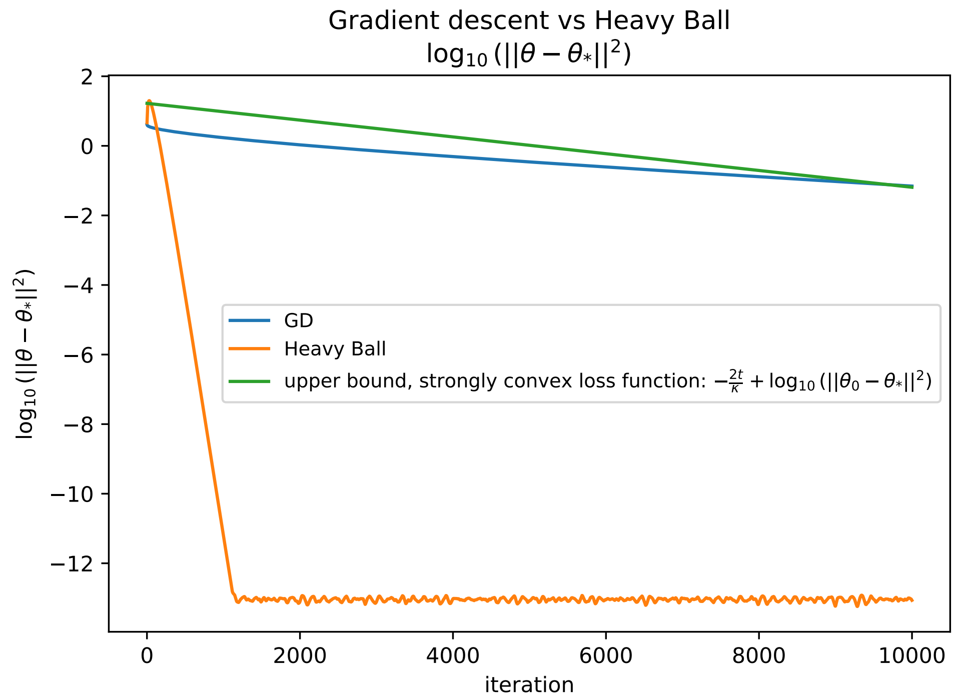
Heavy Ball



1 HEAVY-BALL

In this exercise we study the values of the γ and β parameters in a Heavy-Ball algorithm, which is an acceleration of gradient descent, in the case where f is a least squares loss function.

$$\theta_{t+1} = \theta_t - \gamma \nabla_\theta f(\theta_t) + \beta(\theta_t - \theta_{t-1}) \tag{1}$$

The update $\theta_{t+1} - \theta_t$ is then a combination of the gradient $\nabla_\theta f(\theta_t)$ and of the previous update $\theta_t - \theta_{t-1}$. This method might balance the effet of oscillations in the gradient. We will use these parameters :

$$\gamma = \frac{4}{(\sqrt{L} + \sqrt{\mu})^2} \tag{2}$$

and

$$\beta = \Big(\frac{\sqrt{L} - \sqrt{\mu}}{\sqrt{L} + \sqrt{\mu}}\Big)^2 \tag{3}$$

We keep the same notations as in the former practical sessions dedicated to the least squares problem. μ is the smallest eigenvalue of the Hessian H and L is the largest. Assuming $\mu > 0$ (strongly convex function), we will show that the characteristic convergence time with the heavy-ball momentum term is $\sqrt{\kappa}$ instead of κ.

Let λ be an eigenvalue of H and u_λ a eigenvector for thie eigenvalue. We are interested in the evolution of $\langle \theta_t - \eta^*, u_\lambda \rangle$.

We note

$$a_t = \langle \theta_t - \eta^*, u_\lambda \rangle \tag{4}$$

Exercice 1 : Show that

$$a_{t+1} = (1 - \gamma\lambda + \beta) a_t - \beta a_{t-1} \tag{5}$$

Exercice 2 : Compute the constant-recursive sequence a_t, and show that there exists a constant C_λ that depends on the initial conditions, such that

$$\forall t, a_t \leqslant (\sqrt{\beta})^t C_\lambda \tag{6}$$

https://en.wikipedia.org/wiki/Constant-recursive_sequence

If u_i is a basis of orthonormal vectors with eigenvalues λ_i, we have that

$$\begin{aligned} ||\theta_t - \eta^*||^2 &= \sum_{i=1}^{d} (\langle \theta_t - \eta^*, u_i \rangle)^2 \\ &\leqslant \sum_{i=1}^{d} (\sqrt{\beta})^{2t} C_{\lambda_i} \\ &= (\sqrt{\beta})^{2t} D \end{aligned} \tag{7}$$

with

$$D = \sum_{i=1}^{d} C_{\lambda_i} \tag{8}$$

We can now remark that

$$\begin{aligned} \sqrt{\beta} &= \frac{\sqrt{L} - \sqrt{\mu}}{\sqrt{L} + \sqrt{\mu}} \\ &= \frac{1 - \sqrt{\frac{\mu}{L}}}{1 + \sqrt{\frac{\mu}{L}}} \\ &\leqslant 1 - \sqrt{\frac{\mu}{L}} \\ &= 1 - \frac{1}{\sqrt{\kappa}} \end{aligned} \tag{9}$$

Exercice 3 : Conclude