SoCP - Gearing up for Algorithmic Thinking

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1 Problem 1

Consider a sequence of n numbers $a_1 < a_2 < a_3 < ..... < a_n$ such that no two are equal. We will calculate the number of permutations p of this sequence such that no three indices $i < j < k$ exist such that $a_k < a_i < a_j$.

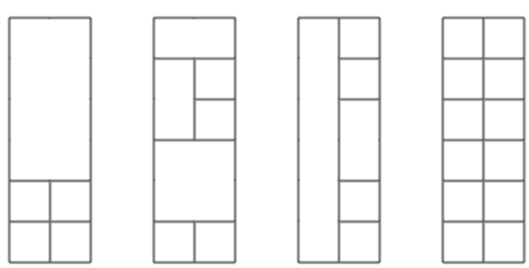
1. We will try to solve this problem in a recursive fashion. Assume that for any n, the number of such permutations is C_n. Let us suppose that the last number(a_n) is placed at index i in the final permutation. For some i, find the number of permutations such that the n^{th} number is at the index - i in terms of C_j s for some j in $1..n-1$.

2. Observe that i can range from 1 to n. Hence, get an expression for C_n in terms of $C_1, C_2, ..., C_{n-1}$ by adding up the terms for i ranging from 1 to n for the expression you got above.

3. (BONUS) Try to look for a closed form for the above recursion. You are encouraged to consult online sources for this.

2 Problem 2

There is a plank of width 2 and length n. This plank needs to be covered with rectangular tiles of width 1 or width 2 and arbitrary length. Possible examples are as follows.



We will calculate the number of such arrangements.

1. Assume that for any n, let us write the number of such arrangements as $A_n = B_n + C_n$ where B_n (lets call it type 1 arrangement) are those arrangements in which a single tile(of width 2) forms the top, whereas C_n (lets call it type 2 arrangement) are those in which 2 tiles(of width 1) together form the top. For example, the first and second examples above from the left are of type 1 and the other two are of type 2. We aim to relate B_n and $C_n$to B_i, C_i (or both) for $i < n$.

2. Consider a valid tiling of a tower of length $n-1$ of type 1. Think about all the possible ways you can extend this to valid arrangements of both type 1 and 2 for length n. Repeat it for type 2. To verify your answer, see whether you can get all examples for say $n = 3$ by applying the ways you came up with on examples with $n = 2$.
   Hint: Try to extend the topmost tile(s) or add a tile or more.

3. Using the above problem, try to write B_n and C_n in terms of B_{n-1} and C_{n-1}.

4. (BONUS) Try to write the above equations in the form of a vector equation. $(B_n, C_n) = (B_{n-1}, C_{n-1}) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. This helps in calculating the actual values faster. We will cover how to do this in SoCP. Stay tuned :P.

3 Problem 3

There are n participants in a tennis tournament such that $n \geq 3$. Every participant plays exactly one match against every other opponents, and more over every participant wins at least one match (and there are no draws). Show that there are three participants A, B, C for which the following holds: A wins against B, B wins against C, C wins against A.

1. Think about what happens for $n = 3, 4, \ldots$ Now consider a game with $n+1$ players and try to find a relation with the game of n players.

2. Alternatively, assume that the given situation is not possible and obtain a contradiction.

3. (BONUS) Generalization of the above problem: Suppose that for every set of ordered players a_i, a_j there exists a sequence of players $x_1, x_2, ..., x_p$ such that a_i wins against x_1, x_1 wins against x_2 and so on till x_p wins against a_j (where p can also be 0). Prove that with the constraints given in the original problem and the one given above, for all $3 \leq k \leq n$ there exits k players $a_1, a_2, a_3, ..., a_k$ such that a_1 wins against a_2, a_2 wins against a_3 and so on till a_k wins against a_1.