Option pricing and hedging with one-step unscented Kalman filtered factors in non-affine stochastic volatility models

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Abstract

New pricing and hedging strategies are proposed for two non-affine auto-regressive conditionally homo- and heteroscedastic stochastic factor models with non-predictable drift which allows to account for leverage effects. We consider a factor dependent exponential linear pricing kernel with stochastic risk aversion parameters and implement both pricing and hedging for these models estimated via the one-step unscented Kalman filter. This technique proves to outperform standard GARCH and Heston-Nandi based strategies in terms of a variety of considered criteria in an empirical exercise using historical returns and options data.

Keywords: autoregressive stochastic volatility models, bivariate Esscher transform, option pricing, local risk minimization hedging, unscented Kalman filter.
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1 Introduction

Empirical findings show strong evidence against several assumptions of the Black-Scholes (1973) option valuation model. The pricing and hedging of index options have been thus extensively studied in the context of stochastic volatility (SV) models in both discrete and continuous time.

In discrete-time settings, there are two main approaches for modeling the well documented volatility smile/smirk. The first direction is represented by the family of Generalized Autoregressive Conditionally Heteroskedastic (GARCH) models introduced by Engle (1982) and Bollerslev (1986), which have become very popular due to their ability to capture several of the "stylized facts" observed in financial markets, such as volatility clustering, fat tails, leverage effects, etc. Duan (1995) proposed a GARCH option pricing model driven by Gaussian innovations and based on a stochastic discount factor (SDF) which contains only a market price of equity risk. Due to its non-affine structure, there are no closed-form solutions available for pricing European style options. An alternative approach was provided by Heston and Nandi (2000), who introduced an affine class of Gaussian GARCH models which admit a semi-closed form expression for the unconditional Laplace transform of the log-asset price process. In order to improve the empirical fit of the Gaussian GARCH model, several extensions have been proposed in the literature, by including skewed and heavy-tailed innovations (see for instance Christoffersen *et al.* (2006), Chorro *et al.* (2012) and Badescu *et al.* (2008)), realized volatility measures (see, for example, Stentoft (2008), Hansen *et al.* (2011), Corsi *et al.* (2013) and Christoffersen *et al.* (2014)), multi-factor volatility dynamics (see Christoffersen *et al.* (2008) and Majewski *et al.* (2015)), variance dependent pricing kernels (see Christoffersen *et al.* (2013), Bormetti *et al.* (2016)) or combinations of the above (see Babaoglu *et al.* (2014) and Badescu *et al.* (2017)). The general finding in the literature is that non-affine GARCH models outperforms the affine counterparts when fitted to asset returns, options and VIX data (see e.g. Hsieh and Rithcken (**?**), Christoffersen *et al.* (**?**), Kanniainen *et al.* (**?**) among others).

The second class consists of the Autoregressive Stochastic Volatility (ARSV) models (see Taylor (1986), Harvey *et al.* (1994) and Taylor (2005)), which can be viewed as discretizations of continuous-time SV models as they allow for separate driving noises for the asset-returns and the volatility/factor processes. Even though they provide more flexible dynamics than their GARCH counterparts[5] have not been as popular in terms of pricing and hedging financial derivatives, mainly due to estimation related issues, and only recently they got some attention by the research community. For example, Darolles *et al.* (2006) introduced the Compound Autoregressive (CAR) process which, equipped with an exponential affine pricing kernel, has been used for derivative valuation by Bertholon *et al.* (2008). The CAR framework has been extended by Khrapov and Renault (2016) who proposed an affine bivariate model for asset returns variance which allows for leverage effect and volatility feedback. Following their ap-

[5]Carnero *et al.* (**?**) showed that ARSV models are more flexible than GARCH models in representing the relationship between the persistence of shocks to the volatility, first-order autocorrelation of squared returns, and excess kurtosis. In particular, they argued that, unlike in the GARCH cases, there is no need for the return noise in an ARSV(1) model to be leptokurtic or the persistence coefficient to be closer to one, in order to allow simultaneously for fat tails and small first order autocorrelation of squared observations. The choice of a simple one-factor Gaussian ARSV model in this paper for pricing and hedging European options is also justified by this finding.

proach, Han (?) proposed a class of affine high-frequency based SV option pricing models (HEAVY-SV) (see also Shepard and Sheppard (?) for GARCH type HEAVY models). Discrete-time affine stochastic volatility models with conditional skewness have been considered in Feunou and Tédongap (2012). They propose an ARSV option pricing model based on conditional Inverse Gaussian returns and autoregressive Gamma latent factors (see also Gourieroux and Jasiak (2006)). They show that these option pricing models outperform existing affine GARCH and continuous time jump diffusion models. However, in their framework the model parameters cannot be estimated using information coming from returns and options, since there is no existing link between the physical and the risk-neutral worlds. More recently, Bormetti *et al.* (?) introduce a class of discrete-time affine SV pricing models which allows the filtering of the latent conditional volatility from both returns and realized variance measures. They showed that their model outperforms the two-component GARCH model of Christoffersen *et al.* (2008), as well as the Realized Variance Autoregressive Gamma model with Heterogeneous parabolic Leverage from Majewski *et al.* (2015)).

The key ingredient in deriving pricing expressions in all aforementioned studies is the affine structure of the underlying models, which allows the recursive computation of the unconditional moment generating functions of asset returns under both physical and risk-neutral pricing measures, provided an appropriate exponential affine stochastic discount factor is used. However, the (affine) constraints typically imposed in the conditional mean return and volatility dynamics are somewhat restrictive. For example, Khrapov and Renault (2016) assumed that both the conditional Laplace transform of the variance process and the bivariate Laplace transform of the asset returns and variance process have an exponential affine form. Although this assumption leads to semi-explicit pricing formulae, there are not many distribution candidates for the driving noise process which satisfy the required condition. For instance, most empirical studies rely on models constructed using a conditional Gaussian distribution for the asset returns and an autoregressive Gamma process for the latent factor. Another drawback of the affine models is that the exponential pricing kernel associated to the pricing methodology is typically based on constant equity and variance risk preference parameters, which is not consistent with empirical findings.[6]

In order to address some of these issues, this paper proposes a non-affine discrete-time SV option pricing model which allows for both stochastic prices of risk and leverage effect. Our main contributions are described in detail below.

First, we consider a simple one-factor non-affine ARSV process constructed based on a conditional Gaussian distribution for the two driving noise processes which govern the return and factor dynamics. The factor process can be equipped with both homoscedatic and heteroscedastic structures. As in the affine modelling literature, we use an exponential pricing kernel which, in our situation, contains stochastic equity and factor risk premiums. We show that having both risk premiums constant at the same time is not consistent with our setting, and that the market price of factor risk has a linear

[6]See Badescu *et al.* (2017) for a discussion in this sense on the variance dependent pricing kernel implementation within a GARCH setup.

dependence of the factor process. Under this setting, we prove that the the risk-neutral bivariate Laplace transform has an exponential quadratic form in the latent factor (e.g. unlike in the affine setting of Khrapov and Renault (2016)). Moreover, our change of measure preserves the conditional distribution of the factor process, but not that of the asset returns.

The second main contribution of the paper relies on the implementation of a novel sequential estimation and calibration procedure to observed historical returns and option prices. Since the unconditional risk-neutral Laplace transform of the asset returns cannot be computed explicitly, and consequently a (spectral) generalized method of moments (GMM) estimation cannot be obtained using standard methods, we propose a sequential estimation procedure based on the information on both asset returns and option prices. First, using historical returns, we estimate the model parameters following the one-step Kalman method. This method consists of performing a likelihood estimation in which the unobserved factor values are treated as unknown parameters in the optimization problem. In the second stage, the pricing kernel parameters are estimated based on the cross section of options written on the same underlying.

Third, we conduct an extensive empirical analysis to test the pricing and hedging performance of our model for a large panel of European put and calls options. The option prices are computed based on Monte-Carlo simulation of weighted payoffs under the real-world measure, where the weights are given by the Radon-Nikodym derivatives of the corresponding measure change. Following the approach developed by Föllmer and Sondermann (1986), the hedging ratios are computed using the local risk minimization with respect to this risk-neutral measure. The benchmark used in our analysis is the Heston and Nandi (2000) affine GARCH model (HNGARCH). Our results show that both ARSV specifications considered consistently outperforms the GARCH benchmark for all classes of moneyness and maturity considered, both in and out-of-sample. For example, when the latent the factor risk is priced, the in-sample overall performance of the HNGARCH model as measured via the Implied Volatility Root Mean Squared Error (IVRMSE) is improved by around 60% by both factor specifications. This improvement is further supported by the empirical results of the out-of-sample exercise, ranging from 37% to 71% for the homoscedastic factor structure, and from 48% to 68% for the heteroscedastic one, depending on the moneyness class considered. Finally, a similar finding is revealed by the hedging exercise in the sense that the ARSV outperforms both the GARCH and the Black-Scholes model for almost all maturities and moneyness groups. The overall improvement over GARCH, as measured by a Normalized Hedging Error (NHE) indicator, is of 28%.

The rest of the paper is organized as follows. In Section 2 we introduce the underlying discrete-time stochastic volatility models. The pricing kernel and risk-neutral derivations are presented in Section 3. The estimation methodology and the numerical results are illustrated in Section 4. Section 5 concludes the paper.

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2 Non-affine one-factor autoregressive SV models

Consider a discrete time economy with the set of trading dates $\mathcal{T} = \{t | t = 0, 1 \ldots, T\}$, where T represents the terminal time. We assume that the market consists of a bond and a stock with corresponding price processes denoted by $B := \{B_t\}_{t\in\mathcal{T}}$ and $S := \{S_t\}_{t\in\mathcal{T}}$, respectively. Let r be the instantaneous constant risk free rate of return and assume that bond price dynamics is given by:

$$\begin{aligned} B_t &= e^{rt}, \ \ t \in \mathcal{T}, \\ B_0 &= 1. \end{aligned}$$

The log-return process, defined by $y := \{y_t\}_{t\in\mathcal{T}} = \{\log S_t/S_{t-1}\}_{t\in\mathcal{T}}$, is determined by the following equation:

$$y_t = \mu + \lambda f_t^2 + f_t \epsilon_t, \quad \epsilon_t \sim \mathbf{N}(0, 1), \tag{2.1}$$

and is driven by a one-dimensional factor process $f := \{f_t\}_{t\in\mathcal{T}}$ whose dynamics is prescribed by two different models. As the first instance we consider an AR(1) process for the dynamics of f, namely

$$f_t = \gamma + \phi f_{t-1} + \omega_t, \quad \omega_t \sim \mathbf{N}(0, \sigma_\omega^2) \tag{2.2}$$

and call (2.1)-(2.2) jointly the one-factor autoregressive stochastic volatility (OFARSV) model. In order to allow for more flexible dynamics of the factor process, we also consider the conditionally heteroscedastic model for f, namely

$$f_t = \gamma + \phi f_{t-1} + f_{t-1}\omega_t, \quad \omega_t \sim \mathbf{N}(0, \sigma_\omega^2) \tag{2.3}$$

and say in this case that y in (2.1) and f in (2.3) are governed by the one-factor autoregressive conditionally heteroscedastic stochastic volatility (OFARCHSV) model.

We use the following assumptions and notations:

(1) The return process $y := \{y_t\}_{t\in\mathcal{T}}$ represents the observable quantity, while $f := \{f_t\}_{t\in\mathcal{T}}$ is an unobservable one-dimensional latent factor process.[7]

(2) The innovations $\epsilon := \{\epsilon_t\}_{t\in\mathcal{T}}$ and $\omega := \{\omega_t\}_{t\in\mathcal{T}}$ are two sequences of i.i.d. Gaussian random variables which are both serially independent and also independent of each other.

(3) The model parameter vector is denoted by $\boldsymbol{\theta} := (\mu, \lambda, \gamma, \phi, \sigma_\omega) \in \mathbb{R}^5$ and is assumed to satisfy standard stationarity conditions.

[7]This setting can be easily extended to the multi-factor case. In order to keep the presentation simple, we restrict our attention to the one-factor setup.

(4) Let (Ω, P) be the probability space under which the OFARSV model (2.1)-(2.2) and the OFARCHSV model in (2.1) and (2.3) have been introduced; we refer to P as the physical (real-world) probability measure. We denote by $\mathcal{F}_t := \sigma(y_s, f_s; s \leq t)$ the sigma-algebra generated by both the return and the factor processes, and introduce the augmented filtration $\mathcal{G}_t := \sigma(y_s, f_s, f_t; s \leq t-1)$.

We notice that y_t is $\mathcal{F}_t$-measurable, but it is not $\mathcal{G}_t$-measurable, while, f_t is measurable with respect to both filtrations. Furthermore, the noise processes ϵ_t is independent of $\mathcal{G}_t$ and w_t is independent of $\mathcal{F}_{t-1}$. From equation (2.1) and from the two considered equations for the factor process, namely (2.2) and (2.3), we can conclude that the asset returns are conditionally Gaussian distributed given $\mathcal{G}_t$, that is,

$$y_t|_{\mathcal{G}_t} \overset{P}{\sim} \mathbf{N}\left(\mu + \lambda f_t^2, f_t^2\right), \tag{2.4}$$

while the latent factor f_t has an $\mathcal{F}_{t-1}$-conditional Gaussian distribution, namely,

$$f_t|_{\mathcal{F}_{t-1}} \overset{P}{\sim} \mathbf{N}\left(m_{t-1}, s_{t-1}^2\sigma_\omega^2\right), \tag{2.5}$$

with $m_t = \gamma + \phi f_t$. Both considered OFARSV and OFARCHSV models are incorporated in (2.5) via setting $s_t := 1$ or $s_t := f_t$ for all $t \in \mathcal{T}$, respectively.

We start by evaluating the joint cumulant generating function of the asset return and the factor process, conditionally on the information set $\mathcal{F}_t$.

Proposition 2.1 *If the asset returns follow the dynamics in (2.1)-(2.2), then the $\mathcal{F}_{t-1}$-conditional bivariate cumulant generating function of y_t and f_t under P is given by:*

$$C^P_{(y_t,f_t)}\left(z_1, z_2|\mathcal{F}_{t-1}\right) := \log \mathrm{E}^P\left[\exp\left(z_1 y_t + z_2 f_t\right)|\mathcal{F}_{t-1}\right] = z_1\mu - \frac{1}{2}\log u(z_1, s_{t-1}) + \frac{v\left(z_2, m_{t-1}, s_{t-1}\right)}{u(z_1, s_{t-1})} - \frac{m_{t-1}^2}{2s_{t-1}^2\sigma_\omega^2}. \tag{2.6}$$

Here, $u(\cdot)$ and $v(\cdot, m_t, s_t)$ are two real-valued functions given by:

$$u(z_1, s_t) = 1 - z_1\left(z_1 + 2\lambda\right)s_t^2\sigma_\omega^2, \tag{2.7}$$

$$v(z_2, m_t, s_t) = \frac{s_t^2\sigma_\omega^2}{2}\left(z_2 + \frac{m_t}{s_t^2\sigma_\omega^2}\right)^2, \tag{2.8}$$

which satisfy the conditions $u(0, s_t) = 1$ and $v(0, m_t, s_t) = m_t^2/(2s_t^2\sigma_\omega^2)$.

Note that $C^P_{(y_t,f_t)}\left(z_1, z_2|\mathcal{F}_{t-1}\right)$ is well defined only for the values z_1 such that $z_1\left(z_1 + 2\lambda\right)s_{t-1}^2\sigma_\omega^2 < 1$ and that, furthermore, it is not an affine function of the factor process. In fact, if we expand m_{t-1} in (2.6), we observe that $C^P_{(y_t,f_t)}\left(z_1,z_2|\mathcal{F}_{t-1}\right)$ has a quadratic dependence with respect to f_{t-1}. We now characterize the univariate conditional cumulant generating functions of y_t and f_t. Indeed, taking $z_1 = 0$ in (2.6), it is easy to verify that the $\mathcal{F}_{t-1}$-conditional cumulant generating function of the latent factor corresponds to that of a Gaussian random variable with mean m_{t-1} and variance $s_{t-1}\sigma_\omega^2$:

$$C_{f_t}^P\left(z_2|\mathcal{F}_{t-1}\right) = C_{(y_t,f_t)}^P\left(0, z_2|\mathcal{F}_{t-1}\right) = z_2 m_{t-1} + \frac{1}{2} z_2^2 s_{t-1}^2 \sigma_\omega^2. \tag{2.9}$$

Similarly, the cumulant generating function of y_t conditionally on $\mathcal{F}_{t-1}$ can be obtained from (2.6) as follows:

$$C_{y_t}^P\left(z_1|\mathcal{F}_{t-1}\right) = C_{(y_t,f_t)}^P\left(z_1, 0|\mathcal{F}_{t-1}\right) = z_1\mu - \frac{1}{2}\log u(z_1, s_{t-1}) + \frac{m_{t-1}^2}{2s_{t-1}^2\sigma_\omega^2}\left(\frac{1}{u(z_1, s_{t-1})} - 1\right). \tag{2.10}$$

Taking the first and the second order derivatives of $C_{y_t}^P\left(z_1|\mathcal{F}_{t-1}\right)$ evaluated at $z_1 = 0$, we obtain the conditional mean and variance of the asset returns under the physical measure:

$$\mathrm{E}^P\left[y_t|\mathcal{F}_{t-1}\right] = \mu + \lambda\left(m_{t-1}^2 + s_{t-1}^2\sigma_\omega^2\right), \tag{2.11}$$

$$\mathrm{Var}^P\left[y_t|\mathcal{F}_{t-1}\right] = m_{t-1}^2 + s_{t-1}^2\sigma_\omega^2 + 2\lambda^2 s_{t-1}^2\sigma_\omega^2\left(2m_{t-1}^2 + s_{t-1}^2\sigma_\omega^2\right). \tag{2.12}$$

The conditional covariance between the asset return process and the latent factor is given by:

$$\mathrm{Cov}^P\left(y_t, f_t|\mathcal{F}_{t-1}\right) = 2\lambda m_{t-1} s_{t-1}^2\sigma_\omega^2. \tag{2.13}$$

Finally, we can compute the covariance between the asset return process and its one-step ahead conditional variance given the information set $\mathcal{F}_{t-1}$ as:

$$\mathrm{Cov}^P\left(y_t, \mathrm{Var}^P\left[y_t|\mathcal{F}_{t-1}\right]|\mathcal{F}_{t-1}\right) = 2\lambda\phi s_{t-1}^2\sigma_\omega^2(1 + 4\lambda^2 s_{t-1}^2\sigma_\omega^2)\left[2\gamma m_{t-1} + \phi\left(2m_{t-1}^2 + s_{t-1}^2\sigma_\omega^2\right)\right]. \tag{2.14}$$

Thus, both parameters λ and ϕ contribute to the leverage effect.

3 Pricing kernel and risk-neutral dynamics

The choice of the risk-neutral measure plays an important role in the pricing and hedging of financial derivatives. In this section we introduce an exponential linear pricing kernel via a bivariate conditional Esscher transform.

First, for any $t \in \mathcal{T}$, we define the following stochastic process $N = \{N_t\}_{t\in\mathcal{T}}$:

$$N_t = \exp\Big(\eta_{1t} y_t + \eta_{2t} f_t - C_{(y_t,f_t)}^P\left(\eta_{1t}, \eta_{2t}|\mathcal{F}_{t-1}\right)\Big). \tag{3.1}$$

Here, $\eta_{1t} = \{\eta_{1t}\}_{t\in\mathcal{T}}$ and $\eta_{2t} = \{\eta_{2t}\}_{t\in\mathcal{T}}$ are two $\mathcal{F}_t$-predictable processes representing the market prices of equity risk and factor risk, respectively. Similar stochastic discount factors have been proposed in the context of discrete-time stochastic volatility models (see e.g. Khrapov and Renault (2016) and Bormetti *et al.* (2016) among others), but only with constant market prices of risk. However, as showed later in this section, it is not possible to have both η_{1t} and η_{2t} constants in our setting.

The pricing measure Q is introduced through the following Radon-Nikodym derivative process denoted by $Z = \{Z\}_{t\in\mathcal{T}}$:

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_T} := Z_T = \prod_{t=1}^{T} N_t = \prod_{t=1}^{T} \exp\left(\eta_{1t} y_t + \eta_{2t} f_t - C^P_{(y_t,f_t)}\left(\eta_{1t},\eta_{2t}|\mathcal{F}_{t-1}\right)\right). \tag{3.2}$$

It is straightforward to check that Q is an equivalent probability measure with respect to P. Indeed, we first notice that Z_t defined in (3.2) with $Z_0 = 1$ is an $\mathcal{F}_t$-martingale:

$$\mathrm{E}^P\left[Z_t|\mathcal{F}_{t-1}\right] = Z_{t-1}\mathrm{E}^P\left[N_t|\mathcal{F}_{t-1}\right] = Z_{t-1}\mathrm{E}^P\left[\exp\left(\eta_{1t}y_t + \eta_{2t}f_t - C^P_{(y_t,f_t)}\left(\eta_{1t},\eta_{2t}|\mathcal{F}_{t-1}\right)\right)|\mathcal{F}_{t-1}\right] = Z_{t-1}.$$

Since Z_T is positive by definition and $\mathrm{E}^Q[1] = \mathrm{E}^P\left[Z_T\right] = \mathrm{E}^P\left[Z_0\right] = 1$, it follows that Q is a well-defined probability measure equivalent to P. In order for Q to be a risk-neutral measure, we need to impose a constraint on the market prices of risk such that the discounted asset prices are martingales after the change of measure. To achieve this, we characterize the conditional bivariate cumulant generating function of the asset returns and latent factor under the pricing measure defined above.

Proposition 3.1 *If the asset returns follow the dynamics in (2.1)-(2.2), then the $\mathcal{F}_{t-1}$-conditional bivariate cumulant generating function of y_t and f_t under Q defined in (3.2) is given by:*

$$C^Q_{(y_t,f_t)}\left(z_1,z_2|\mathcal{F}_{t-1}\right) = z_1\mu - \frac{1}{2}\log\frac{u\left(z_1+\eta_{1t},s_{t-1}\right)}{u\left(\eta_{1t},s_{t-1}\right)} + \frac{v\left(z_2+\eta_{2t},m_{t-1},s_{t-1}\right)}{u\left(z_1+\eta_{1t},s_{t-1}\right)} - \frac{v\left(\eta_{2t},m_{t-1},s_{t-1}\right)}{u\left(\eta_{1t},s_{t-1}\right)}. \tag{3.3}$$

Here, the functions $u(\cdot,s_t)$ and $v(\cdot,m_t,s_t)$ are those defined in (2.7)-(2.8).

Using the above representation, we can re-write the martingale constraint for the discounted asset price, $\mathrm{E}^Q\left[\exp y_t|\mathcal{F}_{t-1}\right] = \exp r$, as $C^Q_{(y_t,f_t)}\left(1,0|\mathcal{F}_{t-1}\right) = r$. Thus, it follows that for any $t\in\mathcal{T}$, the market prices of risk η_{1t} and η_{2t} must satisfy the equation below:

$$\mu - r - \frac{1}{2}h_1\left(1,\eta_{1t},s_{t-1}\right) + v\left(\eta_{2t},m_{t-1},s_{t-1}\right)h_2\left(1,\eta_{1t},s_{t-1}\right) = 0, \tag{3.4}$$

where $h_1\left(\cdot,\eta_{1t},s_{t-1}\right)$ and $h_2\left(\cdot,\eta_{1t},s_{t-1}\right)$ are the two real-valued functions given by:

$$h_1\left(z_1,\eta_{1t},s_{t-1}\right) := \log\frac{u\left(z_1+\eta_{1t},s_{t-1}\right)}{u\left(\eta_{1t},s_{t-1}\right)}, \tag{3.5}$$

$$h_2\left(z_1,\eta_{1t},s_{t-1}\right) := \frac{1}{u\left(z_1+\eta_{1t},s_{t-1}\right)} - \frac{1}{u\left(\eta_{1t},s_{t-1}\right)}. \tag{3.6}$$

Unlike in the univariate conditional Esscher transform case where the market price of risk is uniquely determined by the martingale constraint, here we have an infinite number of pairs (η_{1t},η_{2t}) which solve (3.4). In general, we need to use option prices in order to calibrate for one of these parameters. For example, we notice that using (3.4) we can solve for η_{2t} as a function of η_{1t} as follows:

$$\eta_{2t} = -\frac{m_{t-1}}{s_{t-1}^2 \sigma_\omega^2} \pm \sqrt{\frac{2}{s_{t-1}^2 \sigma_\omega^2 h_2\left(1, \eta_{1t}, s_{t-1}\right)} \left(r - \mu + \frac{1}{2} h_1\left(1, \eta_{1t}, s_{t-1}\right)\right)}. \tag{3.7}$$

If we let η_{1t} to be a constant for any $t \in \mathcal{T}$, this results in having an η_{2t} which depends on f_{t-1}. Therefore, we cannot have both market prices of risk to be constant at the same time. In our empirical applications we shall assume a constant factor preference parameter which will be estimated from observed option quotes, while we let the market price of equity risk to be a stochastic process determined by (3.4). In order to potentially identify the risk-neutral distributions of the asset return process and the latent factor conditional on $\mathcal{F}_t$, we need to evaluate the corresponding cumulant generating functions. This is carried out in the following corollaries.

Corollary 3.1 *If the asset returns follow the dynamics in (2.1)-(2.2), then the $\mathcal{F}_{t-1}$-conditional cumulant generating function of the latent factor f_t under Q defined in (3.2) is given by:*

$$C_{f_t}^Q\left(z_2 | \mathcal{F}_{t-1}\right) = z_2 \frac{s_{t-1}^2 \sigma_\omega^2}{u(\eta_{1t}, s_{t-1})} \left(\eta_{2t} + \frac{m_{t-1}}{s_{t-1}^2 \sigma_\omega^2}\right) + \frac{z_2^2}{2} \frac{s_{t-1}^2 \sigma_\omega^2}{u(\eta_{1t}, s_{t-1})}. \tag{3.8}$$

The proof follows immediately by setting $z_1 = 0$ in (3.3).

We notice that the expression (3.8) corresponds to the cumulant generating function of a Gaussian random variable and hence we can conclude that:

$$f_t\Big|_{\mathcal{F}_{t-1}} \overset{Q}{\sim} \mathbf{N}\left(\frac{s_{t-1}^2 \sigma_\omega^2}{u(\eta_{1t}, s_{t-1})} \left(\eta_{2t} + \frac{m_{t-1}}{s_{t-1}^2 \sigma_\omega^2}\right), \frac{s_{t-1}^2 \sigma_\omega^2}{u(\eta_{1t}, s_{t-1})}\right).$$

Thus, the risk-neutral measure Q preserves the underlying conditional distribution of the latent factor. Moreover, if we set η_{1t} to be constant, $\eta_{1t} = \eta_1$, it follows from (3.7) that f_t has an $AR(1)$ structure under Q.

Corollary 3.2 *If the asset returns follow the dynamics from (2.1)-(2.2), then the $\mathcal{F}_{t-1}$-conditional cumulant generating function of y_t under Q defined in (3.2) is given by:*

$$C_{y_t}^Q\left(z_1 | \mathcal{F}_{t-1}\right) = z_1 \mu - \frac{1}{2} h_1\left(z_1, \eta_{1t}, s_{t-1}\right) + \frac{h_2\left(z_1, \eta_{1t}, s_{t-1}\right)}{h_2\left(1, \eta_{1t}, s_{t-1}\right)} \left(r - \mu + \frac{1}{2} h_1\left(1, \eta_{1t}, s_{t-1}\right)\right), \tag{3.9}$$

where $h_1\left(z_1, \eta_{1t}, s_{t-1}\right)$ and $h_2\left(z_1, \eta_{1t}, s_{t-1}\right)$ are given in (3.5)-(3.6).

The proof follows immediately by replacing $z_2 = 0$ into (3.3) and from using the martingale constraint (3.4).

In this case, we notice that the pricing measure does not preserve the underlying distribution of the asset returns since the above risk-neutral cumulant generating function is not of the same form as that in (2.10). However, in order to perform pricing and hedging of financial derivatives one does not necessarily need to use the asset return dynamics under the martingale measure. In the next section we show how options can be computed under the physical measure by making use of the closed-form expression of the Radon-Nikodym derivative in (3.2).

4 Empirical analysis

In this section we investigate the pricing and hedging performance of the proposed one-factor stochastic volatility models, namely OFARSV presented in (2.1)-(2.2), and OFARCHSV given in (2.1) and (2.3). We implement risk-neutralizing via the exponential factor dependent pricing kernel introduced in Section 3. The model based option prices are computed based on a sequential estimation procedure in which the model parameters are obtained using a one-step unscented Kalman filter that uses historical returns, while the pricing kernel parameter is estimated by maximizing an option likelihood function in which quoted option prices intervene. The hedging ratios are constructed using local-risk minimization strategies under the risk-neutral measure. The pricing and hedging performance of our model is conducted using an extensive dataset of European calls and puts on the S&P 500 index and it is tested relative to the Heston-Nandi GARCH pricing model that is used as a benchmark.

4.1 Data description

We investigate the pricing performance using three datasets of S&P500 European options. The first two contain call option quotes ranging over the period January 1st, 2007–December 31st, 2013. Both datasets comprise contracts with maturities between 20 and 250 days and moneyness between 0.9 and 1.1 and were obtained after applying standard filters similar to those proposed in Bakshi *et al.* (1997). The first dataset, called Sample A, contains 31,968 call and put prices quoted every Wednesday of the reference period considered and is used for the in and out-of-sample analysis, while the second dataset, called Sample B, consists of 32,429 call and put prices recorded every Thursday for the same period and is only used for the out-of-sample performance assessment. The basic features of the datasets which include the number of contracts, average prices, and implied volatilities are illustrated in Tables 1 and 2. The average price and implied volatility for Sample A are \$34.978 and 20.0%, respectively, while the corresponding values for Sample B are \$35.301 and 20.0%, respectively. All options are grouped into six classes of moneyness and four classes of maturities.[8] Finally, for the hedging exercise we use a third dataset, called Sample C, which consists of the same S&P500 put and call options as Sample A but with the moneyness between 0.95 and 1.1 which results in 26,092 prices.

[8]The moneyness is defined as the ratio between the price of the underlying S_0 and the strike price K ($Mo := S_0/K$), so call (respectively, put) options with $Mo < 1$ (respectively, $Mo > 1$) are out-of-money (OTM) and those with $Mo > 1$ (respectively, $Mo < 1$) are in-the-money (ITM).

4.2 Estimation methodology

The model parameters are estimated using a two-stage procedure based on both historical asset return data on S&P500 and the option data described in the previous subsection. In order to proceed with the pricing and hedging exercises, provided T observed returns of S&P 500, one needs, first, to estimate both the model parameters $\boldsymbol{\theta} = (\mu, \lambda, \gamma, \phi, \sigma_\omega)$ and the corresponding unobserved latent factor values, and then, second, to calibrate the pricing kernel parameters using available option prices.

4.2.1 One-step unscented Kalman filter

We present first the general probabilistic state-space or the so called filtering model

$$f_0 \overset{P}{\sim} p\left(f_0\right), \tag{4.1}$$

$$f_t | f_{t-1} \overset{P}{\sim} p(f_t | f_{t-1}), \tag{4.2}$$

$$y_t | f_t \overset{P}{\sim} p(y_t | f_t). \tag{4.3}$$

Let $\mathbf{y}_{i:j} := \left(y_i, y_{i+1}, \ldots, y_j\right)^\top$, and $\mathbf{f}_{i:j} := \left(f_i, f_{i+1}, \ldots, f_j\right)^\top$, $i \leq j \in \mathbb{N}_0$, be the vectors of the observed measurements and the latent states values, respectively. The most important assumptions of this general filtering model which will be used later on are:

- **Markov property of states:** the states $\{f_t\}_{t \in \mathcal{T}}$ form a Markov sequence that is for $\mathbf{f}_{1:t-1}$ and $\mathbf{y}_{1:t-1}$ for any $t \in \mathcal{T}$ it holds that

$$p(f_t | \mathbf{f}_{1:t-1}, \mathbf{y}_{1:t-1}) = p(f_t | f_{t-1}). \tag{4.4}$$

- **Conditional independence of the measurements** with respect to the preceding ones and the preceding states: for $\mathbf{f}_{1:t}$ and $\mathbf{y}_{1:t-1}$, $t \in \mathcal{T}$, it holds that

$$p(y_t | \mathbf{f}_{1:t}, \mathbf{y}_{1:t-1}) = p(y_t | f_t). \tag{4.5}$$

The goal of any filtering method is to compute the posterior distribution $p(\mathbf{f}_{0:T} | \mathbf{y}_{1:T})$ of the states for a given T. One may proceed in a straightforward way and to write it down using Bayes's rule as

$$p(\mathbf{f}_{0:T} | \mathbf{y}_{1:T}) = \frac{p(\mathbf{f}_{0:T}, \mathbf{y}_{1:T})}{p(\mathbf{y}_{1:T})} = \frac{p(\mathbf{y}_{1:T} | \mathbf{f}_{0:T}) p(\mathbf{f}_{0:T})}{p(\mathbf{y}_{1:T})}, \tag{4.6}$$

where $p(\mathbf{y}_{1:T})$ will be referred to as the normalization constant in what follows, and where under assumptions (4.4) and (4.5) the prior distribution of the states (factors) $p(\mathbf{f}_{0:T})$ and the joint likelihood of the measurements (returns) $p(\mathbf{y}_{1:T}|\mathbf{f}_{0:T})$ are given by

$$p(\mathbf{f}_{0:T}) = p(f_0)\prod_{t=1}^{T} p(f_t|f_{t-1}) \tag{4.7}$$

and

$$p(\mathbf{y}_{1:T}|\mathbf{f}_{0:T}) = \prod_{t=1}^{T} p(y_t|f_t) \tag{4.8}$$

respectively. Unfortunately, whenever T becomes large enough, (4.6) turns out to be computationally not tractable. At the same time, under assumptions (4.4) and (4.5), it is possible to construct recursive procedures which require constant number of computations per time step, in order to compute posteriors sequentially and to allow for dynamic states and parameters estimation (see for extensive discussion). One can write the following general Bayesian recursive filtering scheme:

(i) **Initialization**: one selects a prior distribution $p(f_0)$ for the initial state f_0 in (4.1)

(ii) **Prediction (forecasting)**: using the Markovian property of the states (4.4), their predicted distributions $p(f_{t+h}|\mathbf{y}_{1:t})$ for $t > 0$, $t \in \mathcal{T}$, $h \in \mathbb{N}$ satisfy the Chapman-Kolmogorov equation

$$p(f_t|\mathbf{y}_{1:t-1}) = \int p(f_t|f_{t-1})p(f_{t-1}|\mathbf{y}_{1:t-1})df_{t-1} \tag{4.9}$$

(iii) **Update (filtering)** distributions of the states $p(f_t|\mathbf{y}_{1:t})$ for $t > 0$, $t \in \mathcal{T}$: as a direct consequence of Bayes's rule and (4.5) they can be written as:

$$p(f_t|\mathbf{y}_{1:t}) = \frac{1}{Z_t} p(y_t|f_t)p(f_t|y_{1:t-1}), \tag{4.10}$$

where

$$Z_t := p(y_t|\mathbf{y}_{1:t-1}) = \int p(y_t|f_t)p(f_t|\mathbf{y}_{1:t-1})df_t \tag{4.11}$$

and is referred to as the recursive normalization constant.

Some classes of filtering problems (4.1)-(4.3) are known to render closed form solutions of (4.9)-(4.11). In particular, for a linear Gaussian (in the measurement process) filtering problem Kalman filter offers closed form solution, since the posterior is proved to be also Gaussian and hence no approximations are needed. However, when the dynamics of the measurements is nonlinear in terms of the latent variable, the exact inference of filtered posteriors is not available. This is also the case for our proposed OFARSV and OFARCHSV models whose specifications, we recall, are given by

$$\begin{aligned} y_t &= g(f_t, \epsilon_t) := \mu + \lambda f_t^2 + f_t\epsilon_t, \quad \epsilon_t \sim \mathbf{N}(0,1), && (4.12)\\ f_t &= h(f_{t-1}, \omega_t) := \gamma + \phi f_{t-1} + s_{t-1}\omega_t, \quad \omega_t \sim \mathbf{N}(0,\sigma_\omega^2) && (4.13) \end{aligned}$$

where

$$y_t|\mathcal{G}_t \overset{P}{\sim} \mathbf{N}\left(\mu + \lambda f_t^2, f_t^2\right), \tag{4.14}$$

$$f_t|\mathcal{F}_{t-1} \overset{P}{\sim} \mathbf{N}\left(m_{t-1}, s_{t-1}^2 \sigma_\omega^2\right), \tag{4.15}$$

with $m_t = \gamma + \phi f_t$, $s_t := 1$ for the OFARSV specification and $s_t := f_t$ for all $t \in \mathcal{T}$ for the OFARCHSV model, respectively. We notice that our one-factor models are a particular case of the probabilistic model (4.1)-(4.3) where according to (4.14) and (4.15) the distributions of the states (factors) and the measurements (returns) conditionally on the previous factor values are assumed to be Gaussian that is $p(f_t|f_{t-1}) = \phi(f_t; \gamma + \phi f_{t-1}, s_{t-1}^2 \sigma_\omega^2)$ and $p(y_t|f_{t-1}) = \phi(y_t; \mu + \lambda f_t^2, f_t^2)$ with $\phi(\cdot; m, v)$ the probability density function of a Gaussian distribution with parameters m and v.

The two models we consider in this paper are non-affine in the measurement equation (see $g(f_t, \epsilon_t)$ in (4.12)) and are either homoscedastic (OFARSV) or heteroscedastic (OFARCHSV) in the latent factor dynamics (see $h(f_{t-1}, \omega_t)$ in (4.13)). This fact rules out closed form solutions of the filtering problem and calls for numerical approximations. There is a vast literature available on various methods of approximate inference which include: Extended Kalman filter (EKF), Unscented Kalman filter (UKF), sequential Monte Carlo methods, particle filters, MCMC, to name a few (Särkkä (2013)). In this paper we restrict ourselves to the Kalman filtering approach and hence need to consider the two available competitors namely EKF and UKF, both valid in the nonlinear non-Gaussian setup and both resulting in the Gaussian approximations of the filtering distributions $p(f_t|\mathbf{y}_{1:t})$. The core difference between the two filtering strategies has to do with the way the approximate inference is handled. In particular, EKF approximates the nonlinear function $g(f_t, \epsilon_t)$ using the first order Taylor series expansion at maximum a posteriori solution while the UKF uses the unscented transform (Julier and Uhlmann (1997); Julier et al. (2000)) to match the first two moments of the target distributions instead of approximating the nonlinear function $g(f_t, \epsilon_t)$. Moreover, UKF is known to outperform EKF in capturing strong nonlinearities which has been also confirmed in our empirical exercises. We hence adopt the UKF technique as the main estimation and filtering tool for both the OFARSV and OFARCHSV models.

In what follows let $p(\cdot)$ stand for an exact distribution and $\tilde{p}(\cdot)$ for its approximate counterpart. Standard UKF builds upon Gaussian approximations $\tilde{p}(\mathbf{f}_{0:T}|\mathbf{y}_{1:T})$ of the exact posterior $p(\mathbf{f}_{0:T}|\mathbf{y}_{1:T})$ and we will take $\tilde{p}(\cdot) := \phi(\cdot; m, v)$ with Gaussian density parametrized by m and v, unless stated otherwise. In the recursive scheme we hence adopt Gaussian approximation of the filtering distributions (4.10), that is, $\tilde{p}(f_t|\mathbf{y}_{1:t}) = \phi(f_t; m_{t|t}, v_{t|t})$ with $m_{t|t}$ and $v_{t|t}$ denoting the mean and the variance of the filtering distribution for $t > 0$, $t \in \mathcal{T}$. We proceed according to the Bayesian filtering equations as follows:

(i) **Initialization**: $\tilde{p}(f_0) = \phi(f_0; m_{0|0}, v_{0|0})$ where $m_{0|0}$ and $v_{0|0}$ are taken to be equal to the unconditional mean and the unconditional variance of the latent factors, respectively. It easy to verify that one obtains

$$m_{0|0} = \frac{\gamma}{1 - \phi}$$

for both models and

$$v_{0|0} = \begin{cases} \dfrac{\sigma_\omega^2}{1-\phi^2}, & \text{for the OFARSV model,} \\ \dfrac{\sigma_\omega^2\gamma^2}{(1-\phi)^2(1-\phi^2-\sigma_\omega^2)}, & \text{for the OFARCHSV model,} \end{cases}$$

respectively.

(ii) **Prediction (forecasting)**: from (4.15) and (4.4) one has that $p(f_t|f_{t-1}) = \phi(f_t; \gamma + \phi f_{t-1}, s_{t-1}^2\sigma_\omega^2)$ with the appropriate choice of s_{t-1} depending on the model. Consequently, (4.9) can be written as

$$p(f_t|\mathbf{y}_{1:t-1}) = \int \phi(f_t; \gamma + \phi f_{t-1}, s_{t-1}^2\sigma_\omega^2)\phi(f_{t-1}; m_{t-1|t-1}, v_{t-1|t-1})df_{t-1} \tag{4.16}$$

and one obtains that

(a) For the OFARSV model this predictive density can be computed explicitly and is exactly Gaussian, that is, $p(f_t|\mathbf{y}_{1:t-1}) = \phi(f_t; m_{t|t-1}, v_{t|t-1})$ with

$$m_{t|t-1} = \gamma + \phi m_{t-1|t-1}, \tag{4.17}$$

$$v_{t|t-1} = \sigma_\omega^2 + \phi^2 v_{t-1|t-1}. \tag{4.18}$$

(b) For the OFARCHSV model the predictive density $p(f_t|\mathbf{y}_{1:t-1})$ in (4.16) is not analytically tractable and we hence use the Gaussian approximation for the predictive densities of the latent factors, namely $\tilde{p}(f_t|\mathbf{y}_{1:t-1}) = \phi(f_t; m_{t|t-1}, v_{t|t-1})$. The first two moments $m_{t|t-1}$ and $v_{t|t-1}$ are explicitly obtained as follows

$$m_{t|t-1} = \int f_t p(f_t|\mathbf{y}_{1:t-1})df_t = \gamma + \phi m_{t-1|t-1}, \tag{4.19}$$

as in (4.17) and

$$v_{t|t-1} = \int (f_t - m_{t|t-1})^2 p(f_t|\mathbf{y}_{1:t-1})df_t = \sigma_\omega^2(v_{t-1|t-1} + m_{t-1|t-1}^2) + \phi^2 v_{t-1|t-1}. \tag{4.20}$$

(iii) **Update (filtering)** step: the Bayesian equations (4.10) and (4.11) cannot be explicitly computed and we adopt the UKF approach which builds upon the Gaussian approximation of the filtering densities $\tilde{p}(f_t|\mathbf{y}_{1:t}) = \phi(f_t; m_{t|t}, v_{t|t})$ and uses the matching of the first two moments. We first notice that from (4.12) and (4.5) we have $p(y_t|f_t) = \phi(y_t; \mu + \lambda f_t^2, f_t^2)$ for the two models and hence

$$m_{t|t} = \frac{1}{Z_t}\int f_t \phi(y_t; \mu + \lambda f_t^2, f_t^2)p(f_t|\mathbf{y}_{1:t-1})df_t, \tag{4.21}$$

$$v_{t|t} = \frac{1}{Z_t}\int (f_t - m_{t|t})^2 \phi(y_t; \mu + \lambda f_t^2, f_t^2)p(f_t|\mathbf{y}_{1:t-1})df_t, \tag{4.22}$$

where the normalization constant is given by

$$Z_t = p(y_t|\mathbf{y}_{1:t-1}) = \int \phi(y_t; \mu + \lambda f_t^2, f_t^2) p(f_t|\mathbf{y}_{1:t-1}) df_t, \tag{4.23}$$

and where for the OFARSV model the predictive distribution $p(f_t|\mathbf{y}_{1:t-1})$ is exactly Gaussian and for the OFARCHSV only approximately Gaussian, as explained in step (ii) above. Since integrals in (4.21)-(4.23) cannot be computed explicitly, we use the unscented transform to approximate them numerically.

The estimates $\widehat{\mathbf{f}_{1:T}}$ and $\widehat{\boldsymbol{\theta}}$ of the smoothed latent factor values $\mathbf{f}_{1:T}$ and of the parameter vector $\boldsymbol{\theta}$, respectively, allow us to proceed to a second stage in which we calibrate the pricing kernel parameters using quoted option prices.

4.2.2 Pricing kernel calibration using option prices

In the second stage, given the parameter values estimated at the stage of one-step unscented Kalman filering, we calibrate the pricing kernel parameters using the observed option prices. More specifically, we assume a constant factor preference parameter, $\eta_{2t} = \eta_2$, and since the market of price of factor risk η_{1t} depends on η_2 through equation (3.4), we have only one parameter to be estimated at this step.

Following the approach in Trolle and Schwartz (2009), we construct an option likelihood in the following way. For each set of quoted option prices $\mathbf{O}^{\text{mkt}} := \{O_1^{\text{mkt}}, \ldots, O_N^{\text{mkt}}\}$, we define the vega weighted option errors by:

$$e_i := \frac{O_i^{\text{mkt}} - O_i}{\nu_i^{\text{mkt}}}, \quad i = 1, \ldots, N.$$

Here, O_i represents the model option price and ν_i^{mkt} is the Black-Scholes vega corresponding to the quoted option price. Furthermore, we assume that e_i are independent and normally distributed with mean zero and variance $\sigma_e^2 := \frac{1}{N}\sum_{i=1}^{N} e_i^2$. Thus, the log-likelihood function of the option price vector $\mathbf{O}^{\text{mkt}}$ is given by:

$$\log L(\mathbf{O}^{\text{mkt}}, \widehat{\boldsymbol{\theta}}, \eta_2) = \sum_{i=1}^{N} \log L_i(O_i^{\text{mkt}}, \widehat{\boldsymbol{\theta}}, \eta_2) = \log\left(\frac{1}{\nu_i^{\text{mkt}}} f_e\left(\frac{O_i^{\text{mkt}} - O_i}{\nu_i^{\text{mkt}}}\right)\right),$$

where $f_e(\cdot)$ is the probability density function for a Gaussian random variable with mean zero and variance σ_e^2, $\widehat{\boldsymbol{\theta}}$ is the model parameter vector estimated with the one-step UKF, and η_2 is the pricing kernel parameter to be estimated. In the remainder of this subsection, we briefly describe how the option prices are computed.

Note that unlike the situation under the physical measure, there are no explicit expressions available that describe the risk-neutral dynamics of the asset returns under the factor-dependent pricing kernel. Therefore, we evaluate the option prices using Monte-Carlo simulations under P by generating the asset paths according to (2.1)-(2.2) and then weighting the option payoff by the corresponding Radon-Nikodym derivative path. For example, the price of any call option i with $i = 1, \ldots, N$, strike K_i and maturity T_i is given by:

$$O_i = \mathrm{E}^Q \left[\exp\left(-rT_i\right) \max\left(S_{T_i} - K_i, 0\right) | \mathcal{F}_{t-1}\right] = \frac{1}{M} \sum_{j=1}^{M} \exp\left(-rT_i\right) \max\left(S^P_{T_i}(j) - K_i, 0\right) N_{T_i}(j).$$

Here $S^P_{T_i}(j)$, $j = 1, \ldots, M$ represents the j-th simulated path of the asset price under the physical measure, $N_{T_i}(j)$ is the j-th simulated Radon-Nikodym factor given in (3.2), and M is the number of Monte-Carlo paths. In our calibration exercise we use $M = 20,000$. The continuously compounded one-period interest rate r is obtained from the corresponding T-Bill rates adequately interpolated in order to match the option maturity. The starting values for the asset price process and the latent value are provided by the last factor estimates filtered with the one-step unscented Kalman procedure. Furthermore, in order to reduce stochastic noise and in the spirit of Eichler *et al.* (2011), we use the same random numbers for generating the Monte Carlo paths at each step in the maximization of the likelihood. Note that, in general, $\log L(\mathbf{O}^{\mathrm{mkt}}, \widehat{\boldsymbol{\theta}}, \eta_2)$ is a non-convex function of η_2, so we need to use global search solvers in order to make sure that we end up with a global optimal solution.

4.3 Hedging implementation

We follow a hedging strategy based on the minimization of the local risk, as presented in Fölmer *et al.* (2002) and references therein. This hedging implementation is based on the construction of two-instrument portfolios formed with the risk asset and a risk-free bond. We consider a generalized trading strategy denoted by $(\xi^B, \xi^S) = \left\{\left(\xi^B_t, \xi^S_t\right)\right\}_{t\in\mathcal{T}}$, where ξ^B_t is adapted to the filtration $\mathcal{F}_t$ and represents the amount invested in the bond, while ξ^S is a $\mathcal{F}_t$-predictable process quantifying the amount invested in the risky asset. The value process associated to this trading strategy, $V(\xi^B, \xi^S) = \left\{V_t(\xi^B, \xi^S)\right\}_{t\in\mathcal{T}}$, is defined by:

$$V_0(\xi^B, \xi^S) = \xi^B_0, \quad \text{and} \quad V_t(\xi^B, \xi^S) = \xi^B_t \cdot B_t + \xi^S_t \cdot S_t, \quad t \in \mathcal{T}.$$

The cost process associated to this strategy is denoted by $C(\xi^B, \xi^S) = \left\{C_t(\xi^B, \xi^S)\right\}_{t\in\mathcal{T}}$ and is given by:

$$C_t(\xi^B, \xi^S) = V_t(\xi^B, \xi^S) - \sum_{k=1}^{t} \xi^S_k \cdot \left(S_k - S_{k-1}\right), \quad t \in \mathcal{T}.$$

Since under our autoregressive SV model the markets are incomplete, option prices cannot be fully replicated using such self-financing portfolios. There are various optimality criteria proposed in the literature to tackle simultaneously the pricing and hedging of these financial derivatives. In this paper we use the local risk minimization criteria, where the optimization problem is carried out in the risk neutral world.[9] More specifically, we find the trading strategy which solves the following optimization

[9] A detailed discussion on the advantages of using the local risk minimization criteria with respect to a martingale measure is provided in Badescu *et al.* (2014).

problem for any $t \in \mathcal{T}$:

$$(\widehat{\xi}_t^B, \widehat{\xi}_t^S) = \underset{\xi_t^B, \xi_t^S}{\operatorname{argmin}} \, \mathrm{E}^Q \left[\left(\widetilde{C}_{t+1}(\xi^B, \xi^S) - \widetilde{C}_t(\xi^B, \xi^S) \right)^2 \Big| \mathcal{F}_t \right], \quad t \in \mathcal{T}.$$

Here, $\widetilde{C}(\xi^B, \xi^S)_t$ represents the discounted cost of hedging $\widetilde{C}(\xi^B, \xi^S)_t := \exp(-rt) C(\xi^B, \xi^S)_t$. The optimal locally risk minimizing trading strategy for a financial derivative with payoff $H(S_T)$ at maturity is determined by the following recursions:

$$\widehat{\xi}_{t+1}^S = \exp(-r(T-t)) \frac{\mathrm{E}^Q \left[H(S_T) \left(\exp(-r) S_{t+1} - S_t \right) \mid \mathcal{F}_t \right]}{\mathrm{Var}^Q \left[\exp(-r) S_{t+1} - S_t \mid \mathcal{F}_t \right]}, \tag{4.24}$$

$$\widehat{V}_t(\widehat{\xi}^B, \widehat{\xi}^S) = \mathrm{E}^Q \left[\exp(-r(T-t)) H(S_T) \mid \mathcal{F}_t \right], \tag{4.25}$$

Here we assumed that $\widehat{V}_T(\widehat{\xi}^B, \widehat{\xi}^S) = H(S_T)$ and for any $t \in \mathcal{T}$, the optimal allocation in the bond is determined from:

$$\widehat{\xi}_t^B = \frac{1}{B_t} \left(\widehat{V}_t(\widehat{\xi}^B, \widehat{\xi}^S) - \widehat{\xi}_t^S S_t \right), \tag{4.26}$$

where $\widehat{\xi}_t^S$ and $\widehat{V}_t(\widehat{\xi}^B, \widehat{\xi}^S)$ are provided in (4.24)-(4.25). Note that the above scheme requires that the hedging is performed on a daily basis. However, the above recursions can be easily adapted if hedging takes place at lower frequencies than the one at which the asset prices are observed. For example, in our empirical analysis, the optimal portfolio is rebalanced on a weekly basis even though the prices of the underlying asset are quoted daily.

Since there are no closed-form expressions available, the hedging ratios $\widehat{\xi}^S$ are evaluated using Monte-Carlo simulations. Thus, the conditional expectation and variance in (4.24) are estimated using $M = 5,000$ paths, at each time $t \in \mathcal{T}$. Each strategy is initialized using the estimated parameters and the smoothed volatilities coming from the one-step unscented Kalman filter.

4.4 Empirical results

The empirical pricing and hedging performance of the proposed one-factor autoregressive conditionally homoscedastic (OFARSV) and one-factor autoregressive conditionally heteroscedastic (OFARCHSV) stochastic volatility models is assessed in this section. For the pricing exercise we consider as benchmark the affine GARCH model of Heston and Nandi (2000) (HNGARCH) risk-neutralized with the variance dependent pricing kernel of Christoffersen *et al.* (2013)[10]. In the hedging analysis we additionally include in the comparison the Black-Scholes model performance.

[10] Note that although the risk-neutral GARCH model of Heston and Nandi (2000) allows for a semi-closed form expression for the option prices, we implement it using Monte Carlo simulations in a similar fashion as our autoregressive SV models, in order to make the comparison more fair.

4.4.1 Option pricing performance

We carry out an extensive in and out-of-sample pricing performance assessment using the option data sets spelled out in Tables 1 and 2. The model and the pricing kernel parameters are estimated using the sequential estimation procedure described in Section 4.2 and are updated as follows. For the first day in Sample A (Wednesdays), we run the one-step unscented Kalman estimation of the model parameter vector $\boldsymbol{\theta} = (\mu, \lambda, \gamma, \phi, \sigma_\omega)$ using the historical daily returns on S&P 500 over a period of ten years prior to that date, for a total of 2,520 observations. Next, we calibrate the latent factor risk parameter η_2 from the pricing kernel using an options likelihood constructed using all options quotes on that particular Wednesday. Finally, the parameters obtained at this stage are used for the out-of-sample exercise in which we compute the prices quoted the next day (the corresponding Thursday from Sample B), as well as those quoted the next Wednesday from Sample A. This procedure is repeated for the whole dataset from Sample A and the model parameters are reestimated on a monthly basis using a rolling window of 2,520 observations.

In order to assess the performance of our model relative to the Heston and Nandi (2000) affine GARCH option pricing model we report the Implied Volatility Root Mean Squared Error (IVRMSE) measure, defined below:

$$IVRMSE = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(IV - IV^{\mathrm{mkt}}\right)^2} \times 100.$$

Here, the IV and IV^{mkt} represent the Black-Scholes implied volatilities corresponding to the prices associated to the proposed model and to the observed market quotes, respectively.

In Table 3 we present the IVRMSEs for the case in which no market price of factor risk is considered, that is, $\eta_2 = 0$ and our estimates are based solely on the historical return data, while in Table 4 we illustrate the case when η_2 is calibrated using option prices. Both tables contain three modules structured as follows: the“In-Sample Error” box reports the average IVRMSE for each Wednesday of Sample A with parameters calibrated based on option quotes from the same day; the “Next Day Pricing Error” box reports the average IVRMSE for each Thursday of Sample B with parameters calibrated based on option quotes from the previous day; in the box called “Next Week Pricing Error” we report the average IVRMSE for each Wednesday from Sample A using the parameters calibrated based on option quotes from the preceding Wednesday.

The results from Tables 3 and 4 indicate that our one-factor models (both OFARSV and OFARCHSV) consistently outperform the HNGARCH model for the in- and out-of-sample exercises under both risk-neutral measures considered. We now carefully go through the results in Table 3 obtained for the zero price of a factor risk $\eta_2 = 0$ and then assess the improvements achieved using the factor dependent pricing kernel corresponding to calibrated prices of factor risk η_2 in terms of the pricing errors (IVRMSE).

Comparative performance of competing models with $\eta_2 = 0$. By examining Table 3 for the *in-sample scenario*, we notice that the overall IVRMSE across all maturities and all moneyness levels for the HNGARCH process is 8.841, while for the OFARCHSV and OFARSV models it is 6.313 and 5.066, which correspond to around 43% and around 23% improvements for the two models, respectively. More specifically, the gains offered by the one-factor models for options with maturities below 180 days ranges from around 36% to 50% for OFARSV and from around 13% to 46% for OFARCHSV, the effect being more pronounced for short term options. The overall improvement provided by the OFARSV model for the at the money options with $Mo \in [0.975, 1]$ and $Mo \in [1, 1.025]$ is 16% and 37%, respectively. For the options with $Mo \in [0.9, 0.95]$ and $Mo \in [0.95, 0.975]$ OFARSV offers gain of 63% and 55%, while for $Mo \in [1.025, 1.05]$ and $Mo \in [1.05, 1.1]$ these numbers are slightly lower but still reach 53%. Remarkably, the OFARSV model is superior to both its competitors for all the maturity and moneyness values.

For the *out-of-sample analysis* the performance of the proposed one-factor stochastic volatility models with respect to the HNGARCH model is even more pronounced than in the in-sample case. When pricing with the OFARSV model and with the OFARCHSV model vs the HNGARCH benchmark, the overall IVRMSE is reduced by around 45% and 31%, respectively, for the next day and by around 42% and 29%, respectively, for the next week pricing exercises.

Comparative performance of competing models with calibrated η_2. As soon as the latent factor risk is priced, it is obvious from Table 4 that our proposed models strikingly outperform the HNGARCH benchmark both for the in-sample and for the out-of-sample exercises. The pricing differences dramatically increase for all the bins and both one-factor models outperform the HNGARCH under all maturities and all moneyness values Mo. By examining Table 4 for the *in-sample scenario*, we notice that the overall IVRMSE gain with respect to the HNGARCH process for all the maturities and moneyness levels goes up to around 63% and 60% for the OFARSV and OFARCHSV models, respectively. We also emphasize that pricing with the factor depending kernel brings into line the performance of the two proposed models, improves their IVRMSE significantly with respect to the pricing kernel with $\eta_2 = 0$ demonstrating slightly better error differencies for the OFARSV model. In the *out-of-sample exercise* the improvements are even more sound. For instance, for the "Next Day Pricing Error" the overall improvement provided by the OFARSV model for the at the money options with $Mo \in [0.975, 1]$ and $Mo \in [1, 1.025]$ is 37% and 55%, respectively. At the same time for the options with $Mo \in [0.9, 0.95]$ and $Mo \in [0.95, 0.975]$ OFARSV shows the decrease of the pricing errors in comparison to HNGARCH of around 71% and 68%, respectively. Finally, for $Mo > 1.025$ these numbers are slightly lower but still reach 48% on average. The behavior of the pricing errors remains the same for the "Next Week Pricing Error" and both one-factor stochastic volatility models alow for achieving around 50% improvement across maturities and moneyness intervals.

4.4.2 Hedging performance

The hedging portfolio are constructed using the locally risk minimization ratios described in Section 4.3 and are rebalanced on a weekly basis. We use the following normalized hedging error (NHS) to assess the performance of our autoregressive model relative to the HNGARCH model and the Black-Scholes (B-S) model:[11]

$$\mathrm{NHE}(\xi) := \frac{\left| H(S_T) - V_0 - \sum_{i=0}^{K} \widehat{\xi}_{t_{i+1}} \cdot \left(S_{t_{i+1}} - S_{t_i} \right) \right|}{V_0}$$

Here, $H(S_T)$ is the option payoff at expiration, V_0 is the option price at time 0, S_t is the observed value of the underlying at time t, and $\{t_0 = 0, t_1, \ldots, t_K\}$ represent the set of rebalancing dates during the lifetime of the option. The ratios $\widehat{\xi}_t$ are computed using the formulas (4.24)-(4.26). The results are reported in Table 5. As in the option pricing case, we notice that hedging with the one-factor stochastic volatility dynamics is preferred to hedging using the corresponding HNGARCH and B-S ones.

5 Conclusions

In this paper we propose one-factor autoregressive stochastic volatility models with conditionally homoscedastic (OFARSV) and heteroscedastic (OFARSV) factor dynamics and employ them for pricing and hedging European style options. Using an exponential affine pricing kernel which contains both equity and latent factor risk preferences, we derive the risk-neutral generating functions for the asset returns and for the factor process. For both specifications considered the change of measure preserves the conditional distribution of the factor models, but not that of the asset return process.

We provide a detailed empirical analysis to assess the pricing and hedging performance of our models using both historical returns and option quotes written on the S&P 500 index. The implementation is based on a sequential type algorithm where the model parameters are estimated first using the one-step unscented Kalman filter and then pricing kernel parameters are calibrated to the observed market prices. For the hedging part, we construct two-instrument portfolios formed with a riskless asset and the underlying and we compute the hedging ratios using a local risk minimization criterion.

Our numerical results indicate that our one-factor autoregressive SV models consistently outperform the Heston and Nandi GARCH option pricing model for all classes of moneyness and maturity considered and the improvements become even more pronounced whenever the pricing kernel is factor dependent. For example, the one week out-of-sample improvement over the HNGARCH competitor for the OFARCHSV model is around 29% in the case of a zero market price of risk and 51% when the factor risk preference are included in the pricing kernel, while for OFARCHSV these numbers go up to 43% and 54%, respectively. The hedging results also support the findings from the pricing exercise that the one-factor model are superior to the HNGARCH counterpart and to the B-S standard benchmark. Additionally, while in the pricing exercises both for the in-sample and out-of-sample strategies and both risk neutral measures OFARSV shows dominating performance over the model with the OFARCHSV specification, the situation is the opposite when hedging the options with specific maturity and moneyness values.

[11] The Black-Scholes portfolios are constructed using the usual BS delta hedging formulas.

Our framework can be further extended to accommodate for multi-factor dynamics and non-Gaussian factor processes, and a more detailed comparison with more complex GARCH dynamics and other discrete SV models has to be performed.

6 Appendix

6.1 Proof of Proposition 2.1

First, we express the the $\mathcal{F}_{t-1}$-conditional bivariate cumulant generating function (c.g.f.) of y_t and f_t under P in terms of the corresponding conditional c.g.f. of f_t and f_t^2:

$$\begin{aligned} C^P_{(y_t,f_t)}(z_1,z_2|\mathcal{F}_{t-1}) &:= \log \mathrm{E}^P \left[\exp\left(z_1 y_t + z_2 f_t\right)|\mathcal{F}_{t-1}\right] = \log \mathrm{E}^P \left[\exp\left(z_1\left(\mu+\lambda f_t^2\right) + z_1 f_t \epsilon_t + z_2 f_t\right)|\mathcal{F}_{t-1}\right] \\ &= \log \mathrm{E}^P \left[\mathrm{E}^P\left[\exp\left(z_1\left(\mu+\lambda f_t^2\right) + z_1 f_t \epsilon_t + z_2 f_t\right)|\mathcal{G}_t\right]|\mathcal{F}_{t-1}\right] \\ &= \log \mathrm{E}^P \left[\exp\left(z_1\left(\mu+\lambda f_t^2\right) + z_2 f_t\right)\mathrm{E}^P\left[\exp\left(z_1 f_t \epsilon_t\right)|\mathcal{G}_t\right]|\mathcal{F}_{t-1}\right] \\ &= z_1\mu + C^P_{(f_t,f_t^2)}\left(z_2, z_1\lambda + \frac{1}{2}z_1^2|\mathcal{F}_{t-1}\right) \end{aligned} \tag{6.1}$$

Recall that $f_t|_{\mathcal{F}_{t-1}} \overset{P}{\sim} \mathbf{N}\left(m_{t-1}, s_{t-1}^2\sigma_\omega^2\right)$, with $m_t = \gamma + \phi f_t$, and with $s_t := 1$ for the OFARSV model (2.1)-(2.2) and $s_t := f_t$ for the OFARCHSV model in (2.1) and (2.3) for all $t \in \mathcal{T}$, respectively. It immediately follows that f_t^2 has an $\mathcal{F}_{t-1}$-conditional scaled non-central Chi-Square distribution under P with 1 degree of freedom, scaling parameter $s_{t-1}^2\sigma_\omega^2$ and noncentrality parameter $m_{t-1}^2/s_{t-1}^2\sigma_\omega^2$:

$$f_t^2|_{\mathcal{F}_{t-1}} \overset{P}{\sim} \mathbf{sn}\chi^{\mathbf{2}}_{\mathbf{1}}\left(\frac{m_{t-1}^2}{s_{t-1}^2\sigma_\omega^2}, s_{t-1}^2\sigma_\omega^2\right).$$

The conditional c.g.f. of this distribution is given by:

$$C^P_{f_t^2}(z|\mathcal{F}_{t-1}) = -\frac{1}{2}\log\left(1-2zs_{t-1}^2\sigma_\omega^2\right) + \frac{zm_{t-1}^2}{1-2zs_{t-1}^2\sigma_\omega^2}, \quad 2zs_{t-1}^2\sigma_\omega^2 < 1.$$

Finally, the bivariate conditional c.g.f. of f_t and f_t^2 is given by (see Keller-Ressel and Muhle-Karbe (2012) and for the details):

$$C^P_{(f_t,f_t^2)}(z_1,z_2|\mathcal{F}_{t-1}) = -\frac{1}{2}\log\left(1-2z_2s_{t-1}^2\sigma_\omega^2\right) - \frac{m_{t-1}^2}{2s_{t-1}^2\sigma_\omega^2} + \frac{s_{t-1}^2\sigma_\omega^2}{2(1-2z_2s_{t-1}^2\sigma_\omega^2)}\left(z_1 + \frac{m_{t-1}}{s_{t-1}^2\sigma_\omega^2}\right)^2, \quad 2z_2s_{t-1}^2\sigma_\omega^2 < 1.$$

Replacing the above equation into (6.1) and using the notations from (2.7)-(2.8), we find that:

$$C^P_{(y_t,f_t)}(z_1,z_2|\mathcal{F}_{t-1}) = z_1\mu - \frac{1}{2}\log u(z_1,s_{t-1}) + \frac{v\left(z_2,m_{t-1},s_{t-1}\right)}{u(z_1,s_{t-1})} - \frac{m_{t-1}^2}{2s_{t-1}^2\sigma_\omega^2}, \quad z_1\left(z_1+2\lambda\right)s_{t-1}^2\sigma_\omega^2 < 1, \;\; z_2 \in \mathbb{R}.$$

This completes the proof. ∎

6.2 Proof of Proposition 3.1

We compute the conditional bivariate c.g.f. of (y_t, f_t) under the risk neutral measure Q provided in (3.2) as follows:

$$\begin{aligned} C^Q_{(y_t,f_t)}(z_1, z_2|\mathcal{F}_{t-1}) &:= \log \mathrm{E}^Q \left[\exp\left(z_1 y_t + z_2 f_t\right)|\mathcal{F}_{t-1}\right] = \log \mathrm{E}^P \left[\exp\left(z_1 y_t + z_2 f_t\right) N_t|\mathcal{F}_{t-1}\right] \\ &= C^P_{(y_t,f_t)}(z_1+\eta_{1t}, z_2+\eta_{2t}|\mathcal{F}_{t-1}) - C^P_{(y_t,f_t)}(\eta_{1t}, \eta_{2t}|\mathcal{F}_{t-1}) \\ &= z_1\mu - \frac{1}{2}\log\frac{u\left(z_1+\eta_{1t}, s_{t-1}\right)}{u\left(\eta_{1t}, s_{t-1}\right)} + \frac{v\left(z_2+\eta_{2t}, m_{t-1}, s_{t-1}\right)}{u\left(z_1+\eta_{1t}, s_{t-1}\right)} - \frac{v\left(\eta_{2t}, m_{t-1}, s_{t-1}\right)}{u\left(\eta_{1t}, s_{t-1}\right)}. \quad \blacksquare \end{aligned}$$

6.3 Derivation of Bayesian equations (4.9)-(4.10)

We derive the Chapman-Kolmogorov equation (4.9) as follows

$$\begin{aligned} p(f_t|\mathbf{y}_{1:t-1}) &= \int p(f_t, f_{t-1}|\mathbf{y}_{1:t-1}) df_{t-1} = \int p(f_t|f_{t-1}, \mathbf{y}_{1:t-1}) p(f_{t-1}|\mathbf{y}_{1:t-1}) df_{t-1} \\ &= \int p(f_t|f_{t-1}) p(f_{t-1}|\mathbf{y}_{1:t-1}) df_{t-1}, \end{aligned}$$

where in the last equality the Markov property of states (4.4) was used.

Next, we show that (4.10) holds and write

$$\begin{aligned} p(f_t|\mathbf{y}_{1:t}) = p(f_t|\mathbf{y}_{1:t-1}, y_t) &= \frac{p(f_t, \mathbf{y}_{1:t-1}, y_t)}{p(\mathbf{y}_{1:t})} = \frac{p(y_t|f_t, \mathbf{y}_{1:t-1}) p(f_t, \mathbf{y}_{1:t-1})}{p(\mathbf{y}_{1:t})} = \frac{p(y_t|f_t, \mathbf{y}_{1:t-1}) p(f_t|\mathbf{y}_{1:t-1}) p(\mathbf{y}_{1:t-1})}{p(y_t|\mathbf{y}_{1:t-1}) p(\mathbf{y}_{1:t-1})} \\ &= \frac{p(y_t|f_t) p(f_t|\mathbf{y}_{1:t-1})}{p(y_t|\mathbf{y}_{1:t-1})}, \end{aligned} \tag{6.2}$$

where in the last step we used the property of conditional independence of measurements (4.5). Finally, let $Z_t := p(y_t|\mathbf{y}_{1:t-1})$ in (6.3) and under the same assumption (4.5) one obtains

$$Z_t = \int p(y_t, f_t|\mathbf{y}_{1:t-1}) df_t = \int p(y_t|f_t, \mathbf{y}_{1:t-1}) p(f_t|\mathbf{y}_{1:t-1}) df_t = \int p(y_t|f_t) p(f_t|\mathbf{y}_{1:t-1}) df_t$$

which together with (6.3) gives (4.10) and (4.11) as required.

6.4 Computation of unconditional moments (4.9)-(4.10)

We derive the Chapman-Kolmogorov equation (4.9) as follows

$$\begin{aligned} p(f_t|\mathbf{y}_{1:t-1}) &= \int p(f_t, f_{t-1}|\mathbf{y}_{1:t-1}) df_{t-1} = \int p(f_t|f_{t-1}, \mathbf{y}_{1:t-1}) p(f_{t-1}|\mathbf{y}_{1:t-1}) df_{t-1} \\ &= \int p(f_t|f_{t-1}) p(f_{t-1}|\mathbf{y}_{1:t-1}) df_{t-1} \end{aligned}$$

where in the last equality we used the Markov property of states (4.4).

Next, we show that (4.10) holds and write

$$p(f_t|\mathbf{y}_{1:t}) = p(f_t|\mathbf{y}_{1:t-1}, y_t) = \frac{p(f_t, \mathbf{y}_{1:t-1}, y_t)}{p(\mathbf{y}_{1:t})} = \frac{p(y_t|f_t, \mathbf{y}_{1:t-1})p(f_t, \mathbf{y}_{1:t-1})}{p(\mathbf{y}_{1:t})} = \frac{p(y_t|f_t, \mathbf{y}_{1:t-1})p(f_t|\mathbf{y}_{1:t-1})p(\mathbf{y}_{1:t-1})}{p(y_t|\mathbf{y}_{1:t-1})p(\mathbf{y}_{1:t-1})}$$
$$= \frac{p(y_t|f_t)p(f_t|\mathbf{y}_{1:t-1})}{p(y_t|\mathbf{y}_{1:t-1})}, \tag{6.3}$$

where in the last step we used the property of conditional independence of measurements (4.5). Finally, let $Z_t := p(y_t|\mathbf{y}_{1:t-1})$ in (6.3) and under the same assumption (4.5) one obtains

$$Z_t = \int p(y_t, f_t|\mathbf{y}_{1:t-1})df_t = \int p(y_t|f_t, \mathbf{y}_{1:t-1})p(f_t|\mathbf{y}_{1:t-1})df_t = \int p(y_t|f_t)p(f_t|\mathbf{y}_{1:t-1})df_t$$

which together with (6.3) gives (4.10) and (4.11) as required.

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BASIC FEATURES OF THE OPTION PRICING (CALLS AND PUTS) DATASET (WEDNESDAYS)								
		Moneyness S_0/K						**Across**
	Maturities	$[0.900, 0.950]$	$[0.950, 0.975]$	$[0.975, 1.000]$	$[1.000, 1.025]$	$[1.025, 1.050]$	$[1.050, 1.100]$	**Moneyness**
Number of Contracts	$\mathbf{20} \leq T < \mathbf{30}$	1717	1242	1556	1568	1053	1611	8747
	$\mathbf{30} \leq T < \mathbf{80}$	2675	1998	3076	3299	1805	2571	15424
	$\mathbf{80} \leq T < \mathbf{180}$	1086	680	1034	1147	707	1038	5692
	$\mathbf{180} \leq T \leq \mathbf{250}$	398	224	426	501	253	303	2105
Across Maturities		5876	4144	6092	6515	3818	5523	31968
Average Prices	$\mathbf{20} \leq T < \mathbf{30}$	8.327	15.527	25.197	27.132	22.118	13.427	18.321
	$\mathbf{30} \leq T < \mathbf{80}$	13.555	24.681	39.578	41.011	34.110	23.694	30.154
	$\mathbf{80} \leq T < \mathbf{180}$	33.392	53.327	68.829	70.258	61.818	49.957	56.191
	$\mathbf{180} \leq T \leq \mathbf{250}$	53.895	79.004	94.349	95.933	88.601	76.419	82.173
Across Maturities		18.426	29.574	44.700	47.043	39.544	28.528	34.978
Average Implied Volatilities	$\mathbf{20} \leq T < \mathbf{30}$	0.181	0.174	0.184	0.192	0.214	0.243	0.198
	$\mathbf{30} \leq T < \mathbf{80}$	0.175	0.176	0.193	0.199	0.213	0.235	0.198
	$\mathbf{80} \leq T < \mathbf{180}$	0.181	0.193	0.204	0.203	0.215	0.234	0.205
	$\mathbf{180} \leq T \leq \mathbf{250}$	0.188	0.211	0.210	0.210	0.212	0.234	0.210
Across Maturities		0.179	0.180	0.194	0.199	0.214	0.237	0.200

Table 1: Basic features of the Sample A option dataset (Wednesdays). Prices in this dataset correspond to the period January 1st, 2007–December 31st, 2013.

BASIC FEATURES OF THE OPTION PRICING (CALLS AND PUTS) DATASET (THURSDAYS)								
		Moneyness S_0/K						**Across**
	Maturities	$[0.900, 0.950]$	$[0.950, 0.975]$	$[0.975, 1.000]$	$[1.000, 1.025]$	$[1.025, 1.050]$	$[1.050, 1.100]$	**Moneyness**
Number of Contracts	$\mathbf{20} \leq T < \mathbf{30}$	1402	1209	1546	1638	1059	1528	8382
	$\mathbf{30} \leq T < \mathbf{80}$	2907	2155	3018	3553	1832	2780	16245
	$\mathbf{80} \leq T < \mathbf{180}$	1153	697	954	1188	673	1038	5703
	$\mathbf{180} \leq T \leq \mathbf{250}$	398	238	398	514	251	300	2099
Across Maturities		5860	4299	5916	6893	3815	5646	32429
Average Prices	$\mathbf{20} \leq T < \mathbf{30}$	8.835	14.847	24.323	27.783	23.879	14.939	19.275
	$\mathbf{30} \leq T < \mathbf{80}$	14.681	24.583	37.985	41.293	34.131	24.462	30.012
	$\mathbf{80} \leq T < \mathbf{180}$	35.260	55.212	66.922	70.645	63.395	51.311	56.608
	$\mathbf{180} \leq T \leq \mathbf{250}$	56.148	80.564	92.240	96.525	89.900	74.747	82.342
Across Maturities		20.148	29.910	42.731	47.260	40.117	29.493	35.301
Average Implied Volatilities	$\mathbf{20} \leq T < \mathbf{30}$	0.189	0.170	0.179	0.193	0.215	0.245	0.199
	$\mathbf{30} \leq T < \mathbf{80}$	0.176	0.173	0.189	0.199	0.216	0.235	0.198
	$\mathbf{80} \leq T < \mathbf{180}$	0.181	0.196	0.200	0.207	0.217	0.233	0.205
	$\mathbf{180} \leq T \leq \mathbf{250}$	0.188	0.207	0.209	0.210	0.218	0.230	0.209
Across Maturities		0.181	0.178	0.190	0.200	0.216	0.237	0.200

Table 2: Basic features of the Sample B option dataset (Thursdays). Prices in this dataset correspond to the period January 1st, 2007–December 31st, 2013.

OUT OF SAMPLE PRICING ERRORS (IVRMSE in percentage) WITH SEQUENTIALLY ESTIMATED PARAMETERS AND $\eta_2 = 0$																						
		Moneyness S_0/K																		Across		
	Maturities	[0.900, 0.950]			[0.950, 0.975]			[0.975, 1.000]			[1.000, 1.025]			[1.025, 1.050]			[1.050, 1.100]			**Moneyness**		
	Model	**HN**	**SV**	**ARCH**	**HN**	**SV**	**ARCH**	**HN**	**SV**	**ARCH**	**HN**	**SV**	**ARCH**	**HN**	**SV**	**ARCH**	**HN**	**SV**	**ARCH**	**HN**	**SV**	**ARCH**
In-Sample Error	$\mathbf{20 \leq T < 30}$	16.020	4.724	5.591	11.563	3.899	4.256	6.358	3.862	4.200	8.993	4.407	4.561	8.345	5.572	5.898	6.040	7.406	8.011	10.322	5.146	5.618
	$\mathbf{30 \leq T < 80}$	10.731	4.342	5.782	8.352	3.947	5.654	4.916	4.161	5.593	7.872	4.806	5.822	12.864	5.345	6.657	5.618	6.651	7.647	8.467	4.939	6.193
	$\mathbf{80 \leq T < 180}$	7.468	3.842	5.232	5.719	3.975	5.733	3.954	4.702	6.319	6.469	5.258	7.062	12.732	5.406	7.452	8.744	6.036	8.207	7.742	4.956	6.756
	$\mathbf{180 \leq T \leq 250}$	6.063	4.525	6.513	4.308	5.576	7.765	3.908	5.712	8.256	5.570	6.243	8.726	10.537	6.192	8.961	13.298	6.927	9.585	7.617	5.876	8.315
Across Maturities		11.847	4.385	5.684	8.930	4.045	5.438	5.120	4.312	5.647	7.782	4.924	6.070	11.618	5.479	6.797	7.035	6.785	7.978	8.841	5.066	6.313
Next Day Pricing Error	$\mathbf{20 \leq T < 30}$	16.172	4.506	6.252	12.075	3.328	4.268	6.268	3.244	4.062	8.545	4.047	5.024	7.381	5.335	6.335	7.274	7.227	8.266	10.164	4.811	5.878
	$\mathbf{30 \leq T < 80}$	12.418	4.836	5.938	8.663	4.128	5.050	4.915	4.387	5.610	7.709	4.623	5.815	13.236	5.265	6.825	6.270	6.612	7.605	9.045	5.031	6.173
	$\mathbf{80 \leq T < 180}$	9.203	3.749	4.703	6.027	4.140	5.342	3.900	4.687	6.227	6.585	4.827	6.724	12.761	5.265	6.880	8.664	5.997	7.652	8.133	4.829	6.337
	$\mathbf{180 \leq T \leq 250}$	7.197	4.776	6.735	4.798	5.459	7.747	3.807	5.993	8.592	5.620	6.069	8.766	10.723	6.300	8.790	15.384	6.735	9.434	8.382	5.890	8.368
Across Maturities		12.592	4.557	5.855	9.252	4.015	5.093	5.091	4.312	5.634	7.595	4.661	6.089	11.676	5.360	6.859	7.768	6.677	7.901	9.149	5.003	6.300
Next Week Pricing Error	$\mathbf{20 \leq T < 30}$	16.082	4.669	5.307	11.599	3.928	4.060	6.360	3.926	4.159	9.046	4.477	4.551	7.998	5.663	5.926	6.090	7.482	8.025	10.328	5.190	5.542
	$\mathbf{30 \leq T < 80}$	10.856	4.334	5.859	8.378	3.932	5.717	4.932	4.125	5.618	7.937	4.781	5.831	12.967	5.354	6.716	5.623	6.639	7.665	8.530	4.923	6.230
	$\mathbf{80 \leq T < 180}$	7.429	3.821	5.234	5.717	3.982	5.723	3.969	4.705	6.291	6.475	5.246	7.223	12.810	5.448	7.412	9.111	6.020	8.306	7.830	4.953	6.800
	$\mathbf{180 \leq T \leq 250}$	6.136	4.466	6.435	4.372	5.582	7.785	3.925	5.686	8.262	5.649	6.180	8.657	10.501	6.175	8.976	13.426	6.903	9.560	7.668	5.843	8.290
Across Maturities		11.946	4.357	5.630	8.967	4.048	5.419	5.138	4.306	5.641	7.840	4.916	6.092	11.604	5.515	6.816	7.147	6.802	8.008	8.896	5.068	6.314

Table 3: Option pricing results for the one-factor autoregressive stochastic volatility (OFARSV) model, the one-factor autoregressive conditionally heteroscedastic stochastic volatility (OFARCHSV) model, and the Heston and Nandi GARCH (HNGARCH) model computed using the conditional Esscher transform which corresponds to a zero price of factor risk, $\eta_2 = 0$. The parameter values are computed based solely on the asset return daily data and are reestimated every four weeks. The“In-Sample Error” box reports the average IVRMSE for each Wednesday. The “Next Day Pricing Error” box reports the average IVRMSE committed each Thursday using the models whose parameters have been eventually estimated the preceding day. The module marked “Next Week Pricing Error” reports the average IVRMSE for each Wednesday using the models whose parameters have been eventually estimated the Wednesday of the preceding week.

OUT OF SAMPLE PRICING ERRORS (IVRMSE in percentage) WITH SEQUENTIALLY ESTIMATED PARAMETERS AND η_2 OBTAINED BY CALIBRATION

		Moneyness S_0/K																		Across Moneyness		
	Maturities	$[0.900, 0.950]$			$[0.950, 0.975]$			$[0.975, 1.000]$			$[1.000, 1.025]$			$[1.025, 1.050]$			$[1.050, 1.100]$					
	Model	**HN**	**SV**	**ARCH**	**HN**	**SV**	**ARCH**	**HN**	**SV**	**ARCH**	**HN**	**SV**	**ARCH**	**HN**	**SV**	**ARCH**	**HN**	**SV**	**ARCH**	**HN**	**SV**	**ARCH**
In-Sample Error	$\mathbf{20} \leq T < \mathbf{30}$	16.417	3.709	3.293	11.800	2.805	2.153	5.911	2.634	2.305	7.762	3.098	3.273	11.145	4.301	4.595	3.520	6.290	6.652	10.363	4.033	4.039
	$\mathbf{30} \leq T < \mathbf{80}$	10.830	3.120	2.890	8.309	2.135	2.142	4.334	1.973	2.207	7.188	2.286	2.769	12.976	3.008	3.629	4.333	4.253	4.575	8.178	2.871	3.111
	$\mathbf{80} \leq T < \mathbf{180}$	7.152	2.194	2.536	5.242	2.063	2.644	4.129	2.498	3.013	7.471	2.618	3.053	13.604	3.007	3.521	9.078	3.501	3.758	8.089	2.702	3.116
	$\mathbf{180} \leq T \leq \mathbf{250}$	6.105	3.074	4.433	5.153	3.939	5.732	5.363	3.939	5.545	7.566	4.270	5.044	12.563	4.022	5.338	12.994	4.778	6.343	8.506	4.016	5.358
Across Maturities		12.011	3.161	3.083	8.979	2.470	2.564	4.826	2.426	2.743	7.408	2.749	3.171	12.597	3.480	4.028	6.104	4.842	5.239	8.832	3.283	3.571
Next Day Pricing Error	$\mathbf{20} \leq T < \mathbf{30}$	16.578	4.028	3.825	12.315	2.867	2.480	5.837	2.604	2.520	7.738	3.299	3.352	8.218	4.467	4.675	5.186	6.419	6.540	9.994	4.150	4.142
	$\mathbf{30} \leq T < \mathbf{80}$	12.587	3.924	3.413	8.652	2.988	2.533	4.382	3.062	2.703	7.039	3.180	3.043	13.544	3.555	3.726	6.042	4.710	4.741	8.940	3.622	3.426
	$\mathbf{80} \leq T < \mathbf{180}$	9.018	2.777	3.455	5.540	2.804	3.734	4.003	2.911	4.301	7.641	3.228	4.482	13.634	3.398	4.050	8.868	4.054	5.413	8.451	3.236	4.323
	$\mathbf{180} \leq T \leq \mathbf{250}$	7.080	3.260	4.742	5.242	4.130	6.062	5.275	4.204	5.638	7.819	4.457	5.762	13.103	4.310	5.901	13.017	4.607	5.998	8.801	4.169	5.644
Across Maturities		12.767	3.706	3.631	9.299	3.004	3.050	4.808	3.022	3.257	7.373	3.330	3.663	12.320	3.850	4.231	7.010	5.104	5.462	9.128	3.738	3.962
Next Week Pricing Error	$\mathbf{20} \leq T < \mathbf{30}$	16.313	4.376	3.738	11.859	3.517	2.745	6.241	3.283	2.871	8.058	3.576	3.546	10.001	4.565	4.788	3.722	6.408	6.680	10.288	4.447	4.298
	$\mathbf{30} \leq T < \mathbf{80}$	11.040	3.989	3.979	8.400	3.286	3.302	4.496	3.183	3.451	7.301	3.419	3.898	13.287	3.983	4.351	4.312	5.141	5.295	8.333	3.865	4.083
	$\mathbf{80} \leq T < \mathbf{180}$	7.219	3.401	4.831	5.382	3.441	4.106	4.226	3.685	4.673	7.524	3.781	5.085	13.554	3.994	5.437	8.162	4.391	5.940	7.939	3.804	5.073
	$\mathbf{180} \leq T \leq \mathbf{250}$	6.380	3.963	5.652	4.822	4.410	6.362	5.627	4.883	6.434	7.889	4.953	6.607	12.331	4.225	6.224	13.631	5.845	7.230	8.718	4.772	6.427
Across Maturities		12.099	4.007	4.211	9.057	3.454	3.540	5.044	3.441	3.835	7.575	3.660	4.313	12.450	4.170	4.828	5.954	5.456	5.965	8.874	4.087	4.517

Table 4: Option pricing results for the one-factor autoregressive stochastic volatility (OFARSV) model, the one-factor autoregressive conditionally heteroscedastic stochastic volatility (OFARCHSV) model, and the Heston and Nandi GARCH (HNGARCH) model computed using the factor dependent pricing kernel which corresponds to calibrated prices of factor risk η_2. The model parameters are computed based on daily asset returns and are re-estimated every four weeks and the pricing kernel parameter η_2 is estimated based on the observed option quotes. The"In-Sample Error" box reports the average IVRMSE for each Wednesday at the time of pricing the options that have been used to optimize the options likelihood with respect to η_2 the very same Wednesday. The "Next Day Pricing Error" box reports the average IVRMSE committed each Thursday using the models whose parameters have been estimated the preceding day. The module marked "Next Week Pricing Error" reports the average IVRMSE for each Wednesday using the models whose parameters have been estimated the Wednesday of the preceding week..

RESULTS FOR THE 2007-2013 HEDGING EXERCISE							
		Moneyness S_0/K					**Across**
	Maturities	$[0.950, 0.975]$	$[0.975, 1.000]$	$[1.000, 1.025]$	$[1.025, 1.050]$	$[1.050, 1.100]$	**Moneyness**
NHSE Black-Scholes	$\mathbf{20} \leq T < \mathbf{30}$	6.009	2.495	1.407	2.312	1.579	1.875
	$\mathbf{30} \leq T < \mathbf{80}$	2.215	1.971	2.554	1.520	1.768	1.937
	$\mathbf{80} \leq T < \mathbf{180}$	2.363	1.440	1.508	0.968	1.011	1.191
	$\mathbf{180} \leq T \leq \mathbf{250}$	1.047	1.867	0.887	1.122	1.247	1.167
Across Maturities		4.024	1.999	1.989	1.707	1.579	1.802
NHSE HNGARCH	$\mathbf{20} \leq T < \mathbf{30}$	3.901	2.007	1.225	1.423	1.386	1.989
	$\mathbf{30} \leq T < \mathbf{80}$	1.902	1.709	1.880	1.294	1.536	1.664
	$\mathbf{80} \leq T < \mathbf{180}$	1.304	1.441	1.127	1.142	1.158	1.234
	$\mathbf{180} \leq T \leq \mathbf{250}$	1.031	1.954	1.054	1.333	1.357	1.346
Across Maturities		2.034	1.778	1.321	1.298	1.359	1.558
NHSE OFARSV	$\mathbf{20} \leq T < \mathbf{30}$	3.483	1.626	1.299	1.731	1.602	1.948
	$\mathbf{30} \leq T < \mathbf{80}$	2.149	1.742	1.740	1.321	1.475	1.685
	$\mathbf{80} \leq T < \mathbf{180}$	1.838	1.393	1.365	0.991	1.027	1.323
	$\mathbf{180} \leq T \leq \mathbf{250}$	1.128	1.813	1.008	1.173	1.319	1.288
Across Maturities		2.149	1.644	1.353	1.304	1.355	1.561
NHSE OFARCHSV	$\mathbf{20} \leq T < \mathbf{30}$	3.233	1.560	1.284	1.590	1.528	1.839
	$\mathbf{30} \leq T < \mathbf{80}$	2.106	1.730	1.745	1.277	1.459	1.664
	$\mathbf{80} \leq T < \mathbf{180}$	1.687	1.360	1.267	1.009	1.146	1.294
	$\mathbf{180} \leq T \leq \mathbf{250}$	1.090	1.786	0.972	1.191	1.325	1.273
Across Maturities		2.029	1.609	1.317	1.267	1.365	1.517

Table 5: Average normalized hedging square errors (NHSE) associated with portfolios constructed based on the one-factor autoregressive stochastic volatility (OFARSV) model, the one-factor autoregressive conditionally heteroscedastic stochastic volatility (OFARCHSV) model, the Heston-Nandi GARCH (HNGARCH), and the Black-Scholes (BS) model; the hedging strategies for the OFARSV, OFARCHSV, and HN are computed using the local risk minimization criteria, and the BS ones are based on the Black-Scholes delta hedging formula. Each entry in the table has been computed by averaging the normalized hedging errors committed when handling the options contained in the corresponding moneyness-time to maturity bin.