Chapter 3 - Monte Carlo Methods

Introduction to Monte Carlo Methods.

Prof. Alex Alvarez, Ali Raisolsadat

School of Mathematical and Computational Sciences
University of Prince Edward Island

Introduction to Monte Carlo Methods

Monte Carlo methods are a general class of computational algorithms that generate random scenarios to obtain numerical results.

The method was originally developed by Stanislaw Ulam and John von Neumann at the Los Alamos Laboratory, to solve some physical problems that were difficult to solve analytically.

Monte Carlo methods can be used to solve a variety of mathematical problems. **In its most basic form, we use it as a numerical integration method** for integrals with no explicit solution. In this course we will also explore other uses of Monte Carlo methods.

Computing Expected Values

Let $X = (X_1, X_2, \ldots, X_d)$ be a random vector with joint density $f(x_1, x_2, \ldots, x_d)$ and $g : \mathbb{R}^d \to \mathbb{R}$. We want to compute:

$$Eg(X) = \int_{\mathbb{R}^d} g(x)f(x)dx$$

If the integral on the right hand side cannot be solved explicitly, we are only left with numerical methods to get an approximated value of that integral.

To simplify the presentation we will assume next that we are in the one-dimensional case ($d = 1$) but most results apply also in the multidimensional case.

The Law of Large Numbers

The main theoretical justification to use Monte Carlo methods is the Law of large numbers:

Theorem: Let be (X_n) independent and identically distributed(i.i.d.) sequence of random variables with $E(X_n) = \mu$, $Var(X_n) = \sigma^2$ (both finite). Then we have:

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \to \mu$$

with probability 1.

Consequences

If $X_1, X_2, \ldots, X_n$ is a simulated random sample with the same probability law as X

$$\frac{1}{n}\sum_{j=1}^{n} g(X_j) \approx E(g(X)) = \int_{\mathbb{R}} g(x)f(x)dx$$

provided n is large.

In particular if $U_1, U_2, \ldots, U_n$ are independent and uniformly distributed on $[0, 1]$ then:

$$\frac{1}{n}\sum_{j=1}^{n} g(U_j) \approx E(g(U)) = \int_{0}^{1} g(x)dx$$

Monte Carlo Integration Example

Example: Compute by Monte Carlo integration $\int_0^{\pi} \sin x dx$.

Solution: Make the change of variable $y = \frac{1}{\pi} x$, then:

$$\int_0^{\pi} \sin x dx = \pi \int_0^1 \sin(\pi y) dy$$

Algorithm 1 Monte Carlo Estimation of $\pi \int_0^1 \sin(\pi x)\, dx$

1: **Input:** Number of samples n
2: Generate independent random variables $U_1, U_2, \ldots, U_n \sim \text{Uniform}(0, 1)$
3: Compute the empirical mean:

$$\widehat{I}_n = \pi \frac{1}{n} \sum_{j=1}^{n} \sin(\pi U_j)$$

4: **Output:** Monte Carlo estimate $\widehat{I}_n$ of the integral

Monte Carlo Integration Example

Example: Compute by Monte Carlo integration

$$\int_0^1 \int_0^1 \sqrt{x+y}\,e^{xy}\,dxdy$$

Algorithm 2 Monte Carlo Estimator for $\mathbb{E}\big[\sqrt{U+V}\,e^{UV}\big]$

1: **Input:** Number of samples n

2: **Generate** $U_1, U_2, \ldots, U_n \sim \text{Uniform}(0,1)$

3: **Generate** $V_1, V_2, \ldots, V_n \sim \text{Uniform}(0,1)$

4: **Compute:**

$$\text{Empirical Mean} = \frac{1}{n}\sum_{j=1}^{n} \sqrt{U_j + V_j}\,e^{U_j V_j}$$

5: **Output:** Estimated value of $\mathbb{E}\big[\sqrt{U+V}\,e^{UV}\big]$

Homework

1. Compute using a Monte Carlo method $E\left[Z^{2/3}(|Z|+1)\right]$ where Z is a standard normal random variable.
2. Compute by Monte Carlo integration

$$\int_0^1 \int_0^2 (x+2y)\ln(3x+2y+1)dxdy$$

Hint: First make a change of variable so that the region of integration is the square $[0,1]\times[0,1]$