

# Chapter 1 - Random Number Generation

## The Inverse Transform Method

---

Prof. Alex Alvarez, Ali Raisolsadat

School of Mathematical and Computational Sciences  
University of Prince Edward Island

**General Quantile Function:** Let  $F$  be a cumulative distribution function(c.d.f.). Then the inverse of  $F$  is defined as

$$F^{-1}(u) = \inf \{x \in \mathbb{R} | F(x) \geq u\}$$

**Theorem:** Let  $F : \mathbb{R} \rightarrow [0, 1]$  be a c.d.f. and  $F^{-1}$  its inverse. If  $U \sim \text{uniform}[0, 1]$  and we define  $X = F^{-1}(U)$  then  $X$  has c.d.f.  $F$ .

This result is very general (applicable to both continuous and discrete distributions).

**Proof:**

Let  $U \sim \text{Uniform}(0, 1)$  and define the generalized inverse of  $F$  as

$$F^{-1}(u) = \inf\{x \in \mathbb{R} \mid F(x) \geq u\}$$

Let

$$X = F^{-1}(U)$$

Then, for any  $x \in \mathbb{R}$ :

$$P(X \leq x) = P(F^{-1}(U) \leq x)$$

By the definition of the generalized inverse:

$$F^{-1}(U) \leq x \iff U \leq F(x)$$

Hence,

$$P(X \leq x) = P(U \leq F(x)) = F(x)$$

This shows that  $X$  has CDF  $F$ , regardless of whether  $F$  is continuous or discrete.

## Algorithm

1. Generate  $U \sim U[0, 1]$
2. Define  $X = F^{-1}(U)$
3. Return  $X$

## Remarks:

- This method is more suited to continuous distributions, but it can also be applied to discrete distributions (See example 1.18 from the textbook).
- An important limitation of the method is that can only be applied to one-dimensional probability distributions.

**Example:** Generate a sample of 5 random numbers from a continuous random variable with probability density function

$$f(x) = \begin{cases} x^3/4 & \text{if } x \in [0, 2] \\ 0 & \text{otherwise} \end{cases}$$

**Example:** Generate a sample of 5 random numbers from a continuous random variable with probability density function

$$f(x) = \begin{cases} x^3/4 & \text{if } x \in [0, 2] \\ 0 & \text{otherwise} \end{cases}$$

**Solution:** First we find the cumulative distribution function  $F$ .

$$F(x) = \int_{-\infty}^x f(s)ds = \begin{cases} 0 & \text{if } x < 0 \\ x^4/16 & \text{if } 0 \leq x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$$

From here we get  $F^{-1}(x) = \sqrt[4]{16x}$

## Code

### R Code

```
1 n <- 5
2 U <- runif(n, min=0, max=1)
3 X <- (16*U)^(1/4)
4 print(X)
```

### Python Code

```
1 import numpy as np
2
3 n = 5
4 U = np.random.uniform(0, 1, n)
5 X = (16*U)**(1/4)
6 print(X)
```

**Example (Exponential Distribution):** Generate a sample of  $n = 100$  random numbers from the exponential distribution with parameter  $\lambda = 2$  using the inverse transform method.



**Example (Exponential Distribution):** Generate a sample of  $n = 100$  random numbers from the exponential distribution with parameter  $\lambda = 2$  using the inverse transform method.

In general the density for the exponential distribution with parameter  $\lambda$  is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then the cumulative distribution function is  $F(x) = 1 - e^{-\lambda x}$  for  $x \geq 0$ .

The inverse function is  $F^{-1}(x) = -\frac{\ln(1-x)}{\lambda}$

## Code

### R Code

```
1 n <- 100
2 lambda <- 2
3 U <- runif(n, min=0, max=1)
4 X <- -log(1-U)/lambda
5 print(X)
6 hist(X)
```

### Python Code

```
1 import numpy as np
2 import matplotlib.pyplot as plt
3
4 n = 100
5 lambda_ = 2
6 U = np.random.uniform(0, 1, n)
7 X = -np.log(1-U)/lambda_
8 print(X)
9 plt.hist(X, bins=10)
10 plt.show()
```

## Code

### R Code

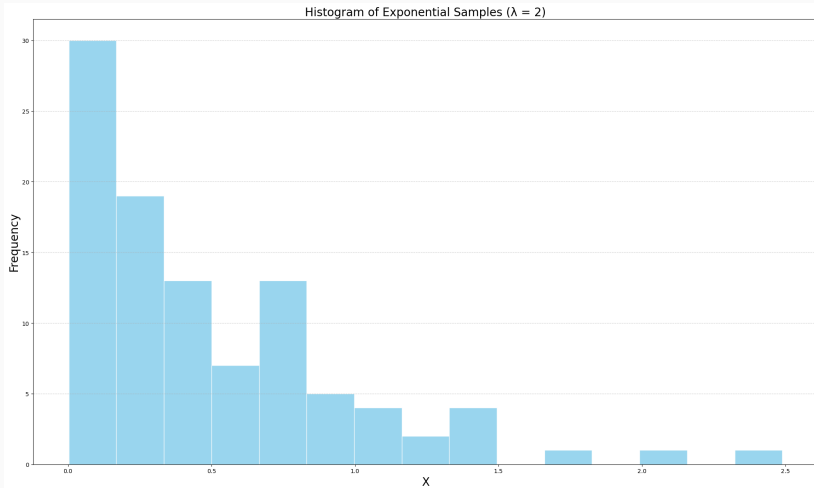
```
1 n <- 100
2 lambda <- 2
3 U <- runif(n, min=0, max=1)
4 X <- -log(1-U)/lambda
5 print(X)
6 hist(X)
```

### Python Code

```
1 import numpy as np
2 import matplotlib.pyplot as plt
3
4 n = 100
5 lambda_ = 2
6 U = np.random.uniform(0, 1, n)
7 X = -np.log(1-U)/lambda_
8 print(X)
9 plt.hist(X, bins=10)
10 plt.show()
```

**Remark:** Of course, without the constraint of using the inverse transform method, we could use R's or Python's built-in exponential generator.

# Envelope Rejection Sampling Example



- The inverse transform method is straightforward, efficient and very general.
- For the generation of continuous random variables, the inverse transform method is the method of choice in most cases, as long as  $F^{-1}$  can be found explicitly.
- Unfortunately, some important distributions do not admit a closed-form solution for  $F^{-1}$  (the normal distribution is an obvious example of this) therefore other methods would have to be applied in those cases.

1. Write down a computer program to generate a sample of 1000 random numbers from the probability distribution with density function

$$f(x) = \frac{3}{2}x^{-5/2}, \quad x \in [1, \infty), \quad 0 \text{ otherwise.}$$

2. Find the explicit inverse of the CDF for the **Weibull distribution**

$$F(x) = 1 - e^{-(x/\lambda)^k}, \quad x \geq 0, \quad x \leq 0$$

3. Find the explicit inverse of the CDF for the **Cauchy distribution**

$$F(x) = \frac{1}{\pi} \arctan\left(\frac{x - x_0}{\gamma}\right) + \frac{1}{2}, \quad \gamma > 0$$

4. Find the explicit inverse of the CDF for the **Logistic distribution**

$$F(x) = \frac{1}{1 + e^{-(x-\mu)/s}}, \quad s > 0$$

and plot both the PDF and CDF. What is the difference between this distribution and the **Normal distribution**?