HACETTEPE UNIVERSITY
ENGINEERING FACULTY
ELECTRICAL AND ELECTRONICS ENGINEERING PROGRAM

2023-2024
SPRING SEMESTER

ELE708
NUMERICAL METHODS IN ELECTRICAL ENGINEERING

HW4

N23239410 – Ali Bölücü

1) Exercises

1.a) 4.2

4.2) What are the eigenvalues and corresponding eigenvectors of following matrix?

$$\begin{bmatrix} 1 & 2 & -4 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$Ax = \lambda . x \Rightarrow (A - \lambda . I) . x = 0 \Rightarrow \det(A - \lambda . I) = 0$$

In order to solve the problem more easily, we take advantage of triangular form.

The eigenvalues are diagonal entries: 1, 2 and 3

for $\underline{\lambda = 1}$;

$$\begin{bmatrix} 1 & 2 & -4 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} - 1 . \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{array}{r} 2x_2 - 4x_3 = 0 \\ x_2 + x_3 = 0 \\ 2x_3 = 0 \end{array} \Bigg\rangle \begin{array}{l} x_3 = 0 \\ x_2 = 0 \\ x_1 = ? \end{array} \quad \text{so, } X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

any multiple of x

for $\underline{\lambda = 2}$;

$$\begin{bmatrix} 1 & 2 & -4 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{array}{r} -x_1 + 2x_2 - 4x_3 = 0 \\ x_3 = 0 \end{array} > \boxed{x_1 = 2x_2} \text{ so, } \quad X = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

any multiple

for $\underline{\lambda = 3}$;

$$\begin{bmatrix} 1 & 2 & -4 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 & -4 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 + 2R_2} \begin{bmatrix} -2 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{array}{l} -2x_1 + 2x_3 = 0 \Rightarrow \boxed{x_3 = x_1} \\ -x_2 + x_3 = 0 \Rightarrow \boxed{x_3 = x_2} \end{array} > X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

any multiple

1.b) 4.3

4.3-)

$$A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$$

a-) what is the characteristic polynomial of A?

$$\text{Det}(A - \lambda . I) = 0$$

$$\downarrow$$

$$\text{Det}\left(\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} - \lambda . \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{bmatrix}\right)$$

$$\Rightarrow (1-\lambda).(1-\lambda) - (1.4) = \underline{\underline{\lambda^2 - 2\lambda - 3}}$$

b-) what are the roots of the characteristic polynominal of A

$$\begin{matrix} \lambda^2 - 2\lambda - 3 & \\ \lambda & -3 \\ \lambda & +1 \end{matrix} \Rightarrow (\lambda - 3).(\lambda + 1) = 0$$

$$\boxed{\begin{matrix} \lambda = 3 \\ \lambda = -1 \end{matrix}}$$

c-) what are the eigen values of A?

$$\lambda_1 = 3 \quad , \quad \lambda_2 = -1$$

d-) What are the eigen vectors of A?

for $\lambda_1 = 3$,

$$\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} - 3.\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} \xrightarrow{R_1 + 2R_2} \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{matrix} x_1 - 2x_2 = 0 \\ \text{if } x_2 = 1 \end{matrix} \quad \text{so, } X = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightarrow \text{any multiple}$$

for $\lambda_2 = -1$,

$$\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} - (-1).\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{matrix} x_1 + 2x_2 = 0 \\ \text{if } x_2 = 1 \end{matrix} \quad \text{so } X, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \rightarrow \text{any multiple}$$

e-) Perform one iteration of power iteration on A, using $X_0 = [1\ 1]^T$ as starting vector.

$$X_k = A \cdot X_{k-1}$$

Ratio

$$X_1 = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \rightarrow 2$$

$$X_2 = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 13 \\ 7 \end{bmatrix} \rightarrow 3{,}5$$

$$X_3 = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 13 \\ 7 \end{bmatrix} = \begin{bmatrix} 41 \\ 20 \end{bmatrix} \rightarrow 2{,}86$$

$$X_4 = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 41 \\ 20 \end{bmatrix} = \begin{bmatrix} 121 \\ 61 \end{bmatrix} \rightarrow 3{,}05$$

$$X_5 = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 121 \\ 61 \end{bmatrix} = \begin{bmatrix} 365 \\ 182 \end{bmatrix} \; 2{,}98$$

Ratio

$$X_6 = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 365 \\ 182 \end{bmatrix} = \begin{bmatrix} 1093 \\ 547 \end{bmatrix} \rightarrow 3{,}0055$$

$$X_7 = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1093 \\ 547 \end{bmatrix} = \begin{bmatrix} 3281 \\ 1640 \end{bmatrix} \rightarrow 2{,}998$$

$$X_8 = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3281 \\ 1640 \end{bmatrix} = \begin{bmatrix} 9841 \\ 1640 \end{bmatrix} \rightarrow 3{,}291$$

The first iteration is $X_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0{,}4 \end{bmatrix}$

f-) To what eigenvector it will ultimately converge ?

The ratio between the vector components are

$X_1 \rightarrow 2{,}5$
$X_2 \rightarrow 1{,}85$
$X_3 \rightarrow 2{,}05$
$X_4 \rightarrow 1{,}98$
$X_5 \rightarrow 2{,}005$

$X_6 \rightarrow 1{,}998$
⋮

The ratio is 2

So it will converge to

$$X = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0{,}5 \end{bmatrix}$$

g-) what eigen value estimate given by the Rayleigh quotient using $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

As we computed on part e.

$$x_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 2 \end{bmatrix} \Big/ \begin{bmatrix} 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 2 \end{bmatrix} = 79/29 = 2{,}72$$

$$x_2 = \begin{bmatrix} 13 \\ 7 \end{bmatrix} \rightarrow \begin{bmatrix} 13 & 7 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 13 \\ 7 \end{bmatrix} \Big/ \begin{bmatrix} 13 & 7 \end{bmatrix} \cdot \begin{bmatrix} 13 \\ 7 \end{bmatrix} = 673/218 = 3{,}087$$

$$x_3 = \begin{bmatrix} 41 \\ 20 \end{bmatrix} \qquad = 6181 / 2081 = 2{,}97$$

$$x_4 = \begin{bmatrix} 121 \\ 61 \end{bmatrix} \qquad = 55267 / 18362 = 3{,}009$$

$$x_5 = \begin{bmatrix} 365 \\ 182 \end{bmatrix} \qquad = 498499 / 166349 = 2{,}99$$

h-) what will inverse iteration will converge ?

The inverse iteration will converge the eisenvector according to lowest eigenvalue

lowest Eisenvalue = -1 so the eisenvetor $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

i-) what if the $\sigma = 2$ for shift ?

$$A.v - \alpha.I.v = (\lambda.v - \alpha I.v$$

$$v = \frac{(A - \alpha I)^{-1}.(\cancel{\lambda. - \alpha})v}{\cancel{\lambda. - \alpha}}$$

$$(A - \alpha.I)^{-1}.v = \boxed{\frac{1}{\lambda - \alpha}}.v \longrightarrow \frac{1}{3-2} = 1$$

→ this value for $\lambda = 3$ more dominant so the answer is $\boxed{\lambda = 3}$

$$\frac{1}{-1-2} = \frac{1}{-3}$$

J-) if QR applied to A, what would it converage, triangular or diagonal ?

$A \neq A^T$ so The answer is triangular.

2) Computer Problems

2.a) 4.5

(a) Use a library routine to compute the eigenvalues of the matrix

$$A = \begin{bmatrix} 9 & 4.5 & 3 \\ -56 & -28 & -18 \\ 60 & 30 & 19 \end{bmatrix}$$

(b) Compute the eigenvalues of the same matrix again, except with the a33 entry changed to 18.95. What is the relative change in magnitudes of the eigenvalues?

(c) Compute the eigenvalues of the same matrix again, except with the a33 entry changed to 19.05. What is the relative change in magnitudes of the eigenvalues?

(d) What conclusion can you draw about the conditioning of the eigenvalues of A? Compute an appropriate condition number or condition numbers to explain this behavior.

```
>> CP_4_5
(a) Eigenvalues of original matrix: [  1.000  0.000 -1.000 ]

(b) Eigenvalues of first modified matrix: [  0.200  0.000 -0.250 ]
    Relative change in largest eigenvalue:  0.800

(c) Eigenvalues of second modified matrix: [  1.422 -1.372  0.000 ]
    Relative change in largest eigenvalue: [  0.422 ]

(d) Eigenvalues are sensitive;
    Condition number of eigenvector matrix: [ 114.688 ]
    Inner products of right and left eigenvectors: [ -0.018  0.046 -0.024 ]
    Condition numbers of eigenvalues: [ 55.671 21.593 41.737 ]
    Angles between right and left eigenvectors: [ 91.029 87.346 91.373 ]
```

As it can be seen from the image, in ill-conditioned system a little change can cause big changes in the result.

The condition number of matrix A is 4 but this irrelevant for the eigenvalues. Thus, we calculated the condition number of eigenvector. And it tells us that matrix is ill-conditioned.

The inner products and the angle between right and left eigenvectors tell us that these are almost orthogonal to each other. Which makes the eigenvalues easily perpetuated by little changes.

2.b) 4.8

Compute all the roots of the polynomial

$$p(t) = 24 - 40t + 35t2 - 13t3 + t4$$

by forming the companion matrix (see Section 4.2.1) and then calling an eigenvalue routine to compute its eigenvalues. Note that the companion matrix is already in Hessenberg form, which you may be able to take advantage of, depending on the specific software you use. Compare the speed and accuracy of the companion matrix method with those of a library routine designed specifically for computing roots of polynomials (see Table 5.2). You may need to experiment with polynomials of larger degree to see a significant difference.

The roots of the polynomial and eigenvalues of the matrix give the same answers. We could not see any accuracy differences in this polynomial.

```
>> CP_4_8
Output of MATLAB function roots:

Roots =

   9.8274 + 0.0000i
   1.8048 + 0.0000i
   0.6839 + 0.9410i
   0.6839 - 0.9410i

Companion matrix:

C =

     0     0     0   -24
     1     0     0    40
     0     1     0   -35
     0     0     1    13

Eigenvalues of companion matrix:

Eigenvalues =

   0.6839 + 0.9410i
   0.6839 - 0.9410i
   1.8048 + 0.0000i
   9.8274 + 0.0000i
```

2.c) 4.10

a) A singular matrix must have a zero eigenvalue, but must a nearly singular matrix have a "small" eigenvalue? Consider a matrix of the form

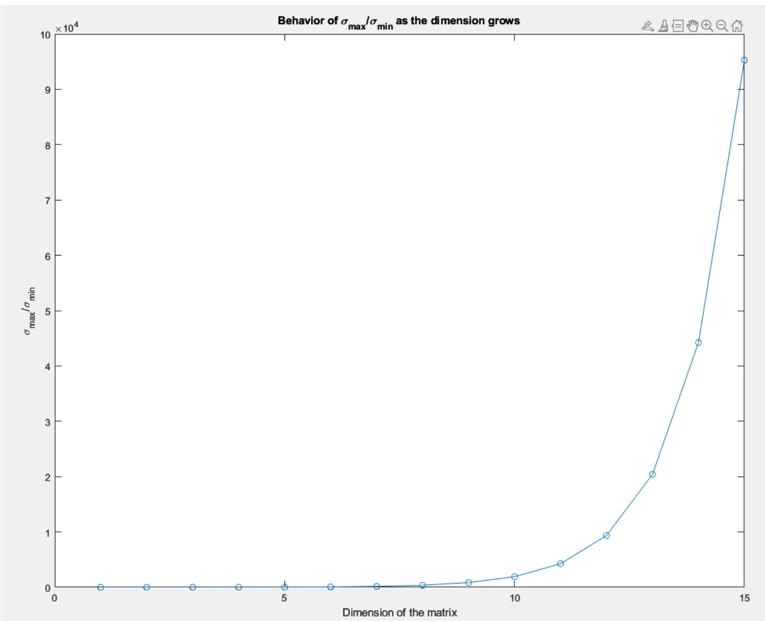
$$\begin{matrix} 1 & -1 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{matrix}$$

whose eigenvalues are obviously all ones. Use a library routine to compute the singular values of such a matrix for various dimensions. How does the ratio σmax/σmin behave as the order of the matrix grows? What conclusions can you draw?

As the ratio increases. The matrix gets closer to being singular and it becomes ill-conditioned.

```
>> CP_4_10
 n min singular value max singular value ratio

 1 min:[  1.0000000000000e+00] || max:[  1.0000000000000e+00] || ratio:[1.0000e+00]
 2 min:[  6.1803398874989e-01] || max:[  1.6180339887499e+00] || ratio:[2.6180e+00]
 3 min:[  3.4729635533386e-01] || max:[  1.8793852415718e+00] || ratio:[5.4115e+00]
 4 min:[  1.8264432359595e-01] || max:[  2.2630774103132e+00] || ratio:[1.2391e+01]
 5 min:[  9.2985333703543e-02] || max:[  2.7363296458404e+00] || ratio:[2.9428e+01]
 6 min:[  4.6761167514514e-02] || max:[  3.2660613276992e+00] || ratio:[6.9846e+01]
 7 min:[  2.3421074823466e-02] || max:[  3.8299348631840e+00] || ratio:[1.6353e+02]
 8 min:[  1.1716426460536e-02] || max:[  4.4148226693743e+00] || ratio:[3.7681e+02]
 9 min:[  5.8590509406860e-03] || max:[  5.0131614643211e+00] || ratio:[8.5563e+02]
10 min:[  2.9296427980878e-03] || max:[  5.6204812731560e+00] || ratio:[1.9185e+03]
11 min:[  1.4648376382548e-03] || max:[  6.2340355985614e+00] || ratio:[4.2558e+03]
12 min:[  7.3242104554303e-04] || max:[  6.8520679296997e+00] || ratio:[9.3554e+03]
13 min:[  3.6621082563223e-04] || max:[  7.4734124289936e+00] || ratio:[2.0407e+04]
14 min:[  1.8310545374335e-04] || max:[  8.0972692811547e+00] || ratio:[4.4222e+04]
15 min:[  9.1552732371274e-05] || max:[  8.7230735886286e+00] || ratio:[9.5279e+04]
```



2.d) 4.13

Consider the generalized eigenvalue problem Kx = λMx derived from the spring-mass system given in Example 4.1 and illustrated in Fig. 4.1. For purposes of this problem, assume the values k1 = k2 = k3 = 1, m1 = 2, m2 = 3, and m3 = 4, in arbitrary units.

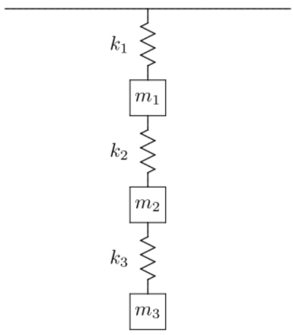


Figure 4.1: Spring-mass system.

$$A = M^{-1} * K \quad and \quad \lambda = w^2$$

So we can find the frequency by finding the eigenvalues of matrix.

```
X =

    0.1653   -0.3960   -0.5620
    0.3116   -0.3494    0.3378
    0.4044    0.2829   -0.0804


Lambda =

    0.0573
    0.5588
    1.3005


Omega =

    0.2394
    0.7475
    1.1404
```