Port-Hamiltonian systems with time-delays

Absolventenseminar WS 22/23

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Motivation

Goal

Formulation of a port-Hamiltonian systems for delay differential-algebraic equations (DDAE)

Image 1: https://www.haus.de/test/haushalt/koerperpflege/duschkoepfe-31078,30.11.21
Image 2: https://www.bbz-arnsberg.de/aktuelles/2019/06/zum-kran-profi-in-zwei-tagen,30.11.21

Port-Hamiltonian formulation for standard systems

standard systems

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t),$$

with $A \in \mathbb{R}^{n\times n}$, B, $C^\top \in \mathbb{R}^{n\times m}$

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port-Hamiltonian system (pH)

$$\dot{x}(t) = (J - R)Hx(t) + Bu(t)$$
$$y(t) = B^\top Hx(t)$$

with $J = -J^\top$, $R = R^\top \geq 0$, $H = H^\top \geq 0$
and *Hamiltonian* $\mathcal{H}(x) = \frac{1}{2}x^\top Hx$

Properties

Properties

- Hamiltonian is explicitly included in system

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- Hamiltonian is explicitly included in system
- stability

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- stability
- structure-preserving with interconnection

Properties

- Hamiltonian is explicitly included in system
- stability
- structure-preserving with interconnection
- passive

Definition

A system is called passive if there exists a state-dependent storage function $\mathcal{H}\colon \mathbb{R}^n \to \mathbb{R}$ such that the dissipation inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(z(t)) \leq y(t)^\top u(t)$$

is satisfied for any $t > 0$.

Equivalences to pH systems of standard systems

Theorem — e.g. Cherifi et al. '22

For a minimal standard system the following are equivalent:

- The system is a pH system
- The system is passive.
- The Kalman-Yakubovich-Popov (KYP) inequality

$$\mathcal{W}(H) := \begin{bmatrix} -A^\top H - HA & C^\top - HB \\ C - B^\top H & 0 \end{bmatrix} \geq 0$$

has a symmetric positive definite solution $H \in \mathbb{R}^{n \times n}$.

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$$\begin{aligned} H\dot{x}(t) &= HAx(t) + HBu(t) \\ y(t) &= \quad Cx(t) \end{aligned}$$

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$$H\dot{x}(t) = \left(\frac{HA - A^\top H^\top}{2} - \frac{-HA - A^\top H^\top}{2} \right) x(t) + Gu(t)$$

$$y(t) = G^\top x(t)$$

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$$\begin{aligned} H\dot{x}(t) &= (J - R)x(t) + Gu(t) \\ y(t) &= G^\top x(t) \end{aligned}$$

Time-delayed system

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + A_1 x(t-\tau) + Bu(t), \\ y(t) &= Cx(t), \end{aligned}$$

with $A_0, A_1 \in \mathbb{R}^{n\times n}$, B, $C^\top \in \mathbb{R}^{n\times m}$

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Idea: Infinite-dimensional representation

Time-delayed system

$$
\begin{aligned}
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\end{aligned}
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with $A_0, A_1 \in \mathbb{R}^{n\times n}$, $B,\ C^\top \in \mathbb{R}^{n\times m}$

Idea: Infinite-dimensional representation

Plan:

- Rewrite as an infinite-dimensional system
- Apply infinite-dimensional KYP
- Transform to pH system with the help of infinite-dimensional KYP

Time-delayed system to infinite-dimensional system

$$
\begin{aligned}
\dot{x}(t) &= A_0 x(t) + A_1 x(t-\tau) + Bu(t), \\
y(t) &= Cx(t), \\
x(t) &= \phi(t) \qquad \text{for } t \in [-\tau, 0]
\end{aligned}
$$

Time-delayed system to infinite-dimensional system

$$
\begin{aligned}
\dot{x}(t) &= A_0 x(t) + A_1 x(t-\tau) + Bu(t), \\
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$$

Hilbert space: $\mathcal{Z}_{n;\tau} := \mathbb{R}^n \times L^2([-\tau,0];\mathbb{R}^n)$ with $\left\langle \begin{bmatrix} x_1 \\ \phi_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ \phi_2 \end{bmatrix} \right\rangle_{\mathcal{Z}_{n;\tau}} := \langle x_1, x_2 \rangle_{\mathbb{R}^n} + \langle \phi_1, \phi_2 \rangle_{L^2}$

Time-delayed system to infinite-dimensional system

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Operator: $\mathcal{A}\colon \operatorname{dom}(\mathcal{A}) \subseteq \mathcal{Z}_{n;\tau} \to \mathcal{Z}_{n;\tau}$, $\mathcal{B}\colon \mathbb{R}^m \to \mathcal{Z}_{n;\tau}$, and $\mathcal{C}\colon \mathcal{Z}_{n;\tau} \to \mathbb{R}^p$

$$
\mathcal{A}\begin{bmatrix} x \\ \phi \end{bmatrix} := \begin{bmatrix} A_0 x + A_1 \phi(-\tau) \\ \frac{\mathrm{d}}{\mathrm{d}t}\phi \end{bmatrix}, \qquad \mathcal{B}u := \begin{bmatrix} Bu \\ 0 \end{bmatrix}, \qquad \mathcal{C}\begin{bmatrix} x \\ \phi \end{bmatrix} := Cx
$$

$$
\operatorname{dom}(\mathcal{A}) := \left\{ \begin{bmatrix} x \\ \phi \end{bmatrix} \in \mathcal{Z}_{n;\tau} \;\middle|\; \begin{array}{l} \phi \text{ is absolutely continuous,} \\ \frac{\mathrm{d}}{\mathrm{d}t}\phi \in L_2([-\tau;0];\mathbb{R}^n), \text{ and } \phi(0) = x \end{array} \right\}
$$

Time-delayed system to infinite-dimensional system

$$\begin{aligned}\dot{x}(t) &= A_0 x(t) + A_1 x(t-\tau) + Bu(t), \\ y(t) &= Cx(t), \\ x(t) &= \phi(t) && \text{for } t \in [-\tau, 0]\end{aligned}$$

Equivalent infinite-dimensional formulation:

$$\begin{aligned}\dot{z} &= \mathcal{A}z + \mathcal{B}u, \\ y &= \mathcal{C}z\end{aligned}$$

Infinite-Dimensional KYP-Inequality

$$\mathcal{W}(\mathcal{Q}) = \begin{bmatrix} -\mathcal{A}^*\mathcal{Q} - \mathcal{Q}\mathcal{A} & \mathcal{C}^* - \mathcal{Q}\mathcal{B} \\ \mathcal{C} - \mathcal{B}^*\mathcal{Q} & 0 \end{bmatrix} \geq 0$$

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Adjoint operator

$$\mathcal{A}^* \begin{bmatrix} q \\ \psi \end{bmatrix} = \begin{bmatrix} A_0^\top q + \psi(0) \\ -\frac{\mathrm{d}}{\mathrm{d}t}(\psi - A_1^\top q \mathbb{1}_{[-\tau,0]}) \end{bmatrix}$$

$$\mathrm{dom}(\mathcal{A}^*) = \left\{ \begin{bmatrix} q \\ \psi \end{bmatrix} \in \mathcal{X} \;\middle|\; \begin{array}{l} \psi - A_1^\top q \mathbb{1}_{[-\tau,0]} \text{ is absolutely continuous,} \\ \frac{\mathrm{d}}{\mathrm{d}t}(z - A_1^\top q) \in L_2([-\tau;0];\mathbb{R}^n) \text{ and } z(-\tau) = A_1^\top q \end{array} \right\}$$

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port-Hamiltonian system

$$\mathcal{Q}\dot{z} = \mathcal{Q}\mathcal{A}z + \mathcal{Q}\mathcal{B}u$$
$$y = \mathcal{C}z$$

Infinite-Dimensional KYP-Inequality

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port-Hamiltonian system

$$\begin{aligned} \mathcal{Q}\dot{z} &= (\mathcal{J} - \mathcal{R})z + \mathcal{Q}\mathcal{B}u \\ y &= \mathcal{C}z \end{aligned}$$

Assumption on the Hamiltonian

$$\mathcal{Q} = \begin{bmatrix} \mathcal{Q}_1 & \mathcal{Q}_2 \\ \mathcal{Q}_2^* & \mathcal{Q}_3 \end{bmatrix}, \qquad \mathcal{H}(x,\phi) = \frac{1}{2}\left\langle \begin{bmatrix} x \\ \phi \end{bmatrix}, \mathcal{Q}\begin{bmatrix} x \\ \phi \end{bmatrix} \right\rangle = \frac{1}{2}x^\top \mathcal{Q}_1 x + x^\top \mathcal{Q}_2 \phi + \frac{1}{2}\int_{-\tau}^{0} \phi(s)^\top (\mathcal{Q}_3\phi)(s)\,\mathrm{d}s$$

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Lyapunov-Krasovskii functional:

$$\mathcal{H}(x,\phi) = \frac{1}{2}x^\top H x + \int_{-\tau}^0 \phi(s)^\top S \phi(s)\mathrm{d}s$$

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Assumptions: $\mathcal{Q}_2 = 0$, $\mathcal{Q}_1 = H \in \mathbb{R}^{n\times n}$, $\mathcal{Q}_3 = S \in \mathbb{R}^{n\times n}$

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$$\mathcal{Q} = \begin{bmatrix} \mathcal{Q}_1 & \mathcal{Q}_2 \\ \mathcal{Q}_2^* & \mathcal{Q}_3 \end{bmatrix}, \qquad \mathcal{H}(x,\phi) = \tfrac{1}{2}\left\langle \begin{bmatrix} x \\ \phi \end{bmatrix}, \mathcal{Q}\begin{bmatrix} x \\ \phi \end{bmatrix} \right\rangle = \tfrac{1}{2}x^\top \mathcal{Q}_1 x + x^\top \mathcal{Q}_2\phi + \tfrac{1}{2}\int_{-\tau}^{0} \phi(s)^\top (\mathcal{Q}_3\phi)(s)\mathrm{d}s$$

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Assumptions: $\mathcal{Q}_2 = 0$, $\mathcal{Q}_1 = H \in \mathbb{R}^{n\times n}$, $\mathcal{Q}_3 = S \in \mathbb{R}^{n\times n}$

$$\mathcal{J}\begin{bmatrix} x \\ \phi \end{bmatrix} := -\tfrac{1}{2}(\mathcal{A}^*\mathcal{Q} - \mathcal{Q}\mathcal{A})\begin{bmatrix} x \\ \phi \end{bmatrix} = -\tfrac{1}{2}\begin{bmatrix} A_0^\top Hx + Sx - HA_0x - H\phi(-\tau) \\ -\frac{\mathrm{d}}{\mathrm{d}t}(2S\phi - A_1^\top Hx\mathbb{1}_{[-\tau,0]}) \end{bmatrix},$$

$$\mathcal{R}\begin{bmatrix} x \\ \phi \end{bmatrix} := -\tfrac{1}{2}(\mathcal{A}^*\mathcal{Q} + \mathcal{Q}\mathcal{A})\begin{bmatrix} x \\ \phi \end{bmatrix} = -\tfrac{1}{2}\begin{bmatrix} A_0^\top Hx + Sx + HA_0x + H\phi(-\tau) \\ -\frac{\mathrm{d}}{\mathrm{d}t}(2S\phi - A_1^\top Hx\mathbb{1}_{[-\tau,0]}) \end{bmatrix}$$

PH formulation for time-delayed systems

Definition

A time-delay system of the form

$$\begin{aligned} H\dot{x}(t) &= (J-R)x(t) - Zx(t-\tau) + Gu(t), \\ y(t) &= G^\top x(t) \end{aligned}$$

with Hamiltonian

$$\mathcal{H}(x|_{[t-\tau,t]}) = \tfrac{1}{2}x(t)^\top Hx(t) + \int_{t-\tau}^{t} x(s)^\top Sx(s)\,\mathrm{d}s$$

is called a port-Hamiltonian (pH) delay system, if $H = H^\top > 0$, $S = S^\top \geq 0$, $J = -J^\top$ and

$$\begin{bmatrix} R-S & \frac{1}{2}Z \\ \frac{1}{2}Z^\top & S \end{bmatrix} \geq 0$$

and symmetric.

Example

$$
\begin{aligned}
H\dot{x}(t) &= (J-R)x(t) - Zx(t-\tau) + Bu(t),\\
y(t) &= B^\top x(t)
\end{aligned}
$$

Example

$$
\begin{aligned}
\dot{x}(t) &= -\alpha x(t) - \beta x(t-\tau) + u(t),\\
y(t) &= x(t)
\end{aligned}
$$

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$$

Set $H = 1,\ R = \alpha,\ Z = \beta,\ B = 1$

Consequences

- $R \geq 0 \implies \alpha \geq 0$
- Find $\eta \geq 0$ such that

$$
\begin{bmatrix} \alpha - \eta & \beta \\ \beta & \eta \end{bmatrix} \geq 0
$$

Example

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Consequences

- $R \geq 0 \implies \alpha \geq 0$
- Find $\eta \geq 0$ such that

$$
\begin{bmatrix} \alpha - \eta & \beta \\ \beta & \eta \end{bmatrix} \geq 0 \iff \text{necessary \& sufficient condition for passivity}
$$

Properties

Lemma (Passivity)

A pH delay system satisfies the dissipation inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(x|_{[t-\tau,t]}) \leq y(t)^\top u(t)$$

along any solution.

Properties

Lemma (Passivity)

A pH delay system satisfies the dissipation inequality

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(x|_{[t-\tau,t]}) \leq y(t)^\top u(t)$$

along any solution.

- With $A_1 = 0$ and setting $S = 0$ we get pH systems of standard systems
- With $\tau = 0$ and setting $S = 0$ we get pH systems of standard systems

Properties

PH delay system:

$$\begin{aligned} H_i\dot{x}_i(t) &= (J_i - R_i)x_i(t) - Z_i x_i(t-\tau) + G_i u_i(t), \\ y_i(t) &= G_i^\top x_i(t) \end{aligned}$$

$i = 1, 2$

Properties

PH delay system:

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\begin{aligned}
H_i \dot{x}_i(t) &= (J_i - R_i)x_i(t) - Z_i x_i(t-\tau) + G_i u_i(t),\\
y_i(t) &= G_i^\top x_i(t)
\end{aligned}
$$

$$i = 1, 2$$

Output-feedback:

$$\widetilde{u} = F\widetilde{y} + w \qquad \widetilde{u} := \begin{bmatrix} u_1^\top & u_2^\top \end{bmatrix}^\top, \widetilde{y} := \begin{bmatrix} y_1^\top & y_2^\top \end{bmatrix}^\top$$

Properties

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Define $\widetilde{x} := \begin{bmatrix} x_1^\top & x_2^\top \end{bmatrix}^\top$ and

$$
\begin{aligned}
\widetilde{H} &:= \operatorname{diag}(H_1, H_2), & \widetilde{J} &:= \operatorname{diag}(J_1, J_2), \\
\widetilde{R} &:= \operatorname{diag}(R_1, R_2), & \widetilde{Z} &:= \operatorname{diag}(Z_1, Z_2), \\
\widetilde{G} &:= \operatorname{diag}(G_1, G_2), & \widetilde{S} &:= \operatorname{diag}(S_1, S_2)
\end{aligned}
$$

Properties

Interconnected system:

$$\begin{aligned}\widetilde{H}\dot{\widetilde{x}}(t) &= (\widetilde{J} - \widetilde{R} + \widetilde{G}^\top F \widetilde{G}^\top)\widetilde{x}(t) + \widetilde{Z}\widetilde{x}(t-\tau) + \widetilde{G}w(t) \\ \widetilde{y}(t) &= \widetilde{G}^\top \widetilde{x}(t)\end{aligned}$$

with Hamiltonian $\widetilde{\mathcal{H}} := \mathcal{H}_1 + \mathcal{H}_2$ given by

$$\widetilde{\mathcal{H}}\left(\widetilde{x}_i\big|_{[t-\tau,t]}\right) = \tfrac{1}{2}\widetilde{x}(t)^\top \widetilde{H}\widetilde{x}(t) + \int_{t-\tau}^{t} \widetilde{x}(s)^\top \widetilde{S}\widetilde{x}(s)\,\mathrm{d}s$$

Properties

Interconnected system:

$$\widetilde{H}\dot{\widetilde{x}}(t) = (\widetilde{J} - \widetilde{R} + \widetilde{G}^\top F \widetilde{G}^\top)\widetilde{x}(t) + \widetilde{Z}\widetilde{x}(t-\tau) + \widetilde{G}w(t)$$
$$\widetilde{y}(t) = \widetilde{G}^\top \widetilde{x}(t)$$

with Hamiltonian $\widetilde{\mathcal{H}} := \mathcal{H}_1 + \mathcal{H}_2$ given by

$$\widetilde{\mathcal{H}}\left(\widetilde{x}_i\big|_{[t-\tau,t]}\right) = \tfrac{1}{2}\widetilde{x}(t)^\top \widetilde{H}\widetilde{x}(t) + \int_{t-\tau}^{t} \widetilde{x}(s)^\top \widetilde{S}\widetilde{x}(s)\,\mathrm{d}s$$

Lemma (Interconnection)

The interconnected system is a pH delay system if

$$\begin{bmatrix} \widetilde{R} - \widetilde{G}\,\mathrm{sym}(F)\widetilde{G}^\top - \widetilde{S} & \frac{1}{2}\widetilde{Z} \\ \frac{1}{2}\widetilde{Z}^\top & \widetilde{S} \end{bmatrix} \geq 0$$

and symmetric e.g. $\mathrm{sym}(F) = 0$, $-\mathrm{sym}(F) \geq 0$.

Necessary condition

Proposition

A necessary condition for a system with $\tau > 0$ to be a pH delay system is

$$\begin{aligned} &\ker(R) \subseteq \ker(S) \subseteq \ker(Z), \\ &\ker(R) \cap \operatorname{Im}(Z) = \{0\}, \text{ and} \\ &\ker(R) \cap \operatorname{Im}(S) = \{0\}. \end{aligned}$$

Sufficient condition

Proposition

Assume $\ker(R) \subseteq \ker(Z)$ and $\ker(R) \cap \operatorname{Im}(Z) = \{0\}$. Let $r := \operatorname{rank}(R)$ and $V_1 \in \mathbb{R}^{n \times r}$ such that $V_1^\top R V_1 = I_r$. If $\|V_1^\top Z V_1\|_2 \leq 1$, then for $S := \frac{1}{2}R \geq 0$ symmetric the condition

$$\begin{bmatrix} R - S & \frac{1}{2}Z \\ \frac{1}{2}Z^\top & S \end{bmatrix} = \begin{bmatrix} R - S & \frac{1}{2}Z \\ \frac{1}{2}Z^\top & S \end{bmatrix}^\top \geq 0$$

is satisfied.

Sufficient condition

Example

$$\dot{x}(t) = -3x(t) - 2\sqrt{2}x(t-\tau) + u(t),$$
$$y(t) = x(t)$$

- $V_1 = 1 \implies \|V_1^\top Z V_1\|_2 = \|Z\|_2 = 2\sqrt{2} > 1$

Sufficient condition

Example

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$$y(t) = x(t)$$

- $V_1 = 1 \implies \|V_1^\top Z V_1\|_2 = \|Z\|_2 = 2\sqrt{2} > 1$
- $S = 1 \implies \begin{bmatrix} R-S & \frac{1}{2}Z \\ \frac{1}{2}Z^\top & S \end{bmatrix} = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix} \geq 0$, symmetric

Comparison with the literature

Lemma e.g. NICULESCU ET AL. '01

If there exist positive definite matrices Q, S such that

$$A_0^\top Q + QA_0 + QA_1 S^{-1} A_1^\top Q + S \leq 0, \tag{2a}$$
$$C = B^\top Q, \tag{2b}$$

then the Lyapunov-Krasovskii functional satisfies the dissipation inequality and hence, the delay equation is passive.

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Lemma

Consider a pH delay system and assume $S > 0$. Then, the pH delay system fulfills the inequality (2a).

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Lemma

Consider a pH delay system and assume $S > 0$. Then, the pH delay system fulfills the inequality (2a).

- pH condition does not require S nonsingular, dissipation inequality easier to verify
- less flexible because explicit choice of H

Summary and Outlook

Time-delay pH systems

- Rewrite time-delay system as infinite dimensional system
- Obtain pH formulation via infinite-dimensional KYP (assuming a special solution)
- Rewrite as time-delay system to obtain pH formulation for time-delay systems
- Present properties, necessary and sufficient conditions

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Open questions

- Compare with pH system for infinite-dimensional systems
- Construction for actual application
- Extend to delay differential-algebraic equations
- Delay in other parts of the system