Chapter 3 - Monte Carlo Methods

Variance Reduction Methods.
Control Variates.

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Control Variates Method Rationale

Suppose that we want to find $\theta = E(g(X))$. If we know of another "simpler" function h that is close to g, such that $E(h(x))$ can be found analytically then we can write

$$E\left(g(X)\right) = E\left[g(X) - h(X)\right] + E\left(h(X)\right)$$

Now, we can use Monte Carlo methods to find $E\left[g(X) - h(X)\right]$.

Because g and h are close, the random variable $g(X) - h(X)$ has a small variance, so the Monte Carlo error should be smaller than the Monte Carlo error for g on its own.

We refer to $h(X)$ as a **Control Variate** for $g(X)$.

Algorithm 1 Monte Carlo Estimation via Control Variates

1: **Input:** Number of samples n, random variable X, target function $g(x)$, control function $h(x)$ with known $E[h(X)]$
2: **for** $j = 1$ to n **do**
3: Generate $X_j \sim$ distribution of X
4: Compute control-adjusted value $Y_j = g(X_j) - h(X_j)$
5: **end for**
6: Compute sample mean $s = \frac{1}{n} \sum_{j=1}^{n} Y_j$
7: **Output:** $\hat{\theta}_n^{CV} = s + E[h(X)]$

The estimator θ_n^{CV} is unbiased and

$$MSE(\theta_n^{CV}) = \frac{1}{n} Var(g(X) - h(x))$$

Control Variates Method Example

Find an estimate of

$$\theta = E\Big[\log(2U^2+1)\Big], \quad U \sim \text{Uniform}(0,1)$$

using the **Control Variates Method**.

The target function is $g(x) = \log(2x^2+1)$, which satisfies $g(0) = 0$ and $g(1) = \log(3) \approx 1.10$.

As a control variate, consider $h(x) = x$. The functions g and h are positively correlated, and

$$E[h(U)] = E[U] = \frac{1}{2}$$

Algorithm

1. Generate $U_1, U_2, \ldots, U_n \sim \text{Uniform}(0,1)$
2. Compute $s = \frac{1}{n}\sum_{j=1}^{n}\Big(\log(2U_j^2+1) - U_j\Big)$
3. Compute the control variates estimator $\hat{\theta}_n^{CV} = s + E[h(U)] = s + \frac{1}{2}$

Remark: The efficiency gain depends on the strength of the correlation between $g(U)$ and $h(U)$. Higher (positive) correlation leads to greater variance reduction.

Control Variates Method – General Approach

There is a more general approach to the control variates method.

As usual we are trying to estimate $\theta = E(g(X))$,$X \sim f$.

If we generate random numbers/vectors $X_1, X_2, \ldots X_n$ with distribution f we can get the basic MC estimator $\hat{\theta}_n = \frac{1}{n}\sum_{j=1}^{n} g(X_j)$.

Consider a function $h(x)$ where $E(h(X)) = \mu_h$ is known and define a new MC estimator that depends on a real parameter c:

$$\hat{\theta}_n^{c,h} = \hat{\theta}_n - c(\hat{\theta}_n^h - \mu_h)$$

where:

$$\hat{\theta}_n^h = \frac{1}{n}\sum_{j=1}^{n} h(X_j)$$

Optimal Estimator

Notice that $\hat{\theta}_n^c$ is unbiased and

$$MSE(\hat{\theta}_n^{c,h}) = Var(\hat{\theta}_n) + c^2 Var(\hat{\theta}_n^h) - 2c \cdot cov(\hat{\theta}_n, \hat{\theta}_n^h)$$

As a function of c we see that its MSE is minimized at

$$c_0 = \frac{cov(\hat{\theta}_n, \hat{\theta}_n^h)}{Var(\hat{\theta}_n^h)} = \frac{Cov(g(X), h(X))}{Var(h(X))}$$

$$MSE\left(\hat{\theta}_n^{c_0,h}\right) = \frac{1}{n}\left(Var(g(X)) - \frac{cov^2(g(X), h(X))}{Var(h(X))}\right)$$

The relative variance reduction using control variates (compared to the basic Monte Carlo estimator) is:

$$\frac{MSE(\hat{\theta}_n^{c_0,h})}{MSE(\hat{\theta}_n)} = \frac{\frac{1}{n}(Var(g(X)) - \frac{cov^2(g(X),h(X))}{Var(h(X))})}{\frac{1}{n}Var(g(X))}$$
$$= 1 - \rho^2(g(X), h(X))$$

where $\rho(g(X), h(X))$ is the correlation coefficient between $g(X)$ and $h(X)$.

Remarks:

- The control variates **optimal estimator** is not worse (it has the same or smaller variance) than the standard Monte Carlo estimator.
- The method works better if there's a strong correlation (either positive or negative) between $g(X)$ and $h(X)$.

Control Variates Remarks

- The choice of c in the estimator is important, and in general we are not able to find the optimal value c_0 analytically. That is because the quantities $Var(h(X))$ and $cov(g(X), h(X))$ are, in general, unknown.
- An alternative is to estimate them from standard statistical methods using an initial (relatively small) sample of size m, and then provide an estimation of c_0 as:

$$\hat{c}_0 = \frac{\sum_{j=1}^{m}(g(X_j) - \hat{\theta}_m)(h(X_j) - \hat{\theta}_n^h)}{\sum_{j=1}^{m}(h(X_j) - \hat{\theta}_n^h)^2}$$

- The choice $c = 1$ corresponds to the basic approach to the control variate method, covered in previous slides.

Homework

Let $\theta = E(g(U))$ with $U \sim U[0,1]$ and $g(x) = e^x$.

1. Find the exact value of θ.
2. Write a computer program to estimate θ using basic Monte Carlo integration.
3. Consider $h(x) = x$. Find the optimal control variates Monte Carlo estimator (essentially finding c_0 theoretically)
4. Write a computer program to estimate θ using the optimal control variates estimator from part (3)
5. How much is the variance reduced by using the control variates estimator, compared to the basic Monte Carlo estimator in your program outcomes?
6. Does the answer to (5) match the theoretical variance reduction given by $1 - \rho^2(g(U), h(U))$?