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A numerical method with the posteriori error estimates for fractional integro-differential equations

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Contents

Fractional integro-differential equations

Fractional integro-differential equation given as

$$D^\alpha y(t) = y(t) + \int_0^t K(t,s)y(s)ds + g(t) \tag{1}$$

subject to initial condition

$$y(0) = c \tag{2}$$

Preliminaries

Definition

[1] Let $0 \leq \alpha$ and $m = \lceil \alpha \rceil$. The fractional derivative of $f(t)$ in the Caputo sense is defined as follows

$$
\begin{aligned}
D_*^{\alpha} f(t) &= D^m J^{m-\alpha} f(t) \\
&= \begin{cases} \dfrac{d^m f(t)}{dt^m} & m = \alpha \\ \dfrac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} f^{(m)}(s) ds & m = \lceil \alpha \rceil . \end{cases}
\end{aligned}
$$

where $\Gamma(\cdot)$ denotes the gamma function .

[1] **podlubny1998fractional**.

Preliminaries

Corollary

[2] *From Def.1 the Caputo derivative satisfies the following result*

$$D_a^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha} \quad 0 \leq \alpha,\ -1 < \gamma,\ 0 < t.$$

Definition

[3] The Hermite polynomials are defined as follows

$$H_n(t) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k n! 2^{n-2k}}{(n-2k)!k!} t^{n-2k} = \sum_{k=0}^{[\frac{n}{2}]} \beta_{n,k} t^{n-2k}. \tag{3}$$

[2] **diethelm2010analysis**, **mainardi2000mittag**, **almeida2019variable**.
[3] **szego1975orthogonal**.

Preliminaries

The approximate solution of the Eq. (1) can be expressed in the following truncated Hermite series form

$$y(t) \cong y_N(t) = \sum_{n=0}^{N} c_n H_n(t^\alpha) \tag{4}$$

where $N \in \mathbb{Z}^+$ and c_n are the unknown Hermite coefficients.
Firstly, the solution of Eq. (1) is indicated in matrix form as follows

$$y_{N,\alpha}(t) = \mathbf{H}_{N,\alpha}(t)\mathbf{C}_N, \tag{5}$$

where

$$\mathbf{H}_{N,\alpha}(t) = [H_0(t^\alpha) H_1(t^\alpha) \cdots H_N(t^\alpha)], \quad \mathbf{C}_N = [c_0\; c_1\; \ldots\; c_N]^T, \quad i = 0, \ldots, N.$$

Numerical Method

Using the Eq. (3), the vector $\mathbf{H}_{N,\alpha}(t)$ is expressed as follows

$$\mathbf{H}_{N,\alpha}(t) = \mathbf{X}_N(t^\alpha)\mathbf{F}_N \tag{6}$$

where

$$\mathbf{X}_N(t^\alpha) = [1 \; t^{1\alpha} \; \ldots \; t^{N\alpha}]$$

$$\mathbf{F}_N^T = \begin{array}{r} \\ \\ \\ \\ \text{N is even} \leftarrow \\ \text{N is odd} \leftarrow \end{array} \begin{bmatrix} \beta_{0,0} & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & \beta_{1,0} & 0 & 0 & \ldots & 0 & 0 \\ \beta_{2,1} & 0 & \beta_{2,0} & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ \beta_{N,N/2} & 0 & \beta_{N,N/2-1} & 0 & \ldots & 0 & \beta_{N,0} \\ 0 & \beta_{N,(N-1)/2} & 0 & \beta_{N,((N-1)/2)-1} & \ldots & 0 & \beta_{N,0} \end{bmatrix}$$

Numerical Method

By substituting Eq. (6) into Eq. (5), we have

$$y_{N,\alpha}(t) = \mathbf{X}_N(t^\alpha)\mathbf{F}_N\mathbf{C}_N. \tag{7}$$

Lemma

The α-th order Caputo fractional derivative of Eq. (1) is constructed in matrix form as

$$D^\alpha y_{N,\alpha}(t) = \mathbf{X}_N(t^\alpha)\mathbf{P}_N\mathbf{F}_N\mathbf{C}_N. \tag{8}$$

Numerical Method

Proof.

With the help of Eq. (7), we have

$$D^{\alpha} y_{N,\alpha}(t) = D^{\alpha} \mathbf{X}_N(t^{\alpha}) \mathbf{F}_N \mathbf{C}_N. \tag{9}$$

From Corollary

$$\underbrace{\begin{bmatrix} D^{\alpha} t^0 \\ D^{\alpha} t^{1\alpha} \\ D^{\alpha} t^{2\alpha} \\ \dots \\ D^{\alpha} t^{N\alpha} \end{bmatrix}}_{(D^{\alpha}\mathbf{X}_N(t^{\alpha}))^T} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{\Gamma(\alpha+1)}{\Gamma(1)} & 0 & 0 & 0 & 0 \\ 0 & \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \frac{\Gamma(N\alpha+1)}{\Gamma((N-1)\alpha+1)} & 0 \end{bmatrix}}_{\mathbf{P}_N^T} \underbrace{\begin{bmatrix} t^0 \\ t^{1\alpha} \\ t^{2\alpha} \\ \dots \\ t^{N\alpha} \end{bmatrix}}_{(\mathbf{X}_N(t^{\alpha}))^T}$$

Numerical Method

Proof.

So,

$$D^\alpha \mathbf{X}_N(t^\alpha) = \mathbf{X}_N(t^\alpha)\mathbf{P}_N. \tag{10}$$

Hence, the desired result is proved by substituting Eq. (10) into Eq. (9). □

Numerical Method

Lemma

The integral part of the Eq. (1) is constructed in matrix form as

$$I_N(t) = \mathbf{X}_N(t)\mathbf{K}_N\mathbf{Q}_N\mathbf{F}_N\mathbf{C}_N. \tag{11}$$

Numerical Method

Proof.

By using Taylor polynomials, the kernel function can be formed by (see[4])

$$K_N(t,s) = \mathbf{X}_N(t)\mathbf{K}_N\mathbf{X}_N^T(s) \tag{12}$$

where

$$\mathbf{K}_N = [k_{ij}], k_{ij} = \frac{1}{i!j!}\frac{\partial^{i+j}K(t,s)}{\partial t^i \partial t^i}.$$

By using Eq. (7) and Eq. (12), the integral part of Eq. (1) is contructed as

$$I(t) = \int_0^t \mathbf{X}_N(t)\mathbf{K}_N\mathbf{X}_N^T(s)\mathbf{X}_N(s^\alpha)dt\mathbf{F}_N\mathbf{C}_N = \mathbf{X}_N(t)\mathbf{K}_N\int_0^t \mathbf{X}_N^T(s)\mathbf{X}_N(s^\alpha)dt\mathbf{F}_N\mathbf{C}_N$$

where

$$\int_0^t \mathbf{X}_N^T(s)\mathbf{X}_N(s^\alpha)\,dt =: \mathbf{Q}_N.$$

□

[4] **karamete2002taylor**.

Numerical method

The fundamental matrix form of Eq. (1) is obtained by gathering and simplifying the matrix relations (7), (8) and (11)

$$\Big\{\mathbf{X}_N(t^\alpha)\mathbf{P}_N\mathbf{F}_N - \mathbf{X}_N(t^\alpha)\mathbf{F}_N - \mathbf{X}_N(t)\mathbf{K}_N\mathbf{Q}_N\mathbf{F}_N\Big\}\mathbf{C}_N = g(t). \tag{13}$$

By using the collocation points defined by

$$t_i = a + \left(\frac{b-a}{N}\right)i \quad \text{and} \quad i = 0, 1, 2, \ldots, N \tag{14}$$

Numerical Method

into Eq. (13), the system of the matrix equations is obtained as follows

$$\left\{\mathbf{X}_N(t_i^\alpha)\mathbf{P}_N\mathbf{F}_N - \mathbf{X}_N(t_i^\alpha)\mathbf{F}_N - \mathbf{X}_N(t_i)\mathbf{K}_N\mathbf{Q}_N\mathbf{F}_N\right\}\mathbf{C}_N = G_N(t_i). \tag{15}$$

or briefly,

$$\mathbf{W_N C_N = G_N} \quad \Leftrightarrow \quad [\mathbf{W_N} : \mathbf{G_N}] \tag{16}$$

such that

$$\mathbf{W_N} = \begin{bmatrix} \mathbf{W}_N(t_0) \\ \mathbf{W}_N(t_1) \\ \vdots \\ \mathbf{W}_N(t_N) \end{bmatrix}, \quad \mathbf{C}_N = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_N \end{bmatrix} \quad \mathbf{G}_N = \begin{bmatrix} \mathbf{g}(t_0) \\ \mathbf{g}(t_1) \\ \vdots \\ \mathbf{g}(t_N) \end{bmatrix}.$$

Numerical Method

Applying the same steps to the initial condition (2) is represented by

$$y_N(0) = \mathbf{X}_N(0)\mathbf{F}_N\mathbf{C}_N = c. \tag{17}$$

Eventually, replacing last row of (16) by (17), the required matrix equation is obtained

$$\widetilde{\mathbf{W}}_{\mathbf{N}}\mathbf{C}_{\mathbf{N}} = \widetilde{\mathbf{G}}_{\mathbf{N}}. \tag{18}$$

Numerical solvability of the Hermite algebraic system

For simplicity, we consider Eq. (1) in the following form

$$D^{\alpha(t)}y + \mathcal{L}y = g \tag{19}$$

According to the given method, the following problem is obtained

$$\left(D^{\alpha(t)} + \mathcal{L}\right) y_N = \left(g\right), i = 0, ..., N \tag{20}$$

where $\mathcal{L}$ is the linear operator.

Theorem

Assume that $(D^{\alpha} + \mathcal{L})^{-1}$ exists and $I_N y$ be the interpolation polynomial approximation of the function $y(t)$, then Eq. (20) uniquely solvable for sufficiently large N .

Numerical solvability of the Hermite algebraic system

Proof.

By multiply both sides of Eq. (20) by I and then sum up $i = 0, \ldots, N$ we have

$$I_N \left(D^\alpha + \mathcal{L}\right) y_N = I_N g.$$

Then we obtain

$$\left(D^\alpha + I_N \mathcal{L}\right) y_N = I_N g$$

so we have,

$$y_N = (D^\alpha + I_N \mathcal{L})^{-1} I_N g \tag{21}$$

where

$$\begin{aligned} D^\alpha + I_N \mathcal{L} &= D^\alpha + \mathcal{L} - \mathcal{L} + I_N \mathcal{L} \\ &= (D^\alpha + \mathcal{L})\Big(I - (D^\alpha + \mathcal{L})^{-1}(\mathcal{L} - I_N \mathcal{L})\Big). \end{aligned}$$

□

Numerical solvability of the Hermite algebraic system

With a Neumann series we have

$$
\begin{aligned}
(D^\alpha + I_N\mathcal{L})^{-1} &= \Big(I - (D^\alpha+\mathcal{L})^{-1}(\mathcal{L} - I_N\mathcal{L})\Big)^{-1}(D^\alpha+\mathcal{L})^{-1} \\
&= \sum_{n=0}^{\infty} \left((D^\alpha+\mathcal{L})^{-1}(\mathcal{L} - I_N\mathcal{L})\right)^k (D^\alpha+\mathcal{L})^{-1} \\
&= \sum_{n=0}^{\infty} a_N^k (D^\alpha+\mathcal{L})^{-1}
\end{aligned}
\tag{22}
$$

where

$$
a_N = (D^\alpha+\mathcal{L})^{-1}(\mathcal{L} - I_N\mathcal{L}).
$$

Numerical solvability of the Hermite algebraic system

Proof.

Here,

$$|| a_N || \leq \frac{|| \mathcal{L} - I_N \mathcal{L} ||}{|| D^\alpha - \mathcal{L} ||}.$$

For N large enough we have

$$|| \mathcal{L} - I_N \mathcal{L} || < || D^\alpha - \mathcal{L} || . \tag{23}$$

Since for $y \in L^2(\Omega)$, we have $I_N y \to y$ as $n \to \infty$. From [5, Lemma 3.2.1], we can deduce

$$|| \mathcal{L} - I_N \mathcal{L} || \to 0 \text{ as } n \to \infty. \tag{24}$$

Then from Eq. (23) and Eq. (24) $|| a_N || < 1$ and so the series is convergent. From Eq. (22) and [6, Theorem 3.1.1] we conclude that the operator $(D^\alpha + I_N \mathcal{L})^{-1}$ exists and is bounded. Thus, the Hermite-collocation solution of Eq. (20) exists and is unique. □

[5] **atkinson2009numerical**.
[6] **atkinson2009numerical**.

Error Estimation

The posteriori error estimation is established as follows (see[7])

$$\| e(t) \| \leq \sup_{0<s\leq t} \left\{ \frac{\| R_N(s) \|}{\mathcal{R}_0} \right\} \tag{25}$$

where

$$R_N(t) = D^{\alpha} y_N(t) - y_N(t) - \int_0^t K(t,s) y_N(s) ds - g(t)$$

$$\mathcal{R}_0(t) := \Gamma(1-\alpha)^{-1} t^{-\alpha} + \lambda.$$

[7] **kopteva2022pointwise**.

Numerical Example

Example

Consider the fractional integro-differential equation

$$\begin{cases} D^{1/3}y(t) = \dfrac{3\sqrt{\pi}t^{7/6}}{4\Gamma(13/6)} - \dfrac{2}{63}t^{9/2}(9+7t^2) + \displaystyle\int_0^t (ts+t^2s^2)y(s)ds \\ y(0) = 1 \end{cases} \tag{26}$$

The exact solution is $y(t) = t^{3/2}$. Note, that y is not a polynomial of t^α.

Numerical example

Table 1: Comparison of the absolute errors for the Example for $N = 5$

t	LS[a]	QS[b]	LQS[c]	Proposed Method
0.2	$9.76000e-03$	$9.76000e-03$	$9.76000e-03$	$1.20443e-04$
0.4	$1.08590e-02$	$4.80300e-03$	$4.86700e-03$	$6.56595e-05$
0.6	$1.18490e-02$	$2.87900e-03$	$3.18400e-03$	$5.54591e-05$
0.8	$1.41180e-02$	$2.63700e-03$	$3.52300e-03$	$5.33538e-05$
1	$1.96700e-02$	$3.27000e-03$	$5.47000e-03$	$9.86918e-05$

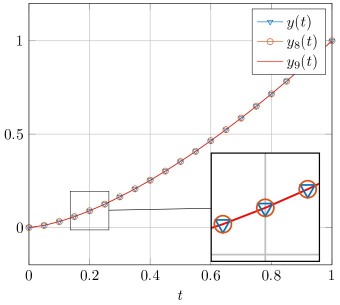
[a] **kumar2017comparative**.
[b] **kumar2017comparative**.
[c] **kumar2017comparative**.

Numerical example

Table 2: Numerical results of the error computations versus N for the Example

N	L_∞	$\mid e_{N,1/3}(0.5) \mid$	$\mid e_{N,1/3}(1) \mid$
4	$5.29335e-03$	$8.52679e-04$	$4.41351e-04$
6	$1.09623e-04$	$1.07048e-05$	$9.84619e-06$
8	$1.54955e-05$	$9.52838e-07$	$1.06680e-06$
10	$4.00965e-06$	$1.69629e-07$	$1.93629e-07$
12	$1.41739e-06$	$4.37179e-08$	$5.00016e-08$
14	$6.73800e-07$	$1.63708e-08$	$1.94436e-08$

Numerical examples



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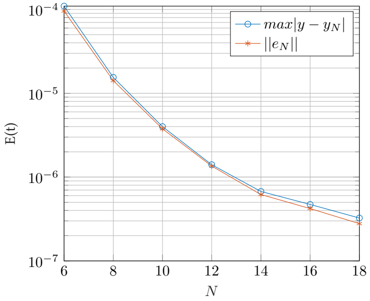

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Figure 1: Comparison of numerical solutions of the Example with the exact solution in terms of N on $[0, 1]$ (a). Logarithmic scaled plot of the error estimation $|| \; e_N \; ||$ and the max absolute error $max \mid y - y_N \mid$ with respect to N for the Example (b).

Numerical example

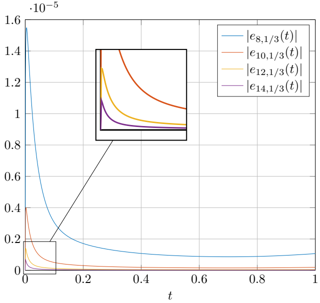

Figure 2: The comparison of the absolute error functions of the Example for $N = 8,\ 10,\ 12$ and 14

Summary

- a matrix-collocation method based on the Hermite polynomials is constructed.
- Solvability of the linear system, error analysis, and convergence analysis based on the posteriori error estimator of the proposed method are also discussed.
- The method's convergence with respect to N is investigated numerically.

Thank you!