Hypocoercivity and hypocontractivity of linear evolution equations

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Absolvent:innenseminar 20.10.22, joint w. F. Achleitner and A. Arnold

Continuous and discrete-time linear dissipative systems.

$$\dot{x} = A_c x,\ t > 0,\ x(0) = x_0,$$

or

$$x_{k+1} = A_d x_k, k = 1, 2, \ldots,\ x_0 \text{ given}.$$

Different stability concepts:
(hypo)coercivity, (semi)-dissipativity, (asymptotic) stability.
(hypo)contractivity, (semi)-contractivity, (asymptotic) stability.
How are they related?

- F. Achleitner, A. Arnold, and V. M., *Hypocoercivity and controllability in linear semi-dissipative ODEs and DAEs.* ZAMM, Zeitschrift f. Angewandte Mathematik und Mechanik, 2021. https://doi.org/10.1002/zamm.202100171
- F. Achleitner, A. Arnold, and V. M., *Hypocoercivity and hypocontractivity concepts for linear dynamical systems*, in preparation, 2022.

Outline

Continuous-time evolution eq.

Given an evolution equation: $\dot{x} = A_c x$.

- ▷ A_c linear operator constant in time.
- ▷ A_c has a unique steady state $A_c x_\infty = 0$,.

Motivating questions:

1. Optimal long-time decay estimate.
   - Exponential decay: $\|x(t) - x_\infty\| \leq c e^{-\mu t} \|x(0) - x_\infty\|$, $t > 0$.
   - (sharp) maximal rate $\mu > 0$.
   - and minimal $c \geq 1$ (uniform for all $x(0)$).
2. Short-time decay estimate.
   - $\|x(t) - x_\infty\| \leq (1 - c t^\alpha + \mathcal{O}(t^{\alpha+1}))\|x(0) - x_\infty\|$, $t > 0$.
   - Relation to spectral properties of operator A.

Definition

Consider evolution equation $\dot{x} = A_c x = (J - R)x$,
$J = -J^H$, $R = R^H \in \mathbb{C}^{n,n}$. $-A_c$ is *coercive* if $-x^H A_c x \geq \kappa \|x\|_2$ for all $x \in \mathbb{C}^n$ and some $\kappa > 0$.

Example: $A_c = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$.

- Evs $\frac{1}{2}(-1 \pm \sqrt{3}i)$,
- decay rate $\frac{1}{2}$,
- $-A_c$ not coercive.

Stability characterization for linear ODEs

Theorem

Consider a constant coefficient system $\dot{x} = A_c x$ with $A_c \in \mathbb{C}^{n,n}$.

- *It is asymptotically stable if all eigenvalues of A_c have negative real part or equivalently there exists solution $P > 0$ of the Lyapunov inequality $PA_c + A_c^H P < 0$.*
- *It is stable if all evs of A_c have nonpositive real part and all evs with real part 0 are semisimple or equivalently there exists solution $P > 0$ of the Lyapunov inequality $PA_c + A_c^H P \leq 0$.*

Typically, instead of evs one should rather use pseudospectra.

- L.N. Trefethen and M. Embree. Spectra and Pseudospectra. Princeton University Press, 2020.

- Notion hypocoercivity was introduced for PDEs $\dot{x} = -Bx$ on Hilbert space $\mathcal{H}$, where linear operator B is not coercive, but solutions still exhibit exponential decay in time.
- More precisely, for hypocoercive operators C there exist constants $\lambda > 0$ and $c \geq 1$, such that

$$\|e^{-Bt}x_0\|_{\widetilde{\mathcal{H}}} \leq c\, e^{-\lambda t}\|x_0\|_{\widetilde{\mathcal{H}}} \qquad \text{for all } x_0 \in \widetilde{\mathcal{H}}\,, \qquad t \geq 0\,,$$

$\widetilde{\mathcal{H}}$ Hilbert space, densely embedded in $(\ker B)^{\perp} \subset \mathcal{H}$.

In the following we use $-B = A_c = J - R$ with J skew-adjoint and R self-adjoint. Splitting may not exist for every operator A_c. If $R \geq 0$ we call the system semi-dissipative.

- C. Villani. Hypocoercivity. *Mem. Amer. Math. Soc.*, 202, 2009.

Lemma (Achleitner, Arnold, M. 2021)

Let $J, R \in \mathbb{C}^{n,n}$ *with* $J^H = -J$ *and* $R^H = R \geq 0$*. T.f.a.e.*

1. *There exists integer* $m \geq 0$ *such that* $\operatorname{rank}[R, JR, \ldots, J^m R] = n$.
2. *There exists integer* $m \geq 0$ *such that* $\mathcal{T}_m := \sum_{j=0}^{m} J^j R (J^H)^j > 0$.
3. *No eigenvector of* J *lies in the kernel of* R*.*
4. $\operatorname{rank}[\lambda I - J, R] = n$ *for every* $\lambda \in \mathbb{C}$*.*

Moreover, the smallest possible m *in 1. and 2. coincide.*

This is controllability of pair (J, R).

Hypercoercive matrices

Definition

A matrix $A_c = J - R \in \mathbb{C}^{n,n}$ with $R \geq 0$ is called hypocoercive if the spectrum of A_c lies in the *open* left half plane.

Definition

Let $J, R \in \mathbb{C}^{n,n}$ with $J^H = -J$ and $R^H = R \geq 0$.
The hypocoercivity index (HC-index) $m_{HC}(A_c)$ of $A_c = J - R$ is the smallest integer m such that $\sum_{j=0}^{m} J^j R (J^H)^j > 0$.
For matrices $A_c = J - R$ that are not hypocoercive we set $m_{HC}(A_c) = \infty$.

Is there a numerically feasible way to check hypocoercivity?

Lemma

Let $J = -J^H$, $R = R^H \in \mathbb{C}^{n,n}$. There exists unitary P, such that

$$PJP^H = \left[\begin{array}{cccccccc|c} J_{1,1} & -J_{2,1}^H & & \cdots & & & 0 & & 0 \\ J_{2,1} & J_{2,2} & -J_{3,2}^H & & & & & & \\ & \ddots & \ddots & \ddots & & & \vdots & & \\ & & J_{k,k-1} & J_{k,k} & -J_{k+1,k}^H & & & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & & & \\ & & & & J_{s-2,s-3} & J_{s-2,s-2} & -J_{s-1,s-2}^H & & \\ 0 & & \cdots & & & J_{s-1,s-2} & J_{s-1,s-1} & & 0 \\ \hline 0 & & & \cdots & & & 0 & & J_{ss} \end{array}\right], \quad PRP^H = \begin{bmatrix} R_1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}$$

w. block sizes $n_1 \geq \cdots \geq n_{s-1} \geq n_s \geq 0$, $n_{s-1} > 0$, $R_1 \in \mathbb{C}^{n_1,n_1}$ nonsingular. If R is nonsingular, then $s = 2$ and $n_2 = 0$. If R is singular, then $s \geq 3$ and the matrices $J_{i,i-1}$, $i = 2, \ldots, s-1$, in the subdiagonal have full row rank and are of the form

$$J_{i,i-1} = \begin{bmatrix} \Sigma_{i,i-1} & 0 \end{bmatrix}, \quad i = 2, \ldots, s-1,$$

with nonsingular matrices $\Sigma_{i,i-1} \in \mathbb{C}^{n_i,n_i}$.

Proof via sequence of singular value decompositions.

HC-index and staircase form

Lemma

Let $A_c = J - R$ be a semi-dissipative matrix transformed to staircase form. Then the matrix A_c is hypocoercive if and only if $n_s = 0$, i.e., the last row and last column in PJP^H are absent, and the HC-index of A_c is $m_{HC} = s - 2$.

We can check hypocoercivity via staircase form and compute HC-index in a numerically stable way. BUT we need rank decisions in finite precision. Perturbation theory for staircase forms difficult. Best to use structure of operator.

Relation between concepts

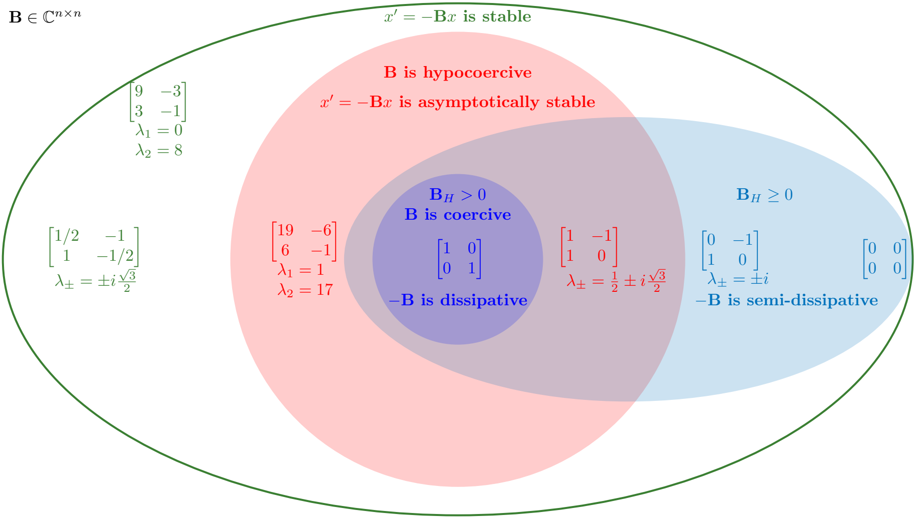

Figure: $-B = A_c = J - R$, $B_H = R$: (hypo)coercive (pink), $R \geq 0$ (blue), and for which solutions of $\dot{x} = A_c x$ are stable (white).

HC-index under perturbations

Set of semi-dissipative matrices is convex, i.e. for two semi-dissipative matrices $A_c = J - R$, $\tilde{A}_c = J_1 - R_1$ and $\delta \geq 0$, also $A_c + \delta \tilde{A}_c = (J + \delta J_1) - (R + \delta R_1)$ are semi-dissipative.
Example:

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; R = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; J_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \; R_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For $\delta \geq 0$, the HC-index of the perturbed matrix is

$$m_{HC}(\tilde{A}_c) = \begin{cases} 1 & \text{if } \delta > 0, \\ 2 & \text{if } \delta = 0. \end{cases}$$

It is easy to decrease the HC-index with arbitrary small perturbations which preserve the structure. But small enough perturbations cannot increase the HC-index.

HC-index and short-time decay

Relation between short-time decay of $\|e^{A_c t}\|_2$ and HC-index.

Theorem

Consider a semi-dissipative Hamiltonian ODE whose system matrix A_c has finite HC-index. Its (finite) HC-index is m_{HC} if and only if

$$\|e^{A_c t}\|_2 = 1 - ct^a + \mathcal{O}(t^{a+1}) \quad \text{for } t \to 0+,$$

where $c > 0$ and $a = 2m_{HC}(A_c) + 1$.

Consider ODE with $A_c = J - R := \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$, $R = \mathrm{diag}(-1, 0)$ singular, $m_{HC}(A_c) = 1$. Evs $(-1 \pm i\sqrt{3})/2$.
The squared *propagator norm* satisfies

$$\begin{aligned}
\|e^{A_c t}\|_2^2 &= \frac{1}{6}\left(\left(\sqrt{-2\cos(\sqrt{3}t) + 14} + \sqrt{-2\cos(\sqrt{3}t) + 2}\right)\sqrt{-2\cos(\sqrt{3}t) + 2} + 6\right)e^{-t} \\
&\sim 1 - \frac{1}{6}t^3 + \mathcal{O}(t^4) \quad \text{for } t \to 0+,
\end{aligned}$$

Kernel of B is one-dimensional and spanned by $[0, 1]^T$.
With this initial condition:

$$\begin{aligned}
\|x(t)\|_2^2 &= \frac{1}{9}e^{-t}\left(\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}\,t\right) - 3\,\cos\left(\frac{\sqrt{3}}{2}\,t\right)\right)^2 + \frac{4}{3}\,e^{-t}\sin\left(\frac{\sqrt{3}}{2}\,t\right)^2 \\
&\sim 1 - \frac{2}{3}t^3 + \mathcal{O}(t^4) \quad \text{for } t \to 0+\,.
\end{aligned}$$

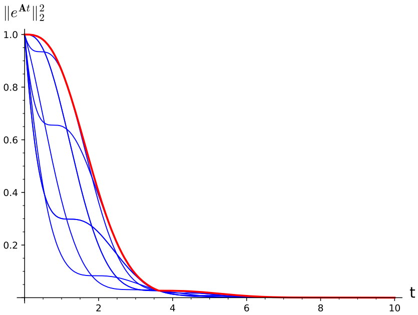

Figure: For the ODE with $A_c = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$, the squared propagator norm ($\|e^{A_c t}\|_2^2 \sim 1 - t^3/6 + \mathcal{O}(t^4)$ for $t \to 0+$), (red line) and the squared norms of a family of solutions (blue lines) are plotted. The squared propagator norm is not continuously differentiable at $t = 2\pi/\sqrt{3}$, it is the envelope of $\|x(t)\|_2^2$ for all solutions with $\|x(0)\|_2^2 = 1$.

Answer to question by Hans Zwart.

Lemma (Achleitner, Arnold, M. 2022)

1. *If A_c is hypocoercive then A_c is invertible and A_c^{-1} is hypocoercive.*
2. *If $A_c = J - R$ is semi-dissipative and invertible then it follows that*
   a. *If $v \in \ker(R)$ then $A_c v \in \ker(A_c^{-1} + A_c^{-H})$.*
   b. $\dim \ker R = \dim \ker(A_c^{-1} + A_c^{-H})$.
   c. *A_c^{-1} is semi-dissipative.*
3. *If A_c is semi-dissipative and hypocoercive then A_c and A_c^{-1} have the same HC-index.*

Outline

Linear discrete-time evolution equation.

$$x_{k+1} = A_d x_k \,, \qquad k = 0, 1, 2, \ldots.$$

Definition

A solution of the discrete-time system is called stable if it is bounded for all k and asymptotically stable if it is stable and converges to 0 for $k \to \infty$. If all solutions are (asymptotically) stable for all initial values x_0 then we call the system (asymptotically) stable.

- The discrete-time system is stable if all eigenvalues of A_d have modulus less or equal than one and the eigenvalues of modulus one are semi-simple or equivalently if there exists a solution $P > 0$ of the discrete Lyapunov (Stein) inequality

$$A_d^H P A_d - P \leq 0.$$

- The discrete-time systerm is *asymptotically stable* if all eigenvalues of A_d have modulus strictly less than one or equivalently if there exists a solution $P > 0$ of the discrete Lyapunov (Stein) inequality

$$A_d^H P A_d - P < 0.$$

Definition

Let $\sigma_{\max}(A_d)$ be the largest singular value (the *spectral norm*) of A_d. We call A_d *contractive* if $\sigma_{\max}(A_d) < 1$; and we call A_d *semi-contractive* if $\sigma_{\max}(A_d) \leq 1$.
A matrix A_d is called *hypocontractive* if all eigenvalues of A_d have modulus strictly less than one.

Consequently, a discrete-time system is asymptotically stable if and only if the system matrix A_d is hypocontractive.

Hypocontractivity and controllability.

Lemma

Let $A_d \in \mathbb{C}^{n,n}$ be semi-contractive. T.f.a.e.

- *There exists an integer $m \geq 0$ such that* $\operatorname{rank}[(I - A_d^H A_d), A_d^H (I - A_d^H A_d), \ldots, (A_d^H)^m (I - A_d^H A_d)] = n$.
- *There exists an integer $m \geq 0$ such that* $D_m := \sum_{j=0}^{m} (A_d^H)^j (I - A_d^H A_d) A_d^j > 0$.
- *No eigenvector of A_d lies in the kernel of $(I - A_d^H A_d)$.*
- $\operatorname{rank}[\lambda I - A_d^H, I - A_d^H A_d] = n$ *for every $\lambda \in \mathbb{C}$, in particular for every eigenvalue λ of A_d.*

Moreover, the smallest m in first two conditions coincide.
This is controllability of the pair $(A_d^H, I - A_d^H A_d)$.
Similar result (in different notation) via observability, O. Staffans, Well-posed linear systems, Cambridge Univ. Press, 2005.

Hypocontractivity index

Definition

For a semi-contractive matrix A_d, we define the hypocontractivity index or discrete HC-index (dHC-index) m_{dHC} as the smallest integer (if it exists) such that the second condition in the Lemma holds. For semi-contractive matrices A_d that are not hypocontractive we set $m_{dHC} = \infty$.

A semi-contractive matrix A_d is contractive iff $m_{dHC} = 0$.
In operator theory $(I - A_d^H A_d)^{\frac{1}{2}}$ is called the *defect operator* of A_d and the closure of its imagethe *defect space* with its dimension being called the *defect index*. The defect operator and its index are a measure for the distance of an operator from being unitary.

Theorem

Let A_d be semi-contractive with finite hypocontractivity index. Its (finite) hypocontractivity index is m_{dHC} if and only if

$$\|A_d^j\|_2 = 1 \text{ for all } j = 1, \ldots, m_{dHC} \text{ , and } \|A_d^{m_{dHC}+1}\|_2 < 1 \ .$$

Polar decomposition is the discrete-time analogue of the additive splitting of a matrix into its Hermitian and skew-Hermitian part.

Lemma (Polar decomposition)

Let $A_d \in \mathbb{C}^{n,n}$.

(a) *There exist positive semi-definite Hermitian matrices P_d, Q_d and a unitary matrix U_d such that*

$$A_d = P_d U_d = U_d Q_d.$$

The factors P_d, Q_d are uniquely determined and if A_d is nonsingular, then $U_d = P_d^{-1} A_d = A_d Q_d^{-1}$ is unique..

(b) *If A_d is real, then P_d, Q_d and U_d may be taken to be real.*

A_d with polar decomp. $A_d = P_d U_d = U_d Q_d$ is semi-contractive iff spectra of P_d or Q_d are contained in $[0, 1]$.

Hypocontractivity via polar factors

Lemma

Let U be a unitary matrix, and H be a semi-contractive Hermitian matrix. T.f.a.e.

- *There exists an integer $m \geq 0$ such that*

$$\operatorname{rank}[(I-H), U^H(I-H), \dots, (U^H)^m(I-H)] = n\,.$$

- *There exists an integer $m \geq 0$ such that*

$$\hat{D}_m := \sum_{j=0}^{m} (U^H)^j (I-H) U^j > 0\,.$$

- *No eigenvector of U lies in the kernel of $I-H$.*
- $\operatorname{rank}[\lambda I - U^H, I-H] = n$ *for every $\lambda \in \mathbb{C}$, in particular for every eigenvalue λ of U^H.*

Lemma (Staircase form for (U, H))

Let U be a unitary matrix, and H be a nonzero semi-contractive Hermitian matrix. Then there exists a unitary matrix P, such that PHP^H and PUP^H are block upper Hessenberg matrices of the form

$$PUP^H = \begin{array}{c} \left[\begin{array}{cccccc|c} U_{1,1} & U_{1,2} & \cdots & & \cdots & U_{1,s-1} & 0 \\ U_{2,1} & U_{2,2} & U_{2,3} & & \cdots & U_{2,s-1} & 0 \\ & \ddots & \ddots & & \ddots & \ddots & \vdots \\ & & U_{s-2,s-3} & & U_{s-2,s-2} & U_{s-2,s-1} & 0 \\ 0 & \cdots & 0 & & U_{s-1,s-2} & U_{s-1,s-1} & 0 \\ \hline 0 & \cdots & & & & 0 & U_{s,s} \end{array}\right] \end{array} \begin{array}{c} n_1 \\ n_2 \\ \vdots \\ n_{s-2} \\ n_{s-1} \\ n_s \end{array},$$

$$P\,H\,P^H = \left[\begin{array}{cccccc|c} H_1 & 0 & \cdots & \cdots & & 0 & 0 \\ 0 & I_{n_2} & 0 & \cdots & & \vdots & \vdots \\ \vdots & 0 & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & & I_{n_{s-1}} & 0 \\ \hline 0 & 0 & \cdots & \cdots & & 0 & I_{n_s} \end{array}\right],$$

where $n_1 \geq n_2 \geq \cdots \geq n_{s-1} \geq n_s \geq 0$, $n_{s-1} > 0$, and $H_1 = H_1^H \in \mathbb{C}^{n_1,n_1}$ is contractive. If H is contractive, then $s = 2$ and $n_2 = 0$. If H is not contractive, then $s \geq 3$, $U_{i,i-1}$, $i = 2, \ldots, s-1$, have full row rank and are of form $U_{i,i-1} = \begin{bmatrix} \Sigma_{i,i-1} & 0 \end{bmatrix}$, $i = 2, \ldots, s-1$, with nonsingular matrices $\Sigma_{i,i-1} \in \mathbb{C}^{n_i,n_i}$.

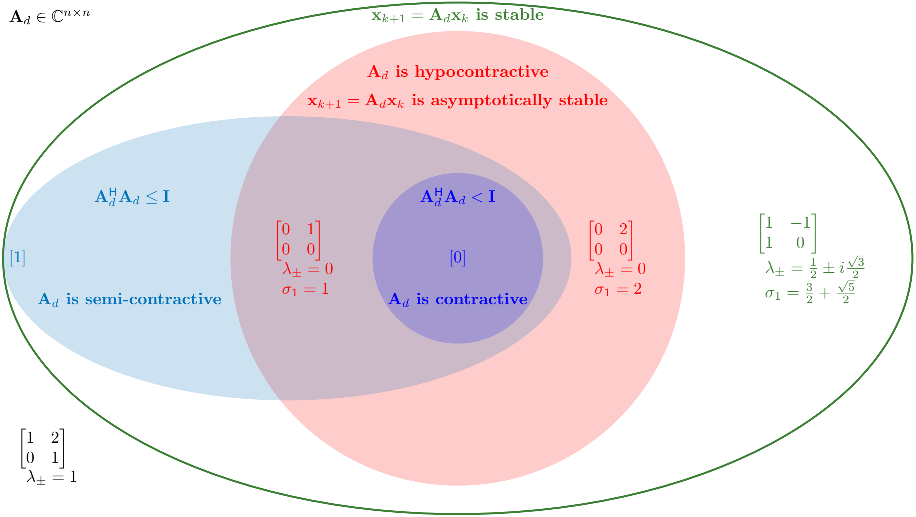

Figure: Relation between matrices A_d which are (semi-)contractive, hypocontractive and those for which the discrete-time system

Continuous vs discrete-time

	continuous-time	discrete-time
system	$\frac{d}{dt}x = A_c x$ for $t \geq 0$	$x_{k+1} = A_d x_k$ for $k = 0, 1, 2, \ldots$
decomp	$A_c = J - R$	polar $A_d = Q_d U_d = U_d P_d$,
asympt. stability	$\mathrm{Re}(\lambda) < 0$ for all evs	$\lvert\lambda\rvert < 1$ for all evs.,
dissip./contract.	$R \geq 0$	evs of Q_d, P_d in $[0, 1]$,
hypocoerc./contr.	$\sum_{j=0}^{m} J^j R (J^H)^j > 0$	$\sum_{j=0}^{m} (A_d^H)^j (I - A_d^H A_d) A_d^j > 0$.

Table: Relation of concepts for continuous and discrete-time systems.

Outline

Stability concepts for DAEs

- Analysis of linear autonomous DAEs $E\dot{x} = Ax + f(t)$ with help of the *Kronecker canonical form*.
- However, one has to be careful with the initial conditions and the regularity of inhomogeneities, they are restricted due to algebraic constraints.

Theorem

Consider constant coefficient system $E\dot{x} = Ax$ with $E, A \in \mathbb{C}^{n,n}$.

- *It is asymptotically stable if the pair (E, A) is regular $(\det(\lambda E - A) \neq 0)$, if all finite eigenvalues of $\lambda E - A$ have negative real part, and all infinite eigenvalues are semisimple.*
- *It is stable if the pair (E, A) is regular, all eigenvalues of A have non-positive real part and all eigenvalues with real part 0 (including ∞) are semisimple.*

Pseudospectra, Lyapunov solution?

- M. Embree and B. Keeler. Pseudospectra of matrix pencils for transient analysis of differential-algebraic equations. SIAM J. Matrix Analysis and Applications 38, 1028-1054, 2017.

Definition

For a (semi-)dissipative matrix $A = J - R \in \mathbb{C}^{n,n}$ with $J = -J^H$, $R = R^H \geq 0$ and $E = E^H \geq 0 \in \mathbb{C}^{n,n}$, the associated DAE is called (semi-)dissipative Hamiltonian DAE.

Theorem (Mehl/M./Wojtylak 2018)

Let $E \in \mathbb{C}^{n,n}$ and $J = -J^H$, $R = R^H \in \mathbb{C}^{n,n}$ be such that $R \geq 0$, $E^H = E \geq 0$. Then the following holds for $P(\lambda) = \lambda E - (J - R)$.

1. *If $\mu \in \mathbb{C}$ is an eigenvalue of $P(\lambda)$ then $\mathrm{Re}(\mu) \leq 0$;*
2. *If $\omega \in \mathbb{R}$ and $\mu = i\omega$ is an eigenvalue of $P(\lambda)$ then μ is semisimple. Moreover, if the columns of $V \in \mathbb{C}^{n,k}$ form a basis of the deflating subspace associated with μ of $\lambda E - J$, then $RV = 0$.*
3. *Kronecker blocks at ∞ are at most of size two.*
4. *The pencil $\lambda E - (J - R)$ is singular iff the kernels of the matrices E, J, and R have a nontrivial intersection.*

- ▷ Mehl, V. M., and Wojtylak, *Linear algebra properties of dissipative Hamiltonian descriptor systems*. SIAM Journal Matrix Analysis and Applications, Vol. 39, 489–1519, 2018.
- ▷ Mehl, V.M., Wojtylak. *Distance problems for dissipative Hamiltonian systems and related matrix polynomials* Linear Algebra and its Application, 2021.

Theorem

Consider a DH system with pencil $P(\lambda) = \lambda E - (J - R)$ *with* $E = E^H \geq 0$, $R = R^H \geq 0$, *and* $J = -J^H$.

- *Asymptotic stability if* $P(\lambda)$ *is regular, finite evs are all in open left half plane, and system is index* 1 *(ev* ∞ *is semisimple)*
- *Stability if* $P(\lambda)$ *is regular, and* ∞ *is semisimple.*
- *Robust stability if distance to instability, to nearest index* 2 *problem, and distance to nearest singular pencil are large.*

Lemma

Consider a semi-dissipative Hamiltonian DAE in staircase form with $\check{A} = \check{J} - \check{R}$. Then there exist nonsingular matrices L, Z such that

$$\hat{E} := L\,\check{E}\,Z =: \begin{bmatrix} \hat{E}_{1,1} & 0 & 0 & 0 & 0 \\ 0 & \hat{E}_{2,2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{A} := L\,\check{A}\,Z =: \begin{bmatrix} 0 & 0 & 0 & I & 0 \\ 0 & \hat{A}_{2,2} & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ -I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The two matrices are partitioned in the same way, with (square) diagonal block matrices of sizes $n_1, n_2, n_3, n_4 = n_1, n_5$. If the matrices $\hat{E}_{1,1}$ and $\hat{E}_{2,2}$ are present, then they are Hermitian positive definite. If $n_1 > 0$, $n_2 > 0$, and $n_3 > 0$, then $\hat{E}_{1,1} = E_{1,1} - E_{2,1}^H E_{2,2}^{-1} E_{2,1}$, $\hat{E}_{2,2} = E_{2,2}$, and $\hat{A}_{2,2} = \check{A}_{2,2} - \check{A}_{2,3}\check{A}_{3,3}^{-1}\check{A}_{3,2}$.

Corollary

Let $E, J, R \in \mathbb{C}^{n,n}$ *satisfy* $E = E^H \geq 0$, $R = R^H \geq 0$ *and* $J = -J^H$. *Consider a regular pencil* $\lambda E - (J - R)$, *and its almost Kronecker form* $\lambda \hat{E} - \hat{A}$. *The DAE-index* ν *of a regular pencil* $\lambda E - (J - R)$ *satisfies*

$$\nu = \begin{cases} 2 & \text{if and only if } n_1 = n_4 > 0, \\ 1 & \text{if and only if } n_1 = n_4 = 0 \text{ and } n_3 > 0, \\ 0 & \text{if and only if } n_1 = n_4 = 0 \text{ and } n_3 = 0. \end{cases}$$

If pencil has $\nu = 2$ *and* $n_2 > 0$, $n_3 > 0$, *then*
$y_1 = 0$, $\check{E}_{22}\dot{y}_2 = \check{A}_{22}y_2$, $y_3 = \check{A}_{3,3}^{-1}\check{A}_{3,2}y_2$,
$y_4 = J_{4,1}^{-H}\left((-J_{2,1}^H - R_{2,1}^H)y_2 + (-J_{3,1}^H - R_{3,1}^H)y_3 - E_{2,1}^H\dot{y}_2\right)$,
leading to restrictions in the initial values.

Definition

A matrix pencil $\lambda E - A$ is called (negative) hypocoercive if the pencil is regular, of DAE-index at most two and the finite eigenvalues of the pencil $\lambda E - A$ have negative real part.

Definition

Consider a linear semi-dissipative Hamiltonian DAE system with a regular pencil $\lambda E - A$ and the unitarily congruent DAE in staircase form. If the underlying implicit ODE is present then the system is said to exhibit non-trivial dynamics. In case of non-trivial dynamics, the *HC-index* m_{HC} of $\lambda E - A$ is defined as the HC-index of the system matrix $(E_{2,2}^{1/2})^{-1}\hat{A}_{2,2}(E_{2,2}^{1/2})^{-1}$, otherwise it is defined as 0.

Consider semi-norm $\|x(t)\|_E = \langle x, Ex \rangle^{\frac{1}{2}}$

Theorem

Consider a semi-dissipative Hamiltonian DAE with a regular, (negative) hypocoercive pencil $\lambda E - A$, DAE-index at most two, non-trivial dynamics, and consistent initial condition $x(0)$. Then its (finite) HC-index is m_{HC}, if and only if

$$\|S(t)\|_E = 1 - ct^a + \mathcal{O}(t^{a+1}) \qquad \textit{for } t \to 0+,$$

where $c > 0$ and $a = 2m_{HC} + 1$, and the propagator semi-norm of the evolution is

$$\|S(t)\|_E := \sup_{\|x(0)\|_E = 1, \textit{for consistent } x(0)} \|x(t)\|_E, \qquad t \geq 0.$$

Problem with semi-norm

For $\varepsilon > 0$, consider linear semi-dissipative Hamiltonian DAE system in staircase form with matrices

$$\check{E} = \begin{bmatrix} 5 & 3 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \check{J} = \begin{bmatrix} 0 & 1 & 1 & \varepsilon \\ -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ -\varepsilon & 0 & 0 & 0 \end{bmatrix}, \check{R} := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

such that $n_1 = n_2 = n_3 = n_4 = 1$.
For given $y_2(0) \in \mathbb{R}$, the solution is

$$y_1(t) = 0,\ y_2(t) = y_2(0)\ e^{-t},\ y_3(t) = -y_2(0)\ e^{-t},\ y_4(t) = -\frac{3}{\varepsilon} y_2(0)\ e^{-t},$$

and $y_4(0) = -3y_2(0)/\varepsilon$ can be large for small $\varepsilon > 0$.
In contrast, the squared weighted semi-norm of this solution satisfies $\|y(t)\|_E^2 = 2(y_2(0))^2 e^{-2t}$ for $t \geq 0$.

Generalized Lyapunov equation

Consider a linear DAE $E\dot{x} = Ax$ with square matrices E, A and an associated *generalized Lyapunov equation*

$$E^H XA + A^H XE = -E^H WE.$$

Theorem

Consider semi-dissipative Hamiltonian DAE with regular $\lambda E - A$ and finite HC-index.
For every W, the generalized Lyapunov equation has an explicit solution via staircase form. For all solutions X the matrix $E^H XE$ is unique. If W is positive (semi-)definite, then every solution X is positive (semi-)definite on the image of the spectral projection on left deflating subspace associated to finite eigenvalues.

- T. Stykel. *Analysis and numerical solution of generalized Lyapunov equations*. PhD thesis, Technische Universität, Berlin, Institut für Mathematik, 2002.
- T. Stykel. Stability and inertia theorems for generalized Lyapunov equations. *Linear Algebra Appl.*, 355:297–314, 2002.
- T. Reis, O. Rendel, and M. Voigt. The Kalman-Yakubovich-Popov inequality for differential-algebraic systems. *Linear Algebra Appl.*, 485:153–193, 2015.

Outline

Hypocoercivity in infinite dimensions

Definition

A linear operator C on a Hilbert space $\mathcal{H}$, with domain $\mathcal{D}(C)$, is said to be *accretive* if the numerical range of C is a subset of the right-half plane, that is, if $\mathrm{Re}\langle Cx, x\rangle \geq 0$ for all $x \in \mathcal{D}(C)$. In this case $-C$ is said to be *dissipative*. And C is called *coercive* if there exists $\gamma > 0$ such that $\langle Cx, x\rangle \geq \gamma \|x\|^2$ for all $x \in \mathcal{D}(C)$.

This should be called semi-dissipative to be consistent.

▷ T. Kato, Perturbation theory for linear operators, Classics in Mathematics, Springer-Verlag, Berlin, 1995.

Theorem

Suppose that A is the infinitesimal generator of the C_0-semigroup $T(t)$ on the Hilbert space $\mathcal{H}$. Then $T(t)$ is exponentially stable if and only if there exists a bounded positive operator $P \in \mathcal{L}(\mathcal{H})$ such that

$$\langle Ax, Px\rangle + \langle Px, Ax\rangle = -\langle x, x\rangle \qquad \text{for all } x \in \mathcal{D}(A)\ .$$

Definition

Let C be a (possibly unbounded) operator on a Hilbert space $\mathcal{H}$ with kernel $\ker C$. Let $\widetilde{\mathcal{H}}$ be a Hilbert space, which is continuously and densely embedded in $(\ker C)^\perp$, endowed with a scalar product $\langle \cdot, \cdot \rangle_{\widetilde{\mathcal{H}}}$ and norm $\| \cdot \|_{\widetilde{\mathcal{H}}}$.
The operator C is called *hypocoercive* on $\widetilde{\mathcal{H}}$ if $-C$ generates a uniformly exponentially stable C_0-semigroup $(e^{-Ct})_{t\geq 0}$ on $\widetilde{\mathcal{H}} \hookrightarrow (\ker C)^\perp$.

▷ C. Villani. Hypocoercivity. *Mem. Amer. Math. Soc.*, 202, 2009.

Lemma

Consider bounded operators $R, J \in \mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ such that R is self-adjoint and nonnegative, and J is skew-adjoint, i.e., $J^* = -J$. Then t.f.a.e.

1. *There exists $m \in \mathbb{N}_0$ such that*

$$\bigcap_{j=0}^{m} \ker\left(R^{1/2} J^j\right) = \{0\} .$$

2. *There exists $m \in \mathbb{N}_0$ such that*

$$\sum_{j=0}^{m} J^j R (J^*)^j > 0$$

Moreover, the smallest possible $m \in \mathbb{N}_0$ (if it exists) coincides.

The Oseen equations describe the flow of a viscous and incompressible fluid at low Reynolds numbers

$$\begin{cases} u_t &= -(b\cdot\nabla)u - \nabla p + \nu\Delta u\,, \quad t>0\,,\\ 0 &= -\operatorname{div} u\,. \end{cases}$$

Oseen equations arise when one linearizes the incompressible or nearly incompressible *Navier-Stokes equations* around constant vector field b.
Oseen equations can be viewed as operator DAE.

Time-dependent, incompressible Oseen equation on 2D torus $\mathbb{T}^2 := (0, 2\pi)^2$ with periodic boundary conditions.

$$\begin{aligned} u_t &= -(b \cdot \nabla)u - \nabla p + \nu \Delta u\,, \quad t > 0\,, \quad \text{on } \mathbb{T}^2\,, \\ 0 &= -\operatorname{div} u\,, \quad t \geq 0\,, \end{aligned}$$

for velocity field $u = u(x, t)$ and pressure $p = p(x, t)$ in $x \in \mathbb{T}^2$ and $t \geq 0$.
Viscosity coefficient $\nu > 0$ and constant drift $b \in \mathbb{R}^2$.

Theorem

Considered as operator DAE the solution $(u(\cdot,t),\ p(\cdot,t))$ of the isotropic Oseen equation converges, as $t \to \infty$, to a constant (in x and t) equilibrium with the exponential decay rate $\mu = \nu$.
The hypocoercivity index is 0.

The model

$$\begin{aligned} u_t &= -(b(x)\cdot\nabla)u - \nabla p + \nu\partial_{x_2}^2 u\,, \quad t>0,\ \text{on}\ \mathbb{T}^2, \\ 0 &= -\operatorname{div} u\,, \quad t\geq 0\,, \end{aligned}$$

prescribes transport with a drift $b(x)\in\mathbb{R}^2$ which may depend on $x\in\mathbb{T}^2$, and diffusion only in x_2.

Theorem

Let $b \in \mathbb{R}^2$ be constant with $b_1 \neq 0$. Then, the operator $C = b \cdot \nabla - \nu \partial_{x_2}^2$ is neither coercive nor hypocoercive in $\widetilde{\mathcal{H}} = \{u \in \mathcal{H} \mid \int_{\mathbb{T}^2} u dx = 0\}$.

Hypocoercivity index

Let $b = \begin{bmatrix} \sin(x_2) \\ 0 \end{bmatrix}$ so consider operator.

In frequency domain the modal dynamics is hypocoercive with hypocoercivity index 1 and we can derive long-term decay hebavior.

Summary and further results

- Concepts of hypocoercivity and hypocontractivity for linear dynamical systems finite and infinite dimensional.
- Shifted HC index, scaled dHC index.
- Transformation invariance properties.
- Bilinear transformations between continuous and discrete time.
- Application to Oseen-type equations.
- F. Achleitner, A. Arnold, and V. Mehrmann. *Hypocoercivity and hypocontractivity concepts for linear dynamical systems*. http://arxiv.org/abs/2204.13033. F. Achleitner, A. Arnold, and V. Mehrmann. *Hypocoercivity in algebraically constrained partial differential equations with application to Oseen equations*. In preparation.

Things to do

- Linear time varying and nonlinear systems.
- Port-Hamiltonian systems with inputs and outputs (behavior setting).
- Extension to infinite dimensions.
- Discrete-time descriptor systems.
- Other operator DAEs.

- V. Mehrmann and B. Unger, *Control of port-Hamiltonian differential-algebraic systems and applications*, http://arxiv.org/abs/2201.06590, 2022.

Summary

- Relation between semi-dissipative/contractive and hypocoercive/hypocontractive systems.
- Controllability of (J, R) (U, H) characterizes hypocoercivity (hypocontractivity).
- HC-indices describes short-time decay.
- Staircase forms alloww to compute HC-indices.

Thank you very much
for your attention
and my sponsors for their support

- Research centers MATHEON, Einstein Center ECMath.
- DFG collaborative Research Centers 910, TRR154.
- DFG priority program 1984

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Let $E, A \in \mathbb{C}^{n,m}$. Then there exist nonsingular matrices $S \in \mathbb{C}^{n,n}$ and $T \in \mathbb{C}^{m,m}$ such that (for all $\mu \in \mathbb{C}$)

$$S(\mu E - A)T = \operatorname{diag}(\mathcal{L}_{\epsilon_1}, \ldots, \mathcal{L}_{\epsilon_p}, \mathcal{M}_{\eta_1}, \ldots, \mathcal{M}_{\eta_q}, \mathcal{J}_{\rho_1}, \ldots, \mathcal{J}_{\rho_r}, \mathcal{N}_{\sigma_1}, \ldots, \mathcal{N}_{\sigma_s}),$$

where the block entries have the following properties:

- Every entry $\mathcal{J}_{\rho_j}$ is a Jordan block of size $\rho_j \times \rho_j$, $\rho_j \in \mathbb{N}$, $\lambda_j \in \mathbb{C}$,

$$\mu \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} - \begin{bmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{bmatrix}.$$

Kronecker canonical form (KCF)

- Every entry $\mathcal{N}_{\sigma_j}$ is a nilpotent block of size $\sigma_j \times \sigma_j$, $\sigma_j \in \mathbb{N}$, (ev ∞)

$$\lambda \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} - \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

- Every entry $\mathcal{L}_{\epsilon_j}$ is a bidiagonal (singular) block of size $\epsilon_j \times \epsilon_j + 1$, $\epsilon_j \in \mathbb{N}_0$, of the form

$$\mu \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}.$$

- Every entry $\mathcal{M}_{\eta_j}$ is a bidiagonal (singular) block of size $\eta_j + 1 \times \eta_j$, $\eta_j \in \mathbb{N}_0$,

$$\mu \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix} - \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix}.$$

The Kronecker canonical form is unique up to permutation of the blocks. The biggest size of $\mathcal{N}_\sigma$ is called index and the sizes of the rectangular blocks are called left and right minimal indices.

Lemma

Let $E, J, R \in \mathbb{C}^{n,n}$ satisfy $E = E^H \geq 0$, $R = R^H \geq 0$ and $J = -J^H$. Then there exists a unitary matrix P, such that

$$\check{E} \quad := \quad P\,E\,P^H =: \begin{bmatrix} E_{1,1} & E_{2,1}^H & 0 & 0 & 0 \\ E_{2,1} & E_{2,2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \check{R} := P\,R\,P^H =: \begin{bmatrix} R_{1,1} & R_{2,1}^H & R_{3,1}^H & 0 & 0 \\ R_{2,1} & R_{2,2} & R_{3,2}^H & 0 & 0 \\ R_{3,1} & R_{3,2} & R_{3,3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\check{J} \quad := \quad P\,J\,P^H =: \begin{bmatrix} J_{1,1} & -J_{2,1}^H & -J_{3,1}^H & -J_{4,1}^H & 0 \\ J_{2,1} & J_{2,2} & -J_{3,2}^H & 0 & 0 \\ J_{3,1} & J_{3,2} & J_{3,3} & 0 & 0 \\ J_{4,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

These three matrices are partitioned in the same way, with (square) diagonal block matrices of sizes $n_1, n_2, n_3, n_4 = n_1, n_5$. In the case that the following blocks are present, $\begin{bmatrix} E_{1,1} & E_{2,1}^H \\ E_{2,1} & E_{2,2} \end{bmatrix}$ is positive definite, and the matrices $J_{4,1}$, $J_{3,3} - R_{3,3}$ are invertible.