HACETTEPE UNIVERSITY
ENGINEERING FACULTY
ELECTRICAL AND ELECTRONICS ENGINEERING PROGRAM

2023-2024
SPRING SEMESTER

ELE708
NUMERICAL METHODS IN ELECTRICAL ENGINEERING

HW6

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1) Exercises

1.a) 6.5

6.5.-) Determine the critical points of each of the fallowing function and characterize each as a min, max, or saddle point. Also determine whether each function has global min or max on $\mathbb{R}^2$.

a-) $x^2 - 4xy + y^2$

$\nabla f(x,y) = \begin{bmatrix} 2x-4y \\ -4x+2y \end{bmatrix} \rightarrow \nabla f(x,y) = 0$ at $\begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow$ critical point

$H_f(x,y) = \begin{bmatrix} 2 & -4 \\ -4 & 2 \end{bmatrix} \rightarrow$ eigen values $\underbrace{\begin{matrix} \lambda_1 = 1+3i \\ \lambda_2 = 1-3i \end{matrix}}_{\text{positive and negative}} \rightarrow H_f(0,0)$ is indefinete

so $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is saddle point

Since $f(x,y)$ has only saddle point, there is no global min or max.

b-) $x^4 - 4xy + y^4$

$\nabla f(x,y) = \begin{bmatrix} 4x^3-4y \\ -4x+4y^3 \end{bmatrix} \rightarrow \nabla f(x,y) = 0$ at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$H_f(x,y) = \begin{bmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{bmatrix} \rightarrow H_f(0,0) = \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix} \rightarrow$ eigen values $\begin{matrix} \lambda_1 = -4 \\ \lambda_2 = 4 \end{matrix} \rightarrow H_f(0,0)$ indefinite

so $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is saddle point.

$\rightarrow H_f(1,1) = \begin{bmatrix} 12 & -4 \\ -4 & 12 \end{bmatrix} \rightarrow$ eigenvalue $\begin{matrix} \lambda_1 = 8 \\ \lambda_2 = 16 \end{matrix} \rightarrow H(1,1)$ is pozitive def

so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is local min

$\rightarrow H(-1,-1) = \begin{bmatrix} 12 & -4 \\ -4 & 12 \end{bmatrix} \rightarrow$ eigenvalues $\begin{matrix} \lambda_1 = 8 \\ \lambda_2 = 16 \end{matrix} \rightarrow H(-1,-1)$ is pozitive def

so $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ is local min

Since $H(x,y)$ is always pozitive def,

$[1, 1]^T$ and $[-1\ -1]^T$ are global min c-) $2x^3 - 3x^2 - 6xy(x-y-1)$

$$\nabla f(x,y) = \begin{bmatrix} 6x^2 - 12xy + 6y^2 - 6x + 6y \\ -6x^2 + 12xy + 6x \end{bmatrix} \to \nabla f(x,y) = 0$$

$\to (y-x)=0$ or $(y-x+1)=0$

$\to x=0$ or $(2y-x+1)=0$

$$\to \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$H_f(x,y) = \begin{bmatrix} 12x - 12y - 6 & -12x + 12y + 6 \\ -12x + 12y + 6 & 12x \end{bmatrix}$$

$H_f(0,0) = \begin{bmatrix} -6 & +6 \\ +6 & 0 \end{bmatrix} \to$ indefinite so $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is saddle point

$H_f(-1,-1) = \begin{bmatrix} -6 & 6 \\ 6 & -12 \end{bmatrix} \to$ Indefinite so $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ is saddle point

$H(0,-1) = \begin{bmatrix} 6 & -6 \\ -6 & 0 \end{bmatrix} \to$ indefinite so $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ is saddle point

$H(1,0) = \begin{bmatrix} 6 & -6 \\ -6 & 6 \end{bmatrix} \to$ pozitive so $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is local minimum

There is no global min or max.

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d-) $(x-y)^4 + x^2 - y^2 - 2x + 2y + 1$

$$\nabla f(x,y) = \begin{bmatrix} 4(x-y)^3 + 2x - 2 \\ -4(x-y)^3 - 2y + 2 \end{bmatrix} \to \nabla f(x,y) = 0 \text{ at } \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$H_f(x,y) = \begin{bmatrix} 12(x-y) + 2 & -12(x-y)^2 \\ -12(x-y)^2 & 12(x-y)^2 - 2 \end{bmatrix} \to H_f(1,1) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \to \text{Indefinite}$$

So $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is saddle point, there is no global min and max

1.b) 6.6

6.6-) Determine the critical points of the Langrangian function for each of the following problems and determine whether each ~~of the~~ constrained minimum, maximum and neither.

a-) $f(x,y) = x^2 + y^2$ subject to $g(x,y) = x + y - 1 = 0$

$$\nabla f(x,y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix} \qquad J_g(x,y) = \nabla g(x,y) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}^T$$

$$\nabla L(x,\lambda) = \begin{bmatrix} 2x + 1 \cdot \lambda \\ 2y + 1 \cdot \lambda \\ x + y - 1 \end{bmatrix} = \begin{bmatrix} \nabla f(x,y) + J_g^T(x,y) \cdot \lambda \\ g(x) \end{bmatrix} \Rightarrow \begin{bmatrix} 0,5 \\ 0,5 \\ 1 \end{bmatrix}$$ critical point

$$B(x,y,\lambda) = \underline{\nabla_{xx} L(x,y,\lambda)} = H_f(x) + \sum_{i=1}^{m} \lambda_i \cdot H_g(x)$$

$$H_f(x,y) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad H_g(x,y) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B(x,y,\lambda) = H_f + \lambda \cdot H_g = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$, The ortogonal matrix to $J_g(x,y)$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$z^T \cdot B \cdot z = \begin{bmatrix} 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 4$$ so $\begin{bmatrix} 0,5 \\ 0,5 \\ 1 \end{bmatrix}$ is constrained minumum.

b-) $f(x,y) = x^3 + y^3$ subject to $g(x,y) = x + y - 1$

① $\nabla f(x,y) = \begin{bmatrix} 3x^2 \\ 3y^2 \end{bmatrix}$ $\nabla g = Jg(x,y) = [1,\ 1]$

② $\nabla L(x,y,\lambda) = \begin{bmatrix} \nabla f(x,y) + \lambda . J^T(x,y) \\ g(x) \end{bmatrix} = \begin{bmatrix} 3x^2 + \lambda \\ 3y^2 + \lambda \\ x + y - 1 \end{bmatrix} \rightarrow \begin{bmatrix} 0{,}5 \\ 0{,}5 \\ -0{,}75 \end{bmatrix}$ critical point $\Rightarrow$

③ $B(x,y,\lambda) = Hf(x,y) + \lambda . Hg(x,y)$

$$= \begin{bmatrix} 6x & 0 \\ 0 & 6y \end{bmatrix} + \lambda . \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 6x & 0 \\ 0 & 6y \end{bmatrix}$$

④ The ortogonal vector to $Jg(x,y) = [1\ 1]$ is $z = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

⑤ $z^T . B . z = [1\ -1] . \begin{bmatrix} 6.0{,}5 & 0 \\ 0 & 6.0{,}5 \end{bmatrix} . \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 6 > 0$ so $\begin{bmatrix} 0{,}5 \\ 0{,}5 \\ -0{,}75 \end{bmatrix}$ is constrained minumum $\Rightarrow$

c-) $f(x,y) = 2x + y$ subject to $g(x,y) = x^2 + y^2 - 1 = 0$

① $\nabla f(x,y) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $J_{g(x,y)} = \begin{bmatrix} 2x & 2y \end{bmatrix}$

② $\nabla L = \begin{bmatrix} 2 + 2x \cdot \lambda \\ 1 + 2y\,\lambda \\ x^2 + y^2 - 1 \end{bmatrix} \Rightarrow \begin{matrix} x = -1/\lambda \\ y = -1/2\lambda \end{matrix} \Rightarrow \left(\frac{1}{\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 1$

$\Rightarrow$ critical points

$\begin{bmatrix} -0{,}894 \\ -0{,}447 \\ 1{,}12 \end{bmatrix}$ and $\begin{bmatrix} -0{,}894 \\ -0{,}447 \\ 1{,}12 \end{bmatrix}$

③ $B(x,y,\lambda) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \lambda \cdot \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2\lambda & 0 \\ 0 & 2\lambda \end{bmatrix}$

④ The ortogonal vector to J_g^T is $z = \begin{bmatrix} y \\ -x \end{bmatrix}$

⑤ $z^T . B . z = \begin{bmatrix} y & -x \end{bmatrix} . \begin{bmatrix} 2\lambda & 0 \\ 0 & 2\lambda \end{bmatrix} . \begin{bmatrix} y \\ -x \end{bmatrix} = y^2 . 2\lambda + x^2 . 2\lambda$

for $\begin{bmatrix} -0{,}894 \\ -0{,}447 \\ 1{,}12 \end{bmatrix} \Rightarrow$ $z^T.B.z = 2{,}24 > 0$ so constrained min

for $\begin{bmatrix} -0{,}894 \\ -0{,}447 \\ -1{,}12 \end{bmatrix} \Rightarrow$ $z^T.B.z = -2{,}24 < 0$ so constrained max d-) $f(x,y) = x^2 + y^2$ subject to $g(x,y) = x.y^2 - 1 = 0$

① $\nabla f(x,y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$, $Jg(x,y) = \begin{bmatrix} y^2 & 2xy \end{bmatrix}$

② $\nabla L_{(x,y,\lambda)} = \begin{bmatrix} 2x + \lambda . y^2 \\ 2y + \lambda . 2xy \\ xy^2 - 1 \end{bmatrix}$ $\Rightarrow x = -1/\lambda$ $\quad x = -\lambda . y^2 / 2 \Rightarrow \lambda^3 = -2$

$\lambda = \underline{\underline{-1,25992 \ldots}}$

$\begin{bmatrix} 0,794 \\ 1,12 \\ \cancel{-0,126} \end{bmatrix}$

③ $B(x,y,\lambda) = \begin{bmatrix} 2 & 2\lambda y \\ 2\lambda y & 2+2\lambda x \end{bmatrix}$

④ $z = \begin{bmatrix} 2x \\ -y \end{bmatrix}$

⑤ $z^T . B . z = 15,1 > 0$ so $\begin{bmatrix} 0,794 \\ 1,12 \\ -1,26 \end{bmatrix}$ is constrained minumum.

2) Computer Problems

2.a) 6.3

Use a library routine, or one of your own design, to find a minimum of each of the following functions on the interval [0, 3]. Draw a plot of each function to confirm that it is unimodal.

(a) f(x) = x4 − 14x3 + 60x2 − 70x.
(b) f(x) = 0.5x2 − sin(x).
(c) f(x) = x2 + 4 cos(x).
(d) f(x) = Γ(x). $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt, \quad x > 0,$

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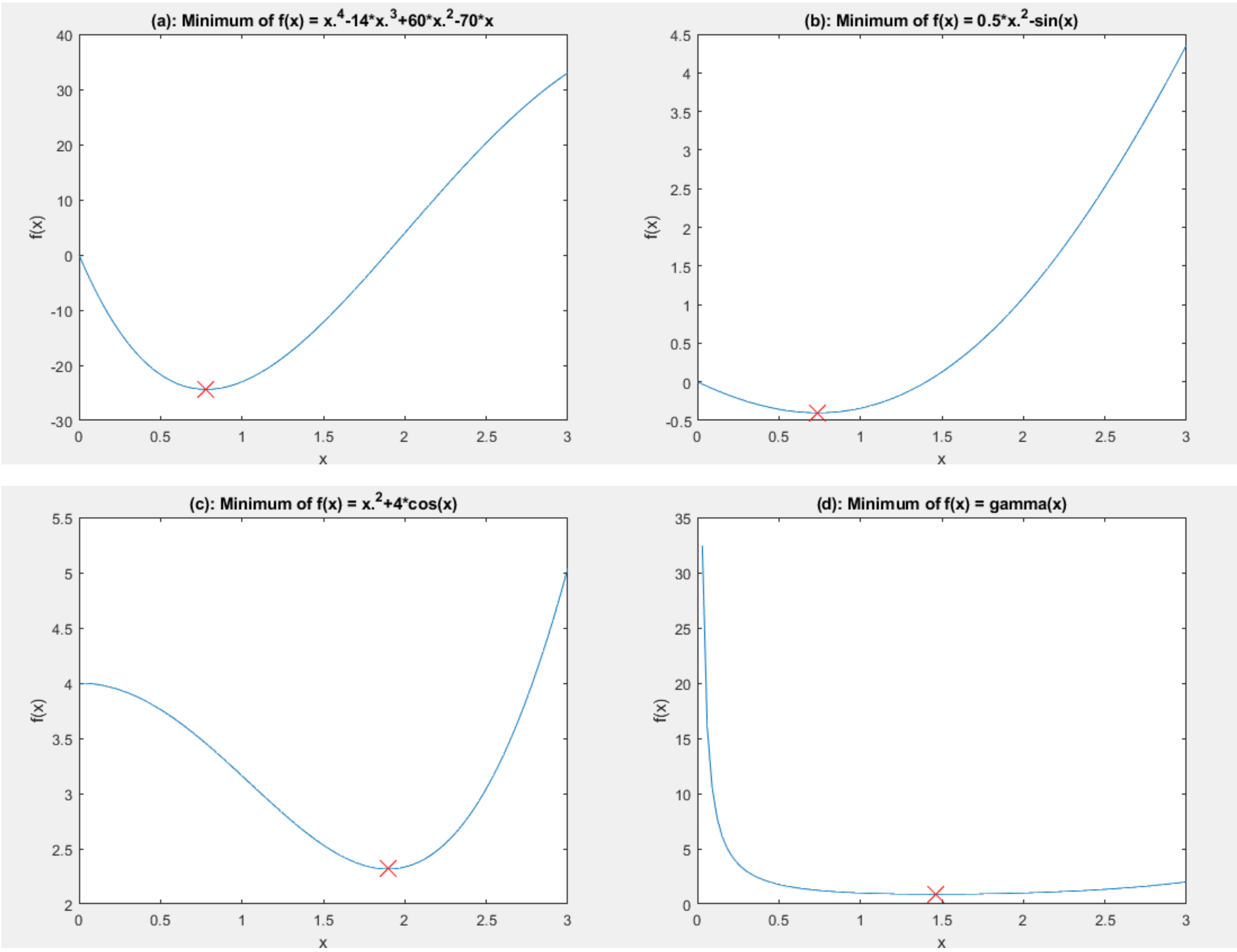
1. The equations are defined.
2. The minimum of equation in bounded interval find by using **fminbnd(equation, 0, 3).**
3. The equation is plotted and then the minimum point is marked.

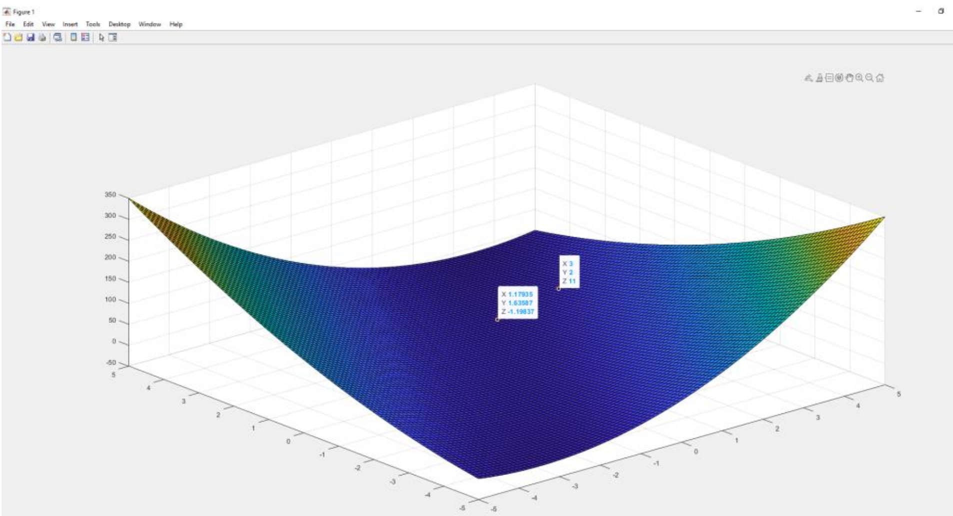


2.b) 6.6

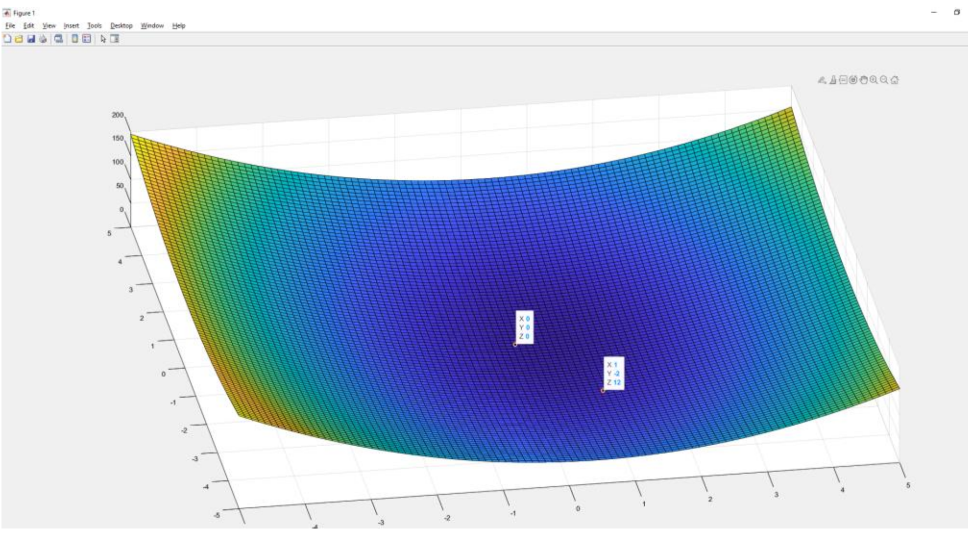
Write a general-purpose line search routine. Your routine should take as input a vector defining the starting point, a second vector defining the search direction, the name of a routine defining the objective function, and a convergence tolerance. For the resulting one-dimensional optimization problem, you may call a library routine or one of your own design. In any case, you will need to determine a bracket for the minimum along the search direction using some heuristic procedure. Test your routine for a variety of objective functions and search directions. This routine will be useful in some of the other computer exercises in this section.

The LineSearch algorithm created by using library routine function ‘`fminbnd`’ `and iteration for each step. By doing these optimal minimum point can be found.`

1)
```
x0 = [3;2];
s = [-5; -1];
y = 5 * x(1)^2 + 2 * x(2)^2 - 7 * x(1) * x(2);
```



2)
```
x0 = [1;-2];
s = [-1; 2];
Y = 5 * X1.^2 + 2 * X2.^2 - X1.*4;
```



2.c) 6.9

Write a program to find a minimum of Rosenbrock's function

f(x, y) = 100(y − x2)2 + (1 − x)2

using each of the following methods:

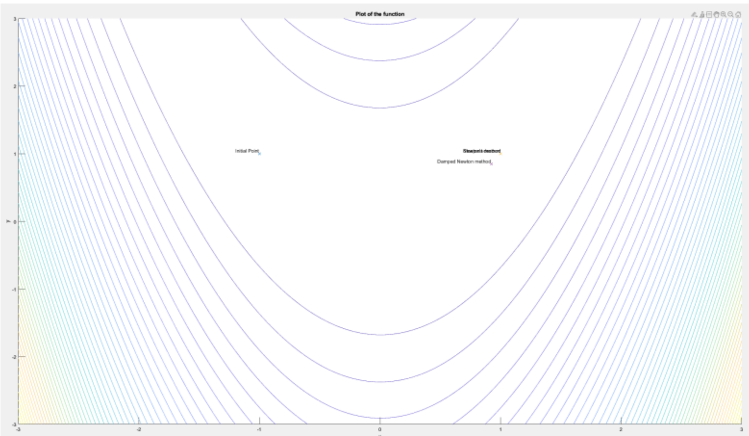
(a) Steepest descent

(b) Newton

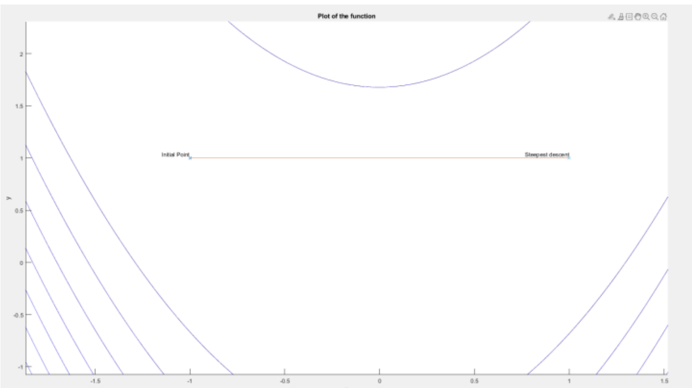
(c) Damped Newton (Newton's method with a line search)

You should try each of the methods from each of the three starting points [−1 1]T, [0 1]T, and [2 1]T. For any line searches and linear system solutions required, you may use either library routines or routines of your own design. Plot the path taken in the plane by the approximate solutions for each method from each starting point.

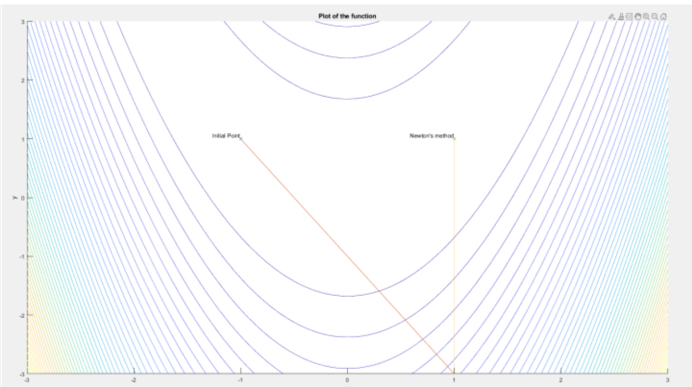
Plot 1, starting point = [−1 1]T



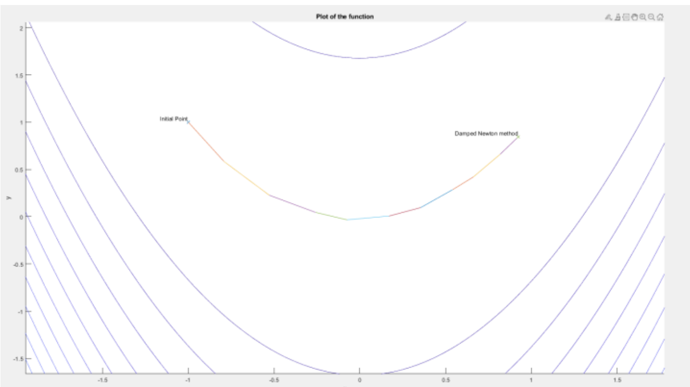
Steepest Descent



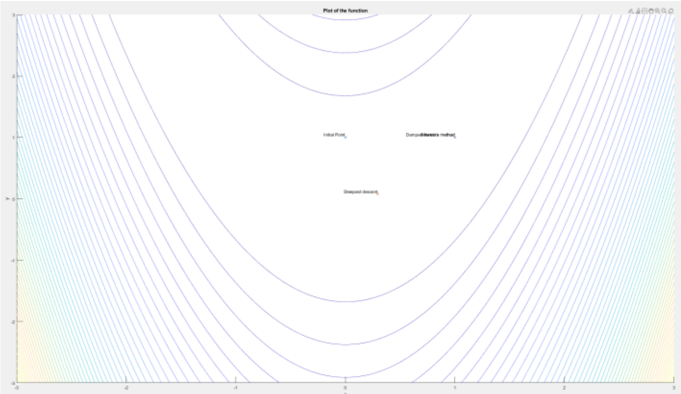
Newton's Method



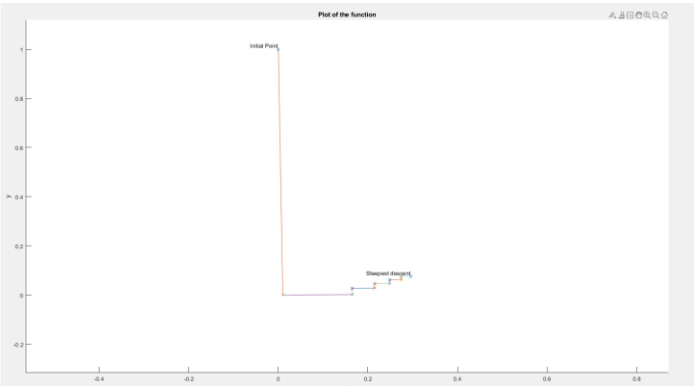
Damped Newton method:



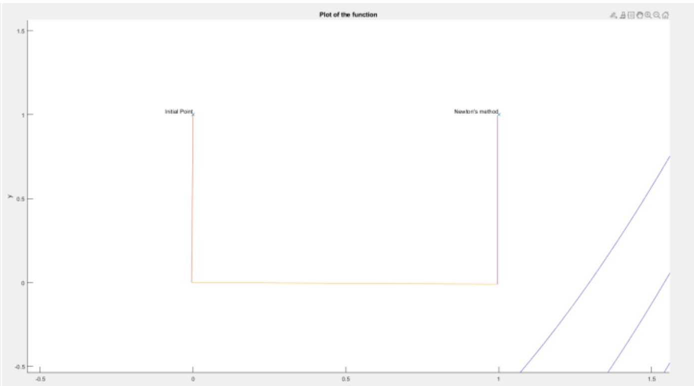
Plot 2, starting point = [0 1]T



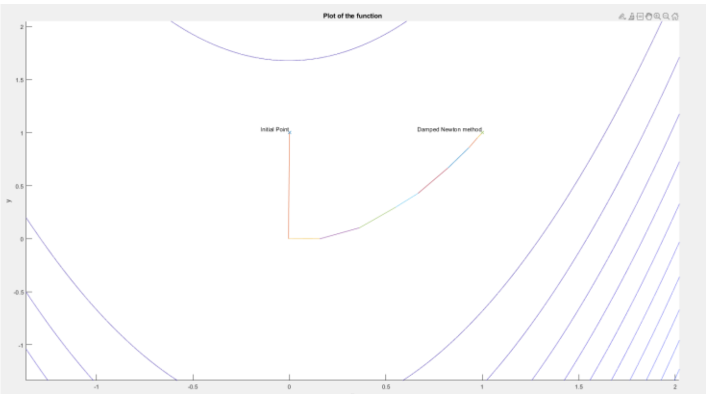
Steepest Descent



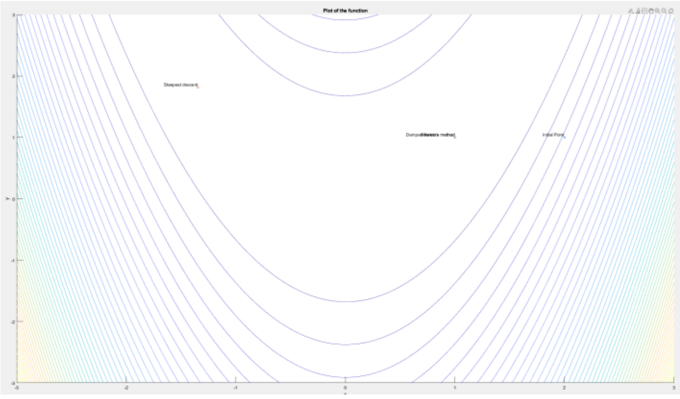
Newton's Method



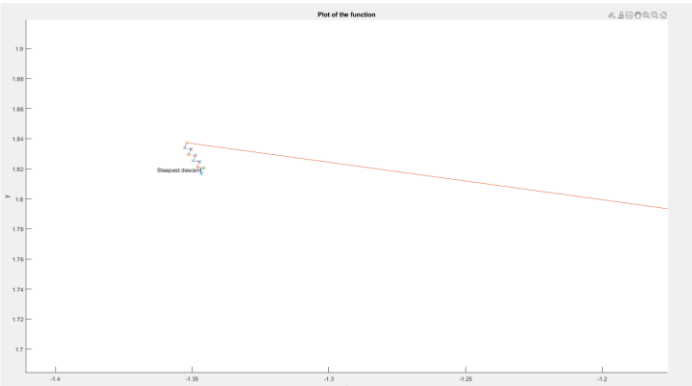
Damped Newton method:



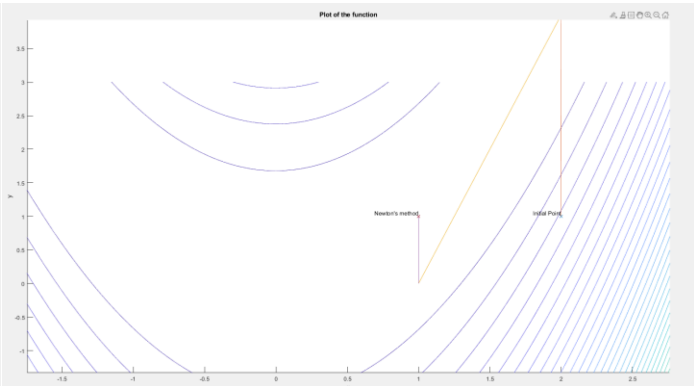
Plot 3, starting point = [2 1]T



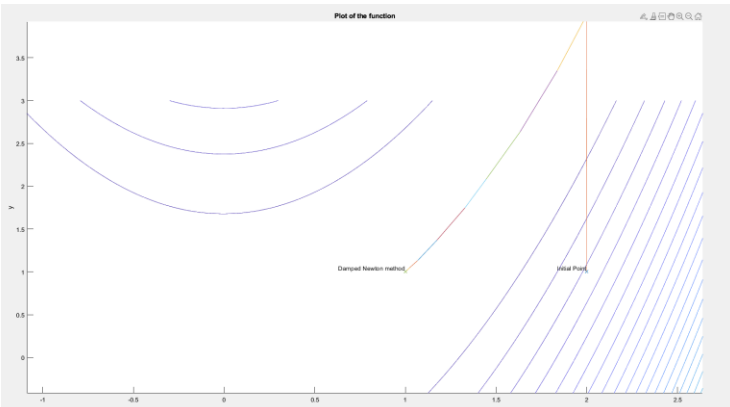
Steepest Descent



Newton’s Method



Damped Newton method:



2.d) 6.14 The concentration of a drug in the bloodstream is expected to diminish exponentially with time. We will fit the model function

y = f(t, x) = x1ex2t

to the following data:

t 0.5 1.0 1.5 2.0

y 6.80 3.00 1.50 0.75

t 2.5 3.0 3.5 4.0

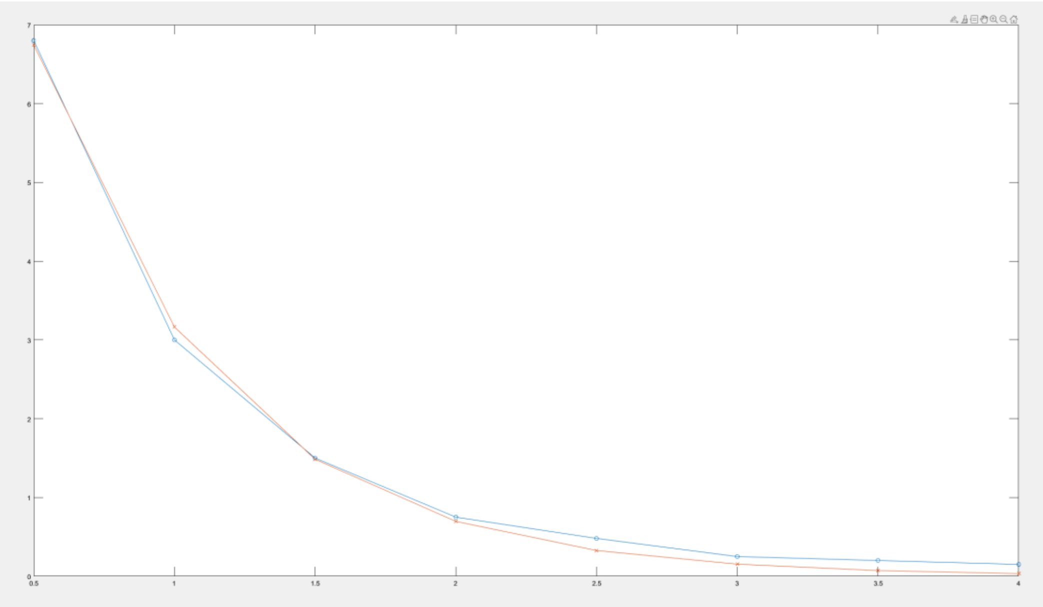
y 0.48 0.25 0.20 0.15

(a) Perform the exponential fit using nonlinear least squares. You may use a library routine or one of your own design, perhaps using the GaussNewton method.

(b) Taking the logarithm of the model function gives log(x1) + x2t, which is now linear in x2. Thus, an exponential fit can also be done using linear least squares, assuming that we also take logarithms of the data points yi. Use linear least squares to compute x1 and x2 in this manner. Do the values obtained agree with those determined in part a? Why?

The blue line is nonlinear least square with curve fitting.

The red line is linear least square.



There is differences between linear and non-linear fit. It is because first one adjust its variables for best fit to the values and the other is adjust for exact fit for the equation.

```
>> CP_6_14
(a) Nonlinear fit:
   14.3766
   -1.5139

(b) Linear fit:

x =

    8.6350
   -1.0967
```