Lecture 1. Introduction: Matrices and Vectors

$$\begin{bmatrix} 9 & 13 & 5 & 2 \\ 1 & 11 & 7 & 6 \\ 3 & 7 & 4 & 1 \\ 6 & 0 & 7 & 10 \end{bmatrix}$$

What is a matrix?

- A matrix is a rectangular array of elements arranged in horizontal rows and vertical columns, and usually enclosed in brackets.

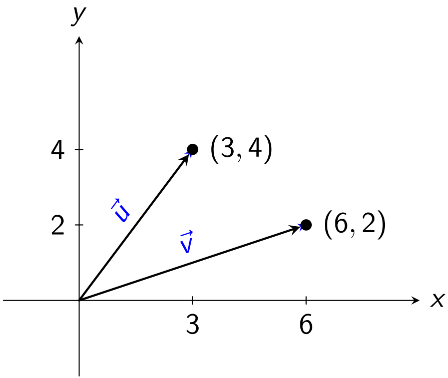
$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 3 \\ -4 & 5 & 9 & 1 \end{bmatrix}$$

  The above is a 3×4 matrix (3 rows and 4 columns).
- In this course, we will learn how to use matrices to solve two (related but) different problems *efficiently*:
  1. systems of linear equations ($A\vec{x} = \vec{b}$), and
  2. eigenvalue and eigenvector problems ($A\vec{x} = \lambda\vec{x}$).

Column Vector $\vec{v}$

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

where v_1 = first component of $\vec{v}$ and v_2 = second component of $\vec{v}$.



$$\vec{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

Scalar Multiplication

$$2\vec{v} = \begin{bmatrix} 2v_1 \\ 2v_2 \end{bmatrix} = \vec{v} + \vec{v}, \quad -\vec{v} = \begin{bmatrix} -v_1 \\ -v_2 \end{bmatrix}$$

The components of $c\vec{v}$ are cv_1 and cv_2. The number c is called a scalar.

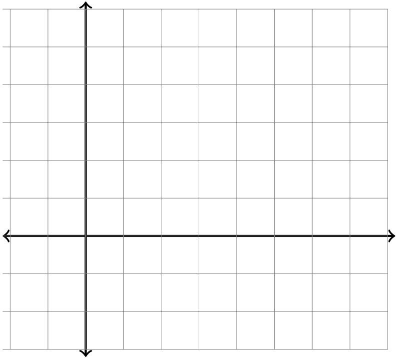
Vector Addition

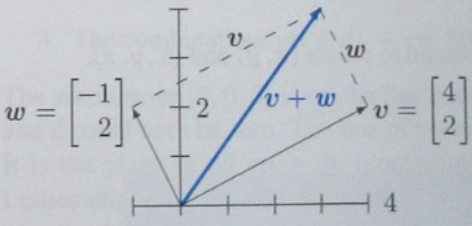
$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \text{ and } \vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \Rightarrow \vec{v} + \vec{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix}$$

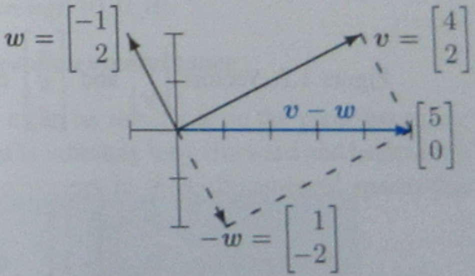
Subtraction follows the same idea: The components of $\vec{v} - \vec{w}$ are $v_1 - w_1$ and $v_2 - w_2$.

Example 1.1. Visualize $\vec{v} + \vec{w}$ and $\vec{v} - \vec{w}$ with

$$\vec{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \quad \text{and} \quad \vec{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$







$$v + w = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad v - w = \begin{bmatrix} 4 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Vectors in Three Dimensions

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

Problem 1.2. Compute $2\vec{v} - 3\vec{w}$

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From now on $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is also written as $\vec{v} = (1, 1, -1)$ to save space. Notice that $\vec{v} = (1, 1, -1)$ is not a row vector $[1 \;\; 1 \; -1]$.

Linear Combination of Vectors

The sum of $c\vec{v}$ and $d\vec{w}$ is a linear combination

$$c\vec{v} + d\vec{w},$$

where c and d are scalars.

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Problem 1.3. Find c and d so that the linear combination $c\vec{v} + d\vec{w}$ produces $\vec{b}$:

$$(a) \quad \vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$$

$$(b) \quad \vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Problem 1.4. Let $\vec{v} = (1, 1, 0)$ and $\vec{w} = (0, 1, 1)$. Find a vector that is not a combination of $\vec{v}$ and $\vec{w}$.

In general, the linear combination of vectors $\vec{v}_1, \vec{v}_2, ..., \vec{v}_n$ is

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots c_n \vec{v}_n$$

where $c_1, ..., c_n$ are arbitrary constants.