Black-Scholes Model

Ali Raisolsadat

AMS 3910 - Final Project

1 Introduction

1.1 Financial Mathematics

It all started when bartering first introduced 6000 BC in Mesopotamia tribes. People saw that they can not produce all their needs in life, therefore they started to exchange what they could offer with things they needed. For example, if a hunter required to get more wood for arrows, then he would go to a lumberjack, and in exchange of more wood, he would offer part of his hunt to the lumberjack. Since the bartering system was hard and time consuming, around 5000 BC people started to use precious objects (such as salt, pelts, wheat, marbles, etc.) and metals (such as gold and silver) as a medium of exchange. For an object to be a medium exchange it must constitute some value which is accepted by two parties who are engaged in bartering. By 700 BC the Lydians were the first people to create currency which today people refer to it as “money”. Many centuries later, the first bank was created by money and land lenders in Florence, Italy. The first form of accounting by using the notion of zero and decimal places, which were introduced by Fibonacci in Europe and Muhammad ibn Musa al-Khwarizmi in Middle East, was used by the Italian bankers. Soon these banks became larger and larger, and their system of lending, borrowing, accounting and record keeping became more complex. These bankers started to seek mathematicians and statisticians so that they could solve their complex problems. The reason for their interest of solving these complex problems was to make the most profit among other banks and companies. They would even hire street gamblers so that they could help them to lower the risk of their investments. As years went by banking problems became more and more complex. Corporations decided to create a new type of bartering called investment. East Indian company was the first company to issue a stock to shareholders, the first type of investment, and as volume of trading shares increased the stock traders would gather in London coffee house to trade stock with each other. “Eventually, they took over the coffee-house and, in 1773, changed its name to the stock exchange. Thus, the first exchange, the London Stock Exchange, was founded” [1]. This is were when finances and mathematics started to combine with each other, and by 1900 the first financial mathematician Louis Bachelier, introduced the concept of stochastic processes, which is widely used in today’s financial world. Mathematical finance uses most branches of mathematics and statistics, such as linear algebra, partial differential equations, probability distributions and stochastic processes (Brownian motion) to solve financial problems faced by investors and corporations. This subject did not become popular until 1970's when Fischer Black, Myron Scholes and Robert Merton introduced the world a partial differential equation which would help corporations, banks and investors to price different market derivatives. They called this PDE Black-Scholes equation,later known as Black-Scholes-Merton equation, which would be the focus of this paper.

1.2 Derivatives Market

The Black-Scholes model is focused on financial market derivatives. Before defining what a derivative is, let us look at some definitions. Many people around the world go to work everyday, and after putting many hours they receive an amount of money as a wage (or salary), compensating them for their spent hours at work. This money that people receive is an asset. They use this money to purchase houses, cars, laptops, and other instruments. All these instruments are assets, because they have a worth or value. Therefore, we can say that an **asset** is something that has a value, and an owner that benefits from this value. A **stock share** refers to ownership certificate of a certain company that has a previously assigned value, and since a stock share has a value, then we can say that a stock is an asset. A house that a person lives in is also an asset, since the house has a value, and it provides shelter for the owner. Same can be said about insurance. People tend to think that insurance is an expense, but it is in fact an asset since it has a value (continuous paid premium) and it protects the owner from any incidents. **Financial securities** are financial instruments such as stock, mutual funds, options (we will cover the definition of options in further chapters), futures, etc which are an asset to the owner. Now that we looked at what is an asset, let us look at what a derivative is.

A **financial derivative** is a financial security that its value depends on another asset and is a reached contract between two or more parties over that asset. Or as [1] says "A derivative is a financial instrument that has a value determined by the price of something else". This asset can also be called an **underlying asset**. The most common underlying assets are government bonds and securities, currencies, indexes (a portion of stock shares) and individual stock shares, and commodities. **Derivative market** is a financial market (like stock market) that focuses on selling and buying derivatives. These derivatives come in different forms. The most famous derivatives are options and futures.

2 Financial Positions

Financial and derivative markets are complex mediums. Therefore, people who work, or do any kind of business through these markets have come up with certain vocabulary which can sometimes be confusing. These certain words and expressions describe a position of an investor regarding their **portfolios**, a set or combination of assets that investor hold in financial markets. This chapter quickly covers two notions that are widely used in financial world and are important to understand for this model. The goal of a regular investor in a time frame of $t \in [0, T]$, is to make profit from a portfolio. If not, then minimize the amount of the losses (this also refers to risk management). Therefore, an investor will sell any asset, if the price of that asset increases by time T, or will buy more assets if the price of that asset is low at time T where the investor believes that after time t the price of that asset will increase. **Long position** refers to a position where the investor is buying an asset or a financial derivative at $t = 0$, which whom he thinks will rise in value by time T. **Short position** refers to a position where the investor sells an asset at $t = 0$ where he believes the price of that asset will fall in near future, and buying the asset with a lower price in future time t. **Payoff** of a portfolio refer to the amount of profit/loss at time T. **Premium** of a financial derivative refers to an initial price of that derivative in the market. Pricing different assets and financial derivatives have been developed drastically since 1900's by introducing Brownian Motion and Stochastic calculus. Many have proposed different mathematical models to set and initial price on assets and financial derivatives. Since derivatives can have really high profits, therefore investors by using mathematical asset pricing models needed to set prices for these derivatives. The whole purpose of Black-Scholes model was to be able give an initial premium (price) to options, which we will see further in chapters on how Fischer Black and Myron Scholes managed to accomplish this goal.

3 Forward Contract

We first look at a simple financial derivative. This simple derivative will familiarize us with derivative markets, and will review different notions in finance. For time $t \in [0, T]$, a **forward contract** or simply a **forward** is a simple contract between two parties where they both agree upon a price at time $t = 0$ for either buying or selling an asset, and they would execute the contract at time T. This contract is on an underlying asset. The party who is willing to buy the underlying asset at time T is in a long position of the forward contract, and the party willing to sell the underlying asset at time T is in a short position of this contract. Now if an investor is in a long position then they would prefer for the asset price to go higher than what both parties agreed upon. This is because the underlying asset will be sold in a lower price than what it is in the market, therefore there will be a profit for the investor. The party in the short position would prefer that the price of the asset to decrease, since they will be paid more in the forward contract than what the price of the underlying asset is in the market.

For time $t \in [0, T]$, we denote F_T as the payoff for the forward contract, S_T

as the price of the asset at time T, and K as the price that both parties agreed upon then we will have the following payoff functions:
Payoff for long position in forward contract:

$$F_T = S_T - K$$

Payoff for short position in forward contract:

$$F_T = K - S_T$$

Please note that this derivative is a contract. This means that both parties are obligated to execute the contract at time T. The following are the payoff diagrams for both long and short positions.

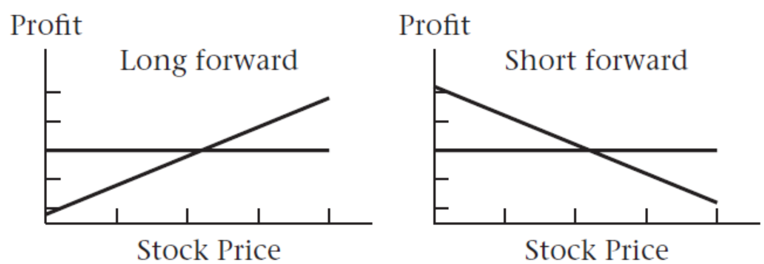

Figure 1: Profit diagrams for both long and short positions[6]

Example 3.1: Suppose that Alex is looking to buy a house in a year. Fortunately for him William has a house worth of \$200000, and is willing to sell it to him in one year. William and Alex decide on a fair price of \$215000, so they both sign a forward contract for one year ($T = 1$). After one year, a realtor checks out William's house and puts a price of \$220000 on the house. Since Alex was in a long position of this forward contract, his payoff will be

$$F_T = S_T - K = 220000 - 215000 = \$5000$$

Therefore he will be paying \$5000 less than the real price of the house in the market. He can save this money and invest it in the house. On the other hand William is in a Short position, so his payoff will be

$$F_T = K - S_T = 215000 - 220000 = -\$5000$$

Therefore William will be losing money in this contract, since the price of the market is higher than what was in the forward contract. Now there was a recession and land values dropped in this one year period. The realtor sets the price of \$190000 for the house according to the market regulations. In this case Alex's payoff will be

$$F_T = S_T - K = 190000 - 215000 = -\$25000$$

and William's payoff will be

$$F_T = K - S_T = 215000 - 190000 = \$25000$$

Therefore Alex will need to pay an extra \$25000 for the house that is worth less hence losing substantial amount of money he could have saved, and William will make \$25000 extra since the forward contract price is higher than the market price.

4 Options

4.1 Introduction to Options

Let us look at another type of derivative that is available to investors and financial institutions. These derivatives can take more than one form, requiring more complex forms of Black-Scholes model, or other complex models. We will look at two basic forms of this derivative. Before continuing, we need to distinguish between payoff and profit/loss. In contract forward, there are no premiums, therefore the payoff is the same as profit/loss. But we will see that these derivatives have an initial price (which we will look at how to find them). Therefore there the profit/loss will be different than the payoff of the derivative.

We looked at a forward contract, which was a contact between two parties agreeing on a price of an underlying asset at $t = 0$ and executing the contract at $t = T$. In these contracts, both parties are **obligated** to follow through with the contract. But what if that was not necessary. What if an investor had the option to either sell or buy an underlying asset when $t = T$? This is where option derivatives come into play. **Options** are financial derivatives that offers an investor the right, but not the obligation, to buy or sell an underlying asset (stock, house, car, insurance, etc.) at a specific price set at an initial time, for a period. This agreed price is called **strike price** and we denote it as **K**. For example, if Alex signs an option contract on William's house with strike price of K at time $t = 0$, then he has the right to buy William's house at time $t = T$,which in this case is one year, but is not obligated to do so. If he decides not to, then he can either sell the option contract to someone else or lose the premium in the process. Therefore we can say that Alex can exercise this option at maturity. **Exercising** an option means to follow through with an option contract at a specified time T mentioned in the contract. We will get more familiar with this notion further in the chapters. **Maturity** time in finance refers to the last payment of a financial instrument.

4.2 Option Styles

There are many different styles of options. In this section we will cover three different types of options. The first and the most simple style of options are **European Options** with an initial premium (price), and can be exercised only at time of maturity. That means, after an investor signs this contract over an underlying asset then after one year ($T = 1$) this investor has the option to either sell or buy an underlying asset, but is not obligated to do so, therefore they may have a payoff of zero. The following chapter goes into detail on two different European options, and different positions an investor can take using these options. A more complicated type of options are **Bermudan Options**, where the owner of this contract have the option to exercise in at a set of number of times (this is always discrete) until maturity time. This means that investor holding this type of contract has the right to exercise this contract until the end of specified time for the option contract. The most complicated options are **American Options**. If we have time $t \in [0, T]$ then an investor holding an American option has the right to exercise any time between $t = 0$ and $t = T$ (continuous time). These options have higher premium than European due to the freedom that provides for the investor. We are only going to focus on European options because different styles of options are very difficult to price using Black-Scholes model and they require to modify this model. This paper is only an introduction to Black-Scholes model.

4.3 European Call Option

We already discussed European options. They give the owner the right, but not the obligation, to either buy or sell an underlying asset at maturity time T, where time $t \in [0, T]$.

Now we look at a European call option. **European Call Options** give the owner the right to buy an underlying asset based on the agreed strike price K at the maturity (expiry) time T. As we discussed before there are two positions when it comes to do investment in derivatives market. We applied these two positions, long and short, to a forward contract. We can also apply these two positions to our call option. An investor can either **long/buy/purchase** or **short/sell/write** a call option. These two positions will result in different payoffs (and profit/loss) just like we saw in forward contract.

Since the call option is over an underlying asset then if the investor long/buy(ing) this option, will need to pay attention to the price of the asset and what is the strike price K is, because call options only allows the investor to exercise the option expiry time T. This means that the payoff for this option can either be zero, when the price of the asset at time T is lower than the strike K leading to the investor not to exercise the option giving a payoff 0, or unlimited profit payoff, when the price of the asset is much higher than the strike K, leading the investor to exercise the option giving a payoff Asset Price at time $T-$ strike price. So we would have the following payoff function:

$$\Lambda C(S_T, K) = max(S_T - K, 0) = (S_T - K)^+$$

where $t \in [0, T]$.

If investor short/sells a call option, investor can have a payoff zero when strike price is higher than the underlying asset price at expiry time T, and unlimited loss when the price of the asset is higher than the strike price. Therefore we would have the following payoff function:

$$\Lambda C(S_T, K) = -max(S_T - K, 0) = -(S_T - K)^+$$

where $t \in [0, T]$.
Now let us look at the payoff diagram of both long and short positions of a European call option. They are as following:

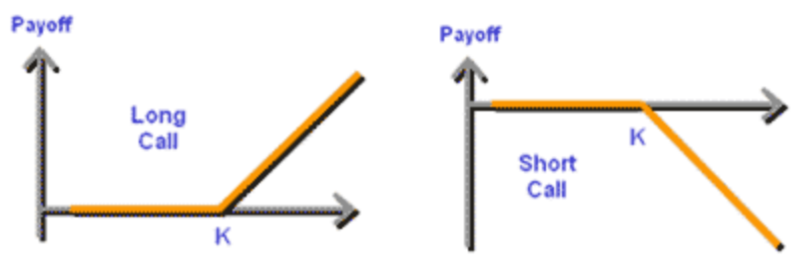

Figure 2: Payoff diagrams for both long and short positions

Now at this point, the reader may ask that why would anyone sell a call option (take a short position on a call option), since there can be either no payoff or unlimited amount of loss? This is where the idea of premium or the initial price of the call comes into play. When an investor purchases a call option (long position) then they will need to pay some amount of money initially since he can have the possibility of unlimited profit. Therefore there is a possibility of loss for the investor in a long position. When an investor writes a call option then he will charge a fair price for the call option because that investor is providing a derivative to another investor. This means means that if the the strike price stays higher than the asset price at maturity time, then there will be some amount of profit for the writer's position (initial price of the call option). Let us look at the profit/loss diagram of both long and short positions of a European call option. They are as following:

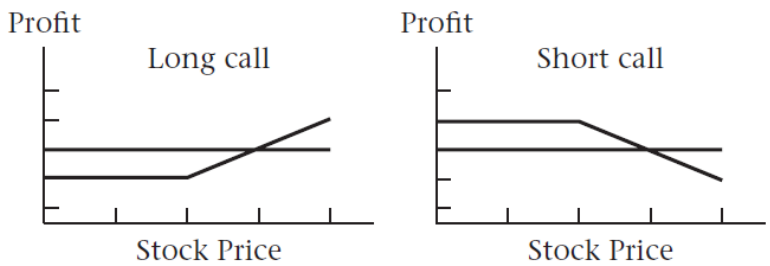

Figure 3: Profit diagrams for both long and short positions[6]

As we mentioned the premium is a fair price that an investor needs to calculate. Black-Scholes model focuses on how to find this fair initial price for European options.

4.4 European Put Option

A **European Put Option** or simply a put is a contract that gives the owner the right to sell an underlying asset (based on the agreed strike price K) at expiry time T. Just like European call option there are two positins for a put option; long put and short put.

When an investor has a long position in a put option, then he can have a positive payoff when the underlying asset price at time T is lower than the agreed strike price K, and payoff of zero when the asset price is the same as the strike price. In this case the investor does not exercise the option

When an investor has a short position in a put option, then he can have a negative payoff when the underlying asset price at time T is lower than the agreed strike price K, and payoff of zero when the asset price is the same as the strike price. This will result in the following payoff functions:
Long put payoff function:

$$\Lambda P(S_T, K) = max(K - S_T, 0) = (K, S_T)^+$$

Short put payoff function:

$$\Lambda P(S_T, K) = -max(K - S_T, 0) = -(K, S_T)^+$$

where $t \in [0, T]$.

Now let us look at the payoff diagram of both long and short positions of a European call option. They are as following:

Figure 4: Payoff diagrams for both long and short positions

Put options are different than call options. They do not provide unlimited payoff, but they provide an insurance for the seller, because it ensures that if the price of an asset falls really low then an investor does not lose more than the price of the asset. This is why there is a premium for long put position. On the other hand, a short position in put option may seem illogical, specially looking at the payoff diagram, but if an investor is "sure" that the price of the asset is going to be more than the strike price at expiry time then he can gain the amount of the premium by coosing this into this position. Let us look at the profit/loss diagrams for both long and short positions in a put option. Please note that profit/loss diagrams an initial price for the derivative just as we saw previously in the call option.

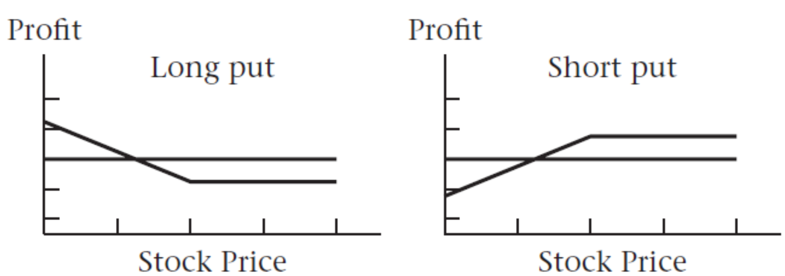

Figure 5: Profit diagrams for both long and short positions[6]

Example 4.4.1: Consider an Apple.Inc stock with an initial price \$205. An investor purchases a European call option over this stock with strike price of \$208, and maturity in one year ($T = 1$). After one year the price of the stock is \$210. What is the investor's payoff? Would the investor exercise this call option?

$$C(S_T, K) = max(210 - 208, 0) = \$2$$

where $T = 1, K = 208, S_T = 210$. In this case the investor will exercise the option and receive a payoff of \$2.

What if the price of the stock falls down to \$200? What will be the payoff in that case? Would investor exercise the option?

$$C(S_T, K) = max(200 - 208, 0) = \$0$$

where $T = 1, K = 208, S_T = 200$. In this case the investor will not exercise the option, and will only lose the premium amount on the call option.

Example 4.4.2: Let the call option above have an initial premium of \$5. What would be the investor's profit/loss if after one year, the stock price is \$215? What about \$200?

$$C(S_T, K) = max(220 - 208, 0) - 5 = \$7$$

$$C(S_T, K) = max(200 - 208, 0) - 5 = -\$5$$

where $T = 1$ and $K = 208$. We see in these two cases, that if the stock is higher than the strike price, then the investor will exercise the option and receive a profit of \$7. On the other hand, if the stock price is lower than K price then the investor will not exercise the option and will only lose the initial amount of the call option.

Example 4.4.3: Consider an investor sells a put option over the same stock as above for \$10. What will be this investor's profit/loss if the price of the stock moves up to \$220? What if it moves down to \$190?

$$P(S_T, K) = -(max(208 - 220, 0) - 10) = \$10$$

$$P(S_T, K) = -(max(208 - 190, 0) - 10) = -\$8$$

where $t \in [0, T]$ and $K = 208$. In the first case the investor will since the stock price did not decrease, therefore the option payoff is zero but since there is a premium the investor will be receiving the when he sells the option.

5 Brownian Motion

In the following sections we will look at different subjects that are commonly used in financial mathematics. We need to review these sections so that we can fully understand the Black-Scholes model. Please note that these sections are not explained in mathematical depth. There are various books written about each section in a very technical detail, that are listed in the references. We first start by reviewing stochastic processes. Stochastic process are one of the important foundations of asset pricing in financial world. After reviewing stochastic processes, we will review two examples of stochastic processes. First example will be random walk, and after introducing random walk, we will discuss Brownian motion and its properties. This will help us to glimpse over Itos process and lemma, which will help us to derive the Black-Scholes Partial differential equation.

5.1 Stochastic Process

In financial markets the price of an asset is constantly changing through time. This constant change in an asset price is the result of economical and political movements in a country. It is hard to create a model that takes all these decisions into consideration, therefore mathematicians use random variables to consider all these constant changes. The concept of stochastic processes was made based on how these random variables change after a specific period of time. To review concept of stochastic process in detail can be very long and rigorous. We do not need the full knowledge of random process. We only need to know the notion so that we can understand Brownian motion better.

A stochastic or random process is a mathematical object usually defined as a collection of random variables. Historically, the random variables were associated with or indexed by a set of numbers, usually viewed as points in time, giving the interpretation of a stochastic process representing numerical values of some system randomly changing over time, such as the growth of a bacterial population, an electrical current fluctuating due to thermal noise, or the movement of a gas molecule. Random changes in financial markets have motivated the extensive use of stochastic processes in finance.This is because of randomness of an asset price in financial market.[3]

We can define a stochastic process as the following[2]:
Let set $\Omega = \{\omega_1, \omega_2, \omega_2, ..., \omega_T\}$ be the possible future possible scenarios for an asset price, where time $t \in [0, T]$. A random process (or a stochastic process) is a collection of random variables that are all defined on a common sample space Ω and indexed by $t \in T$:

$$\{X_t(\omega)\}_{t \in T}$$

where

$$\forall t \in T, X_t : \Omega \to \mathbb{R}$$

5.2 Random Walk

A random walk is a mathematical object and a simple example of random processes, that describes a path consisting of a succession of random steps on some mathematical space such as the integers. An elementary example of a random walk is the random walk on the integer number line, $\mathbb{Z}$, which starts at 0 and at each step moves +1 or 1 with equal probability. Other examples include the path traced by a molecule as it travels in a liquid or a gas, the search path of a foraging animal, the price of a fluctuating stock and the financial status of a gambler can all be approximated by random walk models, even though they may not be truly random in reality.

To define this walk formally, take independent random variables $Z_1, Z_2, Z_3, \ldots$, where each variable is either +1 or 1, with a 50% probability for either value, and set $S_0 = 0$ and

$$S_n = \sum_{j=1}^{n} Z_j$$

The series $\{S_n\}$ is called the simple random walk on $\mathbb{Z}$. This series (the sum of the sequence of -1s and $+1$s) gives the distance walked, if each part of the walk is of length one. The expectation $E(S_n)$ is zero.

5.3 Brownian Motion

Brownian motion is a continuous stochastic process. This model is used in many fields of science, specially in financial mathematics. Mathematicians use this model to price assets, in this case pricing derivatives. This motion is used to derive the Black-Scholes equation. Brownian motion is a continuous stochastic process, $B(t)$, with the following characteristics:

1. $B(0) = 0$
2. $B(t + s) - B(t)$ is normally distributed with mean zero and variance s.
3. $B(t + s_1) - B(t)$ is independent of $B(t) - B(t - s_2)$ where $s_1, s_2 > 0$, in other words non-overlapping increments are independently distributed.
4. $B(t)$ is continuous.

These properties imply that $B(t)$ is a martingale: a stochastic process for which $E[B(t + s)|B(t)] = B(t)$. The process $B(t)$ is also called a diffusion process. Brownian motion is an example of a random walk, which is a stochastic process with independent increments.[6]

We have reviewed the asset pricing model in class that uses Brownian motion to find the price of an asset at a given time. This model consists of a drift term and a Brownian random process term. Let us look at this model closely and see what each term means. We defined this model as the following:

$$\frac{dS_t}{S_t} = \mu dt + \sigma \emptyset \sqrt{dt} \tag{5.3.1}$$

where the term μdt is the drift term and $\sigma \emptyset \sqrt{dt}$ is the random increments of our Brownian motion affected by the volatility of the market.

The term μdt is called the drift. The drift value μ is the mean reverting level, which its purpose is to direct the price of an asset towards the expected value of the asset price. Since we are looking at the price change over time we multiply this drift by $dt = \sqrt{T/n}$, where $n = 0, 1, 2, ..., T$.

In equation(5.3.1), σ is the volatility of the underlying asset in the financial market. To calculate the volatility we need the historical price data of the asset. Volatility is just the square root of the price variance.

The $\emptyset$ is a random variable from a binomial distribution, where $\emptyset$ is either $+1$ or -1 with probability of 0.5. Let us denote this random variable over time t as $Y(t)$. Then we will have the following:

$$\frac{dS_t}{S_t} = \mu dt + \sigma Y(t)\sqrt{dt} \tag{5.3.2}$$

The term $Y(t)\sqrt{dt}$ is the Brownian increments. Brownian motion is an example of random walk, but it takes other environmental variables into consideration, creating a "force" like movement in a random path. In this case, the Brownian increments are affected by $\sqrt{(dt)}$ and volatility σ. Volatility plays an important role in asset pricing. Depending on the value of this parameter, the price can either change drastically or stay constant in a period of time. As $dt \to 0$ then we will have the following:

$$Y(t)\sqrt{dt} = dB(t)$$

Therefore if we have an asset with price S at time $t \in [0, T]$, then we will have the following pricing model:

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dB(t) \tag{5.3.3}$$

$$dS(t) = [S(t)]\mu dt + [S(t)]\sigma dB(t) \tag{5.3.4}$$

Equation (5.3.3) is an *Itô* process. This equation says that the dollar mean and standard deviation of $S(t)$ are $\mu S(t)$ and $\sigma S(t)$, proportional to the level of $S(t)$. Thus, the percentage change in $S(t)$ is normally distributed with instantaneous mean μ and instantaneous variance σ^2. The process in equation (5.3.3) is also known as geometric Brownian motion.[6] We can show equation (5.3.3) in an integral form as the following:

$$S(T) = S(0) + \int_0^T \mu S(t)dt + \int_0^T \sigma dB(t) \tag{5.3.5}$$

The following are the multiplication rules for terms containing dt and $dZ(t)$. This is the case where the increments dt gets really small ($dt \to 0$):

1. $dt \times dB(t) = 0$
2. $dt^2 = 0$
3. $dB(t)^2 = dt$

The reasoning behind these multiplication rules is that the multiplications resulting in powers of dt greater than 1 vanish in the limit.[6]

Let us look at an asset price trajectory, where Brownian motion was used to price this asset and how the volatility affects an asset price trajectory. The starting price for this asset was $S_0 = \$100$ and was held for 1 year ($t \in [0, 1]$). The drift (or rate of return) was $\mu = 0.15$. The asset was priced every business day (250 days in a year). The plots will look like the following: We can see

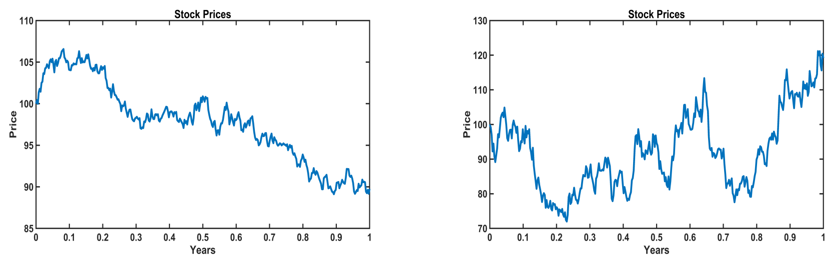

Figure 6: Left plot $\rightarrow \sigma = 0.20$, Right plot $\rightarrow \sigma = 0.40$

how two different volatility affect an asset price. The higher the volatility, the larger $\sigma dB(t)$ value will be, therefore more deviation from the expected value of the actual asset price ($E[S_T]$). The lower the volatility the closer the asset price to the expected value of the asset. Please note that we used constant volatility. This is not the case in financial markets. A stochastic volatility is used in financial markets, where volatility is changed every second. We will not look further into stochastic volatility case (too complex).

5.4 Ito's Lemma

This is the final piece of information (mathematical method) we need to derive the Black-Scholes equation from the Brownian motion. This is an informal definition of the Ito's lemma. The formal definition requires more technical mathematical preview on Ito's process, and limits of random variable sequences. The formal definition is not required to derive the Black- Shcoles equation.

We saw that the equation

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dB(t)$$

is an Ito's process, where B_t is a Brownian motion process.

If $f(t, S)$ is a twice differentiable scalar function, then its Tylor series expansion is

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}dS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}dS^2 + ...$$

However, what if we have a function f which depends not only on a real variable t, but also on a Brownian motion. We can do this by substituting S by $\frac{dS_t}{S_t}$ therefore substituting $(\mu dt + \sigma dB_t)$ by dS

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}[\mu dt + \sigma dB(t)] + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}[\mu^2 + 2\mu\sigma dt dB(t) + \sigma^2 dB(t)^2] + ... \quad (5.4.1)$$

Using the multiplication rules that we mentioned above, as $dt \to 0$, then any term containing $dtdB(t)$ and dt^2 will become zero. Any term $dB(t)^2$ will become dt. Therefore we will have a reduced version of the above taylor series of:

$$df = [\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial S^2}]dt + \sigma\frac{\partial f}{\partial S}dB(t) \quad (5.4.2)$$

~~Now substituting~~

6 Black-Scholes Equation

6.1 Definitions

Now that we have a basic understanding of Brownian motion and Ito's lemma, we proceed to deriving the Black-Scholes equation. Before doing so, let us look at some terms that will be used frequently.

A **financial portfolio** is a group of assets such as stocks, derivatives, commodities, currencies, mutual funds, etc. These portfolios are created and held by investors/financial professionals. Give a maturity time T, a portfolio can be changed, replicated and hedged by investors. Each portfolio owned by the investor has payoff amount at maturity time T

An **arbitrage** happens when a security is purchased in one market and simultaneously sold in another market at a higher price. This provides a completely risk free profit for the investor. Arbitrage exists as a result of market inefficiencies and continuous asset price changes. An intelligent investor always looks for an arbitrage opportunity to exploit in financial markets. Black-Scholes model assumes that the financial markets are arbitrage-free (efficient), meaning that all the prices are fair and law of one price exists in the market.

The risk-free rate of return in market represents the interest an investor would expect from a risk-free investment for a period of time. This is the minimum amount of return an investor expects from any type of investment. The simplest example of risk-free return is when an investor creates a risk-free bank account, which will gain a constant and risk-free interest of r, for a period of time. The risk-free interest rate is important, because investors tend to diversify (hedge) atheir portfolios by investing in a risk-free accounts or bonds, which they will receive constant and risk-free interest. We assume that this value is always given (a known parameter) and create a portfolio using this interest rate.

6.2 Hull's Derivation of Black-Scholes Equation

Let S be an underlying asset which follows the Brownian motion:

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dB(t) \tag{6.2.1}$$

$$dS(t) = S(t)\mu dt + S(t)\sigma dB(t) \tag{6.2.2}$$

where $B(t)$ is the stochastic variable. Let $V(S,t)$ be the value of a European derivative, for time $t \in [0,T]$. We create a portfolio, Φ, by selling the derivative $V(S,t)$ and buying $\frac{\partial V}{\partial S}$ units of the underlying asset. Therefore the value of the portfolio will be:

$$\Phi(t) = V(S,t) - \frac{\partial V}{\partial S}S(t) \tag{6.2.3}$$

Over a small time period of h, we will have the following portfolio value over time $[t, t+h]$:

$$d\Phi(t) = dV(S,t) - \frac{\partial V}{\partial S}dS(t) \tag{6.2.4}$$

We know previously that we can find the payoff value of a European option, $V(S,t)$, at maturity time. To find the value of the option in an earlier time we look to find dV. Since $V(S,t)$ is a twice differentiable scalar function, therefore we use Ito's lemma to find dV such as following:

$$dV(S,t) = [\mu\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2}]dt + \sigma\frac{\partial V}{\partial S}dB(t) \tag{6.2.5}$$

Now substituting the equations (6.2.2) and (6.2.4) into equation (6.2.3) we will have the following:

$$d\Phi(t) = [\mu\frac{\partial V}{\partial S} + \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2}]dt + \sigma\frac{\partial V}{\partial S}dB(t) - \frac{\partial V}{\partial S}[S\mu dt + S\sigma dB(t)] \tag{6.2.6}$$

Simplifying the equation (6.2.6) will give us:

$$d\Phi(t) = [\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}]dt \tag{6.2.7}$$

The first thing we notice is that the drift value μ is disappeared. We also notice is that the stochastic variable $B(t)$ is completely vanished. This means that there is no uncertainty any more, eliminating the risk that could be risen from this variable making it completely risk-free. This means that the rate of return on this portfolio should be equal to the rate of return of a risk-free bank account note that if this is not the case, then there will b arbitrage thus breaking one of the assumptions of the Black-Scholes model. With that established we therefore know that the difference between the the return of the portfolio over time $[t, t+h]$ and the change in portfolio is zero:

$$r\Phi dt - d\Phi = 0$$

Substituting equation (6.2.7) in the above equation we will have:

$$r\Phi dt = [\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}]dt$$

and furthermore substituting the equation (6.2.3) we will get:

$$\begin{aligned} r[V(S,t) - \frac{\partial V}{\partial S}S(t)]dt &= [\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}]dt \\ rV - rS\frac{\partial V}{\partial S} &= \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \\ \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV &= 0 \end{aligned} \tag{6.2.8}$$

Equation (6.2.8) is the infamous Black-Scholes equation. With the assumptions of the Black-Scholes model, this second order partial differential equation holds for any type of European option (in our case calls and puts) as long as its price function $V(S,t)$ is twice differentiable with respect to S

7 Solution to Black-Scholes Equation

Now we look at how to solve the Black-Scholes partial differential equation. The solution to this equation will help us to price a European option. The steps to solving this equation is mathematically advanced, so the reader has the choice to skip to the end, equation (7.0.4), where it shows the solution of this equation.

For simplicity let us denote the equation (6.2.8) as following:

$$rV = V_t + \frac{\sigma^2}{2}S^2 V_{SS} + rSV_S \tag{7.0.1}$$

where $V_t = \frac{\partial V}{\partial t}$, $V_{SS} = \frac{\partial^2 V}{\partial S^2}$ and $V_S = \frac{\partial V}{\partial S}$.

Assume a European call option that gives the owner the right to buy and underlying asset at strike price K at any time t. Then we would have the following boundary condition:

$$\begin{cases} V(0,t) = 0 & \text{for all } t \\ V(S,t) \to S & \text{as } S \to \infty \\ V(S,t) = (S_T - K)^+ & \end{cases} \tag{7.0.2}$$

Now we need to solve the following initial value problem:

$$rV = V_t + \frac{\sigma^2}{2}S^2 V_{SS} + rSV_S$$

with boundary condition

$$\begin{cases} V(0,t) = 0 & \text{for all } t \\ V(S,t) \to S & \text{as } S \to \infty \\ V(S,t) = (S_T - K)^+ & \end{cases}$$

We will be able to do this by transforming the Black-Scholes PDE into a heat equation. The heat equation in one space dimension with Drichlet boundary condition is

$$\begin{cases} u_t = u_{xx} \\ u(x,0) = u_0(x) \end{cases}$$

and its solution is

$$u(x,t) = u_0 * \phi(x,t)$$

where

$$\phi(x,t) = \frac{1}{\sqrt{4\pi t}} e^{\frac{-x^2}{4kt}}$$

Please note that $*$ is the convolution operator. Now let

$$\begin{cases} \tau = \frac{\sigma^2}{2}(T-t) \\ x = ln(\frac{S}{K}) \\ V(S,t) = Ku(x,\tau) \end{cases}$$

By multivariate chain rule we will have:

$$V_S = Ku = K(u_x X_s + u_\tau \tau_S) = \frac{Ku_x}{S} = e^{ln\frac{S}{K}} u_x = e^{-x} u_x$$

$$V_{SS} = -\frac{K\sigma^2}{2} u_\tau$$

$$V_t = -\frac{K\sigma^2}{2} u_\tau$$

Substituting into the original equation (7.0.1) we will have:

$$u_\tau = u_{xx} + (k-1)u_x - kV$$

where $k = \frac{2r}{\sigma^2}$. This is not yet the Drichlet heat equation, therefore we need another substitution.

Let $w(x,\tau) = e^{ax+b\tau} w(x,\tau)$. Then

$$\begin{cases} w_x = e^{ax+b\tau}(ay(x,t) + u_x) \\ w_{xx} = e^{ax+b\tau}(a^2 u(x,\tau) + 2 + u_{xx}) \\ w_\tau = e^{ax+b\tau}(bu(x,\tau) + u_\tau) \end{cases}$$

By substitution we will have:

$$w_\tau = (\alpha^2 + (k-1)a - k - b)w + (2a + k - 1)w_x + w_{xx}$$

To have out heat equation we need the terms $\alpha^2 + (k-1)a - k - b = 0$ and $2a + k - 1 = 0$. Let $a = 1 - \frac{k}{2}$ and $-\frac{(k+1)^2}{4}$. Then we will have the following heat equation with Dirchlet boundary conditions:

$$w_\tau = w_{xx}, x \in \mathbb{R}, \tau \in (0, T\frac{\sigma^2}{2})$$

$$\begin{cases} w(x,0) = max(e^{(k+1)\frac{x}{2}} - e^{(k-1)\frac{x}{2}}, 0), x \in \mathbb{R} \\ w(x,\tau) \to 0, \text{as } x \to \pm\infty, \tau \in (0, T\frac{\sigma^2}{2}) \end{cases} \tag{7.0.3}$$

Solving this heat equation for a European call option will give us the following formula:

$$C(S,t) = S_t N(d_1) - Ke^{-r(T-t)}N(d_2) \tag{7.0.4}$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2} dz$$

$$d_1 = \frac{ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{(T)}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Solving this heat equation for a European put option will give us the following formula:

$$P(S,t) = Ke^{-r(T-t)}N(-d_2) - S_t N(-d_1) \tag{7.0.5}$$

where

$$d_1 = \frac{ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{(T)}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Equations (7.0.4) and (7.0.5) are the two derivative pricing formulas that are being used by many financial specialist around the world[7]. The solution to these equations will give a fair (arbitrage-free) price for either of European options we previously analysed. This price is the same as the premium we discussed in chapter 4. The following are two examples, one for a call option and one for a put option.

Example 7.1 A European call option is made for a security currently trading at \$55 per share with volatility 0.45. The term is 6 months and the strike price is \$50. The risk-free interest rate is 3

Solution. We have $S = 55$, $K = 50$, $r = 0.03$, $\sigma = 0.45$ and $T = \frac{1}{2}$ (since it is only for six months). Therefore we have the following values for d_1 and d_2:

$$d_1 = 0.50577$$

and

$$d_2 = 0.18757$$

and $N(d_1) = 0.6950$, $N(d_2) = 0.5753$ from the normal distribution table. Therefore we would have:

$$C(S,t) = S_t N(d_1) - Ke^{-r(T-t)} N(d_2) = 55(0.6950) - 50e^{-(0.03*0.5)}(0.5753) = \$9.89$$

Example 7.2 A European put option is made for a security currently trading at \$55 per share with volatility 0.45. The term is 6 months and the strike price is \$50. The risk-free interest rate is 3

Solution. We have $S = 55$, $K = 50$, $r = 0.03$, $\sigma = 0.45$ and $T = \frac{1}{2}$ (since it is only for six months). Therefore we have the following values for d_1 and d_2:

$$d_1 = 0.50577$$

and

$$d_2 = 0.18757$$

and $N(d_1) = 0.6950$, $N(d_2) = 0.5753$ from the normal distribution table. Now since are looking for $N(-d_1)$ and $N(-d_2)$, we can just do the following:

$$N(-d_1) = 1 - N(d_1) = 0.3050$$

$$N(-d_2) = 1 - N(d_2) = 0.4247$$

We are allowed do that, because of normal distribution characteristics. Therefore we would have

$$P(S,t) = Ke^{-r(T-t)} N(-d_2) - S_t N(-d_1) = 50e^{-(0.03*0.5)}(0.4247) - 55(0.305) = \$4.14$$

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