Homework: Markov Chains with Continuous State Space

Question 1: Gaussian Transition Density

We generate a Markov chain path of length 40 using the Gaussian transition density (Algorithm 1).

```
T = 40
phi = 0.5
sigma = 1

x = np.zeros(T)
for t in range(1, T):
    x[t] = phi * x[t-1] + np.random.normal(0, sigma)

plt.plot(x, marker="o")
plt.title("Gaussian␣Transition␣Density␣(AR(1)␣Path)")
plt.show()
```

Question 2: Uniform Transition Density

The following is the lgorithm for simulating a Markov chain with the **Uniform transition density**

$$p(x, y) = \frac{1}{2}\,\mathbb{1}_{[x-1,\,x+1]}(y), \quad x, y \in \mathbb{R}$$

Algorithm 1 Simulating a Markov Chain with Uniform Transition Density

1: Set $X_0 = 0$ (or draw from an initial distribution)
2: **for** $i = 1$ to n **do**
3: Generate $X_i \sim \text{Uniform}(X_{i-1} - 1,\ X_{i-1} + 1)$
4: **end for**
5: **return** $(X_0, X_1, \ldots, X_n)$

We now simulate a Markov chain with the uniform transition density

```
T = 40
x = np.zeros(T)

for t in range(1, T):
    x[t] = np.random.uniform(x[t-1]-1, x[t-1]+1)

plt.plot(x, marker="o")
plt.title("Uniform␣Transition␣Density␣Path")
plt.show()
```

Question 3: AR(1) Process and Stationary Distribution

Consider the AR(1) process

$$X_j = \phi X_{j-1} + \varepsilon_j, \qquad \varepsilon_j \overset{\text{i.i.d.}}{\sim} N(0, \sigma^2), \quad |\phi| < 1.$$

(1) Markov property and transition density

Given $X_{j-1} = x$, we have

$$X_j = \phi x + \varepsilon_j.$$

Since ε_j is independent of the past $(X_0, \dots, X_{j-1})$, the conditional law of X_j depends only on $X_{j-1} = x$. Thus $\{X_j\}$ satisfies the Markov property.

Moreover, conditionally on $X_{j-1} = x$,

$$X_j \sim N(\phi x, \sigma^2),$$

so the one-step transition density is

$$p(x, y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y - \phi x)^2}{2\sigma^2} \right).$$

(2) Stationary distribution is Gaussian with $\mu = 0$

Step 1: Iterating the recursion. Starting from

$$X_n = \phi X_{n-1} + \varepsilon_n,$$

we substitute for X_{n-1}:

$$X_n = \phi(\phi X_{n-2} + \varepsilon_{n-1}) + \varepsilon_n = \phi^2 X_{n-2} + \phi \varepsilon_{n-1} + \varepsilon_n.$$

Repeating this process:

$$X_n = \phi^3 X_{n-3} + \phi^2 \varepsilon_{n-2} + \phi \varepsilon_{n-1} + \varepsilon_n.$$

Continuing inductively, we obtain

$$X_n = \phi^n X_0 + \sum_{k=0}^{n-1} \phi^k \varepsilon_{n-k}.$$

In particular, if $X_0 = 0$, then

$$X_n = \sum_{k=0}^{n-1} \phi^k \varepsilon_{n-k}.$$

Step 2: Gaussian structure. This is a finite linear combination of independent Gaussian random variables, so X_n is Gaussian. As $n \to \infty$, the variance converges (since $|\phi| < 1$), so X_n converges in distribution to a Gaussian random variable X_∞. Thus the stationary distribution, if it exists, must be Gaussian.

Alternative argument. If $Y \sim \pi$ is stationary, then with independent $\varepsilon \sim N(0, \sigma^2)$

$$Y \overset{d}{=} \phi Y + \varepsilon.$$

Taking expectations yields

$$\mathbb{E}[Y] = \phi\, \mathbb{E}[Y],$$

so $\mathbb{E}[Y] = 0$. The Gaussian form follows from solving the variance recursion.

(3) Stationary variance

Let $v = \mathrm{Var}(X)$ under stationarity. Then

$$v = \mathrm{Var}(X_j) = \mathrm{Var}(\phi X_{j-1} + \varepsilon_j) = \phi^2 v + \sigma^2,$$

so

$$v = \frac{\sigma^2}{1 - \phi^2}.$$

(4) Special case $\phi = \frac{1}{2},\ \sigma^2 = 1$

Plugging into the variance formula:

$$\mathrm{Var}(X) = \frac{1}{1 - (1/2)^2} = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}.$$

Hence the stationary distribution is

$$\pi \sim N\big(0, \tfrac{4}{3}\big),$$

with stationary standard deviation

$$\sqrt{\tfrac{4}{3}} = \frac{2}{\sqrt{3}} \approx 1.1547.$$