Lecture 2. Matrix-Vector Multiplications and Transformations

Matrix-Vector Multiplication, $A\vec{x}$

The two most effective ways that a matrix A acts on a vector $\vec{x}$ are

1. Dot Product
2. Linear Combination

① Dot product of the rows of A with $\vec{x} = (x_1, x_2)$:

$$A\vec{x} = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 \\ 2x_1 + 4x_2 \\ 3x_1 + 7x_2 \end{bmatrix}$$

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Example 2.1. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$. Compute $A\vec{x}$ using dot products with the rows of A.

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Practice 2.2. Compute $A\vec{x}$, where A is as in the example above and $\vec{x} = (-2, 0, 3)$.

❷ Linear combination of the columns of $A = [\vec{a}_1 \ \vec{a}_2]$ with the components of $\vec{x} = (x_1, x_2)$:

$$A\vec{x} = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix} = x_1 \vec{a}_1 + x_2 \vec{a}_2 = \begin{bmatrix} 2x_1 + 3x_2 \\ 2x_1 + 4x_2 \\ 3x_1 + 7x_2 \end{bmatrix}$$

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Example 2.3. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$. Compute $A\vec{x}$ as a linear combination of the columns of A.

2 Linear combination of the columns of $A = [\vec{a}_1 \ \vec{a}_2]$ with the components of $\vec{x} = (x_1, x_2)$:

$$A\vec{x} = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 4 \\ 7 \end{bmatrix} = x_1 \vec{a}_1 + x_2 \vec{a}_2 = \begin{bmatrix} 2x_1 + 3x_2 \\ 2x_1 + 4x_2 \\ 3x_1 + 7x_2 \end{bmatrix}$$

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Practice 2.4. Compute $A\vec{x}$ as a linear combination of the columns of A, where A is as in the example above and $\vec{x} = (-2, 0, 3)$.

Practice 2.5. Let

$$\vec{s_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{s_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{s_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Find the linear combination $3\vec{s_1} + 4\vec{s_2} + 5\vec{s_3} = \vec{b}$. Then write $\vec{b}$ as a matrix-vector multiplication $\vec{b} = S\vec{x}$ with $\vec{x} = (3, 4, 5)$. Note that the columns of S consist of $\vec{s_1}, \vec{s_2}$, and $\vec{s_3}$.

Summary: Matrix-Vector Multiplication, $A\vec{x}$

The two most effective ways that a matrix A acts on a vector $\vec{x}$.

1 Dot products of the rows of A with $\vec{x} = (x_1, x_2)$:

$$A\vec{x} = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 \\ 2x_1 + 4x_2 \\ 3x_1 + 7x_2 \end{bmatrix}$$

2 Linear combination of the columns of $A = [\vec{a}_1 \; \vec{a}_2]$ with the components of $\vec{x} = (x_1, x_2)$:

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Both ways give the same result.

By now we have seen the operation $A\vec{x} = \vec{b}$; that is,

- Given A and $\vec{x}$, we multiplied A and $\vec{x}$ to produce an output vector $\vec{b}$.

So, A works on $\vec{x}$ to transform it to another vector $\vec{b}$

$$\text{(input)} \ \vec{x} \longrightarrow \boxed{A} \longrightarrow \vec{b} \ \text{(output)}$$

Now, let's look at some special types of matrices A that do simple things on $\vec{x}$ and then we'll think about how to combine them to do complicated things.

Example 2.6. First, let's think about a matrix that doesn't change anything.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$$

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[1]A matrix transformation can be considered as a (linear) function expressed as a matrix.

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Example 2.6. First, let's think about a matrix that doesn't change anything.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Here $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called a identity matrix.

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Example 2.7. Scale transformations [2]

Let $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, $\vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Compute and visualize the operations $A\vec{e_1}$ and $A\vec{e_2}$. Discuss the effect of A on the vector $\vec{x}$.

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[2] $\vec{e_1}$ and $\vec{e_2}$ are called two-dimensional unit vectors.

Example 2.8. Reflection transformations

Let $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Compute and visualize the operations $B\vec{e_1}$ and $B\vec{e_2}$. Discuss the effect of B on the vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Do the same procedure with C and discuss its effects on vectors.

Practice 2.9. Let $M = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$ and $N = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. Discuss the effects of the matrices M and N on the vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Example 2.10. Shear transformations

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\vec{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Compute and visualize the operations $A\vec{e_1}$ and $A\vec{e_2}$. Discuss the effect of A on the vector $\vec{x}$.

Example 2.11. Rotation transformations

What is the transformation matrix for the rotation by 90° counterclockwise?

$(a) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $(b) \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ $(c) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $(d) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

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$$(a) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (b) \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad (c) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad (d) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

In general, the (2×2) counterclockwise rotation matrix is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Practice 2.12. Construct a 2×2 matrix that first reflects points through the vertical y-axis and then perform a horizontal shear transforming $\vec{e_2}$ into $\vec{e_2} + 4\vec{e_1}$ (leaving $\vec{e_1}$ unchanged).

(This practice includes a matrix multiplication, which comes in the next lecture.)

- $A(c\vec{x}) = cA\vec{x}$ for any constant c.
- $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$ for any vectors $\vec{x}$ and $\vec{y}$.

1. Fill in the missing entries of the matrix satisfying the following:

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - 4x_2 \\ x_1 - x_3 \\ -x_2 + 3x_3 \end{bmatrix}$$

2. (Composition of matrix transformations) Let $A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and

$A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Discuss the effect of $A_1 A_2$ on a vector $\vec{x}$ without calculation.
(Hint: $A_1 A_2 \vec{x} = A_1(A_2 \vec{x})$.)

3. Construct a 2×2 matrix that first performs a horizontal shear transforming $\vec{e}_2$ into $\vec{e}_2 + 2\vec{e}_1$ (leaving $\vec{e}_1$ unchanged) and then reflects vectors through the line $y = -x$.