

Chapter 2 - Simulating Statistical Models

Markov Chains on a Continuous State Space.

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Finite state space: So far we considered Markov chains with

$$S = \{1, 2, \dots, M\}$$

Beyond finite: Markov chains can also take values in more general spaces, such as

$$S = \mathbb{R}^d$$

In this case, instead of a transition **matrix**, we will need a transition **density**.

A **transition density** is a function

$$p : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$$

such that:

1. $p(x, y) \geq 0$ for all $x, y \in \mathbb{R}^d$.
2. $\int_{\mathbb{R}^d} p(x, y) dy = 1 \quad \forall x \in \mathbb{R}^d$.

The interpretation of this definition is that x plays the role of the current state of the Markov Chain, so for a fixed value of x , the function $g(y) = p(x, y)$ is the density function of the random variable Y representing the next state of the Markov Chain.

Example 2.29 from the Textbook: Let $X_0 = 0$ and define

$$X_j = \frac{1}{2}X_{j-1} + \varepsilon_j, \quad \varepsilon_j \sim N(0, 1) \text{ i.i.d.}$$

Observation: The sequence X_0, X_1, X_2, \dots is a Markov chain with state space $S = \mathbb{R}$.

Conditional law: Given $X_{j-1} = x$, we have

$$X_j = \frac{x}{2} + \varepsilon_j \sim N\left(\frac{x}{2}, 1\right)$$

Transition density:

$$p(x, y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(y - x/2)^2\right), \quad x, y \in \mathbb{R}$$

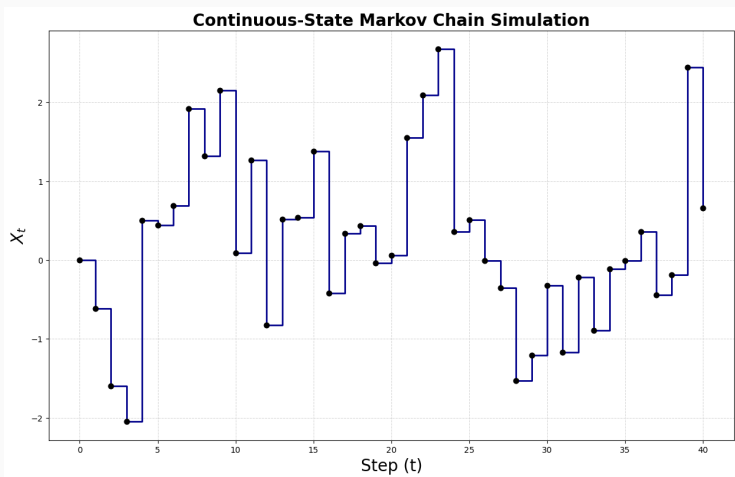
Algorithm 1 Simulating a Markov Chain Path

- 1: Generate X_0 according to the initial distribution
- 2: **for** $i = 1$ to $n - 1$ **do**
- 3: Generate $X_i \in S$ according to the density

$$g(y) = p(X_{i-1}, y)$$

- 4: **end for**
 - 5: **return** (X_0, X_1, \dots, X_n)
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Example: Markov Chain Path



Simulated trajectory of a continuous state space Markov Chain with $n = 40$ with

$$\text{transition density } p(x, y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} (y - x/2)^2\right), \quad x, y \in \mathbb{R}.$$

Example: Uniform Transition Density

Example (from Homework): Let $X_0 = 0$ and define the transition density

$$p(x, y) = \frac{1}{2} \mathbb{1}_{[x-1, x+1]}(y), \quad x, y \in \mathbb{R}$$

Note: The indicator function $\mathbb{1}_A$ is defined as

$$\mathbb{1}_A(y) = \begin{cases} 1, & \text{if } y \in A, \\ 0, & \text{if } y \notin A. \end{cases}$$

Observation: The sequence X_0, X_1, X_2, \dots is a Markov chain with state space $S = \mathbb{R}$.

Conditional law: Given $X_{j-1} = x$, we have

$$X_j \sim \text{Uniform}(x - 1, x + 1)$$

Transition density:

$$p(x, y) = \begin{cases} \frac{1}{2}, & y \in [x - 1, x + 1], \\ 0, & \text{otherwise.} \end{cases}$$

For Markov chains with a continuous state space, we also have the notion of a **stationary distribution**.

A probability density $\pi : \mathbb{R}^d \rightarrow [0, \infty)$ is called a **stationary density** for a Markov chain with transition density p if it satisfies

$$\int_{\mathbb{R}^d} \pi(x) p(x, y) dx = \pi(y), \quad \forall y \in \mathbb{R}^d.$$

Intuition: A stationary density π is an **equilibrium law** for the Markov chain:

- If $X_n \sim \pi$, then $X_{n+1} \sim \pi$ as well.
- The distribution is **invariant** under the dynamics of the chain.
- In the long run, many Markov chains converge to their stationary distribution, regardless of the starting point.

Example (Gaussian AR(1) Chain):

$$X_j = \frac{1}{2}X_{j-1} + \varepsilon_j, \quad \varepsilon_j \sim N(0, 1).$$

- This chain has a stationary distribution:

$$\pi \sim N\left(0, \frac{1}{1-(1/2)^2}\right) = N\left(0, \frac{4}{3}\right).$$

- **Interpretation:** After many steps, the state X_n is approximately $N(0, 4/3)$, no matter the initial X_0 .

- Generate Markov chain paths using the **Gaussian transition density** (Algorithm 1) for 40 time steps.
- Write an algorithm for simulating a Markov chain with the **Uniform transition density**

$$p(x, y) = \frac{1}{2} \mathbb{1}_{[x-1, x+1]}(y), \quad x, y \in \mathbb{R},$$

and implement code to generate a path of length 40.

- **AR(1) problem:** Consider

$$X_j = \phi X_{j-1} + \varepsilon_j, \quad \varepsilon_j \sim N(0, \sigma^2) \text{ i.i.d.}, \quad |\phi| < 1.$$

1. Show that $\{X_j\}$ defines a Markov chain with transition density

$$p(x, y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y - \phi x)^2}{2\sigma^2}\right)$$

2. Prove that if a stationary distribution exists, it must be Gaussian with mean 0.
3. **Hint** For an AR(1) process, the stationary variance satisfies $\text{Var}(X) = \frac{\sigma^2}{1-\phi^2}$
4. Specialize to the case $\phi = \frac{1}{2}$, $\sigma^2 = 1$. Find the stationary distribution and its standard deviation.