This problem is NPC, because we can reduce hamiltonian path on grids to it.

1. dfs. A naive upper bound is $O^*(3^k)$, where k is the number of nonempty cells. We should be able to get more refined bounds, and some related papers are asymptotic number of hamiltonian paths on planar graphs [1], and on grid graphs [2]. An upper bound is the number of self-avoiding walks on the square graph $\mathbb{Z}^2$, which is estimated to be $O^*(2.639^k)$ [4, 5].

other related papers: [6].

my article: https://leetcode-cn.com/problems/path-with-maximum-gold/solution/fu-za-du-fen-xi-wei-shi-yao-dfsbu-hui-ch-0v9f/

2. We can use bitmask DP, or planar separator theorem + meet in the middle.

3. This the (node weighted) longest path problem in planar graphs. $2^{O(\sqrt{k}\log k)}$ [3].

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