Numerical Methods for Finance

Second Order Discretization Schemes for CIR Processes

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Outline

Introduction

- CIR schemes are frequently used.
- However, usual numerical schmees can fail when one tries to simulate them.
- We have implemented a solution proposed in 2008 by Auélien Alfonsi in his paper *High order discretization schemes for the CIR process: application to A ne Term Structure and Heston models* (hal-00143723).
- We have calibrated an dsimulated some results with the Heston model.

Problem setting

Noting X_t^x the solution of the CIR SDE:

$$dX_t^x = a - kX_t^x + \sigma\sqrt{X_t^x}dW_t$$
$$X_0^x = x, x \in \mathbb{R}_+$$

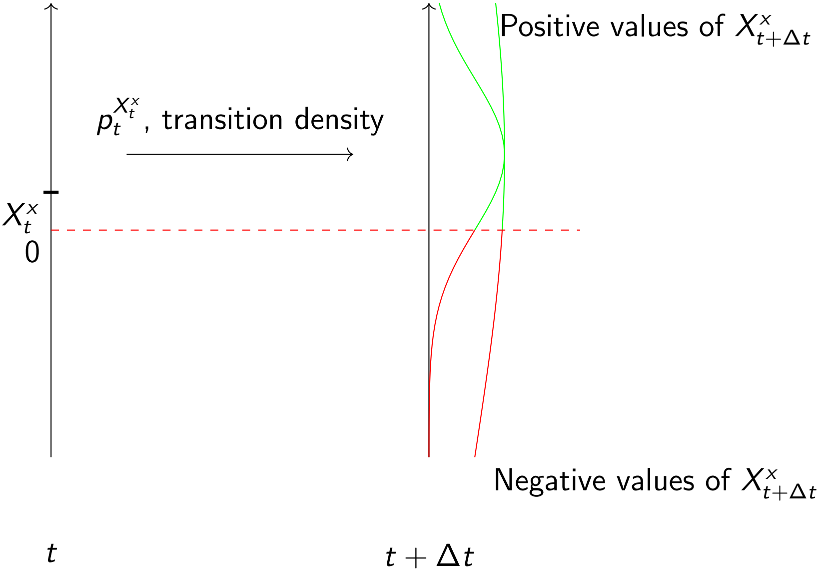
Qualitative analysis: when X is very close to 0, then the SDE becomes approximately

$$dX_t^x \approx a + \sigma\sqrt{X_t^x}dW_t$$

There are two possible regimes:

- If $\sigma << a$: X will mostly stay positive
- If $\sigma >> a$: X may become negative!

How usual numerical schemes can fail



Regime $\sigma << a$ ($\frac{\sigma^2}{4} \leq a$)

How do we keep the process positive and the scheme precise? Replace the original distribution by another with compact support. The higher the degree of precision, the more this variables has to account for the tail behaviour of the subsituted one.
Alfonsi shows [CITE] that we keep the scheme of order ν, if we subsitute the random variable with one that matches the first $2\nu + 1$ moments.
Using discrete random variables, we can control the tail probability and ensure the process stays positive.

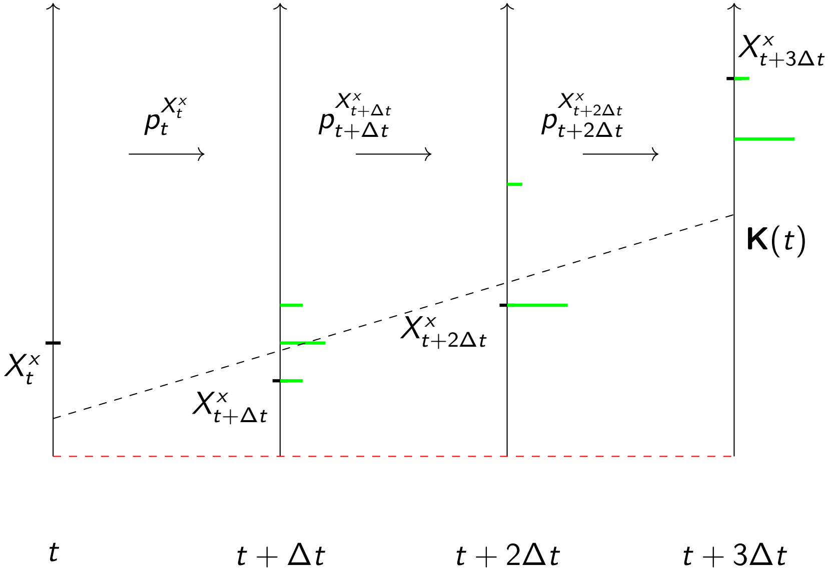
Regime $\sigma >> a$ ($\frac{\sigma^2}{4} > a$)

Alonfsi proves [CITE]:

- When X_t^x is far away from 0, then the scheme in the case $\sigma << a$ is also valid
- When X_t^x is close to 0, we can approximate the process by a *positive* discrete random variable that matches the first two moments and still have a second-order scheme

And we know when to switch between the two, via a threshold **K**.

Algorithm in practice



Dealing with Higher Order Schemes

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Simulation Results

Simulation Results

Using the Heston Model - Heston model

- Mathematical model implementing stochastic volatility.
- Uses an underlying CIR process for the variance.

$$dS_t = \mu S_t dt + \sqrt{\nu_t} S_t dW_t^s$$
$$d\nu_t = \kappa(\theta - \nu_t)dt + \xi\sqrt{\nu_t} dW_t^\nu$$

Using the Heston Model - Heston model

There is one subtlety when generating the random numbers.

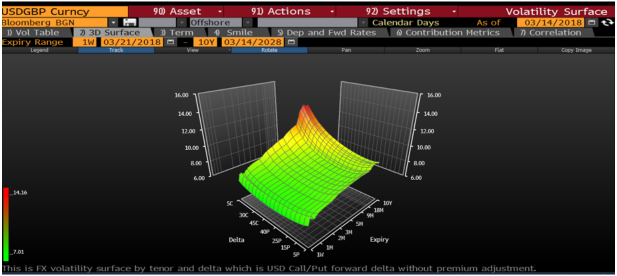
$$\Delta S_t = \mu S_t dt + \sqrt{\nu_t} S_t Z_1$$

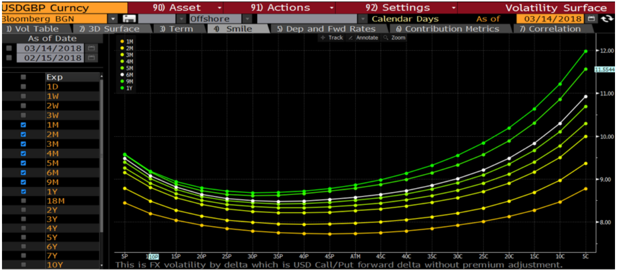
$$\Delta \nu_t = \kappa(\theta - \nu_t) dt + \xi \sqrt{\nu_t} Z_2$$

Using the Heston Model - Calibration

- In order to test the efficiency of the second order scheme, we calibrate the Heston Model using real data. In detail, we use a USDGBP volatility surface which was downloaded in Bloomberg.
- To calibrate the Heston model, we use the most liquid option in this market.

Using the Heston Model - Calibration





Using the Heston Model - Calibration

- In the currency option market, prices are quoted for moneyness levels for different time to expiry periods. These moneyness levels are:
  - At the money level or at the money forward (50 delta dual),
  - Out of the money level at 25 delta dual,
  - In the money level (75 delta dual).
- Delta dual is the first derivative with respect to the strike price.

$$\Delta_{dual} = \frac{\partial BS_{call}}{\partial k} = e^{-r_d}\Phi\left(\frac{ln(S_0/K) + (r_d - r_f - \sigma^2(0.5))T}{\sigma\sqrt{T}}\right)$$

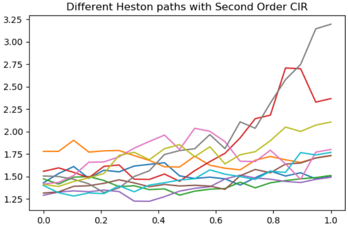
Using the Heston Model - Calibration

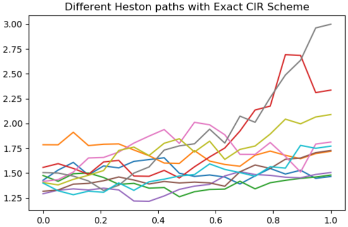
Basically, it is necessary to compute the associated strike for each option in order to compute the following equation:

$$\min_{\theta,\sigma,\rho,\kappa,\eta,\mu} = (BS(\sigma, r, K, S_0, T) - P_{heston}(\theta, \sigma, \rho, \kappa, \eta, \mu, r, K, S_0, T))^2$$

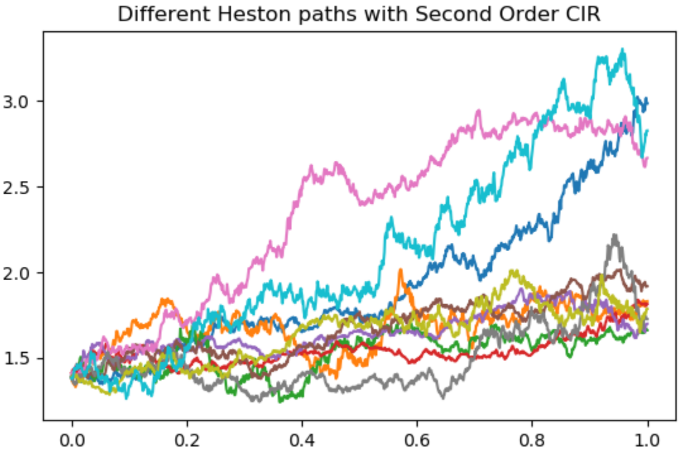
When the option is traded in the OTC market, each smile has an associated level of strike. When the option is traded in an exchange market all smiles have the same level of strike.

Using the Heston Model

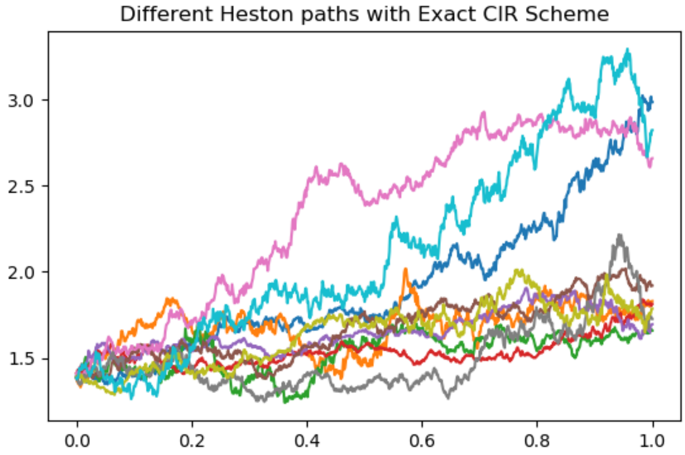




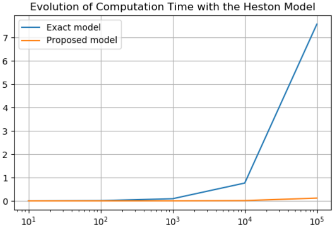
Using the Heston Model

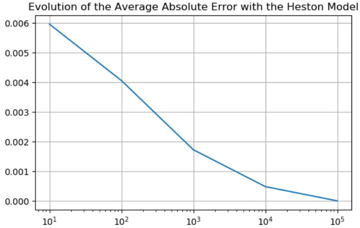


Using the Heston Model



Using the Heston Model





Conclusion

- We have succesfully implemented Aurléien Alfonsi's work on CIR second order discretization schemes.
- We have tried to calibrate and to simulate the Heston model.
- Beyond our work: how would it behave with path-dependent options?

Thank you!

`github.com/tjespel/discretization-cir-processes`