Chapter 2 - Simulating Statistical Models

Markov Chains on a Continuous State Space.

Prof. Alex Alvarez, Ali Raisolsadat

School of Mathematical and Computational Sciences
University of Prince Edward Island

Markov Chains on Different State Spaces

Finite state space: So far we considered Markov chains with

$$S = \{1, 2, \dots, M\}$$

Beyond finite: Markov chains can also take values in more general spaces, such as

$$S = \mathbb{R}^d$$

In this case, instead of a transition **matrix**, we will need a transition **density**.

Transition Density

A **transition density** is a function

$$p : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$$

such that:

1. $p(x, y) \geq 0$ for all $x, y \in \mathbb{R}^d$.
2. $\displaystyle\int_{\mathbb{R}^d} p(x, y)\, dy = 1 \quad \forall x \in \mathbb{R}^d$.

The interpretation of this definition is that x plays the role of the current state of the Markov Chain, so for a fixed value of x, the function $g(y) = f(x, y)$ is the density function of the random variable Y representing the next state of the Markov Chain.

Example: Gaussian Markov Chain

Example 2.29 from the Textbook: Let $X_0 = 0$ and define

$$X_j = \tfrac{1}{2}X_{j-1} + \varepsilon_j, \quad \varepsilon_j \sim N(0,1) \text{ i.i.d.}$$

Observation: The sequence $X_0, X_1, X_2, \ldots$ is a Markov chain with state space $S = \mathbb{R}$.

Conditional law: Given $X_{j-1} = x$, we have

$$X_j = \tfrac{x}{2} + \varepsilon_j \;\sim\; N\big(\tfrac{x}{2}, 1\big)$$

Transition density:

$$p(x,y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}\,(y - x/2)^2 \right), \quad x, y \in \mathbb{R}$$

Generation of Markov Chain Paths (Continuous State Space)

Algorithm 1 Simulating a Markov Chain Path

1: Generate X_0 according to the initial distribution
2: **for** $i = 1$ to $n - 1$ **do**
3: Generate $X_i \in S$ according to the density

$$g(y) = p(X_{i-1}, y)$$

4: **end for**
5: **return** $(X_0, X_1, \ldots, X_n)$

Example: Uniform Transition Density

Example (from Homework): Let $X_0 = 0$ and define the transition density

$$p(x, y) = \frac{1}{2}\,\mathbb{1}_{[x-1,\,x+1]}(y), \quad x, y \in \mathbb{R}$$

Note: The indicator function $\mathbb{1}_A$ is defined as

$$\mathbb{1}_A(y) = \begin{cases} 1, & \text{if } y \in A, \\ 0, & \text{if } y \notin A. \end{cases}$$

Observation: The sequence $X_0, X_1, X_2, \ldots$ is a Markov chain with state space $S = \mathbb{R}$.

Conditional law: Given $X_{j-1} = x$, we have

$$X_j \sim \text{Uniform}(x - 1, x + 1)$$

Transition density:

$$p(x, y) = \begin{cases} \dfrac{1}{2}, & y \in [x - 1, x + 1], \\ 0, & \text{otherwise.} \end{cases}$$

For Markov chains with a continuous state space, we also have the notion of a **stationary distribution**.

A probability density $\pi : \mathbb{R}^d \to [0, \infty)$ is called a **stationary density** for a Markov chain with transition density p if it satisfies

$$\int_{\mathbb{R}^d} \pi(x)\, p(x, y)\, dx \;=\; \pi(y), \quad \forall\, y \in \mathbb{R}^d.$$

Intuition: A stationary density π is an equilibrium law for the Markov chain:

- If $X_n \sim \pi$, then $X_{n+1} \sim \pi$ as well.
- The distribution is **invariant** under the dynamics of the chain.
- In the long run, many Markov chains converge to their stationary distribution, regardless of the starting point.

Example (Gaussian AR(1) Chain):

$$X_j = \tfrac{1}{2}X_{j-1} + \varepsilon_j, \quad \varepsilon_j \sim N(0,1).$$

- This chain has a stationary distribution:

$$\pi \ \sim \ N\left(0, \tfrac{1}{1-(1/2)^2}\right) = N\left(0, \tfrac{4}{3}\right).$$

- **Interpretation**: After many steps, the state X_n is approximately $N(0, 4/3)$, no matter the initial X_0.

Homework

- Generate Markov chain paths using the **Gaussian transition density** (Algorithm 1) for 40 time steps.

- Write an algorithm for simulating a Markov chain with the **Uniform transition density**

$$p(x,y) = \tfrac{1}{2}\,\mathbb{1}_{[x-1,\,x+1]}(y), \quad x,y \in \mathbb{R},$$

  and implement code to generate a path of length 40.
- **AR(1) problem:** Consider

$$X_j = \phi X_{j-1} + \varepsilon_j, \quad \varepsilon_j \sim N(0,\sigma^2) \text{ i.i.d.}, \quad |\phi| < 1.$$

1. Show that $\{X_j\}$ defines a Markov chain with transition density

$$p(x,y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-\phi x)^2}{2\sigma^2}\right)$$

2. Prove that if a stationary distribution exists, it must be Gaussian with mean 0.
3. **Hint** For an AR(1) process, the stationary variance satisfies $\mathrm{Var}(X) = \frac{\sigma^2}{1-\phi^2}$
4. Specialize to the case $\phi = \frac{1}{2},\ \sigma^2 = 1$. Find the stationary distribution and its standard deviation.