Lecture 7. Matrix Factorization: LU Decomposition

The Factorization

- A factorization of a matrix A is an equation that express A as a product of two or more matrices.
- Whereas matrix multiplication involves a *synthesis* of data, matrix factorization is an *analysis* of data.
- In the language of computer science, the expression of A as a product amounts to a *preprocessing* of the data in A, organizing that data into two or more parts whose structures are more useful in some way, perhaps more accessible for computation.
- While there are several factorizations in matrix algebra, we focus on an important factorization widely used in applications, the LU factorization (or decomposition).

LU Factorization (or Decomposition) of a matrix, introduced by Alan Turing in 1948, is the process of factoring a square matrix into two special types of matrices. These two matrices are a **lower triangular** matrix and an **upper triangular** matrix.

A lot of matrix operations are easier for triangular matrices. "Easier" here means that the time-complexity for a computer to calculate the result will be lower. While working with a particular matrix, obtaining its LU decomposition will likely speed things up.

Diagonal and Triangular Matrices

- An $n \times n$ matrix A is said to be upper triangular if its entries below the main diagonal are 0's (i.e., $a_{ij} = 0$ for $i > j$) and lower triangular if its entries above the main diagonal are 0's (i.e., $a_{ij} = 0$ for $i < j$).

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- For example,

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 \\ 6 & 0 & 0 \\ 1 & 4 & 3 \end{bmatrix}$$

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- An $n \times n$ matrix A is diagonal if $a_{ij} = 0$ whenever $i \neq j$. For example, the matrices

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

  are diagonal. A diagonal matrix is both upper triangular and lower triangular.

- **Question.** When is a square upper (or lower) triangular matrix invertible? Justify your answer.

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  - **Solution.** If a square upper (or lower) triangular $n \times n$ matrix has nonzero diagonal entries, then because it is already in echelon form, the matrix is row equivalent to I_n and hence is invertible. Conversely, if the matrix is invertible, it has n pivots (leading entries) on the diagonal and hence the diagonal entries are nonzero.

- Assume that A is an $m \times n$ matrix that can be row reduced to echelon form, without row interchanges. Then A can be written in the form of

$$A = LU$$

where L is an $m \times m$ lower triangular matrix with 1's on the diagonal and U is an $m \times n$ matrix in row echelon form. [1]

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{U}$$

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LU Factorization

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- Such a factorization is called an LU factorization of A. The matrix L is invertible and is called a unit lower triangular matrix.

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- Suppose A can be reduced to an echelon form U using only row replacements that add a multiple of one row to another row below it. In this case, there exist unit lower triangular elementary matrices $E_1, E_2, \ldots, E_p$ such that

$$E_p \cdots E_1 A = U$$

- Then

$$A = (E_p \cdots E_1)^{-1} U = LU$$

  where

$$L = (E_p \cdots E_1)^{-1}$$

- It can be shown that products and inverse of unit lower triangular matrices are also unit lower triangular. Thus L is unit lower triangular.

Example 7.1. Find an LU factorization of

$$\begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}.$$

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Solution.

$$\begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Practice 7.2. Let

$$A = \begin{bmatrix} 6 & 9 \\ 4 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & 3 & 2 & 1 \\ 5 & 0 & 1 & 3 \end{bmatrix}$$

Find an LU factorization of each matrix.

If $A = LU$ where the diagonal entries of L are all 1, then we can multiply row i of matrix U by $1/u_{ii}$ and produce a row echelon matrix U^* with diagonal entries of 1.

We then have a factorization of A as $A = LDU^*$, where the diagonal entries of L and U^* are all 1. (Note: When such a factorization of a matrix exists, it is unique.)

Example 7.3. Let

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 8 & 4 \\ -1 & 3 & 4 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 6 \\ 0 & 0 & -15 \end{bmatrix}$$

Factorize A into LDU.