5-Symbol 8-State and 5-Symbol 6-State Universal Turing Machines*

Shigeru Watanabe

University of Tokyo

1. *Introduction*

It is of interest to design a universal Turing machine smaller than any ever previously published in the literature. According to Shannon's suggestion [1], the product of the number of states and the number of symbols would be an appropriate measure of the size of a Universal Turing machine (U.T.M.). We present here one machine with a product of 40—five symbols and eight states—and one with a product of 30—five symbols and six states—provided that symbols can be printed on the infinitely many squares of the input tape. The former is correctly an ordinary Universal Turing machine, but the latter is a slightly extended one. According to Davis' definition [2] of the Turing machine, the input condition must be expressed as follows: the input tape is always finite but can be extended by a certain given rule (presented below). The machine must have the property that, whenever it is about to run off an end of the tape, a row of new squares in which appear certain given symbols is spliced onto the end of the tape. Some published results are listed in Table 1.

TABLE 1

Symbol	*State*	*Product*	*Reference*
2	M	2M	Shannon [1]
N	2	2N	Shannon [1]
6	12	72	Takahashi [3]
6	10	60	Ikeno [4]
3	17	51	Watanabe [5]
6	7	42	Minsky [6]
5	8	40	Watanabe
5	6	30	Watanabe

Some definitions and symbols concerning a Turing machine are mentioned briefly here:

(1) The tape used is fixed, and the head of the machine can shift to scanning squares.

(2) States of U.T.M.—A, B, C, D, etc.

Symbols of U.T.M.—0, 1, 0′, 1′, and *

(3) A given Turing machine can be transformed into a 2-symbol Turing machine by using Shannon's transformation [1]. Furthermore, the tape of this 2-symbol Turing machine can be modified to one having a left end and an infinite right side. A Turing machine with such a tape is called a "prepared Turing

* Received January, 1961; Revised May, 1961.

machine" (P.T.M.). The transition table of a P.T.M. is shown in Table 2, where the states of the P.T.M. are $q_i = q_1, q_2, \cdots, q_M$ and the symbols are $s_j = 0, 1$; and then, the next symbol s_l, the next direction d, the next state q_k are written in the table element situated at the intersection of the current state q_i and the current symbol s_j in Table 2.

TABLE 2

s_j / q_i	0	1
q_1		
q_2		
q_3		
$\vdots$		

(4) The tape of the U.T.M. is divided into two parts, namely, an I-region having a right end and a W-region having a left end (see Table 3 or 11). The I-region inventories "table elements" (atomic acts or instructions) of the transition table (machine table) of the P.T.M. by using some symbols of the U.T.M. The W-region is the same as the tape of the P.T.M.

(5) In order to perform each instruction, $(q_i, s_j) \rightarrow s_l, d, q_k$, the acts of the U.T.M. consist of the following four items.

(i) To read the symbol on the current square in the W-region, and go to the place which represents the next state of the P.T.M. in the I-region. $(\mathrm{I} \leftarrow \mathrm{W})$

(ii) To go to the place which represents the next state of the P.T.M. in the I-region. $(\mathrm{I} \leftrightarrow \mathrm{I})$

(iii) To read the symbol in the I-region and go to the original square (the current square shown in (i)) in the W-region, and then write the symbol on it. $(\mathrm{I} \rightarrow \mathrm{W})_1$

(iv) To read the symbol of the next direction in the I-region, and go to the W-region, and then move right or left. $(\mathrm{I} \rightarrow \mathrm{W})_2$

2. *5-Symbol 8-State U.T.M.*

The 5-symbol 8-state U.T.M. has a finite I-region. The tape of this U.T.M. consists of a W-region and an I-region which is divided into I_α and I_β, as shown in Table 3.

TABLE 3

I_β	I_α	W

(1) W-REGION. The W-region, which is the same as the tape of the P.T.M., consists of a row of squares on which the symbols 0 or 1 are printed.

(2) I_α-REGION. Each table element of the I_α-region shows each instruction of the P.T.M. by using symbols which are written from right to left, as shown in the following items.

(a) A 0 is written on the first square at the right end of the table element,

(b) On the second square of the table element of this U.T.M., a 0 means the next right shifting and a 1 means the next left shifting of the P.T.M.

(c) On the third square, a 0 means the next symbol 0, and a 1 means the next symbol 1 of the P.T.M.

The specifications of items (b) and (c) are given in Table 4.

TABLE 4

Next symbol and direction of P T M		Symbols in I_α-region of U.T M	
s_l	d	c_{ij}	d_{ij}
0	R	0	0
0	L	0	1
1	R	1	0
1	L	1	1

(d) On the fourth square from the right end of the table element in the I_α-region of the U.T.M., a 0 is written.

(e) A row of 10's, the number of which is equal to the number of 0's which queue between the left side of this table element in the I_α-region and the right side of the table element which represents the next state of the P.T.M. in the I_β-region (see [3]), is arranged on the squares to the left of the symbol 0 specified by (d) in the table element of the U.T.M. That is, the table element (q_i, s_j) of I_α-region can be represented by the following expression, where a_{ij} is equal to the number of 0's between (q_i, s_j) in the I_α-region and $(q_k, 0)$ in the I_β-region: $(10)^{a_{ij}}0c_{ij}d_{ij}\,0$.

(f) When the transition table of the P.T.M. is specified as in Table 2, the table elements in the I_α-region are arranged from right to left as in Table 5.

TABLE 5

$\cdots(q_3,1)(q_3,0)(q_2,1)(q_2,0)(q_1,1)(q_1,0)$

(g) Thus, an I_α-region may be written, for example, as shown in Table 6.

TABLE 6

	------------------------0	$1010\text{---------}10100{\genfrac{}{}{0pt}{}{00}{11}}0$	$1010\text{---------}10100{\genfrac{}{}{0pt}{}{00}{11}}0$

NOTE: ${\genfrac{}{}{0pt}{}{0}{1}}$ means 0 or 1.

(h) When this U.T.M. is actually working in the I_α-region, the squares in each table element are scanned from left to right, although this structure has been explained from right to left.

(3) I_β-REGION.

(a) A 0 is written in the square at the right end of each table element (q_i, s_j) in the I_β-region.

(b) A row of 1's is arranged to the left of this 0. The number of 1's is equal to the number of 0's which queue between the right side of this table element (q_i, s_j) in the I_β-region and the left side of the table element (q_i, s_j) which is contained in the I_α-region and represents the same table element of the P.T.M. corresponding to the table element in the I_β-region.

(c) After $(q_i, 0)$ and $(q_i, 1)$ are thus arranged from right to left for each state q_i in the I_β-region as shown in Table 5, a 0 is added on the square at the left end of the table element $(q_i, 1)$ in I_β-region.

(d) A 0 is placed on the square at the right end of the I_β-region. That is, the table elements $(q_i,1),(q_i,0)$ of the I_β-region (a pair in Table 5) can be represented by the following expression, where b_{ij} is equal to the number of 0's between (q_i, s_j) in the I_β-region and (q_i,s_j) in the I_α-region: $0\,1^{b_{i1}}\,0\,1^{b_{i0}}\,0$.

(e) Thus, an I_β-region may be written, for example, as shown in Table 7.

TABLE 7

----------------0	011---------110	11-------------110	011------------110	11-------------110	0

(4) Prepared tape. Thus, a prepared tape of this machine may be written, for example, as shown in Table 8.

TABLE 8

$$\overbrace{\text{---0011101110011101110 0}}^{I_\beta\text{-region}}\text{-----}\overbrace{1010100\substack{00\\11}01010100\substack{00\\11}0}^{I_\alpha\text{-region}}\text{-----}\overbrace{101001}^{\text{W-region}}\text{---}$$

(5) Input tape. If the initial situation of the P.T.M. is expressed by the initial state q_a in the initial square, the corresponding initial situation of the U.T.M. is expressed by the following two items: (i) the initial state A (see Table 9) on the corresponding initial square in the W-region, and (ii) the treatment that all 0's on the squares of the corresponding table element $(q_a, 0)$ in the I_β-region and to the left of this element are changed to $*$'s as shown in Table 9 corresponding to Table 8.

TABLE 9

$$\text{---**111*111*0111011100--------}1010100\substack{00\\11}01010100\substack{00\\11}0\text{----10100A1---}$$

(6) Transition table. The transition table is given in Table 10.

TABLE 10

	0	1	*	0′	1′
A	* L B	* L C	R	0 L E	1 R
B	0′ L	1′ L	0′ L D	0 R G	1 R H
C	0′ L	1′ L	0′ L B	R G	R H
D	R C	1′ R E	R A	* L	L
E	R D	1′ L F	0′ R	* L D	R
F	0′ L	1′ L	0 R G	0 R B	1 R B
G	R A	R A	0 L F	0 R	1 R
H	L A	L A	1 L F	0 R	1 R

To explain precisely the actions of this machine, we give the transition diagram in Figure 1 and divide Table 10 into two parts as shown in Tables 11 and 12.

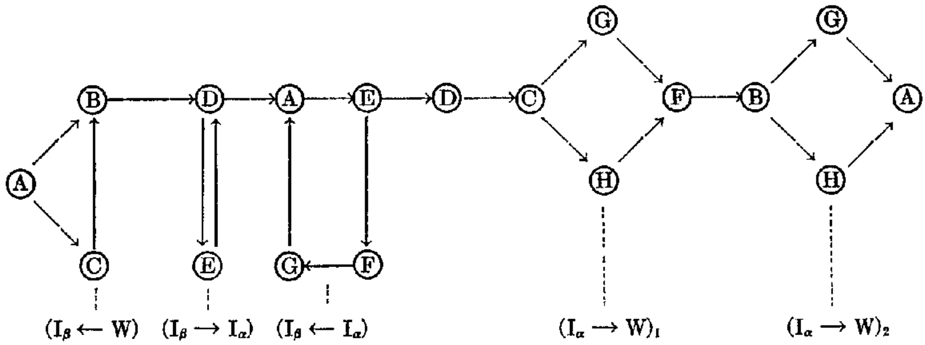

FIG. 1

TABLE 11

	0	1	*	0′	1′
A	* L B	* L C	R	0 L E ③	1 R
B	0′ L ①	1′ L	0′ L D		
C	0′ L	1′ L	0′ L B		
D		1′ R E	R A	* L ②	L
E		1′ L F	0′ R	* L D	R
F	0′ L ④	1′ L	0 R G		
G	R A			0 R	1 R

TABLE 12

	0	1	*	0′	1′
A	* L B ①	* L C		0 L E ⑤	
B				0 R G ⑪	1 R H
C				R G ③	R H
D	R C ⑦				
E	R D ⑥				
F	0′ L	1′ L	⑩	0 R B	1 R B
G	R A	R A	0 L F ⑨	0 R	1 R
H	L A	L A	1 L F	0 R	1 R

Thus, the acts of this machine consist of the following items:

(i) To read the symbol on the current square in the W-region, and go to the I_β-region $(I_\beta \leftarrow W)$. (Box ①)

(ii) From (q_i, s_j) in the I_β-region, go to (q_i, s_j) in the I_α-region $(I_\beta \rightarrow I_\alpha)$. (Boxes ② and ③)

(iii) From (q_i, s_j) in I_α-region, go to $(q_k, 0)$ in I_β-region $(I_\beta \leftarrow I_\alpha)$. (Boxes ④ and ⑤)

(iv) To read the symbol c_{ij} in the I_α-region and go back to the current square in the W-region, and then write c_{ij} on that square $(I_\alpha \rightarrow W)_1$. (Boxes ⑥, ⑦, ⑧, ⑨, ⑩)

(v) To read the symbol d_{ij} in the I_α-region and go back and shift right or left $(I_\alpha \rightarrow W)_2$. (Boxes ⑪ and ⑨)

For example, this machine, which has read "1" in the W-region at the state A, changes the input tape (see Table 9) to the following tape by using box ① of Table 12:

$$---**111*111*0111011100--------1010100\genfrac{}{}{0pt}{}{00}{11}01010C10'0'\genfrac{}{}{0pt}{}{0'0'}{1'1'}0'----1'0'1'0'0'*---$$

$$---**111*1B11'0'0'1'1'1'0'1'1'1'0'0'--------1'0'1'0'1'0'0'\genfrac{}{}{0pt}{}{0'0'}{1'1'}0'1'0'1'0'1'0'0'\genfrac{}{}{0pt}{}{0'0'}{1'1'}0'$$

$$----1'0'1'0'0'*------**11D10'1'1'1'0'0'1'1'1'0'1'1'1'0'0'--------1'0'1'0'1'0'0'\genfrac{}{}{0pt}{}{0'0'}{1'1'}$$

$$0'1'0'1'0'1'0'0'\genfrac{}{}{0pt}{}{0'0'}{1'1'}0'----1'0'1'0'0'*---$$

3. *5-Symbol 6-State U.T.M.*

The 5-symbol 6-state U.T.M. is an example of a small but special Universal Turing machine having infinitely many squares printed. The tape consists of a W-region and an I-region which consists of I_r $(r = 1, 2, 3, \cdots)$, each of which has the same construction, as shown in Table 13.

TABLE 13

	I_3	I_2	I_1	W

(1) W-REGION. The W-region of this U.T.M., which is the same as the tape of the P.T.M., consists of a row of squares on which 0's and 1's are printed.

(2) I-REGION. Each table element of the I-region shows each table element of the transition table of the P.T.M. by using symbols which are written from right to left as follows:

(a) A $*$ is written on the right end of the table element.

(b) c_{ij} and d_{ij} are written on the second and third squares corresponding to s_l and d, as in Table 4.

(c) A 0′ is written on the fourth square from the right end of the table element $(q_i, 0)$. A 0 is written on the fourth square from the right end of the table element $(q_i, 1)$.

(d) A row of 1's, the number of which is equal to the number of the 0′'s that queue between the left side of this table element in the I_r-table and the right side of the table element $(q_k, 1)$, where q_k is the next state of the P.T.M. in the

I_{r+1}-table of the U.T.M., is arranged on the squares to the left of the symbol $0'$ or 0 specified by (c) in the table element of the U.T.M. That is, the table elements $(q_i, 1)(q_i, 0)$ of the I_r-region (a pair in Table 5) can be represented by the following expression, where a_{ij} is equal to the number of $0'$'s between (q_i, s_j) in the I_r-region and $(q_k, 1)$ in the I_{r+1}-region: $1^{a_{i1}}0c_{i1}d_{i1}*1^{a_{i0}}0'c_{i0}\,d_{i0}*$

(e) When the transition table of the P.T.M. is specified as in Table 2, the table elements in the I_k-table are arranged as in Table 5.

(f) The two symbols $0', *$ are written on the square to the left of the I_r-table, corresponding to the atomic action $(q_M, 1) \equiv\ !$ (halt) in the P.T.M.

(3) PREPARED TAPE. Thus, the prepared tape of the U.T.M. may be written, for example, as in Table 14.

TABLE 14

$$\text{---}0'*\text{---}*1110^{00}_{11}*1110'^{00}_{11}*1110^{00}_{11}*1110'^{00}_{11}*1110^{00}_{11}*1110'^{00}_{11}*\text{---}1010\text{---}$$

(4) INPUT TAPE. If the initial situation of the P.T.M. is expressed by the initial state q_a on the initial square, the corresponding initial situation of the U.T.M. is expressed by the following two items: (i) the initial state A (see Table 16) on the corresponding initial square in the W-region, and (ii) the treatment that all $0'$'s and $*$'s on the tape to the right of the $*$ of the table element $(q_a, 1)$ in the I_1-region are changed to 0's, as shown in Table 15 corresponding to Table 14.

TABLE 15

$$\text{---}0'*\text{---}*1110^{00}_{11}*1110'^{00}_{11}*1110^{00}_{11}*1110^{00}_{11}01110^{00}_{11}01110^{00}_{11}0\text{---}\underset{\mathrm{A}}{1}010\text{----}$$

(5) TRANSITION TABLE. The transition table of this U.T.M. is shown in Table 13, where "!" stands for "halt".

TABLE 16

	0	1	*	0	1
A	* L B	* L C		1 R C	1 L D
B	0′ L	1′ L	′ R A	!	1 R D
C	0′ L	1′ L	0′ L B	1′ R E	R F
D	0′ L	1′ L	0′ L	1 R E	1 R F
E	R A	R A	0 L B	0 R	1 R
F	L A	L A	1 L B	0 R	1 R

The transition diagram of this machine is given in Figure 2.

Thus, the acts of this machine consist of the following items.

(i) From the current square on which a symbol s_j is written, go to the table element (q_i, s_j) in the I_r-region. $(I_r \leftarrow W)$

(ii) From (q_i, s_j) in the I_r-region, go to $(q_k, 0)$ in the I_{r+1}-region. $(I_{r+1} \leftarrow I_r)$

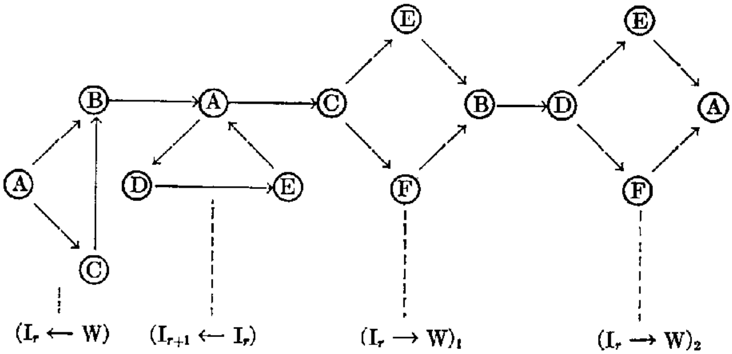

FIG. 2

(iii) To read c_{ij} in (q_i, s_j) in the I_r-region, and go back and write it. $(I_r \rightarrow W)_1$

(iv) To read d_{ij} in (q_i, s_j) in the I_r-region, and go back and shift right or left. $(I_r \rightarrow W)_2$

Note that although the infinite I-region is necessary for the representation of one Turing machine in this U.T.M., infinitely different kinds of transition tables of Turing machines can be written in the I-region—by the method of the diagonal arrangement.

REFERENCES

1. SHANNON, C. E. A Universal Turing Machine with two internal states In *Automata Studies*, Princeton, 1956.
2 DAVIS, M *Computability and Unsolvability*. McGraw-Hill, 1958.
3 TAKAHASHI, H. *Keisankikai II*. Iwanami, Toyko, 1958.
4 IKENO, N. *Proc , Inst Electrical Comm.*, Tokyo, 1960.
5. WATANABE, S On a minimum Universal Turing Machine. MCB Report, Tokyo, 1960.
6 MINSKY, M. A 6-symbol 7-state Universal Turing Machine. M I T., 1960.