HACETTEPE UNIVERSITY
ENGINEERING FACULTY
ELECTRICAL AND ELECTRONICS ENGINEERING PROGRAM

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SPRING SEMESTER

ELE708
NUMERICAL METHODS IN ELECTRICAL ENGINEERING

HW8

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1) Exercises

1.a) 8.1

8.1-)

a) Compute the approximate value of the integral $\int_0^1 x^3 dx$, first by the midpoint rule and the by the trapezoid rule.

a-) Midpoint Rule

$$M(f) = (b-a) \cdot f\left(\frac{a+b}{2}\right)$$
$$= (1-0) \cdot f(0,5)$$
$$= (0,5)^3$$
$$= 0,125 = 1/8$$

Trapezoid Rule

$$T(f) = \frac{b-a}{2} \cdot \left(f(a) + f(b)\right)$$
$$= \frac{1-0}{2} \cdot \left(0^3 + 1^3\right)$$
$$= 0,5 = 1/2$$

b) Use the differences between two results to estimate the error each of them.

b-)

$$E(f) \approx \frac{T(f) - M(f)}{3} = \frac{0,5 - 0,125}{3} = 0,125$$

$$E_M(f) = 0,125 = 1/8$$

$$E_T(f) = -2 \cdot E_M(f)$$
$$= -2 \cdot (0,125)$$
$$= -0,25 = -1/4$$

c-) Combine the two results to obtain the Simpson's Rule approximation to integral.

c-) $S(f) = \frac{2}{3} M(f) + \frac{1}{3} T(f)$

$= \frac{2}{3} \cdot \frac{1}{8} + \frac{1}{3} \cdot \frac{1}{2}$

$= 1/4 = 0{,}25$

d-) Would you expect the latter to be exact for this problem? why?

From the Taylor Series expension, we saw that the error for Simpson's rule depends on the fourth and higher derivatives. So Simpson's rule's degree is three.

In the question, we had polynomial of degree 3.

So Yes. It is exact solution.

1.b) 8.7

8.7-) Derive an open two-point Newton-Cotes quadrature rule for interval $[a,b]$. What are the resulting nodes and weights? What is the degree of resulting rule?

An n-point open Newton-Cotes rule:

$$x_i = a + i(b-a)/n+1 \quad , \quad i = 1,2 \ldots$$

$$\left.\begin{aligned} x_1 &= a + (b-a)/3 = \frac{2a+b}{3} \\ x_2 &= a + 2.(b-a)/3 = \frac{a+2b}{3} \end{aligned}\right\} \underline{\text{nodes}}$$

$$w_1 . 1 + w_2 . 1 = \int_a^b 1\, dx = b - a$$

$$w_1 . \left(\frac{2a+b}{3}\right) + w_2 . \left(\frac{a+2b}{3}\right) = \int_a^b x\, dx = (b^2 - a^2)/2$$

$$\begin{bmatrix} 1 & 1 \\ \frac{2a+b}{3} & \frac{a+2b}{3} \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} b-a \\ (b^2-a^2)/2 \end{bmatrix} \Rightarrow \begin{aligned} R_1 &\rightarrow \frac{a+2b}{3} . R_1 \\ R_1 &\rightarrow R_1 - R_2 \end{aligned}$$

$$= \frac{(b-a)}{3} . w_1 = (b-a) . \left(\frac{a+2b}{3}\right) - \frac{(b^2-a^2)}{2}$$

$$\left.\begin{aligned} \Rightarrow w_1 &= \frac{b-a}{2} \\ w_2 &= \frac{b-a}{2} \end{aligned}\right\} \underline{\text{weights}}$$

$$Q(f) = \frac{b-a}{2} . \left(f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right)\right)$$

polynomial degree of $\underline{\underline{1}}$

1.c) 8.10

8.10-) In a Chebyshev quadrature rule, all the weights are taken to have the same value, thereby eliminating n multiplication in evalauting the resulting quadtature rule, since single weight can be factored out of the ~~elimination~~ summation.

a-) Use the method of underterminal coefficients to determine the nodes and weight for three point Chebyshev quadrature rule on the interval [-1, 1]

b-) What is the resulting degree of the rule?

a-)

$$w.(1+1+1) = \int_{-1}^{1} 1\,dx = 2$$

$$w.(x_1+x_2+x_3) = \int_{-1}^{1} x\,dx = 0$$

$$w.(x_1^2+x_2^2+x_3^2) = \int_{-1}^{1} x^2\,dx = 2/3$$

$$w.(x_1^3+x_2^3+x_3^3) = \int_{-1}^{1} x^3\,dx = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \\ x_1^3 & x_2^3 & x_3^3 \end{bmatrix} \cdot \begin{bmatrix} w \\ w \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2/3 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} &3w = 2 \Rightarrow \boxed{w = 2/3} \\ &\left.\begin{aligned} &x_1 + x_2 + x_3 = 0 \\ &x_1^2 + x_2^2 + x_3^2 = 1 \\ &x_1^3 + x_2^3 + x_3^3 = 0 \end{aligned}\right\} \text{one must be zero} \end{aligned}$$

$$\boxed{x_2 = 0, \quad x_1 = \frac{\sqrt{2}}{2}, \quad x_3 = -\frac{\sqrt{2}}{2}}$$

b-) The resulting answer is quadrature of degree 3.

2) Computer Problems

2.a) 8.1

Since

$$\int_0^1 \frac{4}{1+x^2}\, dx = pi$$

one can compute an approximate value for π using numerical integration of the given function.

(a) Use the midpoint, trapezoid, and Simpson composite quadrature rules to compute the approximate value for π in this manner for various step sizes h. Try to characterize the error as a function of h for each rule, and also compare the accuracy of the rules with each other (based on the known value of π). Is there any point beyond which decreasing h yields no further improvement? Why?

(b) Implement Romberg integration and repeat part a using it.

(c) Compute π again by the same method, this time using a library routine for adaptive quadrature and various error tolerances. How reliable is the error estimate it produces? Compare the work required (integrand evaluations and elapsed time) with that for parts a and b. Make a plot analogous to Fig. 8.4 to show graphically where the integrand is sampled by the adaptive routine.

(d) Compute π again by the same method, this time using Monte Carlo integration with various numbers n of sample points. Try to characterize the error as a function of n, and also compare the work required with that for the previous methods. For a suitable random number generator, see Section 13.5.

(a) Decreasing the h (step size) more than truncation error will have no impact on the answer, other than that decreasing the step size will decrease the error as we can see, but since the answer already very close to number of pi, we can stop decreasing the it for the errors lower than 10^-3.

```
(a) Values and Error of midpoint, trapezoid, and Simpson composite quadrature rules

Values
Stepsize    | Midpoint    | Trapezoid    | Simpson
 1.000e+00 |  6.400e+00 |  3.000e+00   |  5.267e+00
 2.000e-01 |  3.937e+00 |  3.135e+00   |  3.670e+00
 4.000e-02 |  3.302e+00 |  3.141e+00   |  3.248e+00
 8.000e-03 |  3.174e+00 |  3.142e+00   |  3.163e+00
 1.600e-03 |  3.148e+00 |  3.142e+00   |  3.146e+00
 3.200e-04 |  3.143e+00 |  3.142e+00   |  3.142e+00
 6.400e-05 |  3.142e+00 |  3.142e+00   |  3.142e+00
 1.280e-05 |  3.142e+00 |  3.142e+00   |  3.142e+00
 2.560e-06 |  3.142e+00 |  3.142e+00   |  3.142e+00

Errors
Stepsize    | Midpoint    | Trapezoid    | Simpson
 1.000e+00 |  3.258e+00 |  1.416e-01   |  2.125e+00
 2.000e-01 |  7.954e-01 |  6.667e-03   |  5.281e-01
 4.000e-02 |  1.601e-01 |  2.667e-04   |  1.066e-01
 8.000e-03 |  3.200e-02 |  1.067e-05   |  2.133e-02
 1.600e-03 |  6.400e-03 |  4.267e-07   |  4.267e-03
 3.200e-04 |  1.280e-03 |  1.707e-08   |  8.533e-04
 6.400e-05 |  2.560e-04 |  6.827e-10   |  1.707e-04
 1.280e-05 |  5.120e-05 |  2.731e-11   |  3.413e-05
 2.560e-06 |  1.024e-05 |  1.092e-12   |  6.827e-06
```

(b) This rule decrease the error in less iteration than other rules.

```
(b) Repeat using Ramborg rule

Stepsize    | Ramborg | error
 1.0e+00  3.000e+00  1.416e-01
 6.2e-02  3.142e+00  1.169e-08
 3.9e-03  3.142e+00  1.332e-15
 2.4e-04  3.142e+00  1.776e-15
 1.5e-05  3.142e+00  4.441e-15
 9.5e-07  3.142e+00  4.441e-16
```

(c) Adaptive quadrature method quickly calculate better answer when the timings are compared with the other methods.

```
(c) Error and function evaluations:
 quad                              || quadl
 tolerance | error | iter_num | errror | iter_num | timing1    | timing2
 1.0e-01    2.597e-06    13    5.344e-08    18       1.900e-04  1.525e-04
 1.0e-02    2.597e-06    13    5.344e-08    18       1.317e-04  1.139e-04
 1.0e-03    2.597e-06    13    5.344e-08    18       1.152e-04  8.630e-05
 1.0e-04    2.597e-06    13    5.344e-08    18       9.660e-05  8.310e-05
 1.0e-05    5.662e-08    17    5.344e-08    18       2.992e-04  3.036e-04
 1.0e-06    2.933e-08    21    5.344e-08    18       9.510e-05  4.650e-05
```

```
Timing
Stepsize      | Midpoint    | Trapezoid     | Simpson
 1.000e+00 |  4.740e-05 |  3.680e-05   |  4.620e-05
 2.000e-01 |  1.086e-04 |  6.700e-05   |  9.180e-05
 4.000e-02 |  1.059e-04 |  3.150e-05   |  1.017e-04
 8.000e-03 |  3.421e-04 |  3.190e-05   |  3.051e-04
 1.600e-03 |  2.173e-03 |  3.427e-04   |  2.211e-03
 3.200e-04 |  7.438e-03 |  4.960e-05   |  7.164e-03
 6.400e-05 |  3.513e-02 |  8.500e-05   |  3.395e-02
 1.280e-05 |  1.757e-01 |  2.280e-04   |  1.854e-01
 2.560e-06 |  8.714e-01 |  2.231e-03   |  9.061e-01

 (b) Repeat using Ramborg rule

Stepsize      | Ramborg | error      | Timing
 1.0e+00       3.000e+00  1.416e-01,  2.256e-04
 6.2e-02       3.142e+00  1.169e-08,  1.532e-03
 3.9e-03       3.142e+00  1.332e-15,  1.132e-03
 2.4e-04       3.142e+00  1.776e-15,  2.480e-04
 1.5e-05       3.142e+00  4.441e-15,  7.441e-04
 9.5e-07       3.142e+00  4.441e-16,  1.724e-02
```

(d) This method can not give same result with less duration compared to the other methods. It may be a better option for higher order equations.

```
(d) Error in Monte Carlo:
 n        | value     | error    | 1/(sqrt(n) | timing
       1  2.469e+00  7.926e-01  1.000e+00  2.310e-04
      10  3.279e+00  2.163e-01  3.162e-01  1.128e-03
     100  3.104e+00  1.385e-02  1.000e-01  7.275e-04
    1000  3.120e+00  1.801e-04  3.162e-02  4.340e-05
   10000  3.137e+00  6.551e-04  1.000e-02  3.860e-04
  100000  3.142e+00  1.165e-03  3.162e-03  1.078e-03
 1000000  3.142e+00  7.111e-04  1.000e-03  2.326e-02
```

2.b) 8.4

Use numerical integration to verify or refute each of the following conjectures.

(a)

$$\int_0^1 \sqrt{x^3}\, dx = 0.4$$

(b)

$$\int_0^1 \frac{1}{1+10x^2}\, dx = 0.4$$

(c)

$$\int_0^1 \frac{e^{-9x^2}+e^{-1024(x-1/4)^2}}{\sqrt{\pi}}\, dx = 0.2$$

(d)

$$\int_0^{10} \frac{50}{\pi(2500x^2+1)}\, dx = 0.5$$

(e)

$$\int_{-9}^{100} \frac{1}{\sqrt{|x|}}\, dx = 26$$

(f)

$$\int_0^{10} 25e^{-25x}\, dx = 1$$

(g)

$$\int_0^1 \log(x)\, dx = -1$$

Instead of function named “quad”, “integral” is used by the matlab’s advice.

```
f1 = @(x) x.^(3/2)
 Ans : 4.000000e-01

f2 = @(x) 1./(1+10*x.^2);
 Ans : 3.998760e-01

f3 = @(x) exp(-9*x.^2)+exp(-1024*(x-0.25).^2))/sqrt(pi);
 Ans : 1.979130e-01

f4 = @(x) 50./(pi*(2500*x.^2+1))
 Ans : 4.993634e-01

f5 = @(x) 1./sqrt(abs(x))
 Ans : 2.599995e+01

f6 = @(x) 25*exp(-25*x)
 Ans : 1.000000e+00

f7 = @(x) log(x);
 Ans : -1.000000e+00
```