Chapter 1 - Random Number Generation

Conditional Distributions

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Conditional Distributions

Conditional Distribution: Suppose that we we start with a random variable X with a distribution P_X that we know how to sample from. Let A be an event and consider the conditional distribution of X given A $P_{X|X\in A}$.

We will be able to sample from the conditional distribution $P_{X|X\in A}$ by using a version of the rejection sampling algorithm as follows:

Algorithm

1. Generate $X \sim P_X$ (proposal)
2. If $X \in A$ then $Y = X$ (proposal is accepted)
3. If $X \notin A$ then return to step 1
4. Return Y

Example Generate a random sample of size $n = 1000$ from the conditional distribution of $X \sim N(0, 1)$ conditioned on $X \geq 0$

R Code

```r
n <- 1000
counter <- 1

target_sample <- vector();
while (counter < n+1 ) {
  proposal <-rnorm(1)

  if (proposal > 0){
    target_sample[counter] <-
         proposal
    counter <- counter+1
  }
}
hist(TargetDistSample)
```

Python Code:

```python
import numpy as np
import matplotlib.pyplot as plt

n = 1000
counter = 0
target_sample = []
mu, sigma = 0, 1

while counter < n:
    proposal = np.random.normal(loc=
         mu, scale=sigma)
    if proposal > 0:
        target_sample.append(
             proposal)
        counter += 1

plt.hist(target_sample, bins=30)
plt.show()
```

Conditional Distributions Example (Vectorized)

Example: Generate a random sample of size $n = 1000$ from the conditional distribution of $X \sim N(0, 1)$ conditioned on $X \geq 0$

R Code

```r
n <- 1000
mu <- 0
sigma <- 1
samples <-
^^Irnorm(n * 2, mean = mu, sd =
     sigma)
target_sample <-
^^Isamples[samples > 0]
target_sample <-
^^Itarget_sample[1:n]
hist(target_sample, breaks = 30)
```

Python Code

```python
import numpy as np
import matplotlib.pyplot as plt

n = 1000
mu, sigma = 0, 1
samples =
 ^^Inp.random.normal(loc=mu, scale=
     sigma, size=n*2)
target_sample =
^^Isamples[samples > 0]
target_sample = target_sample[:n]

plt.hist(target_sample, bins=30)
plt.show()
```

Remarks

- The described method to generate samples from conditional distributions is very straightforward as long as we know how to sample from the unconditional distribution of X.
- The proposals will be accepted with probability $P(A)$ so the efficiency of the method may not be good if A is an event with very low probability.
- In cases where $P(A)$ is small, this method might not be advisable. Example 1.28 from the textbook covers one of such examples and provides an alternative solution.

Rejection Sampling in Arbitrary Spaces

One of the advantages of rejection sampling is that it can be used to generate random objects in more general spaces, in particular random vectors.

A standard use of the rejection sampling algorithm is related to the generation of uniformly distributed random vectors over some arbitrary subsets of $\mathbb{R}^d$.

For simplicity we will focus on the case $d = 2$ (which can be visualized easily) but these ideas also apply in higher dimensions.

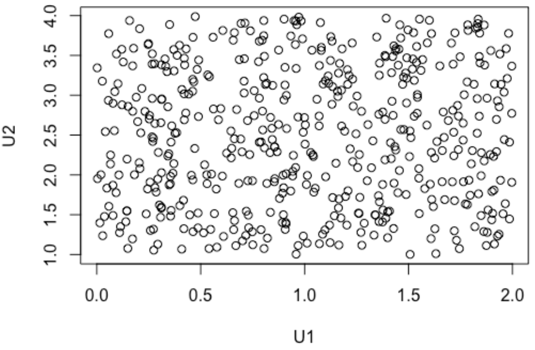
Uniform sampling from a rectangle

If we want to generate a random vector with uniform distribution over the rectangle $[a, b] \times [c, d]$ we can do so directly by generating its two components independently one from the other (each of them uniformly distributed) as follows:

Algorithm:

1. Generate $X \sim U[a, b]$
2. Generate $Y \sim U[c, d]$ (independently of X)
3. Return vector $(X.Y)$

Plot of 500 uniformly distributed random points on $[0, 2] \times [1, 4]$.



Main result

Lemma 1.31 from textbook: Let X be uniformly distributed on a set A, and let B be a set such that $|A \cap B| > 0$. Then the conditional distribution $P_{X|X \in B}$ of X conditioned on the event $X \in B$ coincides with the uniform distribution on $A \cap B$.

Remark: The symbol $|Y|$ refers to the "volume" (or measure) of set Y. For instance, In the case of two dimensions we need that the area of $A \cap B$ is strictly positive.

- The previous Lemma indicates a possible approach (using rejection sampling) to generate random vectors uniformly distributed in some "irregular" subset B of $\mathbb{R}^2$ (and $\mathbb{R}^d$ in general).
- First we would need to start with a set $A \supset B$ so that we can sample uniformly distributed random vectors from A. Then we will reject all the vectors that do not belong to B and keep the generated random vectors in B.
- According to the previous Lemma, this sample will be uniformly distributed on $A \cap B = B$.
- The easiest approach (but not strictly necessary or possible) consists on selecting A with a rectangular shape, as we already know how to generate random vectors from a rectangle.

Rejection Sampling in Arbitrary Spaces Example

Example: Generate a sample of uniformly distributed random vectors on the semicircle defined by $x^2 + y^2 \leq 1$ and $y \geq 0$.

Rejection Sampling in Arbitrary Spaces Example

Example: Generate a sample of uniformly distributed random vectors on the semicircle defined by $x^2 + y^2 \leq 1$ and $y \geq 0$.

Solution: We will use the algorithm described earlier with B being the described semicircle and A being the rectangle $[-1, 1] \times [0, 1]$ which includes B.

Essentially we will start generating uniformly distributed random numbers in the rectangle A and out of those, we will reject the ones that are not in B. Notice that the theoretical probability that a proposal point is accepted is $\pi/4 \approx 0.785$.

Algorithm: Rejection Sampling Inside Unit Semicircle

Algorithm 1 Generate points (X_1, X_2) uniformly inside the semicircle

Input: Integer n
Initialize empty vectors X_1 and X_2
Generate uniform random vectors:
    $U_1 \sim \text{Uniform}(-1, 1)^n$, $U_2 \sim \text{Uniform}(0, 1)^n$
Initialize counter: counter $\leftarrow 1$
for $i = 1$ to n **do**
    if $U_1[i]^2 + U_2[i]^2 < 1$ **then**
        $X_1[\text{counter}] \leftarrow U_1[i]$
        $X_2[\text{counter}] \leftarrow U_2[i]$
        counter $\leftarrow$ counter $+ 1$
    end if
end for
Output: X_1, X_2 (accepted points inside the semicircle)

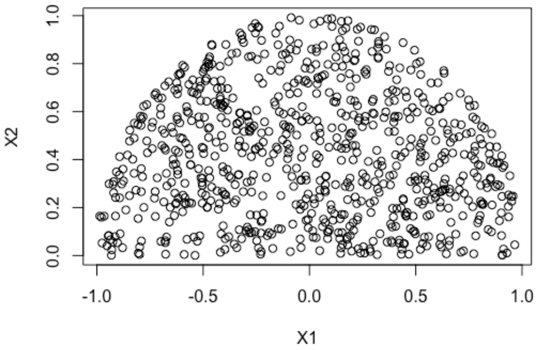
Algorithm: Vectorized Rejection Sampling Inside Unit Semicircle

Algorithm 2 Generate n points (X_1, X_2) uniformly inside the semicircle (vectorized)

1: **Input:** Integer n
2: Generate candidate points:
   $U_1 \leftarrow \text{runif}(n \times 2, -1, 1)$
   $U_2 \leftarrow \text{runif}(n \times 2, 0, 1)$
3: Compute mask for points inside the unit circle:
   $\text{inside} \leftarrow (U_1^2 + U_2^2 < 1)$
4: Keep only the accepted points:
   $X_1 \leftarrow U_1[\text{inside}]; X_1 \leftarrow X_1[1:n]$
   $X_2 \leftarrow U_2[\text{inside}]; X_2 \leftarrow X_2[1:n]$
5: **Output:** X_1, X_2 (vectors of n points inside the semicircle)

Note: In vectorized rejection sampling, we often generate more candidate points than needed (e.g., $n \times 2$) to ensure that enough points satisfy the acceptance condition. Since only a fraction of the candidates fall inside the desired region, oversampling increases the likelihood that we can select exactly n accepted points without looping. The factor 2 is a simple heuristic; larger factors may be needed if the acceptance rate is low.

The code gave us 780 generated uniformly distributed random points on the semicircle $x^2 + y^2 \leq 1, y \geq 0$.



Remarks

- As the previous example shows, this method can be very useful to generate samples that follow the uniform distribution over some irregular sets of $\mathbb{R}^d$.
- If the set B is small compared to the set A then the method may be inefficient.
- It is better to use this method if we have a relatively easy way to check whether a given point belongs to set B . That is easy for a semicircle(previous example) but not so easy if B is a pentagram(a five-pointed star).
- Hard (but not impossible) to use this method if the set B is unbounded, as we won't be able to enclose it in a rectangle A.

Homework

1. Write code for the uniform sampling from a rectangle algorithm, with $X \sim U[0,2]$ and $Y \sim U[1,4]$.
2. Implement code for Algorithms 1 and 2 using the rectangle $[-1,1] \times [0,1]$. If computationally feasible, run both algorithms for $num\_sims = [50, 100, 500, 1000, 2000]$ and plot the difference in their running times.
    - **Hint:** Use built-in timers to measure running time:
        - In Python: `import time; start = time.time(); ...; end = time.time()`
        - In R: `system.time({ ... })`
3. Write a computer program to generate a random sample of size 1000 from the conditional distribution of $X \sim Binomial(n, p)$ (with $n = 10$ and probability of success $p = 0.6$) conditioned on $X \geq 5$.
4. Write a computer program to generate (and plot) a sample of 500 uniformly distributed random points on the set of plane given by $y \geq 0$, $-\pi/2 \leq x \leq \pi/2$, and $y \leq \cos x$.