Is this correct?

1 Given:

$$E_{Killing}[\Psi] = \sum_{x,y,z} vec(J_\Psi)^T vec(J_\Psi) + \gamma vec(J_\Psi^T)^T vec(J_\Psi) = \sum_{x,y,z} \mathcal{L}[\Psi]$$

$$\mathcal{L}[\Psi(x,y,z)] = (1+\gamma)u_x^2 + v_x^2 + w_x^2 + u_y^2 + (1+\gamma)v_y^2 + w_y^2 + u_z^2 + v_z^2 + (1+\gamma)w_z^2 + 2\gamma u_y v_x + 2\gamma u_z w_x + 2\gamma v_z w_y$$

2 Solution

Functional derivatives for $E[\Psi]$:

$$\frac{\partial E_{Killing}}{\partial u} = \frac{\partial \mathcal{L}}{\partial u} - \left(\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial u_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \mathcal{L}}{\partial u_y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \mathcal{L}}{\partial u_z}\right)\right)$$

$$\frac{\partial E_{Killing}}{\partial v} = \frac{\partial \mathcal{L}}{\partial v} - \left(\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial v_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \mathcal{L}}{\partial v_y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \mathcal{L}}{\partial v_z}\right)\right)$$

$$\frac{\partial E_{Killing}}{\partial w} = \frac{\partial \mathcal{L}}{\partial w} - \left(\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial w_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \mathcal{L}}{\partial w_y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \mathcal{L}}{\partial w_z}\right)\right)$$

Solving the first equation:

$$\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial u_x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial (1+\gamma)u_x^2)}{\partial u_x}\right) = \frac{\partial}{\partial x} 2(1+\gamma)u_x = 2(1+\gamma)u_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial \mathcal{L}}{\partial u_y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial (u_y^2 + 2\gamma u_y v_x)}{\partial u_y}\right) = \frac{\partial}{\partial y}(2u_y + 2\gamma v_x) = 2u_{yy} + 2\gamma v_{xy}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial \mathcal{L}}{\partial u_z}\right) = \frac{\partial}{\partial z}\left(\frac{\partial (u_z^2 + 2\gamma u_z w_x)}{\partial u_z}\right) = \frac{\partial}{\partial y}(2u_z + 2\gamma w_x) = 2u_{zz} + 2\gamma w_{xz}$$

$$\frac{\partial E_{Killing}}{\partial u} = \frac{\partial \mathcal{L}}{\partial u} - (2(1+\gamma)u_{xx} + 2u_{yy} + 2\gamma v_{xy} + 2u_{zz} + 2\gamma w_{xz})$$