Chapter 3 - Monte Carlo Methods

Variance Reduction Methods. Antithetic Variates.

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Previously

If $\hat{\theta}_n$ is the Monte Carlo estimator for $\theta = E(g(X))$ and we want $P\left(|\hat{\theta}_n - \theta| \leq \epsilon\right) \geq 1 - \alpha$, then we need to use a number of simulations n such that

$$n \geq \frac{Z^2_{1-\alpha/2}\sigma^2}{\epsilon^2}$$

where Z_α is the α-level percentile of the standard normal distribution and $\sigma^2 = Var(g(X))$.

Then, the accuracy of the estimator depends strongly on $\sigma^2 = Var(g(X))$

Variance Reduction Methods

There are more elaborate variants of the Monte Carlo method that try to deal with this issue. Many of these algorithms are referred to as **variance reduction methods**.

One way of describing variance reduction methods is that they try to generate "smart" random samples for the problem at hand. Some well known variance reduction methods are:

- Antithetic Variates Method
- Control Variates Method
- Importance Sampling

This is not an exhaustive list.

We will study these methods in more details, including their theoretical justification, when to use them, etc.

In this lecture we will cover the **Antithetic Variable Method**

Antithetic Variable Method

Consider two non-independent random variables X and X' with the same distribution. These are some classic examples of such pairs of random variables:

- $U \sim U[0,1]$ and $U' = 1 - U$ have the same distribution
- If f is a density function symmetric around 0, the random variables $X \sim f$ and $X' = -X$ have the the same distribution.

If we consider the random quantity $\frac{g(X)+g(X')}{2}$ we have

$$E\left(\frac{g(X)+g(X')}{2}\right) = E(g(X))$$

and

$$\begin{aligned} Var\left(\frac{g(X)+g(X')}{2}\right) &= \frac{Var(g(X)) + 2Cov(g(X), g(X')) + Var(g(X'))}{4} \\ &= \frac{1}{2}Var(g(X)) + \frac{1}{2}Cov(g(X), g(X')) \end{aligned}$$

Antithetic Variates Method

Comparing the previous expression to the case where X and X' are independent, we have an additional covariance term.

How can we exploit this observation?

Usual Monte Carlo estimator:

$$\hat{\theta}_n = \frac{1}{n}\sum_{j=1}^{n} g(X_j) \quad \text{and} \quad MSE(\hat{\theta}_n) = \frac{1}{n} Var(g(X))$$

Antithetic Variates estimator:

$$\hat{\theta}_n^{AV} = \frac{1}{n}\sum_{j=1}^{n/2} (g(X_j) + g(X_j')) \quad \text{and} \quad MSE(\hat{\theta}_n^{AV}) = \frac{1}{n} Var(g(X))(1+\rho)$$

where ρ is the correlation coefficient between X_j, X_j', and

$$E(\hat{\theta}_n^{AV}) = \theta \quad \text{(unbiased)}$$

If $\rho < 0$ then this estimator is more efficient than the usual Monte Carlo estimator.

Antithetic Variates Algorithm

Algorithm 1 Monte Carlo with Antithetic Variates

1: **Input:** Sample size n (assume n is even)
2: **for** $j = 1$ to $n/2$ **do**
3: Generate an antithetic pair (X_j, X_j')
4: **end for**
5: Compute the estimator:

$$\hat{\theta}_n^{AV} = \frac{1}{n} \sum_{j=1}^{n/2} \left(g(X_j) + g(X_j') \right)$$

6: **Output:** Antithetic variates estimator $\hat{\theta}_n^{AV}$

Remark: For this algorithm to be more efficient than the usual Monte Carlo estimator, $g(X_j)$ and $g(X_j')$ should be negatively correlated. The stronger the negative correlation, the greater the variance reduction and efficiency.

Example: Antithetic Variates for $E(e^U)$

Example: Use the antithetic variates method to estimate $\mathbb{E}[e^U]$ where $U \sim \text{Uniform}(0,1)$, and compare the results with the basic Monte Carlo estimator.

Consider the antithetic pair $(U, 1-U)$. Because the function $g(x) = e^x$ is increasing, the values $g(U)$ and $g(1-U)$ are negatively correlated.

Analytical Results:

$$\mathbb{E}[e^U] = \int_0^1 e^x \, dx = e - 1$$

$$\mathrm{Var}(e^U) = \mathbb{E}[e^{2U}] - (\mathbb{E}[e^U])^2 = \frac{e^2-1}{2} - (e-1)^2$$

$$\mathrm{Cov}(e^U, e^{1-U}) = \mathbb{E}[e^U e^{1-U}] - \mathbb{E}[e^U]\mathbb{E}[e^{1-U}] = e - (e-1)^2$$

Hence, the correlation coefficient is

$$\rho(e^U, e^{1-U}) = \frac{\mathrm{Cov}(e^U, e^{1-U})}{\sqrt{\mathrm{Var}(e^U)\,\mathrm{Var}(e^{1-U})}} \simeq -0.9677$$

showing a strong negative correlation, and a variance reduction by roughly a factor of **30**.

Now let's review Assignment 3 question 2.

Monte Carlo Estimation using Antithetic Variates

We want to estimate

$$\theta = E(e^{1+Z}),$$

where Z is a standard normal random variable ($Z \sim N(0,1)$).

1. Compute the exact analytical value of θ.
2. Write a computer program to estimate θ using the **Basic Monte Carlo** method. Provide a 95% confidence interval for your estimator.
3. Write a computer program to estimate θ using the **Antithetic Variates** method. Provide a 95% confidence interval for this estimator as well.

Hint: Consider the antithetic pair (Z, Z') where $Z' = -Z$.