# CMU 21-122 Notes

November 8, 2022

# 1 Differential Equations

## 1.1 Differential Model

### 1.1.1 Population Models

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

where $P$ is the population and $M$ is the carrring capacity (decrease to words M if it ever exceeds M). $P(t) = M$ and $P(t) = 0$ are two **equlibrium solutions.**

### 1.1.2 The Motion of a spring

$$F = ma \Leftrightarrow m\frac{d^2x}{dt^2} = -kx \Leftrightarrow \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

This is an example of what is called a second-order differential equation because it involves second derivatives.

### 1.1.3 General diferential Equations

In general, a **differential equation** is an equation that contains an unknown function and one or more of its derivatives. The order of a differential equation is the order of the highest derivative that occurs in the equation.A function $f$ is called a **solution** of a differential equation if the equation is satisied when $y = f(x)$ and its derivatives are substituted into the equation.

## 1.2 Slope Field and Euler's Method

### 1.2.1 Slope field

As a guide to sketching the rest of the curve, let's draw short line segments at a number of points $(x, y)$ with slope of the differential function. The result is called a **direction field.**

### 1.2.2 Euler's method

Approximate values for the solution of the initial-value problem $y' = F(x.y)$ , $y(x_0) = y_0$ with step $h$ at $x_n = x_{n-1} + h$ are:

$$y_0 = y_{n-1} + hF(x_{n-1}, y_{n-1})$$

## 1.3 Seperable Functions

A separable equation is a irst-order differential equation in which the expression for $\frac{dy}{dx}$ can be factored as a function of x times a function of y. In other words, it can be written in the form:

$$\frac{dy}{dx} = g(x) f(y) \to \frac{dy}{dx} = \frac{g(x)}{h(y)} \to h(y) \frac{dy}{dx} = g(x)$$

### 1.3.1 Example 1

Find the orthogonal trajectories of the family of curves $x = ky^2$ , where $k$ is an arbitrary constant.

$$\frac{dy}{dx} = \frac{1}{2ky}$$

This differential equation depends on $k$, but we need an equation that is valid for all values of k simultaneously. Since $k = \frac{x}{y^2}$:

$$\frac{dy}{dx} = \frac{1}{2\frac{x}{y^2}y} = \frac{y}{2x}$$

On an orthogonal trajectory the slope of the tangent line must be the negative reciprocal of this slope. Therefore the orthogonal trajectories must satisfy the differential equation:

$$\frac{dy}{dx} = -\frac{y}{2x}$$

$$\int y dy = -\int 2x dx \to \frac{y^2}{2} + x^2 = C$$

## 1.4 Population Growth

### 1.4.1 The law of natural growth

The solution of the initial-value problem

$$\frac{dP}{dt} = kP,\ P(0) = P_0 \to P(t) = P_0 e^{kt}$$

### 1.4.2 The logistic model

The **logistic differential equation:**

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

and the solution is:

$$P(t) = \frac{M}{1 + Ae^{kt}} \text{ where } A = \frac{M - P_0}{p_0}$$

## 1.5 Linear Equations

A first-order **linear** differential equation is one that can be put into the form:

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

where $P$ and $Q$ are continous functions at a given interval. To solve the linear differential equation $y' + P(x)\,y = Q(x)$, multiply both side by the **integrating factor** $I(x) = e^{\int P(x)dx}$ and integrate both sides,

# 2 Parametric Equations and Polar Coordinates

## 2.1 Curves Defined by Parametric Equations

Suppose that x and y are both given as functions of a third variable t, called a parameter, by the equations:

$$x = f(t), \qquad y = g(t) \qquad a \leq t \leq b$$

are called **parametric equations**.

## 2.2 Calculus with Parametric Curves

### 2.2.1 Tangents

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

### 2.2.2 Area

$$A = \int_a^b y dx = \int_\alpha^\beta g(t)\,f'(t)\,dt$$

### 2.2.3 Arc Length

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

### 2.2.4 Surface Area

$$S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

## 2.3 Polar Coordinates

$$x = r\cos\theta \qquad y = r\sin\theta$$
$$r^2 = x^2 + y^2 \qquad \tan x = \tfrac{y}{x}$$

## 2.4 Calculus in Polar Coordinates

### 2.4.1 Area

$$A = \frac{1}{2}r^2\theta = \int_{\alpha}^{\beta} \frac{1}{2}r^2 d\theta$$

### 2.4.2 Arc Length

$$L = \int_{a}^{b} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta = \int_{a}^{b} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

### 2.4.3 Tangents

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta + r\sin\theta} = \tan\theta$$

# 3 Sequence and Series

## 3.1 Sequence

A **sequence** can be thought of as a list of numbers written in a definite order:

$$\{a_n\} = a_1, a_2, a_3, a_4, ..., a_n$$

A sequence $\{a_n\}$has the **limit** L and **we write**:

$$\lim_{n\to\infty} a_n = L$$

if the limit exists, the sequence **converges**, or else the sequence **diverges.**

A sequence is **bounded above** if M:

$$a_n \leq M$$

and is **bounded below** if there is a number such that:

$$M \leq a_n$$

Every bounded, monotonic sequence is convergent.

## 3.2 Series

Given a seires $\sum_{n=1}^{\infty} a_{\mathrm{n}} = a_1 + a_2 + a_3 + ... + a_n$,let $s_n$denotes its partial sum:

$$s_n = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ... + a_n$$

If the sequence is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number then $s_n$ is called **convergent**

$$\sum_{n=1}^{\infty} a_n = s$$

where s is the **sum of series**.

An Example of the series is a geometric series:

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + ... + ar^{n-1}$$

which converges if $|r| < 1$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

else the series is divergent

If the series $\sum_{n=1}^{\infty} a_n$is convergent, then $lim_{n\to\infty} a_n = 0$

## 3.3 Tests

### 3.3.1 The integral test

Suppose $f$ is a continous, positive, decreasing function on$[1, \infty)$ and let $a_n = f(n)$. Then:

1. if $\int_1^{\infty} f(x)\, dx$ is convergent, $\sum_{n=1}^{\infty} a_n$ is convergent
2. if $\int_1^{\infty} f(x)\, dx$ is divergent, $\sum_{n=1}^{\infty} a_n$ is divergent

**The p-series$\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent otherwise.**

**Suppose $f(k) = a_k$ is a continous, positive, decreasing function for $x \geq n$ and $\sum a_n$is convergent. If $R_n = s - s_n$:**

$$\int_{n+1}^{\infty} f(x)\, dx \leq R_n \leq \int_{n}^{\infty} f(x)\, dx$$

### 3.3.2 The Comparison test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms:

1. if $\sum b_n$ is convergent and $a_n \leq b_n$ for all $n$, then $\sum a_n$ is also convergent
2. if $\sum b_n$ is divergent and $a_n \geq b_n$ for all $n$, then $\sum a_n$ is also divergent

### 3.3.3 The Limit comparison test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms, If:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converges or diverges

### 3.3.4 Alternating series test

If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + ...$$

satsifies:

1. $b_{n+1} \leq b_n$ for all $n$
2. $\lim_{n\to\infty} b_n = 0$

Then the series is convergent.

### 3.3.5 Alternating Series Estimation Theorem

If $s = \sum (-1)^{n-1} b_n$ is the sum of an alternating series that converges:

$$|R_n| = |s - s_n| \leq b_{n+1}$$

## 3.4 Absolute convergence/divergence Tests

### 3.4.1 Ratio Tests

1. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$ then the series $\sum a_n$ is absolutely convergent
2. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ then the series $\sum a_n$ is absolutely divergent
3. If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L = 1$ then no conclusion can be drawn

### 3.4.2 Root tests

1. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$ then the series $\sum a_n$ is absolutely convergent
2. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ then the series $\sum a_n$ is absolutely divergent
3. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L = 1$ then no conclusion can be drawn

### 3.4.3 Power series

A power series is a series of the form:

$$\sum_{n=1}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 ...$$

where $x$ is the variable and $c_n$ as the coefficents of the series, But more generally:

$$\sum_{n=1}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3 ...$$

which is called **power series centered at a.**

#### The convergence of power series

1. The series converges only when $x = a$
2. The series converges for all $x$
3. There is a positive number $R$ such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$

### 3.4.4 Differnciation and Integration of Power Series

If the power series $\sum_{n=1}^{\infty} c_n x^n$ has a convergence $R > 0$, then the function $f$ is defined by:

$$f(x) = c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3$$

is defferentiable on the interval $(a - R, a + R)$

1. $f'(x) = c_1 + 2c_2 (x-a) + c_2 (x-a) + 3c_3 (x-a)^2 + ... = \sum_{n=1}^{\infty} nc_n (x-a)^{n-1}$
2. $\int f(x)\, dx = C + c_0 (x-a) + c_1 \frac{(x-a)}{2} + c_2 \frac{(x-a)^3}{3} + ... = C + \sum_{n=1}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$

### 3.5 Taylor/Maclaurin series

Taylor Series:

$$f\left(x\right)=\sum_{n=0}^{\infty}\frac{f^{(n)}\left(a\right)}{n!}\left(x-a\right)^{n}$$

Maclurin Series:

$$f\left(x\right)=\sum_{n=0}^{\infty}\frac{f^{(n)}\left(0\right)}{n!}\left(x\right)^{n}$$

### 3.6 Taylor's Inequality

### 3.7 Some interesting series to remember

1. $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^{n}$
2. $e^{x}=\sum_{n=0}^{\infty}\frac{x^{n}}{n!}$
3. $\sin x=\sum_{n=0}^{\infty}\left(-1\right)^{n}\frac{x^{2n+1}}{(2n+1)!}$
4. $\cos x=\sum_{n=0}^{\infty}\left(-1\right)^{n}\frac{x^{2n}}{(2n)!}$
5. $\tan^{-1}x=\sum_{n=0}^{\infty}\left(-1\right)^{n}\frac{x^{2n+1}}{(2n+1)}$
6. $\ln\left(1+x\right)=\sum_{n=0}^{\infty}\left(-1\right)^{n-1}\frac{x^{n}}{n}$
7. $\left(1+x\right)^{k}=\sum_{n=0}^{\infty}\left(\begin{array}{c}k\\n\end{array}\right)x^{n}$

## 4 3D stuff

### 4.1 3D coordinate systems

#### 4.1.1 Distance Formula in Three Dimensions

The distance $|P_1P_2|$ between points $P_1$ and $P_2$ is:

$$|P_1P_2|=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2+\left(z_2-z_1\right)^2}$$

#### 4.1.2 Sphere

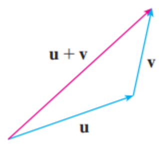
An equation of a sphere with center $C\left(h,k,l\right)$ and radius $r$ is:

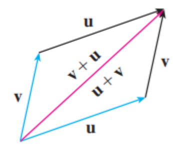
$$\left(x-h\right)^2+\left(y-k\right)^2+\left(z-l\right)^2=r^2$$

## 4.2 Vectors

### 4.2.1 Addition

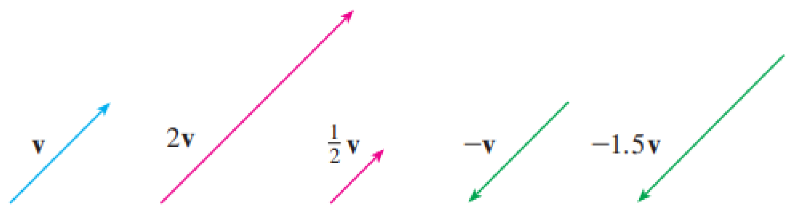
If $u$ and $v$ are the vectors positioned so the the initial point of $v$ is at the terminal point of $v$ is at the terminal point of $u$, then the sum $u + v$ is the vector from the initial point of $u$ to the terminal point of $v$.




### 4.2.2 Scalar Multiplication

If $c$ is a scalar and $v$ is a vector, then the **scalar multiple** $cv$ is the vector whose length is $|c|$times the length of $v$ and whose direction is the same as $v$ if $c > 0$ and is opposite to $v$ if $c < 0$. If $c = 0$ or $v = 0$, then $cv = 0$.




### 4.2.3 Components

Given the points $A\,(x_1, y_1, z_1)$ and $B\,(x_2, y_2, z_2)$,the vector $a$ with representation $\overrightarrow{AB}$ is

$$a = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

**2D Vector length**

$$|a| = \sqrt{a_1^2 + a_2^2}$$

**3D vector length**

$$|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

## 4.3 The products

### 4.3.1 The dot product

If $a =\langle a_1, a_2, a_3\rangle$ and $b =\langle b_1, b_2, b_3\rangle$ ,then the dot product $a \cdot b$:

$$a \cdot b = a_1b_1 + a_2b_2 + a_3b_3$$

### 4.3.2 The cosine theorem

If is the angle between the vectors a and b, then:

$$a \cdot b = |a|\,|b| \cos\theta$$

which lead to the conclusion:

$$\cos\theta = \frac{a \cdot b}{|a|\,|b|}$$

and two vectors a and b are orthogonal if and only if:

$$a \cdot b = 0$$

### 4.3.3 Projections

Scalar projection of $b$ onto $a$:

$$\mathrm{comp}_a b = \frac{a \cdot b}{|a|}$$

Vector projection of $b$ onto $a$:

$$\mathrm{proj}_a b = \left(\frac{a \cdot b}{|a|}\right)\frac{a \cdot b}{|a|} = \frac{a \cdot b}{|a|^2}$$

### 4.3.4 Cross product

If $a =\langle a_1, a_2, a_3\rangle$ and $b =\langle b_1, b_2, b_3\rangle$ ,then the dot product $a \times b$:

$$a \times b = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$$

- **Orthogonality**
  - The vector $a \times b$ is orthogonal to both $a$ and $b$.
- **The sine theorem:**

  $$|a \times b| = |a|\,|b| \sin\theta$$

  Two nonzero vectors $a$ and $b$ are parallel if and only if:

  $$a \times b = 0$$

### 4.3.5 Magnitude of a cross product

The length of the cross product $a \times b$ is equal to the area of parallelogram determined by $a$ and $b$

### 4.3.6 Properties of the Cross Product

1. $(ru) \times (sv) = (rs)(u \times v)$
2. $v \times u = -(u \times v)$
3. $0 \times u = 0$
4. $u \times (v + w) = u \times v + u \times w$
5. $(v + w) \times u = v \times u + w \times u$
6. $u \times (v \times w) = (u \cdot w) v - (u \cdot v) w$

## 4.4 Planes and Lines

### 4.4.1 Lines

A vector equation of L can be described as:

$$r = r_0 + tv$$

### 4.4.2 Parametric Equations for a Line

The standard parametrization of the line through $P_0 (x_0, y_0, z_0)$ parallel to $v = v_1 i + v_2 j + v_3 k$ is

$$x = x_0 + tv_1, \qquad y = y_0 + tv_2, \qquad z = z_0 + t_3, \qquad -\infty < t < \infty$$

### 4.4.3 The line segment from $r_0$ to $r_1$

$$r(t) = (1 - t) r_0 + tr_1$$

### 4.4.4 Planes

The scalar equation of the plane through $P_0 (x_0, y_0, z_0)$ with normal vector $n = \langle a, b, c \rangle$ is:

$$a (x - x_0) + b (y - y_0) + c (z - z_0) = 0$$

### 4.4.5 Distances from a point to the plane

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$