M5MF2 Numerical Methods for Finance

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GENERALISED ARBITRAGE-FREE SVI VOLATILITY SURFACES

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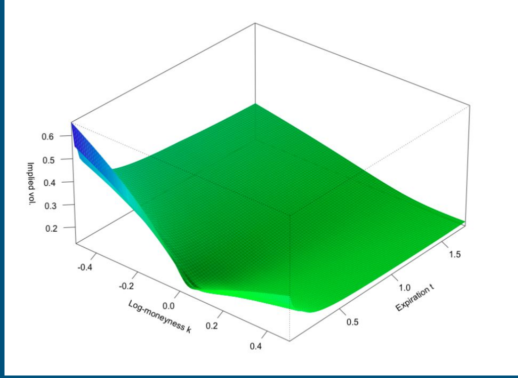
Potential audience of the article:

- Researchers in the corresponding field who seek to exhibit the class of implied volatility surfaces of non-SVI models
- Quantitative analysts in the industry who want to work with more flexible models, since there is a relaxation of some regularity conditions, such as differentiability of total variance
- Traders who want to be aware of the implied volatility in the arbitrage-free models’ setup

Background



Implied Volatility is a key concept in quoting option prices.

An implied volatility surface is a function,

(Time to maturity, Strike) → Implied volatility(Time to maturity, Strike)

The log (forward) moneyness, k, for an option with strike K and expiry t is defined as $log(K/F_t)$

The corresponding implied volatility is denoted by σ(k, t) and the total variance w is defined by $\omega(k,t) = \sigma(k,t)^2 t$

Arbitrage

A dynamic arbitrage opportunity is a costless trading strategy that gives a positive future profit with positive probability and has no probability of a loss.

A static arbitrage opportunity is a dynamic arbitrage opportunity where positions in the underlying at a particular time only can depend on time and actual corresponding price

A volatility surface is free of static arbitrage if and only if the following conditions are satisfied: (i) it is free of calendar spread arbitrage; (ii) each time slice is free of butterfly arbitrage.

Arbitrage-Free Conditions

Theorem 2.2. *If the two-dimensional map* $w : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies*

(i) $w(\cdot, t)$ *is of class* $C^2(\mathbb{R})$ *for each* $t \geq 0$;

(ii) $w(k, t) > 0$ *for all* $(k, t) \in \mathbb{R} \times \mathbb{R}_+^*$;

(iii) $w(k, \cdot)$ *is non-decreasing for each* $k \in \mathbb{R}$;

(iv) for each $(k, t) \in \mathbb{R} \times \mathbb{R}_+^*$, $\mathcal{L}w(k, t)$ *is non-negative;*

(v) $w(k, 0) = 0$ *for all* $k \in \mathbb{R}$;

(vi) $\lim_{k\uparrow\infty} d_+(k, w(k, t)) = -\infty$, *for each* $t > 0$.

Then the corresponding Call price surface $(K, t) \mapsto \mathrm{BS}(K, w(\log(K), t))$ *is free of static arbitrage.*

SVI

There is a variety of widely used models that are used for the surface construction. The stochastic volatility inspired, which is also known as an SVI model, is the model of implied volatility surface that was originally created at Merrill Lynch in 1999.

The model became so popular because of the following properties:

1. It satisfies Lee’s Moment Formula.

2. It is easy to calibrate the SVI model to the market data so that the corresponding implied volatility surface is free of arbitrage.

SVI (extension by Gatheral and Jacquier)

$$w(k,t) \equiv \frac{\theta_t}{2}\left\{1 + \rho k \varphi(\theta_t) + \sqrt{(k\varphi(\theta_t)+\rho)^2 + (1-\rho^2)}\right\}$$

Purpose: Find necessary and sufficient conditions for preventing static arbitrage.

The goal that was achieved by Gatheral and Jacquier is the theorem, which provides sufficient conditions for the implied volatility surface to be free of static arbitrage:

Theorem 2.6. The surface is 'free of static arbitrage' if the following conditions are satisfied:

(1) $\partial_t \theta_t \geq 0$ *for all* $t > 0$;

(2) $\varphi(\theta) + \theta\varphi'(\theta) \geq 0$ *for all* $\theta > 0$;

(3) $\varphi'(\theta) < 0$ *for all* $\theta > 0$;

(4) $\theta\varphi(\theta)(1+|\rho|) < 4$ *for all* $\theta > 0$;

(5) $\theta\varphi(\theta)^2(1+|\rho|) \leq 4$ *for all* $\theta > 0$.

Aims

1. Generalize the framework provided by Gatheral-Jacquier in their paper without specifying the shape of the SVI model by considering implied volatility surfaces of the form $\omega(k,t) = \theta_t \Psi(k\varphi(\theta_t))$, for all $k \in \mathbb{R}$ $t \geq 0$

2. Determine (necessary and sufficient) conditions on the triplet (θ, ϕ, Ψ) to eliminate the static arbitrage.

3. Enlarge the class of arbitrage-free volatility surfaces considering non-SVI models.

Assumptions

The paper states that with the following form of implied volatility surface $\omega(k,t) = \theta_t \Psi(k\varphi(\theta_t))$, for all $k \in \mathbb{R}$ $t \geq 0$ should satisfy the following assumptions in order to preserve the non-arbitrage property:

(i) $\theta \in \mathcal{C}^1(\mathbb{R}_+^* \to \mathbb{R}_+^*)$, is not constant, $\lim_{t\downarrow 0} \theta_t = 0$, and $\theta_\infty := \lim_{t\uparrow\infty} \theta_t$ is well defined in $(0,\infty]$;

(ii) $\varphi \in \mathcal{C}^1(\mathbb{R}_+^* \to \mathbb{R}_+^*)$, and $\lim_{u\uparrow\infty} \varphi(u)$ is well defined in $(0,\infty]$;

(iii) $\Psi \in \mathcal{C}^2(\mathbb{R} \to \mathbb{R}_+^*)$ with $\Psi(0) = 1$ and Ψ is not constant;

(iv) for any $k \in \mathbb{R}$, $\lim_{t\downarrow 0} w(k,t) = 0$.

Calendar Spread Arbitrage

How do we get rid of it?

$w(k, .)$ is non-decreasing for all $k \in \mathbb{R}$

Proposition - 1st Coupling Condition

$$F(z) := z\frac{\Psi'(z)}{\Psi(z)}, \qquad f(u) := u\frac{\varphi'(u)}{\varphi(u)}.$$

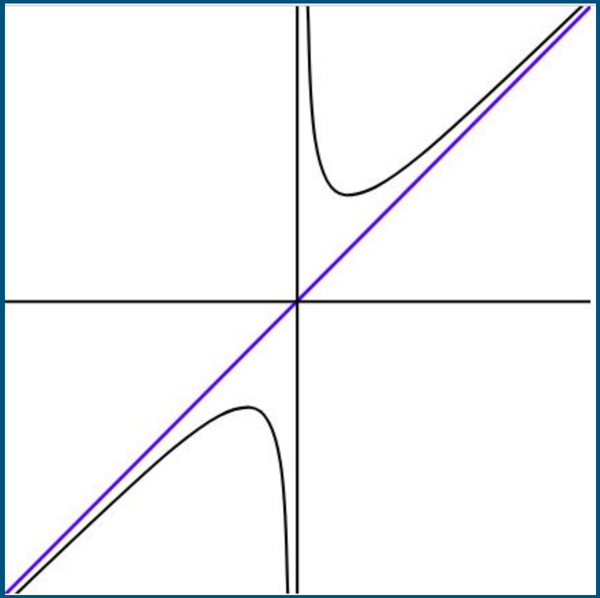
Proposition 3.1. (First coupling condition). The surface is free of calendar spread arbitrage if and only if the following two conditions hold:

(i) θ is non-decreasing;

(ii) $1 + F(z)f(u) \geq 0$ *for any* $z \in \mathbb{R}$ *and* $u \in (0, \theta_\infty)$.

Asymptotically Linear Case

Useful in eliminating calendar spread in SVI



Definition 3.4. The function Ψ is said to be asymptotically linear if $\lim_{z\to\pm\infty} \Psi'(z) =: \alpha_\pm \in \mathbb{R} \setminus \{0\}$.

Proposition 3.6. *If Ψ is asymptotically linear and if there is no calendar spread arbitrage, then the map $u \mapsto u\varphi(u)$ is non-decreasing on $\mathbb{R}_+$.*

Butterfly Arbitrage

Now how do we get rid of this one?

$\mathcal{L}\, w(k,t)$ is non-negative

$$\mathcal{L}w(k,t) := \left(1 - \frac{k\partial_k w(k,t)}{2w(k,t)}\right)^2 - \frac{(\partial_k w(k,t))^2}{4}\left(\frac{1}{w(k,t)} + \frac{1}{4}\right) + \frac{\partial^2_{kk} w(k,t)}{2}, \quad \text{for all } k \in \mathbb{R}, t > 0.$$

Proposition - 2nd Coupling Condition

For any $u \in (0, \theta_\infty]$, define the set

$$\mathcal{Z}_+(u) := \left\{ z \in \mathbb{R} : \frac{1}{4u}\left(\frac{\Psi'(z)^2}{\Psi(z)} - 2\Psi''(z)\right) + \frac{\Psi'(z)^2}{16} > 0 \right\},$$

as well as the function $\Lambda : \{(u, z) : u \in (0, \theta_\infty], z \in \mathcal{Z}_+(u)\} \to \mathbb{R} \cup \{+\infty\}$ by

$$\Lambda(u, z) := \left(\frac{1}{4u}\left(\frac{\Psi'(z)^2}{\Psi(z)} - 2\Psi''(z)\right) + \frac{\Psi'(z)^2}{16}\right)^{-1}\left(1 - \frac{z\Psi'(z)}{2\Psi(z)}\right)^2$$

Proposition 4.1. (Second coupling condition, general formulation). The surface w given in is free of butterfly arbitrage if and only if:

$$(u\varphi(u))^2 \leq \inf_{z \in \mathcal{Z}_+(u)} \Lambda(u, z), \quad \textit{for all } u \in (0, \theta_\infty).$$

Asymptotically Linear Case

$$M_\infty := \lim_{u \uparrow \theta_\infty} u\varphi(u)$$

Proposition 4.2. Assume that Ψ is asymptotically linear and there is no calendar spread arbitrage. Then $\overline{\mathcal{Z}}_+$ is neither empty nor bounded from above. Moreover, there is no butterfly arbitrage if and only if the following two conditions hold (recall that the functions $\mathcal{Z}_+$ *and* Λ are defined in (4.1) and (4.2)):

(i)
$$\begin{aligned} M_\infty^2 &\leq \inf_{z \in \overline{\mathcal{Z}}_- \cap \mathcal{Z}_+(\theta_\infty) \cap \aleph^c} \Lambda(\theta_\infty, z), && \text{if } \theta_\infty < \infty, \\ M_\infty &\leq \inf_{z \in \overline{\mathcal{Z}}_- \cap \aleph^c} \left| \frac{4}{\Psi'(z)} - \frac{2z}{\Psi(z)} \right|, && \text{otherwise;} \end{aligned}$$

(ii)
$$\text{for any } u \in (0, \theta_\infty),\ (u\varphi(u))^2 \leq \inf_{z \in \overline{\mathcal{Z}}_+} \Lambda(u, z).$$

Non-SVI Parameterisation

Proposition 5.1. If the generalised SVI surface is free of static arbitrage, Ψ is asymptotically linear and and $\theta_\infty = \infty$, then there exist $z_+ \geq 0$ and $\kappa \geq 0$ such that for all $z \geq z_+$ the following upper bound holds (with M_∞ defined as in (4.6)):

$$\Psi(z) \leq \kappa^2 + \frac{2z}{M_\infty} - \kappa\sqrt{\kappa^2 + \frac{2z}{M_\infty}}.$$

Model 1

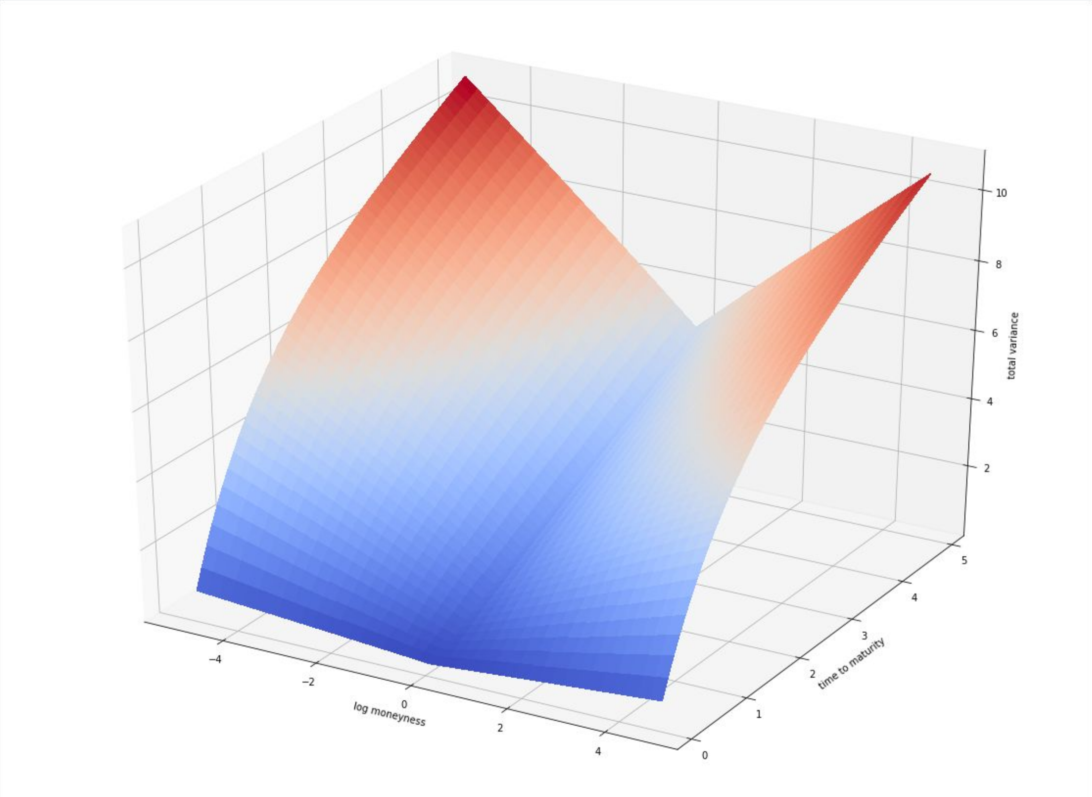
$$\theta_t \equiv t$$

$$\varphi(u) := \begin{cases} \dfrac{1-\mathrm{e}^{-u}}{u}, & \text{if } u > 0, \\ 1, & \text{if } u = 0, \end{cases}$$

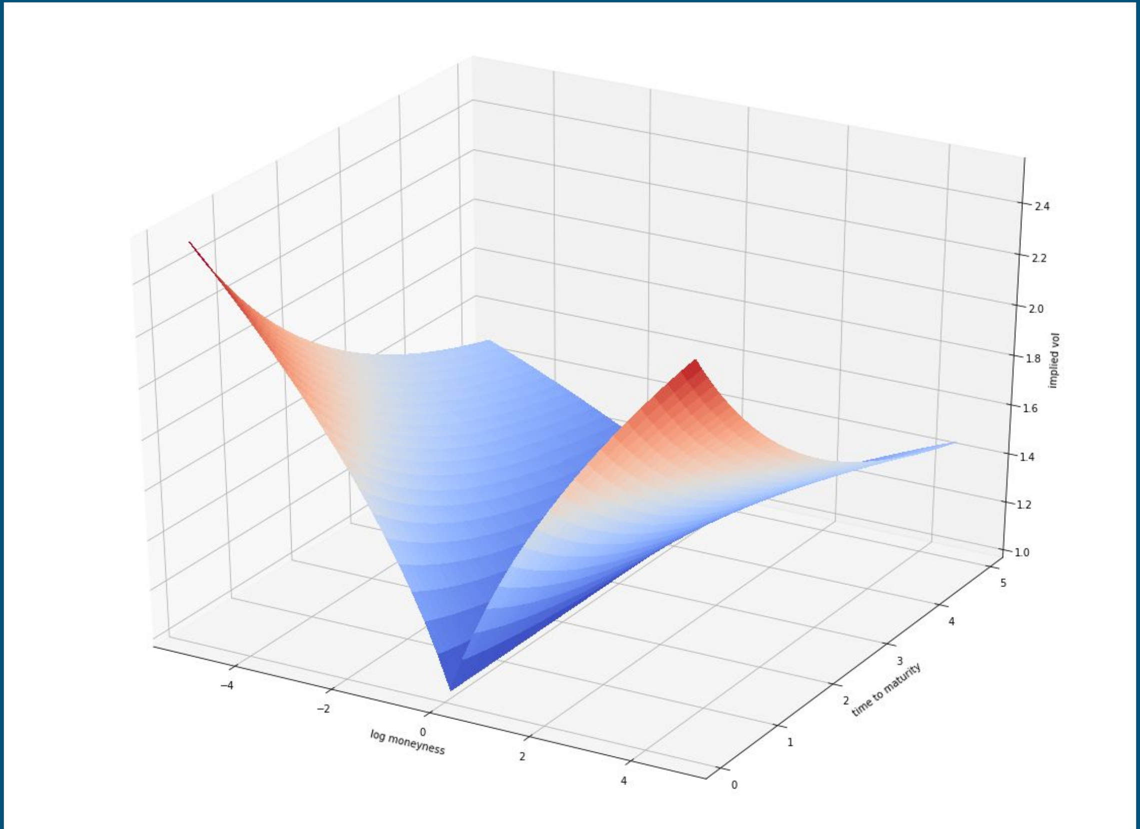
$$\Psi(z) := |z| + \frac{1}{2}\left(1 + \sqrt{1+|z|}\right), \quad \text{for all } z \in \mathbb{R}.$$

$$w(k,t) = k\left(1-\mathrm{e}^{-t}\right) + \frac{\sqrt{t}}{2}\left(\sqrt{t} + \sqrt{k\left(1-\mathrm{e}^{-t}\right)+t}\right), \qquad \text{for all } k \in \mathbb{R}, t \geq 0,$$

Total Variance Surface



Implied Volatility Surface



Model 2

$$\theta_t \equiv t$$

$$\varphi(u) := \begin{cases} \alpha \dfrac{1-\mathrm{e}^{-u}}{u}, & \text{if } u > 0, \\ \alpha, & \text{if } u = 0, \end{cases}$$

$\alpha \in (0, \bar{\alpha})$ $\quad \bar{\alpha} \approx 1.33$

$$\Psi_\nu(z) := \left(1 + |z|^\nu\right)^{1/\nu}, \quad \text{for } z \in \mathbb{R},$$

$\nu \in (1, \infty)$

$$w(k,t) = \theta_t \left(1 + \frac{(1-\mathrm{e}^{-\theta_t})^\nu}{\theta_t^\nu} \alpha^\nu |k|^\nu \right)^{1/\nu}, \quad \text{for all } k \in \mathbb{R}, t > 0,$$

Outlook

Calibrating the Non-SVI models

Propose more formulations

Compare how well they fit with market data

Thank you!