

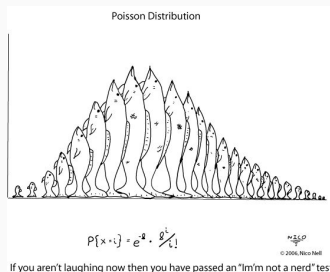
Chapter 2 - Simulating Statistical Models

Poisson Processes.

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Poisson processes are typically used to model the occurrence of events in time. More generally, they can also be used to model the occurrence of events in space.



Example: The arrival times of people to the Emergency Room in a Hospital during a predetermined interval of time $[t_1, t_2]$ is usually modelled as a Poisson Process.

Example: At a future moment in time, the location of each and every fish of a given species in a lake can also be modelled as a Poisson process.

In these two examples notice that the number of arrivals/number of fish in the lake is random. Also the arrival times/location of the fish is random.

In order to study the Poisson Process we need to start first with the Poisson distribution.

A random variable X has **Poisson distribution** with parameter λ if it takes non-negative integer values and its probability mass function is given by:

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!} \text{ for } k = 0, 1, 2, \dots$$

Some properties:

If $X \sim \text{Pois}(\lambda)$ and $Y \sim \text{Pois}(\mu)$ are independent then:

- $E(X) = \lambda$
- $\text{Var}(X) = \lambda$
- $X + Y \sim \text{Pois}(\lambda + \mu)$

Poisson processes can be defined on very general spaces. For our purposes, we will consider that they are defined on subsets of \mathbb{R}^d .

A **Poisson Process** on a set $D \subseteq \mathbb{R}^d$ with intensity function $\lambda : \mathbb{R}^d \rightarrow [0, \infty)$ is a random set of points $\Pi \subseteq D$ such that the following two conditions hold:

- a. If $A \subseteq D$, then $|\Pi \cap A| \sim \text{Pois}(\Lambda(A))$ where $|\Pi \cap A|$ is the number of points of Π in A and
- b. If $A, B \subseteq D$ are disjoint, then $|\Pi \cap A|$ and $|\Pi \cap B|$ are independent.

- The number of points of the Poisson Process that are located in set A is random, moreover $E(|\Pi \cap A|) = \Lambda(A)$
- On average, regions with large values of the intensity function λ will have more concentration of points than regions with small values of λ
- In the particular case where the function λ is constant over a region, the Poisson Process points are uniformly distributed over that region.

Depending on the specific problem there may be different ways to simulate a Poisson process. One of the most straightforward ways to do this is summarized by the following two-step process:

Step 1: Generation of the number of points

Step 2: Generation of the location of the points

Algorithm 1 Generate a Poisson Process

- 1: **Input:** Intensity function $\lambda(\cdot)$, region D
 - 2: Generate $N \sim \text{Poisson}(\Lambda(D))$
 - 3: **for** $i = 1$ to N **do**
 - 4: Generate $X_i \sim 1_D \frac{\lambda(\cdot)}{\Lambda(D)}$
 - 5: **end for**
 - 6: **Output:** Points $\{X_1, X_2, \dots, X_N\}$ forming a Poisson process on D
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Remark: The density function $1_D \lambda(\cdot) / \Lambda(D)$ is defined as

$$1_D \lambda(\cdot) / \Lambda(D) = \begin{cases} \lambda(x) / \Lambda(D) & \text{if } x \in D \\ 0 & \text{if } x \notin D \end{cases}$$

- The previous algorithm feasibility is linked to our ability to generate samples of points in \mathbb{R}^d that follow the density function $1_D \lambda(\cdot) / \Lambda(D)$
- Depending on the intensity function λ this might be difficult.
- In some cases, the use of rejection methods may be necessary.

Example: Generate one sample corresponding to a Poisson process with constant intensity $\lambda = 1$ on the interval $D = [0, 10]$

$$\Lambda(D) = \int_0^{10} \lambda(x) dx = \int_0^{10} 1 \cdot dx = 10$$

Algorithm 2 Generate Poisson Random Points

- 1: **Input:** Interval $[0, 10]$, intensity $\lambda = 1$
 - 2: Generate $N \sim \text{Poisson}(10)$
 - 3: **for** $i = 1$ to N **do**
 - 4: Generate $X_i \sim \text{Uniform}(0, 10)$
 - 5: **end for**
 - 6: **Output:** Set of Poisson points $\{X_1, X_2, \dots, X_N\}$
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Example: Generate one sample corresponding to a Poisson process with intensity

$\lambda(x) = \frac{x}{50} + \frac{3x^2}{100}$ on the interval $D = [0, 15]$ and 0 otherwise.

$$\Lambda(D) = \int_0^{15} \lambda(x) dx = \int_0^{15} \left(\frac{x}{50} + \frac{3x^2}{100} \right) dx = \left(\frac{x^2 + x^3}{100} \right) \Big|_0^{15} = 36$$

Following the previous algorithm we have to generate random numbers that follow the density $\lambda(x)/\Lambda(D)$. In this specific case probably the easiest way to do that is using a rejection algorithm.

A roughly equivalent method (called thinning method) is described in Algorithm 2.41 from the textbook. The thinning method turns a Poisson process with intensity λ into a Poisson process with intensity $\lambda^* \leq \lambda$ by rejecting some of the points.

Objective: Generate a realization of a Poisson process with intensity λ^* . Let $\lambda^* \leq \lambda$ and $\Lambda(D) = \int_D \lambda(x) dx$.

Algorithm 3 Generate a Nonhomogeneous Poisson Process via Thinning

- 1: **Input:** Intensity function $\lambda(x)$, upper bound $\lambda^*(x)$, domain D
 - 2: Generate $N \sim \text{Pois}(\Lambda(D))$
 - 3: Initialize $\Pi \leftarrow \emptyset$
 - 4: **for** $i = 1$ to N **do**
 - 5: Generate $X_i \sim \frac{\lambda(x)}{\Lambda(D)}$
 - 6: Generate $U \sim U[0, 1]$
 - 7: **if** $U < \frac{\lambda^*(X_i)}{\lambda(X_i)}$ **then**
 - 8: $\Pi \leftarrow \Pi \cup \{X_i\}$
 - 9: **end if**
 - 10: **end for**
 - 11: **Output:** Π (set of accepted points)
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In the previous example $\lambda(x) = \frac{x}{50} + \frac{3x^2}{100}$ on $[0, 15]$. This is an increasing function so its maximum is achieved at $x = 15$, and we have $\lambda(15) = 7.05$.

This means that we could start with a Poisson process with intensity $\tilde{\lambda} = 7.05 \geq \lambda(x)$ and apply the thinning method.

$$\tilde{\Lambda}(D) = \int_0^{15} \tilde{\lambda}(x) dx = 7.05 \cdot 15 = 105.75$$

Example: Generate one sample corresponding to a Poisson process with intensity $\lambda(x_1, x_2) = 500x_1$ on the rectangle $D = [0, 2] \times [0, 1]$

$$\Lambda(D) = \int_0^1 \int_0^2 500x_1 dx_1 dx_2 = 1000$$

This means that we will have to generate points on $[0, 2] \times [0, 1]$ according to the density $f(x_1, x_2) = \frac{\lambda(x_1, x_2)}{\Lambda(D)} = \frac{x_1}{2}$ on D and 0 outside of D .

From here we can see that $f(x_1, x_2)$ can be written as the product of $f_1(x_1) = x_1/2$ and $f_2(x_2) = 1$. Notice that f_1 is a density function on $[0, 2]$ and f_2 is a density function on $[0, 1]$.

The generation of random vectors with density f can be done by **independently** generating its components according to densities f_1 and f_2 respectively

Algorithm 4 Generate Bivariate Poisson Process

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1: Input:  $\Lambda(D) = 1000$ 
2: Generate  $N \sim \text{Pois}(1000)$ 
3: for  $i = 1$  to  $N$  do
4:   Generate  $X_1[i] \sim f_1$ 
5:   Generate  $X_2[i] \sim f_2$ 
6: end for
7: Output:  $X = (X_1, X_2)$ 
```

Remarks

- To generate random numbers according to density f_1 we can use the inverse transform method
- f_2 is the uniform distribution density on $[0, 1]$

Generate one sample corresponding to a Poisson process with intensity $\lambda(x_1, x_2) = 30(x_1^2 + x_2^2)$ on the rectangle $D = [0, 3] \times [0, 4]$

Hint: Use the thinning method