

## **Chapter 2 - Simulating Statistical Models**

Poisson Processes.

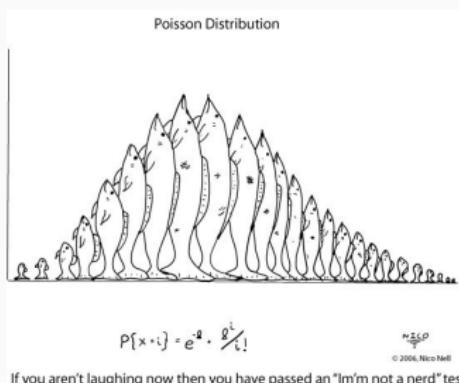
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# Poisson Processes

**Poisson processes** are typically used to model the occurrence of events in time. More generally, they can also be used to model the occurrence of events in space.



**Example:** The arrival times of people to the Emergency Room in a Hospital during a predetermined interval of time  $[t_1, t_2]$  is usually modelled as a Poisson Process.

**Example:** At a future moment in time, the location of each and every fish of a given species in a lake can also be modelled as a Poisson process.

In these two examples notice that the number of arrivals/number of fish in the lake is random. Also the arrival times/location of the fish is random.

In order to study the Poisson Process we need to start first with the Poisson distribution.

A random variable  $X$  has **Poisson distribution** with parameter  $\lambda$  if it takes non-negative integer values and its probability mass function is given by:

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!} \text{ for } k = 0, 1, 2, \dots$$

## Some properties:

If  $X \sim \text{Pois}(\lambda)$  and  $Y \sim \text{Pois}(\mu)$  are independent then:

- $E(X) = \lambda$
- $\text{Var}(X) = \lambda$
- $X + Y \sim \text{Pois}(\lambda + \mu)$

Poisson processes can be defined on very general spaces. For our purposes, we will consider that they are defined on subsets of  $\mathbb{R}^d$ .

A **Poisson Process** on a set  $D \subseteq \mathbb{R}^d$  with intensity function  $\lambda : \mathbb{R}^d \rightarrow [0, \infty)$  is a random set of points  $\Pi \subseteq D$  such that the following two conditions hold:

- If  $A \subseteq D$ , then  $|\Pi \cap A| \sim Pois(\Lambda(A))$  where  $|\Pi \cap A|$  is the number of points of  $\Pi$  in  $A$  and
- If  $A, B \subseteq D$  are disjoint, then  $|\Pi \cap A|$  and  $|\Pi \cap B|$  are independent.

- The number of points of the Poisson Process that are located in set  $A$  is random, moreover  $E(|\Pi \cap A|) = \Lambda(A)$
- On average, regions with large values of the intensity function  $\lambda$  will have more concentration of points than regions with small values of  $\lambda$
- In the particular case where the function  $\lambda$  is constant over a region, the Poisson Process points are uniformly distributed over that region.

Depending on the specific problem there may be different ways to simulate a Poisson process. One of the most straightforward ways to do this is summarized by the following two-step process:

**Step 1:** Generation of the number of points

**Step 2:** Generation of the location of the points

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#### Algorithm 1 Generate a Poisson Process

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- 1: **Input:** Intensity function  $\lambda(\cdot)$ , region  $D$
  - 2: Generate  $N \sim \text{Poisson}(\Lambda(D))$
  - 3: **for**  $i = 1$  to  $N$  **do**
  - 4:     Generate  $X_i \sim 1_D \frac{\lambda(\cdot)}{\Lambda(D)}$
  - 5: **end for**
  - 6: **Output:** Points  $\{X_1, X_2, \dots, X_N\}$  forming a Poisson process on  $D$
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**Remark:** The density function  $1_D \lambda(\cdot)/\Lambda(D)$  is defined as

$$1_D \lambda(\cdot)/\Lambda(D) = \begin{cases} \lambda(x)/\Lambda(D) & \text{if } x \in D \\ 0 & \text{if } x \notin D \end{cases}$$

- The previous algorithm feasibility is linked to our ability to generate samples of points in  $\mathbb{R}^d$  that follow the density function  $1_D \lambda(\cdot)/\Lambda(D)$
- Depending on the intensity function  $\lambda$  this might be difficult.
- In some cases, the use of rejection methods may be necessary.

**Example:** Generate one sample corresponding to a Poisson process with constant intensity  $\lambda = 1$  on the interval  $D = [0, 10]$

$$\Lambda(D) = \int_0^{10} \lambda(x) dx = \int_0^{10} 1 \cdot dx = 10$$

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## Algorithm 2 Generate Poisson Random Points

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- 1: **Input:** Interval  $[0, 10]$ , intensity  $\lambda = 1$
  - 2: Generate  $N \sim \text{Poisson}(10)$
  - 3: **for**  $i = 1$  to  $N$  **do**
  - 4:     Generate  $X_i \sim \text{Uniform}(0, 10)$
  - 5: **end for**
  - 6: **Output:** Set of Poisson points  $\{X_1, X_2, \dots, X_N\}$
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## Poisson Process – Thinning Method Example

**Example:** Generate one sample corresponding to a Poisson process with intensity

$$\lambda(x) = \frac{x}{50} + \frac{3x^2}{100} \text{ on the interval } D = [0, 15] \text{ and 0 otherwise.}$$

$$\Lambda(D) = \int_0^{15} \lambda(x) dx = \int_0^{15} \left( \frac{x}{50} + \frac{3x^2}{100} \right) dx = \left( \frac{x^2}{100} + \frac{x^3}{300} \right) \Big|_0^{15} = 36$$

Following the previous algorithm we have to generate random numbers that follow the density  $\lambda(x)/\Lambda(D)$ . In this specific case probably the easiest way to do that is using a rejection algorithm.

A roughly equivalent method (called thinning method) is described in Algorithm 2.41 from the textbook. The thinning method turns a Poisson process with intensity  $\lambda$  into a Poisson process with intensity  $\lambda^* \leq \lambda$  by rejecting some of the points.

**Objective:** Generate a realization of a Poisson process with intensity  $\lambda^*$ . Let  $\lambda^* \leq \lambda$  and  $\Lambda(D) = \int_D \lambda(x)dx$ .

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### Algorithm 3 Generate a Nonhomogeneous Poisson Process via Thinning

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- 1: **Input:** Intensity function  $\lambda(x)$ , upper bound  $\lambda^*(x)$ , domain  $D$
- 2: Generate  $N \sim \text{Pois}(\Lambda(D))$
- 3: Initialize  $\Pi \leftarrow \emptyset$
- 4: **for**  $i = 1$  to  $N$  **do**
- 5:     Generate  $X_i \sim \frac{\lambda(x)}{\Lambda(D)}$
- 6:     Generate  $U \sim U[0, 1]$
- 7:     **if**  $U < \frac{\lambda^*(X_i)}{\lambda(X_i)}$  **then**
- 8:          $\Pi \leftarrow \Pi \cup \{X_i\}$
- 9:     **end if**
- 10: **end for**
- 11: **Output:**  $\Pi$  (set of accepted points)

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## Poisson Process – Thinning Method Example

In the previous example  $\lambda(x) = \frac{x}{50} + \frac{3x^2}{100}$  on  $[0, 15]$ . This is an increasing function so its maximum is achieved at  $x = 15$ , and we have  $\lambda(15) = 7.05$ .

This means that we could start with a Poisson process with intensity  $\tilde{\lambda} = 7.05 \geq \lambda(x)$  and apply the thinning method.

$$\tilde{\lambda}(D) = \int_0^{15} \tilde{\lambda}(x) dx = 7.05 \cdot 15 = 105.75$$

## Bivariate Poisson Process Example

**Example:** Generate one sample corresponding to a Poisson process with intensity  $\lambda(x_1, x_2) = 500x_1$  on the rectangle  $D = [0, 2] \times [0, 1]$

$$\Lambda(D) = \int_0^1 \int_0^2 500x_1 dx_1 dx_2 = 1000$$

This means that we will have to generate points on  $[0, 2] \times [0, 1]$  according to the density  $f(x_1, x_2) = \frac{\lambda(x_1, x_2)}{\Lambda(D)} = \frac{x_1}{2}$  on  $D$  and 0 outside of  $D$ .

From here we can see that  $f(x_1, x_2)$  can be written as the product of  $f_1(x_1) = x_1/2$  and  $f_2(x_2) = 1$ . Notice that  $f_1$  is a density function on  $[0, 2]$  and  $f_2$  is a density function on  $[0, 1]$ .

The generation of random vectors with density  $f$  can be done by **independently** generating its components according to densities  $f_1$  and  $f_2$  respectively

## Algorithm: Generating Bivariate Poisson Process

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**Algorithm 4** Generate Bivariate Poisson Process

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- 1: **Input:**  $\Lambda(D) = 1000$
- 2: Generate  $N \sim \text{Pois}(1000)$
- 3: **for**  $i = 1$  to  $N$  **do**
- 4:     Generate  $X_1[i] \sim f_1$
- 5:     Generate  $X_2[i] \sim f_2$
- 6: **end for**
- 7: **Output:**  $X = (X_1, X_2)$

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### Remarks

- To generate random numbers according to density  $f_1$  we can use the inverse transform method
- $f_2$  is the uniform distribution density on  $[0, 1]$

Generate one sample corresponding to a Poisson process with intensity  $\lambda(x_1, x_2) = 30(x_1^2 + x_2^2)$  on the rectangle  $D = [0, 3] \times [0, 4]$

**Hint:** Use the thinning method