Lecture 6. Elementary Matrices

Elementary Matrices are the matrix representations of three *elementary row operations* as discuss in Lecture 5.

Practice 6.1 Compute E_1A, E_2A, and E_3A, and describe how these products can be obtained by elementary row operations:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix},\ E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix},\ A = \begin{bmatrix} 2 & -1 \\ 1 & 4 \\ 0 & -2 \end{bmatrix}.$$

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Practice 6.1 illustrates the following general fact about elementary matrices.

- If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where E is the $m \times m$ matrix performing the same row operation on A.

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- If E is obtained by multiplying row i by the scalar k, then E^{-1} is obtained by multiplying row i by the scalar $\frac{1}{k}$.
- E is obtained by adding k times row i to row j, then E^{-1} is obtained by subtracting k times row i from row j, (that is, by adding $-k$ times row i to row j.)

Example 6.3. Find the inverse of $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$.

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- **Solution.** To transform E_1 into I, do $4R_1 + R_3 \to R_3$. The elementary matrix that does this is

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ +4 & 0 & 1 \end{bmatrix}$$

Practice 6.4. Find the inverse of each matrix.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \end{bmatrix}.$$

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- Recall that the procedure of finding A^{-1}

$$[\ A \mid I\ ] \ \Leftrightarrow\ [\ I \mid A^{-1}\ ]$$

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$$\begin{aligned}
[\ A \mid I_n\ ] &\Leftrightarrow [\ E_1A \mid E_1I_n\ ] \\
&\Leftrightarrow [\ E_2(E_1A) \mid E_2(E_1I_n)\ ] \\
&\ \vdots \\
&\Leftrightarrow [\ E_p(E_{p-1}...E_1A) \mid E_p(E_{p-1}...E_1)I_n\ ] \\
&\Leftrightarrow [\ \qquad I_n \qquad \mid E_p(E_{p-1}...E_1I_n)\ ]
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- Therefore,

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$$(E_pE_{p-1}...E_1)^{-1}(E_pE_{p-1}...E_1)A = (E_pE_{p-1}...E_1)^{-1}I_n \Leftrightarrow A = (E_pE_{p-1}...E_1)^{-1}$$

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Example 6.5. Find the elementary matrices such that $A^{-1} = E_p E_{p-1} \ldots E_1$.

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Practice 6.6. Find the elementary matrices such that $A^{-1} = E_p E_{p-1} ... E_1$.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$$