Chapter 2 - Simulating Statistical Models

Markov Chains on a Continuous State Space.

Prof. Alex Alvarez, Ali Raisolsadat

School of Mathematical and Computational Sciences
University of Prince Edward Island

Markov Chains on Different State Spaces

Finite state space: Up to this point, we have worked with Markov chains whose states belong to a finite set

$$S = \{1, 2, \ldots, M\}.$$

In this setting, the transition behavior of the chain is fully described by a *transition matrix* whose (i, j) entry gives the probability of moving from state i to state j in one step.

Beyond finite: Many Markov chains arising in applications take values in much more general spaces. A common example is a chain evolving in a continuous space such as

$$S = \mathbb{R}^d.$$

When the state space is uncountable, a transition matrix is no longer suitable. Instead, we describe the transitions using a *transition density*, which plays the role of the continuous analogue of a matrix row in the finite case.

Transition Density

For Markov chains on a continuous state space $S = \mathbb{R}^d$, transitions are specified by a **transition density**

$$p : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R},$$

where $p(x, y)$ describes the likelihood of moving from state x to a point near y in one step.

The function p must satisfy:

1. **Non-negativity:**
$$p(x, y) \geq 0 \quad \text{for all } x, y \in \mathbb{R}^d.$$
2. **Normalization:** For each fixed x, the function $y \mapsto p(x, y)$ integrates to 1:
$$\int_{\mathbb{R}^d} p(x, y)\, dy = 1.$$

Fix a current state x. Then the function

$$y \mapsto p(x, y)$$

is a probability density for the next state X_{n+1} conditional on $X_n = x$. Thus p is the continuous analogue of a row of the transition matrix in the finite-state case.

Example: Gaussian Markov Chain

Example 2.29 (Textbook): Consider the process defined by

$$X_0 = 0, \qquad X_j = \tfrac{1}{2}\, X_{j-1} + \varepsilon_j,$$

where the noise terms $\varepsilon_j \sim N(0,1)$ are independent and identically distributed.

Markov property: The sequence $(X_j)_{j\geq 0}$ forms a Markov chain with state space $S = \mathbb{R}$, since each update depends only on the previous state and an independent noise term.

Conditional distribution: For any fixed $x \in \mathbb{R}$,

$$X_j \,\big|\, (X_{j-1} = x) \;\sim\; N\big(\tfrac{x}{2}, 1\big)\,.$$

Transition density:

$$p(x,y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\,(y - \frac{x}{2})^2\right), \qquad x, y \in \mathbb{R}.$$

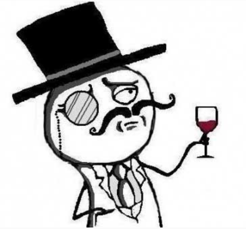
Generation of Markov Chain Paths (Continuous State Space)

Algorithm 1 Simulating a Markov Chain Path

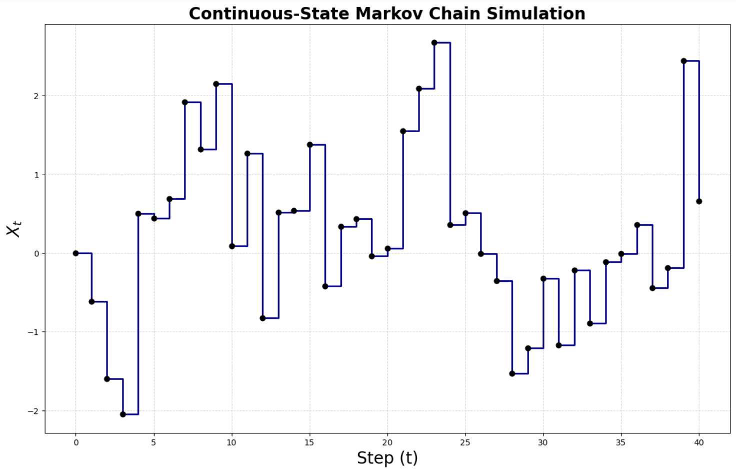
Generate X_0 according to the initial distribution
for $i = 1$ to $n - 1$ **do**
  Generate $X_i \in S$ according to the density

$$g(y) = p(X_{i-1}, y)$$

end for
return $(X_0, X_1, \ldots, X_n)$



Example: Markov Chain Path



Simulated trajectory of a continuous state space Markov Chain with $n = 40$ with transition density $p(x, y) = \frac{1}{\sqrt{2\pi}} \exp\left(- \frac{1}{2} (y - x/2)^2 \right), \quad x, y \in \mathbb{R}$.

Example: Uniform Transition Density

Example (from Homework): Let $X_0 = 0$ and define the transition density

$$p(x, y) = \frac{1}{2}\, \mathbb{1}_{[x-1,\, x+1]}(y), \quad x, y \in \mathbb{R}$$

Note: The indicator function $\mathbb{1}_A$ is defined as

$$\mathbb{1}_A(y) = \begin{cases} 1, & \text{if } y \in A, \\ 0, & \text{if } y \notin A. \end{cases}$$

Observation: The sequence $X_0, X_1, X_2, \ldots$ is a Markov chain with state space $S = \mathbb{R}$.

Conditional law: Given $X_{j-1} = x$, we have

$$X_j \sim \text{Uniform}(x - 1,\, x + 1)$$

Transition density:

$$p(x, y) = \begin{cases} \dfrac{1}{2}, & y \in [x - 1,\, x + 1], \\ 0, & \text{otherwise.} \end{cases}$$

Stationary Distribution in Continuous State Spaces

For Markov chains with a continuous state space, we also have the notion of a **stationary distribution**.

A probability density $\pi : \mathbb{R}^d \to [0, \infty)$ is called a **stationary density** for a Markov chain with transition density p if it satisfies

$$\int_{\mathbb{R}^d} \pi(x)\, p(x, y)\, dx \;=\; \pi(y), \quad \forall\, y \in \mathbb{R}^d.$$

Intuition: A stationary density π is an equilibrium law for the Markov chain:

- If $X_n \sim \pi$, then $X_{n+1} \sim \pi$ as well.
- The distribution is **invariant** under the dynamics of the chain.
- In the long run, many Markov chains converge to their stationary distribution, regardless of the starting point.

Example (Gaussian AR(1) Chain):

$$X_j = \tfrac{1}{2}X_{j-1} + \varepsilon_j, \quad \varepsilon_j \sim N(0,1).$$

- This chain has a stationary distribution:

$$\pi \ \sim \ N\Big(0, \tfrac{1}{1-(1/2)^2}\Big) = N\Big(0, \tfrac{4}{3}\Big).$$

- **Interpretation**: After many steps, the state X_n is approximately $N(0, 4/3)$, no matter the initial X_0.

Homework

- Generate Markov chain paths using the **Gaussian transition density** (Algorithm 1) for 40 time steps.

- Write an algorithm for simulating a Markov chain with the **Uniform transition density**

$$p(x, y) = \frac{1}{2}\, \mathbb{1}_{[x-1,\, x+1]}(y), \quad x, y \in \mathbb{R},$$

  and implement code to generate a path of length 40.
- **AR(1) problem:** Consider

$$X_j = \phi X_{j-1} + \varepsilon_j, \quad \varepsilon_j \sim N(0, \sigma^2) \text{ i.i.d.}, \quad |\phi| < 1.$$

1. Show that $\{X_j\}$ defines a Markov chain with transition density

$$p(x, y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y - \phi x)^2}{2\sigma^2}\right)$$

2. Prove that if a stationary distribution exists, it must be Gaussian with mean 0.
3. **Hint** For an AR(1) process, the stationary variance satisfies $\mathrm{Var}(X) = \frac{\sigma^2}{1-\phi^2}$
4. Specialize to the case $\phi = \frac{1}{2},\ \sigma^2 = 1$. Find the stationary distribution and its standard deviation.