# FUNCTIONAL CENTRAL LIMIT THEOREMS FOR ROUGH VOLATILITY

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ABSTRACT. We extend Donsker's approximation of Brownian motion to fractional Brownian motion with Hurst exponent  $H \in (0,1)$  and to Volterra-like processes. Some of the most relevant consequences of this 'rough Donsker (rDonsker) Theorem' are convergence results for discrete approximations of a large class of rough models. This justifies the validity of simple and easy-to-implement Monte-Carlo methods, for which we provide detailed numerical recipes. We test these against the current benchmark Hybrid scheme [11] and find remarkable agreement (for a large range of values of H). This rDonsker Theorem further provides a weak convergence proof for the Hybrid scheme itself, and allows to construct binomial trees for rough volatility models, the first available scheme (in the rough volatility context) for early exercise options such as American or Bermudan.

#### Introduction

Fractional Brownian motion has a long and famous history in probability, stochastic analysis and their applications to diverse fields [42, 43, 50, 58]. Recently, it has experienced a new renaissance in the form of fractional volatility models in mathematical finance. These were first introduced by Comte and Renault [17], and later studied theoretically by Djehiche and Eddahbi [20], Alòs, León and Vives [3] and Fukasawa [31], and given financial motivation and data consistency by Gatheral, Jaisson and Rosenbaum [35] and Bayer, Friz and Gatheral [8]. Since then, a vast literature has pushed the analysis in many directions [7, 9, 12, 26, 28, 35, 36, 40, 47, 64], leading to theoretical and practical challenges to understand and implement these models. One of the main issues, at least from a practical point of view, is on the numerical side: absence of Markovianity rules out PDE-based schemes, and simulation is the only possibility. However, classical simulation methods for fractional Brownian motion (based on Cholesky decomposition or circulant matrices) are notoriously slow, and faster techniques are needed. The state of the art, so far, is the recent hybrid scheme developed by Bennedsen, Pakkanen and Lunde [11], and its turbocharged version [60]. We rise here to this challenge, and propose an alternative tree-based approach, mathematically rooted in an extension of Donsker's theorem to rough volatility.

Donsker [22] (and later Lamperti [54]) proved a functional central limit for Brownian motion, thereby providing a theoretical justification of its random walk approximation. Many extensions have been studied in the literature, and we refer the interested reader to [24] for an overview. In the fractional case, Sottinen [75] and

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Nieminen [66] constructed–following Donsker's ideas of using iid sequences of random variables—an approximating sequence converging to the fractional Brownian motion, with Hurst parameter H > 1/2. In order to deal with the non-Markovian behaviour of fractional Brownian motion, Taqqu [77] considered sequences of non-iid random variables, again with the restriction H > 1/2. Unfortunately, neither methodologies seem to carry over to the 'rough' case H < 1/2, mainly because of the topologies involved. The recent development of rough paths theory [29, 30, 57] provided an appropriate framework to extend Donsker's results to processed with sample paths of Hölder regularity strictly smaller than 1/2. For  $H \in (1/3, 1/2)$ , Bardina, Nourdin, Rovira and Tindel [4] used rough paths to show that functional central limit theorems (in the spirit of Donsker) apply. This in particular suggests that the natural topology at work for rough fractional Brownian motion is the topology induced by the Hölder norm of the sample paths. Indeed, switching the topology from the Skorokhod one used by Donsker to the (stronger) Hölder topology is the right setting for rough central limit theorems, as we outline in this paper. Recent results [10, 67, 68] provide convergence for (geometric) fractional Brownian motions with general  $H \in (0,1)$  using Wick calculus, assuming that the approximating sequences are Bernoulli random variables. We extend this (Theorem 1.10) to a universal functional central limit theorem, involving general (discrete or continuous) random variables as approximating sequences, only requiring finiteness of moments.

We consider a general class of continuous processes with any Hölder regularity, including fractional Brownian motion with  $H \in (0,1)$ , truncated Brownian semi-stationary processes, Gaussian Volterra processes, as well as rough volatility models recently proposed in the financial literature. The fundamental novelty here is an approximating sequence capable of simultaneously keeping track of the approximated rough volatility process (fractional Brownian motion, Brownian semistationary process, or any continuous path functional thereof) and of the underlying Brownian motion. This is crucial in order to take into account the correlation of the two processes, the so-called leverage effect in financial modelling. While approximations of two-dimensional (correlated) semimartingales are well understood in the standard case, the rough case is so far an open problem. Our analysis easily generalises beyond Brownian drivers to more general semimartingales, emphasising that the subtle, yet essential, difficulties lie in the passage from the semimartingale setup to the rough case. This is the first Monte-Carlo method available in the literature, specifically tailored to two-dimensional rough systems, based on an approximating sequence for which we prove a functional a Donsker-Lamperti-type functional central limit theorem (FCLT). This further provides a pathwise justification of the hybrid scheme by Bennedsen, Lunde and Pakkanen [11], and to develop tree-based schemes, opening the doors to pricing early-exercise options such as American options. In Section 1, we present the class of models we are considering and state our main results. The proof of the main theorem is developed in Section 2 in several steps. We reserve Section 3 to applications of the main result, namely weak convergence of the hybrid scheme, binomial trees as well as numerical examples. We present simple numerical recipes, providing a pedestrian alternative to the advanced hybrid schemes in [11, 60], and develop a simple Monte-Carlo with low implementation complexity, for which we provide comparison charts against [11] in terms of accuracy and against [60] in terms of speed. Reminders on Riemann-Liouville operators and additional technical proofs are postponed to the appendix.

**Notations**: On the unit interval  $\mathbb{I} := [0,1]$ ,  $\mathcal{C}(\mathbb{I})$  and  $\mathcal{C}^{\alpha}(\mathbb{I})$  denote the spaces of continuous and  $\alpha$ -Hölder continuous functions on  $\mathbb{I}$  with Hölder regularity  $\alpha \in (0,1)$ ;  $\mathcal{C}^1(\mathbb{I})$  and  $\mathcal{C}^1_b(\mathbb{I})$  are the continuously differentiable and bounded continuously differentiable functions on  $\mathbb{I}$ , and we use  $C, C_1, C_2$  as strictly positive real constants which may change from line to line, the exact values of which do not matter.

# 1. Weak convergence of rough volatility models

Donsker's invariance principle [22] (also termed 'functional central limit theorem') ensures the weak convergence of an approximating sequence to a Brownian motion in the Skorokhod space. As opposed to the central limit theorem, Donsker's theorem is a pathwise statement which ensures that convergence takes place for all times. This result is particularly important for Monte-Carlo methods, which aim to approximate pathwise functionals of a given process (essential requirement in order to price path-dependent financial securities for example). We prove here a version of Donsker's result, not only in the Skorokhod topology, but also in the stronger Hölder topology, for a general class of continuous stochastic processes.

# 1.1. Hölder spaces and fractional operators. For $\beta \in (0,1]$ , the $\beta$ -Hölder space $\mathcal{C}^{\beta}(\mathbb{I})$ , with the norm

$$||f||_{\beta} := |f|_{\beta} + ||f||_{\infty} = \sup_{\substack{t,s \in \mathbb{I} \\ t \neq s}} \frac{|f(t) - f(s)|}{|t - s|^{\beta}} + \max_{t \in \mathbb{I}} |f(t)|,$$

is a non-separable Banach space [51, Chapter 3]. Following the spirit of Riemann-Liouville fractional operators recalled in Appendix A, we introduce the class of Generalised Fractional Operators (GFO). For any  $\lambda \in (0,1)$  we introduce the intervals

$$\mathfrak{R}^{\lambda} := \Big\{ \alpha \in (-1,1) \text{ such that } \alpha + \lambda \in (0,1) \Big\},$$

$$\mathfrak{R}^{\lambda}_{+} := \mathfrak{R}^{\lambda} \cap (0,1), \, \mathfrak{R}^{\lambda}_{-} := \mathfrak{R}^{\lambda} \cap (-1,0), \text{ and the space } \mathcal{L}^{\alpha} := \big\{ u \mapsto u^{\alpha} L(u) : L \in \mathcal{C}^{1}_{b}(\mathbb{I}) \big\}, \text{ for any } \alpha \in \mathfrak{R}^{\lambda}.$$

**Definition 1.1.** For any  $\lambda \in (0,1)$  and  $\alpha \in \mathfrak{R}^{\lambda}$ , the GFO associated to  $g \in \mathcal{L}^{\alpha}$  is defined on  $\mathcal{C}^{\lambda}(\mathbb{I})$  as

(1.1) 
$$(\mathcal{G}^{\alpha}f)(t) := \begin{cases} \int_0^t f(s) \frac{\mathrm{d}}{\mathrm{d}t} g(t-s) \mathrm{d}s, & \text{if } \alpha \in [0, 1-\lambda), \\ \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t f(s) g(t-s) \mathrm{d}s, & \text{if } \alpha \in (-\lambda, 0). \end{cases}$$

We shall further use the notation  $G(t) := \int_0^t g(u) du$ , for any  $t \in \mathbb{I}$ . Of particular interest in mathematical finance are the following kernels and operators:

Riemann-Liouville: 
$$g(u) = u^{\alpha}$$
, for  $\alpha \in (-1,1)$ ;  
(1.2) Gamma fractional:  $g(u) = u^{\alpha}e^{\beta u}$ , for  $\alpha \in (-1,1)$ ,  $\beta < 0$ ;  
Power-law:  $g(u) = u^{\alpha}(1+u)^{\beta-\alpha}$ , for  $\alpha \in (-1,1)$ ,  $\beta < -1$ .

The following theorem generalises the classical mapping properties of Riemann-Liouville fractional operators first proved by Hardy and Littlewood [38], and will be of fundamental importance in the rest of our analysis. To ease the flow of the paper, we postpone its proof to Appendix C.1.

**Proposition 1.2.** For any  $\lambda \in (0,1)$  and  $\alpha \in \mathfrak{R}^{\lambda}$ , the operator  $\mathcal{G}^{\alpha} : \mathcal{C}^{\lambda}(\mathbb{I}) \to \mathcal{C}^{\lambda+\alpha}(\mathbb{I})$  is continuous.

We develop here an approximation scheme for the following system, generalising the concept of rough volatility introduced in [3, 31, 33] in the context of mathematical finance, where the process X represents the dynamics of the logarithm of a stock price process:

(1.3) 
$$dX_t = -\frac{1}{2}V_tdt + \sqrt{V_t}dB_t, \quad X_0 = 0,$$

$$V_t = \Phi\left(\mathcal{G}^{\alpha}Y\right)(t), \quad V_0 > 0,$$

with  $\alpha \in (-\frac{1}{2}, \frac{1}{2})$ , and Y the (strong) solution to the stochastic differential equation

$$dY_t = b(Y_t)dt + a(Y_t)dW_t, \quad Y_0 \in \mathcal{D}_Y,$$

where  $\mathcal{D}_Y$  denotes the state space of the process Y, usually  $\mathbb{R}$  or  $\mathbb{R}_+$ . The two Brownian motions B and W, defined on a common filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{I}}, \mathbb{P})$ , are correlated by the parameter  $\rho \in [-1, 1]$ , and the functional  $\Phi$  is assumed to be smooth on  $\mathcal{C}^1(\mathbb{I})$ . This is enough to ensure that the first stochastic differential equation is well defined. It remains to formulate the precise definition for  $\mathcal{G}^{\alpha}Y$  (Proposition 1.4) to fully specify the system (1.3) and clarify the existence of solutions. Existence and (strong) uniquess of a solution to the second SDE in (1.4) is guaranteed by the following standard assumption [79]:

**Assumption 1.3.** There exist  $C_b, C_a > 0$  such that, for all  $x, y \in \mathcal{D}_Y$ ,

$$|b(x) - b(y)| \le C_b|x - y|$$
 and  $|a(x) - a(y)| \le C_a \sqrt{|x - y|}$ .

Not only is the solution to (1.4) continuous, but  $(\frac{1}{2} - \varepsilon)$ -Hölder continuous for any  $\varepsilon \in (0, \frac{1}{2})$  as a consequence of the Kolmogorov-Čentsov theorem [16]. Existence and precise meaning of  $\mathcal{G}^{\alpha}Y$  is delicate, and is treated below.

1.2. **Examples.** Before constructing our approximation scheme, let us discuss a few examples of processes within our framework. As a first useful application, these generalised fractional operators render a (continuous) mapping between a standard Brownian motion and its fractional counterpart:

**Proposition 1.4.** For any  $\alpha \in \Re^{1/2}$ , the equality  $(\mathcal{G}^{\alpha}W)(t) = \int_0^t g(t-s) dW_s$  holds almost surely for  $t \in \mathbb{I}$ .

Modulo a constant multiplicative factor  $C_{\alpha}$ , the (left) fractional Riemann-Liouville operator (Appendix A) is identical to the GFO in (1.2), so that the Riemann-Riouville (or Type-II) fractional Brownian motion can be written as  $C_{\alpha}\mathcal{G}^{\alpha}W$ . Furthermore, Proposition 1.2 yields that the Riemann-Liouville operator is continuous from  $\mathcal{C}^{1/2}(\mathbb{I})$  to  $\mathcal{C}^{1/2+\alpha}(\mathbb{I})$  for  $\alpha \in \mathfrak{R}^{1/2}$ . Each kernel in (1.2) gives rise to processes proposed in turbulence modelling and in mathematical finance by Barndorff-Nielsen and Schmiegel [6].

**Example 1.5.** The first example is the rough Bergomi model introduced by Bayer, Friz and Gatheral [8], where

$$V_t = \xi_0(t) \mathcal{E} \left( 2\nu C_H \int_0^t (t-s)^\alpha dW_s \right),$$

with  $V_0, \nu, \xi_0(\cdot) > 0$ ,  $\alpha \in \mathfrak{R}^{1/2}$  and  $\mathcal{E}(\cdot)$  is the Wick stochastic exponential. This corresponds exactly to (1.3) with  $g(u) \equiv u^{\alpha}$ , Y = W and

$$\Phi(\varphi)(t) := \xi_0(t) \exp\left(2\nu C_H \varphi(t)\right) \exp\left\{-2\nu^2 C_H^2 \int_0^t (t-s)^{2\alpha} \mathrm{d}s\right\}.$$

**Example 1.6.** A truncated Brownian semistationary  $(\mathcal{TBSS})$  process is defined as  $\int_0^t g(t-s)\sigma(s)dW_s$ , for  $t \in \mathbb{I}$ , where  $\sigma$  is  $(\mathcal{F}_t)_{t\in\mathbb{I}}$ -predictable with locally bounded trajectories and finite second moments, and  $g:\mathbb{I}\setminus\{0\}\to\mathbb{I}$  is Borel measurable and square integrable. If  $\sigma\in\mathcal{C}_b^1(\mathbb{I})$ , this class falls within the GFO framework.

**Example 1.7.** Bennedsen, Lunde and Pakkanen [12] considered adding a Gamma kernel to the volatility process, which yields the Truncated Brownian semi-stationary (Bergomi-type) model:

$$V_t = \xi_0(t) \mathcal{E} \left( 2\nu C_H \int_0^t (t-s)^{\alpha} e^{-\beta(t-s)} dW_s \right),$$

with  $\beta > 0$ ,  $\alpha \in \Re^{1/2}$ . This corresponds to (1.3) with Y = W, Gamma fractional kernel  $g(u) \equiv u^{\alpha} e^{-\beta u}$  in (1.2),

$$\Phi(\varphi)(t) := \xi_0(t) \exp(2\nu C_H \varphi(t)) \exp\left\{-2\nu^2 C_H^2 \int_0^t (t-s)^{2\alpha} e^{-2\beta(t-s)} ds\right\}.$$

**Example 1.8.** The rough Heston model introduced by Guennoun, Jacquier, Roome and Shi [35] reads

$$Y_t = Y_0 + \int_0^t \kappa(\theta - Y_s) dt + \int_0^t \xi \sqrt{Y_s} dW_s,$$
  

$$V_t = \eta + \int_0^t (t - s)^{\alpha} dY_s,$$

with  $Y_0, \kappa, \xi, \theta > 0$ ,  $2\kappa\theta > \xi^2$  and  $\eta > 0$ ,  $\alpha \in \Re^{1/2}$ . This corresponds exactly to (1.3) with  $g(u) \equiv u^{\alpha}$ ,  $\Phi(\varphi)(t) := \eta + \varphi(t)$ , and the coefficients of (1.4) read  $b(y) \equiv \kappa(\theta - y)$  and  $a(y) \equiv \xi\sqrt{y}$ . This model is markedly different from the rough Heston introduced by El Euch and Rosenbaum [26] (for which the characteristic function is known in semi-closed form). Unfortunately, this second version is out of the scope of our invariance principle.

1.3. The approximation scheme. We now move on to the core of the project, namely an approximation scheme for the system (1.3). The basic ingredient to construct approximating sequences is a family of iid random variables, which satisfies the following assumption:

**Assumption 1.9.** The family  $(\xi_i)_{i\geq 1}$  forms an iid sequence of centered random variables with finite moments of all orders and  $\mathbb{E}[\xi_1^2] = \sigma^2 > 0$ .

Following Donsker [22] and Lamperti [54], we first define, for any  $\omega \in \Omega$ ,  $n \geq 1$ ,  $t \in \mathbb{I}$ , the approximating sequence for the driving Brownian motion B as

(1.5) 
$$B_n(t) := \frac{1}{\sigma \sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} \xi_k + \frac{nt - \lfloor nt \rfloor}{\sigma \sqrt{n}} \xi_{\lfloor nt \rfloor + 1}.$$

As will be explained later, a similar construction holds to approximate the process Y:

$$(1.6) \quad Y_n(t) := \frac{1}{n} \sum_{k=1}^{\lfloor nt \rfloor} b\left(Y_n^{k-1}\right) + \frac{nt - \lfloor nt \rfloor}{n} b\left(Y_n^{\lfloor nt \rfloor}\right) + \frac{1}{\sigma\sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} a\left(Y_n^{k-1}\right) \zeta_k + \frac{nt - \lfloor nt \rfloor}{\sigma\sqrt{n}} a\left(Y_n^{\lfloor nt \rfloor}\right) \zeta_{\lfloor nt \rfloor + 1},$$

where  $Y_n^k := Y_n(t_k)$  and  $\mathcal{T}_n := \{t_k = \frac{k}{n}\}$ . Here  $\{\xi\}_{i=1}^{\lfloor nt \rfloor}$  and  $\{\zeta\}_{i=1}^{\lfloor nt \rfloor}$  satisfy Assumption 1.9, with appropriate correlation structure between the pairs  $\{(\zeta_i, \xi_i)\}_{i=1}^{\lfloor nt \rfloor}$  that will be made precise later. We shall always use  $(\xi_i)$  to denote the sequence generating B and  $(\zeta_i)$  the one generating W. Consequently, we deduce an approximating scheme (up to the interpolating term which decays to zero by Chebyshev's inequality) for X as

$$(1.7) X_n(t) := -\frac{1}{2n} \sum_{k=1}^{\lfloor nt \rfloor} \Phi\left(\mathcal{G}^{\alpha} Y_n\right) (t_k) + \frac{1}{\sigma \sqrt{n}} \sum_{k=1}^{\lfloor nt \rfloor} \sqrt{\Phi\left(\mathcal{G}^{\alpha} Y_n\right) (t_k)} \left(B_n^{k+1} - B_n^k\right)$$

All the approximations above, as well as all the convergence statements below should be understood pathwise, but we omit the  $\omega$  dependence in the notations for clarity. The main result here is a convergence statement about the approximating sequence  $(X_n)_{n\geq 1}$ . As usual in weak convergence analysis [14], convergence is stated in the Skorokhod space  $(\mathcal{D}(\mathbb{I}), \|\cdot\|_{\mathcal{D}})$  of càdlàg processes equipped with the Skorokhod topology.

**Theorem 1.10.** The sequence  $(X_n)_{n\geq 1}$  converges weakly to X in  $(\mathcal{D}(\mathbb{I}), \|\cdot\|_{\mathcal{D}})$ .

The construction of the proof allows to extend the convergence to the case where Y is a d-dimensional diffusion without additional work. The proof of the theorem requires a certain number of steps: we start with the convergence of the approximation  $(Y_n)$  in some Hölder space, which we translate, first into convergence of the stochastic integral in (1.3), then, by continuity of the mapping  $\Phi$ , into convergence of the sequence  $(\Phi(\mathcal{G}^{\alpha}Y_n))$ . All these ingredients are detailed in Section 2 below. Once this is achieved, the proof of the theorem itself is relatively straightforward, as illustrated in Section 2.4.

- 2. Functional Central limit theorems for a family of Hölder continuous processes
- 2.1. Weak convergence of Brownian motion in Hölder spaces. Donsker's classical convergence result was proven under the Skorokhod topology. We concentrate here on convergence in the Hölder topology, due to Lamperti [55]. The standard convergence result for Brownian motion can be stated as follows:

**Theorem 2.1.** For  $\lambda < \frac{1}{2}$ , the sequence  $(B_n)$  in (1.5) converges weakly to a Brownian motion in  $(\mathcal{C}^{\lambda}(\mathbb{I}), \|\cdot\|_{\lambda})$ .

The proof relies on finite-dimensional convergence and tightness of the approximating sequence. Not surprisingly, the tightness criterion [14] in the Skorokhod space  $\mathcal{D}(\mathbb{I})$  and in a Hölder space setting are very different. In fact, the tightness criterion in Hölder spaces is strictly related to Kolmogorov-Čentsov's continuity theorem [16]. Note, in passing, that the approximating sequence (1.5) is piecewise differentiable (in time) for each  $n \geq 1$ , even though its limit is obviously not.

**Theorem 2.2** (Sufficient conditions for weak convergence in Hölder spaces). Let  $Z \in C^{\lambda}(\mathbb{I})$  and  $(Z_n)_{n\geq 1}$  an approximating sequence in the sense that for any sequence  $(\tau_k)_k$  in  $\mathbb{I}$ ,  $(Z_n(\tau_k)_k$  converges in distribution to  $(Z(\tau_k))$  as n tends to infinity.

$$(2.1) \qquad \mathbb{E}\left[|Z_n(t) - Z_n(s)|^{\gamma}\right] \le C|t - s|^{1+\beta}$$

holds for all  $n \geq 1$ ,  $t, s \in \mathbb{I}$ , and some  $C, \gamma, \beta > 0$ . Then  $(Z_n)_{n \geq 1}$  converges weakly to Z in  $C^{\mu}(\mathbb{I})$  for  $\mu < \frac{\beta}{\gamma} \leq \lambda$ .

As pointed out by Račkauskas and Suquet [71], strictly speaking the convergence takes place in the Hölder space  $C_0^{\lambda}(\mathbb{I})$  endowed with the norm  $||f||_{\lambda}^0 := |f|_{\lambda} + |f(0)|$ , for all functions that satisfy

$$\lim_{\delta \downarrow 0} \sup_{\substack{0 < t - s < \delta \\ t \neq c \in \mathbb{T}}} \frac{|f(t) - f(s)|}{(t - s)^{\gamma}} = 0.$$

Then  $(C_0^{\lambda}(\mathbb{I}), \|\cdot\|_{\lambda}^0)$  becomes a separable closed subspace of  $(C^{\lambda}(\mathbb{I}), \|\cdot\|_{\lambda})$  (see [37, 71] for details), and one can then use the simple tightness criterion introduced in Theorem 2.2. Moreover, as the identity map from  $C_0^{\lambda}(\mathbb{I})$  into  $C^{\lambda}(\mathbb{I})$  is continuous, weak convergence in the former implies weak convergence in the latter. To conclude our review of weak convergence in Hölder spaces, the following theorem, due to Račkauskas and Suquet [71] provides necessary and sufficient conditions ensuring convergence in Hölder space:

**Theorem 2.3** (Račkauskas-Suquet [71]). For any  $\lambda \in (0, \frac{1}{2})$ , the sequence  $(B_n)_{n\geq 1}$  in (1.5) converges (pathwise) weakly to a Brownian motion in  $C^{\lambda}(\mathbb{I})$  if and only if  $\mathbb{E}[\xi_1] = 0$  and  $\lim_{t\uparrow\infty} t^{\frac{1}{1-2\lambda}} \mathbb{P}(|\xi_1| \geq t) = 0$ .

We further prove the following result, which allows us to apply Theorem 2.2 on  $\mathbb{I}$  and extend the Hölder convergence result via linear interpolation to a continuous sequence.

**Theorem 2.4.** Let  $Z \in \mathcal{C}^{\lambda}(\mathbb{I})$  and  $(Z_n)_{n\geq 1}$  its corresponding approximation sequence such that  $(Z_n(t))_{t\in\mathcal{T}_n}$  converges in distribution to  $(Z(t))_{t\in\mathcal{T}_n}$  as n tends to infinity. Moreover, if

(2.2) 
$$\mathbb{E}[|Z_n(t_i) - Z_n(t_j)|^{\gamma}] \le C |t_i - t_j|^{1+\beta},$$

for any  $t_i, t_j \in \mathcal{T}_n$  and some  $\beta, \gamma, C > 0$ , then the linear interpolating sequence

$$Z_n(t) := Z_n\left(\frac{\lfloor nt\rfloor}{n}\right) + (nt - \lfloor nt\rfloor)\left(Z_n\left(\frac{\lfloor nt\rfloor + 1}{n}\right) - Z_n\left(\frac{\lfloor nt\rfloor}{n}\right)\right)$$

converges weakly to Z in  $C^{\mu}(\mathbb{I})$  for  $\mu < \frac{\beta}{\gamma} \leq \lambda$ .

*Proof.* For any  $t, s \in \mathbb{I}$ , we can write, letting  $Z_n^k := Z_n(t_k)$ ,

$$\begin{split} \mathbb{E}\left[|Z_n(t) - Z_n(s)|^{\gamma}\right] &= \mathbb{E}\left[\left|Z_n^{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor) \left(Z_n^{\lfloor nt \rfloor + 1} - Z_n^{\lfloor nt \rfloor}\right) - Z_n^{\lfloor ns \rfloor} - (ns - \lfloor ns \rfloor) \left(Z_n^{\lfloor ns \rfloor + 1} - Z_n^{\lfloor ns \rfloor}\right)\right|^{\gamma}\right] \\ &\leq 3^{\gamma - 1} \mathbb{E}\left[\left|Z_n^{\lfloor nt \rfloor} - Z_n^{\lfloor ns \rfloor}\right|^{\gamma} + (nt - \lfloor nt \rfloor)^{\gamma} \left|Z_n^{\lfloor nt \rfloor + 1} - Z_n^{\lfloor nt \rfloor}\right|^{\gamma} + (ns - \lfloor ns \rfloor)^{\gamma} \left|Z_n^{\lfloor ns \rfloor + 1} - Z_n^{\lfloor ns \rfloor}\right|^{\gamma}\right] \\ &\leq C\left(\left(\frac{\lfloor nt \rfloor - \lfloor ns \rfloor}{n}\right)^{1 + \beta} + \frac{(nt - \lfloor nt \rfloor)^{\gamma}}{n^{1 + \beta}} + \frac{(ns - \lfloor ns \rfloor)^{\gamma}}{n^{1 + \beta}}\right) \leq C(t - s)^{1 + \beta}, \end{split}$$

where we used (2.2) and the fact that  $\frac{\lfloor nt\rfloor - \lfloor ns\rfloor}{n} \le 2(t-s)$ ,  $nt - \lfloor nt\rfloor \le 1$  for  $t \ge 0$  and  $\frac{1}{n} \le (t-s)$ . Finally, it is left to prove the case  $\frac{1}{n} > (t-s)$ . There are two possible scenarios here:

• if  $\lfloor nt \rfloor = \lfloor ns \rfloor$ , then

$$\mathbb{E}[|Z_n(t) - Z_n(s)|^{\gamma}] = \mathbb{E}\left[\left|(nt - ns)\left(Z_n^{\lfloor nt \rfloor + 1} - Z_n^{\lfloor nt \rfloor}\right)\right|^{\gamma}\right] \le \frac{C(t - s)^{\gamma}}{n^{1 + \beta - \gamma}} \le C(t - s)^{1 + \beta};$$

• if  $\lfloor nt \rfloor \neq \lfloor ns \rfloor$ , then either  $\lfloor nt \rfloor + 1 = \lfloor ns \rfloor$  or  $\lfloor nt \rfloor = \lfloor ns \rfloor + 1$ . Without loss of generality consider the second case. Then

$$\mathbb{E}\left[\left|Z_{n}(t)-Z_{n}(s)\right|^{\gamma}\right] = \mathbb{E}\left[\left|Z_{n}(t)-Z_{n}^{\lfloor nt\rfloor}+Z_{n}^{\lfloor nt\rfloor}-Z_{n}(s)\right|^{\gamma}\right] \leq 2^{\gamma-1}\mathbb{E}\left[\left|Z_{n}(t)-Z_{n}^{\lfloor nt\rfloor}\right|^{\gamma}+\left|Z_{n}^{\lfloor nt\rfloor}-Z_{n}(s)\right|^{\gamma}\right] \\ \leq C\left((t-s)^{1+\beta}+\mathbb{E}\left[\left|(\lfloor nt\rfloor-ns)\left(Z_{n}^{\lfloor nt\rfloor}-Z_{n}^{\lfloor nt\rfloor-1}\right)\right|^{\gamma}\right]\right),$$

and the result follows as before since  $t - \frac{\lfloor nt \rfloor}{n} < |t-s|$  and  $|s - \frac{\lfloor nt \rfloor}{n}| \le |t-s|$ .

2.2. Weak convergence of Itô diffusions in Hölder spaces. The first important step in our analysis is to extend Donsker-Lamperti's weak convergence from Brownian motion to the Itô diffusion Y in (1.4).

**Theorem 2.5.** The sequence  $(Y_n)_{n\geq 1}$  in (1.6) converges weakly to Y in (1.4) in  $(\mathcal{C}^{\lambda}(\mathbb{I}), \|\cdot\|_{\lambda})$  for all  $\lambda < \frac{1}{2}$ .

*Proof.* Finite-dimensional convergence is a classical result by Kushner [53], so only tightness needs to be checked. In particular, using Theorem 2.4 we need only consider the partition  $\mathcal{T}_n$ . Using  $Y_n^i := Y_n(t_i)$  as above, and without loss of generality assume  $Y_n^0 = 0$  and  $b(Y_n^0) = 0$ , so that

$$\mathbb{E}\left[\left|Y_n^1\right|^{2p}\right] = \mathbb{E}\left[\left|\frac{b\left(Y_n^0\right)}{n} + \frac{a\left(Y_n^0\right)}{\sigma\sqrt{n}}\xi_1\right|^{2p}\right] \le \frac{\widetilde{C}_1}{n^p}.$$

where  $\widetilde{C}_1 := \left(\frac{a(Y_n^0)}{\sigma}\right)^{2p} \mathbb{E}\left(|\xi_1|^{2p}\right)$  is finite by Assumption 1.9. Using Assumption 1.3, we can write

$$\mathbb{E}\left[\left|Y_{n}^{2}\right|^{2p}\right] = \mathbb{E}\left[\left|Y_{n}^{1} + Y_{n}^{2} - Y_{n}^{1}\right|^{2p}\right] = \mathbb{E}\left[\left|Y_{n}^{1} + \frac{1}{n}b\left(Y_{n}^{1}\right) + \frac{1}{\sigma\sqrt{n}}a\left(Y_{n}^{1}\right)\xi_{2}\right|^{2p}\right] \\
\leq \mathbb{E}\left[\left|Y_{n}^{1} + \frac{C_{b}}{n}Y_{n}^{1} + \frac{1}{\sigma\sqrt{n}}C_{a}\sqrt{Y_{n}^{1}}\xi_{2}^{2p-1}\right|^{2p}\right] \\
\leq 2^{2p-1}\left(\mathbb{E}\left[\left|Y_{n}^{1}\right|^{2p}\right]\left(1 + \left(\frac{C_{b}}{n}\right)^{2p}\right) + \mathbb{E}\left[\left|\frac{1}{\sigma\sqrt{n}}C_{a}\sqrt{Y_{n}^{1}}\xi_{2}\right|^{2p}\right]\right) \\
\leq 2^{2p-1}\left(\frac{\widetilde{C}_{1}}{n^{p}}\left(1 + \left(\frac{C_{b}}{n}\right)^{2p}\right) + \frac{C_{a}^{2p}}{\sigma^{2p}n^{p}}\mathbb{E}\left[\left|Y_{n}^{1}\right|^{p}\right]\mathbb{E}\left[\left|\xi_{2}\right|^{2p}\right]\right) \\
\leq 2^{2p-1}\left(\frac{\widetilde{C}_{1}}{n^{p}}\left(1 + \left(\frac{C_{b}}{n}\right)^{2p}\right) + \frac{C_{a}^{2p}}{\sigma^{2p}n^{p}}\mathbb{E}\left[\left|Y_{n}^{1}\right|^{p}\right]\mathbb{E}\left[\left|\xi_{2}\right|^{2p}\right]\right) \leq \frac{\widetilde{C}_{2}}{n^{p}},$$

where  $\widetilde{C}_2 := 4^p \max\left(\widetilde{C}_1\left(1+(C_b)^{2p}\right), \left(\frac{C_a}{\sigma}\right)^{2p} \mathbb{E}\left[\left|Y_n^1\right|^p\right] \mathbb{E}\left[\left|\xi_1\right|^{2p}\right]\right)$  is finite by Assumption 1.9. By induction,  $\mathbb{E}[|Y_n^i-Y_n^j|^{2p}] \leq \widetilde{C}|t_i-t_j|^p$ , which implies the tightness criterion (2.1) for p>1 with  $\gamma=2p$  and  $\beta=p-1$ .  $\square$ 

2.3. Invariance principle for rough processes. We have set the ground to extend our results to processes that are not necessarily  $(1/2 - \varepsilon)$ -Hölder continuous, Markovian nor semimartingales. More precisely, we are interested in  $\alpha$ -Hölder continuous paths with  $\alpha \in (0,1)$ , such as Riemann-Liouville fractional Brownian motion or some  $\mathcal{TBSS}$  processes. A key tool is the Continuous Mapping Theorem, first proved by Mann and Wald [59], which establishes the preservation of weak convergence under continuous operators.

**Theorem 2.6** (Continuous Mapping Theorem). Let  $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$  and  $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$  be two normed spaces and assume that  $g: \mathcal{X} \to \mathcal{Y}$  is a continuous operator. If the sequence of random variables  $(Z_n)_{n\geq 1}$  converges weakly to Z in  $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ , then  $(g(Z_n))_{n\geq 1}$  also converges weakly to g(Z) in  $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ .

Many authors have exploited the combination of Theorems 2.1 and 2.6 to prove weak convergence [70, Chapter IV]. This path avoids the lengthy computations of tightness and finite-dimensional convergence in classical proofs [14]. In fact, Hamadouche [37] already realised that Riemann-Liouville fractional operators are continuous, hence Theorem 2.6 holds under mapping by Hölder continuous functions. In contrast, the novelty here is to consider the family of GFO applied to Brownian motion together with the extension of Brownian motion to Itô diffusions. In fact, minimal changes to the proof of Proposition 1.4 yield the following:

Corollary 2.7. If Y solves (1.4), then 
$$(\mathcal{G}^{\alpha}Y)(t) = \int_0^t g(t-s) dY_s$$
 almost surely for all  $t \in \mathbb{I}$  and  $\alpha \in \mathfrak{R}^{\lambda}$ .

The analogue of Theorem 2.5 for Y follows by continuous mapping along with the fact that  $\mathcal{G}^{\alpha}$  is a continuous operator from  $\left(\mathcal{C}^{\lambda}(\mathbb{I}), \|\cdot\|_{\lambda}\right)$  to  $\left(\mathcal{C}^{\lambda+\alpha}(\mathbb{I}), \|\cdot\|_{\lambda+\alpha}\right)$  for all  $\lambda \in (0,1)$  and  $\alpha \in \mathfrak{R}^{\lambda}$ .

**Theorem 2.8** (Generalised rough Donsker). For  $(Y_n)$  in (1.6), Y its weak limit in  $(\mathcal{C}^{\lambda}(\mathbb{I}), \|\cdot\|_{\lambda})$  for  $\lambda < \frac{1}{2}$ , (2.3)

$$\left(\mathcal{G}^{\alpha}Y_{n}\right)\left(t\right) = \begin{cases} \sum_{i=1}^{\lfloor nt \rfloor} g\left(t-t_{i-1}\right)\left(Y_{n}^{i}-Y_{n}^{i-1}\right) = \sum_{k=1}^{\lfloor nt \rfloor-1} \left[g\left(t-t_{i-1}\right)-g\left(t-t_{i}\right)\right]Y_{n}^{i-1}, & \text{if } t \in \mathcal{T}_{n}, \\ \sum_{i=1}^{\lfloor nt \rfloor} n\left[G\left(t-t_{i-1}\right)-G\left(t-t_{i}\right)\right]\left(Y_{n}^{i}-Y_{n}^{i-1}\right) + nG\left(t-t_{\lfloor nt \rfloor}\right)\left(Y_{n}(t)-Y_{n}^{\lfloor nt \rfloor}\right), & \text{if } t \notin \mathcal{T}_{n}, \end{cases}$$

converges weakly to  $\mathcal{G}^{\alpha}Y$  in  $(\mathcal{C}^{\alpha+\lambda}(\mathbb{I}), \|\cdot\|_{\alpha+\lambda})$  for any  $\alpha \in \mathfrak{R}^{\lambda}$ .

*Proof.* The case  $t \in \mathcal{T}_n$  is trivial to compute applying directly Definition 1.1. Otherwise, recall that the sequence (1.6) is piecewise differentiable in time. For  $\alpha \in \mathfrak{R}_+^{\lambda}$ , integration by parts yields, for  $n \geq 1$  and  $t \in \mathbb{I}$ ,

$$(\mathcal{G}^{\alpha}Y_{n})(t) = \int_{0}^{t} g'(t-s)Y_{n}(s)ds = \int_{0}^{t} g(t-s)\frac{dY_{n}(s)}{ds}ds$$

$$= \frac{1}{\sigma\sqrt{n}} \left[ \sum_{i=1}^{\lfloor nt \rfloor} n \int_{t_{i-1}}^{t_{i}} g(t-s)a\left(Y_{n}^{i-1}\right) \xi_{i}ds + n \int_{t_{\lfloor nt \rfloor}}^{t} g(t-s)a\left(Y_{n}^{\lfloor nt \rfloor}\right) \xi_{\lfloor nt \rfloor + 1}ds \right]$$

$$+ \frac{1}{n} \left[ n \sum_{i=1}^{\lfloor nt \rfloor} \int_{t_{i-1}}^{t_{i}} g(t-s)b\left(Y_{n}^{i-1}\right) ds + n \int_{t_{\lfloor nt \rfloor}}^{t} g(t-s)b\left(Y_{n}^{\lfloor nt \rfloor}\right) ds \right]$$

$$= \sum_{i=1}^{\lfloor nt \rfloor} n \left[ G(t-t_{i-1}) - G(t-t_{i}) \right] \left( Y_{n}^{i} - Y_{n}^{i-1} \right) + nG(t-t_{\lfloor nt \rfloor}) \left( Y_{n}(t) - Y_{n}^{\lfloor nt \rfloor} \right),$$

since G(0) = g(0) = 0. When  $\alpha \in \mathfrak{R}^{\lambda}_{-}$ , similar steps imply

$$(\mathcal{G}^{\alpha}Y_n)(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t g(t-s)Y_n(s)\mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t G(t-s)\frac{\mathrm{d}Y_n(s)}{\mathrm{d}s}\mathrm{d}s$$
$$= \sum_{i=1}^{\lfloor nt \rfloor} n \left[ G(t-t_{i-1}) - G(t-t_i) \right] \left( Y_n^i - Y_n^{i-1} \right) + nG(t-t_{\lfloor nt \rfloor}) \left( Y_n(t) - Y_n^{\lfloor nt \rfloor} \right);$$

when  $\frac{\lfloor nt \rfloor}{n} = t$ , G(0) = 0, and the expression is well defined.

Notice here, that, when  $t \notin \mathcal{T}_n$ , the mean value theorem implies

$$(\mathcal{G}^{\alpha}Y_n)\left(t\right) = \sum_{i=1}^{\lfloor nt \rfloor} g\left(t_i^*\right) \left(Y_n^i - Y_n^{i-1}\right) + g\left(t_{\lfloor nt \rfloor + 1}^*\right) \left(Y_n(t) - Y_n^{\lfloor nt \rfloor}\right),$$

where  $t_i^* \in [t - t_i, t - t_{i-1}]$  and  $t_{\lfloor nt \rfloor + 1}^* \in [0, t - t_{\lfloor nt \rfloor}]$  and we use that G(0) = 0. This expression is closer to the usual left-point forward Euler approximation. For numerical purposes, (2.4) is much more efficient (and a suitable candidate for optimal  $t_i^*$  is given in Section 3.3.1), since the integral G required in (2.3) is not necessarily available in closed form. As could be expected, the Hurst parameter influences the speed of convergence of the scheme. We leave a formal proof to further study, but the following argument provides some intuition about the correct normalising factor: For  $\alpha \in \mathfrak{R}_-^{\lambda}$ , since  $g \in \mathcal{L}^{\alpha}$ , the approximation (2.4) reads, for any  $n \geq 1$ ,

$$\left(\mathcal{G}^{\alpha}Y_{n}\right)(t_{i}) = \frac{1}{n^{1/2+\alpha}} \sum_{k=1}^{i} \left(nt_{i} - (k-1)T\right)^{\alpha} L\left(t_{i} - t_{k-1}\right) \left(Y_{n}^{k} - Y_{n}^{k-1}\right) \sqrt{n}, \quad \text{for } i = 0, \dots, n.$$

Here,  $(nt_i - T(k-1))^{\alpha} \le t_i^{\alpha}$  is bounded for any  $n \ge 1$ , so that the normalisation factor is of order  $n^{-\alpha-1/2}$ . When  $\alpha \in \mathfrak{R}_+^{\lambda}$  we rewrite (2.4) as

$$\left(\mathcal{G}^{\alpha}Y_{n}\right)(t_{i}) = \frac{1}{\sqrt{n}} \sum_{k=1}^{i} \left(t_{i} - t_{k-1}\right)^{\alpha} L\left(t_{i} - t_{k-1}\right) \left(Y_{n}^{k} - Y_{n}^{k-1}\right) \sqrt{n}, \quad \text{for } i = 0, \dots, n,$$

in which case,  $(t_i - t_{k-1})^{\alpha} \leq t_i^{\alpha}$  is bounded for  $n \geq 1$ , and the normalisation factor is of order  $n^{-1/2}$ . This intuition is consistent with the result by Neuenkirch and Shalaiko [65], who found the strong rate of convergence of the Euler scheme to be of order  $\mathcal{O}(n^{-H})$  for  $H < \frac{1}{2}$  for fractional Ornstein-Uhlenbeck. So far, our results hold for  $\alpha$ -Hölder continuous functions; however, for practical purposes, it is often necessary to constrain the

volatility process  $(V_t)_{t\in\mathbb{I}}$  to remain strictly positive at all times. The stochastic integral  $\mathcal{G}^{\alpha}Y$  need not be so in general. However, a simple transformation (e.g. exponential) can easily overcome this fact. The remaining question is whether the  $\alpha$ -Hölder continuity is preserved after this composition.

**Proposition 2.9.** Let  $(Y_n)_{n\geq 1}$  be the approximating sequence (1.6) in  $C^{\lambda}(\mathbb{I})$  for  $\lambda < 1/2$ . Then  $(\Phi(\mathcal{G}^{\alpha}Y_n))$  converges weakly to  $\Phi(\mathcal{G}^{\alpha}Y)$  in  $(C^{\alpha+\lambda}(\mathbb{I}), \|\cdot\|_{\alpha+\lambda})$  for all  $\alpha \in \mathfrak{R}^{\lambda}$ .

Proof. Drábek [23] found necessary and sufficient conditions ensuring that Hölder continuity is preserved under composition (which he calls Nemyckij operators). More precisely, he proved that the composition  $f \circ g$  is continuous from  $(\mathcal{C}^{\lambda}(\mathbb{I}), \|\cdot\|_{\lambda})$  to  $(\mathcal{C}^{\lambda}(\mathbb{I}, \|\cdot\|_{\lambda}))$  if and only if f is of class  $\mathcal{C}^{1}$ . The proof of the proposition then follows by applying the Continuous Mapping Theorem to Theorem 2.8 along with Drábek's continuity property. The following diagram summarises the steps, where  $\lambda < 1/2$ . The double arrows indicate weak convergence, and we indicate next to them the topology in which it takes place.

2.4. Extending the weak convergence to the Skorokhod space and proof of Theorem 1.10. The Skorokhod space of càdlàg processes equipped with the Skorokhod topology has been widely used to prove weak convergence [14]. The Skorokhod space of càdlàg processes equipped with the Skorokhod norm, which we denote  $(\mathcal{D}(\mathbb{I}), \|\cdot\|_{\mathcal{D}})$ , markedly simplifies when we only consider continuous processes (as is the case of our framework with Hölder continuous processes). Billingsley [14, Chapter 3 Section 12] proved that the identity  $(\mathcal{D}(\mathbb{I}) \cap \mathcal{C}(\mathbb{I}), \|\cdot\|_{\mathcal{D}}) = (\mathcal{C}(\mathbb{I}), \|\cdot\|_{\infty})$  always holds. This seemingly simple statement allows us to reduce proofs of weak convergence of continuous processes in the Skorokhod topology to that in the supremum norm, usually much simpler. We start with the following straightforward observation:

**Lemma 2.10.** For any  $\lambda \in (0,1)$ , the identity map is continuous from  $(\mathcal{C}^{\lambda}(\mathbb{I}), \|\cdot\|_{\lambda})$  to  $(\mathcal{D}(\mathbb{I}), \|\cdot\|_{\mathcal{D}})$ .

*Proof.* Since the identity map is linear, it suffices to check that it is bounded. For this observe that  $||f||_{\lambda} = |f|_{\lambda} + \sup_{t \in \mathbb{I}} |f(t)| = |f|_{\lambda} + ||f||_{\infty} > ||f||_{\infty}$ , where  $|f|_{\lambda} > 0$ , which concludes the proof since the Skorokhod norm in the space of continuous functions is equivalent to the supremum norm.

Applying the Continuous Mapping Theorem twice, first with the Generalised fractional operator (Theorem 2.8), then with the identity map, yields the following result directly:

**Theorem 2.11.** The sequence  $(\Phi(\mathcal{G}^{\alpha}Y_n))$  converges weakly to  $\Phi(\mathcal{G}^{\alpha}Y)$  in  $(\mathcal{D}(\mathbb{I}), ||\cdot||_{\mathcal{D}})$  for any  $\alpha \in \Re^{1/2}$ .

The final step in the proof of our main theorem, is to extend weak convergence to the log-stock price. For this, the following result on weak convergence of stochastic integrals  $X \bullet Y := \int X dY$  due to Jakubowski, Memin and Pagès [45], and later generalised to SDEs by Kurtz and Protter [52] is the key ingredient.

**Theorem 2.12.** Let  $(B_n)_{n\geq 1}$  be as in (1.5), N a càdlàg process on  $\mathbb{I}$ , and  $(N_n)_{n\geq 1}$  an approximating sequence such that  $(N_n, B_n)$  converges weakly in  $(\mathcal{D}(\mathbb{I}^2), \|\cdot\|_{\mathcal{D}})$  to (N, B). Then, there exists a filtration  $\mathcal{H}$  under which B is an  $\mathcal{H}$ -continuous martingale and  $(N_n, B_n, N_n \bullet B_n)_{n\geq 1}$  converges weakly to  $(N, B, N \bullet B)$ .

As noted in [52], the Skorokhod topology in  $\mathcal{D}(\mathbb{I}^2)$  is stronger than in  $\mathcal{D}(\mathbb{I}) \times \mathcal{D}(\mathbb{I})$ . In order to use this result, we first need to have the joint convergence of the two correlated driving Brownian motions W and B. Let  $(W_n)_{n\geq 1}$  and  $(W_n^{\perp})_{n\geq 1}$  be two Lamperti sequences with weak limits W and  $W^{\perp}$ , and let  $\overline{\rho}:=\sqrt{1-\rho^2}$ . Donsker's invariance implies that  $(W_n,W_n^{\perp})_{n\geq 1}$  converges weakly to  $(W,W^{\perp})$  in  $(\mathcal{C}^{\lambda}(\mathbb{I}^2),\|\cdot\|_{\lambda})$ , and hence by the Continuous Mapping Theorem with  $f(x,y):=\left(x,\rho x+\sqrt{1-\rho^2}y\right)$ , the sequence  $(W_n,\rho W_n+\overline{\rho}W_n^{\perp})_{n\geq 1}$  converges weakly to  $(W,\rho W+\overline{\rho}W^{\perp})$  in  $(\mathcal{C}^{\lambda}(\mathbb{I}^2),\|\cdot\|_{\lambda})$  for all  $\lambda<\frac{1}{2}$ . Finally, for  $\alpha\in\mathfrak{R}^{\lambda}$ , the first term on the right-hand side of (1.7) converges weakly to  $-\frac{1}{2}\int_0^T\Phi(\mathcal{G}^{\alpha}Y)(s)\mathrm{d}s$  by the Continuous Mapping Theorem, as the integral is a continuous operator from  $(\mathcal{D}(\mathbb{I}),\|\cdot\|_{\mathcal{D}})$  to itself. Since the couple  $(Y_n,B_n)$  converges weakly to (Y,B) in  $(\mathcal{D}(\mathbb{I}^2),\|\cdot\|_{\mathcal{D}})$ , Theorem 2.12 implies that the second term on the right-hand side of (1.7) converges weakly to  $\sqrt{\Phi(\mathcal{G}^{\alpha}Y)} \bullet B$ , and Theorem 1.10 follows.

# 3. Applications

3.1. Weak convergence of the Hybrid scheme. The Hybrid scheme (and its turbocharged version [60]) introduced by Bennedsen, Lunde and Pakkanen [11] is the current state-of-the-art to simulate  $\mathcal{TBSS}$  processes. However, only convergence in mean-square-error was proved, but not weak convergence, which would justify the use of the scheme for path-dependent options. Unless otherwise stated, we shall denote by  $\mathcal{I} := \{t_i = \frac{i}{n}\}_{i=0,...,n}$  the uniform grid on  $\mathbb{I}$ . The framework developed above provides such a convergence result:

**Proposition 3.1.** The sequence  $(\widetilde{\mathcal{G}}^{\alpha}W_n)$  in the Hybrid scheme (defined below in (3.1)) converges to  $\mathcal{G}^{\alpha}W$  in  $(\mathcal{C}^{\alpha+1/2}, \|\cdot\|_{\alpha+1/2})$  for  $\alpha \in \mathfrak{R}^{1/2}$ .

*Proof.* The Hybrid scheme in [11] with  $\kappa \geq 1$  can be written

(3.1) 
$$\widetilde{\mathcal{G}}^{\alpha}W_{n}(t_{i}) := \sum_{k=1}^{(i-\kappa)\vee 0} g(t_{i} - t_{k-1})\xi_{k} + \int_{0\vee t_{i-\kappa}}^{t_{i}} g(t_{i} - s)dW_{s}, \quad i = 0, \dots, n,$$

with  $\xi_k := \int_{t_{k-1}}^{t_k} dW_s \sim \mathcal{N}(0, 1/n)$  Gaussian, hence satisfying the conditions in Theorem 2.1. Comparing (2.4) with (3.1), weak convergence in the former implies weak convergence in the latter, since the error of the Hybrid scheme is smaller. The result then follows by Theorem 2.8

- **Remark 3.2.** Proposition 3.1 may easily be extended to a d-dimensional Brownian motion W (for example for multifactor volatility models), also providing a weak convergence result for the d-dimensional version of the Hybrid scheme recently developed by Heinrich, Pakkanen and Veraart [39].
- 3.2. Application to fractional binomial trees. We consider a binomial setting for the Riemann-Liouville fractional Brownian motion  $\mathcal{G}^{H-1/2}W$  with  $g(u) \equiv u^{H-1/2}$  for  $H \in (0,1)$ , for which Theorem 2.8 provides a

weakly converging sequence. On the partition  $\mathcal{I}$ , with Bernoulli random variables  $\{\zeta_i\}_{i=1}^n$  satisfying  $\mathbb{P}(\zeta_i = 1) = \mathbb{P}(\zeta_i = -1) = \frac{1}{2}$  for all i (justified by Theorem 1.10), the approximating sequence reads

$$(\mathcal{G}^{H-1/2}W_n)(t_i) := \frac{1}{\sqrt{n}} \sum_{k=1}^i (t_i - t_{k-1})^{H-1/2} \zeta_k, \text{ for } i = 0, \dots, n.$$

Figure 1 shows a fractional binomial tree structure for H=0.75 and H=0.1. Despite being symmetric, such trees cannot be recombining due to the (non-Markovian) path-dependent nature of the process. It might be possible, in principle, to modify the tree at each step to make it recombining, following the procedure developed in [2] for Markovian stochastic volatility models. It is not so straightforward though, and requires a dedicated thorough analysis which we leave for future research.

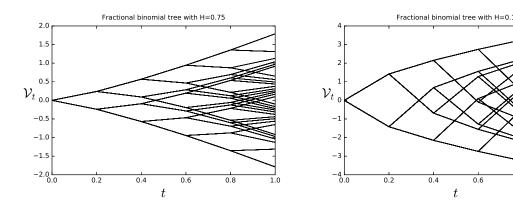


FIGURE 1. Binomial tree for the Riemann-Liouville fractional Brownian motion with n=5 discretisation points for H=0.75 (left) and H=0.1 (right).

3.3. Monte-Carlo. Theorem 1.10 introduces the theoretical foundations of Monte-Carlo methods (in particular for path-dependent options) for rough volatility models. In this section we give a general and easy-to-understand recipe to implement the class of rough volatility models (1.3). For the numerical recipe to be as general as possible, we shall consider the general time partition  $\mathcal{T} := \{t_i = \frac{iT}{n}\}_{i=0,\dots,n}$  on [0,T] with T > 0.

**Algorithm 3.3** (Simulation of rough volatility models).

- (1) Simulate two  $\mathcal{N}(0,1)$  matrices  $\{\xi_{j,i}\}_{\substack{j=1,...,M\\i=1,...,n}}$  and  $\{\zeta_{j,i}\}_{\substack{j=1,...,M\\i=1,...,n}}$  with  $\operatorname{corr}(\xi_{j,i},\zeta_{j,i})=\rho$ ;
- (2) simulate M paths of  $Y_n$  via<sup>1</sup>

$$Y_n^j(t_i) = \frac{T}{n} \sum_{k=1}^i b(Y_n^j(t_{k-1})) + \frac{T}{\sqrt{n}} \sum_{k=1}^i a\left(Y_n^j(t_{k-1})\right) \zeta_{j,k}, \quad i = 1, \dots, n \text{ and } j = 1, \dots, M,$$

and also compute

$$\Delta Y_n^j(t_i) := Y_n^j(t_i) - Y_n^j(t_{i-1}), \quad i = 1, \dots, n \text{ and } j = 1, \dots, M,$$

<sup>&</sup>lt;sup>1</sup>Here,  $Y_n^j(t_i)$  denotes the j-th path  $Y_n$  evaluated at the time point  $t_i$ , which is different from the notation  $Y_n^j$  in the theoretical framework above, but should not create any confusion.

(3) Simulate M paths of the fractional driving process  $((\mathcal{G}^{\alpha}Y_n)(t))_{t\in\mathcal{T}}$  using

$$(\mathcal{G}^{\alpha}Y_n)^j(t_i) := \sum_{k=1}^i g(t_{i-k+1}) \Delta Y_n^j(t_k) = \sum_{k=1}^i g(t_k) \Delta Y_n^j(t_{i-k+1}), \quad i = 1, \dots, n \text{ and } j = 1, \dots, M.$$

The complexity of this step is in general of order  $\mathcal{O}(n^2)$  (see Appendix B for details). However, this step is easily implemented using discrete convolution with complexity  $\mathcal{O}(n \log n)$  (see Algorithm B.4 in Appendix B for details in the implementation). With the vectors  $\mathfrak{g} := (g(t_i))_{i=1,\dots,n}$  and  $\Delta Y_n^j := (\Delta Y_n^j(t_i))_{i=1,\dots,n}$  for  $j=1,\dots,M$ , we can write  $(\mathcal{G}^{\alpha}Y_n)^j(\mathcal{T}) = \sqrt{\frac{T}{n}}(\mathfrak{g}*\Delta Y_n^j)$ , for  $j=1,\dots,M$ , where \* represents the discrete convolution operator.

(4) Use the forward Euler scheme to simulate the log-stock process, for all  $i = 1, \ldots, n, j = 1, \ldots, M$ , as

$$X^{j}(t_{i}) = X^{j}(t_{i-1}) - \frac{1}{2} \frac{T}{n} \sum_{k=1}^{i} \Phi\left(\mathcal{G}^{\alpha} Y_{n}\right)^{j} (t_{k-1}) + \sqrt{\frac{T}{n}} \sum_{k=1}^{i} \sqrt{\Phi\left(\mathcal{G}^{\alpha} Y_{n}\right)^{j} (t_{k-1})} \xi_{j,k}.$$

# Remark 3.4.

- When Y = W, we may skip step (2) and replace  $\Delta Y_n^j(t_i)$  by  $\sqrt{T/n}\zeta_{i,j}$  on step (3).
- Step (3) may be replaced by the Hybrid scheme algorithm [11] only when Y = W.

Antithetic variates in Algorithm 3.3 are easy to implement as it suffices to consider the uncorrelated random vectors  $\zeta_j := (\zeta_{j,1}, \zeta_{j,2}, \dots, \zeta_{j,n})$  and  $\xi_j := (\xi_{j,1}, \xi_{j,2}, \dots, \xi_{j,n})$ , for  $j = 1, \dots, M$ . Then  $(\rho \xi_j + \overline{\rho} \zeta_j, \xi_j)$ ,  $(\rho \xi_j - \overline{\rho} \zeta_j, \xi_j)$ ,  $(-\rho \xi_j - \overline{\rho} \zeta_j, -\xi_j)$  and  $(-\rho \xi_j + \overline{\rho} \zeta_j, -\xi_j)$ , for  $j = 1, \dots, M$ , constitute the antithetic variates, which significantly improves the performance of the Algorithm 3.3 by reducing memory requirements, reducing variance and accelerating execution by exploiting symmetry of the antithetic random variables.

3.3.1. Enhancing performance. A standard practice in Monte-Carlo simulation is to match moments of the approximating sequence with the target process. In particular, when the process is Gaussian, matching first and second moments suffices. We only illustrate this approximation for Brownian motion: the left-point approximation (2.4) (with Y = W) may be modified to match moments as

(3.2) 
$$(\mathcal{G}^{\alpha}W)(t_i) \approx \frac{1}{\sigma\sqrt{n}} \sum_{k=1}^{i} g(t_k^*) \zeta_k, \quad \text{for } i = 0, \dots, n,$$

where  $t_k^*$  is chosen optimally. Since the kernel  $g(\cdot)$  is deterministic, there is no confusion with the Stratonovich stochastic integral, and the resulting approximation will always converge to the Itô integral. The first two moments of  $\mathcal{G}^{\alpha}W$  read

$$\mathbb{E}\left(\left(\mathcal{G}^{\alpha}W\right)(t)\right)=0\qquad\text{and}\qquad\mathbb{V}\left(\left(\mathcal{G}^{\alpha}W\right)(t)\right)=\int_{0}^{t}g(t-s)^{2}\mathrm{d}s.$$

The first moment of the approximating sequence (3.2) is always zero, and the second moment reads

$$\mathbb{V}\left(\frac{1}{\sigma\sqrt{n}}\sum_{k=1}^{j-1}g(t_k^*)\zeta_k\right) = \frac{1}{n}\sum_{k=1}^{j-1}g(t_k^*)^2.$$

Equating the theoretical and approximating quantities we obtain  $\frac{1}{n}g(t_k^*)^2 ds = \int_{t_{k-1}}^{t_k} g(t-s)^2 ds$  for  $k=1,\ldots,n$ , so that the optimal evaluation point can be computed as

(3.3) 
$$g(t_k^*) = \sqrt{n \int_{t_{k-1}}^{t_k} g(t-s)^2 ds}, \quad \text{for any } k = 1, \dots, n.$$

With the optimal evaluation point the scheme is still a convolution so that Algorithm B.4 in Appendix B can still be used for faster computations. In the Riemann-Liouville fractional Brownian motion case,  $g(u) = u^{H-1/2}$ , and the optimal point can be computed in closed form as

$$t_k^* = \left(\frac{n}{2H} \left[ (t - t_{k-1})^{2H} - (t - t_k)^{2H} \right] \right)^{1/(2H-1)}, \quad \text{for each } k = 1, \dots, n.$$

This optimal evaluation point framework is also valid for the Hybrid scheme [11]. The authors originally proposed an optimal evaluation point minimising the mean square error. Nevertheless, we have seen in Proposition 3.1 that the scheme converges weakly already with a left-point approximation, hence the user is free to choose the optimal evaluation point based on criteria different from the mean square error.

3.3.2. Reducing Variance. As Bayer, Friz and Gatheral [8] and Bennedsen, Lunde and Pakkanen [11] pointed out, a major drawback in simulating rough volatility models is the very high variance of the estimators, so that a large number of simulations are needed to produce a decent price estimate. Nevertheless, the rDonsker scheme admits a very simple conditional expectation technique which reduces both memory requirements and variance while also admitting antithetic variates. This approach is best suited for calibrating European type options. We consider  $\mathcal{F}_t^B = \sigma(B_s: s \leq t)$  and  $\mathcal{F}_t^W = \sigma(W_s: s \leq t)$  the natural filtrations generated by the Brownian motions B and W. In particular the conditional variance process  $V_t | \mathcal{F}_t^W$  is deterministic. As discussed by Romano and Touzi [72], and recently adapted to the rBergomi case by McCrickerd and Pakkanen [60], we can decompose the stock price process as

$$\mathbf{e}^{X_{t}} = \mathcal{E}\left(\rho \int_{0}^{t} \sqrt{\Phi\left(\mathcal{G}^{\alpha}Y\right)\left(s\right)} \mathrm{d}B_{s}\right) \mathcal{E}\left(\sqrt{1 - \rho^{2}} \int_{0}^{t} \sqrt{\Phi\left(\mathcal{G}^{\alpha}Y\right)\left(s\right)} \mathrm{d}B_{s}^{\perp}\right) := \mathbf{e}^{X_{t}^{1}} \mathbf{e}^{X_{t}^{2}},$$

and notice that

$$X_{t}|(\mathcal{F}_{t}^{W} \wedge \mathcal{F}_{0}^{B}) \sim \mathcal{N}\left(X_{t}^{1} - (1 - \rho^{2}) \int_{0}^{t} \Phi\left(\mathcal{G}^{\alpha}Y\right)(s) \mathrm{d}s, (1 - \rho^{2}) \int_{0}^{t} \Phi\left(\mathcal{G}^{\alpha}Y\right)(s) \mathrm{d}s\right).$$

Thus  $\exp(X_t)$  becomes log-normal and the Black-Scholes closed-form formulae are valid here (European, Barrier options, maximum,...). The advantage of this approach is that the orthogonal Brownian motion  $B^{\perp}$  is completely unnecessary for the simulation, hence the generation of random numbers is reduced to a half, yielding proportional memory saving. Not only this, but also this simple trick reduces the variance of the Monte-Carlo estimate, hence fewer simulations are needed to obtain the same precision. We present a simple algorithm to implement the rDonsker with conditional expectation and assuming that Y = W.

Algorithm 3.5 (Simulation of rough volatility models with Brownian drivers). Consider the equidistant grid  $\mathcal{T}$ .

- (1) Draw a random matrix  $\{\zeta_{j,i}\}_{\substack{j=1,\dots,M/2\\i=1,\dots,n}}$  with unit variance, and create antithetic variates  $\{-\zeta_{j,i}\}_{\substack{j=1,\dots,M/2\\i=1,\dots,n}}$
- (2) Create a correlated matrix  $\{\xi_{j,i}\}$  as above;
- (3) Simulate M paths of the fractional driving process  $\mathcal{G}^{\alpha}W$  using discrete convolution (see Algorithm B.4 in Appendix B for details in the implementation):

$$(\mathcal{G}^{\alpha}W)^{j}(\mathcal{T}) = \sqrt{\frac{T}{n}}(\mathfrak{g} * \zeta_{j}), \quad j = 1, \dots, M,$$

and store in memory 
$$(1-\rho^2)$$
  $\int_0^T (\mathcal{G}^{\alpha}W)^j(s) ds \approx (1-\rho^2) \frac{T}{n} \sum_{k=0}^{n-1} (\mathcal{G}^{\alpha}W)^j(t_k) =: \Sigma^j$  for each  $j = 1, \dots, M$ ;

(4) use the forward Euler scheme to simulate the log-stock process, for each i = 1, ..., n, j = 1, ..., M, as

$$X^{j}(t_{i}) = X^{j}(t_{i-1}) - \frac{\rho^{2}}{2} \frac{T}{n} \sum_{k=1}^{i} \Phi \left(\mathcal{G}^{\alpha} W\right)^{j} (t_{k-1}) + \rho \sqrt{\frac{T}{n}} \sum_{k=1}^{i} \sqrt{\Phi \left(\mathcal{G}^{\alpha} W\right)^{j} (t_{k-1})} \xi_{j,i};$$

(5) Finally, we have  $X^j(T) \sim \mathcal{N}(X_T^j - \Sigma^j, \Sigma^j)$  for  $j = 1, \ldots, M$ ; we may compute any option using the Black-Scholes formula. For instance a Call option with strike K would be given by  $C^j(K) = \exp(X_T^j)\mathcal{N}(d_1^j) - K\mathcal{N}(d_2^j)$  for  $j = 1, \ldots, M$ , where  $d_1^j := \frac{1}{\sqrt{\Sigma^j}}(X_T^j - \log(K) + \frac{1}{2}\Sigma^j)$  and  $d_2^j = d_1^j - \sqrt{\Sigma^j}$ . Thus, the output of the model would be  $C(K) = \frac{1}{M}\sum_{k=1}^M C^j(K)$ .

The algorithm is easily adapted to the case of general diffusions Y as drivers of the volatility (see Algorithm 3.3 step (2)). Algorithm 3.3 is obviously faster than 3.5, especially when using control variates. Nevertheless, with the same number of paths, Algorithm 3.5 remarkably reduces the Monte-Carlo variance, meaning in turn that fewer simulations are needed, making it very competitive for calibration.

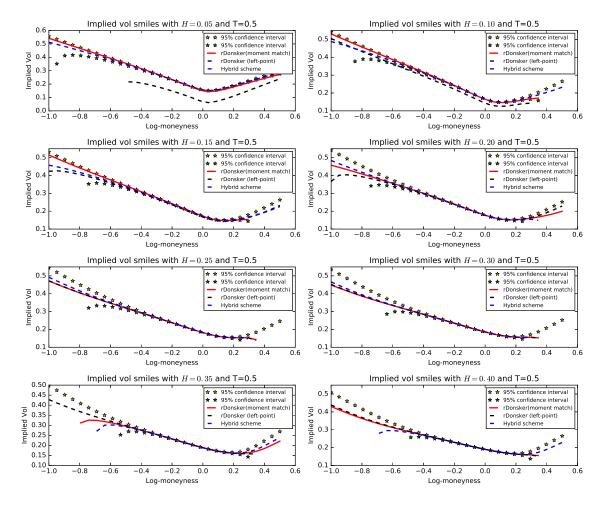


FIGURE 2. Implied volatilities of rDonsker with left-point and variance matching, and in the Hybrid scheme with  $5 \cdot 10^5$  simulations. Conditional expectation and antithetic variates where used in both methods.

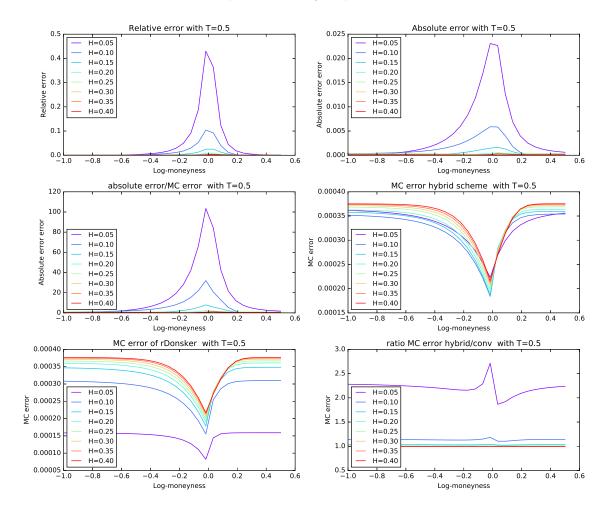


FIGURE 3. Monte-Carlo errors between left-point rDonsker and the Hybrid scheme with  $5 \cdot 10^5$  simulations. Conditional expectation and antithetic variates where used in both methods.

3.4. Numerical example: Rough Bergomi model. Figure 2 shows implied volatilities for different values of H, when using left-point and moment matching optimal evaluation point in the rDonsker scheme and also the Hybrid scheme. In Figures 3 and 4, we give a exhaustive comparison analysis of the errors when using a left-point evaluation and moment matching optimal evaluation. It is obvious from Figure 3 that as H tends to zero, the left-point rDonsker converges too slowly to the required output as opposed to the Hybrid scheme, which was shown in [11] to converge to the output regardless of H and the discretisation grid. This phenomenon is not surprising, since we already discussed that the rate of convergence of the rDonsker scheme is of order  $\mathcal{O}(n^{-H})$ . Nevertheless, for H > 0.15 there is no significant difference between both schemes. In particular, we notice that the biggest error for the rDonsker scheme happens when the options is around-the-money. Now, in Figure 4 we observe how the optimal evaluation point improves substantially the performance of the rDonsker scheme. The relative error and absolute errors are reduced by a factor of 10 when H = 0.05 is very small. This maintains the relative error below 4% for  $H \geq 0.05$ . Specifically, it is worth noticing that from Figure 3 to Figure 4 the behaviour of the Monte-Carlo error when H = 0.05 dramatically changes when using the optimal evaluation point, becoming more similar to the Hybrid scheme.

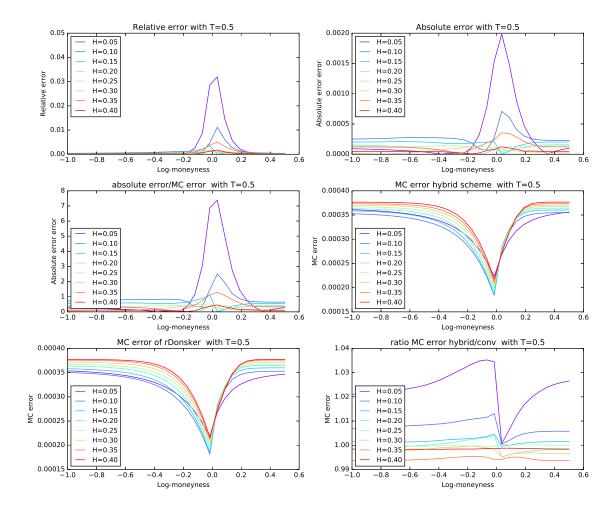


FIGURE 4. Monte-Carlo errors between rDonsker using moment matching evaluation and the Hybrid scheme with  $5 \cdot 10^5$  simulations. Conditional expectation and antithetic variates where used in both methods.

3.5. Speed benchmark against Markovian stochastic volatility models. In this section we benchmark the speed of the rDonsker scheme against the Hybrid scheme and a classical Markovian stochastic volatility model using  $10^5$  simulations and averaging the speeds over 10 trials. For the former ones we simulate the rBergomi model [8], whereas for the latter we use the classical Bergomi [13] model using a forward Euler scheme in both volatility and stock price. All three schemes are implemented in Cython to make the comparisson fair and to obtain C++ like speeds. Figure 5 shows that rDonsker is approximately 2 times slower than the Markovian case whereas the Hybrid scheme is approximately 2.5 times slower. This is of course expected from the complexities of both schemes. However, it is remarkable that the  $\mathcal{O}(n \log n)$  complexity of the FFT stays almost constant with the grid size n and the computational time grows almost linearly as in the Markovian case. We presume that this is the case since n << 10000 is relatively small. Figure 5 also proves that rough volatility models can be implemented very efficiently and are not particularly slower than classical stochastic volatility models.

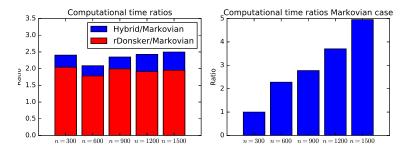


FIGURE 5. Computational time benchmark using Hybrid scheme, rDonsker and Markovian (forward Euler) for different grid sizes n.

3.6. Implementation guidelines and conclusion. Based on the numerical analysis above, we suggest the following guidelines to implement rough volatility models driven by TBSS processes of the form  $\mathcal{G}^{H-1/2}Y$ , for some Itô diffusion Y:

H > 0.1	$H \in [0.05, 0.1]$	H < 0.05
rDonsker	choice depends on error sensitivity	Hybrid scheme

Regarding empirical estimates, Gatheral, Jaisson and Rosenbaum [35] suggest that  $H \approx 0.15$ . Bennedsen, Lunde and Pakkanen [12] give an exhaustive analysis of more than 2000 equities for which  $H \in [0.05, 0.2]$ . On the pricing side, Bayer, Friz and Gatheral [8] and Jacquier, Martini and Muguruza [46] found that calibration routines yield  $H \in [0.05, 0.10]$ . Finally, Livieri, Mouti, Pallavicini and Rosenbaum [56] found evidence in options data that  $H \approx 0.3$ . Despite the diverse ranges found so far, there is a common agreement that H < 1/2.

**Remark 3.6.** The rough Heston model presented by Guennoun, Jacquier, Roome and Shi [35] is out of the scope of the Hybrid scheme. Moreover, any process of the form  $\mathcal{G}^{\alpha}Y$ , for some Itô diffusion Y under Assumptions 1.3 is, in general, out of the scope of the Hybrid scheme. This only leaves the choice of using the rDonsker scheme, for which reasonable accuracy is obtained at least for Hölder regularities greater than 0.05.

3.7. Bushy trees and binomial markets. Binomial trees have attracted a lot of attention from both academics and practitioners, as their apparent simplicity provide easy intuition about the dynamics of a given asset. Not only this, but they are by construction arbitrage free and allow to price path-dependent options, together with their hedging strategy. In particular, early exercise options, in particular Bermudan or American options, are usually priced using trees, as opposed to Monte-Carlo methods. The convergence stated in Theorem 1.10 lays the theoretical foundations to construct fractional binomial trees (note that Bernoulli random variables satisfy the conditions of the theorem). Figure 1 already showed binomial trees for fractional Brownian motion, but we ultimately need trees describing the dynamics of the stock price.

3.7.1. A binary market. We invoke Theorem 1.10 with the independent sequences  $\{\zeta_i\}_{i=1}^n$ ,  $\{\zeta_i^{\perp}\}_{i=1}^n$  such that  $\mathbb{P}(\zeta_i=1)=\mathbb{P}(\zeta_i^{\perp}=1)=\mathbb{P}(\zeta_i=1)=\mathbb{P}(\zeta_i^{\perp}=1)=\mathbb{P}(\zeta_$ 

$$B_n(t_i) = \sqrt{\frac{T}{n}} \sum_{k=1}^{i} \left( \rho \zeta_k + \overline{\rho} \zeta_k^{\perp} \right),$$

$$Y_n(t_i) = \frac{T}{n} \sum_{k=1}^{i} b(Y_n(t_{k-1})) + \sqrt{\frac{T}{n}} \sum_{k=1}^{i} \sigma(Y_n(t_{k-1})) \zeta_k,$$

the approximating sequences to B and Y in (1.3). The approximation for X is then given by

$$X_{n}(t_{i}) = X_{n}(t_{i-1}) - \frac{1}{2} \frac{T}{n} \sum_{k=1}^{i} \Phi\left(\mathcal{G}^{\alpha} Y_{n}\right)(t_{k}) + \sqrt{\frac{T}{n}} \sum_{k=1}^{i} \sqrt{\Phi\left(\mathcal{G}^{\alpha} Y_{n}\right)(t_{k})} \left(\rho \zeta_{k} + \overline{\rho} \zeta_{k}^{\perp}\right).$$

In order to construct the tree we have to consider all possible permutations of the random vectors  $\{\zeta_i\}$  and  $\{\zeta_i^{\perp}\}$ . Since each random variable only takes two values, this adds up to  $4^n$  possible combinations, hence the 'bushy tree' terminology. When  $\rho \in \{-1,1\}$ , the magnitude is reduced to  $2^n$ .

3.8. American options in rough volatility models. There is so far no available scheme for American options (or any early-exercise options for that matter) under rough volatility models, but the fractional trees constructed above provide a framework to do so. In the Black-Scholes model, American options can be priced using binomial trees by backward induction. A key ingredient is the Snell envelope [74] and the following representation by El Karoui [27] ( $\tilde{\mathbb{I}}$  denotes the set of stopping times with values in  $\mathbb{I}$ ):

**Definition 3.7.** Let  $(X_t)_{t\in\mathbb{I}}$  be an  $(\mathcal{F}_t)_{t\in\mathbb{I}}$  adapted process, and  $\tau\in\widetilde{\mathbb{I}}$ . The Snell envelope  $\mathcal{J}$  of X is defined as  $\mathcal{J}(X)(t):=\operatorname{ess\,sup}_{\tau\in\widetilde{\mathbb{I}}}\mathbb{E}(X_\tau|\mathcal{F}_t)$  for all  $t\in\mathbb{I}$ .

In plain words, the Snell envelope of X is the smallest supermartingale that dominates it. Strictly speaking, it is necessary for  $X_{\tau}$  to be uniformly integrable for any  $\tau \in \widetilde{\mathbb{I}}$ . Following [48], an American option is nothing else than the smallest supermartingale dominating its European counterpart:

**Definition 3.8.** Let  $C_t^e(k,T)$  and  $P_t^e(k,T)$  denote European Call and Put prices at time t, with log-strike k and maturity T. Then the American counterparts,  $C_t^a(k,T)$  and  $P_t^a(k,T)$ , are given by

$$C^a_t(k,T) = \mathcal{J}(C^e(k,T))(t) \qquad \text{and} \qquad P^a_t(k,T) = \mathcal{J}(P^e(k,T))(t).$$

Preservation of weak convergence under the Snell envelope map is due to Mulinacci and Pratelli [62], who proved that convergence takes place in the Skorokhod topology only if the Snell envelope is continuous. In our setting, the scheme for American options is fully justified by the following theorem:

**Theorem 3.9.** For V in (1.3), if  $\mathbb{E}\left\{\exp\left(\int_0^t V_s ds\right)\right\}$  is finite, then  $(\mathcal{J}(X_n))_{n\geq 1}$  converges weakly to  $\mathcal{J}(X)$  in the Skorokhod topology.

Proof. Since the sequence  $(X_n)_{n\geq 1}$  converges weakly to X in  $(\mathcal{D}(\mathbb{I}), \|\cdot\|_{\mathcal{D}})$ , for X in (1.3), the theorem follows using the Continuous Mapping Theorem if we can show that  $\mathcal{J}$  is continuous. El Karoui proved [27] proved that the Snell envelope of an optional process, uniformly integrable for all stopping times  $\tau \in \widetilde{\mathbb{I}}$ , is continuous. To prove the proposition, we therefore only need to check uniform integrability of the stock price  $e^X$ . Using the de la Vallée-Poussin theorem, for any  $t \in \mathbb{I}$ ,

$$\mathbb{E}\left(e^{2X_t}\right) = \mathbb{E}\left[\exp\left(-\int_0^t V_s ds + 2\int_0^t \sqrt{V_s} dB_s\right)\right] = \mathbb{E}\left[\mathcal{E}\left(2\int_0^t \sqrt{V_s} dB_s\right) \exp\left(3\int_0^t V_s ds\right)\right].$$

With  $\mathcal{F}_t^W := \sigma(W_s : s \leq t)$  the filtration generated by W (or equivalently by V), the tower property yields

$$\mathbb{E}\left(e^{2X_t}\right) = \mathbb{E}\left[\mathbb{E}\left(e^{2X_t}|\mathcal{F}_t^W\right)\right] = \mathbb{E}\left[\exp\left(3\int_0^t V_s ds\right)\right],$$

by the martingale property of  $\mathcal{E}\left(2\int_0^t \sqrt{V_s} dB_s\right) | \mathcal{F}_t^W$ . Therefore  $\exp(X)$  is a uniformly integrable martingale, and so is  $\exp(X_{\tau \wedge \cdot})$  by Doob's optimal stopping theorem, and the proposition follows.

Corollary 3.10. Theorem 3.9 also holds under the stronger condition  $\mathbb{E}\left(e^{V_t}\right) < \infty$  for all  $t \in \mathbb{I}$ .

*Proof.* Jensen's inequality implies that

$$\mathbb{E}\left[\exp\left(\int_{0}^{t} V_{s} ds\right)\right] \leq \mathbb{E}\left(C_{t} \int_{0}^{t} e^{V_{s}} ds\right) = C_{t} \int_{0}^{t} \mathbb{E}\left(e^{V_{s}}\right) ds$$

for some constant  $C_t > 0$ , and the right-hand side is finite if  $\mathbb{E}(e^{V_s})$  is and the proposition follows.

Mulinacci and Pratelli [62] also gave explicit conditions for the weak convergence to be preserved in the Markovian case. It is trivial to see that the pricing of American options in the rough tree scheme coincides with the classical backward induction procedure. We consider continuously compounded interest rates and dividend yields, denote by r and d.

**Algorithm 3.11** (American options in rough volatility models). On the equidistant grid  $\mathcal{T}$ ,

- (1) construct the binomial tree using the explicit construction in Section 3.7.1 and obtain  $\{S_t^j\}_{t\in\mathcal{T},j=1,\ldots,4^n}$ ;
- (2) the backward recursion for the American with exercise value  $h(\cdot)$  is given by  $h_{t_N} := h(S_{t_N})$  and

$$\widetilde{h}_{t_i} := e^{(d-r)/n} \mathbb{E}\left(\widetilde{h}_{t_{i+1}} | \mathcal{F}_{t_i}\right) \vee h(S_{t_i}), \quad \text{for } i = N-1, \dots, 0,$$

where  $\mathbb{E}(\cdot|\mathcal{F}_{t_i}) = \frac{1}{4} \left( \widetilde{h}_{t_{i+1}}^{++} + \widetilde{h}_{t_{i+1}}^{+-} + \widetilde{h}_{t_{i+1}}^{-+} + \widetilde{h}_{t_{i+1}}^{--} \right)$  and  $\widetilde{h}_{t_i}^{\pm\pm}$  represents the outcome  $(\zeta_i, \zeta_i^{\perp}) = (\pm 1, \pm 1)$  for the driving binomials, following the construction in Section 3.7.1.

(3) finally,  $h_0$  is the price of the American option at inception of the contract.

The main computational cost of the scheme is the construction of the tree in Step 1. Once the tree is constructed, computing American prices for different options is a fast routine.

- 3.8.1. Numerical example: rough Bergomi model. We construct a rough volatility tree for the rough Bergomi model [8] and check the accuracy of the scheme. Figures 6 and 7 show the fractional trees for different values of H and for  $\rho \in \{-1,1\}$ . Both pictures show a markedly different behaviour, but as a common property we observe that as H tends to 1/2, the tree structure somehow becomes simpler.
- 3.8.2. European options. Figure 8 displays volatility smiles obtained using the tree scheme. Even though the time steps are not sufficient for small H, the fit remarkably improves when  $H \geq 0.15$ , and always remains inside the 95% confidence interval with respect to the Hybrid scheme. Moreover, the moment-matching approach from Section 3.3.1 shows a superior accuracy when  $H \leq 0.1$ , but is not sufficiently accurate. In Figure 9 a detailed error analysis corroborates these observations: the relative error is smaller than 3% for  $H \geq 0.15$ .
- 3.8.3. American options. In the context of American options, there is no benchmark to compare our result. However, the accurate results found in the previous section (at least for  $H \geq 0.15$ ) justify the use of trees to price American options. Figure 10 shows the output of American and European Put prices with interest rates equal to r = 5%. Interestingly, the rougher the process (the smaller the H), the larger the difference between in-the-money European and American options.

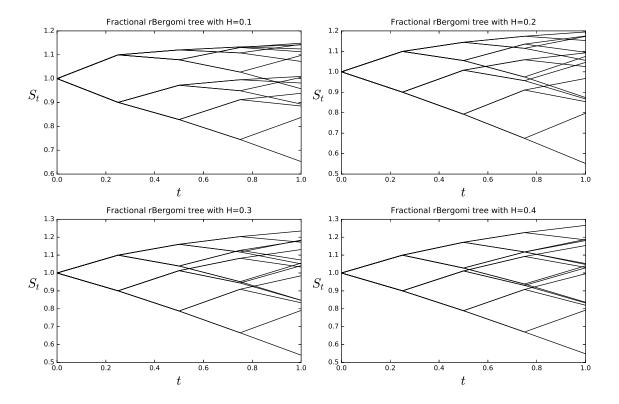


FIGURE 6. rough Bergomi trees for different values of H,  $(\nu, \rho, \xi_0) = (1, -1, 0.04)$  with 5 time steps.

# APPENDIX A. RIEMANN-RIOUVILLE OPERATORS

We review here fractional operators and their mapping properties. We follow closely the excellent monograph by Samko, Kilbas and Marichev [73], as well as some classical results by Hardy and Littlewood [38].

# A.0.1. Riemann-Liouville fractional operators.

**Definition A.1.** For any  $\lambda \in (0,1)$ ,  $\alpha \in \mathfrak{R}^{\lambda}$  the left Riemann-Liouville fractional operator is defined on  $\mathcal{C}^{\lambda}(\mathbb{I})$  as

$$(A.1) \qquad (I^{\alpha}f)(t) := \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} \mathrm{d}s, & \text{for } \alpha \in [0, 1-\lambda), \\ \left(\frac{\mathrm{d}}{\mathrm{d}t} I^{1+\alpha} f\right)(t) = \frac{1}{\Gamma(1+\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-s)^{\alpha} f(s) \mathrm{d}s, & \text{for } \alpha \in (-\lambda, 0). \end{cases}$$

**Theorem A.2.** For any  $f \in C^{\lambda}(\mathbb{I})$ , with  $\lambda \in (0,1)$  and  $\alpha \in \mathfrak{R}^{\lambda}$  the identity

$$(I^{\alpha}f)(t) = \frac{f(0)}{\Gamma(1+\alpha)}t^{\alpha} + \psi(t),$$

holds for all  $t \in \mathbb{I}$ , for some  $\psi \in \mathcal{C}^{\lambda+\alpha}(\mathbb{I})$  satisfying  $|\psi(t)| \leq Ct^{\lambda+\alpha}$  on  $\mathbb{I}$  for some C > 0.

*Proof.* We may easily represent

$$(I^{\alpha}f)(t) = \frac{f(0)}{\Gamma(\alpha)} \int_0^t \frac{\mathrm{d}u}{(t-u)^{1-\alpha}} + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(u) - f(0)}{(t-u)^{1-\alpha}} \mathrm{d}u = \frac{f(0)}{\Gamma(1+\alpha)} t^{\alpha} + \psi(t)$$

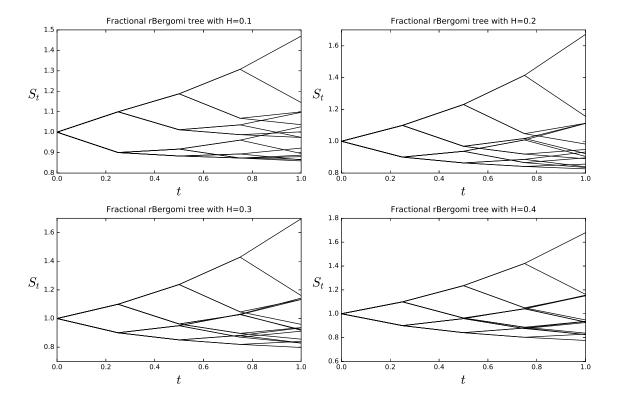


FIGURE 7. rough Bergomi trees for different values of H,  $(\nu, \rho, \xi_0) = (1, 1, 0.04)$  with 5 time steps.

with 
$$\psi(t) := \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(u) - f(0)}{(t - u)^{1 - \alpha}} du$$
. Since  $f \in \mathcal{C}^{\lambda}(\mathbb{I})$ , we obtain  $|\psi(t)| \le \frac{|f|_{\lambda}}{\Gamma(\alpha)} \int_0^t \frac{u^{\lambda}}{(t - u)^{1 - \alpha}} du$ , and hence  $|\psi(t)| \le \frac{\Gamma(2 + \lambda)|f|_{\lambda}}{(1 + \lambda)\Gamma(\alpha + \lambda + 1)} t^{\alpha + \lambda}$ ,

which proves the estimate for  $|\psi|$ . Next, we prove that  $\psi \in \mathcal{C}^{\lambda+\alpha}(\mathbb{I})$ . For this, introduce  $\phi(t) := f(t) - f(0)$  and consider  $t, t + h \in \mathbb{I}$  with h > 0,

$$\psi(t+h) - \psi(t) = \frac{1}{\Gamma(\alpha)} \left( \int_{-h}^{t} \frac{\phi(t-u)}{(u+h)^{1-\alpha}} du - \int_{0}^{t} \frac{\phi(t-u)}{u^{1-\alpha}} du \right)$$

$$= \frac{\phi(t)}{\Gamma(1+\alpha)} \left[ (t+h)^{\alpha} - t^{\alpha} \right] + \frac{1}{\Gamma(\alpha)} \left( \int_{-h}^{0} \frac{\phi(t-u) - \phi(t)}{(u+h)^{1-\alpha}} du \right)$$

$$+ \frac{1}{\Gamma(\alpha)} \left( \int_{0}^{t} \left[ (u+h)^{\alpha-1} - u^{\alpha-1} \right] \left[ \phi(t-u) - \phi(t) \right] du \right) =: J_{1} + J_{2} + J_{3}.$$

We first consider  $J_1$ . If h > t, then

$$|J_1| \le \frac{|f|_{\lambda}}{\Gamma(1+\alpha)} t^{\lambda} \left[ (t+h)^{\alpha} - t^{\alpha} \right] \le Ch^{\lambda+\alpha}.$$

On the other hand, when 0 < h < t, since  $(1+u)^{\alpha} - 1 \le \alpha u$  for u > 0, then

$$|J_1| \le \frac{|f|_{\lambda}}{\Gamma(1+\alpha)} t^{\lambda+\alpha} \left| \left(1 + \frac{h}{t}\right)^{\alpha} - 1 \right| \le Cht^{\lambda+\alpha-1} \le Ch^{\lambda+\alpha}.$$

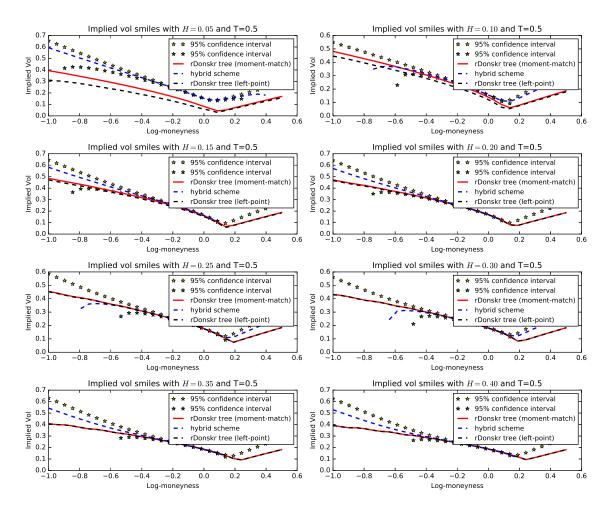


FIGURE 8. rough Bergomi trees for different values of H,  $(\nu, \rho, \xi_0) = (1, -1, 0.04)$  with 24 time steps.

For  $J_2$ , since  $f \in \mathcal{C}^{\lambda}(\mathbb{I})$ , we can write

$$|J_2| \le \frac{|f|_{\lambda}}{\Gamma(\alpha)} \int_{-h}^0 \frac{|u|^{\lambda}}{(u+h)^{1-\alpha}} \le Ch^{\lambda+\alpha}.$$

Finally,

$$|J_3| \le \frac{|f|_{\lambda}}{\Gamma(\alpha)} \int_0^t u^{\lambda} [u^{\alpha - 1} - (u + h)^{\alpha - 1}] du = \frac{|f|_{\lambda}}{\Gamma(\alpha)} h^{\lambda + \alpha} \int_0^{t/h} u^{\lambda} [u^{\alpha - 1} - (u + 1)^{\alpha - 1}] du.$$

Hence, if  $t \leq h$ , then  $|J_3| \leq Ch^{\lambda+\alpha}$ . Likewise, if t > h and  $\lambda + \alpha < 1$ , then  $|J_3| \leq Ch^{\lambda+\alpha}$  since

$$\left|u^{\alpha-1}-(u+1)^{\alpha-1}\right|=u^{\alpha-1}\left[1-\left(1+\frac{1}{u}\right)^{\alpha-1}\right]\leq Cu^{\alpha-2}.$$

Thus  $\psi$  satisfies the  $(\lambda + \alpha)$ -Hölder condition and belongs to  $\mathcal{C}^{\lambda+\alpha}(\mathbb{I})$ .

Corollary A.3. For any  $\lambda \in (0,1)$  and  $\alpha \in \mathfrak{R}^{\lambda}$ ,  $I^{\alpha}$  is a continuous operator from  $C^{\lambda}(\mathbb{I})$  to  $C^{\lambda+\alpha}(\mathbb{I})$ .

*Proof.* It is clear that  $I^{\alpha}$  is a linear operator. Using the estimate in Theorem A.2 we have

$$||I^{\alpha}f||_{\alpha+\lambda} \leq \frac{f(0)}{\Gamma(1+\alpha)}||(\cdot)^{\alpha}||_{\lambda+\alpha} + ||\psi||_{\lambda+\alpha} \leq C_1||f||_{\lambda}||(\cdot)^{\alpha}||_{\lambda+\alpha} + C_2||f||_{\lambda}||(\cdot)^{\alpha+\lambda}||_{\lambda+\alpha} \leq C||f||_{\lambda},$$

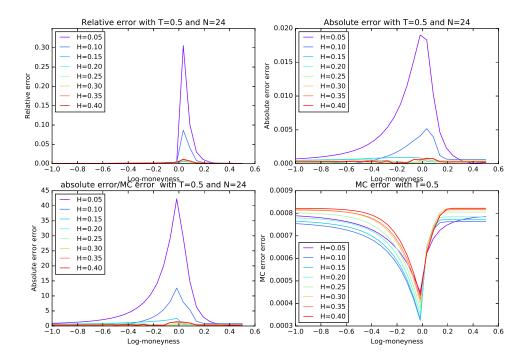


FIGURE 9. Error analysis for the rDonsker moment-match tree for different values of H,  $(\nu, \rho, \xi_0) = (1, -1, 0.04)$  with 24 time steps.

since  $|f|_{\lambda} \leq ||f||_{\lambda}$ ,  $f(0) \leq ||f||_{\lambda}$ . Therefore  $I^{\alpha}$  is also bounded and hence continuous.

**Theorem A.4.** For any  $\lambda \in (0,1)$  and  $\alpha \in \mathfrak{R}^{\lambda}$ , let  $f \in \mathcal{C}^{\lambda}(\mathbb{I})$ . Then  $I^{\alpha}f$  exists,  $I^{-\alpha}I^{\alpha}f = f$  and, for all  $t \in \mathbb{I}$ ,

$$(I^{\alpha}f)(t) = -\frac{\alpha}{\Gamma(1+\alpha)} \int_0^t (t-u)^{\alpha-1} [f(t) - f(u)] du.$$

*Proof.* For  $f \in \mathcal{C}^{\lambda}(\mathbb{I})$ , define, for any  $\varepsilon > 0$  and  $t \in \mathbb{I}$ ,

$$(I_{\varepsilon}^{1+\alpha}f)(t) := \frac{1}{\Gamma(\alpha+1)} \int_{0}^{t-\varepsilon} (t-u)^{\alpha} f(u) du,$$

and note that  $I_0^{1+\alpha} = I^{1+\alpha}$ . Then, we have

$$\Gamma(1+\alpha)\left(\frac{\mathrm{d}}{\mathrm{d}t}I_{\varepsilon}^{1+\alpha}f\right)(t) = \varepsilon^{\alpha}f(t-\varepsilon) + \alpha \int_{0}^{t-\varepsilon} (t-u)^{\alpha-1}f(u)\mathrm{d}u$$

$$= -\alpha \int_{0}^{t-\varepsilon} (t-u)^{\alpha-1}(f(t)-f(u))\mathrm{d}u - \varepsilon^{\alpha}(f(t)-f(t-\varepsilon)).$$
(A.2)

where Hölder continuity implies that  $f(t) - f(u) \leq C(t - u)^{\lambda}$ , so that the integral in (A.2) is well defined. Then, as  $\varepsilon$  tends to zero, the right-hand side of (A.2) tends uniformly to

$$\psi(t) = -\alpha \int_0^t (t - u)^{\alpha - 1} (f(t) - f(u)) du.$$

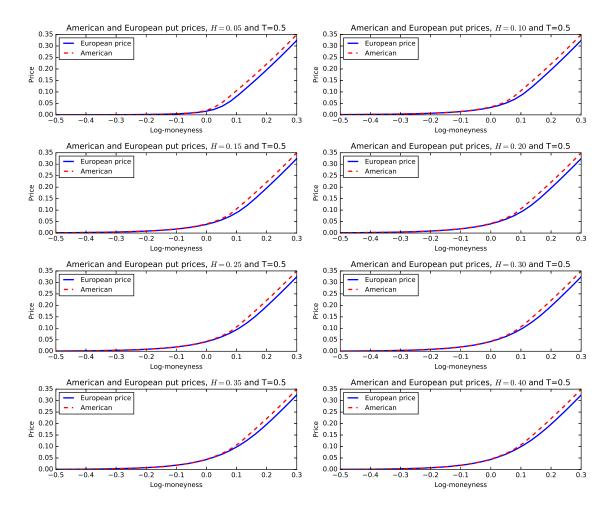


FIGURE 10. American and European Put prices in the rough Bergomi model for different values of H and  $(\nu, \rho, \xi_0) = (1, -1, 0.04)$  with 26 time steps.

Now, for  $t \in \mathbb{I}$ ,

$$(I^{1+\alpha}f)(t) - (I^{1+\alpha}f)(0) = \lim_{\varepsilon \downarrow 0} \left\{ (I_{\varepsilon}^{1+\alpha}f)(t) - (I_{\varepsilon}^{1+\alpha}f)(0) \right\} = \lim_{\varepsilon \downarrow 0} \int_{0}^{t} \left( \frac{\mathrm{d}}{\mathrm{d}u} I_{\varepsilon}^{1+\alpha} f \right) (u) \mathrm{d}u$$
$$= \int_{0}^{t} \lim_{\varepsilon \downarrow 0} \left( \frac{\mathrm{d}}{\mathrm{d}u} I_{\varepsilon}^{1+\alpha} f \right) (u) \mathrm{d}u = \frac{1}{\Gamma(1+\alpha)} \int_{0}^{t} \psi(u) \mathrm{d}u,$$

where the exchange of limit and integral holds since the convergence is uniform and the interval compact. Therefore,  $\Gamma(\alpha+1)(I^{1+\alpha}f)$  is the integral of  $\psi$  and, by the Fundamental Theorem of Calculus,

$$\psi(t) = \Gamma(\alpha + 1) \left(\frac{\mathrm{d}}{\mathrm{d}t} I^{1+\alpha} f\right)(t) = \Gamma(\alpha + 1) (I^{\alpha} f)(t).$$

Therefore it exists and, similarly to Theorem A.2,  $\psi \in \mathcal{C}^{\lambda+\alpha}(\mathbb{I})$ . Finally, since, for  $0 < \beta < 1$ , the equality

$$(I^{\beta}I^{1-\beta}f)(t) = (I^{1-\beta}I^{\beta}f)(t) = (I^{1}f)(t) = \int_{0}^{t} f(u)du$$

holds for all  $t \in \mathbb{I}$ , we conclude that

$$\left(\Gamma(1+\alpha)I^{1+\alpha}f - I^{-\alpha}\psi\right)(t) = \int_0^t \Gamma(1+\alpha)f(u) - (I^{-\alpha}\psi)(u)(t-u)^\alpha du = 0,$$

and hence, by continuity of both f and  $I^{-\alpha}\psi$ ,  $f = I^{-\alpha}I^{\alpha}f$ .

# APPENDIX B. DISCRETE CONVOLUTION

**Definition B.1.** For a, b  $\in \mathbb{R}^n$ , the discrete convolution operator  $*: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$  is defined as

$$(a * b)_i := \sum_{m=0}^{i} a_m b_{i-m}, \quad i = 0, \dots, n-1.$$

When simulating  $\mathcal{G}^{\alpha}W$  on the uniform partition  $\mathcal{T}$ , the scheme reads

$$(\mathcal{G}^{\alpha}W)^{j}(t_{i}) = \sum_{k=1}^{i} g(t_{i} - t_{k-1})\xi_{k} = \sum_{k=1}^{i} g(t_{k})\zeta_{j,k-i+1}, \quad \text{for } i = 1, \dots, n,$$

which has the form of the discrete convolution in Definition B.1. Rewritten in matrix form

$$\begin{pmatrix} g(t_1) & 0 & \cdots & 0 \\ g(t_2) & g(t_1) & \cdots & 0 \\ \vdots & \ddots & \ddots & 0 \\ g(t_n) & g(t_{n-1}) & \cdots & g(t_1) \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_n \end{pmatrix},$$

it is clear that this operator yields a complexity of order  $\mathcal{O}(n^2)$ , which can be improved drastically.

**Definition B.2.** The Discrete Fourier Transform (DFT) of a sequence  $c := (c_0, c_1, ..., c_{n-1}) \in \mathbb{C}^n$  is given by

$$\widehat{f}(\mathbf{c})[j] := \sum_{k=0}^{n-1} c_k \exp\left(-\frac{2i\pi jk}{n}\right), \quad \text{for } j = 0, \dots, n-1,$$

and the Inverse DFT of c is given by

$$f(c)[k] := \frac{1}{n} \sum_{j=0}^{n-1} c_j \exp\left(\frac{2i\pi jk}{n}\right), \quad \text{for } k = 0, \dots, n-1.$$

In general, both transforms require a computational effort of order  $\mathcal{O}(n^2)$ , but the Fast Fourier Transform (FFT) algorithm by Cooley and Tukey [18] exploit the symmetry and periodicity of complex exponentials of the DFT and reduces the complexity of both transforms to  $\mathcal{O}(n \log n)$ .

**Theorem B.3.** For  $a, b \in \mathbb{R}^n$ , the identity  $(a * b) = f(\widehat{f}(a) \bullet \widehat{f}(b))$  holds, with  $\bullet$  the pointwise multiplication.

This in particular implies that the complexity of the discrete convolution is reduced to  $\mathcal{O}(n \log n)$  by FFT.

**Algorithm B.4** (FFT Discrete convolution for  $\mathcal{B}$ ). On the equidistant grid  $\mathcal{T}$ ,

- (1) draw a random matrix  $\{\zeta_{j,i}\}_{\substack{j=1,\dots,M\\i=1,\dots,n}}$  such that  $\mathbb{V}(\zeta_{j,i})=1;$ (2) define the vectors  $\mathfrak{g}:=(g(t_i))_{i=1,\dots,n}$  and  $\zeta_j:=(\zeta_{j,i})_{i=1,\dots,n},$  for  $j=1,\dots,M;$
- (3) using FFT, compute  $\varphi_j := \widehat{f}(\mathfrak{g}) \cdot \widehat{f}(\zeta_j)$ , for  $j = 1, \dots, M$ ;
- (4) simulate M paths of  $(\mathcal{G}^{\alpha}W)$  using FFT, as  $(\mathcal{G}^{\alpha}W)^{j}(\mathcal{T}) = \sqrt{\frac{T}{n}}f(\varphi_{j})$  for  $j = 1, \dots, M$ .

In Step 2 we may replace the evaluation points  $\mathfrak g$  by any optimal evaluation point  $\{g(t_i^*)\}_{i=1}^n$  as in (3.3). Many numerical packages offer a direct implementation of the discrete convolution such as the numpy.convolve function in the NumPy library of Python. The user then only needs to pass the arguments  $\mathfrak g$  and  $\xi_j$  to this function and Steps 3 and 4 are computed automatically (using efficient FFT techniques) by the package. Although the FFT step is the heaviest computation on the simulation of rough volatility models, the actual time grid  $\mathcal T$  is not specially large, i.e. n << 1000. Hence, it is not important to have the fastest possible FFT for very large n, it is much more important for the implementation to be fast on small time grids. In this aspect we find that numpy.convolve is a very competitive implementation.

### APPENDIX C. ADDITIONAL PROOFS

C.1. Proof of Proposition 1.2. Since  $g \in \mathcal{L}^{\alpha}$ , there exists C > 0 such that  $|g(u)| \leq Cu^{\alpha}$ ; hence, for  $t \in \mathbb{I}$ ,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^t |f(s)g(t-s)| \mathrm{d}s \le C \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t |f(s)(t-s)^{\alpha}| \mathrm{d}s.$$

Therefore, for  $f \in \mathcal{C}^{\lambda}(\mathbb{I})$ , the inequalities involving the Riemann-Liouville fractional operator (Appendix A)

(C.1) 
$$(\mathcal{G}^{\alpha}f)(t) \le C(I^{\alpha}f)(t) \le C||f||_{\lambda}$$

hold for  $\alpha \leq 0$  and all  $t \in \mathbb{I}$ . Since Riemann-Liouville operators are continuous (Appendix A), continuity of the GFO follows directly from (C.1) along with linearity. To prove that  $\mathcal{G}^{\alpha}$  belongs to  $\mathcal{C}^{\lambda+\alpha}(\mathbb{I})$ , we may invoke (C.1) and easily adapt Theorem A.4. Similarly, when  $\alpha \geq 0$ , for any  $u \in \mathbb{I}$ ,  $g'(u) = u^{\alpha}L'(u) + u^{\alpha-1}L(u) \leq C_1 + C_2u^{\alpha-1}$ , and the  $\lambda$ -Hölder continuity of f yields, for any  $t \in \mathbb{I}$ ,

$$\int_0^t \frac{\mathrm{d}}{\mathrm{d}t} g(t-s) f(s) \mathrm{d}s \le \frac{C_1}{\lambda+1} t^{\lambda+1} + C_2 \int_0^t (t-s)^{\alpha-1} f(s) \mathrm{d}s \le C_1 + C_2 \int_0^t (t-s)^{\alpha-1} f(s) \mathrm{d}s.$$

Since the time horizon I is compact, the first constant does not affect continuity or mapping properties of the GFO. The second term is bounded by the Riemann-Liouville integral operator (Appendix A), hence continuity and mapping properties follow as before by straightforward modification of Theorem A.2.

C.2. **Proof of Proposition 1.4.** Since the paths of Brownian motion are 1/2-Hölder continuous, existence (and continuity) of  $\mathcal{G}^{\alpha}W$  is guaranteed for all  $\alpha \in \mathfrak{R}^{1/2}$ . When  $\alpha \in \mathfrak{R}^{1/2}_+$ , the kernel is smooth and square integrable, so that Itô's product rule yields (since g(0) = 0)

$$(\mathcal{G}^{\alpha}W)(t) = \int_0^t \frac{\mathrm{d}}{\mathrm{d}t} g(t-s)W(s)\mathrm{d}s = \int_0^t g(t-s)\mathrm{d}W_s,$$

and the claim holds. For  $\alpha \in \mathfrak{R}^{1/2}_{-}$ , and any  $\varepsilon > 0$ , introduce the operator

$$\left(\mathcal{G}_{\varepsilon}^{1+\alpha}f\right)(t):=\int_{0}^{t-\varepsilon}g(t-s)f(s)\mathrm{d}s, \quad \text{for all } t\in\mathbb{I},$$

which satisfies  $\frac{\mathrm{d}}{\mathrm{d}t}\lim_{\varepsilon\downarrow0}\left(\mathcal{G}_{\varepsilon}^{1+\alpha}f\right)(t)=\left(\mathcal{G}^{1+\alpha}f\right)(t)$  pointwise. Now, for any  $t\in\mathbb{I}$ , almost surely,

(C.2) 
$$\left( \mathcal{G}_{\varepsilon}^{1+\alpha} W \right)(t) = g(\varepsilon)W(t-\varepsilon) + \int_{0}^{t-\varepsilon} \frac{\mathrm{d}}{\mathrm{d}t} g(t-s)W(s) \mathrm{d}s = \int_{0}^{t-\varepsilon} g(t-s) \mathrm{d}W_{s}.$$

Then, as  $\varepsilon$  tends to zero, the right-hand side of (C.2) tends to  $\int_0^t g(t-s) dW_s$ , and furthermore, the convergence is uniform. On the other hand, the equalities

$$(\mathcal{G}_0^{1+\alpha}W)(t) - (\mathcal{G}_0^{1+\alpha}W)(0) = \lim_{\varepsilon \downarrow 0} \left[ (\mathcal{G}_\varepsilon^{1+\alpha}W)(t) - (\mathcal{G}_\varepsilon^{1+\alpha}W)(0) \right] = \lim_{\varepsilon \downarrow 0} \int_0^t \left( \frac{\mathrm{d}}{\mathrm{d}s} \mathcal{G}_\varepsilon^{1+\alpha}W \right)(s) \mathrm{d}s$$
$$= \int_0^t \lim_{\varepsilon \downarrow 0} \left( \frac{\mathrm{d}}{\mathrm{d}s} \mathcal{G}_\varepsilon^{1+\alpha}W \right)(s) \mathrm{d}s = \int_0^t \left( \int_0^s g(s-u) \mathrm{d}W_u \right) \mathrm{d}s,$$

hold since convergence is uniform on compacts, and the fundamental theorem of calculus concludes the proof.

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