HACETTEPE UNIVERSITY
ENGINEERING FACULTY
ELECTRICAL AND ELECTRONICS ENGINEERING PROGRAM

2023-2024
SPRING SEMESTER

ELE708
NUMERICAL METHODS IN ELECTRICAL ENGINEERING

HW3

N23239410 – Ali Bölücü

1) Exercises

1.a) 3.1

3.1-) if a vertical beam has a downward force applied its lower end, the amount by which it streches will be proportinial to the magnitude of the force. Thus the total length of y;

$y = x_1 + x_2 . t$

- t → force applied
- x_2 → proportionality constant
- x_1 → original lenght

t	10	15	20
y	11,6	11,85	12,25

a) Set up overdetermined 3x2 sytem

b-) Is the system consistent?, If not calculate each possible values, Is there a reason to select a re

c-) Set up normal equation and solve it to obtain the least square solition. Compare with b.

---

a-) $y = x_1 + x_2 . t = \begin{bmatrix} 1 & t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$Ax = \begin{bmatrix} 1 & 10 \\ 1 & 15 \\ 1 & 20 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cong \begin{bmatrix} 11,6 \\ 11,85 \\ 12,25 \end{bmatrix} = b$$

---

b-) No, It is not consistent, the equation looks like linear but the increase in y and t is not linear.

Solving by using 1 and 2 equation $\Rightarrow \begin{bmatrix} 1 & 10 \\ 1 & 15 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 11,6 \\ 11,85 \end{bmatrix} \Rightarrow \begin{matrix} x_1 = 11,1 \\ x_2 = 0,05 \end{matrix}$

Solving by using 1 and 3 equation $\Rightarrow \begin{bmatrix} 1 & 10 \\ 1 & 20 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 11,6 \\ 12,25 \end{bmatrix} \Rightarrow \begin{matrix} x_1 = 10,95 \\ x_2 = 0,065 \end{matrix}$

Solving by using 2 and 3 equation $\Rightarrow \begin{bmatrix} 1 & 15 \\ 1 & 20 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 11,85 \\ 12,25 \end{bmatrix} \Rightarrow \begin{matrix} x_1 = 10,65 \\ x_2 = 0,08 \end{matrix}$

The aveyure of $x_1 = (11,1 + 10,95 + 10,65)/3 = 10,9$

The avegare of $x_2 = (0,05 + 0,065 + 0,08)/3 = 0,065$

> closest pair is the second

c-) $A^T . A . x = A^T . b$

$$\begin{bmatrix} 1 & 1 & 1 \\ 10 & 15 & 20 \end{bmatrix} \cdot \begin{bmatrix} 1 & 10 \\ 1 & 15 \\ 1 & 20 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 & 45 \\ 45 & 725 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \Bigg| \quad \begin{bmatrix} 1 & 1 & 1 \\ 10 & 15 & 20 \end{bmatrix} \cdot \begin{bmatrix} 11,6 \\ 11,85 \\ 12,25 \end{bmatrix} = \begin{bmatrix} 35,7 \\ 538,75 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 3 & 45 \\ 45 & 725 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 35,7 \\ 538,75 \end{bmatrix}$$

$$\Downarrow$$

$x_1 = 10,925$
$x_2 = 0,065$ > The average values we computed in part b is similiar.

1.b) 3.4

3,4) In fitting straight line $y = x_0 + x_1 . t$ to the three data points $(t_i, y_i) = (0,0), (1,0), (1,1)$ is the least square solution is unique and why?

As it mentioned, "3.2 Existing and Uniqueness", the solution is unique if and only if A has full column rank.

$$A . x \cong b \Rightarrow \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cong \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\downarrow \; S_3 = S_3 - S_2$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \text{rank}(A) = 2 \text{ and } n = 2$$

so its okey, <u>it has unique solition</u>.

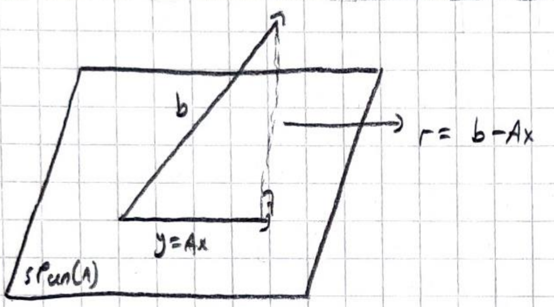
1.c) 3.5

3.5) Let x be the solition to the linear least square problem $Ax \cong b$, where

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

let $r = b - Ax$ be the corresponding residual vector. which of the following three vectors is a possible value for r? Why?

a) $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ b) $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$ c) $\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$



, the residue must be orthogonal to the A

thus,

$$0 = A^T \cdot r = A^T \cdot (b - Ax)$$

a) $A^T \cdot r = 0 \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & 6 \end{bmatrix}$ → not orthogonal X

b) $A^T \cdot r = 0 \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 4 \end{bmatrix}$ → only orthogonal to first column X

c) $A^T \cdot r = 0 \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$ → fully orthogonal ✓

2) Computer Problems

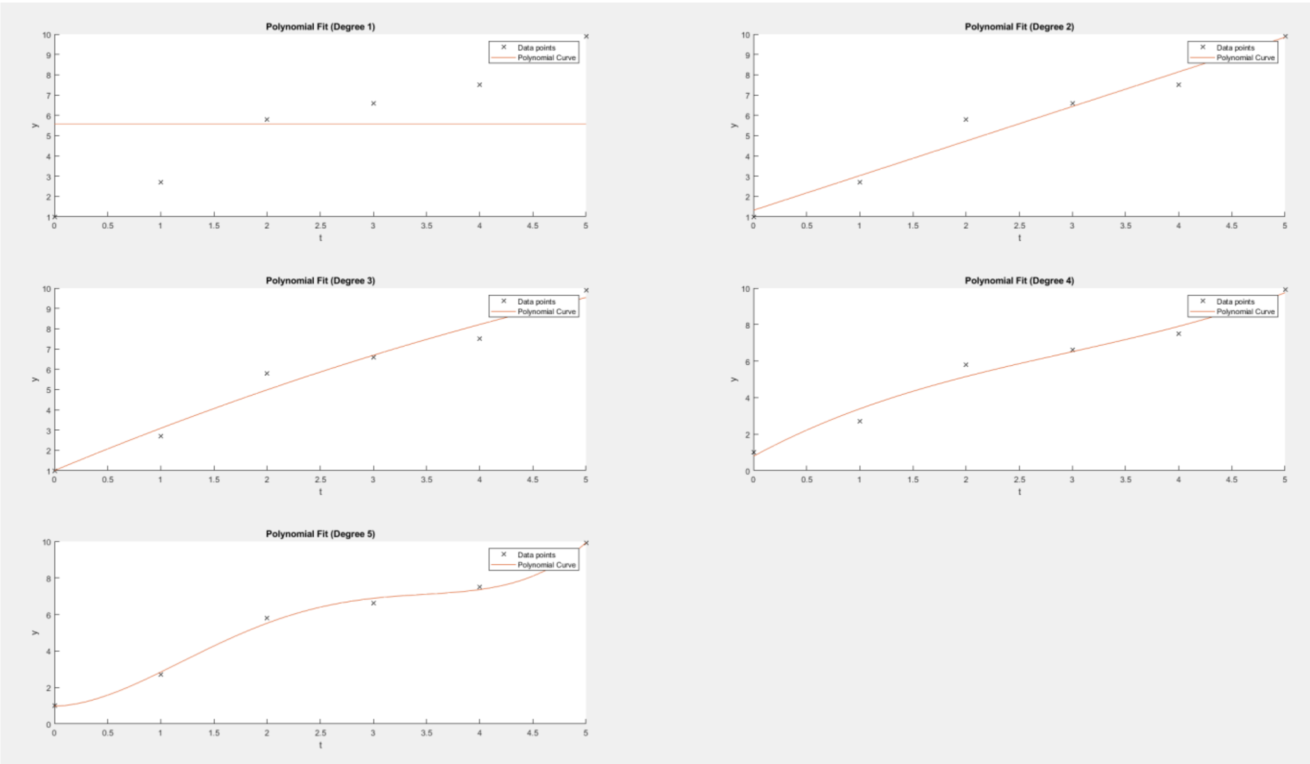
2.a) 3.3 For n = 0, 1, . . . , 5, fit a polynomial of degree n by least squares to the following data:

$\boldsymbol{t}$: $\{\mathbf{0.0, 1.0, 2.0, 3.0, 4.0, 5.0}\}$

$\boldsymbol{y}$: $\{\mathbf{1.0, 2.7, 5.8, 6.6, 7.5, 9.9}\}$

Make a plot of the original data points along with each resulting polynomial curve (you may make separate graphs for each curve or a single graph containing all of the curves). Which polynomial would you say captures the general trend of the data better? Obviously, this is a subjective question, and its answer depends on both the nature of the given data (e.g., the uncertainty of the data values) and the purpose of the fit. Explain your assumptions in answering.

The polynomial curves shown in figure below, in each step the curve fit the data more and more. The last curve would be the best fit yet since the data could have some error, making the curve overfit may not be the best idea. In order to avoid this I would say 3rd or 4th curve would be better choice.



2.b) 3.2 A common problem in surveying is to determine the altitudes of a series of points with respect to some reference point. The measurements are subject to error, so more observations are taken than are strictly necessary to determine the altitudes, and the resulting overdetermined system is solved in the least squares sense to smooth out errors. Suppose that there are four points whose altitudes x1, x2, x3, x4 are to be determined. In addition to direct measurements of each xi with respect to the reference point, measurements are also taken of each point with respect to all of the others. The resulting measurements are:

$$
\begin{array}{rr}
x_1 = 2.95 & x_2 = 1.74 \\
x_3 = -1.45 & x_4 = 1.32 \\
x_1 - x_2 = 1.23 & x_1 - x_3 = 4.45 \\
x_1 - x_4 = 1.61 & x_2 - x_3 = 3.21 \\
x_2 - x_4 = 0.45 & x_3 - x_4 = -2.75
\end{array}
$$

Set up the corresponding least squares system Ax ~= b and use a library routine, or one of your own design, to solve it for the best values of the altitudes. How do your computed values compare with the direct measurements?

The answers shown in figure below. Since there was no big difference between the direct values and the values with a reference point. The answer close to direct measurements.

Workspace

```
>> CP3_2
x1 = 2.96
x2 = 1.746
x3 = -1.46
x4 = 1.314
>>
```

2.c) 3.4 a) Solve the following least squares problem using any method you like:

$$\begin{bmatrix} 0.16 & 0.10 \\ 0.17 & 0.11 \\ 2.02 & 1.29 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.26 \\ 0.28 \\ 3.31 \end{bmatrix}$$

(b) Now solve the same least squares problem again, but this time use the slightly perturbed right-hand side

$$b = \begin{bmatrix} 0.27 \\ 0.25 \\ 3.33 \end{bmatrix}$$

c) Compare your results from parts a and b. Can you explain this difference?

---

a)

```
a):
Normal Equation:
    1.0000
    1.0000

SVD:
    1.0000
    1.0000
```

b)

```
b):
Normal Equation:
    7.0089
   -8.3957

SVD:
    7.0089
   -8.3957
```

c) In this problem we can see that it is an ill-conditioned system. The sensitivity depends on both cond(A) and the angle. Even though the angle is small, the cond number increases the sensitivity.

```
c)
condA = 1097.5387
cos_theta_1 = 1
cos_theta_2 = 0.99998
cos_theta = 0.01123
delx_over_x = 7.8862
delb_over_b = 0.01123
bound = 0.1384
```

2.d) Codes for 1.1 and 1.4

2.d.i) 3.1

```
function CP3_1()

    t = [0; 1; 2; 3; 4; 5];
    y = [1; 2.7; 5.8; 6.6; 7.5; 9.9];
    m = size(t,1);
    A = ones(m,1);
    p = 51;
    ts = linspace(0,5,p)';


    for n = 1:5
        subplot(3,2,n);
        hold on;
        plot(t, y, 'kx');
        x = A(:,1:n)\y;
        ys(1:p,n) = x(n);

        for k = 2:n
            ys(:,n) = ys(:,n).* ts+x(n-k+1);
        end


        plot(ts, ys(:,n));
        title(['Polynomial Fit (Degree ', num2str(n), ')']);
        xlabel('t'); ylabel('y');
        legend('Data points', 'Polynomial Curve');
        hold off;
        A(:,n+1) = A(:,n) .*t;
    end
end
```

2.d.ii) 3.2

Editor - CP3_2.m *

CP3_1.m | CP3_2.m *

```matlab


function CP3_2()

    x1 = 2.95;
    x2 = 1.74;
    x3 = -1.45;
    x4 = 1.32;

    d12 = 1.23;
    d13 = 4.45;
    d14 = 1.61;
    d23 = 3.21;
    d24 = 0.45;
    d34 = -2.75;

    A = [1,  0,  0,  0;
         0,  1,  0,  0;
         0,  0,  1,  0;
         0,  0,  0,  1;
         1, -1,  0,  0;
         1,  0, -1,  0;
         1,  0,  0, -1;
         0,  1, -1,  0;
         0,  1,  0, -1;
         0,  0,  1, -1];

    b = [x1; x2; x3; x4; d12; d13; d14; d23; d24; d34];

    x = A\b;

    disp(['x1 = ', num2str(x(1))]);
    disp(['x2 = ', num2str(x(2))]);
    disp(['x3 = ', num2str(x(3))]);
    disp(['x4 = ', num2str(x(4))]);
end
```

2.d.iii) 3.4

```
function CP3_4()

A = [0.16, 0.10;
    0.17, 0.11;
    2.02, 1.29;];

b1 = [0.26; 0.28; 3.31];
x1_ne = A \ b1;
[U, S, V] = svd(A, 'econ');
x1_svd = V * (S' * inv(S * S')) * U' * b1;
disp('a):');
disp('Normal Equation:');
disp(x1_ne);
disp('SVD:');
disp(x1_svd);



b2 = [0.27; 0.25; 3.33];
x2_ne = A \ b2;
[U, S, V] = svd(A, 'econ');
x2_svd = V * (S' * inv(S * S')) * U' * b2;
disp('b):');
disp('Normal Equation:');
disp(x2_ne);
disp('SVD:');
disp(x2_svd);

%psudo_inverse_A = inv(transpose(A)*A)*transpose(A)

condA = cond(A);
cos_theta_1 = norm(A*x1_ne) / norm(b1);
cos_theta_2 = norm(A*x2_ne) / norm(b2);
cos_theta = norm(b2 - b1) / norm(b1);

delta_x_over_x = norm(x2_ne - x1_ne) / norm(x1_ne);
delta_b_over_b = norm(b2 - b1) / norm(b1);
bound = condA * cos_theta * delta_b_over_b;

disp('c)');
disp(['condA = ', num2str(condA)]);
disp(['cos_theta_1 = ', num2str(cos_theta_1)]);
disp(['cos_theta_2 = ', num2str(cos_theta_2)]);
disp(['cos_theta = ', num2str(cos_theta_2)]);

disp(['delx_over_x = ', num2str(delta_x_over_x)]);
disp(['delb_over_b = ', num2str(delta_b_over_b)]);
disp(['bound = ', num2str(bound)]);


end
```