Lecture 5. The Inverse of a Matrix and Row Reduction Method

- Recall that the multiplicative inverse of a number such as 5 is $1/5$ or 5^{-1}. This inverse satisfies the equations

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- An $n \times n$ matrix A is said to be invertible if there is an $n \times n$ matrix C such that

$$CA = I_n \quad \text{and} \quad AC = I_n.$$

In this case, C is the inverse of A and denoted by A^{-1}. That is, it satisfies

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- An invertible matrix is called a nonsingular matrix. A matrix that is **not** invertible is called a singular matrix.

Example 5.1. Let $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ and $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$. Show that $C = A^{-1}$.

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Example 5.2. Let A be the 2 matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}.$$

Show that A has no inverse.

Algorithm for Finding A^{-1}

- **Algorithm.** Let A be an $n \times n$ matrix. Set up the augmented matrix $[A|I_n]$. Use elementary row operations to convert $[A|I_n]$ into its row equivalent matrix $[I_n|B]$:

$$[A|I_n] \Leftrightarrow [I_n|B].$$

  Then, $B = A^{-1}$.

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  2. Replace a row by a nonzero multiple of itself. We indicate that the row R_i is replaced by kR_i (where $k \neq 0$) by writing

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  2. Replace a row by a nonzero multiple of itself. We indicate that the row R_i is replaced by kR_i (where $k \neq 0$) by writing

$$kR_i \rightarrow R_i$$

  3. Replace a row by the sum of a nonzero multiple of another row and itself. We indicate that R_j is replaced by the sum of kR_i and R_j (where $k \neq 0$) by writing

$$kR_i + R_j \rightarrow R_j$$

Recall

$$[A|I_n] \Leftrightarrow [I_n|A^{-1}].$$

Example 5.3. Find the inverse of the matrix, if it exists. $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$

$$[A|I_n] \Leftrightarrow [I_n|A^{-1}].$$

Practice 5.4. Find the inverse of the matrix, if it exists.

$$(a)\ B = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}, \quad (b)\ C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad (c)\ D = \begin{bmatrix} -2 & -7 & -9 \\ 2 & 5 & 6 \\ 1 & 3 & 4 \end{bmatrix}$$

A special inverse formula of 2×2 matrices:
Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. If $d = a_{11}a_{22} - a_{21}a_{12} \neq 0$, then

$$A^{-1} = \frac{1}{d} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

Use the inverse formula of 2×2 matrices to find A^{-1} of the matrix $A = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$.

Example 5.5. Let A and B the 2×2 matrices

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -2 \\ 1 & -1 \end{bmatrix}.$$

1. Compute $AB, (AB)^{-1}, A^{-1}B^{-1}$, and $B^{-1}A^{-1}$. What do you observe?
2. Calculate $(2A)^{-1}$. What is the relationship between $(2A)^{-1}$ and A^{-1}?
3. Compute $(A^T)^{-1}$ and $(A^{-1})^T$. What do you see?

Properties of Matrix Inverse

Theorem 5.6 Let A and B be invertible $(n \times n)$ matrices and α a nonzero constant. Then,

(a) A^{-1} is unique.

(b) A^{-1} is nonsingular and $(A^{-1})^{-1} = A$.

(c) AB is nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$.

(d) $(A^k)^{-1} = (A^{-1})^k$.

(e) $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$.

(f) $(A^T)^{-1} = (A^{-1})^T$

Through a systematic procedure of row operations, we can simplify an augmented matrix and carry it to row-echelon form or reduced row-echelon form, defined below. In the following, the term leading entry refers to the first nonzero entry of a row when scanning the row from left to right.

Definition 5.7 An augmented matrix is in row-echelon form if

1. All rows having only zero entries are at the bottom.
2. The leading entry (that is, the left-most nonzero entry) of every nonzero row, called the pivot, is on the right of the leading entry of every row above.

Remark: Some texts add the condition that the leading entry must be 1 while others require this only in reduced row echelon form.

Definition 5.7 An augmented matrix is in reduced row-echelon form if

1. It is in row echelon form.
2. The leading entry in each nonzero row is 1.
3. Each column containing a leading 1 has zeros in all its other entries.

Example 5.8

The following augmented matrices are not in row-echelon form (and therefore also not in reduced row-echelon form).

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 4 & -6 \\ 4 & 0 & 7 \end{array}\right], \quad \left[\begin{array}{ccc|c} 0 & 2 & 3 & 3 \\ 1 & 5 & 0 & 2 \\ 7 & 5 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

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Example 5.9 The following augmented matrices are in row-echelon form, but not in reduced row-echelon form.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 6 & 5 & 8 & 2 \\ 0 & 0 & 1 & 2 & 7 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{ccc|c} 1 & 3 & 5 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Example 5.10

The following augmented matrices are in reduced row-echelon form.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

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[1] We will study this concept later.

Example 5.10

The following augmented matrices are in reduced row-echelon form.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

One way in which the row-echelon form of a matrix is useful is in identifying the *pivot positions* and *pivot columns* of the matrix.

Definition 5.11 A pivot position in a matrix is the location of a leading entry in the row-echelon form of a matrix. A pivot column is a column that contains a pivot position.

Definition 5.12. The rank of a matrix is equal to the number of linearly independent [1] rows (or columns) in it. Technically, the rank is the number of pivot columns (equivalently, the number of leading entries.)

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