

# Numerical Methods for Finance

## Second Order Discretization Schemes for CIR Processes

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# Outline

- 1 Discretizing CIR Processes
- 2 Schemes for the CIR process
- 3 Using the Heston Model

# Introduction

- What is the problem
- What does it matter?
- How we solved it = What we are going to tell you

- 1 Discretizing CIR Processes
- 2 Schemes for the CIR process
- 3 Using the Heston Model

# Problem setting

Noting  $X_t^x$  the solution of the CIR SDE:

$$\begin{aligned}dX_t^x &= a - kX_t^x + \sigma\sqrt{X_t^x}dW_t \\ X_0^x &= x, x \in \mathbb{R}_+\end{aligned}$$

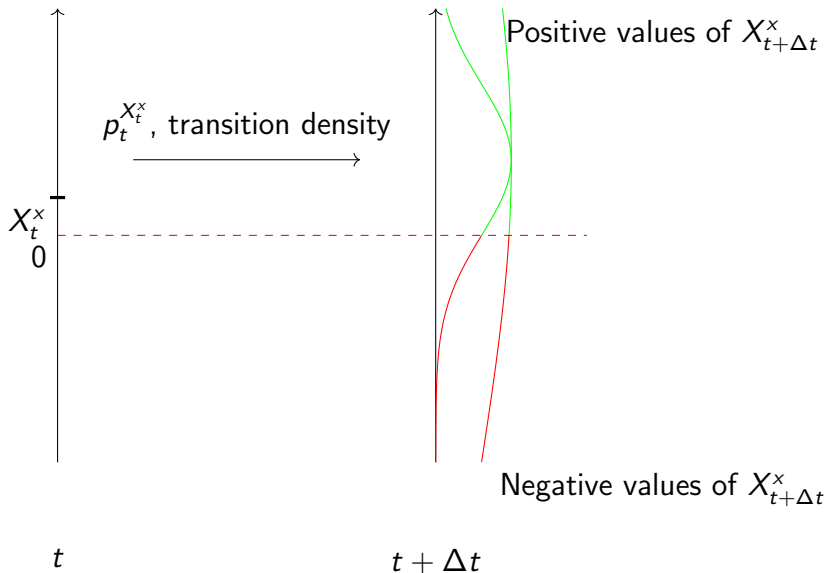
Qualitative analysis: when  $X$  is very close to 0, then the SDE becomes approximately

$$dX_t^x \approx a + \sigma\sqrt{X_t^x}dW_t$$

There are two possible regimes:

- If  $\sigma \ll a$ :  $X$  will mostly stay positive
- If  $\sigma \gg a$ :  $X$  may become negative!

# How usual numerical schemes can fail



Regime  $\sigma \ll a$  ( $\frac{\sigma^2}{4} \leq a$ )

How do we keep the process positive and the scheme precise?  
Replace the original distribution by another with compact support. The higher the degree of precision, the more this variable has to account for the tail behaviour of the substituted one.

Alfonsi shows [CITE] that we keep the scheme of order  $\nu$ , if we substitute the random variable with one that matches the first  $2\nu + 1$  moments.

Using discrete random variables, we can control the tail probability and ensure the process stays positive.

Regime  $\sigma \gg a \left( \frac{\sigma^2}{4} > a \right)$

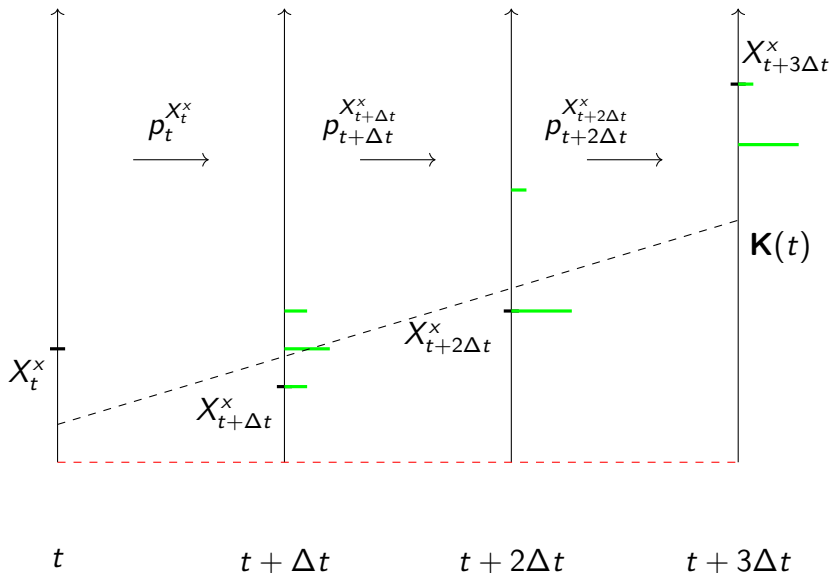
Alonfsi proves [CITE]:

- When  $X_t^x$  is far away from 0, then the scheme in the case  $\sigma \ll a$  is also valid
- When  $X_t^x$  is close to 0, we can approximate the process by a *positive* discrete random variable that matches the first two moments and still have a second-order scheme

And we know when to switch between the two, via a threshold **K**.



# Algorithm in practice



- 1 Discretizing CIR Processes
- 2 Schemes for the CIR process
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# Schemes for the CIR process

CIR process:

$$dr_t = a - kr_t + \sigma\sqrt{r_t}dW_t$$

The process at  $T$  features a transition noncentral chi-square distribution with  $\frac{4a}{\sigma^2}$  degrees of freedom and non-centrality parameter  $\frac{4kr_te^{-kt}}{(1-e^{-kt})\sigma^2}$ .

Exact simulation methods exist.

Why would we want to explore other schemes?

# Schemes for the CIR process

Simulation of  $dr(t) = \alpha(b - r(t)) dt + \sigma\sqrt{r(t)} dW(t)$   
on time grid  $0 = t_0 < t_1 < \dots < t_n$  with  $d = 4b\alpha/\sigma^2$

Case 1:  $d > 1$

for  $i = 0, \dots, n-1$

$$c \leftarrow \sigma^2(1 - e^{-\alpha(t_{i+1}-t_i)})/(4\alpha)$$

$$\lambda \leftarrow r(t_i)(e^{-\alpha(t_{i+1}-t_i)})/c$$

generate  $Z \sim N(0, 1)$

generate  $X \sim \chi_{d-1}^2$

$$r(t_{i+1}) \leftarrow c[(Z + \sqrt{\lambda})^2 + X]$$

end

Case 2:  $d \leq 1$

for  $i = 0, \dots, n-1$

$$c \leftarrow \sigma^2(1 - e^{-\alpha(t_{i+1}-t_i)})/(4\alpha)$$

$$\lambda \leftarrow r(t_i)(e^{-\alpha(t_{i+1}-t_i)})/c$$

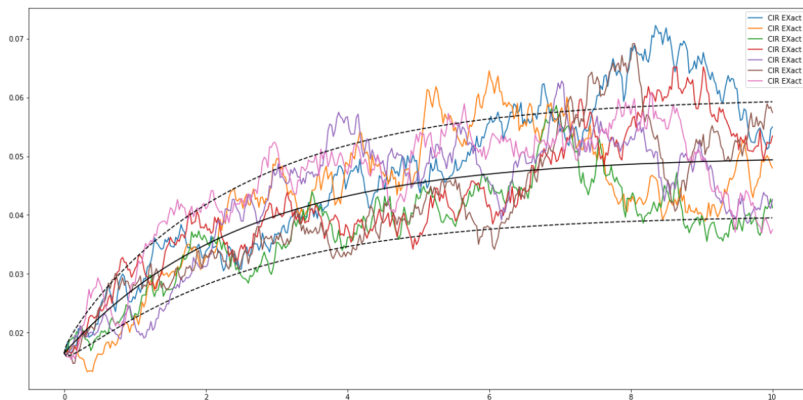
generate  $N \sim \text{Poisson}(\lambda/2)$

generate  $X \sim \chi_{d+2N}^2$

$$r(t_{i+1}) \leftarrow cX$$

end

# Schemes for the CIR process



# Schemes for the CIR process

We can simulate the process exactly provided we can sample from the non chi-square distribution which implies sampling from a Gamma distribution.

$$2X \sim \chi_\nu \text{ where } X \sim \text{gamma}(\nu/2, 1).$$

Methods for sampling a  $\text{gamma}(a, 1)$  usually distinguish between  $a \leq 1$  and  $a > 1$ . In our case  $a = \frac{2a}{\sigma^2}$ .

Exact schemes are drastically too slow if one has to simulate the process along a time-grid which occurs when pricing path-dependent options.

# Schemes for the CIR process

Alternatives:

Euler-Maruyama scheme:

$$X_{t+1}^n = X_t^n + \frac{1}{n}(a - kX_t^n) + \frac{\sigma}{\sqrt{n}}\sqrt{X_t^n}Z_t^n$$

Other schemes:

$$X_{t+1}^n = X_t^n + \frac{1}{n}(a - kX_t^n) + \frac{\sigma}{\sqrt{n}}\sqrt{(X_t^n)^+}Z_t^n$$

$$X_{t+1}^n = |X_t^n + \frac{1}{n}(a - kX_t^n) + \frac{\sigma}{\sqrt{n}}\sqrt{X_t^n}Z_t^n|$$

## Second order schemes for the CIR process

$$\sigma^2 \leq 4a$$

In this case, the *Ninomiya-Victoir* scheme for the CIR process writes:

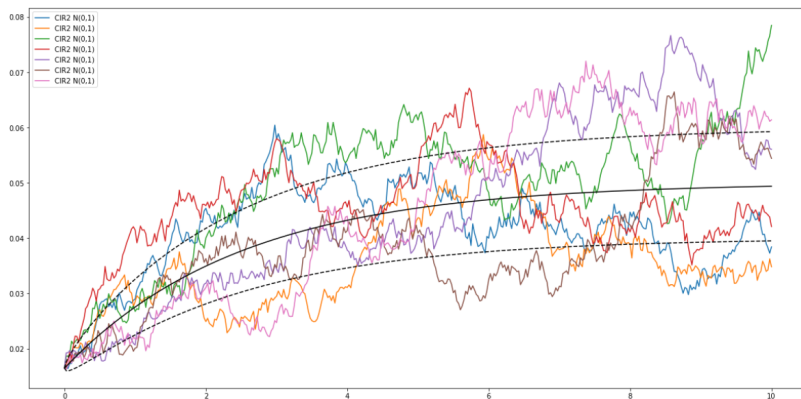
$$X_{t+1}^n = \varphi(X_t^n, \Delta, \sqrt{\Delta}Z)$$

where

$$\begin{aligned} \varphi(x, t, w) = & e^{-\frac{kt}{2}} \left( \sqrt{(a - \sigma^2/4)\phi(t/2)} + e^{-\frac{kt}{2}}x + \frac{\sigma}{2}w \right)^2 \\ & + (a - \sigma^2/4)\phi(t/2) \end{aligned}$$



# Second order schemes for the CIR process



## Second order schemes for the CIR process

$$\sigma^2 > 4a$$

In this case, the *Alfonsi* scheme for the CIR process writes:

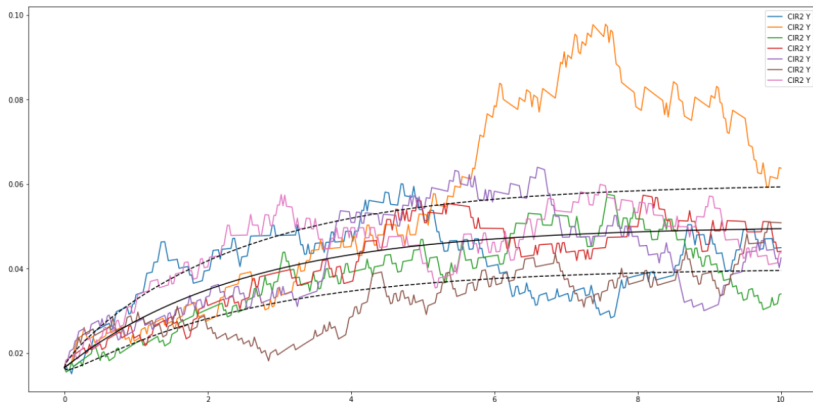
$$X_{t+1}^n = \varphi(X_t^n, \Delta, \sqrt{\Delta} Y), \quad X_t^n \geq 1_{\{\sigma^2 > 4a\}} K_2(\Delta)$$

$$X_{t+1}^n = 1_{\{U \leq \pi(\Delta, X_t^n)\}} A_1(\Delta, X_t^n) + 1_{\{U > \pi(\Delta, X_t^n)\}} A_2(\Delta, X_t^n),$$

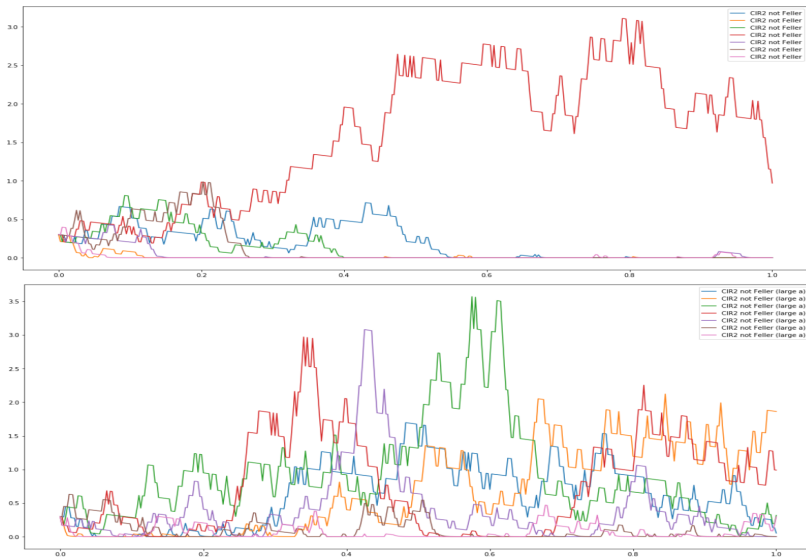
$$X_t^n < 1_{\{\sigma^2 > 4a\}} K_2(\Delta)$$

Where  $Y$  is a discrete bounded variable that fits the five first moments of a standard Gaussian variable.

# Second order schemes for the CIR process



# Second order schemes for the CIR process



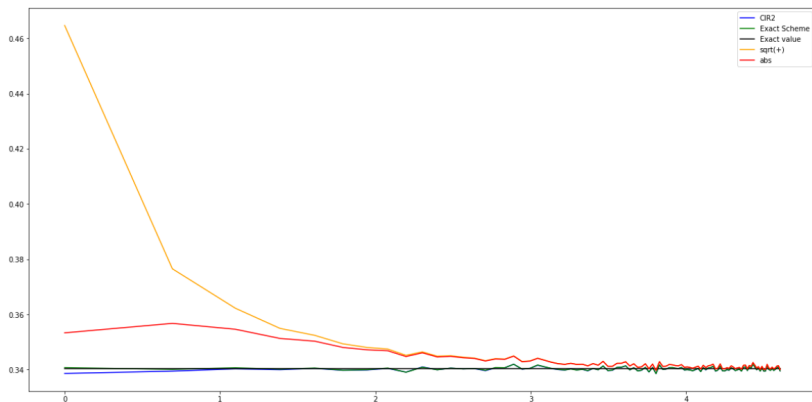
# Simulations for the CIR process

Similarly as in the paper we want to illustrate the convergence of the second order scheme and some of its possible variations.

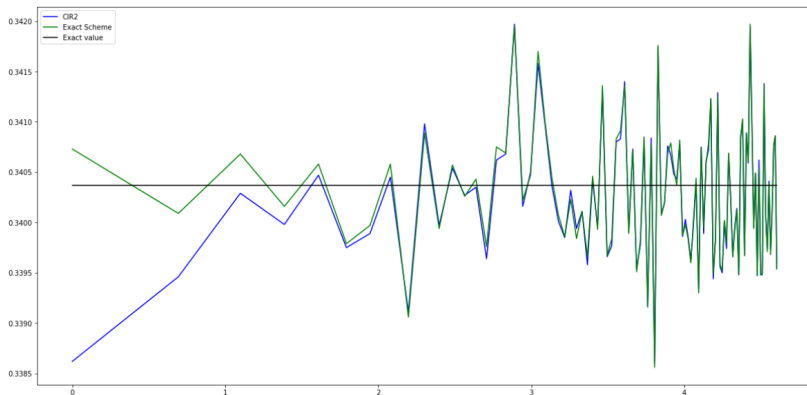
In order to do this we will compute the quantity  $\mathbb{E}[e^{-X_T^n}]$  given by the different schemes and fixing the number of paths = 100000.

To compute the theoretical value  $\mathbb{E}[e^{-X_T}]$  recall that the CIR process has a noncentral chi-square future distribution with  $\frac{4a}{\sigma^2}$  degrees of freedom and non-centrality parameter  $\frac{4kx_te^{-kt}}{(1-e^{-kt})\sigma^2}$ , so we can use the moment generating function of a chi-square distribution.

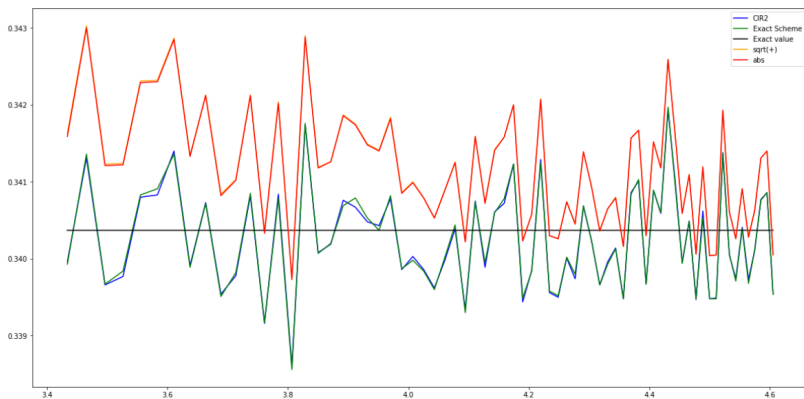
# Simulations for the CIR. Feller condition satisfied.



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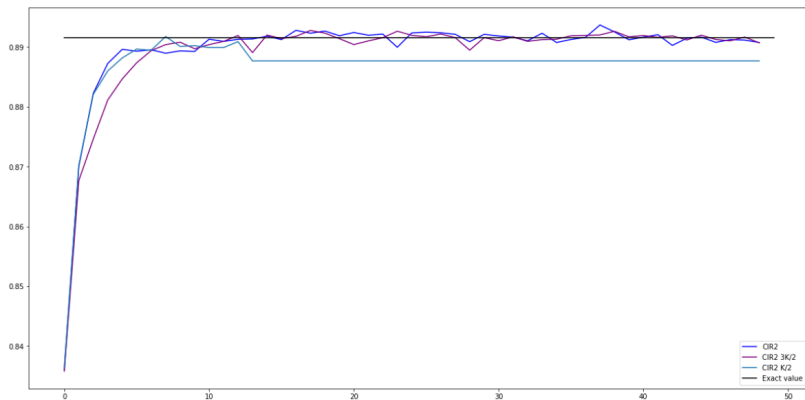


# Simulations for the CIR. Feller condition satisfied.

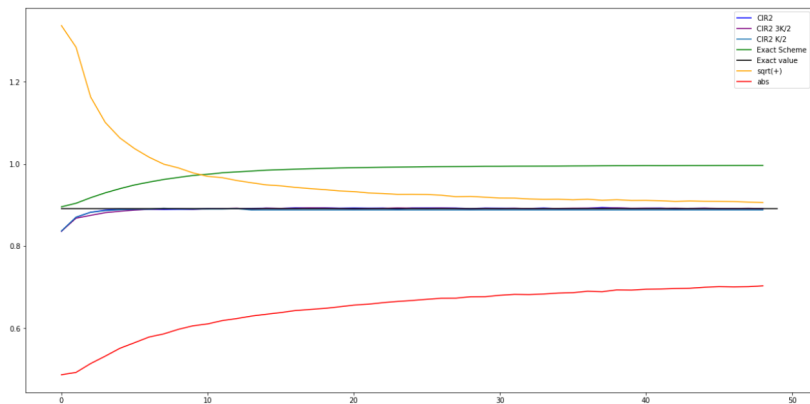




# Simulations for the CIR. Feller cond. not satisfied.



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# Using the Heston Model - Heston model

- Mathematical model implementing stochastic volatility.
- Uses an underlying CIR process for the variance.

$$\begin{aligned}dS_t &= \mu S_t dt + \sqrt{\nu_t} S_t dW_t^s \\d\nu_t &= \kappa(\theta - \nu_t)dt + \xi \sqrt{\nu_t} dW_t^\nu\end{aligned}$$

# Using the Heston Model - Heston model

There is one subtlety when generating the random numbers.

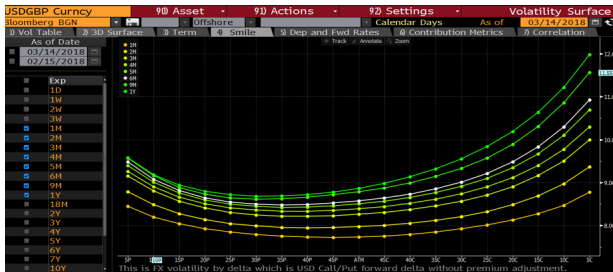
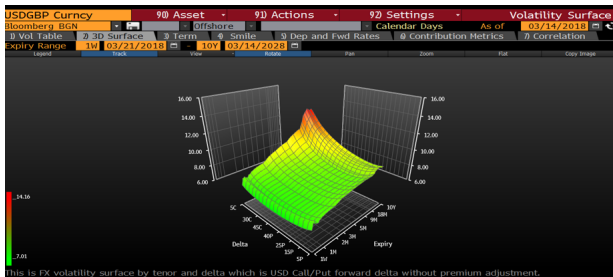
$$\Delta S_t = \mu S_t dt + \sqrt{\nu_t} S_t Z_1$$

$$\Delta \nu_t = \kappa(\theta - \nu_t)dt + \xi \sqrt{\nu_t} Z_2$$

# Using the Heston Model - Calibration

- In order to test the efficiency of the second order scheme, we calibrate the Heston Model using real data. In detail, we use a USDGBP volatility surface which was downloaded in Bloomberg.
- To calibrate the Heston model, we use the most liquid option in this market.

# Using the Heston Model - Calibration



# Using the Heston Model - Calibration

- In the currency option market, prices are quoted for moneyness levels for different time to expiry periods. These moneyness levels are:
  - ▶ At the money level or at the money forward (50 delta dual),
  - ▶ Out of the money level at 25 delta dual,
  - ▶ In the money level (75 delta dual).
- Delta dual is the first derivative with respect to the strike price.

$$\Delta_{dual} = \frac{\partial BS_{call}}{\partial k} = e^{-r_d} \Phi \left( \frac{\ln(S_0/K) + (r_d - r_f - \sigma^2(0.5))T}{\sigma\sqrt{T}} \right)$$



# Using the Heston Model - Calibration

Basically, it is necessary to compute the associated strike for each option in order to compute the following equation:

$$\min_{\theta, \sigma, \rho, \kappa, \eta, \mu} = (BS(\sigma, r, K, S_0, T) - P_{heston}(\theta, \sigma, \rho, \kappa, \eta, \mu, r, K, S_0, T))^2$$

When the option is traded in the OTC market, each smile has an associated level of strike. When the option is traded in an exchange market all smiles have the same level of strike.

# Using the Heston Model

# Conclusion

- First thing we did
- Second thing we did
- Third thing we did

# Thank you!

`github.com/tjespel/discretization-cir-processes`