathbf{M}_{con}$ is non-empty.*

*Following [7], we consider the following parametric limit:*

$$\begin{aligned} \lim_n \alpha^n_0 n &= 0, \\ \lim_n \alpha^n_1 \sqrt{n} &= \xi > 0, \\ \lim_n c^n &= c \in \mathbb{R}, \\ \lim_n n\left(\alpha^n_1(1 + (c^n)^2) + \beta^n_1 - 1\right) &= \eta. \end{aligned} \tag{12}$$

*Let $Y^n_{k/n} = \log(S^n_{k/n})$, $k \geq 1$. The weak limit of the pair of processes $(Y^n, v^n)$ is given by the solution $(Y, v)$ of*

$$
\begin{aligned}
dY_t &= (\lambda\sqrt{v_t} - \frac{1}{2}v_t)dt + \sqrt{v_t}dB^1_t &\quad (13)\\
dv_t &= \eta v_t dt - 2\xi c v_t dB^1_t + \sqrt{2}\xi v_t dB^2_t, &\quad (14)
\end{aligned}
$$

*in which $B^1$ and $B^2$ are independent Brownian motions.*

*The price process $S^n$ converges weakly to the process given by $S = exp(Y)$. Sin [28] showed that $S$ is a strict local martingale if and only if $c < 0$. The NGARCH model given by (11) therefore has an asymp. $(\mathbf{Q}_n)$-bubble if and only if $c < 0$ by Proposition 6.1.*

*More generally, assume that the $\epsilon_k$ have a finite cumulant generating function $\kappa$ under some EMM $\mathbf{Q}_n$, and finite first four moments: $\mu_i = \mathbf{E}_{\mathbf{Q}_n}\epsilon^i_k$, $i \leq 4$. Suppose that the dynamics of $S^n$ under $\mathbf{Q}_n$ are given by*

$$
\begin{aligned}
R^n_{k/n} &= -\kappa\left(\sqrt{\tfrac{1}{n}v^n_{k/n}}\right) + \sqrt{\tfrac{1}{n}v^n_{k/n}}\epsilon_k\\
v^n_{k/n} &= \bar{\alpha}^n_0 + \bar{\alpha}^n_1 v^n_{(k-1)/n}\left(\epsilon_{k-1} - \bar{c}^n\right)^2 + \bar{\beta}^n_1 v^n_{(k-1)/n},
\end{aligned}
$$

*with $\bar{\alpha}^n_0, \bar{\alpha}^n_1, \bar{c}^n, \bar{\beta}^n_1 \in \mathbb{R}$. A similar risk-neutral modeling approach was proposed by Barone-Adesi et al. [2]. By Proposition 3.3 of [1], $(v^n, S^n)$ converges weakly to the solution $(v, S)$ of*

$$
\begin{aligned}
dS_t &= -\tfrac{1}{2}v_t S_t dt + \sqrt{v_t}S_t dB^1_t\\
dv_t &= \bar{\eta}v_t dt + (\mu_3 - 2\bar{c})\bar{\xi}v_t dB^1_t + \sqrt{\mu_4 - \mu_3^2 - 1}\bar{\xi}v_t dB^2_t,
\end{aligned}
$$

*when*

$$
\begin{aligned}
&\lim_n \bar{\alpha}^n_0 n = 0, \lim_n \bar{\alpha}^n_1\sqrt{n} = \bar{\xi} > 0, \lim_n \bar{c}^n = \bar{c} \in \mathbb{R},\\
&\qquad \text{and } \lim_n n\left(\bar{\alpha}^n_1(1 + (\bar{c}^n)^2) + \bar{\beta}^n_1 - 1\right) = \bar{\eta}.
\end{aligned}
$$

*The process then has a $(\mathbf{Q}_n)$-asymp. bubble if and only if $\mu_3 - 2\bar{c} > 0$.*

# A Proofs

The following simple lemma is used on numerous occasions:

**Lemma A.1** *Suppose $(\alpha^k_n)_{1\leq n,k<\infty}$ and $(x^k_n)_{1\leq n,k<\infty}$ are double-sequences in $\mathbb{R}$ for which*

$$\lim_k \lim_n \alpha^k_n =: \alpha < \infty, \quad \lim_k \lim_n x^k_n =: x < \infty.$$

*Then, there is $(k_n)_{n\geq 1}$, a sequence in $\mathbb{N}$ diverging to infinity as $n \to \infty$ for which*

$$\lim_n \alpha^{k_n}_n = \alpha, \text{ and } \lim_n x^{k_n}_n = x. \quad (15)$$

*Proof:*

For fixed $k$, let $\alpha^k = \lim_n \alpha^k_n$ and $x^k = \lim_n x^k_n$. Let $n_0 = 0$. For $k \geq 1$, take $n_k$ large enough ($n_k > n_{k-1}$) such that

$$|\alpha^k_n - \alpha^k| \leq 1/k \text{ and } |x^k_n - x^k| \leq 1/k$$

for $n \geq n_k$. Since $n_k \to \infty$ as $k \to \infty$, we can find a unique $k_n$ for each $n \geq 1$ for which

$$n_{k_n} \leq n < n_{k_n+1}.$$

It is clear that $(k_n)_n$ is a sequence in $\mathbb{N}$ diverging to infinity as $n \to \infty$. Moreover,

$$|\alpha_n^{k_n} - \alpha| \leq 1/k_n + |\alpha^{k_n} - \alpha| \text{ and } |x_n^{k_n} - x| \leq 1/k_n + |x^{k_n} - x|,$$

so that (15) is clearly satisfied. $\square$

To prove Theorem 3.1, we need the following two lemmas.

**Lemma A.2** *Let $(\mathbf{Q}_n) \in \mathbf{M}_{con}$ and $H = (h^n) \in \mathcal{H}$. Assume that $W^n$ is a $\mathbf{Q}_n$-local martingale that satisfies (6) and (7), and for which $W_T^n = h^n$, for each $n$. Let $\beta > 0$. The following two statements are equivalent:*

- *(A) There exists a positive $\mathbf{Q}_n$-martingale $\widetilde{W}^n$ such that*

$$\lim_{n\to\infty} \mathbf{P}_n(\widetilde{W}_T^n \geq h^n) = 1 \tag{16}$$

  *and* $\limsup_n \widetilde{W}_0^n \leq \beta$.

- *(B)* $\inf_{(N_n)\to\infty} \limsup_{n\to\infty} \left(W_0^n - N_n \mathbf{Q}_n\left(W_T^{n,*} \geq N_n\right)\right) \leq \beta$.

*Proof:* (A) $\Rightarrow$ (B):

By hypothesis, the sequences $(\mathbf{Q}_n)$ and $(\mathbf{P}_n)$ are mutually contiguous. Equation 16 then implies that $\lim_n q_n = 0$, where $q_n := \mathbf{Q}_n\left(W_T^n > \widetilde{W}_T^n\right)$. (Recall that $W_T^n = h^n$.)

For $n \geq 1$, define the following quantities and sets:

$$\begin{aligned}
M_n &= \begin{cases} (q_n)^{-\frac{1}{2}}, & \text{if } q_n > 0; \\ n, & \text{otherwise,} \end{cases} \\
\Delta_t^n &= W_t^n - \widetilde{W}_t^n, \\
\tau_n &= \inf\{t \geq 0 : W_t^n > M_n\} \wedge T, \\
B_n &= \{W_T^{n,*} > M_n\}, \quad A_n = \{W_T^n > \widetilde{W}_T^n\}, \\
\beta_n &= W_0^n - \widetilde{W}_0^n.
\end{aligned}$$

Equation 7 implies that $W^n_{\cdot\wedge\tau_n}$ is a martingale, by Lemma 2.1 of [13]. The following relations hold:

$$\begin{aligned}
\beta_n &= \mathbf{E}_{\mathbf{Q}_n}\left(\Delta_{\tau_n}^n\right) \leq \mathbf{E}_{\mathbf{Q}_n}\left(W_{\tau_n}^n \mathbf{1}_{B_n} + W_T^n \mathbf{1}_{B_n^c \cap A_n} + (W_T^n - \widetilde{W}_T^n)\mathbf{1}_{B_n^c \cap A_n^c}\right) \\
&\leq M_n \mathbf{Q}_n(B_n) + C_n + M_n \mathbf{Q}_n(B_n^c \cap A_n) + \mathbf{E}_{\mathbf{Q}_n}\left(\Delta_T^n \mathbf{1}_{B_n^c \cap A_n^c}\right) \\
&\leq M_n \mathbf{Q}_n(B_n) + C_n + \frac{1}{M_n},
\end{aligned}$$

in which

$$C_n = \sup_n \mathbf{E}_{\mathbf{Q}_n}\left(\mathbf{1}_{\{\tau_n < T\}} |\Delta W_{\tau_n}^n|\right).$$

By taking the limit as $n \to \infty$, we find

$$\limsup_{n\to\infty} \left(W_0^n - M_n \mathbf{Q}_n\left(W_T^{n,*} \geq M_n\right)\right) \quad \leq \quad \limsup_n \widetilde{W}_0^n \leq \beta$$

since $\lim_n C_n = 0$ due to (7).

(B) $\Rightarrow$ (A): For all $k \geq 1$ we can find a sequence $(\hat{N}_n^k)_{n\geq 1}$ converging to infinity such that

$$\limsup_{n\to\infty} \left(W_0^n - \hat{N}_n^k \mathbf{Q}_n\left(W_T^{n,*} \geq \hat{N}_n^k\right)\right) \leq \beta + 1/k.$$

By Lemma A.1, one can construct a sequence $N_n$ converging to infinity such that

$$\limsup_{n\to\infty}\left(W_0^n - N_n \mathbf{Q}_n\left(W_T^{n,*}\geq N_n\right)\right)\leq\beta.$$

Define the payoff $\widetilde{W}_T^n := h^n \mathbf{1}_{B_n^c}$ with $B_n := \{W_T^{*,n}\geq N_n\}$, and the $\mathbf{Q}_n$-martingale $\widetilde{W}_t^n := \mathbf{E}_{\mathbf{Q}_n}\left(h^n\mathbf{1}_{B_n^c}\middle|\mathcal{F}_t^n\right)$. Then,

$$W_0^n-\widetilde{W}_0^n = E_{\mathbf{Q}_n}\left(W_{\tau_n}^n\mathbf{1}_{B_n}+W_T^n\mathbf{1}_{B_n^c}-h^n\mathbf{1}_{B_n^c}\right)\geq N_n\mathbf{Q}_n\left(B_n\right)-C_n, \tag{17}$$

in which $\tau_n := \inf\{t\geq 0 : W_t^n > N_n\}\wedge T$, since $\widetilde{W}_T^n = W_T^n\mathbf{1}_{B_n^c}$. From Equation (17), we deduce the inequality:

$$\limsup_n \widetilde{W}_0^n \leq \beta.$$

The fact that $N_n\to\infty$ implies that $\mathbf{Q}_n\left(W_T^{n,*}\geq N_n\right)\to 0$, by (6), from which we can conclude that $\mathbf{Q}_n\left(\widetilde{W}_T^n\geq h^n\right)\to 1$, and $\mathbf{P}_n\left(\widetilde{W}_T^n\geq h^n\right)\to 1$ by contiguity. □

**Lemma A.3** *Let $(\mathbf{Q}_n)\in\mathbf{M}_{con}$ and $H=(h^n)\in\mathcal{H}$. Then,*

$$\inf_{(g^n)\in[H]_0}\limsup_{n\to\infty}\mathbf{E}_{\mathbf{Q}_n}g^n = \inf_{(N_n)\to\infty}\limsup_{n\to\infty}\mathbf{E}_{\mathbf{Q}_n}\left(h^n\mathbf{1}_{\{h^n\leq N_n\}}\right).$$

*Proof:* For all sequences $N_n\to\infty$, $\mathbf{Q}_n(h^n\leq N_n)\to 1$ so that $(h^n\mathbf{1}_{\{h^n\leq N_n\}})\in[H]_0$. Therefore,

$$\inf_{(g^n)\in[H]_0}\limsup_{n\to\infty}\mathbf{E}_{\mathbf{Q}_n}g^n \leq \inf_{(N_n)\to\infty}\limsup_{n}\mathbf{E}_{\mathbf{Q}_n}\left(h^n\mathbf{1}_{\{h^n\leq N_n\}}\right).$$

Let $(g^n)\in[H]_0$. Consider the quantity $r_n=\mathbf{E}_{\mathbf{Q}_n}\left(h^n\mathbf{1}_{\{h^n\leq N_n\}}-g^n\right)$. Then,

$$\begin{aligned} r_n &\leq \mathbf{E}_{\mathbf{Q}_n}\left((h^n-g^n)\mathbf{1}_{\{h^n\leq N_n\}}\right)\\ &\leq \mathbf{E}_{\mathbf{Q}_n}\left((h^n-g^n)\mathbf{1}_{\{h^n\leq N_n\}}\mathbf{1}_{\{g^n<h^n\}}\right)\\ &\leq N_n\mathbf{Q}_n\left(g^n<h^n\right)\to 0\end{aligned}$$

when $N_n := \left(\mathbf{Q}_n\left(g^n<h^n\right)\vee n^{-1}\right)^{-\frac{1}{2}}$, for example. Since $\mathbf{Q}_n\left(g^n<h^n\right)\to 0$, the sequence $N_n\to\infty$ and the following inequality is then readily verified:

$$\inf_{(N_n)\to\infty}\limsup_n\mathbf{E}_{\mathbf{Q}_n}\left(h^n\mathbf{1}_{\{h^n\leq N_n\}}\right)\leq\limsup_n\mathbf{E}_{\mathbf{Q}_n}g^n.$$

□

*Proof of Theorem 3.1:*

Since each intermediary market is complete, every non-negative $\mathbf{Q}_n$-martingale represents an admissible wealth process. For $(g^n)\in[H]_0$, the sequence of martingales $G_t^n=\mathbf{E}_{\mathbf{Q}_n}\left(g^n|\mathcal{F}_t^n\right)$ asymptotically superreplicates $(h^n)$. It is therefore clear that $\pi(H)=\inf_{(g^n)\in[H]_0}\limsup_{n\to\infty}\mathbf{E}_{\mathbf{Q}_n}g^n$. The first part of Theorem 3.1 is then a consequence of Lemma A.3.

Assume that $(W_t^n)_{t\leq T}$ is a $\mathbf{Q}_n$-local martingale that satisfies (6) and (7), and for which $W_T^n=h^n$, for each $n$. Let

$$\beta=\inf_{(N_n)\to\infty}\limsup_{n\to\infty}\left(W_0^n-N_n\mathbf{Q}_n\left(W_T^{n,*}\geq N_n\right)\right).$$

Applying Lemma A.2, there exists a sequence of $\mathbf{Q}_n$-martingales $\widetilde{W}^n$ which asymptotically superreplicates $(h^n)$ and such that $\limsup_n\widetilde{W}_0^n\leq\beta$. Hence, $\pi(H)\leq\limsup_n\widetilde{W}_0^n\leq\beta$.

To obtain the inverse inequality, let $\epsilon > 0$ and $\hat{W}^{n,\epsilon}$ a $\mathbf{Q}_n$-martingale such that $\pi(H) \geq \limsup_n \hat{W}_0^{n,\epsilon} - \epsilon$, and $\mathbf{P}_n(\hat{W}_T^{n,\epsilon} \geq h^n) \to 1$. Consider $\beta = \limsup_n \hat{W}_0^{n,\epsilon}$, and apply Lemma A.2 to find that

$$\pi(H) \geq \beta - \epsilon \geq \inf_{(N_n)\to\infty} \limsup_{n\to\infty} \left(W_0^n - N_n \mathbf{Q}_n \left(W_T^{n,*} \geq N_n\right)\right) - \epsilon.$$

Since $\epsilon$ is arbitrary, the result follows.

□

**Proposition A.4** *When $h^n$ is uniformly bounded by a constant independent of $n$,*

$$\sup_{(\mathbf{Q}_n)\in\mathbf{M}_{con}} \inf_{(g^n)\in[H]_0} \limsup_n \mathbf{E}_{\mathbf{Q}_n}(g^n) = \sup_{(\mathbf{Q}_n)\in\mathbf{M}_{con}} \limsup_n \mathbf{E}_{\mathbf{Q}_n}(h^n).$$

*Proof:*
Suppose there exists $M$ such that $h^n \leq M$, all $n \geq 1$. Let $(\mathbf{Q}_n) \in \mathbf{M}_{con}$. Any sequence $(g^n) \in [H]_0$ that satisfies $0 \leq g^n, h^n \leq M$ also satisfies

$$\mathbf{E}_{\mathbf{Q}_n}(g^n) \quad\geq\quad \mathbf{E}_{\mathbf{Q}_n}\left(h^n - h^n 1_{\{g^n<h^n\}}\right) \geq \mathbf{E}_{\mathbf{Q}_n}(h^n) - M\mathbf{Q}_n(g^n < h^n).$$

Therefore,

$$\limsup_n \mathbf{E}_{\mathbf{Q}_n}(g^n) \quad\geq\quad \limsup_n \mathbf{E}_{\mathbf{Q}_n}(h^n)$$

since $\mathbf{Q}_n(g^n < h^n) \to 0$. Because $(g^n)$ and $(\mathbf{Q}_n)$ are arbitrary, the result follows.

□

We now prove the following equivalent representation of $\pi_d(H)$:

**Proposition A.5**

$$\pi_d(H) = \inf\{\limsup_n x^n : \forall\epsilon > 0, \phi \in \mathcal{Y} \quad \mathcal{K}_0(x^n) \cap [H]_\epsilon^\phi \neq \emptyset\}, \tag{18}$$

*with*

$$[H]_\epsilon^\phi = \{(g_n) : g_n \in \mathcal{F}_T^n, \text{ and } \mathbf{E}_n\phi((g_n)_-) \leq 1, \liminf_{n\to\infty} \mathbf{P}_n(g^n \geq h^n - \epsilon) > 1 - \epsilon\}.$$

*Moreover, for all $\phi \in \mathcal{Y}$, $\epsilon_0 > 0$, there exists $\epsilon < \epsilon_0$ for which the following implication is verified:*

$$\Big(f^n \in (h^n + B^{\phi^\epsilon,n}) \text{ for all } n \text{ larger than some } k \geq 1\Big) \Rightarrow (f^n) \in [H]_\epsilon^\phi$$

*in which $\phi^\epsilon : x \mapsto \phi(x)/\phi(\epsilon)^2$.*

*Proof:*
It is obvious that $\pi_d(H)$ is greater or equal to the right side of (18) since $[H]_0^\phi \subset [H]_\epsilon^\phi$, for all $\epsilon > 0$. So let $(x^n)$ a sequence in $\mathbb{R}$ such that for all $k \geq 1$ and $\phi \in \mathcal{Y}$ there exists $\tilde{f}^{k,n} \in K_0^n(x^n)$ such that:

1. $\mathbf{E}_n\phi((\tilde{f}^{k,n})_-) \leq 1$, all $n \geq 1$,
2. $\lim_n \sup_{m\geq n} \mathbf{P}_m\left(\tilde{f}^{k,m} + \frac{1}{k} \leq h^m\right) \leq \frac{1}{k}$.

Define $\tilde{x}_n^k = x^n + \frac{1}{k}$, so that $\tilde{f}^{k,n} + \frac{1}{k} \in K_0^n(\tilde{x}_n^k)$. By Lemma A.1, there exists $(k_n) \to \infty$ for which $f^n := \tilde{f}^{k_n,n} + \frac{1}{k_n}$ satisfies

1. $\mathbf{E}_n\phi((f^n)_-) \le 1$, all $n \ge 1$,

2. $\limsup_n \mathbf{P}_n\left(f^n \le h^n\right) = 0$,

and $\limsup_n x^n = \limsup_n \tilde{x}_n^{k_n}$. We can therefore conclude that $(f^n) \in \mathcal{K}_0(\tilde{x}_n^{k_n}) \cap [H]_0^\phi$ and $\pi_d(H) \le \limsup_n x^n$. Equation 18 is then readily verified.

To prove the second part of the proposition, let $\phi \in \mathcal{Y}$, $\epsilon_0 > 0$. Take $\epsilon$ small enough that $\phi(\epsilon) < \max(1, \epsilon)$. For all $n$ greater or equal to some $k \ge 1$, assume $f^n = h^n + b^n$ , $b^n \in B^{\phi^\epsilon, n}$. This means that $(f^n)_- \le |b^n|$ and $(h^n - f^n)_+ \le |b^n|$. In particular, $\mathbf{E}_n\phi((f^n)_-) \le \phi(\epsilon)^2 < 1$ and

$$\mathbf{P}_n((h^n - f^n)_+ \ge \epsilon) \le \frac{1}{\phi(\epsilon)}\mathbf{E}_n\phi((h^n - f^n)_+) \le \phi(\epsilon) < \epsilon.$$

In particular, we find that $(f^n) \in [H]_\epsilon^\phi$.

□

*Proof of Theorem 4.3:*

Let

$$V_0 = \sup_{(\mathbf{Q}_n)\in\mathbf{M}_{con}} \inf_{(g^n)\in[H]_0} \limsup_n \mathbf{E}_{\mathbf{Q}_n}\left(g^n\right).$$

We first prove the inequality:

$$V_0 \le \pi_d(H). \tag{19}$$

Let $(\mathbf{Q}_n) \in \mathbf{M}_{con}$ and $\epsilon > 0$, and define $Z_n := \frac{d\mathbf{Q}_n}{d\mathbf{P}_n}, n \ge 1$. By Lemma 4.5 of [21], there exists $\phi \in \mathcal{Y}$ such that

$$\sup_n \mathbf{E}_n\phi(Z_n) \le \epsilon.$$

Define $\phi^* : y \mapsto \sup_{x\ge 0}\left(xy - \phi(x)\right)$, the convex conjugate of $\phi$, which is also in $\mathcal{Y}$. Let $(x^n)_n$ a sequence in $\mathbb{R}$ such that there exists a sequence $(f^n) \in \mathcal{K}_0(x^n)$ for which $f^n = x^n + (X^n \cdot S^n)_T$ with $X^n$ admissible, and $(f^n) \in [H]_0^\psi$, in which $\psi = \frac{1}{\epsilon}\phi^*$. In this case, $\mathbf{E}_n\phi^*((f^n)_-) \le \epsilon$ $(n \ge 1)$ and $((f^n)_+)_n \in [H]_0$.

Since $(X^n \cdot S^n)$ is a local martingale bounded from below, $\mathbf{E}_{\mathbf{Q}_n}\left(f^n\right) \le x^n$ by Theorem 2.9 of [6]. Furthermore,

$$\mathbf{E}_{\mathbf{Q}_n}\left((f^n)_-\right) = \mathbf{E}_n\left(Z_n(f^n)_-\right) \le \mathbf{E}_n\phi(Z_n) + \mathbf{E}_n\phi^*((f^n)_-),$$

by the Fenchel-Young inequality. We can therefore conclude that

$$\limsup_n \mathbf{E}_{\mathbf{Q}_n}\left((f^n)_-\right) \le 2\epsilon.$$

Then,

$$\limsup_n x^n \quad \ge \quad \limsup_n \mathbf{E}_{\mathbf{Q}_n}\left(f^n\right) \ge -2\epsilon + \limsup_n \mathbf{E}_{\mathbf{Q}_n}\left((f^n)_+\right) \ge \inf_{(g^n)\in[H]_0} \limsup_n \mathbf{E}_{\mathbf{Q}_n} g^n - 2\epsilon.$$

Since $(x^n), \epsilon$ and the sequence $(\mathbf{Q}_n)_{n\ge 1}$ are arbitrary, Inequality 19 readily follows.

(a) <u>Suppose the sequence $(h^n)$ is uniformly bounded by a constant $M > 0$.</u> We prove that $\pi_d(H) \le V_0$.

Suppose there exists $\eta > 0$ such that

$$V_0 + \eta \quad < \quad \pi_d(H). \tag{20}$$

Let $x = V_0 + \eta/2$. By Proposition A.5, for all sequences $(x^n)$ that satisfy $\limsup_n x^n = x$, we can find $\psi \in \mathcal{Y}$, $\epsilon_0 > 0$, such that for all $0 < \epsilon < \epsilon_0$,

$$\mathcal{K}_0(x^n) \cap [H]_\epsilon^\psi = \emptyset.$$

In particular, there exists a subsequence $(x^{n_k})_{k\geq 1}$ such that $(x^n + C^n) \cap (h^n + B^{\psi^\epsilon,n}) = \emptyset$ for all $n \in \{n_k : k \geq 1\}$, where $\psi^\epsilon = \psi/\psi(\epsilon)^2$ for some $\epsilon > 0$, by the second part of Proposition A.5.

Following Klein [21], for $\phi \in \mathcal{Y}$, define $V^{\phi,n} = (B^{\phi,n})^o$, the polar set of $B^{\phi,n}$. By Lemma 5.2 of [21], there exists $\phi \in \mathcal{Y}$ for which $V^{\phi,n} \subset B^{\psi^\epsilon,n} \cap L^\infty(\Omega^n, \mathcal{F}^n_T, \mathbf{P}_n)$, all $n \geq 1$.

Fix $n \in \{n_k : k \geq 1\}$. Then, $(x^n + C^n + \widehat{V^{\phi,n}}) \cap \{h^n\} = \emptyset$, where $\widehat{V^{\phi,n}}$ is the interior of $V^{\phi,n}$ with respect to the topology of uniform convergence (Mackey topology) of $L^\infty(\Omega^n, \mathcal{F}^n_T, \mathbf{P}_n)$. Since $x^n + C^n + \widehat{V^{\phi,n}}$ is open, we can apply the Hahn-Banach theorem and find $k^\phi_n \in L^1(\Omega^n, \mathcal{F}^n_T, \mathbf{P}_n)$ such that $E_n|k^\phi_n| = 1$, and for all $f^n \in C^n$ and $v^n \in \widehat{V^{\phi,n}}$ we have

$$x^n + \mathbf{E}_n\left(k^\phi_n\left(f^n + v^n\right)\right) \quad \leq \quad \mathbf{E}_n\left(k^\phi_n h^n\right) \leq M. \tag{21}$$

Since $0 \in \widehat{V^{\phi,n}}$ and $\lambda f^n \in C^n$ for all $\lambda > 0$, $\sup_{f^n\in C^n} \mathbf{E}_n\left(k^\phi_n f^n\right) \leq 0$. Since $-L^\infty_+ \subset C^n$, $k^\phi_n \geq 0$ is the density of a measure $\mathbf{Q}^\phi_n$ absolutely continuous with respect to $\mathbf{P}_n$, for all $n \in \{n_k : k \geq 1\}$. Furthermore,

$$\sup_{v^n\in V^{\phi,n}} \mathbf{E}_n\left(k^\phi_n v^n\right) \quad \leq \quad M.$$

Therefore $\frac{1}{M}k^\phi_n \in B^{\phi,n}$ by the definition of $V^{\phi,n}$. From this we find that $(\mathbf{Q}^\phi_{n_k}) \lhd (\mathbf{P}_{n_k})$ as $k \to \infty$ by Lemma 4.5 of [21]. Let $(\mathbf{Q}'_n) \in \mathbf{M}_{con}$, and define $\mathbf{Q}^\phi_n = \mathbf{Q}'_n$ for $n \notin \{n_k : k \geq 1\}$ to have $(\mathbf{Q}^\phi_n) \lhd (\mathbf{P}_n)$ as $n \to \infty$.

For $0 < \xi < 1$, define $\mathbf{Q}^{\phi,\xi}_n = (1-\xi)\mathbf{Q}^\phi_n + \xi\mathbf{P}_n$. Then, $(\mathbf{Q}^{\phi,\xi}_n) \lhd \rhd (\mathbf{P}_n)$ and

$$V_0 + \eta/2 \quad = \quad \limsup_n x^n \leq \frac{1}{1-\xi} \limsup_n \mathbf{E}_{\mathbf{Q}^{\phi,\xi}_n}\left(h^n\right),$$

due to (21). By taking the supremum over all contiguous sequences,

$$V_0 + \eta/2 \quad \leq \quad \frac{1}{1-\xi} \sup_{(\mathbf{Q}_n)\in\mathbf{M}_{con}} \limsup_n \mathbf{E}_{\mathbf{Q}_n}\left(h^n\right) = \frac{1}{1-\xi}V_0,$$

from Proposition A.4. A contradiction arises when $\xi$ is small enough. Consequently, $\pi_d(H) \leq V_0$.

Hence, $\pi(H) = V_0$ when $(h^n)$ is uniformly bounded.

(b) <u>If $(h^n)$ is not uniformly bounded</u>, let $k \geq 1$ and define $\overline{h}^{n,k} := h^n \wedge k$.

Since $V_0 \geq \sup_{(\mathbf{Q}_n)\in\mathbf{M}_{con}} \limsup_n \mathbf{E}_{\mathbf{Q}_n}\left(\overline{h}^{n,M}\right)$, there exists a sequence $(\tilde{x}^k_n)$ such that $\limsup_n \tilde{x}^k_n = x \leq V_0$ and for all $\phi \in \mathcal{Y}$ there exists a sequence $(\tilde{f}^{k,n}) \in \mathcal{K}_0(\tilde{x}^k_n)$ for which

$$\lim_n \mathbf{P}_n\left(\tilde{f}^{k,n} \geq \overline{h}^{n,k}\right) = 1$$

and

$$\mathbf{E}_n\phi((\tilde{f}^{k,n})_-) \leq 1 \quad (n \geq 1).$$

By Lemma A.1, there exists $(k_n) \to \infty$ such that $f^n := \tilde{f}^{k_n,n}$ satisfies

$$\lim_n \mathbf{P}_n\left(f^n \geq \overline{h}^{n,k_n}\right) = 1$$

and

$$\mathbf{E}_n\phi((f^n)_-) \leq 1 \quad (n \geq 1),$$

and $x^n := \tilde{x}^{k_n}_n$ satisfies

$$\limsup_n x^n = x.$$

Furthermore,

$$\mathbf{P}_n\left(f^n \geq h^n\right) \quad \geq \quad \mathbf{P}_n\left(f^n \geq h^n \wedge k_n\right) - \mathbf{P}_n\left(h^n \geq k_n\right)$$

which converges to 1 as $n \to \infty$ since $\sup_n \mathbf{P}_n(h^n > k_n) \to 0$ as $n \to \infty$ (see Definition 2.1). Consequently, $\pi(H) \leq x \leq V_0$.

$\square$

*Proof of Proposition 2.10:*

First, observe that $\limsup_n \pi_\epsilon^n(h^n)$ is a decreasing function of $\epsilon$. Therefore,

$$\lim_{\epsilon \downarrow 0} \limsup_n \pi_\epsilon^n(h^n)$$

exists and is finite.

Let $\eta > 0$, and $(\epsilon_k)_k$ a sequence of positive numbers converging to zero when $k$ goes to infinity. Define $\alpha^{k,n} = \pi_{\epsilon_k}^n(h^n) + \eta$, $\alpha^k = \limsup_n \pi_{\epsilon_k}^n(h^n)$ and $\alpha = \lim_k \alpha^k$.

For all $k \geq 1$ there exists a sequence $(\tilde{W}^{k,n}) \in \mathcal{K}_0(\alpha^{k,n})$ that satisfies

$$\mathbf{P}_n\left(\tilde{W}_T^{k,n} \leq h^n\right) \leq \epsilon_k.$$

By Lemma A.1, there exists $(k_n) \to \infty$ such that $W^n := \tilde{W}^{k_n,n}$ and $x^n := \alpha^{k_n,n}$ satisfy

$$\lim_n \mathbf{P}_n\left(W_T^n \leq h^n\right) = 0 \text{ and } \limsup_n x^n = \alpha.$$

Therefore, $\pi(H) \leq \alpha = \lim_{\epsilon \downarrow 0} \limsup_n \pi_\epsilon^n(h^n) + \eta$, and

$$\pi(H) \leq \lim_{\epsilon \downarrow 0} \limsup_n \pi_\epsilon^n(h^n)$$

since $\eta$ is arbitrary.

The proof of the inequality $\pi(H) \geq \lim_{\epsilon \downarrow 0} \limsup_n \pi_\epsilon^n(h^n)$ is straightforward.

$\square$

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