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JOURNAL OF Econometrics

Journal of Econometrics 131 (2006) 253–284

www.elsevier.com/locate/econbase

Option valuation with conditional skewness

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Available online 25 February 2005

Abstract

Index option prices differ systematically from Black–Scholes prices. Out-of-the-money put prices (and in-the-money call prices) are relatively high compared to the Black–Scholes price. Motivated by these empirical facts, we develop a new discrete-time dynamic model of stock returns with inverse Gaussian innovations. The model allows for conditional skewness as well as conditional heteroskedasticity and a leverage effect. We present an analytic option pricing formula consistent with this stock return dynamic. An extensive empirical test of the model using S&P500 index options shows that the new inverse Gaussian GARCH model's performance is superior to a standard existing nested model for out-of-the money puts.

JEL classification: G12

Keywords: GARCH; Out-of-sample; Jumps; Discrete-time model; Continuous-time limit

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0304-4076/$ - see front matter
doi:10.1016/j.jeconom.2005.01.010

0. Introduction

There is extensive empirical evidence that index option prices systematically differ from Black–Scholes prices.[1] Out-of-the-money put prices (and in-the-money call prices) are relatively high compared to the Black–Scholes price. This stylized fact is often represented by the well-known "volatility smirk". One interesting approach to capturing these deviations from the Black–Scholes formula is to incorporate models of conditional heteroskedasticity (Hull and White, 1987; Scott, 1987; Heston, 1993a). At the empirical level, a number of papers have demonstrated that these models significantly improve upon the performance of the Black–Scholes model (e.g., see Bakshi et al., 1997; Bates, 2000; Ding and Granger, 1996; Engle and Mustafa, 1992; Jones, 2003; Pan, 2002). Moreover, several papers have demonstrated that the performance of option valuation models with conditional heteroskedasticity can be further improved by including a so-called leverage parameter (Nandi, 1998; Heston and Nandi, 2000; Chernov and Ghysels, 2000; Christoffersen and Jacobs, 2004).

The combination of leverage parameters and stochastic volatility or conditional heteroskedasticity captures the stylized fact that volatility increases relatively more when the stock price drops (Black, 1976; Christie, 1982). This increases the probability of a large loss and consequently the value of out-of-the-money put options. Equivalently, the implications of the leverage effect can be understood by realizing that it generates negative skewness in stock returns.[2]

While dynamic volatility models are intuitively and theoretically appealing, existing discrete-time models may not be sufficiently flexible to explain observed option biases, even with leverage parameters included. This is particularly the case for options with short maturities. While the leverage parameter creates negative skewness in multi-period returns, single-period innovations are Gaussian in these models, and therefore standard models cannot explain the strong biases in short-term options.[3] We therefore suggest a complementary approach to generate skewness in the return distribution by modeling the conditional innovations to returns using a distribution with nonzero third moment.

This paper develops a discrete time model for spot prices that introduces conditional skewness into short-term spot returns in addition to conditional heteroskedasticity and a leverage effect. We model the conditional return innovation using an inverse Gaussian distribution, and we combine the modeling of these nonstandard conditional innovations with a fairly standard model of time-varying conditional volatility that also contains a leverage effect. The resulting return dynamic, which we call inverse Gaussian GARCH (or IG GARCH), is able to capture skewness in short-term as well as long-term returns, and the conditional

[1] See for example Ait-Sahalia and Lo (1998), Bakshi et al. (1997), Bates (1996), Christoffersen and Jacobs (2004), Das and Sundaram (1999), Dumas et al. (1998), Garcia et al. (2003), and Jackwerth (2000).

[2] See Heston (1993a) for an in-depth analysis of the impact of the leverage parameter on option prices.

[3] This critique formally applies only to discrete-time models. Diffusion models with jumps in stock returns and/or volatility may be able to capture deviations from normality.

skewness reinforces the effects of the leverage parameter. We present a closed-form option pricing formula consistent with this return dynamic.

The discrete-time IG GARCH process has two interesting continuous-time limits. One limit is the standard stochastic volatility model of Heston (1993a). The other is a pure jump process with stochastic intensity. Using these limit results, an equivalent motivation for our model is therefore that it generalizes standard stochastic volatility models by allowing for ''jumps'' and other fat-tailed negative movements in short-term stock returns, which is particularly useful for explaining the biases in short-term options (Carr and Wu, 2003). However, while jumps in stock returns reduce the bias in option prices, they cannot adequately address the magnitude of this bias (Bates, 1996). In other words, introducing jumps in returns works in a qualitative sense, because it reduces the biases, but not in a quantitative sense, because part of the bias remains. We therefore need a modeling approach that reinforces the effects of jumps in returns. One of the continuous-time limits of our model additionally contains jumps in volatility, and allows these jumps to be (negatively) correlated with jumps in stock returns.[4] Compared to jumps in returns, jumps in volatility can have a larger impact on option prices because the volatility process is very persistent whereas the return process is not. With jumps in volatility, a series of moderate negative ''jumps'' in stock returns can dramatically increase volatility for a prolonged period. Small changes in the distribution of stock returns can therefore have a potentially large impact on option values.

We implement the model empirically using data on S&P500 index options. We compare the model's performance to the Black–Scholes model as well as to a number of benchmarks. We find that our model performs well compared to these benchmarks. The most relevant benchmark for our model is the Heston and Nandi (2000) model, which is a model with conditional heteroskedasticity, a leverage effect and Gaussian innovations, and which is nested in our model. In-sample the inverse Gaussian(IG) model improves upon the performance of the Heston–Nandi(HN) model in every respect, which is not surprising because it nests the HN model. Out-of-sample, the model's performance is mixed. While the IG model improves on the HN model for the valuation of out-of-the-money puts, this is not necessarily the case for other options. When keeping the model parameters constant for up to ten weeks, the valuation of the IG model improves on that of the HN model. However, when keeping the parameter estimates constant for longer periods, the IG model is outperformed by the HN model. We therefore conclude that the benefits of modeling conditional skewness and jump processes are mixed, and depend on the use of the models; the richer parameterization of these models is helpful in-sample and for some out-of-sample assessments, but a more parsimonious parameterization may be preferable dependent on the strategy one pursues. Presumably many strategies that use options only rely on keeping the parameters constant for short out-of-sample periods, even though changing the model parameters implies a theoretical inconsistency.

[4]See e.g. Duffie et al. (2000), Pan (2002), Eraker (2004), Eraker et al. (2003), and Duan et al. (2002).

The model is presented in the next two sections. Subsequently we present empirical assessments of the model using S&P500 index returns and options prices. A final section concludes. Some technical material on the inverse Gaussian distribution is relegated to the appendix.

1. The stock price dynamics

1.1. The inverse Gaussian distribution

The basic building block for this model is a random variable with an inverse Gaussian distribution. We shall use draws from this distribution as building blocks for a discrete-time stochastic process with conditional heteroskedasticity.[5] The inverse Gaussian density has positive support ($y>0$); its distribution function with parameter δ is given by

$$\begin{aligned} P(y;\delta) &= \int_0^y \frac{\delta}{\sqrt{2\pi z^3}}\, \mathrm{e}^{-\left(\sqrt{z}-\delta/\sqrt{z}\right)^2/2}\, \mathrm{d}z \\ &= N\left(\frac{-\delta}{\sqrt{y}}+\sqrt{y}\right)+\mathrm{e}^{2\delta}N\left(\frac{-\delta}{\sqrt{y}}-\sqrt{y}\right). \end{aligned} \tag{1}$$

Straightforward (albeit tedious) integration gives a generalization of the moment generating function:

$$\mathrm{E}[\exp(\phi y+\theta/y)] = \frac{\delta}{\sqrt{\delta^2-2\theta}}\exp\left(\delta-\sqrt{(\delta^2-2\theta)(1-2\phi)}\right). \tag{2}$$

and the moments:

$$\mathrm{E}[y]=\delta, \quad \mathrm{Var}[y]=\delta, \tag{3}$$

$$\mathrm{E}[1/y]=1/\delta+1/\delta^2, \quad \mathrm{Var}[1/y]=1/\delta^3+2/\delta^4,$$

$$\mathrm{Skew}[y]=\mathrm{E}[(y-\mathrm{E}[y])^3]/\mathrm{Var}[y]^{3/2}=3/\mathrm{Sqrt}(\delta),$$

$$\mathrm{Kurt}[y]=\mathrm{E}[(y-\mathrm{E}[y])^4]/\mathrm{Var}[y]^2-3=15/\delta, \mathrm{Cov}[y,1/y]=-1/\delta.$$

1.2. The IG GARCH process

We combine the conditional skewness of the inverse Gaussian distribution with GARCH-type dynamics that contain conditional heteroskedasticity and a leverage effect.[6] This gives the model flexibility to capture moneyness effects for short-term as

[5] For more detail see Johnson et al. (1994, Chapter 15). For an intuitive derivation of the inverse Gaussian distribution see Whitmore and Seshadri (1987).

[6] See Nelson (1991) for another approach that alters the GARCH dynamic and introduces conditional nonnormality.

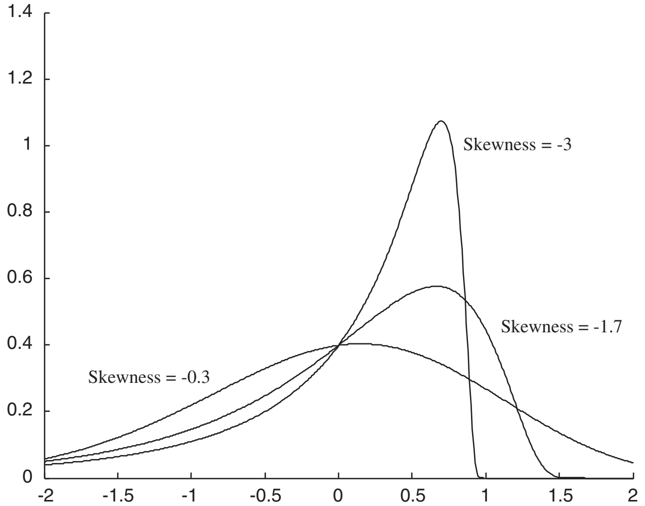

Fig. 1. Standardized reflected inverse Gaussian densities with varying skewness.

well as long-term options. The new dynamic model specifies returns on a spot asset price at time t, $S(t)$, and the conditional variance of return $h(t+\Delta)$ as

$$\log(S(t+\Delta)/(S(t)) = r\Delta + vh(t+\Delta) + \eta y(t+\Delta), \tag{4a}$$

$$h(t+\Delta) = w + bh(t) + cy(t) + ah(t)^2/y(t), \tag{4b}$$

where, given the available information at time t, $y(t+\Delta)$ has an inverse Gaussian conditional distribution with degrees of freedom parameter $\delta(t+\Delta) = h(t+\Delta)/\eta^2$. As $\delta(t+\Delta)$ represents the first moment in the IG distribution the zero-mean stock return innovation is given by $y(t+\Delta) - h(t+\Delta)/\eta^2$. Consider the conditional mean dynamic in (4a). For the inverse Gaussian distribution to display negative skewness, we need a negative value of η. Given an offsetting positive value of $vh(t+\Delta)$ in Eq. (4a), the dynamic can therefore generate positive as well as negative returns centered around a small positive value. Fig. 1 illustrates how a standardized inverse Gaussian distribution can display negative skewness given appropriate parameter values. As the degrees of freedom parameter approaches infinity the standardized inverse Gaussian distribution converges to the standard Gaussian distribution. We label this model the IG GARCH(1,1), because it consists of combining an inverse Gaussian distribution with a GARCH(1,1) type volatility dynamic in (4b).[7] We can use the moments of the inverse Gaussian in (3) to show that the conditional mean and variance of the spot process (4) are linear functions of the current variance.

From Eq. (4a) the conditional mean and variance can be derived as

$$\mathrm{E}_t[\log(S(t+\Delta)/S(t))] = r\Delta + (v + \eta^{-1})h(t+\Delta), \tag{5a}$$

[7]For more complex dynamics one can nest higher-order GARCH processes by adding lagged disturbances to the dynamics of the conditional variance.

$$\mathrm{Var_t}[\log(S(t+\varDelta))] = h(t+\varDelta),$$

and from Eq. (4b) we can compute variance persistence as the coefficient on $h(t+\varDelta)$ in

$$\mathrm{E_t}[h(t+2\varDelta)] = w + \eta^4 a + (a\eta^2 + b + c/\eta^2)h(t+\varDelta), \tag{5b}$$

It can also be seen from (5b) that the unconditional variance is $(w+\eta^4 a)/(1 - a\eta^2 - b - c/\eta^2)$.

The variance of variance is given from Eq. (4b) by

$$\mathrm{Var_t}[h(t+2\varDelta)] = 2a^2\eta^8 + (c/\eta - \eta^3 a)^2 h(t+\varDelta), \tag{5c}$$

and the leverage effect can be quantified as the conditional covariance between returns and variance as in

$$\mathrm{Cov_t}[\log(S(t+\varDelta)/S(t)), h(t+2\varDelta)] = (c/\eta - \eta^3 a)h(t+\varDelta). \tag{5d}$$

Notice that the coefficient on the conditional variance term in (5c) is the square of the corresponding coefficient in (5d). Thus, if there is no leverage effect then the variance of variance is constant in this model.[8]

Finally, the conditional skewness of the return is

$$\mathrm{Skew_t}[\log(S(t+\varDelta)/S(t))] = 3/\mathrm{Sqrt}(\delta(t+\varDelta)) = 3\eta(h(t+\varDelta))^{-1/2}. \tag{5e}$$

Although the functional form appears quite different, the IG GARCH is closely related to existing GARCH processes. By taking the limit as η approaches zero and using the following parameterization,

$$v = \lambda - \eta^{-1}, \quad w = \omega, \quad a = \alpha/\eta^4, \tag{6}$$

$$b = \beta + \alpha\gamma^2 - 2\alpha/\eta^2 + 2\alpha\gamma/\eta, \quad c = \alpha - 2\eta\alpha\gamma,$$

the IG GARCH(1,1) converges to the Heston and Nandi (2000) asymmetric GARCH model with normal disturbances $z(t+\varDelta)$

$$\log(S(t+\varDelta)/S(t)) = r\varDelta + \lambda h(t+\varDelta) + \sqrt{h(t+\varDelta)}z(t+\varDelta), \tag{7a}$$

$$h(t+\varDelta) = \omega + \beta h(t) + \alpha(z(t) - \gamma\sqrt{h(t)})^2. \tag{7b}$$

For a given η, the parameterization in (6) matches the first two conditional moments of the IG GARCH model in Eq. (5) with those in the HN model. In addition, by letting η approach zero, the skewness vanishes and the standardized inverse Gaussian innovation $(y(t+\varDelta) - \delta(t+\varDelta))/\mathrm{Sqrt}(\delta(t+\varDelta))$ from Eq. (4a) converges to the Gaussian innovation $z(t+\varDelta)$ from Eq. (7a). Finally, the parameters in Eq. (6) make Eq. (7b) a second-order Taylor series approximation to Eq. (4b). Therefore the IG GARCH model nests the HN model as a limiting case when η approaches zero.

[8]Note that this features of the IG GARCH model are shared by the Heston and Nandi (2000) model but not by the NGARCH model considered by Duan (1995) and Christoffersen and Jacobs (2004) nor by the continuous time stochastic volatility model in Heston (1993a).

1.3. Continuous-time limits

While the IG GARCH(1,1) is a discrete model that is readily implementable with discrete data, it has two interesting continuous-time limits. First, consider the case where η approaches zero. As shown above, in this case the IG GARCH model converges to the Heston and Nandi (2000) Gaussian GARCH model. As shown in Heston and Nandi (2000), this model converges to Heston's (1993a) square-root model as a diffusion limit.[9] Consider letting the time interval Δ shrink to zero in the HN model and define $v(t) = h(t)/\Delta$ to be the variance per unit of time. Then let $\omega(\Delta) = (\kappa\theta - 1/4\sigma^2)\Delta^2, \beta = 0, \alpha(\Delta) = 1/4\sigma^2\Delta^2, \gamma(\Delta) = 2/(\sigma\Delta) - \kappa/\sigma$. As the time interval shrinks the variance per unit of time converges weakly to the square-root diffusion process and we obtain the Heston (1993a) model

$$\mathrm{d}\log(S(t)) = (r + \lambda v(t))\,\mathrm{d}t + \sqrt{v(t)}\,\mathrm{d}z(t), \tag{8}$$

$$\mathrm{d}v(t) = \kappa(\theta - v(t)) + \sigma\sqrt{v(t)}\,\mathrm{d}z(t),$$

where $z(t)$ is a Wiener process. See Heston and Nandi (2000, Appendix B) for more details regarding this limit.

Second, we can let the time interval shrink with the alternative parameter limits

$$a(\Delta) = 0, \quad b(\Delta) = 1 - b\Delta, \tag{9}$$

$$c(\Delta) = c\Delta, \quad w(\Delta) = w\Delta^2,$$

where we now rely on the IG GARCH parameters directly. Again letting $v(t) = h(t)/\Delta$ represent the variance per unit of time we obtain a pure jump process as the time interval shrinks

$$\mathrm{d}\log(S(t)) = (r + vv(t))\,\mathrm{d}t + \eta\,\mathrm{d}y(t), \tag{10a}$$

$$\mathrm{d}v(t) = (w - bv(t))\,\mathrm{d}t + c\,\mathrm{d}y(t), \tag{10b}$$

where $y(t)$ is a pure-jump inverse Gaussian process with degrees of freedom $\delta(t) = v(t)/\eta^2$ in the interval $[t, t + \mathrm{d}t]$. The stock price converges to a pure jump process with stochastic intensity.[10] To provide some more intuition for the dynamics of this process, Fig. 2 shows how an inverse Gaussian random walk converges to a Wiener process as the skewness parameter converges to zero.

Note that this continuous-time limit contains jumps in volatility in addition to jumps in returns, and allows these jumps to be (negatively) correlated. Compared to jumps in returns, jumps in volatility can have a larger impact on option prices because the volatility process is very persistent whereas the return process is not. With jumps in volatility, a series of moderate negative "jumps" in stock returns can

[9]See also Nelson (1990).

[10]The inverse Gaussian process has been investigated by Barndorff-Nielsen and Levendorskii (2000), Jensen and Lunde (2001), and Bollerslev and Forsberg (2002). See also the excellent overview of related processes in Barndorff-Nielsen and Shephard (2001).

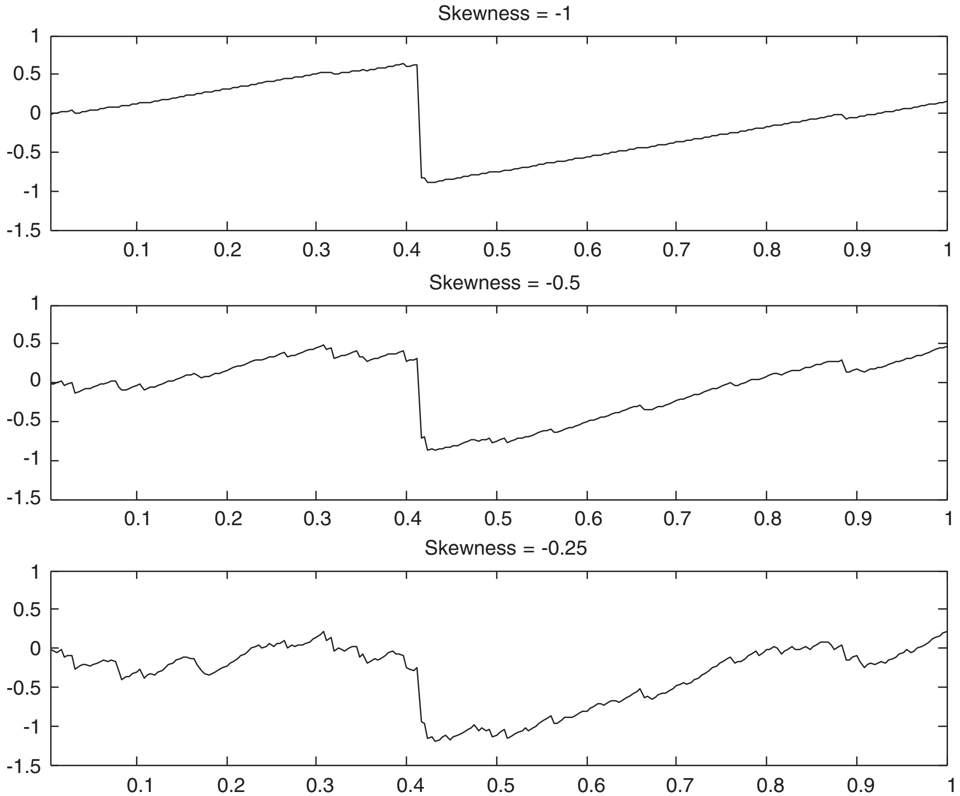

Fig. 2. Standardized inverse Gaussian random walks with varying skewness.

dramatically increase volatility for a prolonged period. Small changes in the distribution of stock returns can therefore have a potentially large impact on option values.

In summary, the continuous-time limits suggest that the dynamic process (4) displays remarkable flexibility: it is able to capture both diffusion processes and pure jump processes dependent on the degree of skewness or kurtosis of returns.

2. Option valuation

Option valuation in the IG GARCH model requires additional assumptions beyond the absence of arbitrage. According to Eq. (4) the spot price $S(t+\Delta)$ equals $\exp(\mu + \eta y(t+\Delta))$, where $\mu = \ln(S(t)) + r\Delta + \nu h(t+\Delta)$ and $y(t+\Delta)$ has an inverse Gaussian distribution with degrees of freedom δ. Hence the conditional density for $S(t+\Delta)$ given the information at time t is log-inverse-Gaussian displaced by μ and scaled by η

$$p(S(t+\Delta)) = \frac{\delta\sqrt{\eta}}{\sqrt{2\pi}(\ln(S(t+\Delta)) - \mu)^{3/2} S(t+\Delta)} \exp\Big(\{\sqrt{(\ln(S(t+\Delta)) - \mu)/\eta} \\ -\delta/\sqrt{(\ln(S(t+\Delta)) - \mu)/\eta}\}^2\Big). \tag{11}$$

In this discrete-time framework, the stock price can jump to an infinite set of values in a single period and consequently one cannot uniquely value options through the absence of arbitrage with respect to a finite number of securities (e.g., a stock and a bond).

Although absence of arbitrage does not identify unique option values, it establishes the existence of a "risk-neutral" probability density $p^*(S(t+\Delta))$ under which conditional expected returns equal the riskless interest rate. The ratio of the risk-neutral probability density to the true probability density is called the state-price density. In order to value options one must characterize this state-price density. Rubinstein (1976) and Brennan (1979) originally used the log-normal distribution with power utility to describe the state-price density, and called this a "risk-neutral valuation relationship". Subsequent literature shows that we can also derive risk neutral valuation relationships for other combinations of utility functions and distributional assumptions.[11] Heston (1993b) and Stutzer (1996, Eq. (6)) combine power utility and exponential utility with log-exponential distributions. We proceed along the lines of Gerber and Shiu (1993, 1994) and Heston (2004) who specifically illustrate this with the inverse Gaussian distribution.

Assumption. The conditional state-price density takes the power form

$$p^*(S(t+\Delta))/p(S(t+\Delta)) = \tilde{\beta}(S(t+\Delta)/S(t))^{\tilde{\gamma}}. \tag{12}$$

This is the functional form of the state-price density used by Rubinstein (1976) to derive the Black–Scholes formula in a discrete setting with log-normality. Intuitively it says the relative risk aversion does not depend on the magnitude of the return innovation. In particular risk is priced identically across the range of possible returns and does not have "crash-phobia" for extreme positive or negative shocks. The preference parameters $\tilde{\beta}$ and $\tilde{\gamma}$ need not explicitly enter the option formula. Instead one can substitute the interest rate and spot price by imposing the condition that the current values of a bond and stock must equal their discounted expected state-prices

$$e^{-r\Delta} = \mathrm{E}[1 \times \tilde{\beta}(S(t+\Delta)/S(t))^{\tilde{\gamma}}], \\ S(t) = \mathrm{E}[S(t+\Delta) \times \tilde{\beta}(S(t+\Delta)/S(t))^{\tilde{\gamma}}]. \tag{13}$$

The appendix solves these equations for $\tilde{\beta}$ and $\tilde{\gamma}$ and shows the resulting risk-neutral conditional distribution $p^*(S(t+\Delta))$ remains log-inverse-Gaussian with

[11] See for example Vankudre (1986), Smith (1987), and Camara (2003).

corresponding scaling parameter η^* and degrees of freedom parameter $\delta^*(t+\Delta)$

$$\eta^* = v^2\eta^3/(1+1/2v^2\eta^3)^2, \tag{14}$$
$$\delta^*(t+\Delta) = \delta(t+\Delta)\sqrt{\eta/\eta^*}.$$

Since the conditional variance in Eq. (4) is related to the degrees of freedom according to the relation $h(t+\Delta)=\eta^2\delta(t+\Delta)$, we can define the conditional risk-neutral variance $h^*(t+\Delta)=\eta^{*2}\delta^*(t+\Delta)=h(t+\Delta)(\eta^*/\eta)^{3/2}$. Substituting this expression and the risk-neutral parameters (14) into the original process (4) allows us to characterize the risk-neutral dynamics as an IG GARCH process with different parameters.

Proposition 1. *Under the risk-neutral probabilities the stock price follows the process*

$$\log(S(t+\Delta)) = \log(S(t)) + r\Delta + v^*h^*(t+\Delta) + \eta^*y^*(t+\Delta), \tag{15a}$$

$$h^*(t+\Delta) = w^* + bh^*(t) + c^*y^*(t) + a^*h^*(t)^2/y^*(t), \tag{15b}$$

where

$$v^* = v(\eta^*/\eta)^{-3/2}, y^*(t+\Delta) = y(t+\Delta)(\eta^*/\eta)^{-1},$$

$$w^* = w(\eta^*/\eta)^{3/2}, c^* = c(\eta^*/\eta)^{5/2}, a^* = a(\eta^*/\eta)^{-5/2},$$

and $y^*(t+\Delta)$ *has an inverse Gaussian distribution with parameter* $\delta^*(t+\Delta) = h^*(t+\Delta)/\eta^{*2}$.

The risk-neutral dynamic in (15) contains six parameters v^*, w^*, b, c^*, a^* and η^*. Appendix C characterizes the risk-neutral dynamic in a different but equivalent way, which clarifies that in fact η^* is a function of the other parameters. The risk-neutral dynamic therefore only contains five independent parameters. This is analogous to the Black–Scholes formula where the true drift parameter of the stock price is eliminated from the (risk-neutral) pricing formula. In our model option values depend only on the history of the stock price, as summarized by the conditional volatility, and on the five risk-neutral parameters v^*, w^*, b, c^*, and a^*.

To provide some intuition for the pricing in this model, consider the discounted expected payoff, which yields a simple two-parameter generalization of the Black and Scholes (1973) formula.[12]

Proposition 2. *The value of a one-day call option with strike price K is*

$$C = SP\left(\frac{\log(K/S) - r\Delta - v^*h^*}{\eta^{**}}; \delta^{**}\right)$$
$$- Ke^{-r\Delta}P\left(\frac{\log(K/S) - r\Delta - v^*h^*}{\eta^*}; \delta^*\right) \quad \text{for } \eta < 0, \tag{16a}$$

[12]The two-parameter formula in Eq. (16) ultimately depends only on parameters of the true distribution and is more parsimonious than the three-parameter expression of Gerber and Shiu (1993, 1994). This formula also appears in Heston (2004).

and

$$C = S\left[1 - P\left(\frac{\log(K/S) - r\Delta - v^* h^*}{\eta^{**}}; \delta^{**}\right)\right] - Ke^{-r\Delta}\left[1 - P\left(\frac{\log(K/S) - r\Delta - v^* h^*}{\eta^*}; \delta^*\right)\right] \quad \text{for } \eta > 0, \tag{16b}$$

where $\eta^{**} = \eta^*/(1-2\eta^*), \delta^{**} = \delta^*\sqrt{1-2\eta^*}, h^*$ *is the one period variance, and where* $P(.)$ *represents the inverse Gaussian distribution function. The formula converges to the Black–Scholes formula as the degrees of freedom parameter* δ *gets large, which is not surprising given the convergence result for the inverse Gaussian distribution.*

Given the risk-neutral probability distribution and characteristic function (see Appendix A), we can value a call option using the inversion formula of Heston and Nandi (2000) or Bakshi and Madan (2000). This "closed-form" solution is very convenient in empirical work. It avoids the use of Monte Carlo techniques, which in many applications are potentially slower and less accurate.

Proposition 3. *At time* t, *a European call option with strike price* K *that expires at time* T *is worth*

$$\begin{aligned} C &= e^{-r(T-t)} \mathrm{E}_t^*[\mathrm{Max}(S(T) - K, 0)] \\ &= S(t)\left(\frac{1}{2} + \frac{e^{-r(T-t)}}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{K^{-\mathrm{i}\phi} f^*(\mathrm{i}\phi + 1)}{\mathrm{i}\phi}\right]\mathrm{d}\phi\right) \\ &\quad - Ke^{-r(T-t)}\left(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{K^{-\mathrm{i}\phi} f^*(i\phi)}{\mathrm{i}\phi}\right]\mathrm{d}\phi\right), \end{aligned} \tag{17}$$

where $f*(\mathrm{i}\phi)$ *denotes the characteristic function of the risk-neutral version of the process given in Appendix A.*

Fig. 3 provides some intuition for the potential benefits of the IG GARCH model. All pictures represent the price of European call options with an exercise price of \$100 and have moneyness defined as $S(t)/K$ on the horizontal axis. The two top pictures show the difference (in dollars) between either the Heston–Nandi and Black–Scholes prices or the IG GARCH and Black–Scholes prices, for an option with twenty days to maturity. It can be seen that both the Heston–Nandi and inverse Gaussian GARCH models lower the price of out-of-the-money calls and increase the price of in-the-money calls vis-à-vis the Black–Scholes model. The effect is stronger for the IG GARCH, and is also stronger the more negative the skewness parameter η^*. To provide some more perspective for these results, note that the Black–Scholes price is \$0.54 when the current stock price is \$95. It can be seen from Fig. 3 that the difference in prices between the IG GARCH and Black–Scholes model is more than 10 cents, which is substantial on a 54 cent option.

The middle two pictures in Fig. 3 indicate that for long maturities (1 yr) the effects on option values as a function of moneyness are similar to short maturities although of course the dollar values are larger now. The bottom two pictures present results for the

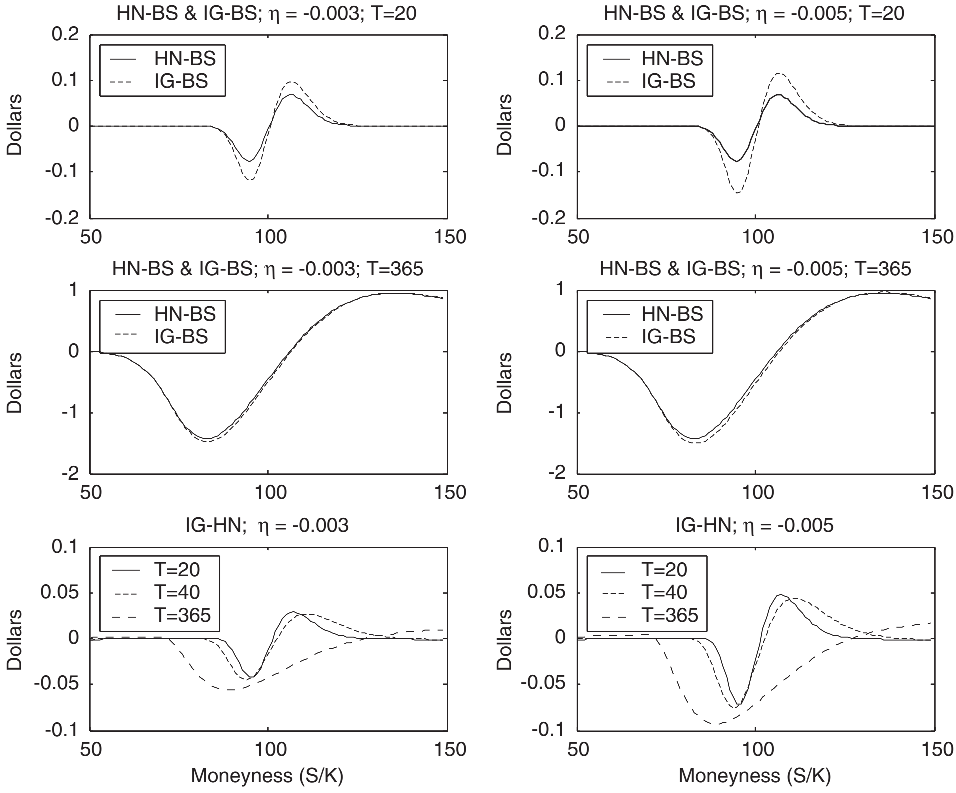

Fig. 3. Comparing Heston–Nandi and IG GARCH option prices.

difference between the IG and HN models. The interesting observation is that the maximum dollar effect of the IG GARCH model does not differ much by maturity, but that the maturity slightly affects at which moneyness the model has maximum impact.

3. Empirical results

3.1. Options data

We now turn to the empirical results on the IG GARCH model. We also provide empirical results on the Heston and Nandi (2000) model (HN), which is nested in the IG GARCH model and therefore provides an interesting benchmark. We implement these models using a data set on S&P500 European call options (SPX) from the CBOE. The liquidity in the SPX option market is relatively high, and this market has therefore been analyzed by a number of researchers.[13]

[13]See for example Bakshi et al. (1997), Chernov and Ghysels (2000), Dumas et al. (1998), Heston and Nandi (2000) and the references therein.

Table 1
In-sample options data (1990–1992)

	$DTM < 20$	$20 < DTM < 80$	$80 < DTM < 180$	*Total*
Panel A. Number of call option contracts				
$S/X < 0.975$	85	982	1057	2124
$0.975 < S/X < 1.00$	132	510	332	974
$1.00 < S/X < 1.025$	132	490	327	949
$1.025 < S/X < 1.05$	114	468	295	877
$1.05 < S/X < 1.075$	109	402	252	763
$1.075 < S/X$	204	725	603	1532
Total	776	3577	2866	7219
Panel B. Average call price				
	$DTM < 20$	$20 < DTM < 80$	$80 < DTM < 180$	*All*
$S/X < 0.975$	0.94	2.66	7.23	4.87
$0.975 < S/X < 1.00$	2.70	7.36	15.89	9.64
$1.00 < S/X < 1.025$	8.01	12.97	22.07	15.41
$1.025 < S/X < 1.05$	15.43	19.90	28.40	22.18
$1.05 < S/X < 1.075$	23.15	27.14	34.89	29.13
$1.075 < S/X$	41.17	43.25	50.84	45.96
All	18.26	17.97	23.71	20.28
Panel C. Average implied volatility from call options				
$S/X < 0.975$	0.1789	0.1649	0.1861	0.1760
$0.975 < S/X < 1.00$	0.1653	0.1772	0.2066	0.1856
$1.00 < S/X < 1.025$	0.1939	0.1990	0.2258	0.2076
$1.025 < S/X < 1.05$	0.2353	0.2270	0.2437	0.2337
$1.05 < S/X < 1.075$	0.3031	0.2562	0.2639	0.2655
$1.075 < S/X$	0.4543	0.3261	0.3058	0.3352
All	0.2773	0.2224	0.2310	0.2317

We split up our data into an in-sample and an out-of-sample period; the models are estimated using the in-sample data only. Table 1 gives an overview of the in-sample data, which consist of option contracts on 156 Wednesdays in the period January 2, 1990–December 31, 1992.[14] We restrict attention to option contracts with maturities between 7 and 180 days and we apply a number of standard filters to the data.[15] The resulting in-sample data set consists of 7219 Wednesday closing quotes. The average call option price for the entire sample is \$20.28 and the average implied Black–Scholes volatility is 23.17%. The well-known post-1987 volatility smirk is evident from Table 1C. The deep in-the-money call option implied volatility is more than 45% for options with less than 20 days to maturity compared with less than 18% for the corresponding out-of-the-money call options. The smirk does flatten

[14]We use Wednesdays only to keep the computations manageable. The same approach was taken by Dumas et al. (1998), Heston and Nandi (2000), and Christoffersen and Jacobs (2004).

[15]We use the same filters as Bakshi et al. (1997).

somewhat with maturity; for options with more than 80 days to maturity the implied volatility is more than 30% for deep in-the-money calls and less than 19% for out-of-the-money calls.

When calculating the option prices one has to account for dividends paid on the stocks in the index. This is dealt with in different ways in the literature. For example, Jiang and van der Sluis (2000) assume a constant dividend yield. We follow Bakshi et al. (1997) and discount future dividends paid out during the life of each option from the current index value corresponding to the option. We use the Treasury yield term-structure in the discounting.

3.2. Estimating the models from returns under the true statistical probability distribution

Because the HN and IG GARCH option valuation models are derived starting from the return dynamics, differences between the models' ability to price options should also be apparent from their ability to fit the dynamics of the underlying asset. In this section we compare the models' ability to fit the returns on the underlying S&P500 index. In addition to the HN and IG GARCH valuation models, we also present estimation results on two nested models, which we label the homoskedastic Gaussian and the homoskedastic IG model, respectively. These models eliminate the GARCH effects from the dynamics in (4) and (7) by imposing $b = a = c = 0$ and $\beta = \alpha = \gamma = 0$, respectively. The resulting homoskedastic models are two- and three-parameter models. A comparison of the fit of these two models indicates the importance of conditional skewness in the absence of heteroskedasticity and volatility clustering.

Table 2 contains the maximum likelihood point estimates of the parameters for the Gaussian and IG models as well as their standard errors, obtained under the true statistical probability distribution.[16] The models are estimated on daily total index returns from CRSP for the period January 3, 1989 through December 20, 2001. We use a time series that is longer than the option dataset because it is well known that it is difficult to estimate return dynamics precisely using short time series.

Perhaps more interesting than the individual parameter estimates are the model properties reported in the right side of Table 2. All models imply an unconditional volatility of approximately 15% per year. The HN model parameter estimates as well as the IG GARCH estimates imply a daily variance persistence of approximately 0.962.

The leverage coefficient reported in Table 2 is the coefficient on conditional variance in the leverage effect in (5b). Thus for the IG GARCH model we report $(c/\eta - \eta^3 a)$, and the corresponding term in the HN model reported is $-2\alpha\gamma$. Notice that the leverage coefficients are negative and quite similar across models.

The log-likelihoods allow several interesting tests. First, it is clear that in the Gaussian case the improvement in fit from the two-parameter homoskedastic model

[16]The standard errors are calculated from the outer product of the vector of gradients evaluated at the ML parameter values.

Table 2
Maximum likelihood estimates of the statistical models of returns. Sample: daily S&P 500 returns from January 3, 1989–December 30, 2001

Gaussian models	λ	ω	β	α	γ		*Persistence*	*Annualized volatility*	*Leverage coefficient*	*Log likelihood*
Homoskedastic	3.106E + 00	9.520E − 0.5						0.1549		10531.6
Std errors	1.799E + 00	1.297E − 06								
Heston–Nandi	2.772E + 00	3.038E − 09	9.026E − 01	3.660E − 06	1.284E + 02		0.9629	0.1577	−9.399E − 04	10872.4
Std errors	1.826E + 00	2.336E − 04	9.370E − 03	4.841E − 07	1.731E + 01					
IG models	v	w	b	a	c	η	*Persistence*	*Annualized volatility*	*Leverage coefficient*	*Log likelihood*
Homoskedastic	3.587E + 03	9.487E − 05				−2.790E − 04		0.1546		10539.2
Std errors	4.604E + 02	1.352E − 06				3.590E − 05				
GARCH	1.625E + 03	3.768E − 10	−1.933E + 01	2.472E + 07	4.142E − 06	−6.162E − 04	0.9620	0.1538	−9.380E − 04	10911.6
Std errors	1.714E + 00	8.621E − 05	5.240E − 06	2.186E + 01	1.356E − 09	1.356E − 09				

to the five parameter HN model is statistically significant at conventional significance levels. The same is true for the improvement in fit of the six-parameter IG GARCH model over the three-parameter homoskedastic IG model. Second, while the HN and IG GARCH models appear to be fairly similar in terms of the above characteristics, their log-likelihoods are dramatically different. The IG model has one extra parameter and provides a much better fit. The increase of over 39 points in log-likelihood going from the HN to the IG model is very large and statistically significant at conventional significance levels. The S&P500 return data thus strongly favor the IG specification. Third, the homoskedastic IG model improves on the fit of the homoskedastic Gaussian model and the extra parameter is significantly estimated. The increase of 7.6 in the log-likelihood is significant at conventional significance levels.

3.3. Estimating the risk neutral models using option prices

While the return-based estimation in Table 2 is interesting, it is well known that for the purpose of option valuation, parameters estimated from option prices are preferable to parameters estimated from the underlying returns (see for instance Chernov and Ghysels, 2000). We therefore estimate the HN and IG models using option prices.

The implementation of these dynamic models is important and deserves comment. First consider the implementation of the Heston and Nandi (2000) model in (7), which has the risk-neutral dynamic

$$\log(S(t+\Delta)) = \log(S(t)) + r\Delta - 0.5h^*(t+\Delta) + \sqrt{h^*(t+\Delta)}z^*(t+\Delta), \tag{18a}$$

$$h^*(t+\Delta) = \omega + \beta h^*(t) + \alpha(z^*(t) - \gamma^*\sqrt{h^*(t)})^2, \tag{18b}$$

Substituting (18b) in (18a) we get conditional variance in terms of observed returns

$$\begin{aligned} h^*(t+\Delta) = \omega + \beta h^*(t) + (\alpha/h^*(t))(&\log(S(t)/S(t-\Delta)) \\ &- r\Delta + 0.5h^*(t) - \gamma^* h^*(t))^2, \end{aligned} \tag{19}$$

The model is implemented by initializing $h^*(0)$ using the unconditional variance $(\omega+\alpha)/(1-\beta-\alpha\gamma^{*2})$ and then using the GARCH updating in (18). We start the iteration 250 trading days before the first option date to allow for the model to find the right conditional variance.[17] We then obtain estimates of the (risk-neutral) parameters ω, α, β and γ^* by minimizing the mean squared error based on the difference between actual option prices and model values using NLS.

$$\mathrm{MSE} = \frac{1}{N}\sum_{i,t}^{N}(C_{i,t} - C_{i,t}(\omega,\alpha,\beta,\gamma^*))^2.$$

[17] In all experiments we set $r\Delta$ equal to 0.05/365 in our use of the dynamic in (7). When valuing options we follow the established tradition of allowing for a time varying term structure of risk free interest rates even though the model strictly speaking assumes constant interest rates.

Now consider the IG model in (4) with its risk-neutral dynamic (15). Substituting (15b) in (15a) yields

$$\begin{aligned} h^*(t+\Delta) = {} & w^* + bh^*(t) + (c^*/\eta^*)(\log(S(t+\Delta)/S(t)) - r\Delta - v^*h^*(t)) \\ & + (a^*h^*(t)^2\eta^*)/(\log(S(t+\Delta)/S(t)) - r\Delta - v^*h^*(t)). \end{aligned} \tag{20}$$

Again, we initialize $h^*(0)$ using the unconditional variance $(w^* + \eta^{*4}a^*)/(1 - a^*\eta^{*2} - b^* - c^*/\eta^{*2})$ and then we use the GARCH updating rule. Minimizing the dollar-based mean squared error gives us estimates of the five independent parameters w^*, η^*, a^*, b^* and c^*. It must be noted that the updating rules (19) and (20) will also be used in the out-of-sample analysis below. The discrete-time GARCH formula enables daily valuation of out-of-sample option values in a straightforward way, which is a strength of this type of model.

Table 3 shows the nonlinear least squares (NLS) estimates of the four models analyzed in Table 2, obtained by minimizing the squared dollar option pricing error, using data for 156 Wednesdays in the 1990–1992 period. Panel A shows the estimates using all the 7219 option contracts in the sample whereas Panel B only uses the 776 options with at most 20 days to maturity.

The parameter estimates in Table 3 refer to the risk neutral representation of the models. Note that all four models contain one less parameter compared to their representation under the true statistical distribution in Table 2. As demonstrated by Heston and Nandi (2000), the HN model performs quite well in valuing SPX options. The root mean squared error (RMSE) for the 7219 contracts with an average price of \$20.28 (Table 1B) is \$1.0043 for the HN model. The IG GARCH model performs slightly better with an RMSE of \$0.9586, which represents a 4.77% improvement. When estimating the models on the short-term contracts, the RMSE is of course lower for both models. The RMSE for the HN model is now \$0.6072 versus \$0.5702 for the IG model, which represents a 6.5% improvement. The relative improvement in fit resulting from the extra parameter in the IG specification is thus larger for shorter-maturity options.[18]

Table 3 allows for a number of other conclusions from comparing the four models as well as from a comparison with the results in Table 2. First, the improvement in fit resulting from adding three extra parameters in the GARCH models vis-à-vis the homoskedastic models is always substantial, but the improvement is smaller in Panel B. Second, the improvement in fit resulting from one extra parameter in the IG models is larger for the homoskedastic models. Perhaps related to this, the estimate of η^* is much larger (in absolute value) for the homoskedastic model. Third, the leverage coefficient is still negative and quite similar across models when looking at options across all maturities (Panel A). When considering only the short-term options in Panel B, the leverage coefficient is considerably larger (in absolute value) in the IG model than in the Gaussian model. Fourth, for the GARCH models the persistence of the variance is higher under the risk neutral than under the true

[18]These differences are clearly small compared with the typical bid-ask spread.

Table 3
Risk neutral parameters from options

Gaussian models	ω	β	α	γ^*		*Persistence*	*Annualized volatility*	*Leverage coefficient*	*Option RMSE*
Panel A. Sample: 7219 *S&P* 500 *wednesday close call options. January* 1, 1990–*December* 31, 1992									
Black–Scholes	1.108E − 04						0.1671		2.0185
Std errors	5.827E − 07								
Heston–Nandi	4.853E − 15	5.771E − 01	2.386E − 07	1.329E + 03		0.9985	0.2003	−6.341E − 04	1.0043
Std errors	6.382E − 09	7.856E − 03	1.478E − 11	1.236E + 01					
IG models	w^*	b	a^*	c^*	η^*	*Persistence*	*Annualized volatility*	*Leverage coefficient*	*Option RMSE*
Homoskedastic	8.704E − 05				−3.983E − 02		0.1481		1.7346
Std errors	5.174E − 07				8.934E − 04				
GARCH	4.852E − 15	4.824E − 01	2.454E + 04	1.473E − 06	−1.848E − 03	0.9975	0.1685	−6.422E − 04	0.9586
Std errors	7.475E − 09	5.508E − 04	1.715E + 02	1.249E − 11	9.390E − 09				
Panel B. Sample: 776 *contracts with less than* 20 *days to maturity*									
Gaussian models	ω	β	α	γ^*		*Persistence*	*Annualized volatility*	*Leverage coefficient*	*Option RMSE*
Black–Scholes	1.054E − 04						0.1630		0.7894
Std errors	2.215E − 06								
Heston–Nandi	2.380E − 15	7.787E − 01	3.088E − 07	8.415E + 02		0.9974	0.1727	−5.197E − 04	0.6072
Std errors	4.600E − 08	2.506E − 02	3.861E − 11	4.800E + 01					
IG models	w^*	b	a^*	c^*	η^*	*Persistence*	*Annualized volatility*	*Leverage coefficient*	*Option RMSE*
Homoskedastic	7.745E − 05				−9.519E − 03		0.1397		0.7417
Std errors	1.552E − 06				8.902E − 03				
GARCH	8.364E − 14	1.245E − 01	5.690E + 04	2.666E − 06	−2.081E − 03	0.9867	0.1422	−7.688E − 04	0.5702
Std errors	1.132E − 07	7.197E − 03	5.390E + 02	3.017E − 09	1.603E − 05				

Table 4
In-sample MSE by maturity and moneyness parameters estimated using all contracts

	$DTM<20$	$20<DTM<80$	$80<DTM<180$	*All*
Panel A. HN model MSE				
$S/X<0.975$	0.2149	0.8469	1.3420	1.0680
$0.975<S/X<1.00$	0.4781	1.1612	1.3124	1.1201
$1.00<S/X<1.025$	0.3454	0.9464	1.0996	0.9156
$1.025<S/X<1.05$	0.2984	0.7860	1.9689	0.7841
$1.05<S/X<1.075$	0.4596	0.9023	1.1601	0.9242
$1.075<S/X$	0.4065	1.0799	1.3162	1.0832
All	0.3789	0.9508	1.2511	1.0086
Panel B. Inverse Gauss GARCH model MSE				
$S/X<0.975$	0.1891	0.7123	1.0143	0.8417
$0.975<S/X<1.00$	0.4378	1.0857	1.2118	1.0409
$1.00<S/X<1.025$	0.3224	0.9050	1.0939	0.8890
$1.025<S/X<1.05$	0.2660	0.7351	1.0471	0.7790
$1.05<S/X<1.075$	0.4147	0.8261	1.2793	0.9170
$1.075<S/X$	0.4094	1.0251	1.2942	1.0490
All	0.3550	0.8711	1.1318	0.9191
Panel C. Ratio of inverse Gauss ARCH to Heston–Nandi MSEs				
$S/X<0.975$	0.8799	0.8411	0.7558	0.7881
$0.975<S/X<1.00$	0.9157	0.9350	0.9234	0.9292
$1.00<S/X<1.025$	0.9334	0.9563	0.9947	0.9710
$1.025<S/X<1.05$	0.8913	0.9352	1.0807	0.9935
$1.05<S/X<1.075$	0.9022	0.9156	1.1028	0.9922
$1.075<S/X$	1.0070	0.9492	0.9833	0.9684
All	0.9369	0.9162	0.9047	0.9113

statistical representation in Table 2. This is true when estimating the models using all maturities, and also when restricting the sample to maturities of at most 20 days.

While Table 3 only reports the aggregate fit of the two GARCH models, Table 4 shows the fit in mean squared error across maturity and moneyness bins. Table 4 uses the parameter estimates based on all the available contracts. Note that generally the dollar fit is better for the cheaper short-term (left column) and out-of-the-money call options (top rows). A comparison of the two models is most easily done using Table 4C which reports the ratio of the MSEs of the two models. The bottom row (marked "All") indicates that the IG model outperforms the HN model for all three maturity bins but mostly so for the longer-term options. The rightmost column (also marked "All") indicates that the IG model outperforms the HN model for all moneyness bins. Most of the improvement comes from the out-of-the-money calls, and to some degree from the deep in-the-money calls, which from put-call parity correspond to deep out-of-the-money puts.

Table 5 provides additional evidence on these improvements in fit. Each panel presents model bias, which is defined as average market price minus average model

Table 5
In-sample bias by maturity and moneyness parameters estimated using all contracts

	$DTM<20$	$20<DTM<80$	$80<DTM<180$	*All*
Panel A. HN model bias				
$S/X<0.975$	−0.0647	−0.3739	−0.4294	−0.3891
$0.975<S/X<1.00$	−0.2006	−0.4044	−0.2551	−0.3259
$1.00<S/X<1.025$	0.0041	−0.0721	0.0456	−0.0209
$1.025<S/X<1.05$	0.3127	0.3040	0.2627	0.2912
$1.05<S/X<1.075$	0.5297	0.5680	0.5970	0.5721
$1.075<S/X$	0.5414	0.7888	0.8073	0.7631
All	0.2222	0.0933	0.0667	0.0966
Panel B. Inverse Gauss GARCH model bias				
$S/X<0.975$	0.0166	−0.3273	−0.2803	−0.2901
$0.975<S/X<1.00$	−0.1922	−0.4500	−0.2113	−0.3337
$1.00<S/X<1.025$	−0.0964	−0.1672	0.0910	−0.0684
$1.025<S/X<1.05$	0.2415	0.2040	0.2950	0.2395
$1.05<S/X<1.075$	0.4776	0.4770	0.6175	0.5235
$1.075<S/X$	0.5318	0.7374	0.7899	0.7307
All	0.1951	0.0528	0.1334	0.1001
Panel C. Black–Scholes model bias				
$S/X<0.975$	−0.0326	−0.7787	−0.7919	−0.7554
$0.975<S/X<1.00$	−0.3369	−0.6579	−0.0284	−0.3998
$1.00<S/X<1.025$	0.1171	0.0793	0.9226	0.3751
$1.025<S/X<1.05$	0.4720	0.8004	1.5793	1.0197
$1.05<S/X<1.075$	0.6294	1.1167	2.1564	1.3905
$1.075<S/X$	0.5785	1.1693	2.0436	1.4348
All	0.2689	0.1705	0.5921	0.3484

price, across moneyness and maturities. Panel C indicates that the Black–Scholes price is too high for deep out-of-the-money calls and too low for deep in-the-money calls. Fig. 3 suggests that the HN and IG GARCH models can potentially address these biases, and Table 5 confirms that this is indeed the case for these data. The biases have the same sign but are much smaller in absolute value in Panel A for the HN model, and smaller yet in Panel B for the IG GARCH model. Interestingly, however, most biases are still present with the same sign.

Fig. 4 illustrates the performance of the models from an implied volatility perspective. It shows the average implied volatility smile for options with less than 20 days to maturity during the 1990–1992 period (solid line). It also depicts the average volatility smile implied from the model based option values from the HN GARCH (dashed) and IG GARCH models (dotted) using the parameter estimates from Table 3B. The figure illustrates that the IG model captures the volatility smile slightly better than the HN model. Unfortunately neither model is fully able to capture the steepness of the smile present in the data. The conclusions are similar from longer maturities but the difference in implied volatility between the two models is smaller.

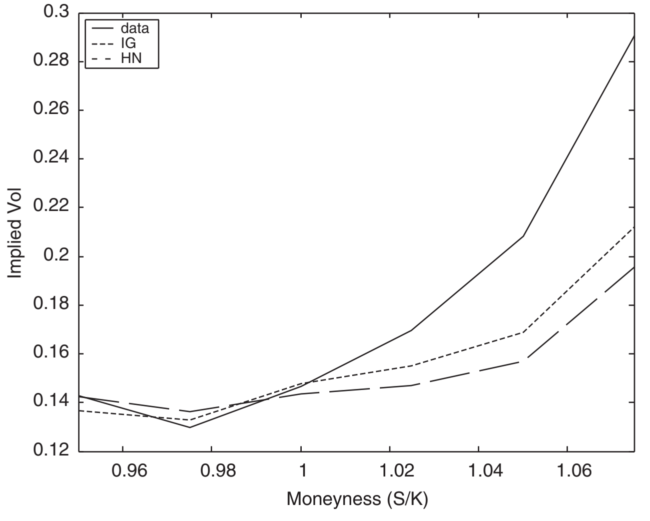

Fig. 4. Average implied volatility smiles. Data and models.

3.4. Out-of-sample analysis

The true test of any estimated model is its out-of-sample performance. We therefore evaluate the models' performance using option data on 52 additional Wednesdays corresponding to the 1993 calendar year.[19] We apply the same filters to these data and again restrict attention to options with maturities between 7 and 180 days. The basic features of the 2985 contracts in the out-of-sample data set are reported across maturity and moneyness bins in Table 6. Notice again the steep smirk for short-term options. Fig. 5 graphs the average implied volatility for the 208 Wednesdays in the in-sample and the out-of-sample period. It is clear that the data display substantial volatility clustering and that there are substantial changes in implied volatility over the four-year period. For the interpretation of the out-of-sample results, it is important to note that the implied volatility in the out-of-sample period is lower than in the in-sample period, and that volatility seems to have been trending downward over the period 1990–1993.

Table 7 shows the overall out-of-sample MSE for the HN and IG models during the out-of-sample period. Each model is evaluated using the parameter estimates from the 1990–1992 sample period. Notice that overall the more parsimonious HN model performs much better than the IG model out-of-sample. This conclusion holds when all options are included (Panel A) but also when the sample is limited to options with less than 20 days to maturity (Panel B). It is also interesting that in

[19]When calcuating option values out-of-sample, we update information on interest rates and stock index values but we use the parameters from the in-sample estimation period. For details on the out-of-sample implementation in this type of discrete time models, see Heston and Nandi (2000), and Christoffersen and Jacobs (2004).

Table 6
Out of sample options data (1993)

	DTM < 20	20 < *DTM* < 80	80 < *DTM* < 180	*Total*
Panel A. Number of call option contracts				
$S/X < 0.975$	3	295	281	579
$0.975 < S/X < 1.00$	46	249	115	410
$1.00 < S/X < 1.025$	53	230	118	401
$1.025 < S/X < 1.05$	47	222	115	384
$1.05 < S/X < 1.075$	43	195	97	335
$1.075 < S/X$	97	463	316	876
Total	289	1654	1042	2985
Panel B. Average call price				
	DTM < 20	20 < *DTM* < 80	80 < *DTM* < 180	*All*
$S/X < 0.975$	0.45	1.65	3.81	2.69
$0.975 < S/X < 1.00$	1.69	5.60	11.86	6.92
$1.00 < S/X < 1.025$	7.88	12.35	19.21	13.77
$1.025 < S/X < 1.05$	17.69	20.90	27.27	22.42
$1.05 < S/X < 1.075$	27.74	29.71	35.59	31.16
$1.075 < S/X$	49.11	49.80	56.38	52.10
All	25.21	23.11	27.93	24.99
Panel C. Average implied volatility from call options				
$S/X < 0.975$	0.1044	0.1067	0.1187	0.1125
$0.975 < S/X < 1.00$	0.1106	0.1205	0.1413	0.1253
$1.00 < S/X < 1.025$	0.1463	0.1438	0.1596	0.1488
$1.025 < S/X < 1.05$	0.2031	0.1722	0.1722	0.1775
$1.05 < S/X < 1.075$	0.2689	0.2003	0.1957	0.2077
$1.075 < S/X$	0.4586	0.2838	0.2433	0.2885
All	0.2725	0.1833	0.1773	0.1898

Panel A the deterioration in the HN model going from in-sample to out-of-sample is minor, whereas the deterioration in the IG model is substantial.

3.5. Interpretation

Table 7 suggests that the IG GARCH model performs poorly out-of-sample. However, Table 8 and Figs. 6 and 7 show that one has to be cautious when investigating the out-of-sample performance of the IG GARCH model. Table 8 analyzes the out-of-sample performance of the HN and the IG GARCH models by maturity and moneyness. Table 8C shows the ratio of the IG to HN model MSEs. Considering the rightmost column we see that the out-of-sample performance of the IG model is particularly poor for the longer-term options. Considering the bottom row we see that the IG model is actually better than the HN model for the deep in-the-money call options. Whereas the in-sample results in Table 4 showed

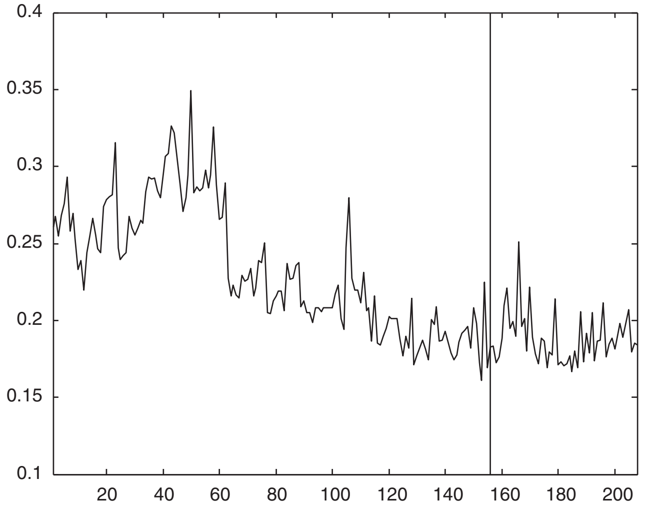

Fig. 5. Weekly average Black–Scholes implied volatility in- and out-of-sample.

Table 7
In-sample and out-of-sample RMSE

Gaussian models	RMSE	
	In-sample (1990–1992)	Out-of-sample (1993)
Panel A. Sample: S&P 500 Wednesday close call options		
Black–Scholes	2.0185	2.7325
Heston–Nandi	1.0043	1.0778
IG models		
Homoskedastic	1.7346	2.5079
GARCH	0.9586	1.3611
Panel B. Sample: contracts with less than 20 days to maturity		
Gaussian models		
Black–Scholes	0.7894	1.0998
Heston–Nandi	0.6072	0.1893
IG models		
Homoskedastic	0.7417	1.0815
GARCH	0.5702	0.8438

improvements in IG over HN for both in-the-money and out-of-the-money options the conditional skewness in the IG model only has an beneficial effect out-of-sample for the deepest in-the-money calls (and equivalently the deepest out-of-the-money puts). We therefore conclude that the out-of-sample performance of the IG GARCH

Table 8
Out-of-sample MSE by maturity and moneyness parameters estimated using all contracts

	$DTM<20$	$20<DTM<80$	$80<DTM<180$	*All*
Panel A. HN model MSE				
$S/X<0.975$	0.0528	1.0567	1.1755	1.1091
$0.975<S/X<1.00$	0.2815	1.9720	2.6395	1.9696
$1.00<S/X<1.025$	0.2459	1.2249	1.8870	1.2903
$1.025<S/X<1.05$	0.3580	0.5323	1.2696	0.7318
$1.05<S/X<1.075$	0.4340	0.4326	0.7333	0.5199
$1.075<S/X$	1.3154	1.3343	0.9482	1.1929
All	0.6547	1.1516	1.3179	1.1616
Panel B. Inverse Gauss GARCH model MSE				
$S/X<0.975$	0.1140	1.9767	2.4794	2.2110
$0.975<S/X<1.00$	0.5762	3.5674	5.0556	3.6492
$1.00<S/X<1.025$	0.4089	2.4676	3.8295	2.5962
$1.025<S/X<1.05$	0.3021	0.9580	2.5487	1.3541
$1.05<S/X<1.075$	0.3594	0.4874	1.3025	0.7070
$1.075<S/X$	1.3154	1.1897	0.8820	1.0926
All	0.7120	1.7518	2.3303	1.8531
Panel C. Ratio of inverse Gauss GARCH to Heston–Nandi MSEs				
$S/X<0.975$	2.1598	1.8707	2.1092	1.9935
$0.975<S/X<1.00$	2.0473	1.8090	1.9154	1.8528
$1.00<S/X<1.025$	1.6627	2.0145	2.0294	2.0121
$1.025<S/X<1.05$	0.8439	1.7996	2.0075	1.8504
$1.05<S/X<1.075$	0.8281	1.1266	1.7762	1.3599
$1.075<S/X$	1.0000	0.8916	0.9302	0.9159
All	1.0874	1.5212	1.7681	1.5953

model is satisfactory for addressing one important remaining problem in option valuation, the valuation of out-of-the-money puts. However, the model fails out-of-sample along a number of other dimensions.

Fig. 6 shows the cumulative RMSEs as a function of time in the in- and out-of-sample periods. The top left panel of Fig. 6 shows the cumulative in-sample RMSEs for the two models. Both models have relatively stable RMSEs over time and the RMSE for the IG model (solid line) tends to be below or very close to the HN model (dashed). The bottom left panel reports the ratio of the cumulative in-sample RMSEs. It appears that the IG model improvements come from early in the sample as well as around week number 100. The full in-sample RMSE ratio is $.9586/1.0043 = .9545$.

More interesting are the corresponding figures for the 52-week out-of-sample period. The top right panel shows the cumulative RMSE for the two models and the bottom panel shows the ratio of the RMSEs. It is quite striking how the in-sample improvement in IG over HN continues 10 weeks into the out-of-sample period. Beyond 10 weeks out-of-sample, both models deteriorate substantially; however, the

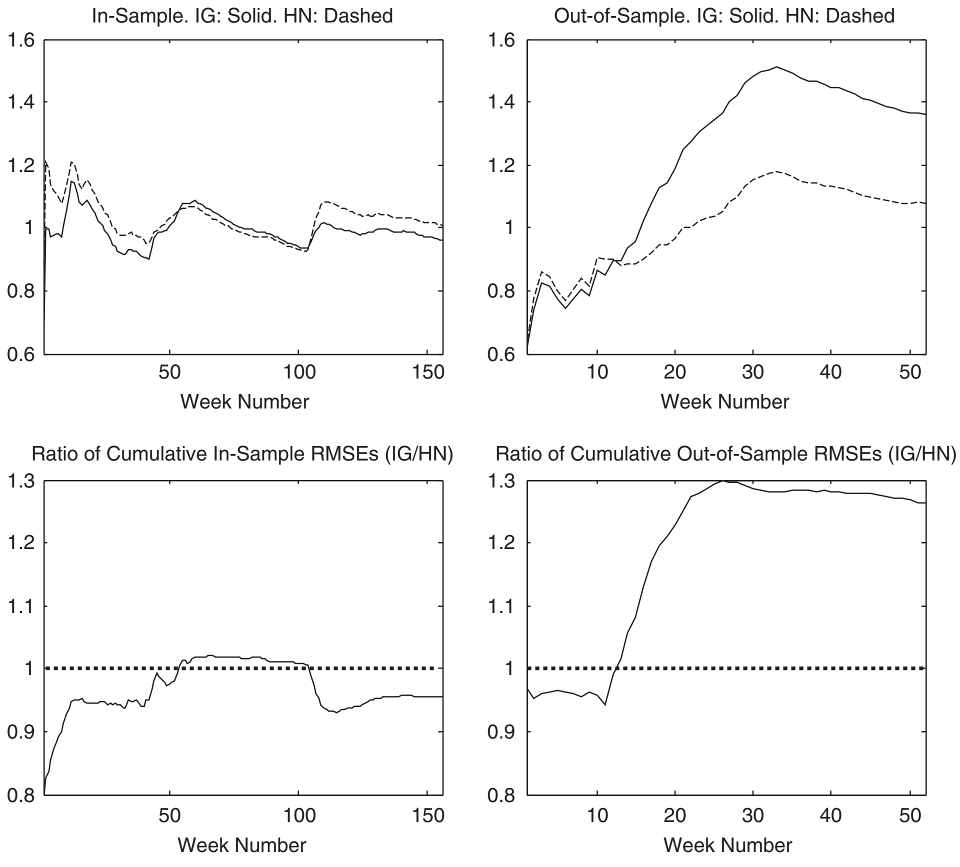

Fig. 6. Cumulative RMSE across weeks. Heston–Nandi and inverse Gaussian GARCH models.

performance of the IG model deteriorates much more than that of the HN model. Consequently, the ratio of the RMSEs increases sharply from week 10 through week 25 until it reaches its full out-of-sample average of $1.3611/1.0778 = 1.2629$. One possible interpretation of these findings is that the conditional skewness parameter η^* in the IG model is difficult to estimate, perhaps because it is truly dynamic. Keeping the IG parameters constant over long periods in an out-of-sample analysis puts heavy demands on the models and causes its performance to deteriorate. More generally, it must of course be noted that this type of long out-of-sample valuation exercise may simply be too ambitious for this type of models, as may be evidenced by the deteriorating RMSE as a function of the forecast horizon.[20]

[20]Bates (2003) argues that, due to the persistence in the implied volatility surface, short out-of-sample periods will tend to favor heavily parameterized models which also perform well in sample. We concur with this view and assess the out-of-sample performance of each model keeping the parameters fixed for up to a year.

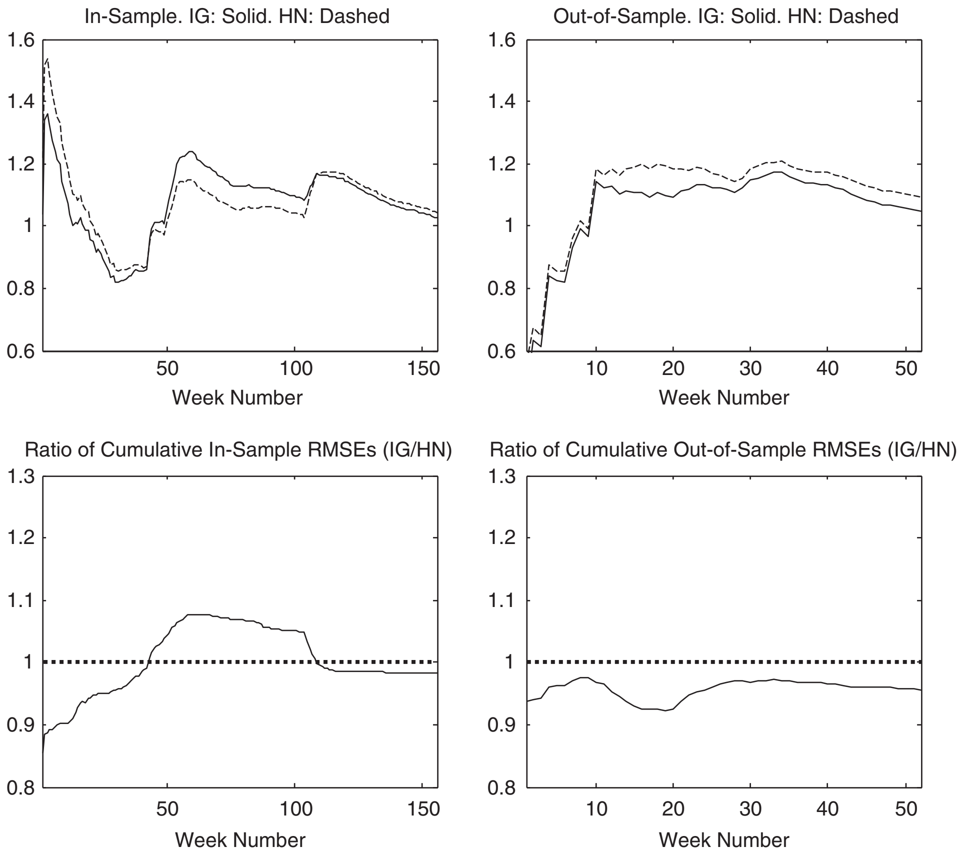

Fig. 7. Cumulative RMSE across weeks. Heston–Nandi and inverse Gaussian GARCH models. Deep in-the-money call options only.

Fig. 7 repeats the analysis in Fig. 6, but only deep-in-the-money calls are included. We see that the overall good out-of-sample performance of the IG GARCH model for deep in-the-money calls (out-of-the-money puts) is confirmed regardless of the out-of-sample horizon.

Fig. 8 provides further background evidence that helps us understand the relative performance of the HN and IG models. The two top panels report the evolution of the conditional standard deviation over time for the two models. Note that we have days on the horizontal axis, instead of weeks as in Fig. 6. The reason is that we use option prices on Wednesdays (once a week), but we use daily information on returns to update the volatility dynamic. It is clear that the sample paths of the volatilities are very similar, even though the bottom left picture indicates that the differences are more pronounced in the out-of-sample period (the beginning of this period is indicated by the vertical line). The bottom right picture plots the difference between the volatilities. While the difference is small in the in-sample period, there is a time

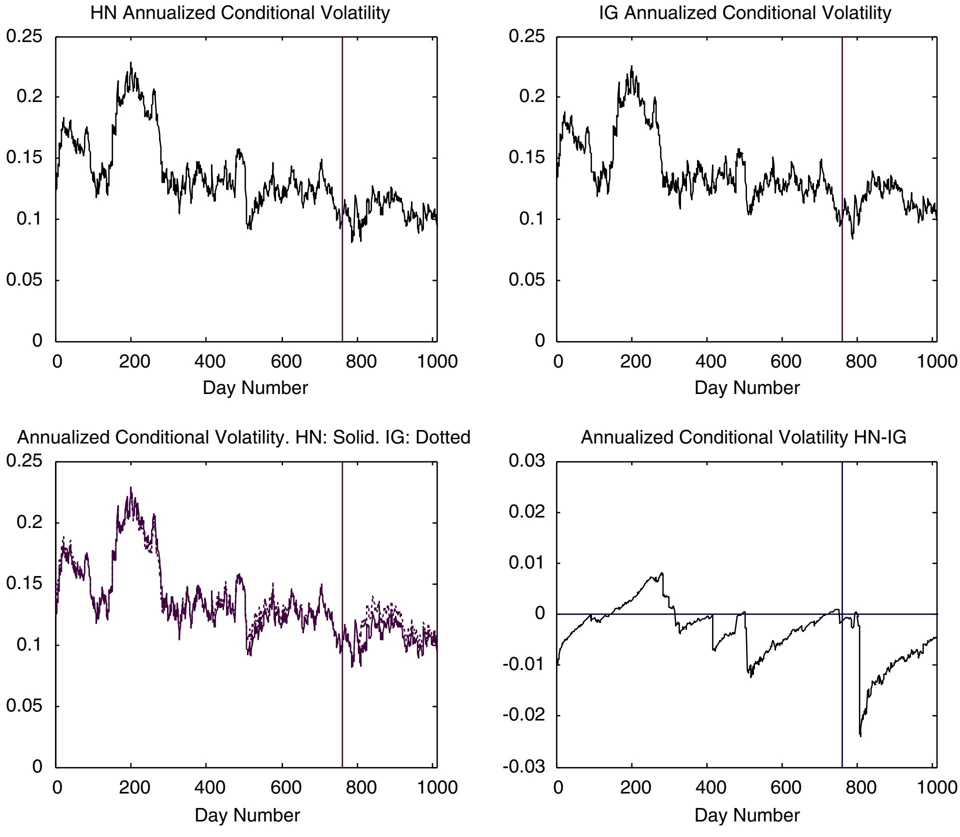

Fig. 8. Volatility paths for the Heston–Nandi and inverse Gaussian GARCH models.

period in the out-of-sample period where the IG volatility suddenly becomes about 2% higher than the HN volatility. This difference in volatility disappears rather slowly, because of the persistence in the estimated processes. This difference in estimated volatility between the two processes coincides with the RMSE differences observed in Fig. 6.

These empirical results are of substantial interest. The literature on continuous-time processes with jumps is growing fast and a large number of papers investigate the impact of jumps in returns and volatility on option prices. However, many of these papers analyze the importance of jumps in-sample, and others estimate model parameters exclusively from returns data or using a rather limited set of option prices (e.g., see. Bates, 2000; Pan, 2002; Duan et al., 2002). Our discrete-time model contains some of the jump models analyzed in these papers as a limit, and it has the advantage for out-of-sample pricing that it is more parsimoniously parameterized. The findings in this paper should therefore be of interest for this literature, because even though the pricing of out-of-the money puts out-of-sample is quite satisfactory, this is not the case for other options. Also, it should be reason for concern that the performance of the model worsens with the out-of-sample horizon relative to a more parsimonious standard model. The empirical analysis in Eraker (2004) has certain similarities with ours, in the sense that he investigates the out-of-sample performance of jump models while holding the parameters fixed for long periods. Interestingly, he finds that the performance of the models worsens with the out-of-sample horizon, but not necessarily more so than that of a standard continuous time model without jumps. In parallel to our result, he finds that adding various types of jumps to a stochastic volatility model does not improve the model's ability to fit observed option values very much.

4. Summary and directions for future work

This paper presents a new option valuation model with analytical solutions based on a return dynamic that contains conditional skewness as well as conditional heteroskedasticity and a leverage effect. We call this model the IG GARCH model. The model nests a standard GARCH model, which contains Gaussian innovations, and the empirical comparison between our new model and the standard GARCH model investigates the importance of modeling conditional skewness. Because the model has a diffusion limit as well as a pure jump limit, such a comparison is also indicative of the incremental value of modeling jumps in returns and volatility in addition to stochastic volatility. Our empirical results are mixed: on the positive side, our new model achieves a better fit than standard models in-sample and up to ten weeks out-of-sample. Also, it performs well out-of-sample for deep in-the-money call options (deep out-of-the-money puts). On the negative side, it performs worse than the standard GARCH model for longer out-of-sample periods and for several other types of options.

Acknowledgements

We would like to thank Tim Bollerslev, Peter Carr, Sanjiv Das, Frank Diebold, Bjorn Eraker, Rob Engle, Eric Ghysels, Nour Meddahi, George Tauchen, Haluk Unal, seminar participants at HEC Montreal, the 14th Annual Derivatives Conference, the Triangle Econometrics Workshop, University of Maryland, the Western Finance Association, and two anonymous referees for helpful comments. Christoffersen and Jacobs would like to thank FQRSC, SSHRC and IFM2 for financial support. John Kozlowski provided expert research assistance. Any remaining errors are ours.

Appendix A. Derivation of the generating function

The dynamics of volatility (4) are particularly convenient because they yield an easily calculated generating function for the spot price. We guess the generating function takes the form

$$f(t;\phi) = \mathrm{E_t}[(S(T)^{\phi}] = S(t)^{\phi}\exp(A(t) + B(t)h(t+\Delta)), \tag{A.1}$$

where it is understood that $f(t;\phi) = f(t;T,\phi), A(t) = A(t;T,\phi), B(t) = B(t;T,\phi)$. At maturity $t = T$ the coefficients must satisfy

$$A(T) = B(T) = 0. \tag{A.2}$$

Applying the law of iterated expectations using the dynamics in Eq. (4) shows

$$\begin{aligned} f(t;\phi) &= \mathrm{E_t}[f(t+\Delta;\phi)] \\ &= \mathrm{E_t}[S(t)^{\phi}\exp(\phi(r\Delta + vh(t+\Delta) + \eta y(t+\Delta)) + A(t+\Delta) \\ &\quad + B(t+\Delta)(w + bh(t+\Delta) + cy(t+\Delta) + ah(t+\Delta)^2/y(t+\Delta))]. \end{aligned} \tag{A.3}$$

Solving this expectation and equating coefficients demonstrates

$$A(t) = A(t+\Delta) + \phi r\Delta + wB(t+\Delta) - \tfrac{1}{2}\ln(1 - 2a\eta^4 B(t+\Delta)), \tag{A.4}$$

$$\begin{aligned} B(t) &= bB(t+\Delta) + \phi v + \eta^{-2} \\ &\quad - \eta^{-2}\sqrt{(1 - 2a\eta^4 B(t+\Delta))(1 - 2cB(t+\Delta) - 2\eta\phi)}, \end{aligned}$$

Appendix B. Derivation of the risk-neutral distribution and process

According to Eq. (4) the spot price $S(t+\Delta)$ equals $\exp(\mu + \eta y(t+\Delta))$, where $\mu = \ln(S(t)) + r\Delta + vh(t+\Delta)$ and $y(t+\Delta)$ has an inverse Gaussian distribution with degrees of freedom δ. Hence the density for $S(t+\Delta)$ is log-inverse-Gaussian

$$\begin{aligned} p(S(t+\Delta)) &= \frac{\delta\sqrt{\eta}}{\sqrt{2\pi}(\ln(S(t+\Delta)) - \mu)^{3/2}S(t+\Delta)} \\ &\quad \times \exp\left(\left\{\sqrt{(\ln(S(t+\Delta)) - \mu)/\eta} - \delta/\sqrt{(\ln(S(t+\Delta)) - \mu)/\eta}\right\}^2\right). \end{aligned} \tag{B.1}$$

The easiest way to derive the result is to assume the risk-neutral density satisfies

$$p^*(S(t+\Delta)) = p(S(t+\Delta))\tilde{\beta}(S(t+\Delta)/S(t))^{\tilde{\gamma}} \tag{B.2}$$

The current values of a bond and stock must equal their discounted expected state-prices

$$e^{-r\Delta} = \mathrm{E}[1 \times \tilde{\beta}(S(t+\Delta)/S(t))^{\tilde{\gamma}}], \tag{B.3}$$

$$S(t) = \mathrm{E}[S(t+\Delta) \times \tilde{\beta}(S(t+\Delta)S(t))^{\tilde{\gamma}}].$$

Using the generating function, the solution is

$$\tilde{\gamma} = (\eta^{-1} - (1 + v^2\eta^3/2)^2/(v^2\eta^3))/2, \tag{B.4}$$

$$\tilde{\beta} = \exp(-\tilde{\gamma} r\Delta - \tilde{\gamma} v h(t+\Delta) - \delta + \delta\sqrt{1-2\tilde{\gamma}}).$$

Substituting these values into Eq. (B.2) shows the risk-neutral distribution is log-inverse-Gaussian with corresponding risk-neutral parameters η^* and $\delta^*(t+\Delta)$.

Appendix C. An alternative representation of the return dynamic

To provide more intuition for the risk neutralization, consider the parameter combination $\theta(t) = \delta(t)\eta^{1/2} = h(t)\eta^{3/2}$. We can rewrite the dynamics (4a), (4b) in terms of this parameter

$$\log(S(t+\Delta)/(S(t)) = r\Delta + v_\theta \theta(t+\Delta) + \varepsilon(t+\Delta), \tag{C.1}$$

$$\theta(t+\Delta) = w_\theta + b\theta(t) + c_\theta \varepsilon(t) + a_\theta \theta(t)^2/\varepsilon(t),$$

where

$$\theta(t) = \eta^{-3/2} h(t), v_\theta = \eta^{3/2} v, \varepsilon(t) = \eta y(t),$$
$$w_\theta = \eta^{-3/2} w, c_\theta = \eta^{-5/2} c, a_\theta = \eta^{5/2} a,$$

and $\varepsilon(t)$ has an inverse Gaussian distribution with scale parameter η and degrees of freedom parameter $\delta(t) = \theta(t)\eta^{-1/2}$.

Interestingly $\theta(t)$ is the same in the true and risk-neutral probabilities. Moreover, the dynamic (C.1) still holds under the risk-neutral measure with the same parameters $v_\theta, w_\theta, b, c_\theta$, and a_θ. The only difference between the true and risk-neutral dynamic is that the innovation $\varepsilon^*(t)$ is distributed inverse Gaussian with scale parameter

$$\eta^* = v_\theta^2/(1 + \tfrac{1}{2} v_\theta^2)^2. \tag{C.2}$$

If we define the risk-neutral variance as $h^*(t) = \eta^{*3/2}\theta(t)$ then the risk-neutral dynamics in Eq. (15a), (15b) follow directly from the dynamics of $\theta(t)$. It can also be seen that (C.2) is equivalent to (14). However, (C.2) provides us with extra intuition because it can now be seen that the option value depends on the path of spot prices, the riskless rate, and the five parameters $v_\theta, w_\theta, b, c_\theta$, and a_θ, which means that compared with the parameterization of the dynamic in (4), one parameter has been eliminated from the pricing formula. The option value can be expressed without the parameter η^*, just as the Black–Scholes formula does not depend on the true statistical mean return, and in both cases this is due to the presence of the riskless rate in the pricing formula.

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