Direct - Sampling:

$$I(\gamma) = \int_0^1 dx \; x^{\gamma}$$

$$= \int_0^1 dx \; \underbrace{\pi(x)}_{=1} \; \underbrace{O[x]}_{x^{\gamma}}$$

this will be bad because there are some regions with high importance very rare

$\longrightarrow x = \mathrm{ran}(0,1)$

To Calculate $\int_0^1 dx \, x^{\gamma}$, we can sample x with probabilities $(dx \, x^{\gamma})$ and sum the values of x.

OR we can do $\int dO \; O \; \pi(O)$ and calculate the mean value of O, given probabilities $\pi(O)$

↳ cont)

For $\gamma > 0$:

$$\int_0^1 dx \quad x^\gamma \quad = \quad \int_0^1 d\vartheta \quad \vartheta \quad \pi(\vartheta)$$

with $\vartheta = x^\gamma$ (in general this is a difficult transform)

$$dx \cdot \underbrace{x^\gamma}_{\vartheta} = d\vartheta \quad \vartheta \quad \pi(\vartheta)$$

$$\ldots \quad \pi(\vartheta) = \frac{1}{\gamma} \, \vartheta^{\left(\frac{1}{\gamma} - 1\right)}$$

$\longrightarrow$

$\boxed{\text{For } \quad \gamma < 0}$:

$$\vartheta = x^{\gamma}$$

$$x = 0 \longrightarrow \vartheta = \infty$$

$$x = 1 \longrightarrow \vartheta = 1.$$

$$\int_0^1 dx \; x^{\gamma} \overset{!}{=} \int_1^{\infty} d\vartheta \; \vartheta \; \pi(\vartheta)$$

$$\pi(\vartheta) = -\frac{1}{\gamma} \, \vartheta^{-\left(1 - \frac{1}{\gamma}\right)}$$

(extra -1 from flipping
the integration limits)

cont.
$\longrightarrow$

- For $\gamma = -3/4$, $\pi(\theta) = \frac{4}{3}\,\theta^{-7/3}$

$$\langle \theta \rangle \propto \int_1^\infty d\theta \;\; \theta \cdot \theta^{-7/3}$$

$$\propto \left[\theta^{-1/3} \right]_1^\infty$$

$$= \text{finite} \qquad (\text{okay}).$$

- Second moment (and variance) is infinite:

$$\int_1^\infty d\theta \;\; \theta^2 \; \pi(\theta)$$

$$\propto \int_1^\infty d\theta \;\; \theta^2 \; \theta^{-7/3}$$

$$= \text{infinite}$$

→ Central limit theorem doesn't hold

(there are some rare events, but are very important, which affects the averaging

Now for cumulative distributions

(essentially probability for x to be less than x', instead of equal to),

we have theorems that hold which don't require finite variable and mean.

e.g. take $\gamma = 3/4$.

$$\int_0^1 dx \; x^{3/4} \rightarrow \int_0^1 d\theta \; \theta \; \pi(\theta)$$

$$\pi(\theta) = \frac{4}{3} \theta^{1/3}$$

Cumulative dist:

$$F(x) = \int_0^x dx' \, (x')^{3/4}$$

$$= x^{4/3} \qquad F_N.$$

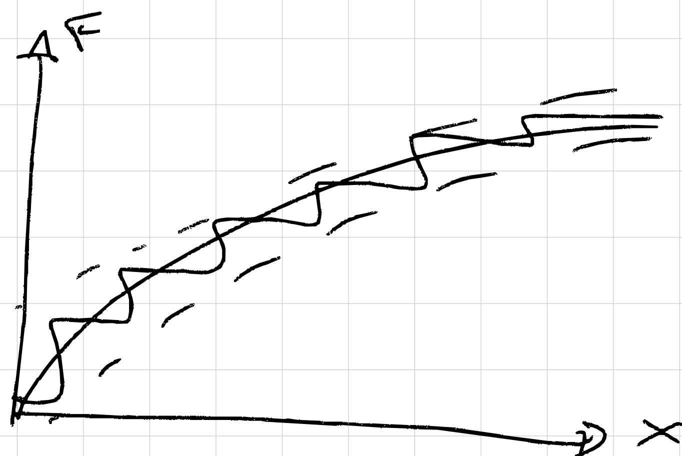
numerically easily can find the statistical cumulative distribution if we can sample the original.

cont $\rightarrow$

Theorem (1):

(Glivenko-Cantelli thm.)

$$\sup_x \left| \hat{F}_N(x) - F(x) \right| \to 0 \text{ for } N \to \infty$$



sup is the maximum distance between the two.

Theorem (2) : Inequality.

(Does not require the mean or variance to be finite... very strong theorem)

$$P\left(\sup_{x} \left| \hat{F}_N(x) - \hat{F}(x) \right| > \epsilon \right) \leq 2e^{-2N\epsilon^2}$$

- For fixed ϵ, at a given N, there is a known fraction $2e^{-2N\epsilon^2}$ for the inequality to be violated

(if we sample $\hat{F}_N(x)$ multiple times, the inequality holds for some known fraction of the time)