Stat-4110-Statistical Simulation-Winter 2024
Assignment 1

Instructions:

- Provide **fully justified** answers for the problems in the assignment. You must include any theoretical work, the code, and outcomes of the code (preferably in the form of histograms, plots, etc.).
- Submit your written/typed answers by **Friday February 16th at 9:00 pm**
- Except for exceptional circumstances, there will be a penalty (20% for each extra day) if the assignment is submitted after the deadline.
- Your submitted answers must reflect your own work. Please refer to the course outline regarding the use of large language models such as Chat GPT in this course.

Problems

1. [10 marks] Consider a random variable X taking values $i = 1, 2, ...$ according to the probability mass function

$$p(X = i) = \frac{1}{\sqrt[3]{i}} - \frac{1}{\sqrt[3]{i+1}}$$

   Write a computer program to generate a random sample of size $n = 50$ according to that distribution.

2. [10 marks] Consider a random variable X with density $f(x) = C \cdot 2^{-x}$ on $[1, \infty)$ and 0 otherwise, where $C \in \mathbb{R}$ is a constant.

   a) Find the value of C.
   b) Write a computer program that uses the inverse transform method to generate a random sample of size $n = 1000$ with the same distribution as X.
   c) Provide a histogram of the random numbers generated with your program.

3. [10 marks] Consider a random variable X with density $f(x) = \dfrac{Cx}{1 + x^{5/2}}$ on $[0, 2]$ and 0 otherwise, where $C \approx 1.313$

   a) Write a computer program that uses a rejection algorithm to generate a random sample of size $n = 1000$ with the same distribution as X.

   b) Plot the density function $f(x)$ as well as a histogram of the random numbers generated with your program (the histogram should have a shape similar to the function).

c) Find the theoretical efficiency of your algorithm, that is how many "proposals" (on average) are needed to generate a single random number with the same distribution as X?.

d) Compare your result in part b) with the empirical outcomes from your program. In other words, when you run your program, how many proposals do you need in total to generate the sample of size $n = 1000$ with the same distribution as X?

4. [10 marks] Write a computer program to generate a sample of 1000 random numbers from the mixture distribution of the three distributions:
Distribution 1: $N(-2, 3)$
Distribution 2: $N(6, 4)$ and
Distribution 3: $N(10, 9)$ with weights $\omega_1 = 0.3$, $\omega_2 = 0.2$ and $\omega_3 = 0.5$ respectively.

   Create a histogram of the corresponding sample.

   Restriction: For the solution of this problem you are **not** allowed to use built-in functions to generate normal random numbers (for example the function "rnorm" in **R**). For this problem you can use functions that generate uniformly distributed random numbers (such as the function "runif" in **R**).

5. [10 marks] Consider a Poisson process with intensity $\lambda(x) = 40\sin(2x)$ on the interval $D = [0, \pi/2]$ and 0 otherwise.

   a) How many points are expected to be in a random sample from this process?

   b) Write a computer program to generate one sample corresponding to this Poisson process, and provide some plot(s) or other outcome(s) of this program.

   Penalty: Part b) can be solved using either an efficient algorithm that generates the random sample directly, or some variation of a more inefficient rejection-type algorithm (such as the thinning method studied in classes). In order to earn full marks in this question, it is required that you do not use a rejection-type algorithm. This means that you will be penalized if you use some rejection algorithm.