Chapter 2 - Simulating Statistical Models

Poisson Processes.

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Poisson Processes

Poisson processes are typically used to model the occurrence of events in time. More generally, they can also be used to model the occurrence of events in space.

Example: The arrival times of people to the Emergency Room in a Hospital during a predetermined interval of time $[t_1, t_2]$ is usually modelled as a Poisson Process.

Example: At a future moment in time, the location of each and every fish of a given species in a lake can also be modelled as a Poisson process.

In these two examples notice that the number of arrivals/number of fish in the lake is random. Also the arrival times/location of the fish is random.

In order to study the Poisson Process we need to start first with the Poisson distribution.

A random variable X has **Poisson distribution** with parameter λ if it takes non-negative integer values and its probability mass function is given by:

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!} \text{ for } k = 0, 1, 2. \dots.$$

Some properties:

If $X \sim \text{Pois}(\lambda)$ and $Y \sim \text{Pois}(\mu)$ are independent then:

- $E(X) = \lambda$
- $Var(X) = \lambda$
- $X + Y \sim \text{Pois}(\lambda + \mu)$

Poisson Processes

Poisson processes can be defined on very general spaces. For our purposes, we will consider that they are defined on subsets of $\mathbb{R}^d$.

A **Poisson Process** on a set $D \subseteq \mathbb{R}^d$ with intensity function $\lambda : \mathbb{R}^d \longrightarrow [0, \infty)$ is a random set of points $\Pi \subseteq D$ such that the following two conditions hold:

a. If $A \subseteq D$, then $|\Pi \cap A| \sim Pois(\Lambda(A))$ where $|\Pi \cap A|$ is the number of points of Π in A and
b. If $A, B \subseteq D$ are disjoint, then $|\Pi \cap A|$ and $|\Pi \cap B|$ are independent.

Remarks

- The number of points of the Poisson Process that are located in set A is random, moreover $E(|\Pi \cap A|) = \Lambda(A)$
- On average, regions with large values of the intensity function λ will have more concentration of points than regions with small values of λ
- In the particular case where the function λ is constant over a region, the Poisson Process points are uniformly distributed over that region.

Depending on the specific problem there may be different ways to simulate a Poisson process. One of the most straightforward ways to do this is summarized by the following two-step process:

Step 1: Generation of the number of points
Step 2: Generation of the location of the points

Algorithm 1 Generate a Poisson Process

1: **Input:** Intensity function $\lambda(\cdot)$, region D
2: Generate $N \sim \text{Poisson}(\Lambda(D))$
3: **for** $i = 1$ to N **do**
4: Generate $X_i \sim 1_D \dfrac{\lambda(\cdot)}{\Lambda(D)}$
5: **end for**
6: **Output:** Points $\{X_1, X_2, \dots, X_N\}$ forming a Poisson process on D

Remark: The density function $1_D\lambda(\cdot)/\Lambda(D)$ is defined as

$$1_D\lambda(\cdot)/\Lambda(D) = \begin{cases} \lambda(x)/\Lambda(D) & \text{if } x \in D \\ 0 & \text{if } x \notin D \end{cases}$$

Remarks

- The previous algorithm feasibility is linked to our ability to generate samples of points in $\mathbb{R}^d$ that follow the density function $1_D\lambda(\cdot)/\Lambda(D)$
- Depending on the intensity function λ this might be difficult.
- In some cases, the use of rejection methods may be necessary.

Poisson Process Example

Example: Generate one sample corresponding to a Poisson process with constant intensity $\lambda = 1$ on the interval $D = [0, 10]$

$$\Lambda(D) = \int_0^{10} \lambda(x)dx = \int_0^{10} 1 \cdot dx = 10$$

Algorithm 2 Generate Poisson Random Points

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Input: Interval [0, 10], intensity λ = 1
Generate N ~ Poisson(10)
for i = 1 to N do
    Generate X_i ~ Uniform(0, 10)
end for
Output: Set of Poisson points {X_1, X_2, ..., X_N}
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Poisson Process – Thinning Method Example

Example: Generate one sample corresponding to a Poisson process with intensity $\lambda(x) = \frac{x}{50} + \frac{3x^2}{100}$ on the interval $D = [0, 15]$ and 0 otherwise.

$$\Lambda(D) = \int_0^{15} \lambda(x)dx = \int_0^{15} \left(\frac{x}{50} + \frac{3x^2}{100}\right) dx = \left(\frac{x^2 + x^3}{100}\right)\Big|_0^{15} = 36$$

Following the previous algorithm we have to generate random numbers that follow the density $\lambda(x)/\Lambda(D)$. In this specific case probably the easiest way to do that is using a rejection algorithm.

A roughly equivalent method (called thinning method) is described in Algorithm 2.41 from the textbook. The thinning method turns a Poisson process with intensity λ into a Poisson process with intensity $\lambda^* \leq \lambda$ by rejecting some of the points.

Thinning Method

Objective: Generate a realization of a Poisson process with intensity λ^*. Let $\lambda^* \leq \lambda$ and $\Lambda(D) = \int_D \lambda(x)dx$.

Algorithm 3 Generate a Nonhomogeneous Poisson Process via Thinning

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Input: Intensity function $\lambda(x)$, upper bound $\lambda^*(x)$, domain D
Generate $N \sim \text{Pois}(\Lambda(D))$
Initialize $\Pi \leftarrow \emptyset$
for $i = 1$ to N **do**
    Generate $X_i \sim \dfrac{\lambda(x)}{\Lambda(D)}$
    Generate $U \sim U[0, 1]$
    if $U < \dfrac{\lambda^*(X_i)}{\lambda(X_i)}$ **then**
        $\Pi \leftarrow \Pi \cup \{X_i\}$
    end if
end for
Output: Π (set of accepted points)

In the previous example $\lambda(x) = \frac{x}{50} + \frac{3x^2}{100}$ on $[0, 15]$. This is an increasing function so its maximum is achieved at $x = 15$, and we have $\lambda(15) = 7.05$.

This means that we could start with a Poisson process with intensity $\tilde{\lambda} = 7.05 \geq \lambda(x)$ and apply the thinning method.

$$\tilde{\Lambda}(D) = \int_0^{15} \tilde{\lambda}(x)dx = 7.05 \cdot 15 = 105.75$$

Example: Generate one sample corresponding to a Poisson process with intensity $\lambda(x_1, x_2) = 500x_1$ on the rectangle $D = [0, 2] \times [0, 1]$

$$\Lambda(D) = \int_0^1 \int_0^2 500x_1 dx_1 dx_2 = 1000$$

This means that we will have to generate points on $[0, 2] \times [0, 1]$ according to the density $f(x_1, x_2) = \dfrac{\lambda(x_1, x_2)}{\Lambda(D)} = \dfrac{x_1}{2}$ on D and 0 outside of D.

From here we can see that $f(x_1, x_2)$ can be written as the product of $f_1(x_1) = x_1/2$ and $f_2(x_2) = 1$. Notice that f_1 is a density function on $[0, 2]$ and f_2 is a density function on $[0, 1]$.

The generation of random vectors with density f can be done by **independently** generating its components according to densities f_1 and f_2 respectively

Algorithm 4 Generate Bivariate Poisson Process

Input: $\Lambda(D) = 1000$
Generate $N \sim \text{Pois}(1000)$
for $i = 1$ to N **do**
    Generate $X_1[i] \sim f_1$
    Generate $X_2[i] \sim f_2$
end for
Output: $X = (X_1, X_2)$

Remarks

- To generate random numbers according to density f_1 we can use the inverse transform method
- f_2 is the uniform distribution density on $[0, 1]$

Generate one sample corresponding to a Poisson process with intensity $\lambda(x_1, x_2) = 30(x_1^2 + x_2^2)$ on the rectangle $D = [0, 3] \times [0, 4]$

Hint: Use the thinning method