A Closed-Form GARCH Option Pricing Model

Steven L. Heston and Saikat Nandi

Federal Reserve Bank of Atlanta
Working Paper 97-9
November 1997

Abstract: This paper develops a closed-form option pricing formula for a spot asset whose variance follows a GARCH process. The model allows for correlation between returns of the spot asset and variance and also admits multiple lags in the dynamics of the GARCH process. The single-factor (one-lag) version of this model contains Heston's (1993) stochastic volatility model as a diffusion limit and therefore unifies the discrete-time GARCH and continuous-time stochastic volatility literature of option pricing. The new model provides the first readily computed option formula for a random volatility model in which current volatility is easily estimated from historical asset prices observed at discrete intervals. Empirical analysis on S&P 500 index options shows the single-factor version of the GARCH model to be a substantial improvement over the Black-Scholes (1973) model. The GARCH model continues to substantially outperform the Black-Scholes model even when the Black-Scholes model is updated every period and uses implied volatilities from option prices, while the parameters of the GARCH model are held constant and volatility is filtered from the history of asset prices. The improvement is due largely to the ability of the GARCH model to describe the correlation of volatility with spot returns. This allows the GARCH model to capture strike-price biases in the Black-Scholes model that give rise to the skew in implied volatilities in the index options market.

JEL classification: G13

Key words: GARCH, options, closed-form

The authors gratefully acknowledge the participants of the workshop at the Atlanta Fed for helpful comments. The views expressed here are those of the authors and not necessarily those of the Federal Reserve Bank of Atlanta or the Federal Reserve System. Any remaining errors are the authors' responsibility.

Please address questions regarding content to Steven L. Heston, John M. Olin School of Business, Washington University in St. Louis, Campus Box 1133, One Brookings Drive, St. Louis, Missouri 63130-4899, 314/935-6329, 314/935-6359 (fax), heston@wuolin.wustl.edu or Saikat Nandi, Federal Reserve Bank of Atlanta, 104 Marietta Street, N.W., Atlanta, Georgia 30303-2713, 404/614-7094, 404/521-8810 (fax), saikat.u.nandi@atl.frb.org.

Questions regarding subscriptions to the Federal Reserve Bank of Atlanta working paper series should be addressed to the Public Affairs Department, Federal Reserve Bank of Atlanta, 104 Marietta Street, N.W., Atlanta, Georgia 30303-2713, 404/521-8020. The full text of this paper may be downloaded (in PDF format) from the Atlanta Fed's World-Wide Web site at http://www.frbatlanta.org/publica/work_papers/.

A Closed-Form GARCH Option Pricing Model

Introduction

Since Black and Scholes (1973) originally developed their option pricing formula, researchers have developed models that incorporate stochastic volatility (see Heston (1993) and references therein). These models have been successful in market option prices as in Melino and Turnbull (1990), Knoch (1992), Nandi (1996), Bakshi, Cao, and Chen (1997), Bates (1996 a,b). The two types of volatility models have been continuous-time stochastic volatility models and discrete-time GARCH models.

Continuous-time stochastic volatility models are effective for option pricing, but can be difficult to implement. Although these models assume that volatility is observable, it is very difficult to filter a continuous volatility variable from discrete observations. One alternative is to use implied volatilities computed from option prices. But empirically, this approach requires estimating many volatilities, one for every date and is computationally burdensome in a long time series of options records. In any case, continuous-time models must be augmented with non-trivial volatility estimation techniques.

GARCH models have been very popular and effective for modeling the volatility dynamics in many asset markets. GARCH option pricing models have the inherent advantage that volatility is observable from discrete asset price data and only a few parameters need to be estimated even in a long time series of options records. Unfortunately, existing GARCH models do not have closed-form solutions for option prices. These models are typically solved by simulation (Engle and Mustafa (1992), Amin and Ng (1993), Duan (1995)) that can be slow and computationally intensive for empirical work.[1]

This paper develops a closed-form solution for option prices for a GARCH model that allows for correlation between returns of the spot asset and variance as well as multiple lags in

[1] More recently, Ritchken and Trevor (1997) provide a lattice approximation and Duan, Gauthier and Simonato (1997) provide a series approximation (based on an Edgeworth expansion) to a GARCH process with single lags in the variance dynamics.

the time series dynamics of the variance process. The single factor (one lag) version of the model reconciles the discrete approach with the continuous-time approach to option pricing by including Heston's (1993) model as a diffusion limit. In the Black-Scholes model option prices are functions of the current spot asset price, while in the GARCH model option prices are functions of current and lagged spot prices. Except for this difference the models are operationally similar. Empirical tests of the single factor version of the model in the S&P 500 index options market show that the GARCH model has substantially smaller pricing error than the Black-Scholes model. This is true even though the Black-Scholes model has the advantage of using implied volatilities from option prices and is updated every period, in contrast to the GARCH model whose time invariant parameters are held constant and the variance is filtered from the history of asset prices, instead of being implied from option prices.

Section 1 describes the GARCH process and presents the option formula. Section 2 applies it to the S&P500 index option data and Section 3 concludes. The appendix contains detailed calculations and derivations of the option formula.

1. The Model

The model has two basic assumptions. The first assumption is that the log-spot price follows a particular GARCH process.

Assumption 1: The spot asset price, $S(t)$ (including accumulated interest or dividends) follows the following process over time steps of length Δ:

$$\log(S(t)) = \log(S(t-\Delta)) + r + \lambda h(t) + \sqrt{h(t)}\,z(t), \qquad (1)$$

$$h(t) = \omega + \sum_{i=1}^{p} \beta_i h(t-i\Delta) + \sum_{i=1}^{q} \alpha_i \left(z(t-i\Delta) - \gamma_i \sqrt{h(t-i\Delta)}\right)^2,$$

where r is the continuously compounded interest rate for the time interval Δ and $z(t)$ is a standard normal disturbance. h(t) is the conditional variance of the log return between t - Δ and t and is known from the information set at time t - Δ. Equation (1) can also be thought of as a juxtaposition of the NGARCH and VGARCH models of Engle and Ng (1993). The conditional variance h(t) appears in the mean as a return premium. This allows the average spot return to depend on the level of risk.[2] In particular limiting cases the variance becomes constant. As the α_i parameters approach zero, this process approaches an autoregressive moving average process for the logarithm of the spot price. As the α_i and β_i parameters approach zero, it is equivalent to the Black-Scholes model observed at discrete intervals.

This paper will focus on the first-order process (p=q=1). The first-order process is stationary with finite mean and variance if $\beta+\alpha\gamma^2 < 1$.[3] In this model one can directly observe h(t+Δ) as a function of the spot price

$$h(t+\Delta) \;=\; \omega + \beta_1 h(t) + \alpha_1 \frac{(\log(S(t))-\log(S(t-\Delta))-r-\lambda h(t)-\gamma_1 h(t))^2}{h(t)}. \tag{2}$$

The γ_1 parameter results in asymmetric influence of shocks; a large positive shock, z(t), affects the variance differently than a large negative z(t). In general the variance process h(t) and the spot return are correlated

$$\mathrm{Cov}_{t-\Delta}[h(t+\Delta),\log(S(t))] \;=\; -2\,\alpha_1\gamma_1 h(t). \tag{3}$$

A positive value for γ_1 results in negative correlation between spot returns and variance. This is consistent with the leverage effect documented by Christie (1982) and others. γ_1 controls the skewness or the asymmetry of the distribution of the log returns and the distribution is symmetric

---

[2] The functional form of this return premium $\lambda h(t)$ prevents arbitrage by ensuring that the spot asset earns the riskless interest rate when the variance equals zero.

[3] In the multiple factor case one must add the additional condition that the polynomial roots of $x^p - \sum_{i=1}^{p}(\beta_i+\alpha_i\gamma_i^2)x^{p-i}$ lie inside the unit circle.

if γ_1 is zero. α_1 determines the kurtosis of the distribution and a zero value implies a deterministic (globally) time varying variance.

Although equation (1) refers to a stochastic process observed at a prescribed time interval Δ, it has an interesting continuous-time limit. Let $\omega(\Delta) = (\kappa\theta\text{-}½\sigma)\Delta$, $\beta_1(\Delta) = 1\text{-}(\kappa\text{+}½\sigma)\Delta$, $\alpha_1(\Delta) = ½\sigma\Delta$, and $\gamma_1(\Delta) = \Delta^{-½}$. Straightforward calculations show the mean and variance of the process are

$$E_{t-\Delta}[h(t+\Delta)-h(t)] = \kappa(\theta-h(t))\Delta, \tag{4}$$
$$Var_{t-\Delta}[h(t+\Delta)] = \sigma^2 h(t)\Delta+½\sigma\Delta^2.$$

This process has a continuous-time diffusion limit following Foster and Nelson (1994). As the observation interval Δ shrinks the variance process h(t) converges weakly to the following square-root process of Feller (1951), Cox, Ingersoll and Ross (1985), and Heston (1993)

$$dh = \kappa(\theta-h)dt + \sigma\sqrt{h}dz, \tag{5}$$

where z(t) is a Wiener process. Consequently the option pricing model (1) contains Heston’s (1993) continuous time stochastic volatility model as a special case.

The second assumption of the model concerns the pricing of options and other derivative securities. The spot price has a conditionally lognormal distribution over a single period. Since variance is not stochastic over this interval we assume that the Black-Scholes-Rubinstein formula holds.

<u>Assumption 2</u>: The value of a call option one period prior to expiration obeys the Black-Scholes-Rubinstein formula.

Black-Scholes prices do not follow from absence of arbitrage in a discrete model. Instead one must justify the assumption with other arguments such as those of Rubinstein (1976), Brennan (1979), and Duan (1995). Since all these models agree on the price of one period options, Assumption 2 is the most benign pricing assumption possible within our framework.

Assumptions 1 and 2 allow us to derive the prices of all contingent claims that can be written as functions of the spot asset price. Since long-term options are functions of S(t) and $h(t+\Delta)$, and $h(t+\Delta)$ can be written as a function of S(t) in equation (2), this includes options of all maturities.[4] Equation (1) is algebraically equivalent to

$$\log(S(t)) = \log(S(t-\Delta)) + r - ½h(t) + \sqrt{h(t)}z^*(t), \qquad (6)$$

$$h(t) = \omega + \sum_{i=1}^{p}\beta_i h(t-i\Delta) + \sum_{i=2}^{q}\alpha_i\Big(\frac{\varepsilon(t-i\Delta)}{\sqrt{h(t-i\Delta)}}-\gamma_i\sqrt{h(t-i\Delta)}\Big)^2 + \alpha_1\big(z^*(t-\Delta)-\gamma_1^*\sqrt{h(t-\Delta)}\big)^2,$$

where

$$z^*(t) = z(t)+(\lambda+½)\sqrt{h(t)},$$

$$\gamma_1^* = \gamma_1+\lambda+½.$$

Cox and Ross (1976) showed if the Black-Scholes formula holds then the risk-neutral distribution of the asset price is lognormal with mean $S(t-\Delta)e^r$. In other words, $z^*(t)$ has a standard normal distribution under the risk-neutral probabilities. We formalize this property as the following proposition.

Proposition 1: The risk-neutral process takes the same GARCH form as equation (1) with λ replaced by -½ and γ_1 replaced by $\gamma_1^* = \gamma_1+\lambda+½$.

The proof of this proposition is trivial by noting that $z^*(t)$, γ_1^* and λ as defined above make the one period return from investing in the spot asset equal to the risk free rate. We proceed

[4] Breeden and Litzenberger (1979) showed call options span the space of contingent claims.

to solve for the generating function of the GARCH process (1) and use it to produce option prices. Let $f(\phi)$ denote the conditional generating function of the asset price

$$f(\phi) = E_t[S(T)^{\phi}]. \tag{7}$$

This is also the moment generating function of the logarithm of S(T). The function $f(\phi)$ depends the parameters and state variables of the model, but these arguments are suppressed for notational convenience. We shall use the notation $f^*(\phi)$ to denote the generating function for the risk-neutral process (6).

Proposition 2: The generating function takes the log-linear form

$$f(\phi) = S(t)^{\phi} \times \tag{8}$$

$$\exp\Big(A(t;T,\phi)+\sum_{i=1}^{p} B_i(t;T,\phi)h(t+2\Delta-i\Delta)+\sum_{i=1}^{q-1} C_i(t;T,\phi)(z(t+\Delta-i\Delta)-\gamma_i\sqrt{h(t+\Delta-i\Delta)})^2\Big)$$

where,

$$A(t;T,\phi) = A(t+\Delta;T,\phi)+\phi r+B_1(t+\Delta;T,\phi)\omega-\tfrac{1}{2}\ln(1-2\alpha_1 B_1(t+\Delta;T,\phi)),$$

$$B_1(t;T,\phi) = \phi(\lambda+\gamma_1)-\tfrac{1}{2}\gamma_1{}^2+\beta_1 B_1(t+\Delta;T,\phi)+\frac{\tfrac{1}{2}(\phi-\gamma_1)^2}{1-2\alpha_1 B_1(t+\Delta;T,\phi)}$$

in the one factor case (p=q=1) and these coefficients can be computed recursively from the boundary conditions:

$$A(T-\Delta;T,\phi) = \phi r,$$

$$B_1(T-\Delta;T,\phi) = \lambda\phi+\tfrac{1}{2}\phi^2$$

The appendix derives the recursion formulas for the coefficients $A(t;T,\phi)$, $B_i(t;T,\phi)$, and $C_i(t;T,\phi)$

in the general case i.e., for any p and q.

Since the generating function of the spot price is the moment generating function of the logarithm of the spot price, $f(i\phi)$ is the characteristic function of the logarithm of the spot price. One can calculate probabilities and risk-neutral probabilities following Feller (1971) or Kendall and Stuart (1977).

Proposition 3: If the characteristic function of the log spot price is $f(i\phi)$ then

$$E[Max(S(T)-K,0)] = \tag{9}$$

$$f(1)\left(\tfrac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{K^{-i\phi}f(i\phi+1)}{i\phi f(1)}\right]d\phi\right) - K\left(\tfrac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{K^{-i\phi}f(i\phi)}{i\phi}\right]d\phi\right),$$

where Re[] denotes the real part of a complex number. An option price is simply the discounted expected value of the payoff, Max(S(T)-K,0) calculated using the risk-neutral probabilities.

Corollary: At time t, a European call option with strike price K that expires at time T is worth

$$C = e^{-r(T-t)}E_t^*[Max(S(T)-K,0)] = \tag{10}$$

$$\tfrac{1}{2}S(t) + \frac{e^{-r(T-t)}}{\pi}\int_0^\infty Re\left[\frac{K^{-i\phi}f^*(i\phi+1)}{i\phi}\right]d\phi - Ke^{-r(T-t)}\left(\tfrac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{K^{-i\phi}f^*(i\phi)}{i\phi}\right]d\phi\right).$$

This completes the option pricing formula.[5] In contrast to the Black-Scholes formula, this formula is a function of the current asset price, S(t), and the conditional variance, $h(t+\Delta)$. Since $h(t+\Delta)$ is a function of the observed path of the asset price, the option formula is effectively a function of current and lagged asset prices. In contrast to continuous-time models, volatility is a

[5] The integrands converge very rapidly and the integration can be very efficiently handled using any integration routine, including quadrature methods. We used Romberg's method in Press et al. (1992).

readily observable function of historical asset prices and need not be estimated with other procedures.

The next section describes the empirical performance of the one factor (p=q=1) version of the model on the S&P 500 index options market.

2. Empirical Analysis

The empirical analysis starts with a descriptions of the options data. It proceeds to estimate the GARCH model with time series data on index returns and with options data. It then presents in-sample and out-of-sample comparisons of the GARCH model with the Black-Scholes (henceforth BS) model.

Description of Data

Intraday data on S&P 500 index options traded on the Chicago Board Options Exchange (CBOE) are used to test the model. The intraday data set is sampled every Wednesday for the years 1992, 1993 and 1994 to create the data set that we work on. The market for S&P 500 index options is the second most active index options market in the United States and, in terms of open interest in options, it is the largest. Unlike options on the S&P 100 index, there are no wild card features that can complicate valuation. Also there is a very active market for the S&P 500 futures, making the replication of the index much easier. Thus, according to Rubinstein (1994) it is one of the best markets for testing a European option pricing model.

The minimum tick for series that trades below $3 is 1/16 and for all other series the tick is 1/8. Strike prices are spaced 5 points apart for near months and 25 points for far away months. The options expire in the three near-term months in addition to the months from the quarterly cycle of March, June, September, December.

The following rules are applied to filter data needed for the empirical test.

1) An option of a particular moneyness and maturity is represented only once in the sample. In other words, although the same option may be quoted again (with same or different index levels) in our time window, only the first record of that option is included in our sample.

2) A transaction has to satisfy the no-arbitrage relationships as outlined in Propositions 1 and 2 of Cox and Rubinstein (1985, p. 129-133).

3) Only option records in which the ratio of index to strike lies between 0.8 and 1.2 are included in the sample. This excludes some very deep out-of-the-money and deep in-the-money options that are either infrequently traded and/or have low enough prices as for the bid-ask spread to constitute a major portion of the price.
4) Similarly, only options with number of days to expiration between 10 and 150 included. Very short term options have substantial time decay that could interfere with one's being able to isolate the volatility parameters. Very long term options are not included because they are not actively traded.

One must note that the S&P 500 is a value weighted index and the bigger stocks that trade more frequently constitute the bulk of the index level. Since intraday data and not the end-of-the-day option prices is used, the problem with the index level being somewhat stale may not be severe enough to undermine an estimation procedure. In theory, one could possibly overcome this problem by using implied index levels from the put-call parity equation. However, this is conditional on put call parity holding as an equality and in the presence of transactions costs (bid-ask spreads that are non-negligible), the equality becomes an inequality. Thus the implied index levels from the put-call parity equation may not equal the true index level. Also, even if one assumes away transactions costs, it is very difficult to create a sample of sufficient size by creating matched pairs of puts and calls because the level of the S&P 500 index changes quite frequently through the day. As a result, the index levels as reported in the data set are used in parameter estimation. The daily dividend yield on the S&P 500 index are taken from Data Stream. For the risk free rate, the bond equivalent yields of the closest (to the maturity of the option) to mature T bill are used from the Federal Reserve Bank of New York.

Estimation

The empirical analysis focuses mainly on the single factor (one lag) version of the GARCH model. Also we set $\Delta = 1$, as daily index returns are used to model the evolution of volatility. Unlike continuous time stochastic volatility models wherein the volatility process is unobservable, all the parameters that enter our pricing formula can be easily estimated directly from the history of asset prices asset through a maximum likelihood estimation (MLE) as has been done in Bollerslev (1986) and many others.[6] Due to the importance of the skewness parameter, $\gamma_{1,}$ we performed estimation with a unrestricted model and with a symmetric GARCH model in which γ_1 was constrained to equal zero. Table 1(a) shows the maximum likelihood estimates of the GARCH model, both when γ_1 is non-zero and when it is restricted to zero, on the daily S&P 500 levels closest to 2:30 P.M. from 01/08/92 to 12/30/94. The skewness parameter γ_1 is substantially positive indicating that shocks to returns and volatility are negatively correlated. Using a likelihood ratio test, the symmetric version is easily rejected implying that the negative correlation between returns and volatility is a feature of the S&P 500 time series. The daily annualized volatility series are shown in Figure 1 and 2 for the unrestricted and restricted/symmetric versions of the model.[7]

While, one could use the parameter estimates from the above MLE (using historical asset prices) in the option pricing formula, the information that one would be using is historical and

[6] The procedure sets h(0) equal to the sample variance of the changes in the logarithm of S(t). Due to the strong mean reversion of volatility, all results were insensitive to the starting value of h(0).

[7] Unreported results show that various symmetric and asymmetric GARCH specifications of Engle and Ng (1993) produce similar results to our symmetric and asymmetric models, respectively.

could be very different from the expectations embedded in option prices about the future evolution of the asset price. In contrast, the information in option prices is forward looking. Since our model has closed-form solutions for option prices, a natural candidate for parameter estimation that uses the information in option prices is the minimization of sum of squared errors via a non-linear least squares procedure. This is the approach we take in this paper and has been used by other authors including Bates (1996 (a), (b)), Bakshi, Cao and Chen (1997), Nandi (1996) and others in the context of continuous time stochastic volatility option pricing models.

The option price at time t is not only a function of the parameters that enter the Black-Scholes formula, but also of the parameters that drive the time varying variance process, namely α_1, β_1, γ_1^*, ω, and the level of the conditional variance, h(t+1). However, the fact that in a GARCH setting, h(t+1) need not be estimated as a parameter and instead is known from the history of asset prices considerably simplifies our estimation procedure as compared to the continuous time stochastic volatility models. This is because if the daily volatility is a parameter to be estimated (as has been the case in continuous time models) and the sample consists of a long time series of option prices, the number of parameters increases proportionately with the number of days in the sample. Also volatility is a random variable and the asymptotic properties of estimation of a random variable in a classical framework are not known. In contrast, in our GARCH setting, only a finite number of time invariant parameters need to be estimated irrespective of the sample size and asymptotic properties (in a least squares context) of these estimates are well known. The next paragraph details the procedure used to estimate the parameters (in-sample).

Let $e_{i,t}$ denote the model error in pricing option i at time t, i.e., $e_{i,t}$ is the difference between the model price of option and the market price of that option at time t.

Then our criterion function is

$$\sum_{t=1}^{T}\sum_{i=1}^{N_t} e(i,t)^2,$$

where T denotes the number of weeks (Wednesdays) in the sample and N_t is the number of options traded on the Wednesday of week t. The criterion function needs to be minimized over α_1, β_1, γ_1, ω and λ. Note that in order to minimize the above criterion function, we also need h(t+1) for each t. However, at each t, h(t+1) is a function of α_1, β_1, γ_1, ω, λ and the history of stock prices and is known at time t (given these time invariant parameters). To evaluate the criterion function, we first compute h(t+1) from its times series dynamics (conditional on the above parameters) and then compute option prices. For the BS model, a single implied volatility can be estimated by minimizing the above criterion fucntion.

As mentioned before, we are using intraday data sampled every Wednesday to test the model. The NLS procedure is carried out on the first six months of this weekly Wednesday data for the years 1992, 1993 and 1994. Although the options data are weekly, the conditional variance, h(t+1), which is relevant for option prices at time t is drawn from the daily evolution of index returns and not from the weekly evolution. Specifically we use the daily index levels closest to 2:30 P.M. The starting variance, h(0), is kept fixed at the in-sample estimate of the variance i.e., the variance for the first six months of 1992, 1993 and 1994 respectively. Out-of-sample option prices are computed in the second six months of each year using parameter estimates from the first six months. When computing out-of-sample option prices, the time series h(t) is generated from the in-sample parameter estimates and the index dynamics.

In-Sample Model Comparison

The parameter estimates for the three years, 1992, 1993, and 1994 and the average in-sample absolute pricing errors appear Table 1(b). The risk-neutral skewness parameter, γ_1^* as implied by the options data is always positive. This indicates that shocks to variance and asset returns are negatively correlated, imparting negative skewness in the risk neutral distribution of asset returns. The long run annual volatility (standard deviation, not variance) implied by the options data for the three years are 11.9%, 9.3% and 16.16% respectively.

The average in-sample absolute pricing errors for the three years, 92, 93 and 94 are 7.01%, 6.52% and 8.76% respectively if the in-sample estimation is carried out in the way described previously i.e., filtering each day's variance, h(t+1), from the time series of asset returns for the first six months of each year and holding constant the time invariant parameters over this period. One could further lower the pricing errors by computing an implied value of h(t+1) from option prices instead of filtering it from the time series of spot prices.[8] In contrast, the average in-sample pricing errors of the BS, in which an implied volatility is estimated from the first six months are 11.04%, 11.41 % and 16.48 % for 92, 93 and 94 respectively. Thus on an average, over the three years, the GARCH model reduces the percentage option pricing error by around 45 percent and a likelihhod ratio test easily rejects the BS model (a nested version of the GARCH) in favor of the GARCH for all three years. However, what is more interesting (and encouraging) is that the GARCH model improves over the popular, updated version of the BS model in which an implied volatility from option prices is re-estimated every Wednesday, instead of being held constant over the sample period. The average percentage pricing errors for this updated version of the BS model (for the first six months) are 9.03 %, 10.3% and 11.41% for 92, 93 and 94 respectively. In other words, the GARCH model fits option prices better as a function of the history of spot prices, even when the BS model is updated/re-calibrated to options data every day. However, the updated version of the BS is not a nested version of the GARCH and a

[8] We have informally done this type of estimation for a few days and indeed the pricing errors are lower than those reported here. This should not be surprising because one essentially has more parameters to fit the option prices.

doing a formal statistical test to compare the two is not possible. Instead, one has to look at the differences in option pricing errors to do any comparison.

The improvements of the GARCH model relative to both versions of the BS model could due to a term structure of volatility and/or due to moneyness effects that create a volatility "skew". In order to explore this issue, we re-estimated the GARCH model by restricting γ_1^* to zero so that we have conditional heteroskedasticity (as α_1 is non-zero) with no skewness in the risk neutral distribution (i.e., a symmetric GARCH model). As expected, the pricing errors increased considerably relative to the unrestricted GARCH model. The average percentage pricing errors are 10.92%, 11.12% and 14.74%, respectively, for the years 1992, 1993 and 1994. Comparing the symmetric version of the GARCH to the two versions of the BS, we find that the symmetric version of the GARCH model dominates the non updated version of the BS. This is expected because this version of the BS model is a restricted version of the symmetric GARCH model. However, the restricted/symmetric version of the GARCH is inferior to the updated version of the BS model. This tells us that skewness effects in the risk neutral distribution are very important in the S&P 500 index options market, a point also noted by Bates (1996 -b) in the context of continuous time stochastic volatility models.[9]

Out-of-Sample Model Comparison

Having estimated the time invariant parameters in-sample from the first six months of each year, we turn to out-of-sample pricing performance of the unrestricted GARCH model for the next six months of the three years under consideration. In computing out-of-sample option prices for the second half of a particular year, we keep the time invariant parameters fixed at their in-sample estimates for that year and obtain the conditional variance, h(t+1), from the dynamics of asset returns. We actually use h(t) instead of h(t+1) to compute out-of-sample prices on day t since h(t+1) is not known until the end of the day t. The average out-of-sample absolute pricing

[9] As mentioned earlier, the true time series skewness parameter may differ from risk-neutral skewness parameter that affects option prices. It is possible that the time series of spot returns has no skewness, while option prices reflect a nonzero skewness effect.

errors for the three years are 9.4%, 7.9% and 11.58% respectively. As expected, they are higher than their in-sample counterparts. In order to compute out-of-sample prices under the updated version of the BS model, we estimate an implied volatility on the previous trading day (prior to every Wednesday) by minimizing the sum of squared errors between model and market option prices and use the estimated implied volatility to price options every Wednesday. By doing this we allow the BS model to have extra flexibility, while the GARCH model is still subject to its time invariant parameters being held constant at the in-sample estimates. The results reported in Tables 2 (a,b,c) for the years 92,93 and 94 clearly show that GARCH maintains its pricing superiority over the BS model even in an out-of-sample context. The average percentage pricing errors under the updated BS are 13.42%, 16.66% and 16.07% and exceed the percentage pricing errors under the GARCH model, by anywhere between 27 percent and 52 percent. Therefore, it is not the case that the GARCH model is superior just because it has more parameters. Its superior performance stems from its ability to generate the appropriate skewness through the correlation between returns and volatility in the risk neutral distribution of the underlying asset's returns. The superior out-of-sample performance of the GARCH model is especially encouraging in the context of the results reported in Dumas, Fleming and Whaley (1996) who find that time varying volatility models based on the implied binomial trees of Derman and Kani (1994), Dupiere (1994), and Rubinstein (1994) under perform the BS model in out-of-sample tests.

Looking at the out-of-sample pricing errors (absolute) by moneyness and maturity as reported in Tables 2(a,b,c), we find that most of the improvements in the GARCH model vis-a-vis the BS model emanate from out-of-the-money options, especially out-of-the-money puts. For example, for the year 1993, the average percentage errors for deep out-of-the-money puts that have between 30 to 90 days to mature are 62.8% under the updated BS and 15.9% under GARCH. In terms of maturity only , we find that the absolute pricing errors under GARCH (as well as the BS) decrease with an increase in maturity. Short-term (less than 30 days to expire) out-of-the-money options are the most difficult to price under both GARCH and BS, although the amount of mispricing under GARCH is substantially lower. The reduction in pricing errors with a lengthening of the maturity is not unexpected as the distribution of the underlying asset approaches the gaussian distribution as maturity increases (by the Central Limit Theorem). Figures 3 (a), 3 (b) and 3 (c) show the graphs of the raw (not absolute) percentage pricing errors (by moneyness) for three different maturities. These figures clearly show that the GARCH model does underprice options with low strike prices, although the amount of underpricing is much lower than the BS model.

3. Conclusions

This paper presents a closed-form solution for option prices when the variance of the spot asset follows a GARCH process with arbitrary lags and is correlated with the returns of the spot asset. Empirical analysis of the one factor version of the GARCH model on the S&P 500 index options data shows substantial pricing improvements over the BS model even if the BS model is allowed to be updated every period and its volatility is implied from option prices, while the GARCH model is not updated and its volatility is filtered from the history of asset prices.

Although this paper has focused on the single factor (one lag) GARCH model, one could estimate the model with multiple lags. Since Assumption 2 states that the GARCH model is identical to the BS model for one period options, additional lags will probably not improve pricing of short term options. Instead one would perhaps need to incorporate a conditionally non-gaussian distribution of shocks to asset returns that may have the potential to address biases in short-term option prices.

One can also extend the model to bond options by assuming that (continuously compounded) interest rates follow an autoregressive moving-average process with the GARCH effects of equation (1). This leads to a family of log-linear bond models, whose continuous time limits nest the diffusion models of Vasicek (1977), Heston (1990) and others.

Appendix: Derivation of the Generating Function and Option Formulas

Proof of Proposition 2:

<u>Derivation of the Generating Function</u>:

Let $x(t) = \log(S(t))$ and let $f(t;T,\phi)$ be the conditional generating function of $S(T)$, or equivalently the conditional moment generating function of $x(T)$

$$f(t;T,\phi) = E_t[\exp(\phi x(T))]. \tag{A1}$$

We shall guess that the moment generating function takes the log-linear form

$$f(t;T,\phi) = \tag{A2}$$

$$\exp\left(\phi x(t)+A(t;T,\phi)+\sum_{i=1}^{p} B_i(t;T,\phi)h(t+2\Delta-i\Delta)+\sum_{i=1}^{q-1} C_i(t;T,\phi)(z(t+\Delta-i\Delta)-\gamma_i\sqrt{h(t+\Delta-i\Delta)})^2\right),$$

and solve using the method of undetermined coefficients.

Conditional normality at time $T-\Delta$ implies

$$A(T-\Delta;T,\phi) = \phi r, \quad B_1(T-\Delta;T,\phi) = \lambda\phi+\tfrac{1}{2}\phi^2, \tag{A3}$$

$$B_i(T-\Delta;T,\phi) = C_{i-1}(T-\Delta;T,\phi) = 0, \quad \text{for } i > 1.$$

By iterated expectations,

$$f(t;T,\phi) = E_t\big[f(t+\Delta;T,\phi)\big] = E_t\Big[\exp\Big(\phi x(t+\Delta)+A(t+\Delta;T,\phi) \tag{A4}$$

$$+ \sum_{i=1}^{p} B_i(t+\Delta;T,\phi)h(t+3\Delta-i\Delta) + \sum_{i=1}^{q-1} C_i(t+\Delta;T,\phi)(z(t+2\Delta-i\Delta)-\gamma_i\sqrt{h(t+2\Delta-i\Delta)})^2\Big)\Big].$$

Substituting the dynamics in equation (1) shows

$$f(t;T,\phi) = E_t\Big[\exp\Big(\phi(x(t)+r+\lambda h(t+\Delta)+\sqrt{h(t+\Delta)}z(t+\Delta))+A(t+\Delta;T,\phi) \tag{A5}$$

$$+ B_1(t+\Delta;T,\phi)(\beta_1 h(t+\Delta)+\alpha_1(z(t+\Delta)-\gamma_1\sqrt{h(t+\Delta)})^2)$$

$$+ B_1(t+\Delta;T,\phi)\Big(\omega + \sum_{i=1}^{p-1}\beta_{i+1}h(t+2\Delta-i\Delta) + \sum_{i=1}^{q-1}\alpha_{i+1}(z(t+2\Delta-i\Delta)-\gamma_{i+1}\sqrt{h(t+2\Delta-i\Delta)})^2\Big)$$

$$+ \sum_{i=1}^{p-1}B_{i+1}(t+\Delta;T,\phi)h(t+2\Delta-i\Delta) +$$

$$C_1(t+\Delta;T,\phi)(z(t+\Delta)-\gamma_1\sqrt{h(t+\Delta)})^2 + \sum_{i=1}^{q-2} C_{i+1}(t+\Delta;T,\phi)(z(t+\Delta-i\Delta)-\gamma_{i+1}\sqrt{h(t+\Delta-i\Delta)})^2\Big)\Big].$$

Rearranging terms shows

$$f(t;T,\phi) = E_t\Big[\exp\Big(\phi(x(t)+r)+A(t+\Delta;T,\phi)+B_1(t+\Delta;T,\phi)\omega \tag{A6}$$

$$+ (B_1(t+\Delta;T,\phi)\alpha_1+C_1(t+\Delta;T,\phi))\Big(z(t+\Delta)-(\gamma_1-\frac{\phi}{2(B_1(t+\Delta;T,\phi)\alpha_1+C_1(t+\Delta;T,\phi))})\sqrt{h(t+\Delta)}\Big)^2$$

$$+ (\phi\lambda+B_1(t+\Delta;T,\phi)\beta_1+\phi\gamma_1-\frac{\phi^2}{4(B_1(t+\Delta;T,\phi)\alpha_1+C_1(t+\Delta;T,\phi))})h(t+\Delta)$$

$$+ B_1(t+\Delta;T,\phi)\sum_{i=1}^{p-1}\beta_{i+1}h(t+2\Delta-i\Delta) + \sum_{i=1}^{p-1}B_{i+1}(t+\Delta;T,\phi)h(t+2\Delta-i\Delta)$$

$$+ B_1(t+\Delta;T,\phi)\sum_{i=1}^{q-1}\alpha_{i+1}(z(t+2\Delta-i\Delta)-\gamma_{i+1}\sqrt{h(t+2\Delta-i\Delta)})^2)$$

$$+ \sum_{i=1}^{q-2} C_{i+1}(t+\Delta;T,\phi)(z(t+\Delta-i\Delta)-\gamma_{i+1}\sqrt{h(t+\Delta-i\Delta)})^2)].$$

A useful result is that for a standard normal variable z

$$E[\exp(a(z+b)^2)] = \exp(-½\ln(1-2a)+ab^2/(1-2a)). \tag{A7}$$

Substituting this result and equating terms in the L.H.S. and R.H.S. shows

$$A(t;T,\phi) = \tag{A8}$$

$$A(t+\Delta;T,\phi)+\phi r+B_1(t+\Delta;T,\phi)\omega-½\ln(1-2\alpha_1B_1(t+\Delta;T,\phi)-2C_1(t+\Delta;T,\phi)),$$

$$B_1(t;T,\phi) =$$

$$\phi(\lambda+\gamma_1)-½\gamma_1^2+\beta_1B_1(t+\Delta;T,\phi)+B_2(t+\Delta;T,\phi)+\frac{½(\phi-\gamma_1)^2}{1-2\alpha_1B_1(t+\Delta;T,\phi)-2C_1(t+\Delta;T,\phi)},$$

$$B_i(t;T,\phi) = \beta_iB_1(t+\Delta;T,\phi) + B_{i+1}(t+\Delta;T,\phi), \quad \text{for } 1<i\leq p,$$

$$C_i(t;T,\phi) = \alpha_{i+1}B_1(t+\Delta;T,\phi) + C_{i+1}(t+\Delta;T,\phi), \quad \text{for } 1<i\leq q-1.$$

One can use equation (A8) to calculate the coefficients recursively starting with equation (A3).

Proof of Proposition 3:

Let $f(\phi)$ denote the moment generating function of the probability density, $p(x)$, where x is the logarithm of the terminal asset price. Let $p^*(x)$ be an adjusted probability density defined by $p^*(x) = \exp(x)p(x)/f(1)$. It is easy to see that it is a valid probability density because it is non-negative and $f(1) = E_t[\exp(x(T))]$. The moment generating function for $p^*(x)$ is

$$\int_{-\infty}^{\infty} \exp(\phi x) p^*(x)dx = \frac{1}{f(1)} \int_{-\infty}^{\infty} \exp((\phi+1)x)p(x)dx = \frac{f(\phi+1)}{f(1)}. \tag{A9}$$

Since the spot asset price is exp(x), then the expectation of a call option payoff separates into two terms with probability integrals.

$$E[\mathrm{Max}(e^x\text{-}K)] = \int_{\ln(K)}^{\infty} \exp(x)p(x)dx - K \int_{\ln(K)}^{\infty} p(x)dx. \tag{A10}$$

$$= f(1) \int_{\ln(K)}^{\infty} p^*(x)dx - K \int_{\ln(K)}^{\infty} p(x)dx.$$

Note that $f(i\phi)$ is the characteristic function corresponding to $p(x)$ and $f(i\phi+1)/f(1)$ is the characteristic function corresponding to $p^*(x)$. Feller [1966] and Kendall and Stuart [1977] show how to recover the "probabilities" from the characteristic functions

$$\int_{\ln(K)}^{\infty} p(x)dx = ½ + \frac{1}{\pi} \int_{0}^{\infty} \mathrm{Re}\left[\frac{e^{-i\phi\ln(K)} f(i\phi)}{i\phi}\right] d\phi, \tag{A11}$$

and similarly the other integral of $p^*(x)$. Substituting equation (A11) into the expression (A10) proves the proposition and noting that under the risk neutral distribution, $S(t) = e^{-r(T-t)} f^*(1)$ (where $f^*(1)$ is the terminal asset price) demonstrates the corollary.

References

Amin, Kaushik and Victor Ng, 1993, "ARCH Processes and Option valuation", Working Paper, University of Michigan.

Bakshi, Gurdip, Charles Cao and Zhiwu Chen, 1997, Empirical Performance of Alternative Option Pricing Models, Forthcoming, Journal of Finance.

Bates, David., 1996 (a) , "Jumps & Stochastic Volatility: Exchange Rate Processes Implicit in Deutschemark Options", Review of Financial Studies 9, 69-107.

Bates, David., 1996 (b) , "Post -'87 Crash Fears in the S&P 500 Futures Options", Working Paper, University of Iowa.

Black, Fisher and Myron Scholes, 1973, "The Pricing of Options and Corporate Liabilities," Journal of Political Economy 81, 637-659.

Bollerslev, Tim., 1986, "Generalized Autoregressive Conditional Heteroskedasticity," Journal of Econometrics 31, 307-327.

Breeden, Douglas T. and Robert H. Litzenberger, 1978, "Prices of State-Contingent Claims Implicit In Options Prices," Journal of Business 5, 621-652.

Brennan, Michael, 1979, "The Pricing of Contingent Claims in Discrete Time Models", Journal of Finance, 34, 53-68.

Christie, Andrew A., 1982, "The Stochastic Behavior of Common Stock Variances: Value, Leverage and Interest Rate Effects," Journal of Financial Economics 10, 407-432.

Cox, John and Stephen Ross, 1976, "The Valuation of Options for Alternative Stochastic Processes," Journal of Financial Economics 3, 145-166.

Cox, John, Jonathan Ingersoll and Stephen Ross, 1985, "A Theory of the Term Structure of Interest Rates", Econometrica, 53, 385-407.

Cox, John and Mark Rubinstein, 1985, "Options Markets", Prentice Hall Inc.

Derman, Emanuel and Iraz Kani, 1994, "Riding on the Smile", Risk 7, 32-39.

Duan, Jin-Chuan, 1995, "The GARCH Option Pricing Model," Mathematical Finance 5, 13-32.

Duan, Jin-Chuan, Genevieve Gauthier and Jean-Guy Simonato, 1997, “An Analytical Approximation for the GARCH Option Pricing Model”, Working Paper, Hong Kong University of Science and Technology.

Dumas, Bernard, Jeff Fleming, and Robert. Whaley, 1996, ``Implied Volatility Functions: Empirical Tests", Working Paper, Jones Graduate School of Administration, Rice University.

Dupiere, Bruno, 1994, “Pricing with a Smile”, Risk, 7, 18-20.

Engle, Robert, 1982, “Autoregressive Conditional Heteroskedasticity,” with Estimates of the Variance of U. K. Inflation,” Econometrica 50, 987-1008.

Engle, Robert, and Chowdhury Mustafa, 1992, “Implied ARCH Models from Options Prices,” Journal of Econometrics 52, 289-311.

Engle, Robert and Victor Ng, 1993, “Measuring and Testing the Impact of News on Volatility,” Journal of Finance 43, 1749-1778.

Feller, William, 1966, An Introduction to Probability Theory and Its Applications, Volume 2, Wiley and Sons, New York.

Heston, Steven L., 1990, “Testing Continuous Time Models of the Term Structure of Interest Rates”, Working Paper, Carnegie Mellon University.

Heston, Steven L., 1993, “A Closed-Form Solution for Options with Stochastic Volatility, with Applications to Bond and Currency Options," Review of Financial Studies 6.

Kendall, Maurice, and Alan Stuart, 1977, The Advanced Theory of Statistics, Volume 1, Macmillan Publishing Co., Inc., New York.

Knoch, Hans J., 1992, "The Pricing of Foreign Currency Options With Stochastic Volatility," Ph.D. Dissertation , Yale School of Organization and Management.

Melino, Angelo and Stuart Turnbull, 1990, "The Pricing of Foreign Currency Options With Stochastic Volatility," Journal of Econometrics, 45, 239-265.

Nandi, Saikat, 1996, “Pricing and Hedging Index Options under Stochastic Volatility: An Empirical Examination,” , Working Paper, 96-9, Federal Reserve Bank of Atlanta.

Nelson, Daniel, 1991, “Conditional Heteroskedastictiy in Asset Returns: A New Approach,” Econometrica 59, 347-370.

Nelson, Daniel and Douglas Foster, 1994, “Asymptotic Filtering Theory for Univariate ARCH Models,” Econometrica 62, 1-41.

Press, W., S. Teukolsky, W. Vetterling & B. Flannery, 1992, ``Numerical Recipes in C - The Art of Scientific Computing'', Cambridge University Press.

Ritchken, Peter and Rob Trevor, 1997, Pricing Options Under Generalized GARCH and Stochastic Volatility Processes, Working Paper, Case Western Reserve University.

Rubinstein, Mark., 1985, ``Nonparametric Tests of Alternative Option Pricing Models Using All Reported Trades and Quotes on the 30 Most Active Option Classes from August 23, 1976 through August 31, 1978'', Journal of Finance, 40, 455-480.

Rubinstein, Mark, 1976, “The Valuation of Uncertain Income Streams and the Pricing of Options,” Bell Journal of Economics and Management Science 7, 407-425.

Rubinstein, Mark., 1994, ``Implied Binomial Trees'', Journal of Finance, 69, 771-818.

Vasicek, Oldrich, 1977, “An Equilibrium Characterization of the Term Structure”, Journal of Financial Economics, 5, 177-188.

TABLE 1 (a)

Maximum Likelihood Estimates of the **GARCH** model with p=q=1 and Δ = 1 (day):

$$\log(S(t)) = \log(S(t-\Delta)) + r + \lambda h(t) + \sqrt{h(t)}z(t),$$

$$h(t) = \omega + \sum_{i=1}^{p}\beta_i h(t-i\Delta) + \sum_{i=1}^{q}\alpha_i\left(z(t-i\Delta)-\gamma_i\sqrt{h(t-i\Delta)}\right)^2$$

The daily 2:30 P.M. (closest to) index levels from **01/08/92 - 12/30/94** are used. No. of Observations = **755**. Asymptotic standard errors appear in parentheses. θ is the annualized long run volatility (standard deviation) implied by the parameter estimates. $\beta_1 + \alpha_1\gamma_1^2$ measures the degree of mean reversion in that $\beta_1 + \alpha_1\gamma_1^2 = 1$ implies that the variance process is integrated.

	α_1	β_1	γ_1	ω	λ	θ	$\beta_1 + \alpha_1\gamma_1^2$	Log Likelihood
GARCH	1.0e-6 (0.03e-6)	0.589 (0.007)	421.39 (11.01)	5.02e-6 (0.19e-6)	0.205 (0.228)	8.02 %	0.766	3503.7
GARCH $\gamma_1 = 0$	1.0e-6 (0.04e-6)	0.922 (0.013)		1.63e-6 (0.43e-6)	0.732 (0.22)	9.2%	0.922	3492.4

TABLE 1(b)

Parameter estimates and pricing errors from the in-sample estimation, minimizing the sum of squared errors between model and market option prices, every Wednesday. Asymptotic standard errors appear in parentheses. BS (constant) is the version of the Black-Scholes in which a single implied volatility is estimated over the first six months of a year while in BS (updated), an implied volatility is estimated every Wednesday. θ is the annualized long run volatility under GARCH (the single implied volatility under BS (constant)), while $\beta_1 + \alpha_1\gamma_1^2$ measures the degree of mean reversion in that $\beta_1 + \alpha_1\gamma_1^2 = 1$ implies that the variance process is integrated. $\gamma_1^* = \gamma_1 + \lambda + 1/2$ measures the skewness of the risk neutral distribution. If $\gamma_1^* = 0$, then $\gamma_1 = -(\lambda + 1/2)$. SSE is the sum of squared errors between model and market option prices. Average pricing error is the root mean squared pricing error divided by the mean option price of the sample.

	α_1	β_1	γ_1	ω	λ	θ	$\beta_1 + \alpha_1\gamma_1^2$	Average Error (pricing)	SSE	No. of Obs.
1992										
BS Constant						14.5 %		11.04 %	822.6	
GARCH $\gamma_1^* = 0$	7.4e-6 (6.31e-6)	0.734 (0.063)	-81.94 (55.91)	8.3e-6 (5.9e-6)	81.44 (55.91)	11.9%	0.783	10.92%	804.1	
BS Updated								9.03 %	550.4	
GARCH	3.1e-6 (0.64e-6)	0.122 (0.067)	487.87 (68.32)	4.9e-6 (0.6e-6)	0.85 (0.56)	11.9 %	0.859	7.01 %	353.6	874
1993										
BS Constant						11.8 %		11.41 %	639.6	
GARCH $\gamma_1^* = 0$	1.1e-7 (2.6e-7)	0.97 (0.006)	416.3 (354.9)	7.0e-11 (1.38e-9)	-416.8 (354.9)	5.05 %	0.988	11.12%	607.9	
BS Updated								10.3 %	521.3	
GARCH	2.5e-6 (0.19e-6)	0.216 (0.067)	508.02 (37.85)	2.5e-6 (0.17e-6)	2.36 (1.95)	9.3 %	0.855	6.52 %	209.4	848
1994										
BS Constant						12.3 %		16.48 %	1700.2	
GARCH $\gamma_1^* = 0$	1.7e-6 (0.21e-6)	0.956 (0.001)	64.46 (166.8)	1.59e-8 (2.39e-8)	-64.96 (166.8)	16.8%	0.963	14.74%	1360.8	
BS Updated								11.41 %	814.69	
GARCH	4.4e-7 (0.33e-7)	0.003 (0.0006)	1466.7 (62.15)	1.1e-6 (0.05e-6)	1.24 (0.85)	16.2 %	0.949	8.76 %	481.2	906

Table 2 - a

Out-of-Sample Percentage Pricing Errors, Every Wednesday, 07/92 - 12/92

Out-of-sample average mispricing of different categories of options by moneyness and maturity. Mispricing for an option is defined to be | Model - Market |/Market. Average mispricing of a particular category is the sum of the individual mispricings in that category divided by the total number of observations in that category. Moneyness is defined as (strike/spot)-1. RMSE for the entire sample is defined to be $\frac{\sqrt{\text{SSE/Sample Size}}}{\text{Mean Option Price}}$, where SSE is the total sum of squared errors. BSC is BS model in which volatility is held fixed at the value implied from the option prices of the first six months. BSF is the BS model for which an implied volatility is estimated every previous trading day.

RMSE: BSC = 14.79%, BSF = 13.42%, GARCH =9.4%

Total number of Observations = 897

Calls	Moneyness	# Obs.	BSC	BSF	GARCH	Puts	Moneyness	# Obs.	BSC	BSF	GARCH
(0-30 days)	< -0.04	13	0.018	0.023	0.012	(0-30 days)	< -0.04	8	0.497	0.653	0.321
	(-0.04,-0.01)	41	0.049	0.041	0.056		(-0.04,-0.01)	34	0.29	0.353	0.245
	(-0.01,0.01)	27	0.238	0.152	0.150		(-0.01,0.01)	27	0.257	0.069	0.053
	(0.01,0.04)	25	0.566	0.328	0.264		(0.01,0.04)	28	0.108	0.069	0.052
	> 0.04	0					> 0.04	6	0.026	0.035	0.016
Calls	< -0.04	17	0.019	0.035	0.023	Puts	< -0.04	96	0.368	0.523	0.197
(30-90 days)	(-0.04,-0.01)	48	0.055	0.048	0.061	(30-90 days)	(-0.04,-0.01)	84	0.153	0.183	0.171
	(-0.01,0.01)	64	0.155	0.095	0.122		(-0.01,0.01)	55	0.161	0.102	0.131
	(0.01,0.04)	76	0.445	0.26	0.21		(0.01,0.04)	44	0.117	0.063	0.071
	> 0.04	32	0.795	0.471	0.326		> 0.04	24	0.047	0.049	0.03
Calls	< -0.04	4	0.048	0.078	0.019	Puts	< -0.04	32	0.251	0.40	0.11
(> 90 days)	(-0.04,-0.01)	10	0.033	0.077	0.031	(> 90 days)	(-0.04,-0.01)	24	0.071	0.142	0.081
	(-0.01,0.01)	14	0.079	0.048	0.076		(-0.01,0.01)	13	0.063	0.063	0.069
	(0.01,0.04)	20	0.216	0.082	0.154		(0.01,0.04)	12	0.1	0.034	0.075
	> 0.04	14	0.771	0.424	0.407		> 0.04	5	0.093	0.051	0.061

Table 2 - b

Out-of-Sample Percentage Pricing Errors, Every Wednesday, 07/93 - 12/93

Out-of-sample average mispricing of different categories of options by moneyness and maturity. Mispricing for an option is defined to be | Model - Market |/Market. Average mispricing of a particular category is the sum of the individual mispricings in that category divided by the total number of observations in that category. Moneyness is defined as (strike/spot)-1. RMSE for the entire sample is defined to be $\frac{\sqrt{\text{SSE/Sample Size}}}{\text{Mean Option Price}}$, where SSE is the total sum of squared errors. BSC is BS model in which volatility is held fixed at the value implied from the option prices of the first six months. BSF is the BS model for which an implied volatility is estimated every previous trading day.

RMSE: BSC = 14.48%, BSF = 14.71%, GARCH =7.92%

Total number of Observations = 1012

Calls	Moneyness	# Obs.	BSC	BSF	GARCH	Puts	Moneyness	# Obs.	BSC	BSF	GARCH
(0-30 days)	< -0.04	12	0.041	0.045	0.026	(0-30 days)	< -0.04	11	0.723	0.815	0.308
	(-0.04,-0.01)	47	0.053	0.062	0.041		(-0.04,-0.01)	46	0.257	0.319	0.187
	(-0.01,0.01)	43	0.196	0.132	0.127		(-0.01,0.01)	44	0.191	0.128	0.11
	(0.01,0.04)	18	0.616	0.361	0.264		(0.01,0.04)	34	0.083	0.056	0.041
	> 0.04	0					> 0.4	3	0.023	0.015	0.013
Calls (30-90 days)	< -0.04	10	0.041	0.05	0.027	Puts (30-90 days)	< -0.04	105	0.492	0.628	0.159
	(-0.04,-0.01)	38	0.034	0.059	0.025		(-0.04,-0.01)	110	0.131	0.212	0.091
	(-0.01,0.01)	77	0.105	0.069	0.059		(-0.01,0.01)	81	0.124	0.069	0.088
	(0.01,0.04)	74	0.504	0.292	0.135		(0.01,0.04)	57	0.11	0.068	0.045
	> 0.04	15	0.854	0.491	0.175		> 0.04	11	0.078	0.057	0.034
Calls (>90 days)	< -0.04	2	0.067	0.082	0.041	Puts	< -0.04	44	0.418	0.552	0.169
	(-0.04,-0.01)	16	0.051	0.101	0.032		(-0.04,-0.01)	33	0.105	0.221	0.039
	(-0.01,0.01)	18	0.036	0.081	0.025		(-0.01,0.01)	23	0.046	0.059	0.042
	(0.01,0.04)	10	0.217	0.091	0.078		(0.01,0.04)	9	0.092	0.054	0.059
	> 0.04	12	0.663	0.338	0.184		> 0.04	9	0.087	0.056	0.045

Table 2 - c

Out-of-Sample Percentage Pricing Errors, Every Wednesday, 07/94 - 12/94

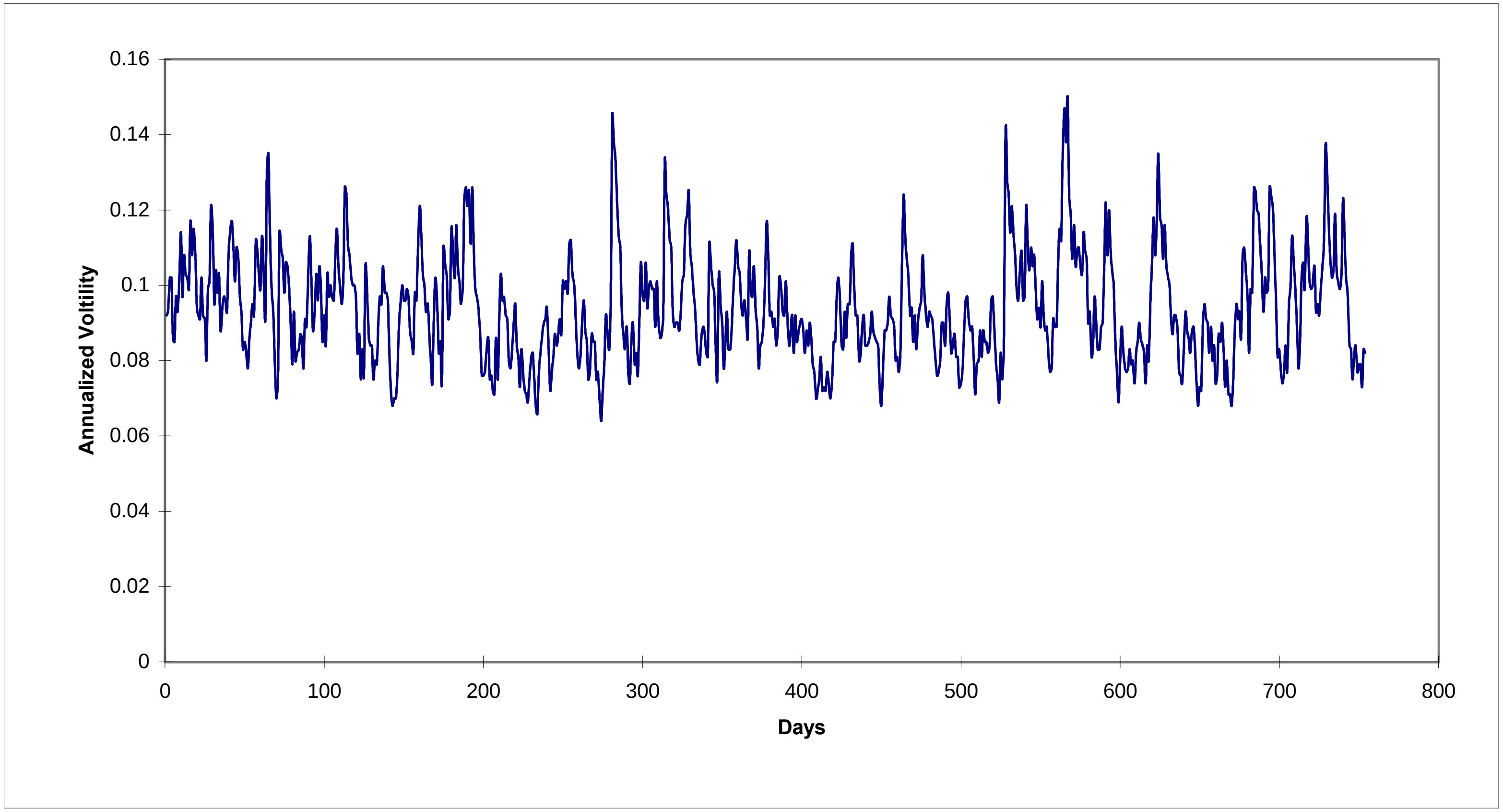
Out-of-sample average mispricing of different categories of options by moneyness and maturity. Mispricing for an option is defined to be | Model - Market |/Market. Average mispricing of a particular category is the sum of the individual mispricings in that category divided by the total number of observations in that category. Moneyness is defined as (strike/spot)-1. RMSE for the entire sample is defined to be $\frac{\sqrt{\text{SSE/Sample Size}}}{\text{Mean Option Price}}$, where SSE is the total sum of squared errors. BSC is BS model in which volatility is held fixed at the value implied from the option prices of the first six months. BSF is the BS model for which an implied volatility is estimated every previous trading day.

RMSE: BSC = 16.35%, BSF = 16.07%, GARCH =11.58%

Total number of Obsevations = 1287

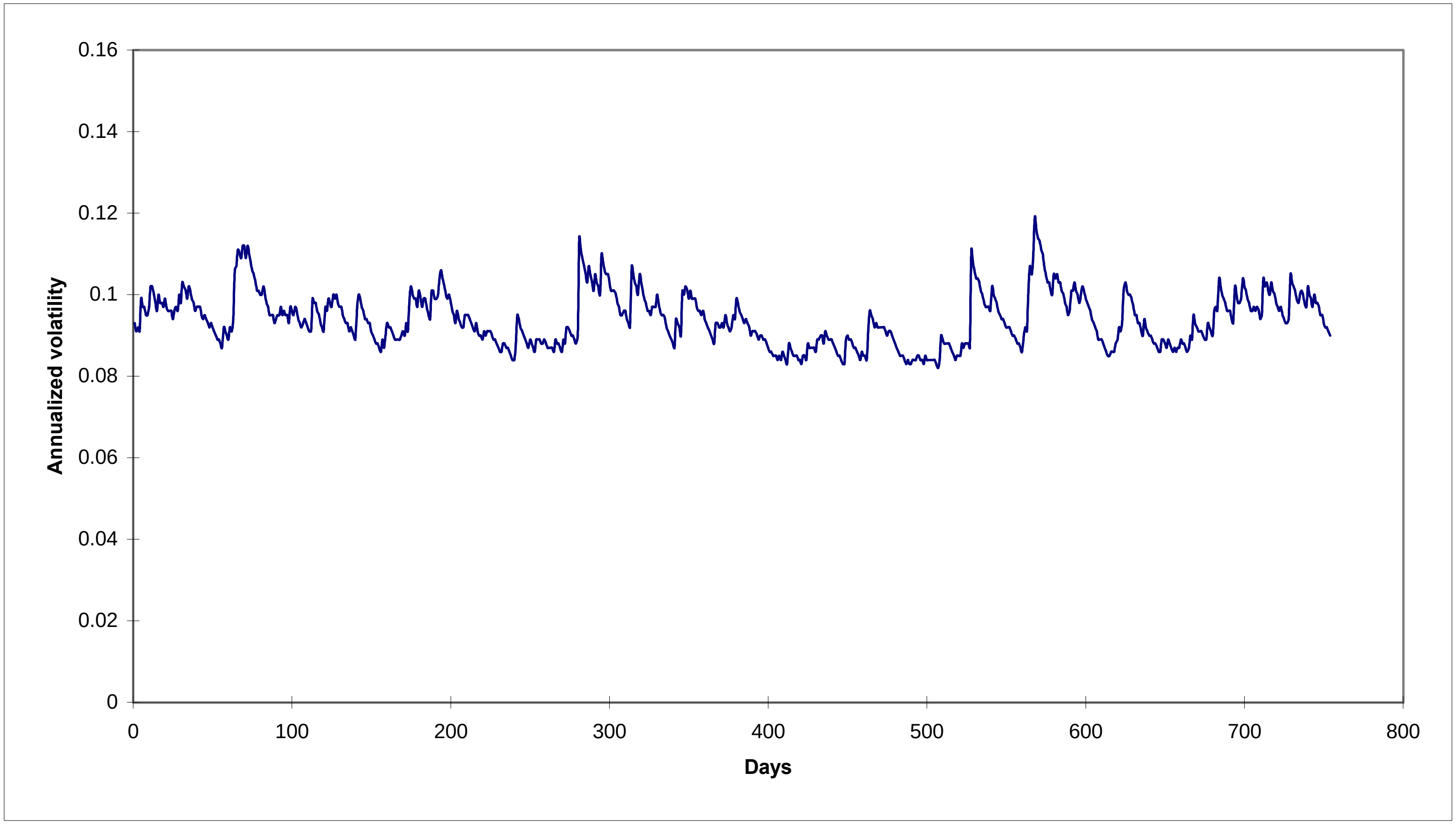
Calls	Moneyness	# Obs.	BSC	BSF	GARCH	Puts	Moneyness	# Obs.	BSC	BSF	GARCH
(0-30 days)	< -0.04	13	0.028	0.028	0.019	(0-30 days)	< -0.04	22	0.775	0.723	0.518
	(-0.04,-0.01)	58	0.064	0.061	0.038		(-0.04,-0.01)	53	0.339	0.312	0.231
	(-0.01,0.01)	44	0.217	0.233	0.122		(-0.01,0.01)	42	0.167	0.179	0.131
	(0.01,0.04)	25	0.417	0.51	0.236		(0.01,0.04)	47	0.067	0.082	0.052
	> 0.04	0					>0.04	15	0.011	0.013	0.01
Calls (30-90 days)	< -0.04	16	0.051	0.044	0.033	Puts (30-90days)	< -0.04	137	0.625	0.556	0.324
	(-0.04,-0.01)	53	0.084	0.079	0.050		(-0.04,-0.01)	115	0.247	0.206	0.149
	(-0.01,0.01)	73	0.136	0.141	0.085		(-0.01,0.01)	85	0.129	0.132	0.093
	(0.01,0.04)	103	0.415	0.505	0.180		(0.01,0.04)	90	0.083	0.099	0.057
	> 0.04	47	0.625	0.805	0.322		> 0.04	48	0.033	0.042	0.026
Calls (>90 days)	< -0.040	4	0.077	0.069	0.044	Puts (>90 days)	< -0.040	36	0.521	0.452	0.336
	(-0.04,-0.01)	8	0.089	0.081	0.063		(-0.04,-0.01)	34	0.234	0.193	0.191
	(-0.01,0.01)	11	0.124	0.143	0.063		(-0.01,0.01)	27	0.131	0.122	0.121
	(0.01,0.04)	20	0.182	0.238	0.093		(0.01,0.04)	17	0.066	0.088	0.041
	>0.04	20	0.558	0.711	0.256		>0.04	24	0.035	0.038	0.039

Figure 1



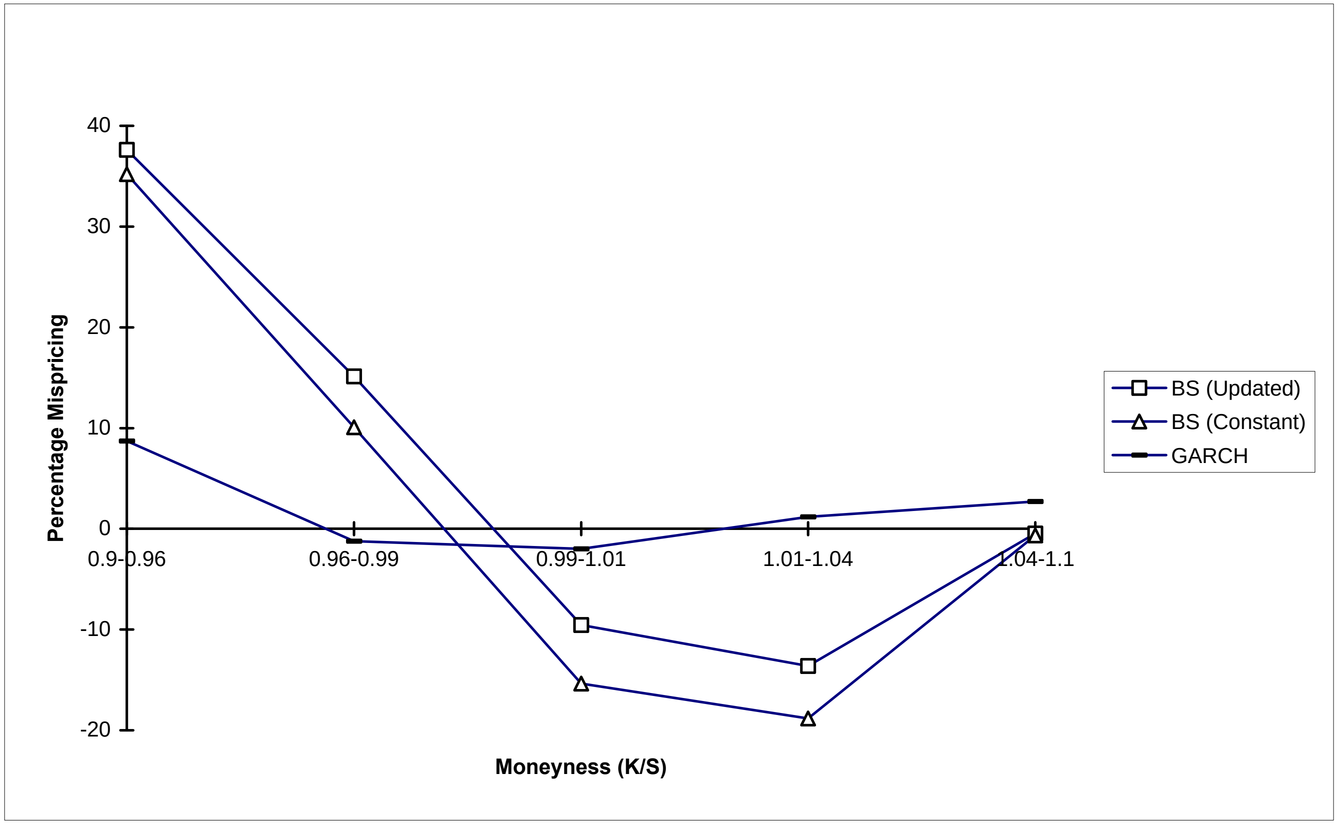
Shows the daily volatility (annualized) from the daily, closest to 2:30 P.M., SP 500 index levels from the GARCH model (unresrticted): 01/08/92-12/30/94

Figure 2



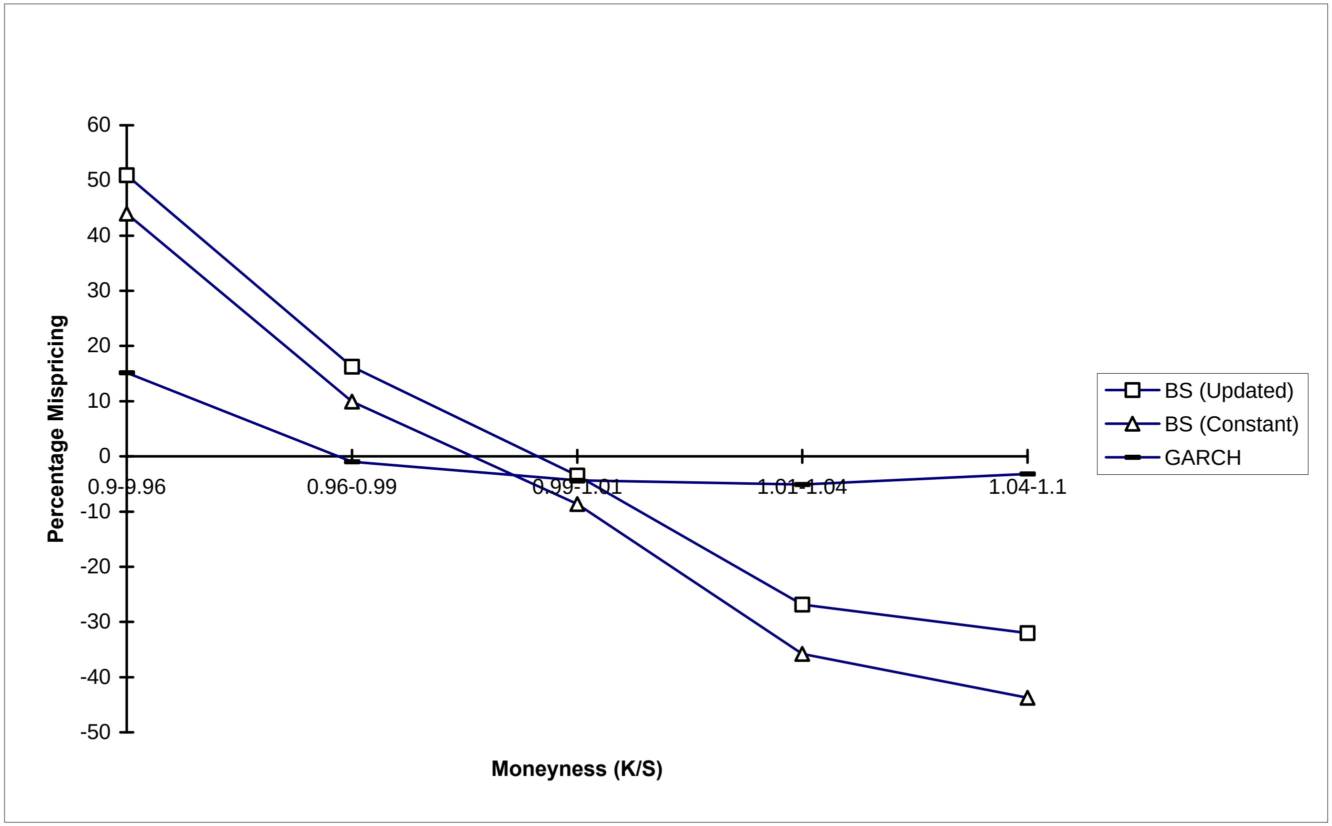
Shows the daily volatility (annualized) from the daily, closest to 2:30 P.M. SP 500 index levels from the resrtricted/symmetric GARCH model (correlation between returns and volatility is zero) : 01/08/92-12/30/94

Figure 3 (a)



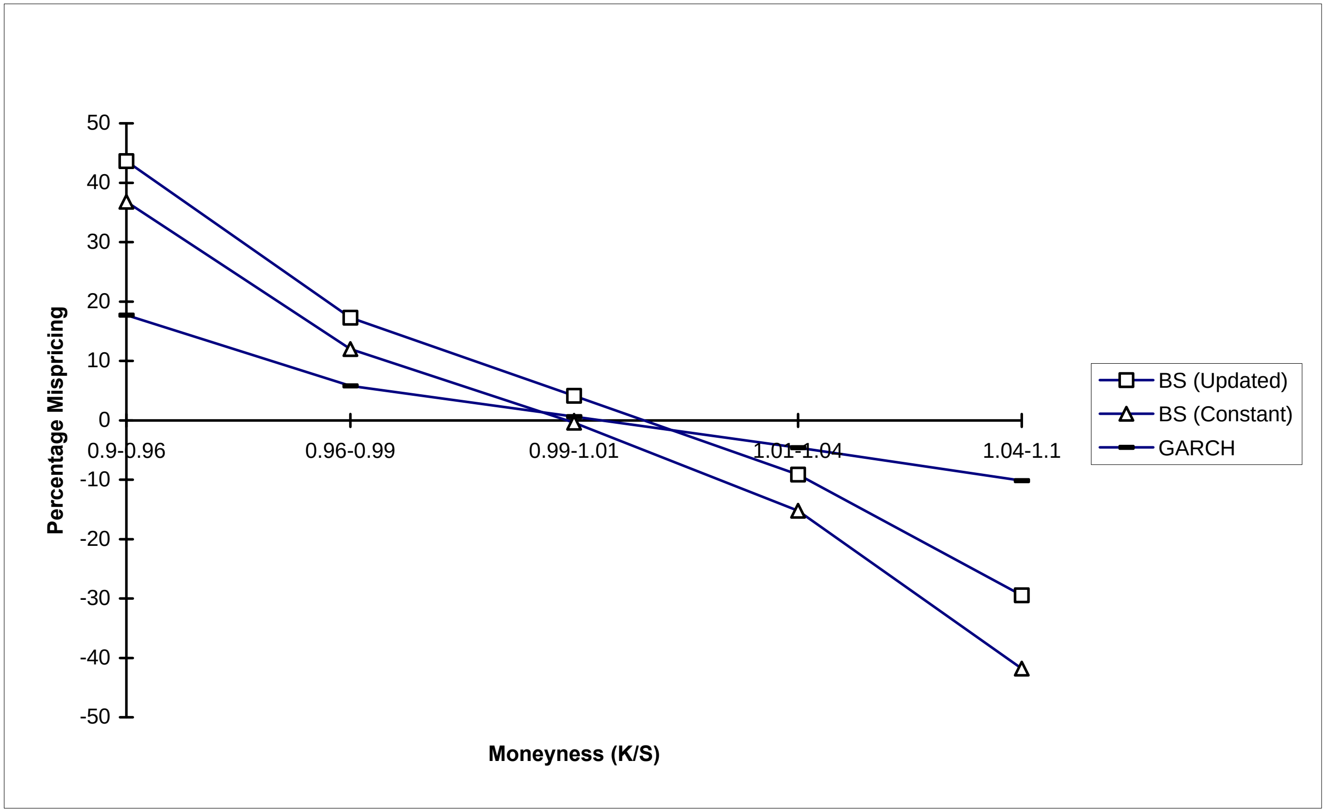
Out-of-sample mispricing (%) by moneyness (K/S) for three years (92,93,94) combined. Maturity of 30 days or less. Monyness categories are, K/S: 0.9-0.96, 0.96-0.99, 0.99-1.01, 1.01-1.04, 1.04 -1.1.

Figure 3 (b)



Out-of-sample mispricing (%) by moneyness (K/S) for three years (92,93,94) combined. Maturity: 31-90 days. Moneyness categories are, K/S: 09-0.96,0.96-0.99,0.99-1.01,1.01-1.04,1.04-1.1

Figure 3 (c)



Out-of-sample mispricing (%) by moneyness (K/S) for three years (92,93,94) combined. Maturity of 91 days or more. Moneyness categories are, K/S: 0.9-0.96, 0.96-0.99, 0.99-1.01, 1.01-1.04, 1.04-1.1