Lecture 4. Matrix Algebra and Properties

Let's denote $\mathcal{O}$ as a zero matrix; for example,

$$\mathcal{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

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Answer. Yes, it is possible. For example, if $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $AB = \mathcal{O} = BA$.

Identity Matrix I_m is a $m \times m$ **square matrix** whose diagonal entries are all 1's and 0's elsewhere; for example,

$$I_6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 4.1. Let A be a 3×2 matrix and let B be a 10×10 matrix. Compute the followings.

(a) AI_2

(b) I_3A

(c) BI_{10} and $I_{10}B$

Properties of Matrix Multiplication

Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

a. $A(BC) = (AB)C$ (associative law of multiplication)

b. $A(B + C) = AB + AC$ (left distributive law)

c. $(B + C)A = BA + CA$ (right distributive law)

d. $r(AB) = (rA)B = A(rB)$
    for any scalar r

e. $I_m A = A = A I_n$ (identity for matrix multiplication)

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Practice 4.2. Let $A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$. Construct matrices B and C such that $AB = AC$ and yet $B \neq C$.

If A is an $n \times n$ square matrix, then A^k denotes the product of k copies of A.

$$A^k = \underbrace{A \cdots A}_{k}$$

if k is a positive integer. Thus, $A^1 = A$, and it is convenient to define $A^0 = I_n$.

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Example 4.3. If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then

$$A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} =$$

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and, in general,

$$A^n = \begin{bmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{bmatrix} \quad \text{for all } n \geq 1.$$

Practice 4.4. Let $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Compute the sequence of powers of B, B^k for all $k \geq 1$.

If A is a square matrix and r and s are nonnegative integers, then

1. $A^r A^s = A^{r+s}$
2. $(A^r)^s = A^{rs}$

Transpose of a Matrix

The transpose of a matrix results from "flipping" the rows and columns. Given an $m \times n$ matrix A, its transpose, written A^T, is the $n \times m$ matrix whose entries are given by

$$(A^T)_{ij} = (A)_{ji}$$

Example 4.5. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $B = \begin{bmatrix} -5 & 2 \\ 1 & -3 \\ 2 & 7 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 0 & -4 & 1 \\ 5 & -2 & 6 & 3 \end{bmatrix}$.

Find A^T, B^T, C^T.

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Practice 4.6. Let $A = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$. Compute $(A\vec{x})^T$, $\vec{x}^T A^T$, $\vec{x}\vec{x}^T$, and $\vec{x}^T \vec{x}$. Is $A^T \vec{x}^T$ defined?

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$
b. $(A + B)^T = A^T + B^T$
c. For any scalar r, $(rA)^T = rA^T$
d. $(AB)^T = B^TA^T$

A square matrix A is symmetric if $A^T = A$.

Example 4.7. Determine which of the matrices

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 5 & 0 \\ 2 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

is symmetric. Are the matrices $A^T A$ and $B B^T$ symmetric?

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Theorem 4.8. If A is any $m \times n$ matrix, then $A^T A$ and AA^T are always symmetric whether or not A is symmetric.

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