Lecture 3. Matrix-Matrix Multiplications

Notations

Let A be an $m \times n$ matrix.

$$\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Equivalently,

$$A = [a_{ij}]$$

or

$$A = [\vec{a}_1 \;\; \vec{a}_2 \;\; \ldots \;\; \vec{a}_n]$$

- a_{ij} = the scalar entry in the ith row and jth column of A.

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$$A = [\vec{a}_1 \ \vec{a}_2 \ \ldots \ \vec{a}_n]$$

- a_{ij} = the scalar entry in the ith row and jth column of A.
- The columns of A are vectors in $\mathbb{R}^m$ and are denoted by $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n$, where

$$\vec{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

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- The diagonal entries in an $m \times n$ matrix $A = [a_{ij}]$ are $a_{11}, a_{22}, a_{33}, \dots$.

Dot Product

$$[a_1 \;\; a_2 \;\; \ldots \;\; a_n] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{k=1}^{n} a_k b_k = {\color{blue}a}_{\color{red}k} {\color{blue}b}_{\color{red}k}$$

For example, $[7 \;\, \text{-}4 \;\, 5 \;\, 1] \begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \end{bmatrix} =$

Dot Product

$$[a_1 \; a_2 \; \ldots \; a_n]\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{k=1}^{n} a_k b_k = a_k b_k$$

For example, $[7 \; \text{-}4 \; 5 \; 1]\begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \end{bmatrix} = 7(3) + (-4)(2) + 5(1) + 1(4) = 21 - 8 + 5 + 4 = 22$

Matrix Multiplication as a Dot Product

dot product

$$\begin{array}{cc} & \text{columns=3} \\ \text{rows=2} & \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \end{array} \quad \times \quad \begin{array}{cc} \text{columns=2} & \\ \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} & \text{rows=3} \end{array} \quad = \quad \begin{bmatrix} \boxed{58} & 64 \\ 139 & 154 \end{bmatrix}$$

First matrix | Second matrix | Product

$$[1\ 2\ 3] \bullet (7, 9, 11) = 1x7 + 2x9 + 3*11 = 58$$

dot product

$$\begin{bmatrix} \boxed{1 \quad 2 \quad 3} \\ 4 \quad 5 \quad 6 \end{bmatrix} \times \begin{bmatrix} 7 & \boxed{8} \\ 9 & \boxed{10} \\ 11 & \boxed{12} \end{bmatrix} = \begin{bmatrix} 58 & \boxed{64} \\ 139 & 154 \end{bmatrix}$$

First matrix | Second matrix | Product

$$[1 \quad 2 \quad 3] \bullet (8, 10, 12) = 1x8 + 2x10 + 3x12 = 64$$

Example 3.1. Compute AB, BA, AC, and CA, where

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 6 \\ 0 & -2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 2 & 0 & -4 \\ 5 & -2 & 6 \end{bmatrix}$$

In general,

Suppose $A = [a_{ik}]$ and $B = [b_{kj}]$ are matrices such that the number of columns of A is equal to the number of rows of B; say, A is an $m \times p$ matrix and B is a $p \times n$ matrix. Then the product AB is the $m \times n$ matrix whose ij-entry is obtained by multiplying the ith row of A by the jth column of B. That is,

$$\begin{bmatrix} a_{11} & \dots & a_{1p} \\ \cdot & \dots & \cdot \\ a_{i1} & \dots & a_{ip} \\ \cdot & \dots & \cdot \\ a_{m1} & \dots & a_{mp} \end{bmatrix} \begin{bmatrix} b_{11} & \dots & b_{1j} & \dots & b_{1n} \\ \cdot & \dots & \cdot & \dots & \cdot \\ \cdot & \dots & \cdot & \dots & \cdot \\ \cdot & \dots & \cdot & \dots & \cdot \\ b_{p1} & \dots & b_{pj} & \dots & b_{pn} \end{bmatrix} = \begin{bmatrix} c_{11} & \dots & c_{1n} \\ \cdot & \dots & \cdot \\ \cdot & c_{ij} & \cdot \\ \cdot & \dots & \cdot \\ c_{m1} & \dots & c_{mn} \end{bmatrix}$$

where $c_{ij} = a_{ik} b_{kj}$.

- The product AB is not defined if A is an $m \times p$ matrix and B is a $q \times n$ matrix, where $p \neq q$.

If B has k columns $\vec{b}_1, \vec{b}_2, ..., \vec{b}_k$, then the columns of AB are $A\vec{b}_1, A\vec{b}_2, ..., A\vec{b}_k$:

$$AB = A\,[\vec{b}_1 \;\; \vec{b}_2 \;\; \cdots \;\; \vec{b}_k] = [A\vec{b}_1 \;\; A\vec{b}_2 \;\; \cdots \;\; A\vec{b}_k]$$

Example 3.2. Compute AB, BA, AC, and CA using the matrix-vector multiplications, where $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 5 & 6 \\ 0 & -2 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 0 & -4 \\ 5 & -2 & 6 \end{bmatrix}$

If B has k columns $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_k$, then the columns of AB are $A\vec{b}_1, A\vec{b}_2, \ldots, A\vec{b}_k$:

$$AB = A\,[\vec{b}_1\ \ \vec{b}_2\ \ \cdots\ \ \vec{b}_k] = [A\vec{b}_1\ \ A\vec{b}_2\ \ \cdots\ \ A\vec{b}_k]$$

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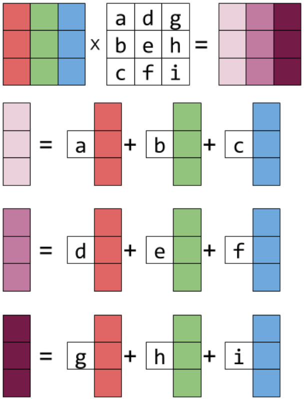
Warning. In general, $AB \neq BA$.



Practice 3.3. Compute AB and CA using the matrix-vector multiplications, where

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 0 \\ 5 & -1 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$