

Chapter 3 - Monte Carlo Methods

Monte Carlo applications to Statistical Inference.
Confidence Intervals.

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Definition: A confidence interval with confidence coefficient $1 - \alpha$ for a parameter θ is a random interval $[U, V] \subset \mathbb{R}$ where $U = U(X)$ and $V = V(X)$ are functions of the random sample $X = (X_1, X_2, \dots, X_n)$ such that:

$$P_{\theta}(\theta \in [U(X), V(X)]) \geq 1 - \alpha$$

for all $\theta \in \Theta$. The subscript θ on the probability P indicates that the random sample X is assumed to be distributed according to the distribution with parameter θ

Remark: For the purpose of today's class, θ is a one-dimensional parameter. Some of these ideas can also be considered in multidimensional parametric spaces.

Confidence Intervals

In many cases we can construct (analytically) confidence intervals $[U(X), V(X)]$ that satisfy the definition above.

One well known example of that is the case in which X_1, X_2, \dots, X_n are i.i.d. with distribution $\sim N(\mu, \sigma^2)$ where the variance σ^2 is known.

Let $\hat{\mu}(X) = \frac{1}{n} \sum_{i=1}^n X_i$. The confidence interval for the parameter μ given by

$$U(X) = \hat{\mu}(X) - Z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}} \quad V(X) = \hat{\mu}(X) + Z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}$$

satisfies that $P_\mu(\mu \in [U(X), V(X)]) = 1 - \alpha$.

We can do this because we know the exact distribution of our point estimator $\hat{\mu}(X)$.

This interval estimator for the mean can be extended to other situations as long as n is large. We might need to replace σ with s (the standard deviation of the sample X). That is because for large values of n , the Central Limit Theorem applies.

In other situations, for instance if the data is not normally distributed, or if n is small it might be more complicated to construct confidence intervals.

Today we will see how we can use Monte Carlo simulations to assess whether a confidence interval $[U, V]$ is appropriate.

For simplicity we will still consider a confidence interval of the mean parameter of some distribution.

Example

Example 3.38 Consider $X = (X_1, X_2, \dots, X_n)$ i.i.d. Poisson distributed with unknown parameter $\lambda \in [0.1, 1]$. Assume that $n = 10$; meaning that Central Limit Theorem approximations do not apply.

Consider the approximated confidence interval for the mean parameter λ given by:

$$U(X) = \hat{\lambda}(X) - t_{1-\alpha/2}^{(n-1)} \frac{s}{\sqrt{n}} \quad V(X) = \hat{\lambda}(X) + t_{1-\alpha/2}^{(n-1)} \frac{s}{\sqrt{n}}$$

where $t_{1-\alpha/2}^{(n-1)}$ is the percentile at level $1 - \alpha/2$ of the Student's t distribution with $n-1$ degrees of freedom, and s is the sample standard deviation. Use $\alpha = 0.05$.

Problem: Do we actually have that

$$P_\lambda(U(X) \leq \lambda \leq V(X)) \geq 0.95$$

for all $\lambda \in [0.1, 1]$?

Algorithm:

1. For fixed λ generate samples $\{X^{(j)}\}_{j=1,2,\dots,N}$ of size 10 and compute $U^{(j)}$, $V^{(j)}$ for each sample.
2. Approximate

$$P_\lambda(U(X) \leq \lambda \leq V(X)) \approx \frac{1}{N} \sum_{j=1}^N 1_{[U^{(j)}, V^{(j)}]}(\lambda)$$

3. Repeat steps 1 and 2 for as many values of λ as needed

Algorithm 1 Monte Carlo Estimation of Coverage Probability for Poisson Mean

1: **Input:** Sample size n , number of simulations N , grid $\{\lambda_i\}_{i=1}^K$
2: **for** $i = 1$ to K **do**
3: Initialize counter $k \leftarrow 0$
4: **for** $j = 1$ to N **do**
5: Generate $X_{ij1}, \dots, X_{ijn} \sim \text{Poisson}(\lambda_i)$
6: Compute sample mean $\bar{X}_{ij} = \frac{1}{n} \sum_{t=1}^n X_{ijt}$
7: Compute sample standard deviation

$$s_{ij} = \sqrt{\frac{1}{n-1} \sum_{t=1}^n (X_{ijt} - \bar{X}_{ij})^2}$$

8: Compute margin of error $d_{ij} = t_{0.975, n-1} s_{ij} / \sqrt{n}$
9: **if** $\bar{X}_{ij} - d_{ij} \leq \lambda_i \leq \bar{X}_{ij} + d_{ij}$ **then**
10: $k \leftarrow k + 1$
11: **end if**
12: **end for**
13: $P_i = k/N$
14: **end for**
15: **Output:** $\{P_i\}_{i=1}^K$

From Nested Loops to Vectorized Mathematics

Nested-loop formulation:

For each λ_i , $i = 1, \dots, K$ and each simulation $j = 1, \dots, N$:

$$X_{ij1}, \dots, X_{ijn} \sim \text{Poisson}(\lambda_i)$$

$$\bar{X}_{ij} = \frac{1}{n} \sum_{t=1}^n X_{ijt}, \quad s_{ij} = \sqrt{\frac{1}{n-1} \sum_{t=1}^n (X_{ijt} - \bar{X}_{ij})^2}$$

$$d_{ij} = s_{ij} \cdot \frac{t_{0.975, n-1}}{\sqrt{n}}, \quad I_{ij} = \begin{cases} 1, & \text{if } \bar{X}_{ij} - d_{ij} \leq \lambda_i \leq \bar{X}_{ij} + d_{ij} \\ 0, & \text{otherwise} \end{cases}$$

$$P_i = \frac{1}{N} \sum_{j=1}^N I_{ij}$$

Matrix/vectorized representation:

$$\mathbf{X} = [X_{ijt}] \in \mathbb{R}^{K \times N \times n}, \quad X_{ijt} \sim \text{Poisson}(\lambda_i)$$

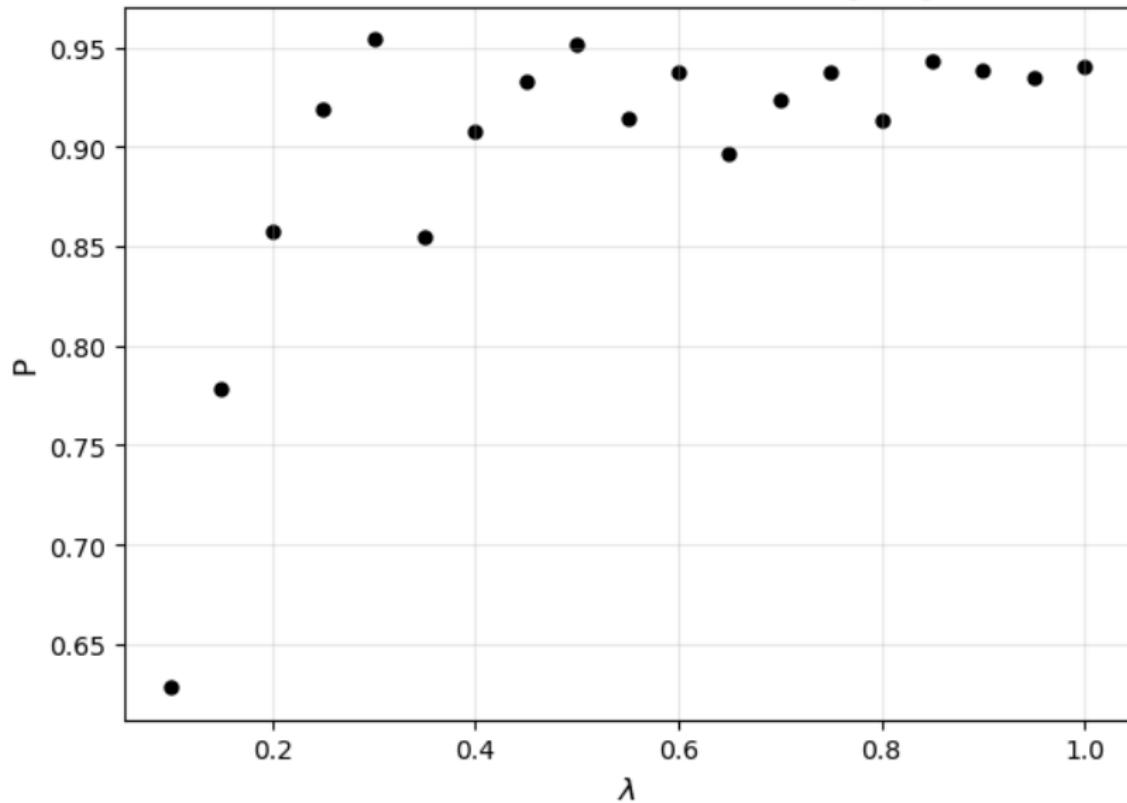
$$\bar{\mathbf{X}} = \text{mean}(\mathbf{X}, \text{axis} = 3), \quad \mathbf{S} = \text{sd}(\mathbf{X}, \text{axis} = 3)$$

$$\mathbf{D} = \mathbf{S} \cdot \frac{t_{0.975, n-1}}{\sqrt{n}}, \quad \boldsymbol{\Lambda} = [\lambda_i]_{i=1}^K \in \mathbb{R}^{K \times 1}$$

$$I_{ij} = \mathbb{1}\{\bar{X}_{ij} - d_{ij} \leq \lambda_i \leq \bar{X}_{ij} + d_{ij}\}$$

$$P_i = \frac{1}{N} \sum_{j=1}^N I_{ij}$$

Confidence Coefficients for $\lambda \in [0, 1]$



- We can see that for small values of λ , the confidence interval will contain the actual value of λ much less frequently than the desired 95% of the time. This indicates that proper 95% confidence intervals must be larger.
- A more detailed analysis would be needed in order to decide how to much increase should we increase that interval.
- Not necessarily the confidence interval has to be symmetric around the point estimator.

Perform an analysis similar to what we did in **Example 3.38** but now using samples of size $n = 8$ from the exponential distribution with mean $\mu \in [5, 10]$. Do you think that U and V as defined in **Example 3.38** can be used as an appropriate 95% confidence interval for $\mu \in [5, 10]$?

Reminder: Random variables with exponential distributions with intensity λ have expected value $\mu = 1/\lambda$.