Homework: Markov Chains

To simulate the next state in a Markov Chain using a uniform random variable:

1. Suppose the current state is i, with transition probabilities

$$P(i \to 1) = p_1, \quad P(i \to 2) = p_2, \quad \dots, \quad P(i \to m) = p_m,$$

   where $\sum_{j=1}^{m} p_j = 1$.

2. Generate $U \sim \text{Uniform}(0, 1)$.

3. Construct the cumulative probabilities:

$$\text{cum\_probs} = \big(p_1,\ p_1 + p_2,\ \dots,\ p_1 + p_2 + \cdots + p_m\big).$$

4. Determine the next state by checking which interval U falls into:
   - If $0 \le U < p_1$, choose state 1.
   - If $p_1 \le U < p_1 + p_2$, choose state 2.
   - Continue similarly until all states are covered.

This works because the transition probabilities partition the unit interval $[0, 1]$ into disjoint segments, and U selects one segment according to the correct probabilities.

Python Code

```python
import numpy as np
import matplotlib.pyplot as plt

n = 40
P = np.array([[0.2, 0.5, 0.3],
              [0.1, 0.3, 0.6],
              [0.5, 0.1, 0.4]])
X = np.zeros(n+1, dtype=int)
U = np.random.rand()
X[0] = 2 #start at state 2

for i in range(n):
    u = np.random.uniform(0, 1)    # uniform random number
    p1, p2, p3 = P[X[i]-1]         # transition probabilities

    if u < p1:
        X[i+1] = 1
    elif u < p1 + p2:
        X[i+1] = 2
    else:
        X[i+1] = 3

# plot path
plt.step(range(n+1), X, where='post')
plt.xlabel("Time step")
plt.ylabel("State")
plt.title("Markov Chain Path (3 states, n=40)")
plt.show()
```

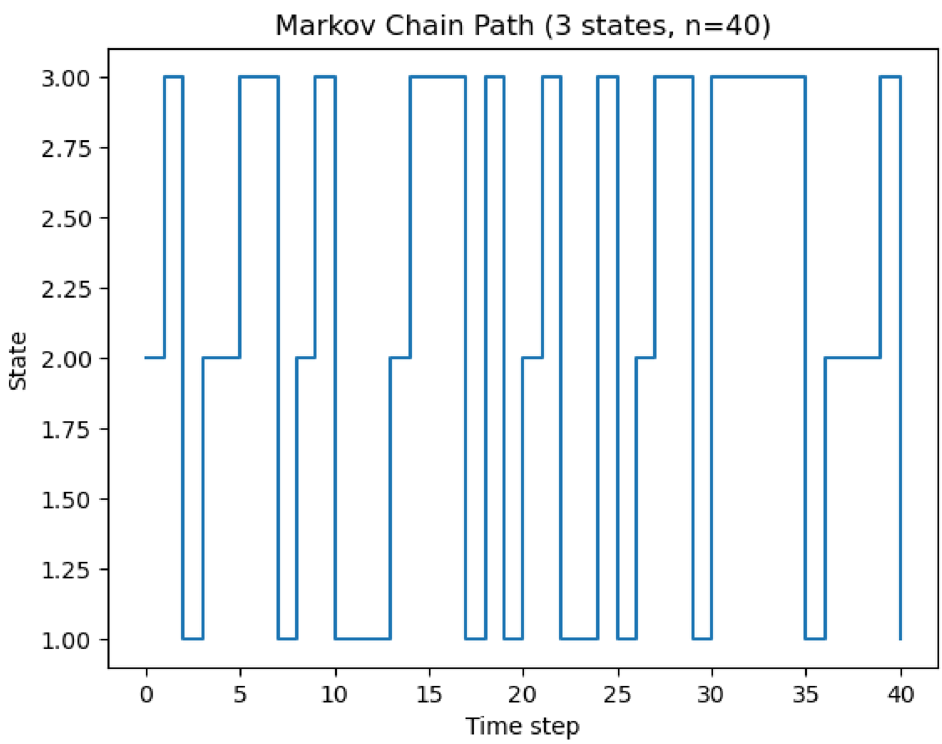

Figure 1: A simulated path of the Markov chain with $n = 40$.