Discrete-Time Signals and Systems

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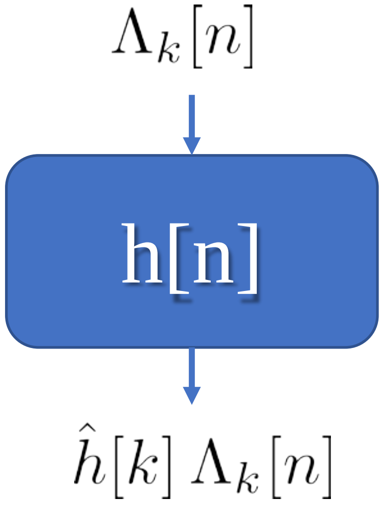
Last Lecture Summary

- Eigenfunctions of LTI systems:



Assuming that the signal $x[n]$ can be written as,

$$x[n] = \sum_k \hat{x}[k]\Lambda_k[n],$$

where the set $\{\Lambda_k[n]\}_k$ is orthonormal, i.e., it satisfies the following conditions:

1. Normalization:

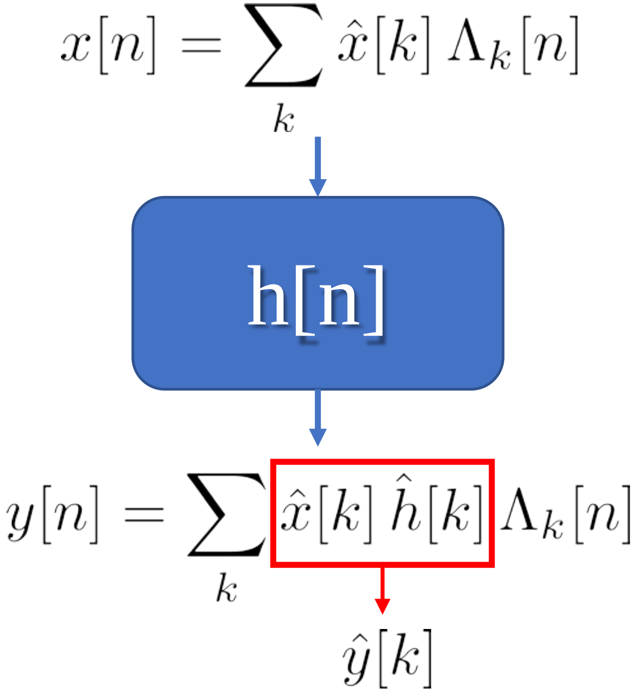
$$\sum_n |\Lambda_k[n]|^2 = 1, \qquad \forall k,$$

2. Orthogonality:

$$\sum_n \Lambda_k[n]\Lambda_m^*[n] = \delta[k-m],$$

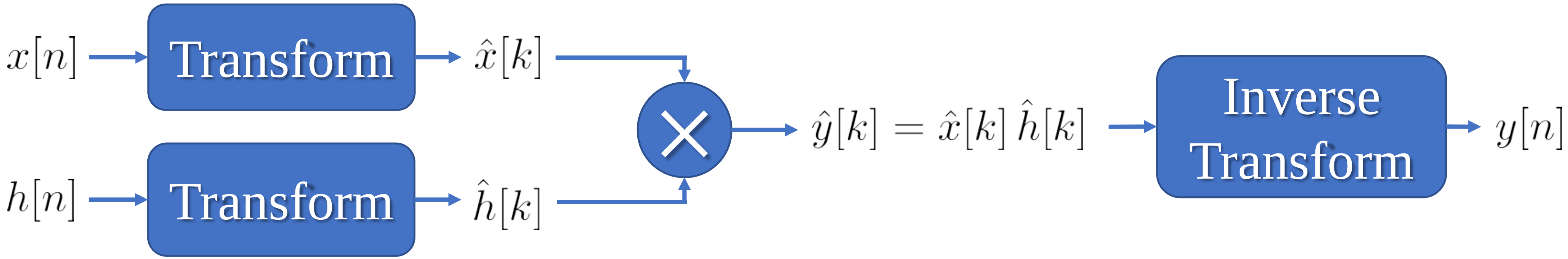
then, $\hat{x}[k]$ can be evaluated as follows,

$$\hat{x}[k] = \sum_n x[n]\Lambda_k^*[n].$$

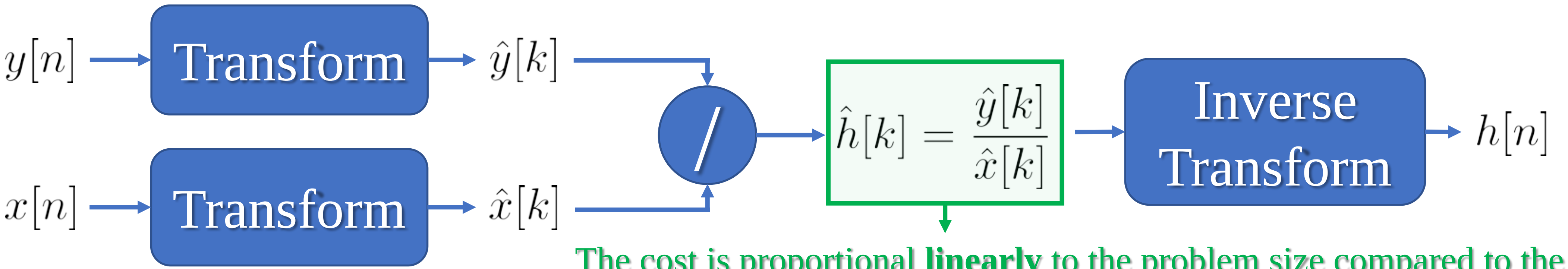


Last Lecture Summary

- Analysis Problems Solution Using Eigenfunctions Decomposition:



- Design Problems Solution Using Eigenfunctions Decomposition:



The cost is proportional **linearly** to the problem size compared to the deconvolution whose cost is proportional the problem size **cube**.

Last Lecture Summary

- Fourier analysis uses complex sinusoid ($e^{j\Omega_k n}$) functions because they are:

1. Eigenfunctions for any LTI system:

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]e^{j\Omega_k(n-m)} = \underbrace{e^{j\Omega_k n}}_{\Lambda_k[n]} \underbrace{\sum_{m=-\infty}^{\infty} h[m]e^{-j\Omega_k m}}_{\text{constant, independent on n}} = \hat{h}[k]e^{j\Omega_k n}.$$

2. Orthogonal:

$$\sum_{n=-\infty}^{\infty} e^{j\Omega_k n}e^{-j\Omega_m n} = \sum_{n=-\infty}^{\infty} e^{j(\Omega_k-\Omega_m)n} = 2\pi\delta_D(\Omega_k-\Omega_m).$$

Periodicity / Time	Aperiodic	Periodic
Continuous	Fourier Transform/ Inverse Fourier Transform	Fourier Series/ Inverse Fourier Series
Discrete	DT Fourier Transform/ Inverse DT Fourier Transform	DT Fourier Series/ Inverse DT Fourier Series

Discrete-Time Fourier Series

Agenda

- In this lecture, we will:
  - review the properties of discrete-time periodic signals,
  - understand the properties of discrete-time complex sinusoids,
  - understand discrete-time Fourier series,
  - understand discrete-time Fourier series properties,
  - and understand radix-2 Fast Fourier Transform (FFT) algorithm.

So, let's get started ☺

Revision

- Periodic Signal Definition:

Definition: The signal $x[n]$ is periodic signal if,

$$x[n] = x[n+N],$$

where $N \in \mathbb{Z}$ is called the period of $x[n]$.

Definition: The fundamental period of the periodic signal $x[n]$ is the smallest integer N that satisfies,

$$x[n] = x[n+N].$$

Example: Find the fundamental period of the following signals:

1. $x_1[n] = \cos(\frac{2\pi}{6}n)$.

   Fundamental Period $= 6$.

2. $x_2[n] = \cos(\frac{2\pi}{6}n) + \cos(\frac{2\pi}{10}n)$.

   Fundamental Period $\leq LCM(6, 10)$,
   Fundamental Period $= 30$.

3. $x_3[n] = \sum_{m=-\infty}^{\infty} z_1[n-6m] + z_2[n-10m]$,

   where $z_1[n] = \delta[n] - \delta[n-1] + \delta[n-2]$, and $z_2[n] = -\delta[n] + \delta[n-1] - \delta[n-2] + \delta[n-3] - \delta[n-4]$.

   Fundamental Period $\leq LCM(6, 10)$,
   Fundamental Period $= 15$.

Discrete-Time Complex Sinusoids

- Properties of Discrete-Time Complex Sinusoid Signals:

Propeties:

If $x[n] = e^{j\Omega n}$ is periodic with period N, then:

1. $\Omega = \frac{2\pi k}{N}$, where $k \in \mathbb{Z}$.

Proof:

If $x[n]$ is periodic with period N, then it satisfies,

$$x[n] = x[n+N],$$
$$e^{j\Omega n} = e^{j\Omega(n+N)},$$
$$e^{j\Omega n} = e^{j\Omega n}e^{j\Omega N},$$
$$x[n] = x[n]e^{j\Omega N}.$$

Hence, $x[n]$ is periodic if $e^{j\Omega N} = 1$, which happens if $\Omega N = 2\pi k$, where $k \in \mathbb{Z}$. Therefore, $\Omega = \frac{2\pi k}{N}$.

Discrete-Time Complex Sinusoids

- Properties of Discrete-Time Complex Sinusoid Signals:

Propeties:

If $x[n] = e^{j\Omega n}$ is periodic with period N, then:

1. $\Omega = \frac{2\pi k}{N}$, where $k \in \mathbb{Z}$.
2. There are at most N discrete-time periodic complex sinusoid with period N.

Proof:

The signal $x_k[n] = e^{j\frac{2\pi k}{N}n}$ is periodic with period N for all $m \in \mathbb{Z}$. If we substitute k with $k + \ell N$, where $\ell \in \mathbb{Z}$, we get,

$$\begin{aligned} x_{k+\ell N}[n] &= e^{j\frac{2\pi(k+\ell N)}{N}n}, \\ &= e^{j\frac{2\pi k}{N}n} e^{j2\pi\ell n}, \\ &= e^{j\frac{2\pi k}{N}n}, \\ &= x_k[n]. \end{aligned}$$

Hence, $x_k[n]$ repeats itself every $k = N$. Accordingly, there are at most N unique discrete-time complex sinusoid with period N.

Discrete-Time Complex Sinusoids

- Properties of Discrete-Time Complex Sinusoid Signals:

Propeties:

If $x[n] = e^{j\Omega n}$ is periodic with period N, then:

1. $\Omega = \frac{2\pi k}{N}$, where $k \in \mathbb{Z}$.
2. There are at most N discrete-time periodic complex sinusoid with period N.
3. The set $\{e^{\frac{j2\pi kn}{N}}\}_{k=0}^{N-1}$ is orthogonal in the range $n \in [0, N-1]$.

Proof:

The orthogonality of the set $\{e^{j\frac{2\pi k}{N}n}\}_{k=0}^{N-1}$ in the range $n \in [0, N-1]$ can be shown as follows,

$$\begin{aligned}
\sum_{n=0}^{N-1} e^{\frac{j2\pi kn}{N}} e^{-\frac{j2\pi mn}{N}} &= \sum_{n=0}^{N-1} e^{\frac{j2\pi(k-m)n}{N}}, \\
&= \sum_{n=0}^{N-1} \left(e^{\frac{j2\pi(k-m)}{N}}\right)^n, \\
&= \frac{1 - e^{j2\pi(k-m)}}{1 - e^{\frac{j2\pi(k-m)}{N}}}, \\
&= N\delta[k-m].
\end{aligned}$$

Discrete-Time Fourier Series

- The Discrete-Time Fourier Series and Inverse Fourier Series:

Theorem:

The set $\{e^{j\frac{2\pi k}{N}n}\}_{k=0}^{N-1}$ can represent any bounded periodic discrete-time $x[n]$ with a fundamental period of N as follows,

$$\boxed{x[n] = \frac{1}{N}\sum_{k=0}^{N-1}\hat{x}[k]e^{\frac{j2\pi kn}{N}}}, \rightarrow \text{Inverse Fourier Series}$$

where $\hat{x}[k]$ is given as follows,

$$\boxed{\hat{x}[k] = \sum_{n=0}^{N-1}x[n]e^{-\frac{j2\pi kn}{N}}.} \rightarrow \text{Fourier Series}$$

Discrete-Time Fourier Series

- The Discrete-Time Fourier Series and Inverse Fourier Series:

Proof:

Assuming that $\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \hat{x}[k]e^{\frac{j2\pi kn}{N}}$, where $\hat{x}[k] = \sum_{n=0}^{N-1} x[n]e^{-\frac{j2\pi kn}{N}}$, we want to prove that $\tilde{x}[n] = x[n]$. First, we check if $\tilde{x}[n]$ is periodic with period N as follows,

$$\begin{aligned}
\tilde{x}[n-N] &= \sum_{k=0}^{N-1} \hat{x}[k]e^{\frac{j2\pi k(n-N)}{N}} \\
&= \sum_{k=0}^{N-1} \hat{x}[k]e^{\frac{j2\pi kn}{N}} \underbrace{e^{-j2\pi kn}}_{=1} = \tilde{x}[n].
\end{aligned}$$

Discrete-Time Fourier Series

- The Discrete-Time Fourier Series and Inverse Fourier Series:

Proof:

Second, we prove that $\tilde{x}[n] = x[n]$ in the range $n \in [0, N-1]$ as follows,

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \hat{x}[k] e^{\frac{j2\pi nk}{N}} = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{m=0}^{N-1} x[m] e^{\frac{-j2\pi mk}{N}} e^{\frac{j2\pi nk}{N}},$$

$$= \sum_{m=0}^{N-1} x[m] \frac{1}{N}\sum_{k=0}^{N-1} e^{\frac{j2\pi(n-m)k}{N}} = \sum_{m=0}^{N-1} x[m]\delta[n-m] = x[n]. \qquad \blacksquare$$

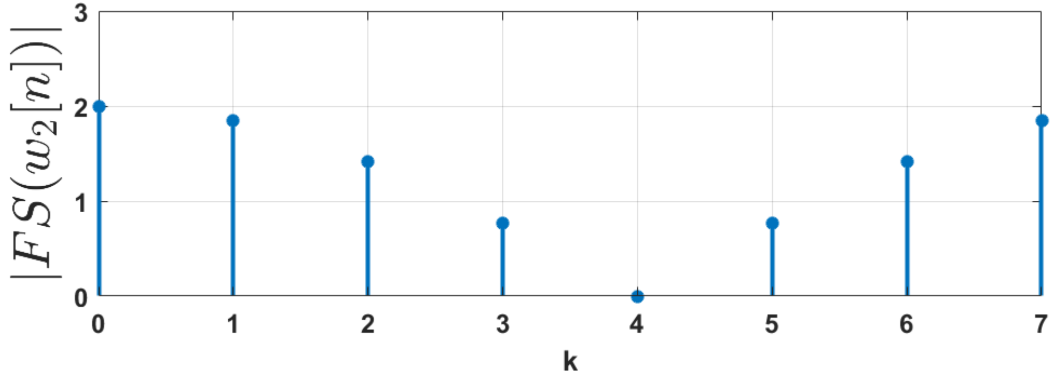
Discrete-Time Fourier Series

- Example:

Find the discrete Fourier series of $x[n] = \sum_{m=-\infty}^{\infty} w_M[n - mN]$, where $w_M[n] = u[n] - u[n-M]$ and $M \in [1, N-1]$.

$$\begin{aligned}
\hat{x}[k] &= \sum_{n=0}^{N-1} w_M[n] e^{\frac{-j2\pi kn}{N}}, \\
&= \sum_{n=0}^{M-1} e^{\frac{-j2\pi kn}{N}}, \\
&= \frac{1 - e^{\frac{-j2\pi kM}{N}}}{1 - e^{\frac{-j2\pi k}{N}}},
\end{aligned}$$

$$\begin{aligned}
\hat{x}[k] &= \frac{e^{\frac{-j\pi kM}{N}}}{e^{\frac{-j\pi k}{N}}} \frac{\sin\left(\frac{\pi kM}{N}\right)}{\sin\left(\frac{\pi k}{N}\right)}, \\
&= e^{\frac{-j\pi(M-1)k}{N}} \frac{\sin\left(\frac{\pi kM}{N}\right)}{\sin\left(\frac{\pi k}{N}\right)}, \quad k \in [0, N-1].
\end{aligned}$$



Discrete-Time Fourier Series

- The Cost of Discrete-Time Fourier Series and Inverse Fourier Series:

  - Fourier Series:

$$\hat{x}[k] = \sum_{n=0}^{N-1} x[n] e^{-\frac{j2\pi kn}{N}}$$

Needs N^2 Complex Multiplication Operations

  - Inverse Fourier Series:

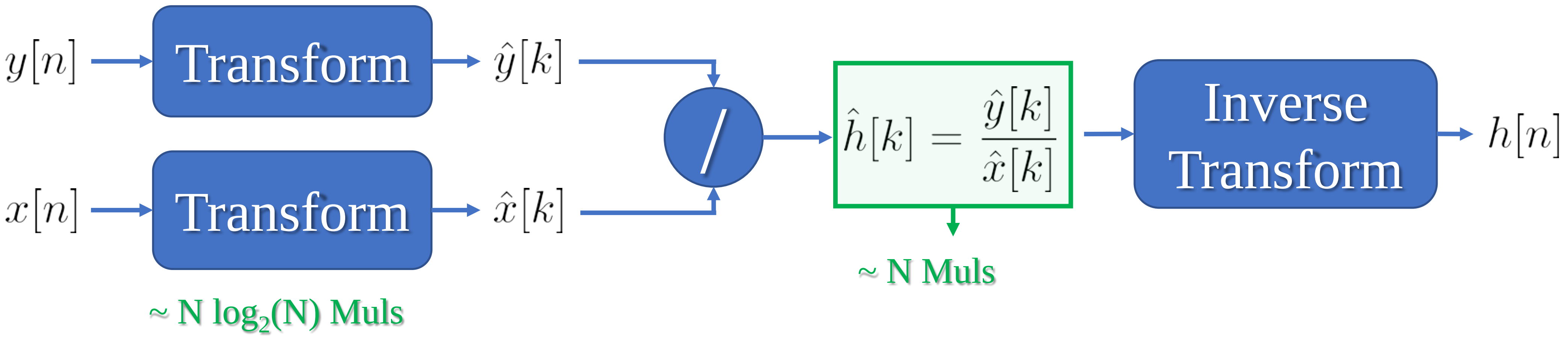
$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} \hat{x}[k] e^{\frac{j2\pi kn}{N}}$$

Needs N^2 Complex Multiplication Operations

Takeaway: Using Fourier series reduces the complexity of design problems from being proportional to the problem size **cube** to be probational to the problem size **square**.

Discrete-Time Fourier Series

- The Cost of Discrete-Time Fourier Series and Inverse Fourier Series:
  - Using Fast Fourier Transform (FFT) algorithm reduces the complexity of Fourier Series and inverse Fourier series from N^2 to $N\log_2(N)$.



Takeaway: Using Fourier series reduces the complexity of design problems from being proportional to the problem size **cube** to be probational to **the problem size times its logarithm.**

Discrete-Time Fourier Series Properties

1. Periodicity:

Theorem: If the discrete Fourier series $\hat{x}[k]$ is given by:

$$\hat{x}[k] = \sum_{n=0}^{N-1} x[n] e^{\frac{-j2\pi kn}{N}},$$

where $x[n]$ is the discrete-time signal, then $\hat{x}[k]$ is periodic with period N.

Proof: We check the periodicity of $\hat{x}[k]$ as follows,

$$\begin{aligned}
\hat{x}[k-N] &= \sum_{n=0}^{N-1} x[n] e^{\frac{-j2\pi(k-N)n}{N}}, \\
&= \sum_{n=0}^{N-1} x[n] e^{\frac{-j2\pi kn}{N}} e^{j2\pi n}, \\
&= \sum_{n=0}^{N-1} x[n] e^{\frac{-j2\pi kn}{N}} = \hat{x}[k].
\end{aligned}$$

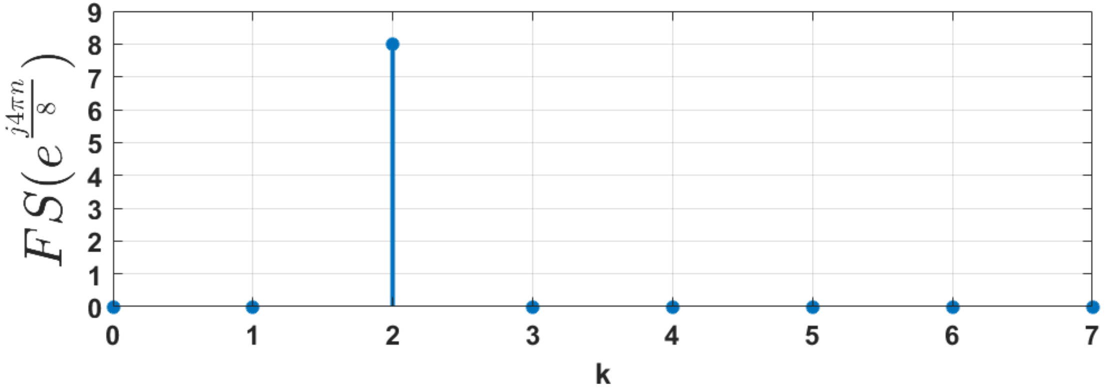
Discrete-Time Fourier Series Properties

1. Periodicity:

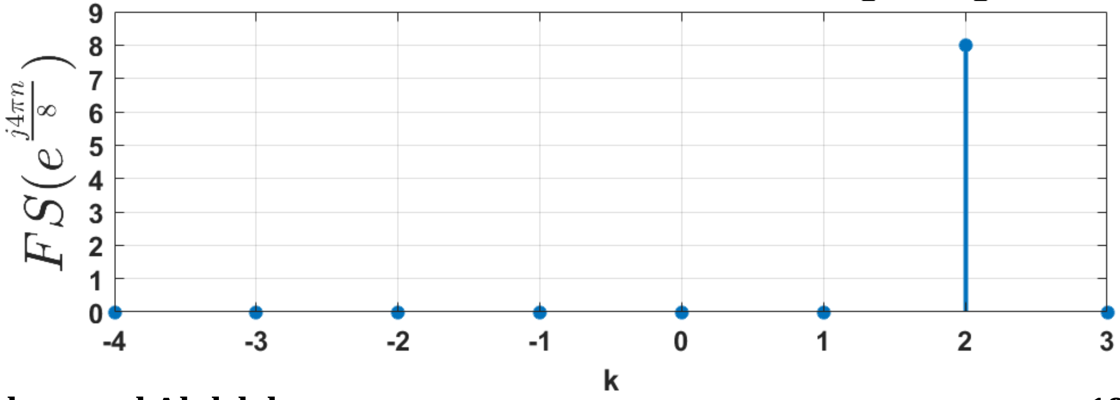
Find the discrete Fourier series of $x[n] = e^{\frac{j2\pi mn}{N}}$, where $m \in [0, N-1]$.

$$\begin{aligned}\hat{x}[k] &= \sum_{n=0}^{N-1} e^{\frac{j2\pi mn}{N}} e^{\frac{-j2\pi kn}{N}}, \\ &= \sum_{n=0}^{N-1} e^{\frac{-j2\pi (k-m)n}{N}}, \\ &= N \sum_{\ell=-\infty}^{\infty} \delta[k - m - \ell N], \quad k \in \mathbb{Z}, \\ &= N\delta[k-m], \quad k \in [0, N-1].\end{aligned}$$

- Convention #1: $k \in [0, N-1]$



- Convention #2: If N is even: $k \in [-\frac{N}{2}, \frac{N}{2} - 1]$
  If N is odd: $k \in [-\frac{N-1}{2}, \frac{N-1}{2}]$



Discrete-Time Fourier Series Properties

2. Time Reversal:

Theorem: If the discrete Fourier series of the signal $x[n]$ is $\hat{x}[k]$, then the discrete Fourier series of the signal $x[-n]$ is $\hat{x}[-k]$.

Proof: The discrete Fourier series of the signal $x[-n]$ can be evaluated as follows,

$$\begin{aligned}
\text{DFS}(x[-n]) &= \sum_{n=0}^{N-1} x[-n] e^{\frac{-j2\pi kn}{N}}, \\
&= \sum_{n=0}^{N-1} x[-n] e^{\frac{-j2\pi(-k)(-n)}{N}}, \\
&= \sum_{n=-N+1}^{0} x[n] e^{\frac{-j2\pi(-k)n}{N}}, \\
&= \sum_{n=0}^{N-1} x[n] e^{\frac{-j2\pi(-k)n}{N}} = \hat{x}[-k].
\end{aligned}$$

Discrete-Time Fourier Series Properties

3. Complex Conjugate:

Theorem: If the discrete Fourier series of the signal $x[n]$ is $\hat{x}[k]$, then the discrete Fourier series of the signal $x^*[n]$ is $\hat{x}^*[-k]$.

Proof: The discrete Fourier series of the signal $x^*[n]$ can be evaluated as follows,

$$\begin{aligned}
\mathrm{DFS}(x^*[n]) &= \sum_{n=0}^{N-1} x^*[n] e^{\frac{-j2\pi kn}{N}}, \\
&= \left(\sum_{n=0}^{N-1} x[n] e^{\frac{-j2\pi(-k)n}{N}} \right)^*, \\
&= (\hat{x}[-k])^* = \hat{x}^*[-k].
\end{aligned}$$

2+3. Time Reversal + Complex Conjugated:

Find the discrete Fourier series of $x[n] = e^{\frac{-j2\pi mn}{N}}$, where $m \in [0, N-1]$.

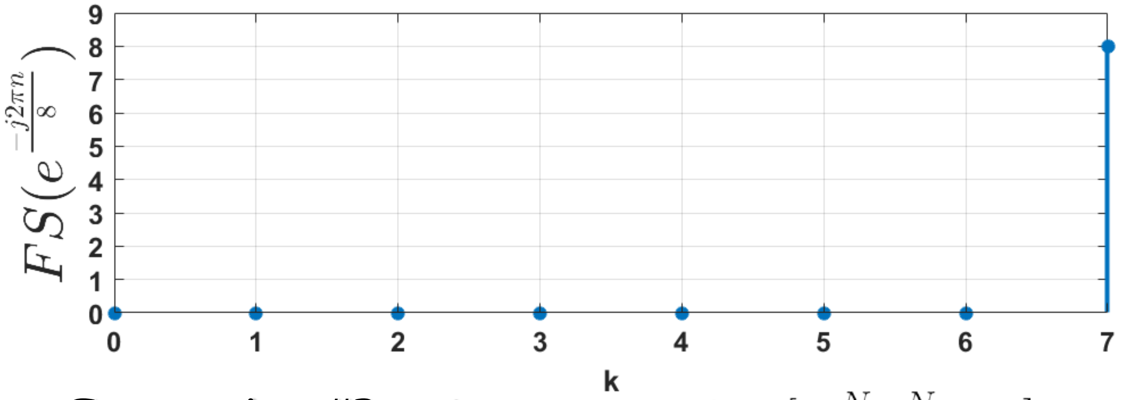
$$\because \text{DFS}(e^{\frac{j2\pi mn}{N}}) = N \sum_{\ell=-\infty}^{\infty} \delta[k - m - \ell N],$$

$$\therefore \text{DFS}(e^{\frac{-j2\pi mn}{N}}) = N \sum_{\ell=-\infty}^{\infty} \delta[-k - m - \ell N],$$

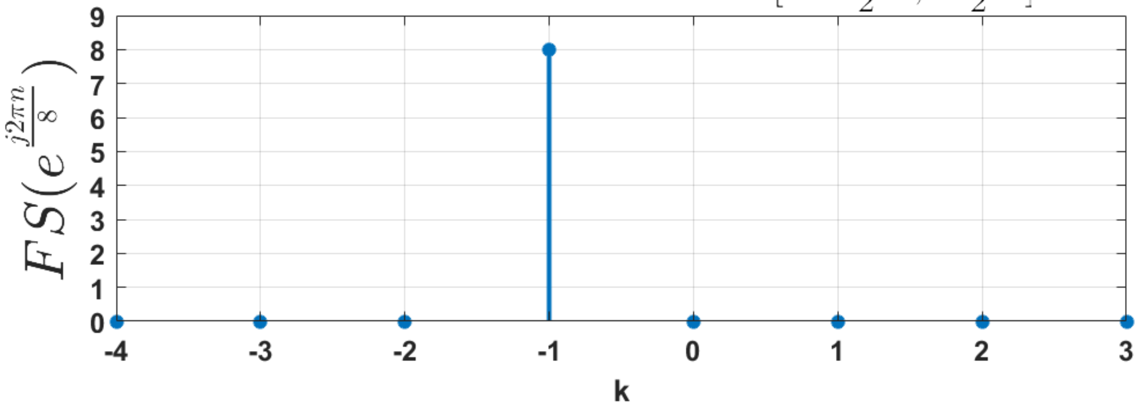
$$= N\delta[-k - m],\ k \in [-N+1, 0],$$

$$= N\delta[(N-k) - m],\ k \in [0, N-1].$$

- **Convention #1:** $k \in [0, N-1]$



- **Convention #2:** If N is even: $k \in [-\frac{N}{2}, \frac{N}{2} - 1]$
  If N is odd: $k \in [-\frac{N-1}{2}, \frac{N-1}{2}]$



Discrete-Time Fourier Series Properties

4. Linearity:

Theorem: If $\hat{x}_1[k]$ and $\hat{x}_2[k]$ are the discrete-time Fourier series of the periodic signals $x_1[n]$ and $x_2[2]$ (whose periods equal N), respectively, then the discrete-time Fourier series of the signal $x_3[n] = a_1 x_1[n] + a_2 x_2[n]$ is given by,

$$\hat{x}_3[n] = a_1\hat{x}_1[n] + a_2\hat{x}_2[n],$$

where $a_1 \in \mathbb{C}$ and $a_2 \in \mathbb{C}$.

Proof: The signal $\hat{x}_3[k]$ can be calculated as follows,

$$\begin{aligned}\hat{x}_3[k] = &\sum_{n=0}^{N-1}(a_1x_1[n] + a_2x_2[n])e^{\frac{-j2\pi kn}{N}}, \\ =&a_1\sum_{n=0}^{N-1}x_1[n]e^{\frac{-j2\pi kn}{N}} + a_2\sum_{n=0}^{N-1}x_2[n]e^{\frac{-j2\pi kn}{N}}, \\ =&a_1\hat{x}_1[k] + a_2\hat{x}_2[k].\end{aligned}$$

Discrete-Time Fourier Series Properties

4. Linearity:

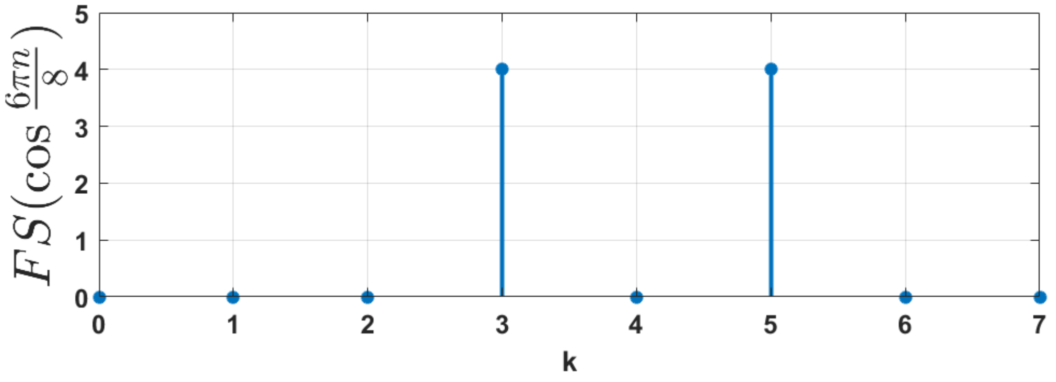
Find the discrete Fourier series of $x[n] = \cos(\frac{2\pi mn}{N})$, where $m \in [0, N-1]$.

$$\because \cos\left(\frac{2\pi mn}{N}\right) = \frac{e^{\frac{j2\pi mn}{N}}}{2} + \frac{e^{\frac{-j2\pi mn}{N}}}{2},$$

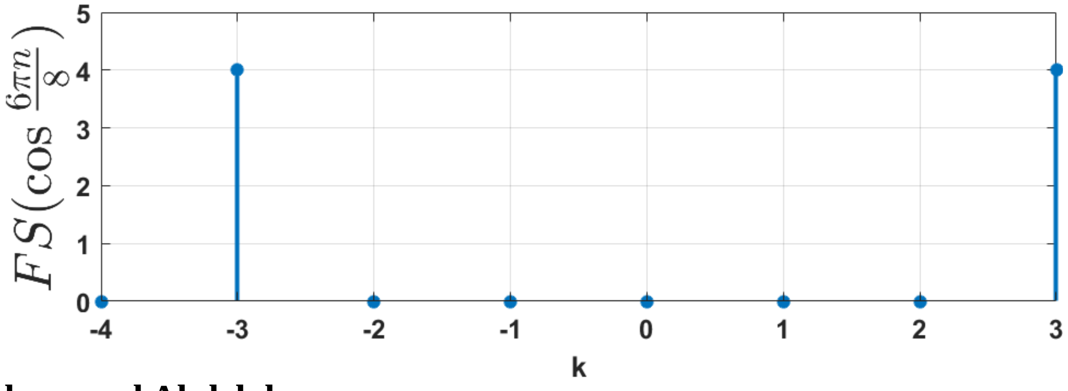
$$\therefore \hat{x}[k] = \frac{N\delta[k-m]}{2} + \frac{N\delta[(N-k)-m]}{2},$$

where $k \in [0, N-1]$.

- **Convention #1:** $k \in [0, N-1]$



- **Convention #2:** If N is even: $k \in [-\frac{N}{2}, \frac{N}{2} - 1]$
  If N is odd: $k \in [-\frac{N-1}{2}, \frac{N-1}{2}]$



4. Linearity:

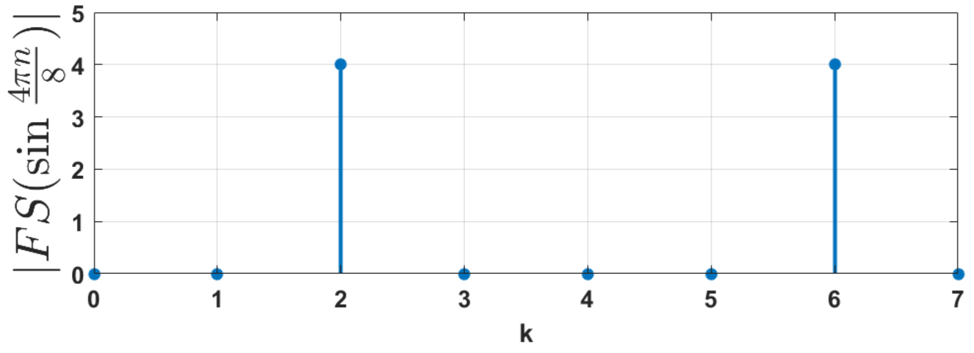
Find the discrete Fourier series of $x[n] = \sin(\frac{2\pi mn}{N})$, where $m \in [0, N-1]$.

$$\because \sin\left(\frac{2\pi mn}{N}\right) = \frac{e^{\frac{j2\pi mn}{N}}}{2j} - \frac{e^{\frac{-j2\pi mn}{N}}}{2j},$$

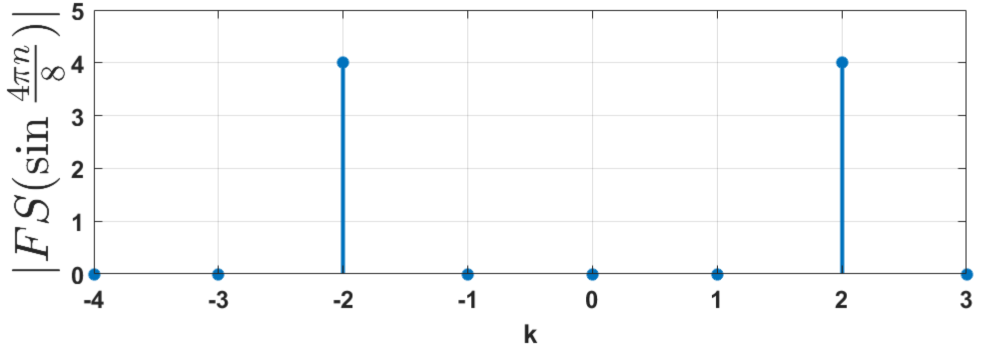
$$\therefore \hat{x}[k] = \frac{N\delta[k-m]}{2j} - \frac{N\delta[(N-k)-m]}{2j},$$

where $k \in [0, N-1]$.

- **Convention #1:** $k \in [0, N-1]$



- **Convention #2:** If N is even: $k \in [-\frac{N}{2}, \frac{N}{2}-1]$
  If N is odd: $k \in [-\frac{N-1}{2}, \frac{N-1}{2}]$



Discrete-Time Fourier Series Properties

5. Time Shift:

Theorem: If $\hat{x}_1[k]$ is the discrete-time Fourier series of the periodic signals $x_1[n]$ (whose period equals N), then the discrete-time Fourier series of the signal $x_2[n] = x_1[n-m]$ is given by,

$$\hat{x}_2[n] = \hat{x}_1[k]e^{\frac{-j2\pi mk}{N}},$$

where $m \in \mathbb{Z}$.

Proof: The signal $\hat{x}_2[k]$ can be calculated as follows,

$$\begin{aligned}
\hat{x}_2[k] &= \sum_{n=0}^{N-1} x_1[n-m]e^{\frac{-j2\pi kn}{N}}, \\
&= \sum_{\tilde{n}=-m}^{N-m-1} x_1[\tilde{n}]e^{\frac{-j2\pi k(\tilde{n}+m)}{N}}, \\
&= \sum_{\tilde{n}=0}^{N-1} x_1[\tilde{n}]e^{\frac{-j2\pi k\tilde{n}}{N}} e^{\frac{-j2\pi km}{N}} = \hat{x}_1[k]e^{\frac{-j2\pi km}{N}}.
\end{aligned}$$

Discrete-Time Fourier Series Properties

6. Circular Frequency Shift:

Theorem: If $\hat{x}_1[k]$ is the discrete-time Fourier series of the periodic signals $x_1[n]$ (whose period equals N), then the discrete-time Fourier series of the signal $x_2[n] = x_1[n]e^{\frac{j2\pi mn}{N}}$ is given by,

$$\hat{x}_2[n] = \hat{x}_1[k-m],$$

where $m \in \mathbb{Z}$.

Proof: The signal $\hat{x}_2[k]$ can be calculated as follows,

$$\begin{aligned}\hat{x}_2[k] &= \sum_{n=0}^{N-1} x_1[n]e^{\frac{j2\pi mn}{N}} e^{\frac{-j2\pi kn}{N}}, \\ &= \sum_{n=0}^{N-1} x_1[n]e^{\frac{-j2\pi(k-m)n}{N}}, \\ &= \hat{x}_1[k-m].\end{aligned}$$

Discrete-Time Fourier Series Properties

7. Duality:

Theorem: If $\hat{x}_1[k]$ is the discrete-time Fourier series of the periodic signals $x_1[n]$ (whose period equals N), then the discrete-time Fourier series of the signal $x_2[n] = \hat{x}_1[n]$ is given by,

$$\hat{x}_2[k] = N x_1[-k].$$

Proof: The signal $\hat{x}_2[k]$ can be calculated as follows,

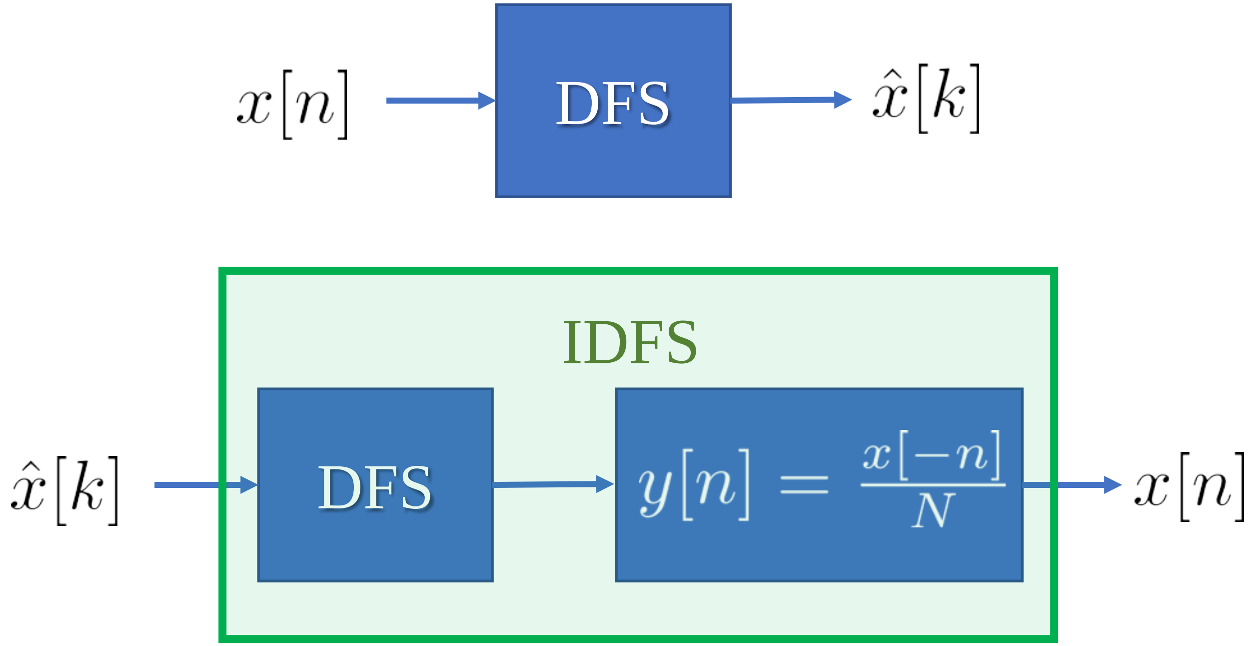
$$\begin{aligned}
\hat{x}_2[k] &= \sum_{n=0}^{N-1} \hat{x}_1[n] e^{\frac{-j2\pi kn}{N}}, \\
&= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x_1[m] e^{\frac{-j2\pi nm}{N}} e^{\frac{-j2\pi kn}{N}} = \sum_{m=0}^{N-1} x_1[m] \sum_{n=0}^{N-1} e^{\frac{-j2\pi n(k+m)}{N}}, \\
&= \sum_{m=0}^{N-1} x_1[m] N \delta[m-(N-k)] = N x_1[N-k] = N x_1[-k].
\end{aligned}$$

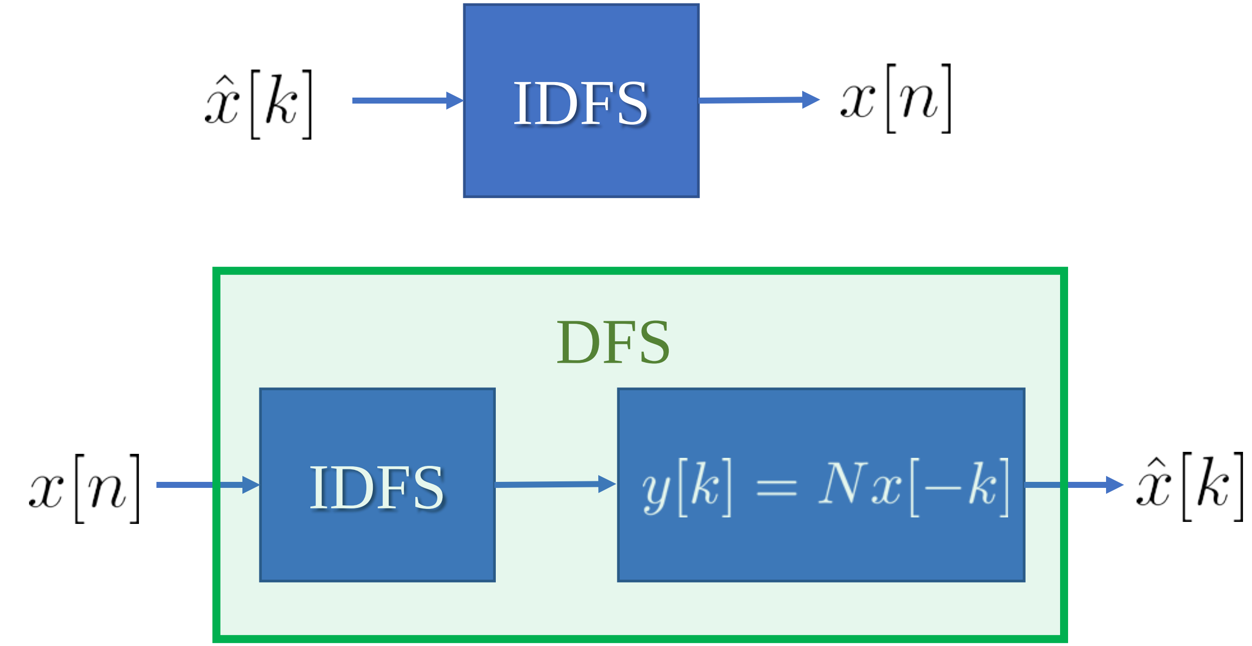
Discrete-Time Fourier Series Properties

7. Duality: The duality theorem shows how the discrete-time Fourier series is closely related to the inverse discrete-time Fourier series:

$$x[n] \overset{\text{DFS}}{\Longrightarrow} \hat{x}[k]$$
$$\hat{x}[k] \overset{\text{DFS}}{\Longrightarrow} Nx[-n]$$

$$\hat{x}[k] \overset{\text{IDFS}}{\Longrightarrow} x[n]$$
$$x[n] \overset{\text{IDFS}}{\Longrightarrow} \frac{1}{N}\hat{x}[-k]$$





Circular Convolution

Definition: Given the signala $x[n]$ and $h[n]$ (where $n \in [0, N-1]$), the N-point circular convolution between $x[n]$ and $h[n]$ is given as follows,

$$\begin{aligned} y[n] &= x[n] \, \textcircled{N} \, h_N[n], \\ &= \sum_{m=0}^{N-1} x[m]h[(n-m)_N], \\ &= \sum_{m=0}^{N-1} h[m]x[(n-m)_N], \end{aligned}$$

where $(n-m)_N = (n-m) \, \mathrm{Mod} \, N.$

Circular Convolution Application

- Linear Convolution for Periodic Signals:

Theorem: If the signal $x[n]$ is periodic with a fundamental period of N, then the signal $y[n] = x[n] \circledast h[n]$ is also periodic with a fundamental period less than or equal to N.

Proof:

$\because$ $x[n]$ is periodic with period N, then $x[n-m] = x[n-N-m]$, where $m \in \mathbb{Z}$.
$\because$ $y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$, then,

$$\begin{aligned} y[n-N] &= \sum_{m=-\infty}^{\infty} h[m]x[n-N-m] \\ &= \sum_{m=-\infty}^{\infty} h[m]x[n-m] = y[n]. \end{aligned}$$

Therefore, $y[n]$ is periodic with a fundamental period less than or equal to N.$\blacksquare$

Circular Convolution Application

- Linear Convolution for Periodic Signals:
  - Therefore, it is enough to compute one period of y[n] to know the whole signal.

- Example:

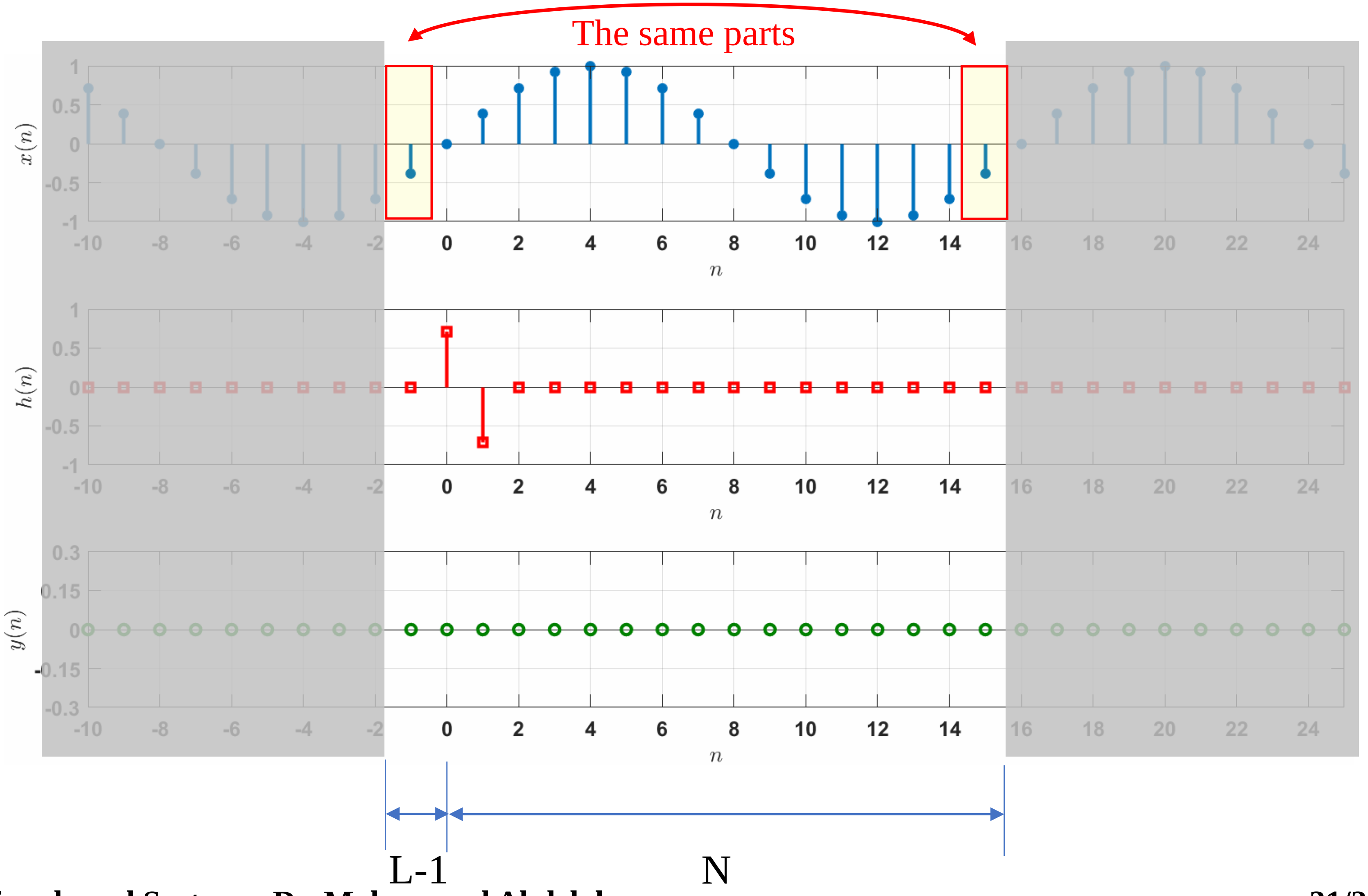
$$x[n] = \sin(\frac{\pi}{8}n)$$

$$h[n] = \frac{\delta[n] - \delta[n-1]}{\sqrt{2}}$$

$$y[n] = ?$$

$$y_N[n] = \sum_{m=-L+1}^{N-1} x[m]h[n-m]$$

where $y_N[n]$ is valid from $n \in [0, N-1]$.



Circular Convolution Application

- Linear Convolution for Periodic Signals:

  - Steps:

Given that $x[n]$ is periodic with a period of N, then the steps are:

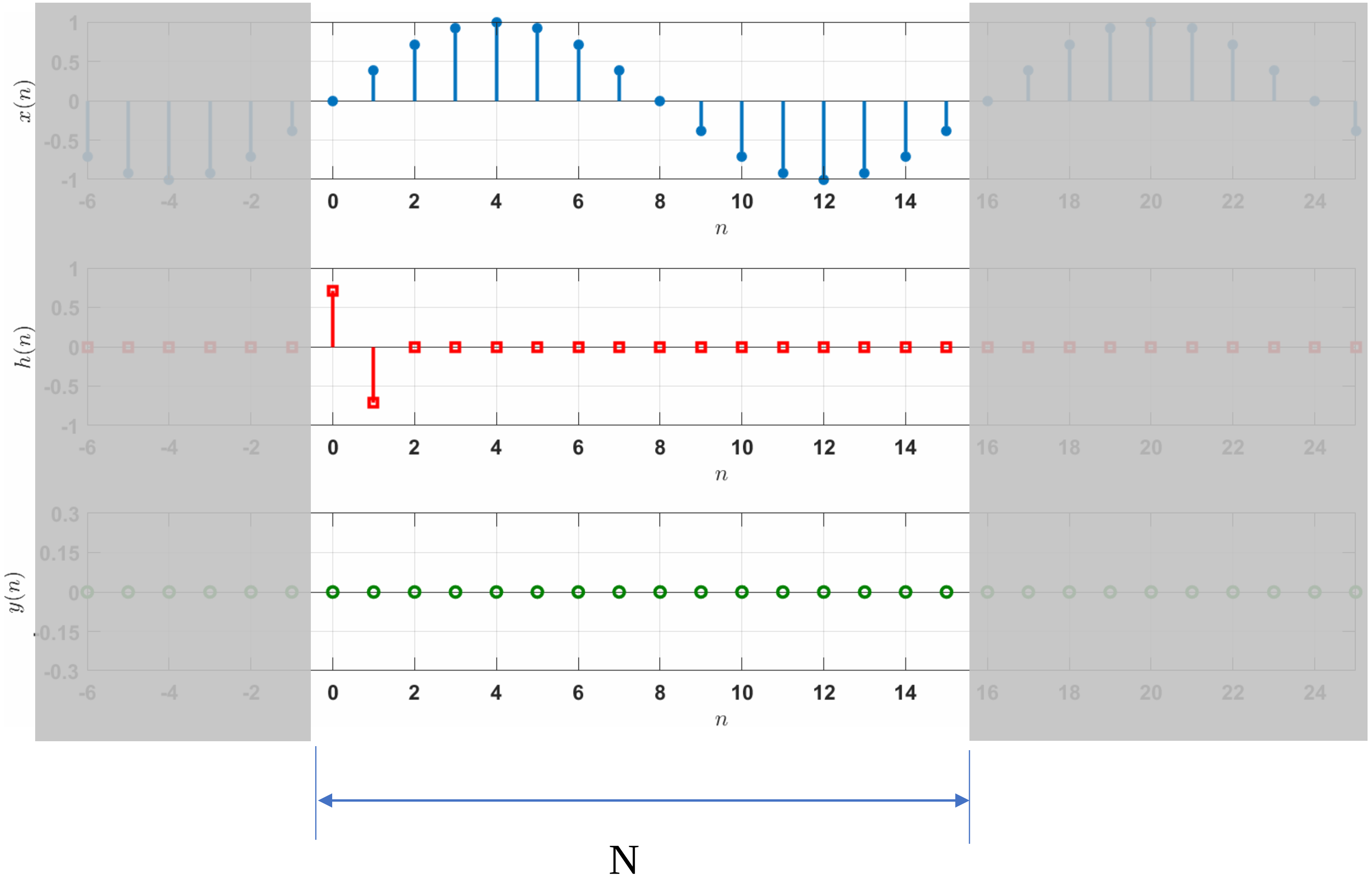
1. $\tilde{h}[n] = \sum_{k=-\infty}^{\infty} h[n-kN]$.

2. $y[n] = \sum_{m=0}^{N-1} x[m]\,\tilde{h}[(n-m)_N]$,

$$= \sum_{m=0}^{N-1} \tilde{h}[m]\,x[(n-m)_N],$$

$$= x[n] \boxed{(N)} \tilde{h}[n].$$

Circular Convolution



Circular Convolution

- Example:

  - If:

    Given:

    $$x[n] = \sin(\frac{\pi}{16}n)w_{16}(n)$$

    where the window function:

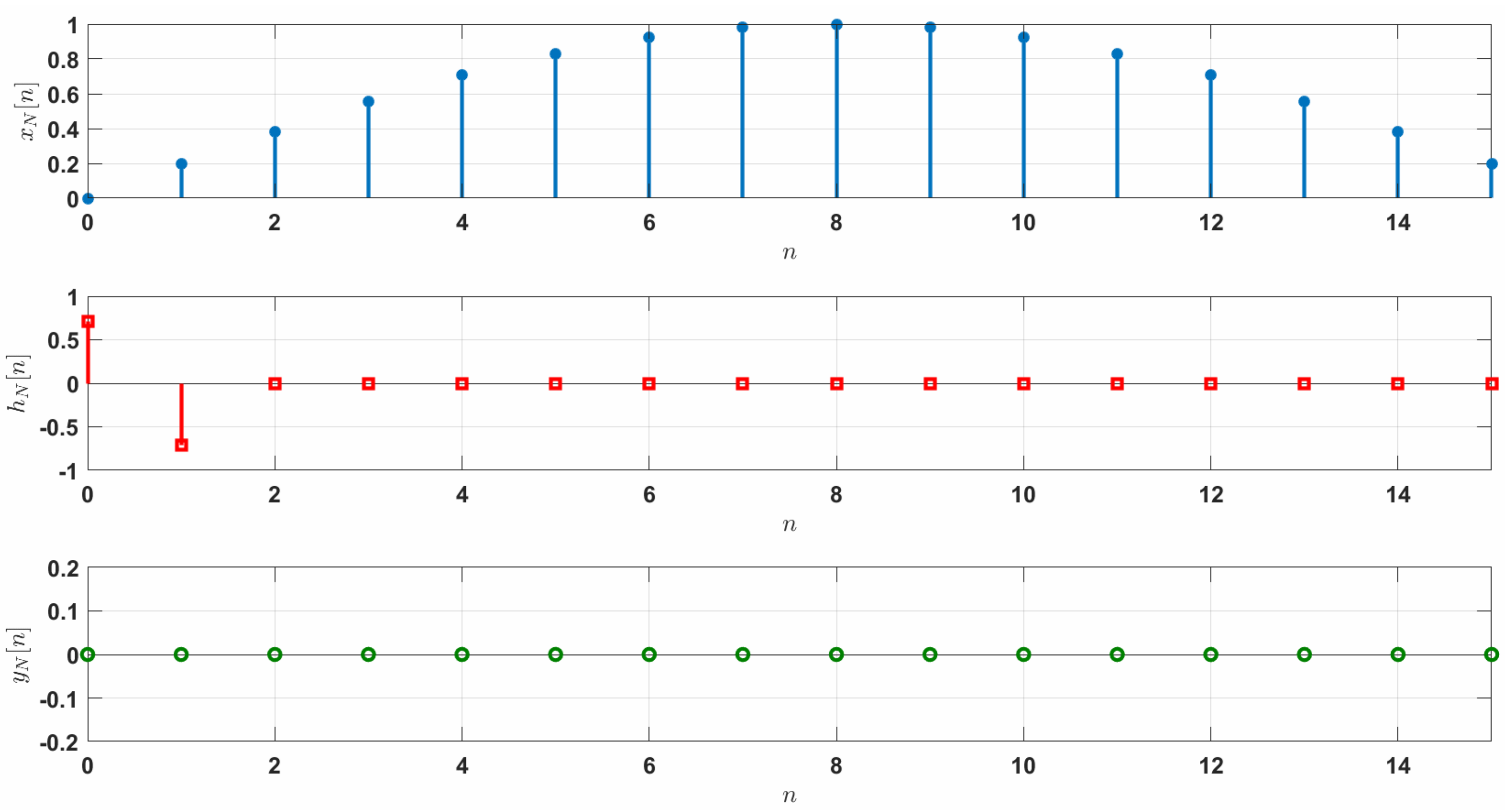
    $$w_N = u[n] - u[n-N]$$

    and

    $$h_N[n] = \delta[n] - \delta[n-1]$$

  - Calculate:

    $$y[n] = x[n] \, (16) \, h[n]$$



Circular Convolution Operation in Matrix Form

1. Form #1:

$$y_N[n] = x[n] \, Ⓝ \, h[n] = \sum_{m=0}^{N-1} x[m] h[(n-m)_N]$$

$$\underbrace{\begin{bmatrix} y_N[0] \\ y_N[1] \\ \vdots \\ y_N[N-2] \\ y_N[N-1] \end{bmatrix}}_{\underbrace{\mathbf{y}_N}_{N\times 1}} = \underbrace{\begin{bmatrix} h[0] & h[N-1] & \dots & h[2] & h[1] \\ h[1] & h[0] & h[N-1] & \dots & h[2] \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h[N-2] & \dots & h[1] & h[0] & h[N-1] \\ h[N-1] & h[N-2] & \dots & h[1] & h[0] \end{bmatrix}}_{\underbrace{\mathbf{H}_N}_{N\times N}} \underbrace{\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-2] \\ x[N-1] \end{bmatrix}}_{\underbrace{\mathbf{x}_N}_{N\times 1}}$$

Circular Convolution Operation in Matrix Form

2. Form #2:

$$y_N[n] = h[n] \circledN x[n] = \sum_{m=0}^{N-1} h[m] x[(n-m)_N]$$

$$\underbrace{\underbrace{\begin{bmatrix} y_N[0] \\ y_N[1] \\ \vdots \\ y_N[N-2] \\ y_N[N-1] \end{bmatrix}}_{\mathbf{y}_N}}_{N\times 1} = \underbrace{\underbrace{\begin{bmatrix} x[0] & x[N-1] & \dots & x[2] & x[1] \\ x[1] & x[0] & x[N-1] & \dots & x[2] \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x[N-2] & \dots & x[1] & x[0] & x[N-1] \\ x[N-1] & x[N-2] & \dots & x[1] & x[0] \end{bmatrix}}_{\mathbf{X}_N}}_{N\times N} \underbrace{\underbrace{\begin{bmatrix} h[0] \\ h[1] \\ \vdots \\ h[N-2] \\ h[N-1] \end{bmatrix}}_{\mathbf{h}_N}}_{N\times 1}$$

Discrete-Time Fourier Series Properties

8. Circular Convolution:

Theorem: If $\hat{x}_1[k]$ and $\hat{x}_2[k]$ are the discrete-time Fourier series of the periodic signals $x_1[n]$ and $x_2[n]$ (whose period equals N), respectively, then the discrete-time Fourier series of the signal $x_3[n] = x_1[n] \, \textcircled{N} \, x_2[n]$ is given by,

$$\hat{x}_3[k] = \hat{x}_1[k]\hat{x}_2[k].$$

Proof: The signal $\hat{x}_3[k]$ can be calculated as follows,

$$\begin{aligned}
\hat{x}_3[k] &= \sum_{n=0}^{N-1} \left(x_1[n] \, \textcircled{N} \, x_2[n]\right) e^{\frac{-j2\pi kn}{N}} = \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x_1[m]x_2[n-m]e^{\frac{-j2\pi kn}{N}}, \\
&= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} x_1[m]x_2[n-m]e^{\frac{-j2\pi k(n-m)}{N}} e^{\frac{-j2\pi km}{N}} = \sum_{m=0}^{N-1} x_1[m]e^{\frac{-j2\pi km}{N}} \sum_{n=0}^{N-1} x_2[n-m]e^{\frac{-j2\pi k(n-m)}{N}}, \\
&= \sum_{m=0}^{N-1} x_1[m]e^{\frac{-j2\pi km}{N}} \sum_{\tilde{n}=-m}^{N-m-1} x_2[\tilde{n}]e^{\frac{-j2\pi k\tilde{n}}{N}} = \sum_{m=0}^{N-1} x_1[m]e^{\frac{-j2\pi km}{N}} \sum_{\tilde{n}=0}^{N-1} x_2[\tilde{n}]e^{\frac{-j2\pi k\tilde{n}}{N}} = \hat{x}_1[k]\hat{x}_2[k].
\end{aligned}$$

Discrete-Time Fourier Series Properties

9. Multiplication:

Theorem: If $\hat{x}_1[k]$ and $\hat{x}_2[k]$ are the discrete-time Fourier series of the periodic signals $x_1[n]$ and $x_2[n]$ (whose period equals N), respectively, then the discrete-time Fourier series of the signal $x_3[n] = x_1[n]x_2[n]$ is given by,

$$\hat{x}_3[k] = \frac{1}{N}\hat{x}_1[k] \, Ⓝ \, \hat{x}_2[k].$$

Proof: Using the duality property, stated as $\text{IDFS}(x[n]) = \frac{1}{N}\hat{x}[-k]$, we can show that:

Discrete-Time Fourier Series Properties

9. Multiplication:

Because $\mathrm{DFS}(\underbrace{z_1[n]}_{\mathrm{IDFS}(x_1[k])} \; \textcircled{N} \; \underbrace{z_2[n]}_{\mathrm{IDFS}(x_2[k])}) = \underbrace{\mathrm{DFS}(z_1[n])}_{x_1[k]} \; \underbrace{\mathrm{DFS}(z_2[n])}_{x_2[k]}$, then:

DFS

$$x_1[n] \; x_2[n] \rightarrow \boxed{\mathrm{IDFS}} \rightarrow \mathrm{IDFS}(x_1[n]) \, \textcircled{N} \, \mathrm{IDFS}(x_2[n]) \rightarrow \boxed{y[k] = Nx[-k]} \rightarrow \hat{x}_3[k]$$

DFS

$$x_1[n] \; x_2[n] \rightarrow \boxed{\mathrm{IDFS}} \rightarrow \frac{\hat{x}_1[-k])}{N} \, \textcircled{N} \, \frac{\hat{x}_2[-k])}{N} \rightarrow \boxed{y[k] = Nx[-k]} \rightarrow \hat{x}_3[k]$$

9. Multiplication:

DFS

$$x_1[n]\, x_2[n] \longrightarrow \boxed{\text{IDFS}} \longrightarrow \frac{1}{N^2}\sum_{m=0}^{N-1} \hat{x}_1[-m]\, \hat{x}_2[-k+m] \longrightarrow \boxed{y[k] = Nx[-k]} \longrightarrow \hat{x}_3[k]$$

DFS

$$x_1[n]\, x_2[n] \longrightarrow \boxed{\text{IDFS}} \longrightarrow \frac{1}{N^2}\sum_{\tilde{m}=0}^{N-1} \hat{x}_1[\tilde{m}]\, \hat{x}_2[-k-\tilde{m}] \longrightarrow \boxed{y[k] = Nx[-k]} \longrightarrow \hat{x}_3[k]$$

$$\hat{x}_3[k] = \frac{1}{N}\sum_{\tilde{m}=0}^{N-1} \hat{x}_1[\tilde{m}]\, \hat{x}_2[k-\tilde{m}] = \frac{1}{N}\hat{x}_1[k] \circledN \hat{x}_2[k]$$

■

Radix-2 Fast Fourier Transform Algorithm

- The straightforward implementation of Fourier series equation requires N^2 complex multiplication operations.
- However, James Cooley and John Tukey found a way to reduce the required number of complex multiplications from N^2 to be proportional to $N \times \log_2(N)$.
- The Fast Fourier Transform (FFT) uses divide-and-conquer algorithm to recursively break down the discrete-time Fourier series operation into smaller ones.
- There are two types of radix-2 FFT algorithms: Decimation-In-Time (DIT) and Decimation-In-Frequency (DIF).
- In this lecture, we will address the DIF FFT algorithm.

Radix-2 Fast Fourier Transform Algorithm

We start by the discrete-time Fourier series equation, as follows,

$$
\begin{aligned}
\hat{x}[k] &= \sum_{n=0}^{N-1} x[n] e^{\frac{-j2\pi kn}{N}}, \\
&= \sum_{n=0}^{\frac{N}{2}-1} x[n] e^{\frac{-j2\pi kn}{N}} + \sum_{n=\frac{N}{2}}^{N-1} x[n] e^{\frac{-j2\pi kn}{N}}, \\
&= \sum_{n=0}^{\frac{N}{2}-1} x[n] e^{\frac{-j2\pi kn}{N}} + \sum_{\tilde{n}=0}^{\frac{N}{2}-1} x[\tilde{n} + \frac{N}{2}] e^{\frac{-j2\pi k\tilde{n}}{N}} e^{j\pi k}, \\
&= \sum_{n=0}^{\frac{N}{2}-1} \left(x[n] + (-1)^k x[n + \frac{N}{2}] \right) e^{\frac{-j2\pi kn}{N}},
\end{aligned}
$$

Radix-2 Fast Fourier Transform Algorithm

Then we can divide the frequency index k into even and odd groups, as follows,

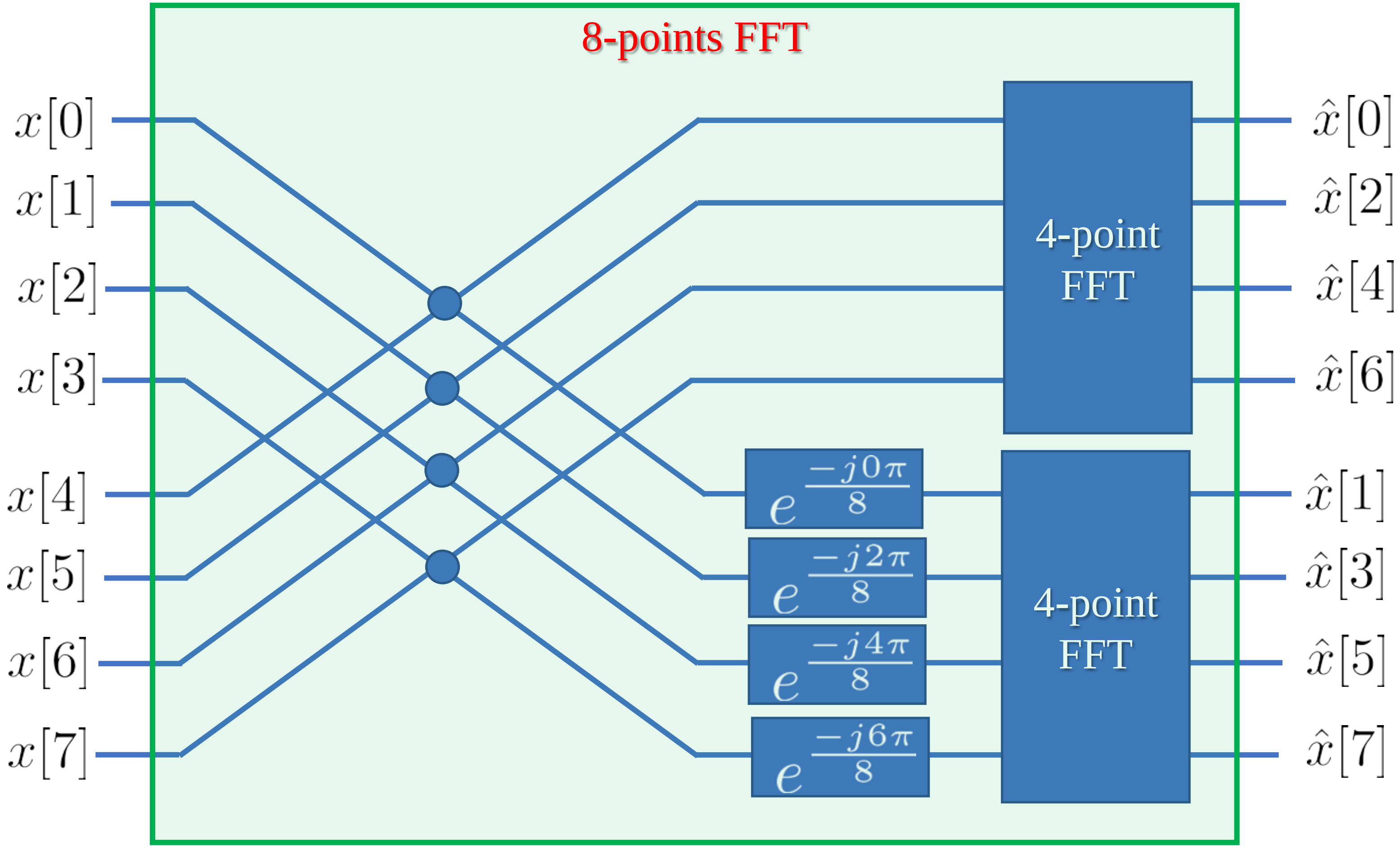
$$\hat{x}[2\tilde{k}] = \sum_{n=0}^{\frac{N}{2}-1} \left(x[n] + (-1)^{2\tilde{k}} x[n + \frac{N}{2}] \right) e^{\frac{-j2\pi 2\tilde{k}n}{N}},$$

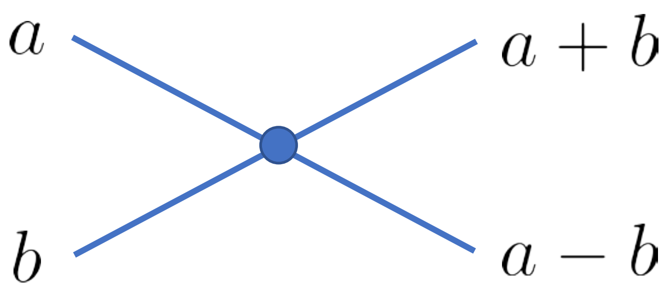
$$= \sum_{n=0}^{\frac{N}{2}-1} \underbrace{\left(x[n] + x[n + \frac{N}{2}] \right)}_{\tilde{x}_1[n]} e^{\frac{-j2\pi \tilde{k}n}{\frac{N}{2}}} = \mathrm{DFS}_{\frac{N}{2}}(\tilde{x}_1[n]).$$

$$\hat{x}[2\tilde{k}+1] = \sum_{n=0}^{\frac{N}{2}-1} \underbrace{\left(x[n] - x[n + \frac{N}{2}] \right) e^{\frac{-j2\pi n}{N}}}_{\tilde{x}_2[n]} e^{\frac{-j2\pi \tilde{k}n}{\frac{N}{2}}} = \mathrm{DFS}_{\frac{N}{2}}(\tilde{x}_2[n]).$$

Radix-2 Fast Fourier Transform Algorithm

- 8-point FFT example:





Butterfly operation

- **Complexity:**

In terms of the number of complex multipications C:

$$C_{8\text{-FFT}} = \frac{8}{2} + 2 \times C_{4\text{-FFT}}$$
$$C_{4\text{-FFT}} = \frac{4}{2} + 2 \times C_{2\text{-FFT}}$$
$$C_{2\text{-FFT}} = 0$$

Hence:

$$C_{8\text{-FFT}} = \frac{8}{2} + 2\left(\frac{4}{2}\right)$$
$$C_{8\text{-FFT}} = \frac{8}{2} + \frac{8}{2} = 8.$$

Radix-2 Fast Fourier Transform Algorithm

- In general, the number of complex multiplications of N-point FFT can be evaluated as follows:

$$
\begin{aligned}
C_{N\text{-FFT}} &= \frac{N}{2} + 2\left(C_{\frac{N}{2}\text{-FFT}}\right), \\
&= \frac{N}{2} + 2\left(\frac{N}{4} + 2\left(C_{\frac{N}{4}\text{-FFT}}\right)\right), \\
&= \frac{N}{2} + \frac{N}{2} + 4\left(C_{\frac{N}{4}\text{-FFT}}\right), \\
&= \frac{N}{2} + \frac{N}{2} + 4\left(\frac{N}{8} + 2\left(C_{\frac{N}{8}\text{-FFT}}\right)\right), \\
&= \frac{N}{2} + \frac{N}{2} + \frac{N}{2} + 8\left(C_{\frac{N}{8}\text{-FFT}}\right), \\
&= \underbrace{\frac{N}{2} + \cdots + \frac{N}{2}}_{(\log_2(N)-1)\ \text{terms}} + \frac{N}{2}\left(\cancel{C_{\frac{N}{N/2}\text{-FFT}}}\right) = \frac{N}{2}\left(\log_2(N) - 1\right).
\end{aligned}
$$

0

Amazing!!