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GÖDEL NUMBERINGS OF PARTIAL RECURSIVE FUNCTIONS

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In § 1 we present conceptual material concerning the notion of a Gödel numbering of the partial recursive functions. § 2 presents a theorem about these concepts. § 3 gives several applications. The material in § 1 and § 2 grew out of attempts by the author to find routine solutions to some of the problems discussed in § 3. The author wishes to acknowledge his debt in § 2 to the fruitful methods of Myhill in [M] and to thank the referee for an abbreviated and improved version of the proof for Lemma 3 in § 2.

1. The notion of a Gödel numbering. In the literature of mathematical logic, "Gödel numbering" usually means an effective correspondence between integers and the well-formed formulas of some logical calculus. In recursive function theory, certain such associations between the non-negative integers and instructions for computing partial recursive functions have been fundamental. In the present paper we shall be concerned only with numberings of the latter, more special, sort. By *numbers* and *integers* we shall mean non-negative integers. Our notation is, in general, that of [K]. If ϕ and ψ are two partial functions, $\phi = \psi$ shall mean that $(\forall x)[\phi(x) \simeq (\psi x)]$, i.e., that ϕ and ψ are defined for the same arguments and are equal on those arguments. We consider partial recursive functions of one variable; applications of the paper to the case of several variables, or to the case of all partial recursive functions in any number of variables, can be made in the usual way using the coordinate functions $(a)_i$ of [K, p. 230]. It will furthermore be observed that we consider only concepts that are invariant with respect to general recursive functions; more limited notions of Gödel numbering, taking into account, say, *primitive* recursive structure, are beyond the scope of the present paper.

Intuitively, a Gödel numbering is an association of numbers with partial recursive functions such that the following three conditions hold:

i) we are able effectively to tell whether or not a number is associated with a partial recursive function, i.e., the set of numbers associated is recursive;

ii) there is an effective procedure such that given any number associated with a function, we can find instructions for effectively computing that function;

iii) there is an effective procedure such that given instructions for effectively computing a partial recursive function, we can find an integer associated with that function.

A first step toward making this intuitive notion precise is to say what

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we mean by "instructions for effectively computing a partial recursive function." For the moment let us, somewhat arbitrarily, identify the idea of such instructions with the formal concept of *system of equations* of [K, § 56]; and let us take, as partial recursive function resulting from such a system, the function of one variable correlated with that system under the enumeration theorem of [K]. As a next step, let us correlate with each system of equations the integer which encodes it under the arithmetization of [K, § 56]. We now have one particular correspondence of integers with instructions which, in turn, gives an association of integers with partial recursive functions that will immediately be seen to satisfy the intuitive requirements i–iii. In fact these requirements are exceeded in that we have, for satisfying ii and iii, a single one-one correspondence between numbers and instructions. Let E be the set of integers correlated with systems of equations under this correspondence. (Of course the association of numbers with functions is not one-one, since each function can be given by $\aleph_0$ different systems of equations.) Let us now, for reasons of minor technical convenience, go to a new correspondence given by a recursive one-one mapping of the members of E onto the set of all integers. More specifically, define k by: $k(0) = \mu z[z \in E]$, and $k(x+1) = \mu z[z > k(x) \text{ and } z \in E]$. Then the new correspondence correlates, with any integer x, the system of equations correlated with $k(x)$ under the previous correspondence. This gives us a new association of integers with partial recursive functions which we call the *standard numbering*. Again, it satisfies the intuitive requirements i–iii and again it goes beyond them in that it gives a single one-one correspondence for ii and iii. For the remainder of the paper, ϕ_i ($i = 0, 1, 2, \ldots$) shall denote the partial recursive function whose standard number is i. (Note that the standard numbering happens not to be the 'Gödel numbering' of the Kleene enumeration theorem (see [K, § 65]), though the latter will, of course, come within the scope of our ultimate definition.)

In a general and obvious sense, each standard number can be identified with its corresponding system of equations. This leads us to give the following sequence of definitions as a formal counterpart to requirements i–iii.

Definition 1. A **numbering** π *is a mapping of a recursive set of integers* D_π, *called the* **domain** *of* π, onto *the set of partial recursive functions.*

Definition 2. A numbering π *is* ***semi-effective*** *if there exists a partial recursive function of two variables* Φ *such that for every* $i \in D_\pi$, $\Phi(i, x)$ *is identical, as a partial function of* x, *with* πi. *Any such* Φ *determines the numbering. We shall say that* Φ ***describes*** π.

Definition 3. A numbering π *is* ***fully effective*** *if there exist a partial recursive function of two variables* Φ *and a recursive function* f *such that:* Φ *describes* π, f *takes all values in* D_π; *and, for all* i, $\Phi(f(i), x)$ *is identical, as a partial function of* x, *with* ϕ_i.

Example. That the standard numbering is fully effective is immediate from the enumeration theorem. From cardinalities, we see that there are numberings which are not semi-effective. The following is an example of a semi-effective numbering that is not fully effective. For any i, we map the number $2i$ into the partial recursive function which is undefined at zero and is $\simeq\phi_i$ for argument $\neq 0$, and we map the number $2i+1$ into the partial recursive function which takes the value $(i)_0$ at argument zero and is $\simeq\phi_{(i)_1}$ for argument $\neq 0$. ($\langle(i)_0, (i)_1\rangle$ is the i^{th} ordered pair in a certain effective ordering, with repetitions, of all ordered pairs of integers; see [K, p. 230].) Since every partial recursive function appears, this is a numbering. By routine construction, it is easily shown to be semi-effective. If it were fully effective, however, we would be able effectively to tell, for any system of equations, whether or not the associated partial recursive function of one variable is defined for argument zero. The latter is impossible since it would enable us, again by a routine construction, to give a recursive characteristic function for every recursively enumerable set and thus violate the hierarchy theorem of [K, p. 283]. (Equivalently: it would enable us effectively to solve the *halting problem* for Turing machines, which is known to be recursively unsolvable; see [D].)

On the basis of Church's Thesis ([K, § 62]), Definitions 1, 2, and 3 will be seen to express progressively the requirements i, ii, and iii. This suggests identifying the notion of a Gödel numbering with the formal concept of fully effective numbering. Before doing so, however, we try to recast the formal concept in an invariant form, independent of whichever Gödel numbering is first chosen as a standard. This is accomplished in the following definition and theorem.

Definition 4. Two numberings, ρ and π, are **equivalent** *if there exist a recursive function g mapping D_ρ into D_π and a recursive function h mapping D_π into D_ρ such that $\rho = \pi g$ on D_ρ and $\pi = \rho h$ on D_π.*

The reader can easily verify that this gives an equivalence relation over the class of all numberings. The following theorem is now intuitively clear and can be formally verified by routine methods.

THEOREM. *A numbering is fully effective if and only if it is equivalent to the standard numbering.*

We thus have the invariance we desired since out concept has been reformulated as an equivalence class. The main definition follows.

Definition 5. A **Gödel numbering** *is a numbering equivalent to the standard numbering.*

While this definition as an equivalence class is *invariant*, it is not *intrinsic*. That is to say, it still depends on the initial choice of *some* member of the equivalence class. It is of interest to find an intrinsic definition, if possible. This is accomplished as follows.

Definition 6. A numbering ρ is **derivable** *from a numbering π if there* *exists a recursive function g mapping D_ρ into D_π, such that $\rho = \pi g$ on D_ρ.*

Intuitively, "ρ derivable from π" means that π is at least as general as ρ in the sense that we can effectively go from any ρ number to a π number for the same function. We abbreviate "ρ derivable from π" as "$\rho \leqq \pi$". $\leqq$ is transitive and reflexive, and, furthermore, [$\rho \leqq \pi$ and $\pi \leqq \rho$] if and only if ρ and π are equivalent under Definition 4. $\leqq$ thus defines a partial ordering over the equivalence classes of numberings; we call it the *partial order of numberings*. If we restrict the relations of equivalence and of $\leqq$ to the semi-effective numberings, we get another partial ordering, the *partial order of semi-effective numberings.*

THEOREM. *The partial order of semi-effective numberings possesses a unique maximal element, and this element is the class of Gödel numberings.*

PROOF. It is enough to show that for any numbering ρ, ρ is semi-effective if and only if ρ is derivable from the standard numbering. Let Φ describe the standard numbering. Assume ρ is derivable from the standard numbering by g; then $\Phi(g(i), x)$ as function of i and x describes ρ and ρ is semi-effective. Conversely, assume Φ_ρ describes ρ; then substituting a numeral for the first variable in the system of defining equations for Φ_ρ will yield a system of defining equations and hence a standard number for the corresponding function under the ρ numbering. This concludes the proof.

The theorem says simply that a Gödel numbering is a semi-effective numbering from which every semi-effective numbering is derivable. Since the definitions of semi-effectiveness and of derivability depend only on the abstract classes of recursive functions and partial recursive functions, we have the intrinsic characterization that we sought. The characterization requires no initial choice of a numbering and enables us, furthermore, to dispense entirely with our initial step of interpreting "instructions" as expressions of a particular formalism, (systems of equations in our case).

Remark. The partial orders mentioned above are of some independent interest. We list several facts.

a) Each of the orders is an upper semi-lattice. For if π_1 and π_2 are given numberings, then the numbering ρ will represent their least upper bound, where $\rho(2i) = \pi_1 i$ and $\rho(2i+1) = \pi_2 i$, $i = 0, 1, 2, \ldots$. If π_1 and π_2 are semi-effective, so is ρ.

b) The semi-lattice of semi-effective numberings is a sub-ordering of the semi-lattice of numberings. This follows from the last theorem and from the fact that a numbering is derivable from a semi-effective numbering only if it is semi-effective.

c) The semi-lattice of numberings can, in turn, be identified with an appropriate sub-ordering of the lattice of mass problems described by Médvédév in [Me]. For to each numbering π can be correlated the mass problem represented by the class of all functions f such that $\pi i = \phi_{f(i)}$

for all i. The correspondence is easily seen to be well defined and appropriately invariant.

d) If Definitions 1, 5, and 6 are altered by replacing the set of all partial recursive functions by some other fixed countably infinite set, a semi-lattice is generated that is isomorphic to the semi-lattice of numberings. The semi-lattice of numberings can therefore be viewed as a generalization "to infinitely many truth values" of the semi-lattice of degrees of unsolvability with respect to *many-one* (sometimes called *strong*) reducibility, (see [K&P]). If the fixed countably infinite range of the numberings is replaced by a finite range of two elements, then the semi-lattice of numberings is replaced by the usual many-one semi-lattice with the degrees of the empty and universal sets deleted.

e) It is easy to show incomparable semi-effective numberings. Greatest lower bounds for pairs of incomparable semi-effective numberings can be shown to exist in certain cases. The semi-lattice of semi-effective numberings contains a sub-ordering isomorphic to the lattice of finite sets of integers, where $\supset$ (set containment) is correlated with $\leqq$, where the empty set is correlated with the class of Gödel numberings, and where greatest lower bounds are preserved but least upper bounds are not.

f) The following questions are open. Is either of the numbering semi-lattices a lattice? If not, is it still true that any two numberings possess a lower bound?

g) The following example illustrates e above. Let C_Λ be the class of Gödel numberings. Let $C_{\{0\}}$ be the class of numberings equivalent to the non-fully effective numbering defined in the Example following Definition 3. Let $C_{\{1\}}$ be the equivalence class of a numbering defined in a similar way, but using argument 1 in place of argument 0. Let $C_{\{0,1\}}$ be the class of a numbering defined in a similar way using arguments 0 and 1 together in place of argument 0, i.e., all possible combinations of behavior are explicitly associated with these arguments as we progress through the numbering. In a similar way C_A can be defined for any finite set A. It is easy to show, for instance, that $C_{\{0\}}$ and $C_{\{1\}}$ are incomparable, that $C_{\{0,1\}}$ is a greatest lower bound for $C_{\{0\}}$ and $C_{\{1\}}$, and that C_Λ is *not* the least upper bound for $C_{\{0\}}$ and $C_{\{1\}}$. More generally, $C_{A\cup B}$ is the greatest lower bound of C_A and C_B, but $C_{A\cap B}$, although an upper bound, is not the least upper bound if A and B are incomparable.

2. The isomorphism of Gödel numberings. For the standard numbering we saw that there is an effective one-one correspondence between numbers and systems of equations. A natural question now presents itself: can such a correspondence be found for any Gödel numbering? This can be more invariantly formulated as the problem: given any two Gödel numberings, is there a recursive one-one correspondence carrying one numbering onto the other? An affirmative answer is given in the following isomorphism theorem. The methods to be used do not directly carry over to the case of semi-effective numberings that are not Gödel numberings, and the answer to the problem for the latter remains unknown. Our approach will be similar, in outline, to Myhill's approach to the isomorphism of creative sets in [M, Theorem 19]. Some of the difficulties encountered, however, will be essentially different.

THEOREM. *Given any two Gödel numberings ρ and π, there exists a recursive function f such that f gives a one-one mapping of D_ρ* onto *D_π and such that $\rho = \pi f$ on D_ρ.*

PROOF. If f is a recursive function mapping D_ρ one-one onto D_π so that $\rho = \pi f$ on D_ρ, then the function f', where $f' = f^{-1}$ on D_π and $f' = 0$ elsewhere, is a recursive function which maps D_π one-one onto D_ρ so that $\pi = \rho f'$ on D_π; furthermore, if ρ and ρ' are numberings and if f_1 maps D_ρ onto D_π as in the theorem and f_2 maps D_π onto $D_{\rho'}$ as in the theorem, it is immediate that $f_2 f_1$ maps D_ρ onto $D_{\rho'}$ as in the theorem. It will therefore be enough to show that the theorem holds when ρ is any Gödel numbering and π is the standard numbering.

We take ρ to be any Gödel numbering and π to be the standard numbering; we take g and h (by Definition 5) to be recursive functions such that g maps D_ρ into D_π, h maps D_π into D_ρ, $\rho = \pi g$ on D_ρ, and $\pi = \rho h$. We obtain the proof of our theorem through three lemmas. Lemmas 2 and 3 use specific properties of the standard numbering. D_π is, of course, the set of all integers.

LEMMA 1. *If g' and h' are recursive functions with the properties that g' maps D_ρ* one-one *into D_π, h' maps D_π* one-one *into D_ρ, $\rho = \pi g'$ on D_ρ, and $\pi = \rho h'$, then there exists a recursive function f such that f maps D_ρ* one-one *onto D_π and $\rho = \pi f$ on D_ρ.*

From Lemma 1 it follows that if g and h can be made into functions one-one on D_ρ and D_π, the theorem holds. The proof of the lemma uses the methods of the proofs of [M, Theorems 17 and 18]. We present our proof in outline and refer the reader to [M] for further detail.

PROOF OF LEMMA 1. We define a *finite correspondence* between D_ρ and D_π to be a finite (possibly empty) sequence of ordered pairs $\langle x_1, y_1\rangle$, $\langle x_2, y_2\rangle, \ldots, \langle x_n, y_n\rangle$ such that

$$\left.\begin{array}{l} x_j \in D_\rho \text{ and } y_j \in D_\pi, \\ i \neq j \text{ implies } x_i \neq x_j \text{ and } y_i \neq y_j, \\ \rho x_j \text{ and } \pi y_j \text{ are the same partial function,} \end{array}\right\} \text{for all } 1 \leqq i, j \leqq n.$$

We then prove the following SUBSIDIARY LEMMA: *there is a uniform* *effective method such that if* $\langle x_1, y_1\rangle, \ldots; \langle x_n, y_n\rangle$ *is a finite correspondence and if* $x' \in D_\rho$ *and* $x' \neq x_j$ *for* $1 \leqq j \leqq n$, *then we can find a* y' *such that* $\langle x_1, y_1\rangle, \ldots, \langle x_n, y_n\rangle, \langle x', y'\rangle$ *is a finite correspondence.* The proof of this subsidiary lemma is as follows. (Convention: values of $j, j_1, j_2, \ldots$ are taken between 1 and n inclusive.) Consider $g(x')$ and see if $g(x') \neq y_j$ for all j; if so, set $y' = g(x')$. If not, and $g(x') = y_{j_1}$ for some j_1, consider $g(x_{j_1})$ and see if $g(x_{j_1}) \neq y_j$ for all j; if so, set $y' = g(x_{j_1})$. If not, and $g(x_{j_1}) = y_{j_2}$ for some j_2, consider $g(x_{j_2})$ and see if $g(x_{j_2}) \neq y_j$ for all j; etc. Since g is one-one on D_ρ and since the given correspondence is finite, this procedure must end with the determination of an appropriate y'. Note that the one-one property of g insures $j_1 \neq j_2 \neq \ldots$. From the symmetry of the conditions, the Subsidiary Lemma also holds with "$x' \in D_\rho$ and $x' \neq x_j$" replaced by "$y' \in D_\pi$ and $y' \neq y_j$" and "y' such that" replaced by "x' such that" in its statement; for this form, the proof uses the function h.

Using the recursiveness of D_ρ and D_π and making alternate applications of the two forms of our Subsidiary Lemma, we can effectively enumerate a sequence of ordered pairs such that: at every stage, the ordered pairs enumerated up to that stage constitute a finite correspondence; and every member of D_ρ eventually appears as the first member of some ordered pair and every member of D_π eventually appears as the second member of some ordered pair. Together with the condition that $f(x) = 0$ for $x \notin D_\rho$, the ordered pairs of this enumeration define a recursive function f on D_ρ with the desired properties. This completes the proof of Lemma 1.

LEMMA 2. *If* g *is a recursive function mapping* D_ρ *into* D_π *such that* $\rho = \pi g$ *on* D_ρ, *then there exists a recursive function* g' *which gives a* one-one *mapping of* D_ρ *into* D_π *such that* $\rho = \pi g'$ *on* D_ρ.

PROOF OF LEMMA 2. There exists a recursive function of two variables Ψ such that for fixed i and varying x, $\Psi(i, x)$ takes as values a succession of distinct integers each of which is a standard number for ϕ_i. A definition for Ψ can be obtained by setting up a uniform effective method by which any system of equations can be successively enlarged, through the inclusion of irrelevant or redundant equations, without alteration of the partial recursive function defined by the system. The construction is straightforward and we omit details. g' can now be obtained from g by the equations

$$\begin{aligned} g'(0) &= g(0), \\ g'(x+1) &= \Psi(g(x+1), t(x+1)), \\ t(x+1) &= \mu y[\Psi(g(x+1), y) > g'(x)]. \end{aligned}$$

g' is defined for every argument since $\Psi(i, x)$ has an infinite range for each i. g' is one-one since it is monotone increasing. $\rho = \pi g'$ on D_ρ since $\rho = \pi g$ on D_ρ and since, for all j, $g'(j)$ and $g(j)$ are standard numbers for the same function (by the definition of Ψ). This concludes the proof of Lemma 2.

LEMMA 3. *If h is a recursive function mapping D_π into D_ρ such that $\pi = \rho h$, then there exists a recursive function h' which gives a one-one mapping of D_π into D_ρ such that $\pi = \rho h'$.*

PROOF OF LEMMA 3. The situation of Lemma 2 is not symmetric, since π is distinguished as the standard numbering; therefore the proof of Lemma 2 does not directly apply.

A similar construction can be made, however, if we can obtain a recursive function of two variables Ψ^* with the following properties: for fixed i and varying x, $\Psi^*(i, x)$ takes as values a succession of distinct integers each of which is a standard number for ϕ_i; and furthermore, given fixed i, then for any j and k, $h\Psi^*(i, j) \neq h\Psi^*(i, k)$ if $j \neq k$. For we can then define h' by the equations

$$h'(0) = h(0),$$
$$h'(x+1) = h\Psi^*(x+1, t(x+1)),$$
$$t(x+1) = \mu y[h\Psi^*(x+1, y) > h'(x)].$$

h' will be defined for every value since $\Psi^*(i, x)$ has infinite range for each i; h' will be one-one since it is monotone increasing; and $\pi = \rho h'$, since $\pi = \rho h$ and from the properties of Ψ^*, $h'(j)$ and $h(j)$ are ρ numbers for πj, $j = 0, 1, \ldots$.

It remains to construct the function Ψ^*. Our construction makes two applications of the recursion theorem, [K, Theorem XXVII]. We restate this theorem in the following fixed-point form: there exists a uniform effective method by which, given any recursive function f, an integer n can be found such that $\phi_{f(n)} = \phi_n$.

To find Ψ^*, it suffices to find a uniform effective method for doing the following: given any finite sequence $i_0, i_1, \ldots, i_r$ such that $\pi i_0 = \pi i_1 = \ldots = \pi i_r$, to get an integer i_{r+1} such that $\pi i_r = \pi i_{r+1}$ and $h(i_{r+1}) \notin \{h(i_0), h(i_1), \ldots, h(i_r)\}$. The recursion theorem gives such a method. For define the partial recursive function

$$\psi(x) \simeq \begin{Bmatrix} \phi_{i_0}(x) \text{ if } h(t) \notin \{h(i_0), \ldots, h(i_r)\}, \\ \text{undefined otherwise} \end{Bmatrix}$$

with t as a parameter. From this definition one can obtain a standard number for ψ as a recursive function of t. Applying the recursion theorem, we have an n such that

$$\phi_n(x) \simeq \begin{Bmatrix} \phi_{i_0}(x) \text{ if } h(n) \notin \{h(i_0), \ldots, h(i_r)\} \\ \text{undefined otherwise} \end{Bmatrix}.$$

Now if, in fact, $h(n) \notin \{h(i_0), \ldots, h(i_r)\}$, we take $i_{r+1} = n$ and are done. If, on the other hand, $h(n) \in \{h(i_0), \ldots, h(i_r)\}$, we know that ϕ_{i_0} is everywhere undefined. In this case we go on to make a second application of the recursion theorem yielding a fixed point m such that

$$\phi_m(x) \simeq \begin{Bmatrix} 0 \text{ if } h(m) \in \{h(i_0), \ldots, h(i_r)\} \\ \text{undefined otherwise} \end{Bmatrix}.$$

To avoid contradiction, ϕ_m must be everywhere undefined and $h(m) \notin \{h(i_0), \ldots, h(i_r)\}$. We hence set $i_{r+1} = m$.

By repeated applications of this procedure, values of Ψ^* can be inductively computed for any given arguments. To make the above semi-formal construction entirely formal is a matter of straightforward though fairly extensive detail. We omit it. This concludes the proof of Lemma 3.

Lemmas 1, 2, and 3 provide us with the desired function f, and hence the proof of the isomorphism theorem is complete. The constructions of the three lemmas give, in fact, a uniform effective way of obtaining f from g and h. From this a corollary follows.

COROLLARY. *There exists a partial recursive functional f, in partial functions g and h, such that if ρ and π are equivalent by g and h (under Definition 4), then f is a recursive function and ρ and π are isomorphic under f in the sense of the above theorem.*

3. Applications. Two applications of § 1 and § 2 are now described.

Application I. Equivalent definitions of effectively computable function. It is known that the concepts of Post normal system and Turing machine are equivalent ways of approaching the partial recursive functions; see [K, p. 320] and [D]. With an appropriate effective coding of machines into integers and an appropriate effective coding of normal systems into integers, this equivalence becomes the equivalence, in the sense of Definition 4, of two Gödel numberings of the partial recursive functions. Hence we can conclude by the isomorphism theorem that there exists an effective one-one correspondence mapping the Turing machines onto the normal systems in such a way that corresponding entities define the same partial recursive function. Similar conclusions hold with respect to systems of equations, λ-definability schemata, semi-Thue systems, Thue systems, etc. (See [K] and [D].)

Application II. Creative sets. (Notation: "$\{x|\ldots x\ldots\}$" for "the set of integers x such that $\ldots x\ldots$"; "$\phi \to B$" for "B is the range of ϕ"; "$\phi \to y$" for "y is *in* the range of ϕ"; and "$\bar{C}$" to denote $\{x|x \notin C\}$.) A set C is *creative* (see [P]) if there exist a partial recursive function ϕ such that $\phi \to C$ and a partial recursive function ψ such that for any number n, if $\phi_n \to B$ and $B \subset \bar{C}$, then $\psi(n)$ is defined, $\psi(n) \in \bar{B}$ and $\psi(n) \in \bar{C}$; such a ψ is called a *productive* partial function for C. In [M], Myhill shows that any two creative sets, call them C_1 and C_2, are isomorphic in the sense that there exists a recursive function f, one-one and *onto* the integers (a 'recursive permutation'), such that for all x, $x \in C_1 \equiv f(x) \in C_2$.

THEOREM. *If ρ is a Gödel numbering then $\{x|x \in D_\rho$ and $\rho x \to x\}$ is creative. Conversely if C is a creative set, then there is a Gödel numbering ρ such that $C = \{x|x \in D_\rho$ and $\rho x \to x\}$, and ρ can be put into recursive one-one correspondence with the standard numbering.*

PROOF. The last clause is immediate from the isomorphism theorem. It remains to show C creative if and only if $C = \{x|x \in D_\rho \text{ and } \rho x \to x\}$ for some Gödel numbering ρ.

We take π to be the standard numbering. Let ρ be any given Gödel numbering, and define $C = \{x|x \in D_\rho \text{ and } \rho x \to x\}$. g and h are taken to be recursive functions associated with ρ and π as in Definitions 4 and 5. Then $C = \{x|x \in D_\rho \text{ and } \pi g(x) \to x\}$. By the methods of [P], a partial recursive function ϕ exists such that $\phi \to C$; and, taking h as productive function, we see that C is creative.

Conversely, let C be a given creative set, and take $C_0 = \{x|\phi_x \to x\}$. By Myhill's theorem, C and C_0 are isomorphic, and we have a recursive function f which is one-one and *onto* the integers such that for all x, $x \in C \equiv f(x) \in C_0$. Thus for all x, $x \in C. \equiv .\phi_{f(x)} \to f(x). \equiv .\pi f(x) \to f(x)$.

Now consider the partial recursive function $f\phi_i$ defined by composition of f with ϕ_i, $i = 0, 1, \ldots$. Since f is recursive, a simple method can be described for carrying a system of equations for ϕ_i into a system of equations for $f\phi_i$; we hence have a recursive function s such that $\phi_{s(i)} = f\phi_i$ for all i. This relationship can be expressed as the commutation relation

$$\pi s = f\pi.$$

Similarly, since f^{-1} is a recursive function, we can get a recursive function t such that

$$\pi t = f^{-1}\pi.$$

We now define

$$g = tf$$

and

$$h = f^{-1}s.$$

The desired numbering is defined by

$$\rho = \pi g.$$

Since ρ is defined for all integers, we take D_ρ to be the set of all integers. To verify that ρ is a Gödel numbering, we must show (by Definition 5) that $\pi = \rho h$.

We argue from the identities displayed above:

$$\rho h = \pi g f^{-1} s = \pi t f f^{-1} s = \pi t s = f^{-1}\pi s = f^{-1} f \pi = \pi.$$

Finally we observe that for all x,

$$\rho x \to x. \equiv .\pi g(x) \to x. \equiv .\pi t f(x) \to .x. \equiv .f^{-1}\pi f(x) \to x$$
$$. \equiv .\pi f(x) \to f(x). \equiv .f(x) \in C_0. \equiv .x \in C.$$

Hence $C = \{x|\rho x \to x\}$, and the proof is completed.

COROLLARY. *C is a creative set if and only if there is a Gödel numbering ρ whose domain is all the integers such that $C = \{x|\rho x \to x\}$.*

Remark. The representation theorem and corollary of Application II are of interest in that the creative sets first studied by Post [P] were, in effect, defined as above with the use of particular Gödel numberings. Creative sets can also be obtained from Gödel numberings by definitions of the form $C = \{x|\rho x$ is defined for argument $x\}$, (see [K]); relevant forms of the above theorem and corollary can be proved in a similar way.

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