جامعة الشطرة

كلية الهندسة – قسم الحاسبات

تقرير رياضيات

تطبيقات التكامل

Limit

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What is a limit?

The General form of a limit statement is

$$\lim_{x \to something} f(x)=something\ else,$$

And means "when x does something, f(x) does something else"

There are 5 possibilities for "something" and three for "Something else", for 15 cases in all, but you should not try to memorize all 5 cases separately. you should understand the overall idea of a limit, and then plug that idea into each case «overall» Limit (Just called "the limit"):

How to find $\lim_{x \to c}$ f(x)

If f(x) is not piece wise:

Factor and simplify F(x). Now, Find F(c).

(lesson 4: Finding limits Analytically)

- case 1: f(c)=#, then $\lim_{x \to c}$ f(x)=f(c)

  - Remember that $\frac{0}{non\ zero\#} = 0$

- case 2: f(c)= $\frac{non\ zero\ \#}{0}$, then $\lim_{x \to c}$ f(x)= ∞, -∞, or DNE

  -To decide between ∞, -∞ and DNE

Look at the one-sided limits.

((Properties of Limits))

In these rules let “a”, “A” and “B” be real numbers and “g” be

Function such that. $\lim_{x \to a}$ f(x)= A and $\lim_{x \to a}$ g(x)=B

Limit of a constant:

$$\lim_{x \to c} k=K$$

Limit of a constant times a function:

$$\lim_{x \to a} [K.F(x)] = K. \lim_{x \to a} F(x)$$

$$= K.A$$

Limit of the sum or difference of functions:

$$\lim_{x \to a} [F(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$

$$= A \pm B$$

Limit of the product of functions:

$$\lim_{x \to a}[F(x).g(x)] = [\lim_{x \to a} f(x)]\ [\ \lim_{x \to a} f(x)][\ \lim_{x \to a}\ g(x)]$$

$$A.B$$

Limit of the quotient of functions (B≠0):

$$\lim_{x \to a} \frac{F(x)}{g(x)} = \frac{\lim_{x \to a} F(x)}{\lim_{x \to a} g(x)} = \frac{A}{B}$$

Limit of a polynomial Function:

$\lim_{x \to a}$ P(x)=P(a)

Limit of a function raised to an exponent (K is any real number):

$\lim_{x \to a}$ [f(x)]x = [$\lim_{x \to a}$ f(x)]x = Ax

Equality of the limit of two functions:

$\lim_{x \to a}$ f(x)= $\lim_{x \to a}$ g(x) if f(x) = g(x) for all x ≠ a

Limit of an exponential expression with a function

As the exponent (b is any real number):

$\lim_{x \to a}$ bf(x) = $b^{\lim_{x \to a} f(x)}$ = bA

Limits at infinity (n is any positive real number):

$\lim_{x \to \infty} [\frac{1}{\frac{n}{x}}] = 0$ and $\lim_{x \to \infty} [\frac{1}{\frac{n}{x}}] = 0$

5.2. Limits from graphs

Finding limits by looking at graphs is usually easy and this is how we begin. Eventually, we will need to be able to find limits directly from expressions defining functions, but starting with graphs is a good way to build intuition.

In order to find $\lim_{x\to a} f(x)$ by looking at the graph of the function f, one looks at the heights of the points on the graph corresponding to values of x that get successively closer to a from either side. If those heights approach a particular value L, then $\lim_{x\to a} f(x) = L$. Otherwise, $\lim_{x\to a} f(x)$ does not exist.

Sometimes it happens that the heights approach one value as x gets close to a from the left and a *different* value as x gets close to a from the right. When this happens, we still say that $\lim_{x\to a} f(x)$ does not exist. It is convenient to have notations for these "one-sided limits":

$\lim\limits_{x\to a^-} f(x)$ is "the limit of $f(x)$ as x approaches x from the left,"

$\lim\limits_{x\to a^+} f(x)$ is "the limit of $f(x)$ as x approaches x from the right."

We can now restate the condition by saying that the "two-sided" limit $\lim_{x\to a} f(x)$ exists if and only if both of the one-sided limits exist and equal the same value L, in which case $\lim_{x\to a} f(x) = L$.

5.2.1 Example Find the indicated limits by looking at the graph.

(a) $\lim_{x \to 3} f(x)$

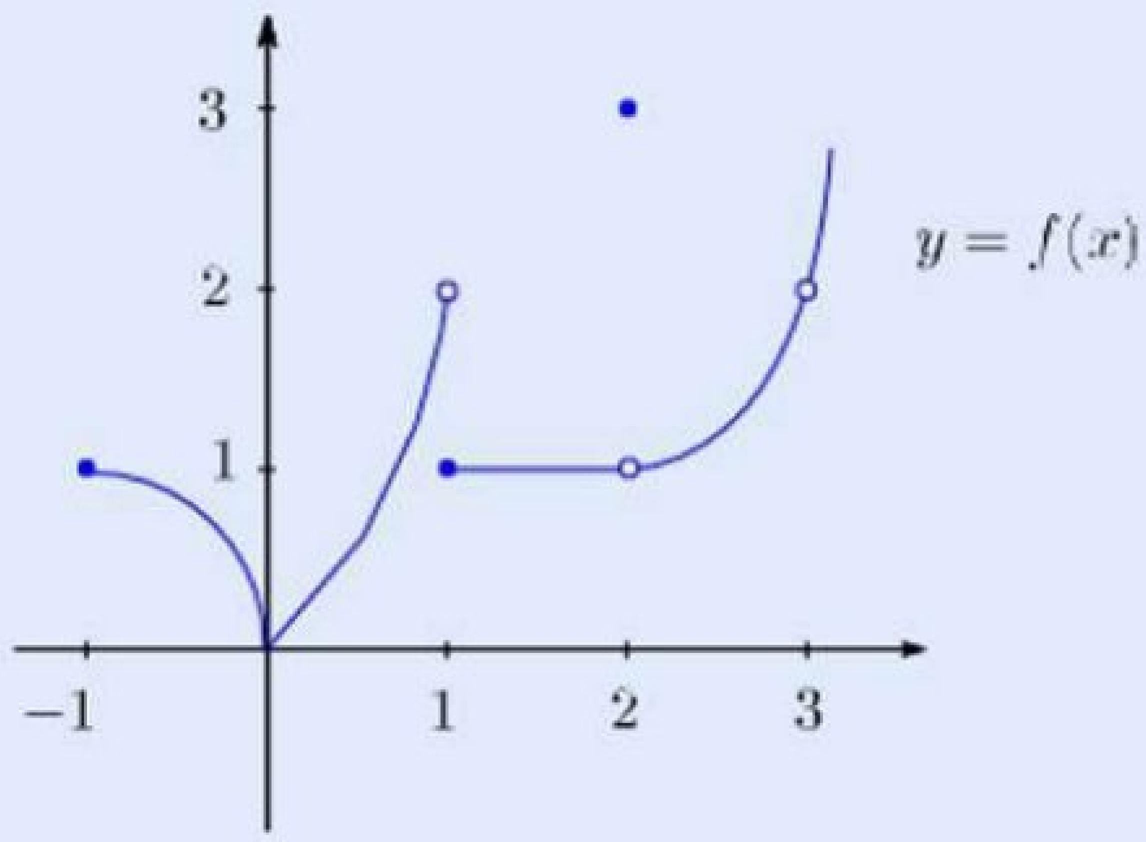
(b) $\lim_{x \to 2} f(x)$

(c) $\lim_{x \to 1^-} f(x)$

(d) $\lim_{x \to 1^+} f(x)$

(e) $\lim_{x \to 1} f(x)$

(f) $\lim_{x \to 0} f(x)$



Solution

(a) $\lim_{x \to 3} f(x) = 2$. (As x gets close to 3 from either side, the height of the graph approaches the value 2.)

(b) $\lim_{x \to 2} f(x) = 1$. (The height of the graph right at $x = 2$ is 3 (indicated by the dot), but this does not enter in since we are letting x get close to 2 without actually equaling 2.)

(c) $\lim_{x \to 1^-} f(x) = 2$. (As x gets close to 1 from the left, the height of the graph approaches the value 2. It is easy to make a mistake here and read 1^- as —1.)

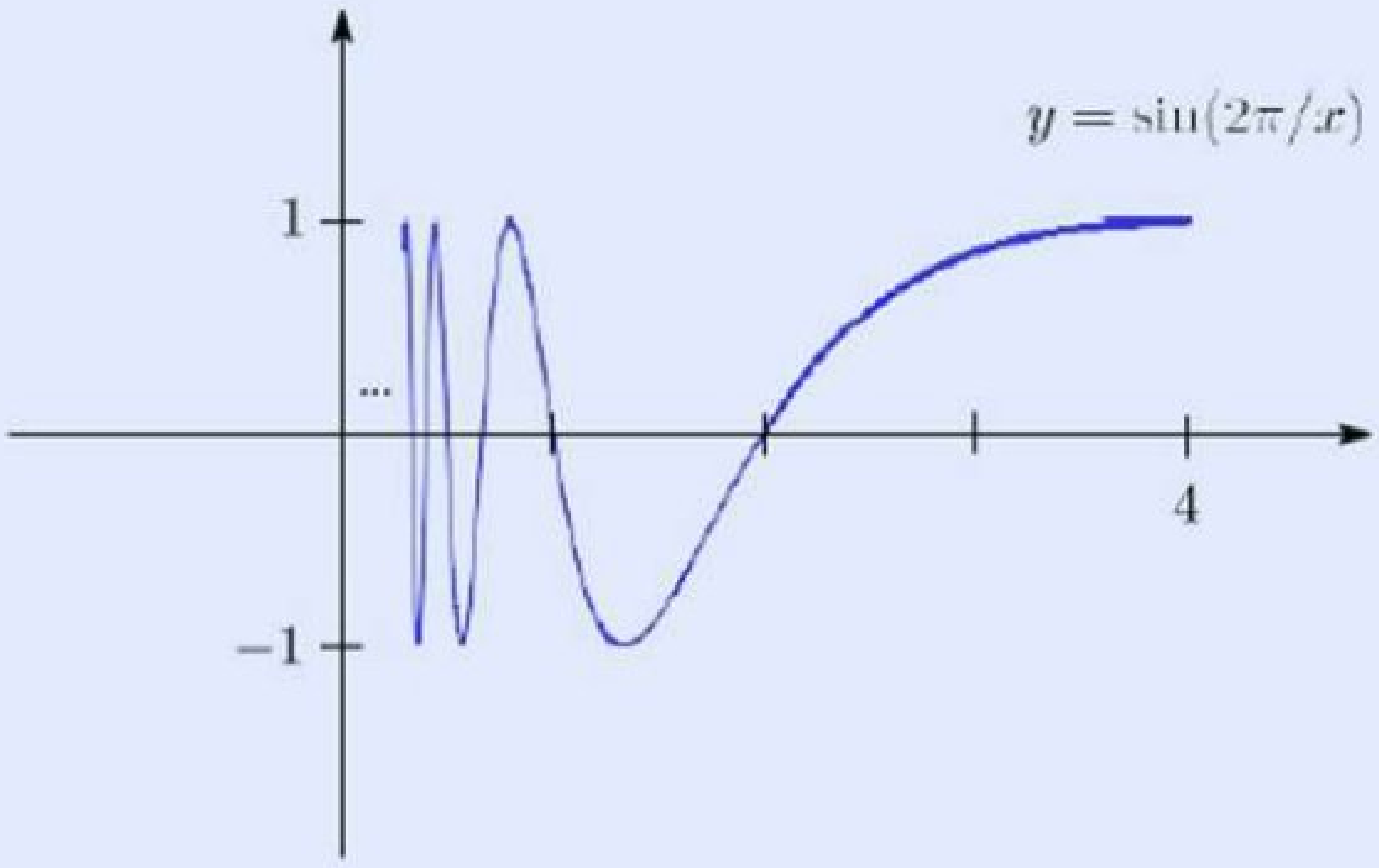
(d) $\lim_{x \to 1^+} f(x) = 1$. (The heights approach, in fact equal, 1 as x gets close to 1 from the right.)

(e) $\lim_{x \to 1} f(x)$ does not exist. (The one-sided limits are not the same.)

(f) $\lim_{x \to 0} f(x) = 0$. (The heights approach 0 as x gets close to 0.)

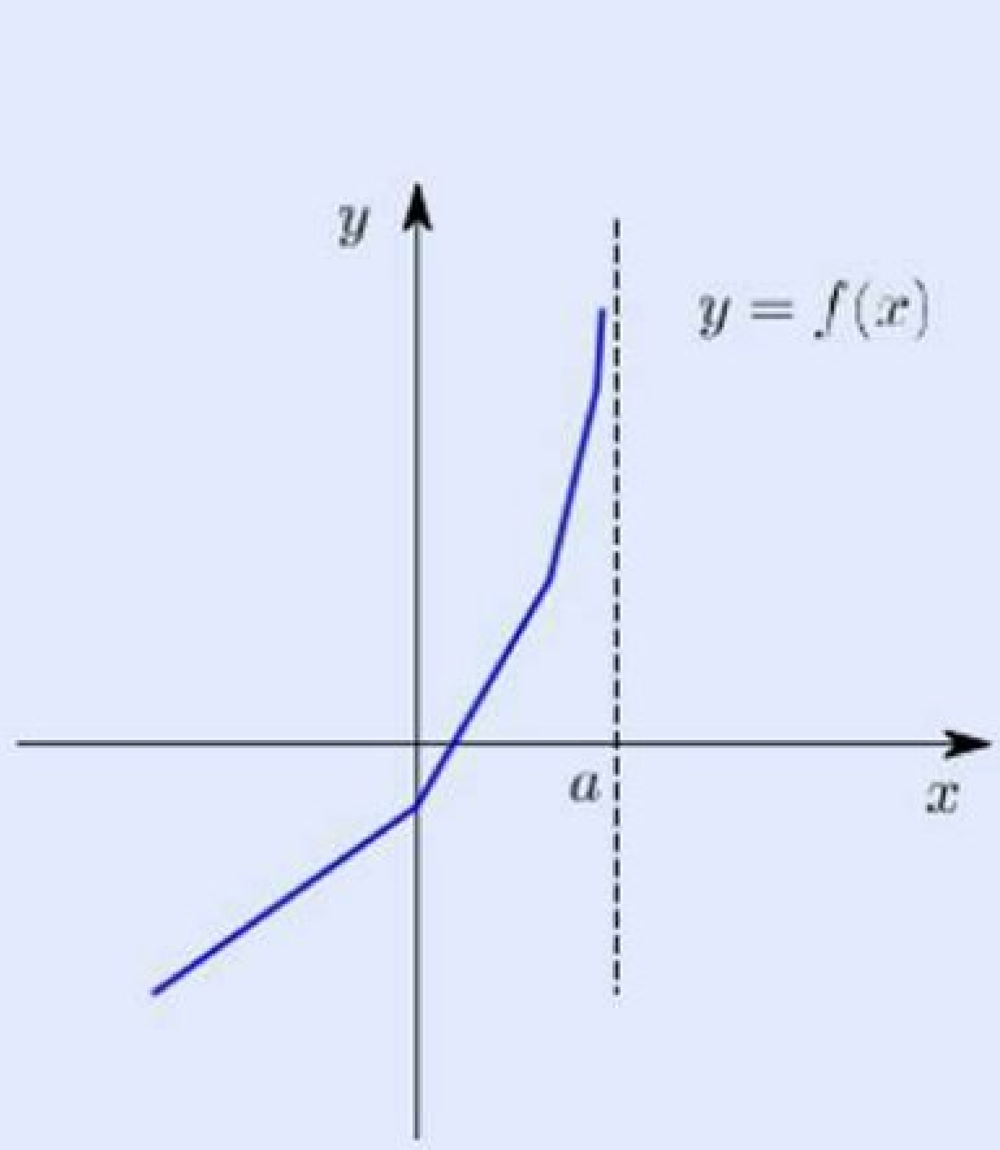
□

5.2.2 Example Find $\lim_{x \to 0^+} \sin(2\pi/x)$ by looking at the graph:

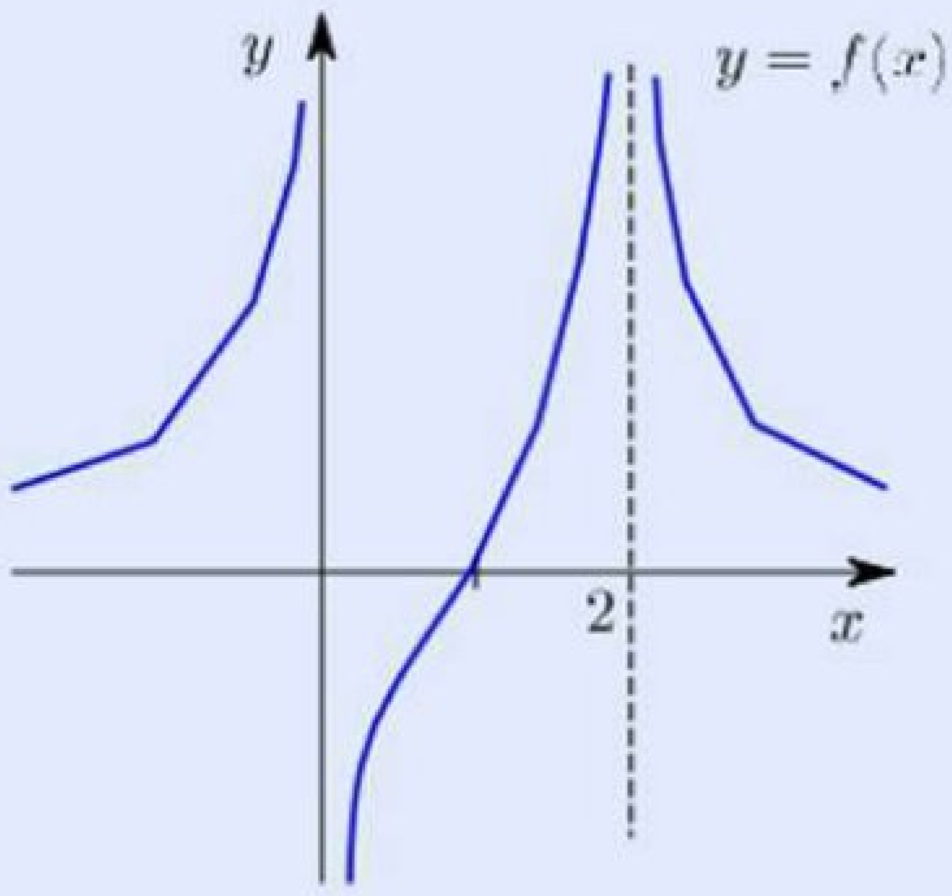


Solution As x approaches 0 from the right, the height of the graph oscillates back and forth between —1 and 1 and does not approach any particular value. Therefore, $\lim_{x \to 0^+} \sin(2\pi/x)$ does not exist. □

In the graph pictured below, as x gets closer to a from the left, the height of the graph $f(x)$ ever increases. This is indicated by writing $\lim_{x \to a^-} f(x) = \infty$. Since a particular value for the height is not approached, the limit does not exist; however, by writing ∞ we can still provide some information about the behavior of the function as a is approached from the left.



5.2.3 Example Find the indicated limits by looking at the graph.



(a) $\lim_{x \to 0^+} f(x)$

(b) $\lim_{x \to 0^-} f(x)$

(c) $\lim_{x \to 0} f(x)$

(d) $\lim_{x \to 2} f(x)$

Solution

(a) $\lim_{x \to 0^+} f(x) = -\infty$. (As x gets close to 0 from the right, the height of the graph ever decreases.)

(b) $\lim_{x \to 0^-} f(x) = \infty$. (As x gets close to 0 from the left, the height of the graph ever increases.)

(c) $\lim_{x \to 0} f(x)$ does not exist. (Since the one-sided limits are not the same, the two-sided limit does not exist (and we make no attempt to provide further information).)

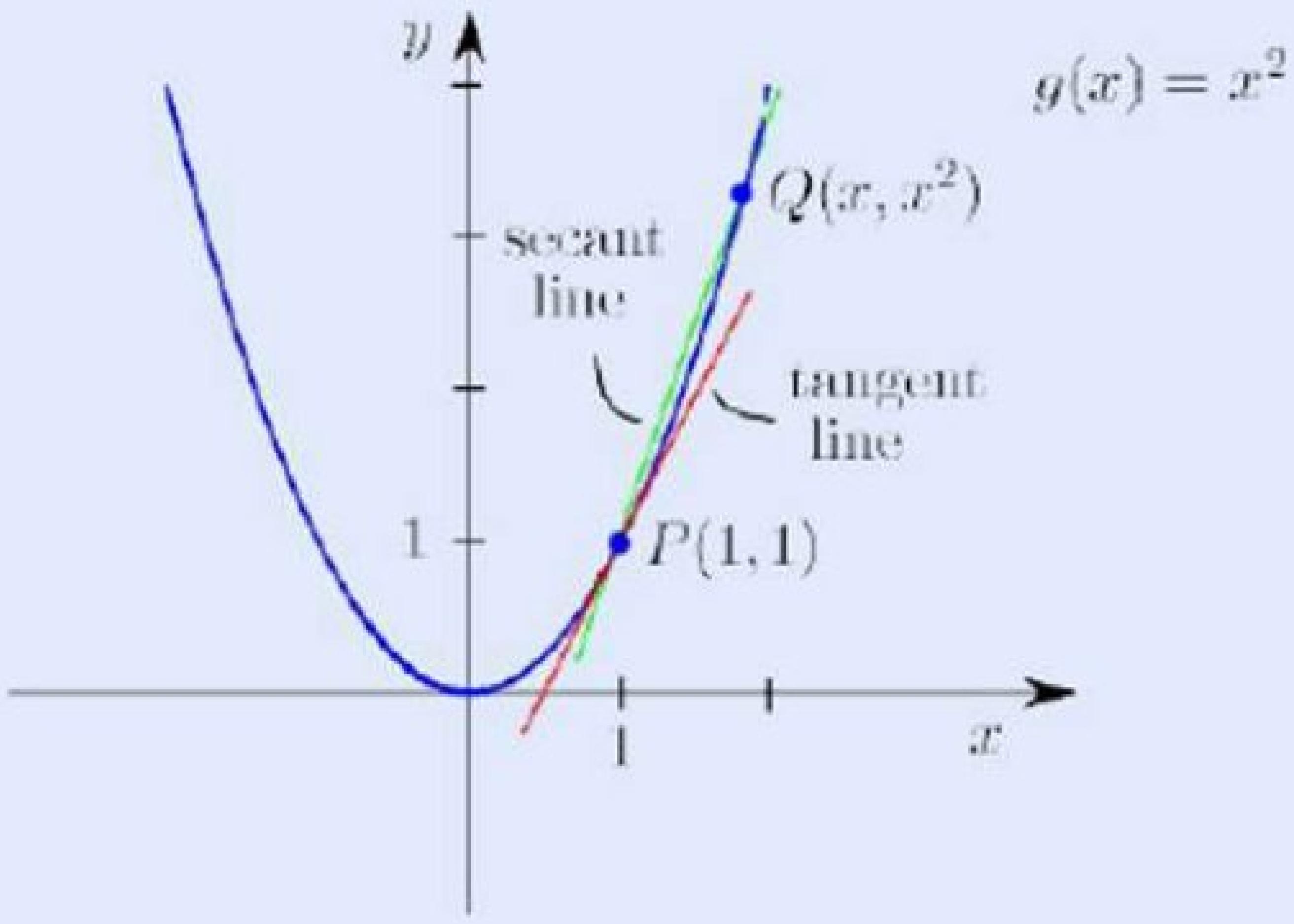
(d) $\lim_{x \to 2} f(x) = \infty$.

5.3. Slope of tangent line

The intuitive notion of a limit given above is enough to allow for a simple example to show the idea behind calculus.

5.3.1 Example Find the slope of the line tangent to the graph of $g(x) = x^2$ at the point $(1, 1)$.

Solution In order to use the formula for slope given in 1.1 we need two points. Certainly $P\,(1, 1)$ is one point on the tangent line, but there is no obvious way to come up with a second point. However, the line $\overline{PQ}$, called a secant line, is not far from being the tangent line, and we can find its slope by using the two points $P(1, 1)$ and $Q(x, x^2)$.



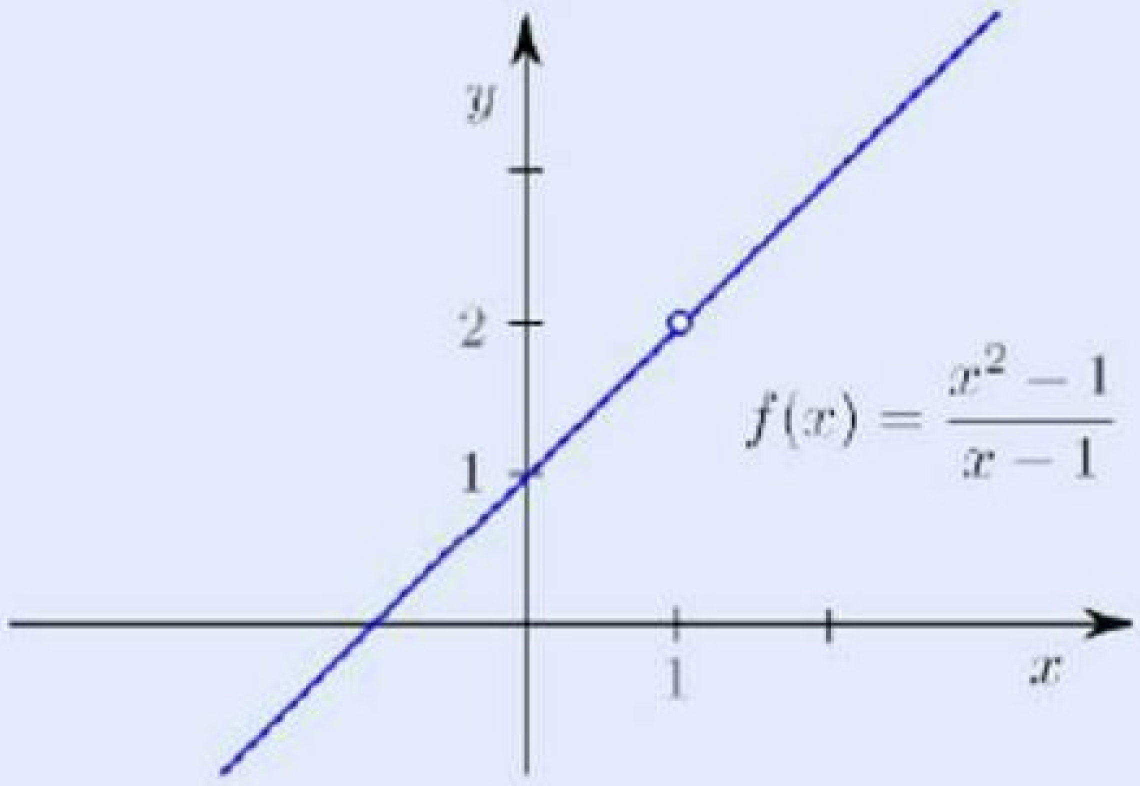
Calling this slope $f(x)$, our formula gives

$$f(x) = \frac{x^2 - 1}{x - 1}.$$

This is an approximation to the desired quantity, namely, the slope of the tangent line, and this approximation gets better as x gets closer to 1, so we have

$$\text{Slope of tangent} = \lim_{x \to 1} f(x).$$

Now, $f(x)$ is undefined at $x = 1$ (due to the 0 it produces in the denominator), but for any other x, we have $f(x) = x + 1$. (Reason: If $x \neq 1$, then we can factor the numerator and cancel $x - 1$ with the denominator; this cancelation is really multiplication of numerator and denominator by the reciprocal of $x - 1$, so it is a valid step if $x \neq 1$ since $x - 1 \neq 0$ in this case.) The graph of f is a line with a hole where $x = 1$:



From the graph, we see that $\lim_{x\to 1} f(x) = 2$. We conclude that the tangent line has slope 2. $\square$

There are two things to note about this example:

It was not carefully stated what was meant by the line tangent to the curve at (1, 1). If this tangent line is *defined* to be the line through (1, 1) with slope $\lim_{x\to 1} f(x)$ (notation as in the solution), then this rigorous definition agrees with our intuition that the tangent line should be the line that best fits the curve at the point (1, 1).

In order to find the slope of the line tangent to the graph of g at the point (1, 1), we ended up looking at the graph of the secant slope function f . Here is the connection: For a given value x, the height of the graph of f above x is the slope of the secant line $\overline{PQ}$, where Q is (x, x^2). For example, the height of the graph of f above $x = 2$ is 3, and this is the slope of the secant line with Q equal to (2, 4).