Chapter 3 - Monte Carlo Methods

Monte Carlo Applications to Statistical Inference.
Point Estimators.

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Standard Error of an Estimator

In this lecture we will see how to use Monte Carlo methods to estimate the standard error of an estimator.

A reminder that our framework consists on having some observations $x = (x_1, x_2, ..., x_n)$ that are considered a random sample of a random variable $X = (X_1, X_2, ..., X_n) \sim P_\theta$, where $\theta \in \Theta$ is unknown.

Let $\hat{\theta} = \hat{\theta}(X)$ be an estimator of θ. One important quantity that we study on estimators is their standard deviation.

$$s.e._\theta(\hat{\theta}) = stdev\left(\hat{\theta}(X)\right) = \sqrt{Var\left(\hat{\theta}(X)\right)}$$

Standard Error of an Estimator

In some cases we can estimate $s.e._\theta(\hat{\theta})$ easily.

For example, for estimators of the form $\hat{\theta} = \frac{1}{n}\sum_{j=1}^{n} X_j$ we can estimate their standard deviation as

$$\widehat{s.e.}_\theta(\hat{\theta}) = \frac{\hat{s}}{\sqrt{n}}$$

where $\hat{s}$ is the sample standard deviation of the observations $(x_1, x_2, .., x_n)$.

(we used this in previous sections of this chapter to give a confidence interval of a Monte Carlo estimator)

However, for estimators $\hat{\theta}$ that are not given as a sample mean, estimating their standard error may be more complicated.

Estimating the Standard Error

For a given value of θ, if we are able to generate N samples: $\left\{x^{(j)}\right\}_{j=1,2,\ldots,N}$ of the random variable X we could estimate the standard error of the estimator $\hat{\theta}$ as:

$$\widehat{s.e.}_{\theta}(\hat{\theta}) = \sqrt{\frac{1}{N}\sum_{j=1}^{N}\left(\hat{\theta}(x^{(j)}) - \overline{\hat{\theta}^{(\cdot)}}\right)^2}$$

where $\overline{\hat{\theta}^{(\cdot)}} = \frac{1}{N}\sum_{j=1}^{N}\hat{\theta}(x^{(j)})$

If necessary, we can do this for a range of values of θ to get a better idea of the standard error of the estimator as a function of θ.

Example

Assume that we would like to estimate an unknown parameter θ from a sample of 20 random numbers that are distributed according to the uniform distribution on $[0, \theta]$. Two unbiased estimators for θ are proposed and we would like to find out which of these two estimators has a smaller variance.

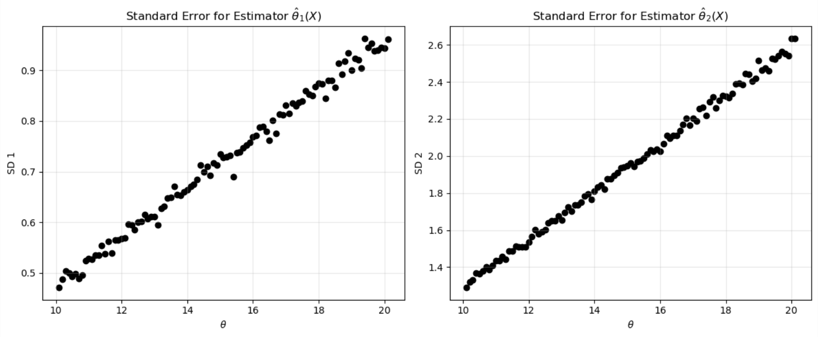
Estimator 1: $\hat{\theta}_1(X) = \frac{21}{20}\max(X_1, X_2, ...X_{20})$

Estimator 2: $\hat{\theta}_2(X) = \frac{2}{N}\sum_{j=1}^{20} X_j$

Use Monte Carlo methods to estimate the standard error for both estimators, for values of θ on $[10, 20]$.

Algorithm 1 Monte Carlo Estimation of Estimator Variability

Input: Number of simulations N, sample size m, parameter grid $\{\theta_j\}_{j=1}^{J}$
for $j = 1$ to J **do**
    Initialize empty vectors for estimator values
    for $i = 1$ to N **do**
        Generate $U_{i1}, \ldots, U_{im} \sim \text{Uniform}(0,\, \theta_j)$
        Compute estimators $\hat{\theta}_{1i} = a_1 \max(U_i) + b_1$ and $\hat{\theta}_{2i} = a_2\, \overline{U_i} + b_2$
    end for
    Compute standard deviations: $\texttt{sd1}_j = \text{sd}(\hat{\theta}_1), \quad \texttt{sd2}_j = \text{sd}(\hat{\theta}_2)$
end for
Output: Standard deviation profiles $\{\texttt{sd1}_j, \texttt{sd2}_j\}_{j=1}^{J}$



Note: Estimator 1 would be preferable as it has a small standard error.

Homework

Assume that we would like to estimate the unknown parameter μ from a sample of 10 random numbers that are distributed according to the normal distribution $N(\mu, 1)$. We don't have the full information about these observations, we only have the min, median and max.

Two unbiased estimators for μ are proposed and we would like to find out which of these two estimators has a smaller variance.

Estimator 1: $\hat{\mu}_1(X) = \dfrac{\min(X) + \max(X)}{2}$

Estimator 2: $\hat{\mu}_2(X) = median(X)$

Use Monte Carlo methods to estimate the standard error for both estimators, for values of μ on $[-5, 5]$