Numerical Methods for Finance

Second Order Discretization Schemes for CIR Processes

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Outline

1. Discretizing CIR Processes

2. Schemes for the CIR process

3. Using the Heston Model

Introduction

- CIR schemes are frequently used.
- However, usual numerical schmees can fail when one tries to simulate them.
- We have implemented a solution proposed in 2008 by Auélien Alfonsi in his paper *High order discretization schemes for the CIR process: application to Affine Term Structure and Heston models* (hal-00143723).
- We have calibrated an dsimulated some results with the Heston model.

1 Discretizing CIR Processes

2 Schemes for the CIR process

3 Using the Heston Model

CIR Processes

Noting X_t^x the solution of the CIR (Cox-Ingersoll-Ross) SDE:

$$\begin{aligned} dX_t^x &= a - kX_t^x + \sigma\sqrt{X_t^x}dW_t \\ X_0^x &= x, x \in \mathbb{R}_+ \end{aligned}$$

- The main difficulty is to disctrize the process around 0, where since the square root is not Lipschitzian.
- Euler and Milstein schemes are usually not well defined, and generate negative values.

CIR Processes - Why it matters

- CIR processes are widely used in the industry.
- Issues with their discretization have been notices. Several works have been published on that matter, notably Ninomiya and Victoir's.
- Aurélien Alfonsi presents very efficient schemes for general affine diffusions, without any restriction on the parameters.

CIR Processes - Why it matters

Noting X_t^x the solution of the CIR (Cox-Ingersoll-Ross) SDE:

$$\begin{aligned} dX_t^x &= a - kX_t^x + \sigma\sqrt{X_t^x}dW_t \\ X_0^x &= x, x \in \mathbb{R}_+ \end{aligned}$$

Large values of σ:

- No problem when the CIR process represents the short interest rate.
- Issues when the CIR process respresents default intensity in credit risk or the stock volatility like in the Heston model.

How usual numerical schemes can fail

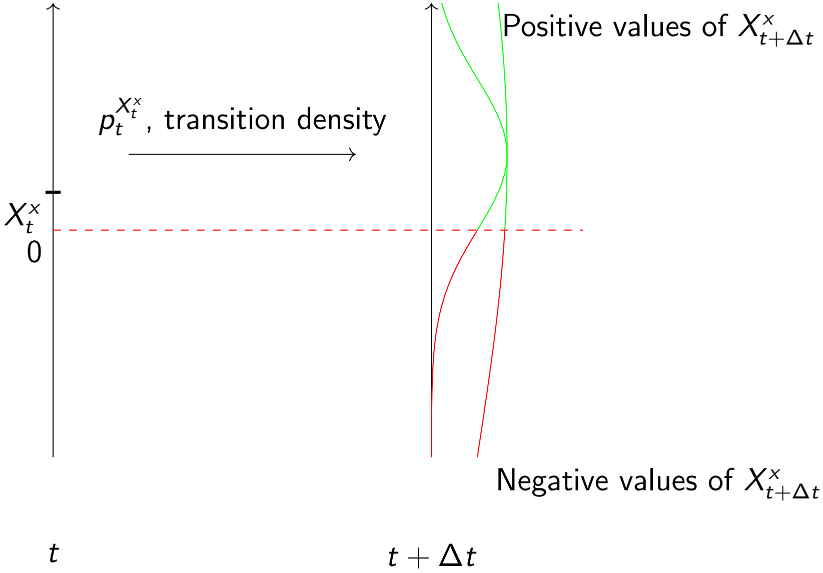
Qualitative analysis: when X is very close to 0, then the SDE becomes approximately

$$dX_t^x \approx a + \sigma \sqrt{X_t^x} dW_t$$

There are two possible regimes:

- If $\sigma << a$: X will mostly stay positive
- If $\sigma >> a$: X may become negative!

How usual numerical schemes can fail



Positive values of $X^x_{t+\Delta t}$

$p^{X^x_t}_t$, transition density

X^x_t

0

Negative values of $X^x_{t+\Delta t}$

t

$t + \Delta t$

Regime $\sigma << a$ ($\frac{\sigma^2}{4} \leq a$)

How do we keep the process positive and the scheme precise? Replace the original distribution by another with compact support. The higher the degree of precision, the more this variables has to account for the tail behaviour of the subsituted one.
Alfonsi shows [CITE] that we keep the scheme of order ν, if we subsitute the random variable with one that matches the first $2\nu + 1$ moments.
Using discrete random variables, we can control the tail probability and ensure the process stays positive.

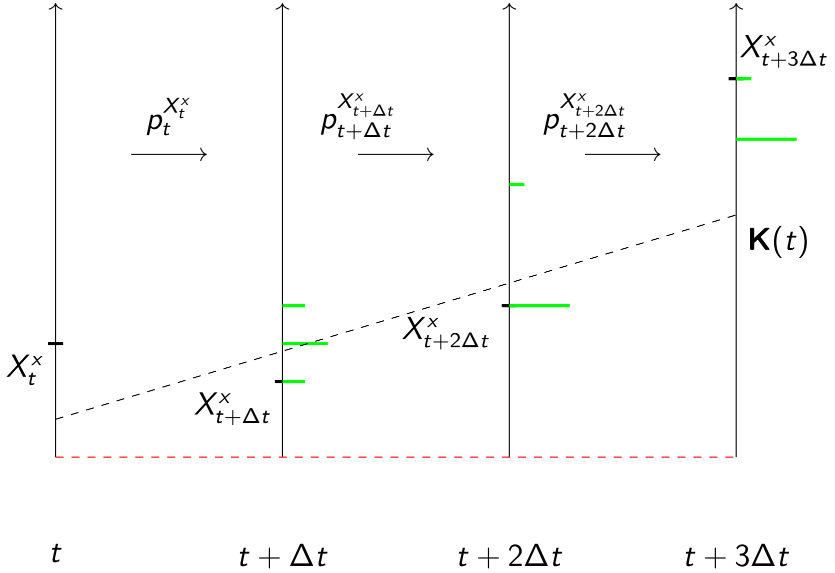
Regime $\sigma >> a$ ($\frac{\sigma^2}{4} > a$)

Alonfsi proves [CITE]:

- When X_t^x is far away from 0, then the scheme in the case $\sigma << a$ is also valid
- When X_t^x is close to 0, we can approximate the process by a *positive* discrete random variable that matches the first two moments and still have a second-order scheme

And we know when to switch between the two, via a threshold **K**.

Algorithm in practice



1. Discretizing CIR Processes

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Schemes for the CIR process

CIR process:

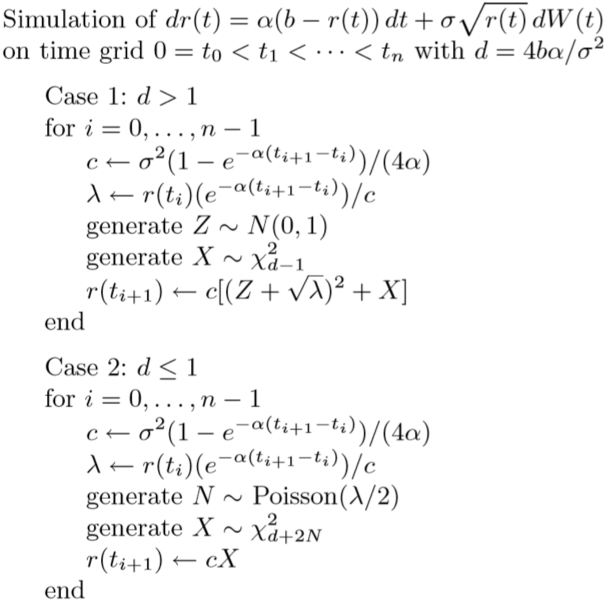
$$dr_t = a - kr_t + \sigma\sqrt{r_t}dW_t$$

The process at T features a transition noncentral chi-square distribution with $\frac{4a}{\sigma^2}$ degrees of freedom and non-centrality parameter $\frac{4kr_te^{-kt}}{(1-e^{-kt})\sigma^2}$.

Exact simulation methods exist.

Why would we want to explore other schemes?

Schemes for the CIR process

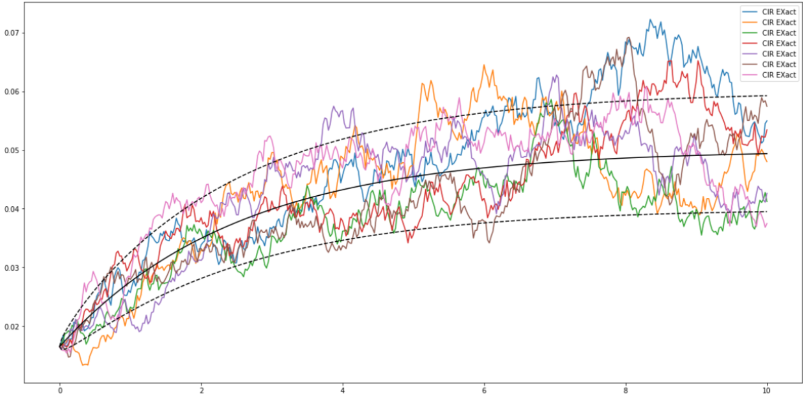


Simulation of $dr(t) = \alpha(b - r(t))\,dt + \sigma\sqrt{r(t)}\,dW(t)$ on time grid $0 = t_0 < t_1 < \cdots < t_n$ with $d = 4b\alpha/\sigma^2$

```
Case 1: $d > 1$
for $i = 0, \ldots, n-1$
    $c \leftarrow \sigma^2(1 - e^{-\alpha(t_{i+1}-t_i)})/(4\alpha)$
    $\lambda \leftarrow r(t_i)(e^{-\alpha(t_{i+1}-t_i)})/c$
    generate $Z \sim N(0,1)$
    generate $X \sim \chi^2_{d-1}$
    $r(t_{i+1}) \leftarrow c[(Z + \sqrt{\lambda})^2 + X]$
end
```

```
Case 2: $d \leq 1$
for $i = 0, \ldots, n-1$
    $c \leftarrow \sigma^2(1 - e^{-\alpha(t_{i+1}-t_i)})/(4\alpha)$
    $\lambda \leftarrow r(t_i)(e^{-\alpha(t_{i+1}-t_i)})/c$
    generate $N \sim \text{Poisson}(\lambda/2)$
    generate $X \sim \chi^2_{d+2N}$
    $r(t_{i+1}) \leftarrow cX$
end
```

Schemes for the CIR process



Schemes for the CIR process

We can simulate the process exactly provided we can sample from the non chi-square distribution which implies sampling from a Gamma distribution.

$2X \sim \chi_\nu$ where $X \sim gamma(\nu/2, 1)$.

Methods for sampling a $gamma(a, 1)$ usually distinguish between $a \leq 1$ and $a > 1$. In our case $a = \frac{2a}{\sigma^2}$.

Exact schemes are drastically too slow if one has to simulate the process along a time-grid which occurs when pricing path-dependent options.

Schemes for the CIR process

Alternatives:

Euler-Maruyama scheme:

$$X_{t+1}^n = X_t^n + \frac{1}{n}(a - kX_t^n) + \frac{\sigma}{\sqrt{n}}\sqrt{X_t^n}Z_t^n$$

Other schemes:

$$X_{t+1}^n = X_t^n + \frac{1}{n}(a - kX_t^n) + \frac{\sigma}{\sqrt{n}}\sqrt{(X_t^n)^+}Z_t^n$$

$$X_{t+1}^n = |X_t^n + \frac{1}{n}(a - kX_t^n) + \frac{\sigma}{\sqrt{n}}\sqrt{X_t^n}Z_t^n|$$

Second order schemes for the CIR process

$$\sigma^2 \leq 4a$$

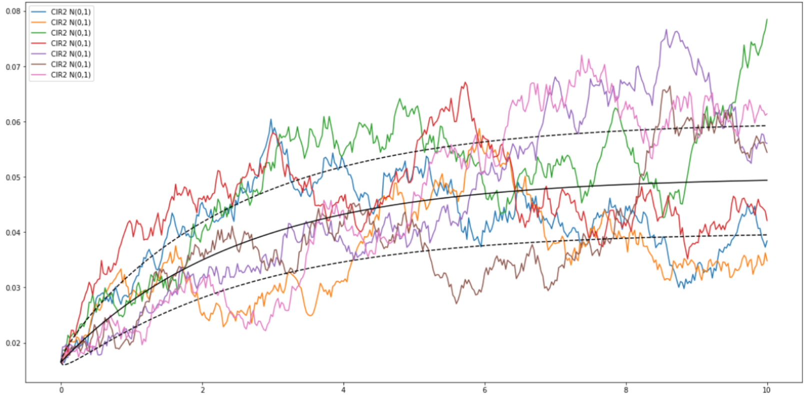
In this case, the *Ninomiya-Victoir* scheme for the CIR process writes:

$$X_{t+1}^n = \varphi(X_t^n, \Delta, \sqrt{\Delta} Z)$$

where

$$\varphi(x,t,w) = e^{-\frac{kt}{2}} \left(\sqrt{(a - \sigma^2/4)\phi(t/2) + e^{-\frac{kt}{2}} x} + \frac{\sigma}{2} w\right)^2 + (a - \sigma^2/4)\phi(t/2)$$

Second order schemes for the CIR process



Second order schemes for the CIR process

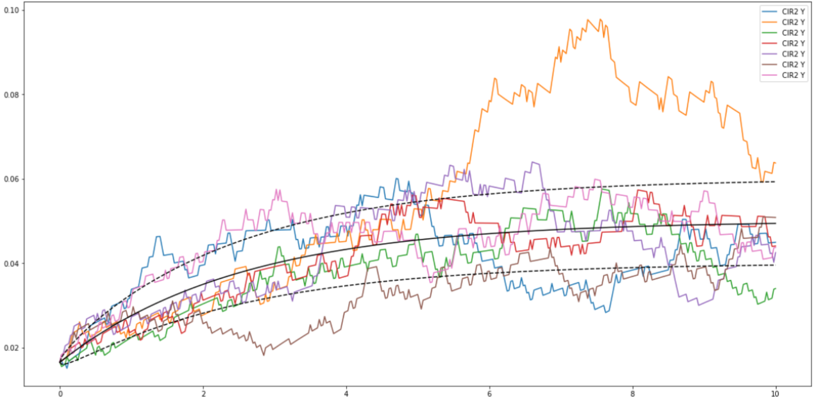
$\sigma^2 > 4a$

In this case, the *Alfonsi* scheme for the CIR process writes:

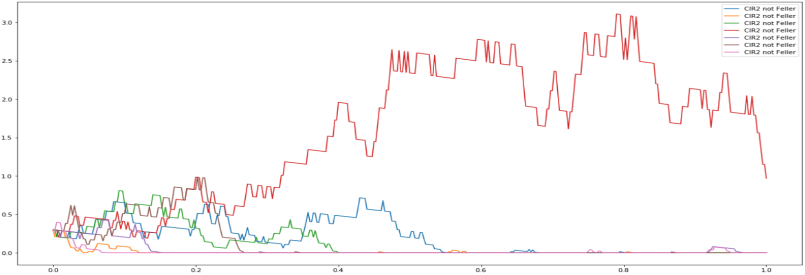
$$\begin{aligned}
X_{t+1}^n =& \varphi(X_t^n, \Delta, \sqrt{\Delta} Y), & X_t^n \geq 1_{\{\sigma^2 > 4a\}} K_2(\Delta) \\
X_{t+1}^n =& 1_{\{U \leq \pi(\Delta, X_t^n)\}} A_1(\Delta, X_t^n) + 1_{\{U \ > \pi(\Delta, X_t^n)\}} A_2(\Delta, X_t^n), & \\
& & X_t^n < 1_{\{\sigma^2 > 4a\}} K_2(\Delta)
\end{aligned}$$

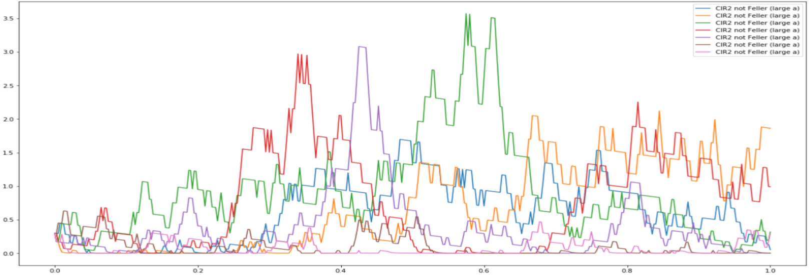
Where Y is a discrete bounded variable that fits the five first moments of a standard Gaussian variable.

Second order schemes for the CIR process



Second order schemes for the CIR process





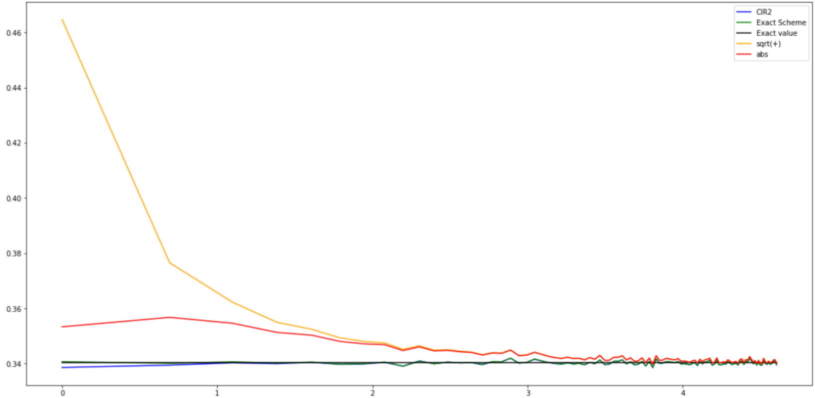
Simulations for the CIR process

Similarly as in the paper we want to illustrate the convergence of the second order scheme and some of its possible variations.

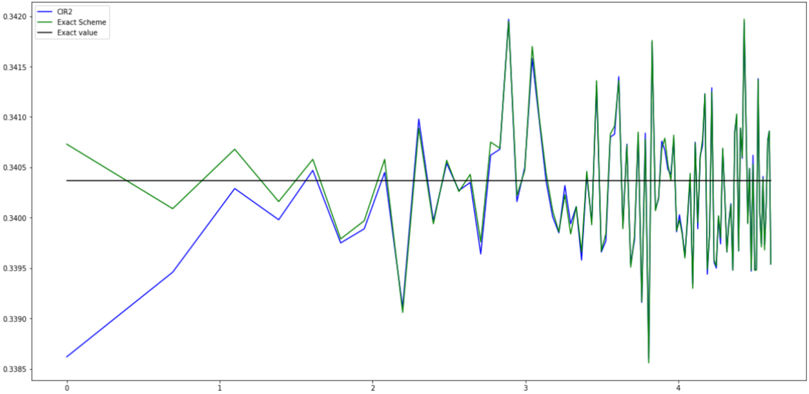
In order to do this we will compute the quantity $\mathbb{E}[e^{-X_T^n}]$ given by the different schemes and fixing the number of paths $= 100000$.

To compute the theoretical value $\mathbb{E}[e^{-X_T}]$ recall that the CIR process has a noncentral chi-square future distribution with $\frac{4a}{\sigma^2}$ degrees of freedom and non-centrality parameter $\frac{4kx_t e^{-kt}}{(1-e^{-kt})\sigma^2}$, so we can use the moment generating function of a chi-square distribution.

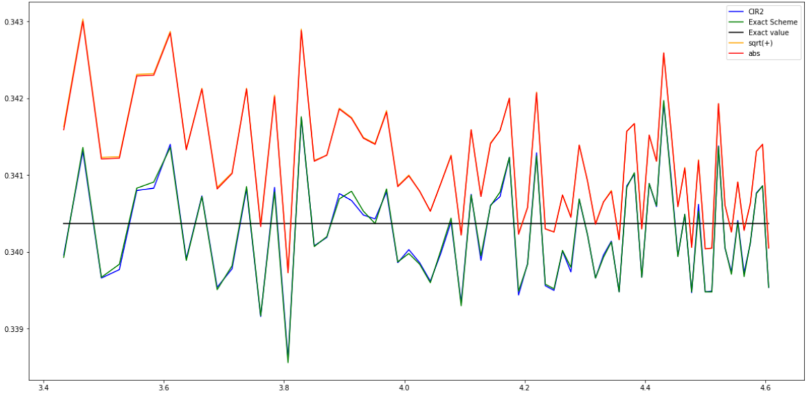
Simulations for the CIR. Feller condition satisfied.



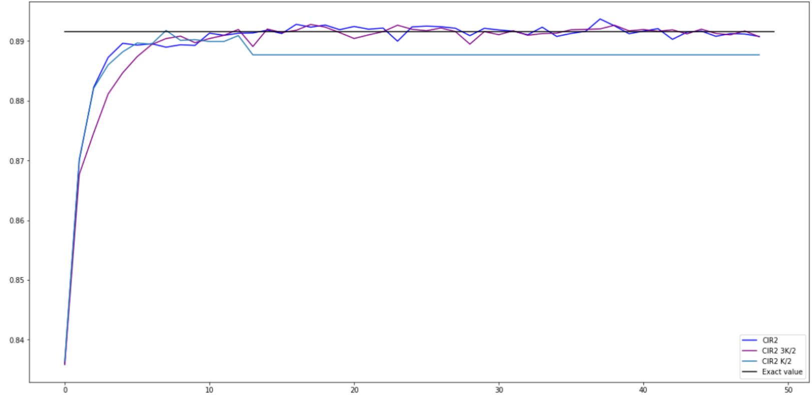
Simulations for the CIR. Feller condition satisfied.



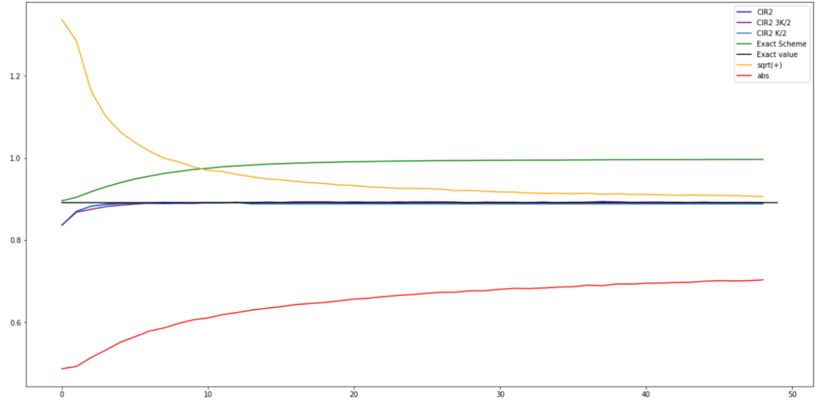
Simulations for the CIR. Feller condition satisfied.



Simulations for the CIR. Feller cond. not satisfied.



Simulations for the CIR. Feller cond. not satisfied.



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Using the Heston Model - Heston model

- Mathematical model implementing stochastic volatility.
- Uses an underlying CIR process for the variance.

$$dS_t = \mu S_t dt + \sqrt{\nu_t} S_t dW_t^s$$

$$d\nu_t = \kappa(\theta - \nu_t) dt + \xi \sqrt{\nu_t} dW_t^\nu$$

Using the Heston Model - Heston model

There is one subtlety when generating the random numbers.

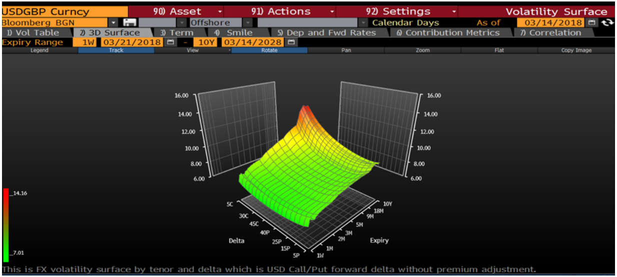
$$\Delta S_t = \mu S_t dt + \sqrt{\nu_t} S_t Z_1$$

$$\Delta \nu_t = \kappa(\theta - \nu_t) dt + \xi \sqrt{\nu_t} Z_2$$

Using the Heston Model - Calibration

- In order to test the efficiency of the second order scheme, we calibrate the Heston Model using real data. In detail, we use a USDGBP volatility surface which was downloaded in Bloomberg.
- To calibrate the Heston model, we use the most liquid option in this market.

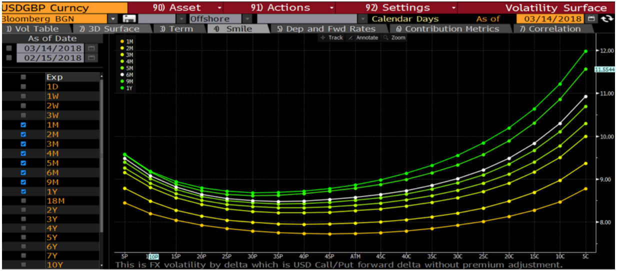
Using the Heston Model - Calibration



As of 03/14/2018

Expiry Range 1W 03/21/2018 - 10Y 03/14/2028

This is FX volatility surface by tenor and delta which is USD Call/Put forward delta without premium adjustment.



As of 03/14/2018

As of Date

□ 03/14/2018

□ 02/15/2018

	Exp
□	1D
□	1W
□	2W
□	3W
☑	1M
☑	2M
☑	3M
☑	4M
☑	5M
☑	6M
☑	9M
☑	1Y
□	18M
□	2Y
□	3Y
□	4Y
□	5Y
□	6Y
□	7Y
□	10Y

This is FX volatility by delta which is USD Call/Put forward delta without premium adjustment.

Using the Heston Model - Calibration

- In the currency option market, prices are quoted for moneyness levels for different time to expiry periods. These moneyness levels are:
  - At the money level or at the money forward (50 delta dual),
  - Out of the money level at 25 delta dual,
  - In the money level (75 delta dual).
- Delta dual is the first derivative with respect to the strike price.

$$\Delta_{dual} = \frac{\partial BS_{call}}{\partial k} = e^{-r_d}\Phi\left(\frac{ln(S_0/K) + (r_d - r_f - \sigma^2(0.5))T}{\sigma\sqrt{T}}\right)$$

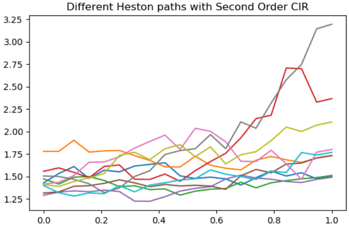
Using the Heston Model - Calibration

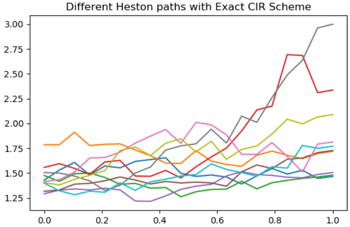
Basically, it is necessary to compute the associated strike for each option in order to compute the following equation:

$$\underset{\theta,\sigma,\rho,\kappa,\eta,\mu}{min} = (BS(\sigma, r, K, S_0, T) - P_{heston}(\theta, \sigma, \rho, \kappa, \eta, \mu, r, K, S_0, T))^2$$

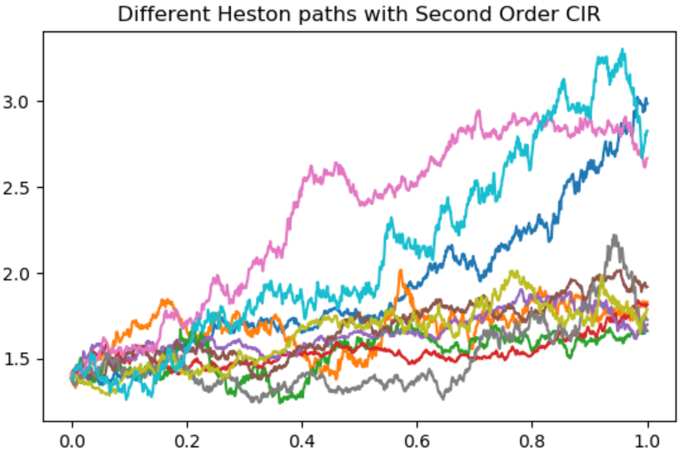
When the option is traded in the OTC market, each smile has an associated level of strike. When the option is traded in an exchange market all smiles have the same level of strike.

Using the Heston Model

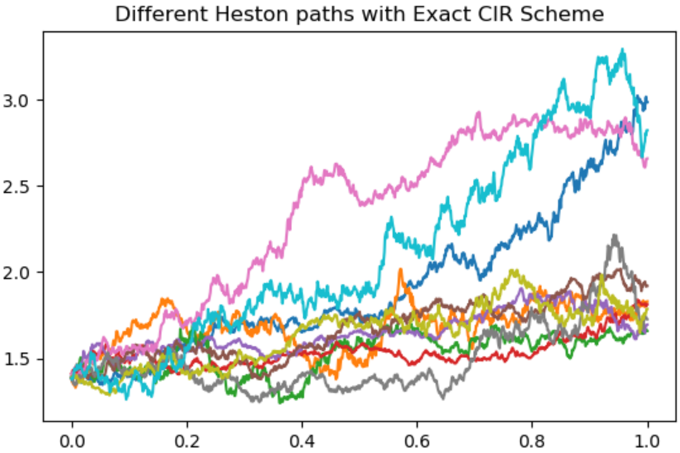

Different Heston paths with Second Order CIR


Different Heston paths with Exact CIR Scheme

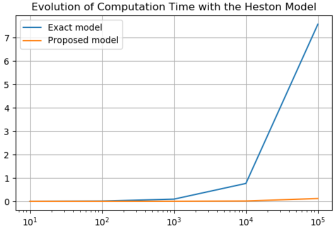
Using the Heston Model

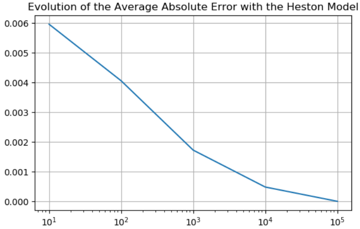


Using the Heston Model



Using the Heston Model





Conclusion

- We have succesfully implemented Aurléien Alfonsi's work on CIR second order discretization schemes.
- We have tried to calibrate and to simulate the Heston model.
- Beyond our work: how would it behave with path-dependent options?

Thank you!

`github.com/tjespel/discretization-cir-processes`