Spectral approximation of Lyapunov operator equations with applications in high dimensional non-linear feedback control

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Overview

Setting

Preliminaries

We define the trajectory starting in x_0 as

$$\begin{cases} x(0;x_0) = x_0 & \text{for } x_0 \in \Omega \\ \frac{\mathrm{d}}{\mathrm{d}t}x(t;x_0) = f(x(t;x_0)) & \text{for } (t,x_0) \in [0,\infty) \times \Omega \end{cases}$$

- $\Omega \subseteq \mathbb{R}^n$ the state space
- $f : \Omega \mapsto \mathbb{R}^n$ the dynamics of the system
- $g : \Omega \mapsto \mathbb{R}_+$ the running cost

Goal

Compute the *Lyapunov function*

$$v : \Omega \to \mathbb{R}_+ \cup \{\infty\}, \qquad x_0 \mapsto \int_0^\infty g(x(t;x_0))\,\mathrm{d}t.$$

A simple example

Linear quadratic system

Assume $\Omega = \mathbb{R}^n$,

$$f(x) = Ax \quad \text{and} \quad g(x) := x^\top Qx$$

where P solves the **Lypunov** equation:

$$A^\top P + PA + Q = 0$$

Then:

$$v(x) = x^\top Px$$

Feedback law

Let Assume the dynamic and cost depend on a control, e.g.

$$f\colon \Omega \times U_{\mathrm{ad}} \to \mathbb{R}^n$$
$$\text{and} \quad g\colon \Omega \times U_{\mathrm{ad}} \to \mathbb{R}_+.$$

For a feedback law $u\colon \Omega \to U_{\mathrm{ad}}$ we define

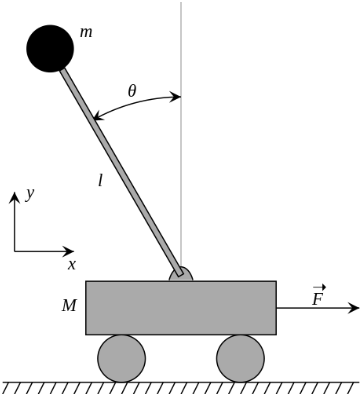
$$f_u(x) := f(x, u(x)) \quad \text{and} \quad g_u(x) := g(x, u(x)).$$

We obtain x_u, v_u. Find

$$u_{\mathrm{opt}} := \operatorname*{arg\,min}_{u\colon \Omega \to U_{\mathrm{ad}}} \|v_u\|_{\mathrm{L}^1(\Omega)}.$$

Second example:

- State space
  $\Omega := [-\pi, \pi) \times \mathbb{R} \ni (\theta, \dot{\theta})$
- Control space
  $U_{\text{ad}} := \mathbb{R}^1 \ni F.$



- **Goal:** Stabilize the pendulum in the origin.

Modelling:

- Physics leads to

$$f(x,\alpha) := \begin{pmatrix} x_2 \\ \dfrac{\frac{g}{l}\sin(x_1) - \frac{m_r}{2} x_2^2 \sin(2x_1) - \frac{m_r}{m \cdot l}\cos(x_1)\alpha}{\frac{4}{3} - m_r \cos^2(x_1)} \end{pmatrix}$$

- We use the following cost

$$g(x,\alpha) := \underbrace{\|x-0\|_2^2}_{\text{steer to the origin}} + \underbrace{\alpha^2}_{\text{steer as little as possible}}$$

- The controller has very little time
  $\Rightarrow$ find feedbacklaw u_{opt}.

Alternative characterization

One can show:

$$f(x)^\top \nabla v(x) + g(x) = 0 \qquad \forall x \in \Omega$$

Challenges

- State space $\Omega \subseteq \mathbb{R}^n$ high dimensional, e.g.
  - Quadcopter $n = 12$,
  - PDE discretization $n \approx 10^2 - 10^6$.

  But: Polynomials of low degree should be enough.
- First order PDE $\Rightarrow$ No regularization effect.
- Often used approach:
  Compute trajectories for samples + interpolation.

Composition operator [R. Singh, J. Manhas]

Let $T:\Omega\to\Omega$ with some assumptions and

$$\begin{array}{rccl} C_T: & \mathrm{L}^p(\Omega) & \to & \mathrm{L}^p(\Omega), \\ & \phi & \mapsto & \phi\circ T \end{array}$$

Properties [R. Singh, J. Manhas]

- If $p=\infty$, then
$$\|C_T\|_{\mathcal{L}(\mathrm{L}^\infty(\Omega))}=1.$$
- No composition operator is compact.

No spectral approximation !

Weighted spaces

Let $w\colon \Omega \to \mathbb{R}_+$ such that $\inf_{x\in\Omega} w(x) > 0$, we define

$$\|\phi\|_{\mathrm{L}^p_w(\Omega)} := \begin{cases} \left(\int_\Omega |\phi(x)|^p w(x)\,\mathrm{d}x\right)^{1/p} & \text{for } 1 \le p < \infty \\ \operatorname*{ess\,sup}_{x\in\Omega} |f(x)w(x)| & \text{for } p = \infty \end{cases}$$

and $\mathrm{L}^p_w(\Omega) := \left\{\phi\colon \Omega \to \mathbb{R} \;\middle|\; \|\phi\|_{\mathrm{L}^p_w(\Omega)} < \infty\right\} =: X$.

Dual pairing

Adjusted dual pairing for $1 \le p < \infty$, $\frac{1}{p} + \frac{1}{p^*} = 1$.

$$\langle \phi, \psi \rangle_{X,X^*} := \begin{cases} \int_\Omega \phi(x)\psi(x)w(x)\,\mathrm{d}x & \text{for } 1 < p < \infty \\ \int_\Omega \phi(x)\psi(x)w(x)^2\,\mathrm{d}x & \text{for } p = 1 \end{cases}$$

Motivation

Let $X = \mathrm{L}^p_w(\Omega)$ a Banach space and

- the *transformation* $T(t)$ be given as

$$T(t)\colon \Omega \to \Omega, \quad x_0 \mapsto x(t; x_0).$$

- $S^*(t) := C_{T(t)}$ is bounded, e.g. $S^*(t) \in \mathcal{L}(X^*)$.

Idea

Assume for the cost g it holds: $g \in X^*$, then

$$v := \int_0^\infty S^*(t) g \,\mathrm{d}t \quad \Leftrightarrow \quad v(x_0) = \int_0^\infty g(x(t; x_0))\mathrm{d}t$$

$\boxed{\text{Does not really work this way !}}$

Definition [M. Tucsnak, G. Weiss]

A strongly continuous semigroup on X is a map

$$S: \quad [0,\infty) \quad \to \quad \mathcal{L}(X)$$

such that

- The semigroup property holds, i.e.

$$S(t+s) = S(t)S(s) \qquad \text{for all } s,t \in [0,\infty)$$

- $S(0) = \mathrm{I}_X$.
- $S(t)$ is continuous, i.e.

$$\lim_{t\to 0} S(t)\phi = \phi \qquad \forall \phi \in X$$

$\boxed{\text{Generalization of the matrix exponential.}}$

Semigroup

Assume that for a *weighting* $w\colon \Omega \to \mathbb{R}_+ \cup \{\infty\}$

- Ω has a $\mathcal{C}^1$ boundary and

$$\langle \nu(x), f(x) \rangle \leq 0 \qquad \text{for all } x \in \partial\Omega.$$

- the dynamic satisfies $f_i \in \mathrm{L}^\infty_w(\Omega)$.
- the weight increase is bounded, i.e.

$$\sup_{x \in \Omega} -\frac{f(x)^\top \nabla w(x)}{w(x)} = \omega_0 < \infty$$

Then $S^*(t)\colon \mathrm{L}^\infty_w(\Omega) \to \mathrm{L}^\infty_w(\Omega)$ defines a weak-* continuous semigroup.

$\boxed{S(t) \text{ is a strongly continuous semigroup in } \mathrm{L}^1_w(\Omega)}$

Stability of $S(t)$

$S(t)\colon \mathrm{L}^1_w(\Omega) \to \mathrm{L}^1_w(\Omega)$ is exponentially stable if

$$\sup_{x\in\Omega} -\frac{f(x)^\top \nabla w(x)}{w(x)} = \omega_0 < 0.$$

Linear system

Let $\Omega = B_r(0)$ and define

$$f(x) := Ax \quad \text{and} \quad w(x) := \frac{1}{\|x\|},$$

where $A \in \mathbb{R}^{n\times n}$ and $\lambda_{\max}\left(A + A^\top\right) < 0$. Then

$$\omega_0 = \frac{1}{2}\lambda_{\max}\left(A + A^\top\right).$$

Rewriting the cost as an operator

Decomposition of the cost

Assume the cost $g\colon \Omega \to \mathbb{R}_+$ has the following decomposition

$$g(x) = \sum_{i=1}^{\infty} g_i(x)^2 \qquad \forall x \in \Omega$$

with $g_i \in \mathrm{L}_w^\infty(\Omega) = X^*$ and $\sum_{i=1}^\infty \|g_i\|_{X^*}^2 < \infty$.

Cost operator

Define the *nuclear* operator G as

$$\begin{array}{rlcl} G\colon & \mathrm{L}_w^1(\Omega) & \to & \mathrm{L}_w^\infty(\Omega), \\ & \phi & \mapsto & \sum_{i=1}^\infty \langle \phi, g_i \rangle_{X,X^*} g_i \end{array}$$

Make the idea work

Integrate in the space of *nuclear* operators:

$$\langle \phi, \psi \rangle_P := \int_0^\infty \underbrace{\langle S(t)\phi, GS(t)\psi \rangle_{X,X^*}}_{=:\langle S(t)\phi, S(t)\psi\rangle_G} \,\mathrm{d}t \qquad \text{for } \phi, \psi \in X$$

Operator Lyapunov equation [M. Tucsnak, G. Weiss]

One can show that $\langle \cdot, \cdot \rangle_P$ is the minimal solution to

$$\langle A\phi, \psi \rangle_P + \langle \phi, A\psi \rangle_P + \langle \phi, \psi \rangle_G = 0 \qquad \forall \phi, \psi \in \mathcal{D}(A),$$

where A is the generator of $S(t)\colon \mathrm{L}^2_{w^2}(\Omega) \to \mathrm{L}^2_{w^2}(\Omega)$.

$$\boxed{\text{Also } P \text{ is } \textit{nuclear} \Rightarrow \text{compact !}}$$

Some properties of $\langle\cdot,\cdot\rangle_P$

Let $\eta_{\varepsilon,z} \in \mathrm{L}^1_w(\Omega) =: X$ be a Dirac sequence. Then

$$v(z) = \lim_{\varepsilon\to 0} \langle \eta_{\varepsilon,z}, \eta_{\varepsilon,z}\rangle_P \qquad \text{for } z \in \Omega$$

under some assumptions. Furthermore

$$\langle\phi,\psi\rangle_P = \sum_{i=1}^{\infty} \langle\phi, p_i\rangle_{X,X^*} \langle\psi, p_i\rangle_{X,X^*} \qquad \text{for } p_i \in X^* = \mathrm{L}^\infty_w(\Omega).$$

Sum of squares solution

We define the *sum of squares solution* as

$$v\colon \Omega \to \mathbb{R}_+, \qquad z \mapsto \sum_{i=1}^{\infty} p_i(z)^2.$$

Adjoint equation

$S^*(t)\colon \mathrm{L}^2_{w^2}(\Omega) \to \mathrm{L}^2_{w^2}(\Omega)$ is a strongly continuous semigroup. Let us define:

$$\langle \phi, \psi \rangle_R := \int_0^\infty \langle S^*\phi, S^*\psi \rangle_P \,\mathrm{d}t \qquad \text{for } \phi, \psi \in \mathrm{L}^2_{w^2}(\Omega).$$

There exists $r_i \in \mathrm{L}^2_{w^2}(\Omega)$ such that:

$$\langle \phi, \psi \rangle_R = \sum_{i=1}^\infty \langle \phi, r_i \rangle_{\mathrm{L}^2_{w^2}(\Omega)} \langle \psi, r_i \rangle_{\mathrm{L}^2_{w^2}(\Omega)} \qquad \forall \phi, \psi \in \mathrm{L}^2_{w^2}(\Omega).$$

Note: Not necessarily $r_i \in \mathrm{L}^\infty_w(\Omega)$!

Fast decaying eigenvalues

If $f, g \in \mathcal{C}^\infty(\Omega)$ then there exists $\alpha, \beta > 0$ s.t.

$$\|p_n\|_{\mathrm{L}^2_{w^2}(\Omega)}, \|r_n\|_{\mathrm{L}^2_{w^2}(\Omega)} \in \mathcal{O}(\exp(-\beta n^\alpha)).$$

Finite rank manifold

For $k \in \mathbb{N}$ define $\mathcal{M} \subseteq \mathrm{HS}(\mathrm{L}^2_{w^2}(\Omega))$

$$\mathcal{M}_k := \left\{ \phi, \psi \mapsto \sum_{n=1}^{k} \langle \phi, p_i \rangle \langle \psi, p_i \rangle \;\middle|\; \langle p_i, p_j \rangle = \lambda_i \delta_{i,j} \text{ with } \lambda_i > 0 \right\}$$

$$\boxed{\mathrm{HS}(\mathrm{L}^2_{w^2}(\Omega)) \text{ has a Hilber structure}}$$

" $L = A \otimes I + I \otimes A^\top$ "

Define the following linear map

$$L\colon \mathcal{D}(L) \subseteq \mathcal{N}_1(\mathrm{L}^2_{w^2}(\Omega)) \to \mathcal{N}_1(\mathrm{L}^2_{w^2}(\Omega)),$$
$$\langle \phi, \psi \rangle_{L(\tilde{R})} \mapsto \langle A^* \phi, \psi \rangle_{\tilde{R}} + \langle \phi, A^* \psi \rangle_{\tilde{R}} \qquad \forall \phi, \psi \in \mathcal{D}(A^*)$$

Motivation in $\mathbb{R}^n$

To solve $L^\top p = g$, with $p, g \in \mathbb{R}^n$, we solve

$$r, p = \operatorname*{arg\,min}_{\tilde{r}, \tilde{p}} \|L\tilde{r} - \tilde{p}\|_2^2 + (L\tilde{r})^\top L\tilde{r} - 2\tilde{r}^\top g$$

Note, that $p = L^{-\top} g$ and $r = L^{-1} p$ is the unique minimizer.

Minimization

Approximate P, R by

$$P_{k,k'}, R_{k,k'}$$

$$= \underset{\tilde{P}\in\mathcal{M}_k, \tilde{R}\in\mathcal{M}_{k'}}{\arg\min} \underbrace{\|L(\tilde{R})\|_{HS}^2 - 2\left\langle \tilde{R}, G \right\rangle_{\mathrm{HS}} + \|L(\tilde{R}) + \tilde{P}\|_{HS}^2}_{=:J(P,R)}$$

and

$$J(P_{k,k'}, R_{k,k'}) \approx -\|P\|_{\mathrm{HS}}^2 \neq -\|P\|_1 = -\|v\|_{\mathrm{L}^1_w(\Omega)}$$

Note: We cannot set $P := -L(R)$ directly, since $-L(R)$ is not necessarily a positive definite operator.

Assume, that $J(P,R) \approx -\|P\|_1$.

Control

For a given feedback control $u \in U_{\mathrm{ad}}$ define

- $f_u(x) := f(x, u(x))$.
- $g_u(x) := \sum_{i=1}^{\infty} g_i(x)^2 + \sum_{i=1}^{\infty} h_i(u(x))^2$.
- A_u, G_u and J_u constructed as before.

$$
\begin{aligned}
-\int_\Omega v(x)w(x)\ \mathrm{d}x &= \lim_{k,k'\to\infty} \max_{u\in U_{\mathrm{ad}}} \min_{\tilde P\in\mathcal{M}_k, \tilde R\in\mathcal{M}_{k'}} J_u(P,R) \\
&= \lim_{k,k'\to\infty} \min_{\tilde P\in\mathcal{M}_k, \tilde R\in\mathcal{M}_{k'}} \underbrace{\max_{u\in U_{\mathrm{ad}}} J_u(P,R)}_{\text{can be resolved analytically}}
\end{aligned}
$$

Numerical method

1. Discretize, e.g., by

   polynomials and tensor trains.
2. Compute low rank solution to the Lyapunov equation, e.g., by

   Riemannian optimization.
3. Compute spectral decomposition.
4. Obtain v by sum of squares.

Control

- Lyapunov equation $\rightarrow$ Non-linear operator equation.
- Policy iteration

  update control $\rightarrow$ linearize $\rightarrow$ solve Lyapunov equation $\rightarrow$
  update control $\rightarrow \ldots$

Nonlinear system

Problem [Beard et al '96]

Define with $\alpha = -1.35219$, $\beta = 0.41421$.

- $\Omega = [-1, 1]^2$ discretized with poly. degree 10.
- TT-ranks $(1, 11, 6)$
- Control

$$u(x) := 3x_1^5 + 3x_1^2 x_2 - x_2 + \beta x_1 + \alpha(x_1^3 + x_2)$$

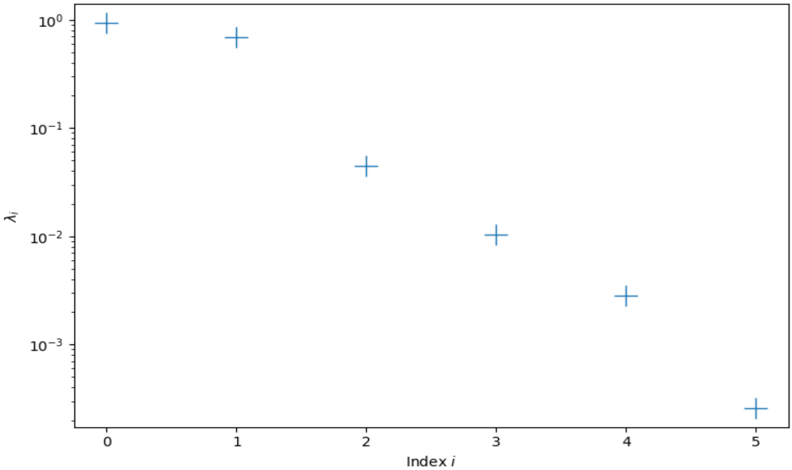
- Dynamics

$$f(x) = \begin{pmatrix} -x_1^3 - x_2 \\ x_1 + x_2 + u(x) \end{pmatrix}$$

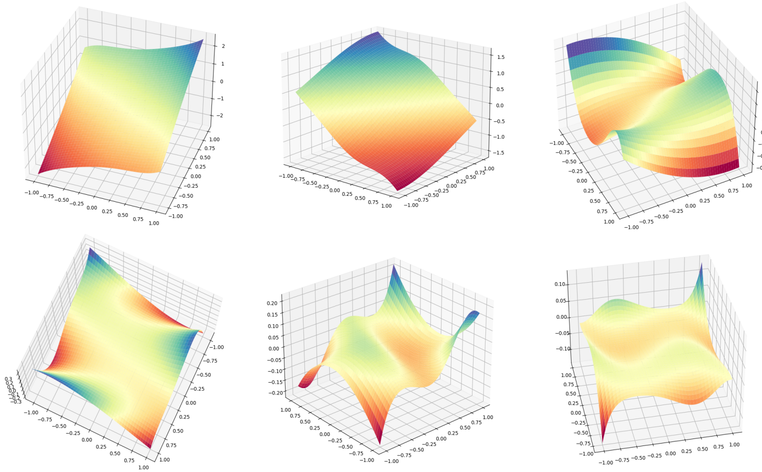
- Cost

$$g(x) = x_1^2 + x_2^2 + u(x)^2$$

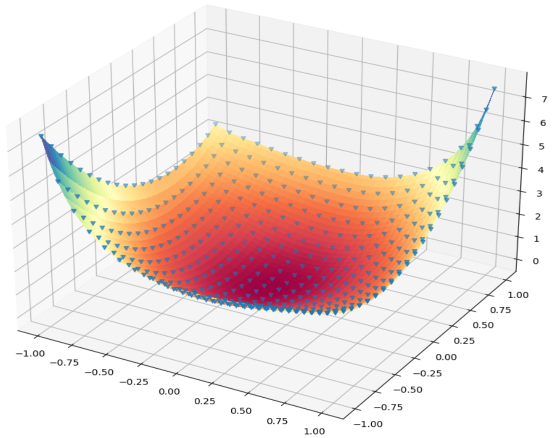
Eigenvalue decay



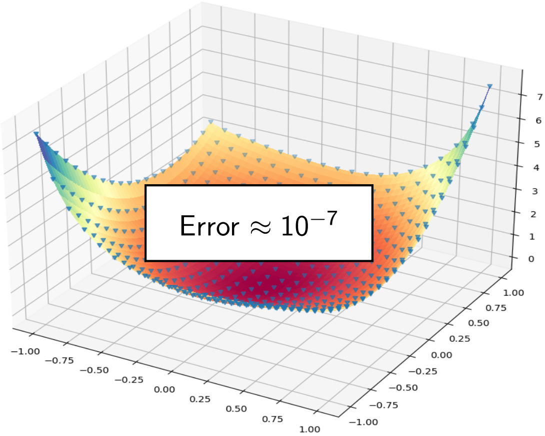
Eigenfunctions v_i



Lyapunov function



Lyapunov function



Summary

- Cost function $\Rightarrow$ cost operator.
- Lyapunov function for a non-linear system
  $\Rightarrow$ linear operator equation.
- Weighted $\mathrm{L}^p(\Omega)$ spaces ensure solvability.
- Rapid eigenvalue decay for smooth g_i and f expected
  $\Rightarrow$ Low rank approximation.
- Numerically solved by Riemannian optimization.

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Thank you for your attention !

Bibliography

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