DOI: 10.1111/risa.13884

ORIGINAL ARTICLE

Risk-layering and optimal insurance uptake under ambiguity: With an application to farmers exposed to drought risk in Austria

Corina Birghila[1] | Georg Ch. Pflug[2] | Stefan Hochrainer-Stigler[3]

[1] Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, Ontario, Canada

[2] ISOR, University of Vienna, Vienna, Austria

[3] IIASA, International Institute for Applied Systems Analysis, Laxenburg, Austria

Correspondence
Corina Birghila, Department of Statistics and Actuarial Science, University of Waterloo, 200, University Ave W., Waterloo, ON, N2L 3G1 - Canada.
Email: corina.birghila@uwaterloo.ca

Funding information
Vienna Graduate School on Computational Optimization (VGSCO), Grant/Award Number: FWF-Project No. W1260-N35; International Institute for Applied Systems Analysis

Abstract

Many risks we face today will very likely not stay the same over time. For example, it is expected that climate change will alter future risks of natural disaster events considerably and, as a consequence, current risk management and governance strategies may not be effective anymore. Large ambiguities arise if future climate change impacts should be taken into account for analyzing risk management options today. Risk insurance, while albeit only one of many risk management actions possible, plays an important role in current societies for dealing with extremes. A natural starting point for our analysis is therefore the question of how ambiguity may be incorporated in a world with changing risks. To shed light on this question, we study how ambiguity can affect the uptake of insurance and risk mitigation within a risk-layer approach where each layer is quantified using distortion risk measures that should reflect the risk aversion of a decisionmaker toward extreme losses. Importantly, we obtain a closed-form solution for such a problem statement which allows an efficient numerical implementation. We apply this model to a case study of drought risk for Austrian farmers and address the question how ambiguity will affect the risk layers of different types of farmers and how subsidies may help to deal with current and future risks. We found that especially for small-scale farmers the consequences of increasing risk and model ambiguity are pronounced and subsidies are especially needed in this case to cover the high-risk layer.

KEYWORDS
Ambiguity, Austria, drought, optimal insurance uptake, risk-layering, subsidies

We study how ambiguity affects the insurance and mitigation in a risk-layer approach that reflects the attitude of a decisionmaker. We examine the effects of subsidies on drought insurance for farms in Austria.

1 | INTRODUCTION

Risk analysis can be understood to cover risk assessment, risk measurement, risk communication, risk management and policy-related aspects (SRA, 2018b). One of the goals of risk analysis is to improve communication about risks and enhance decision making in real-life situations (Aven, 2020). In that regard, one very active applied research area of risk analysis (from a broad-based perspective as mentioned above) is about extremes (especially in the context of natural disasters) (GAR, 2015) and systemic events (Renn et al., 2020). These risks will not stay the same over time and it is expected that climate change (among other global changes) will alter future risks of such events (e.g., natural hazards) considerably and, as a consequence, current risk management and governance strategies may not be effective anymore (IPCC, 2012).

Regarding related risk management and governance options to address such risks, at least three main strategies can be identified, including risk informed strategies, precautionary strategies, and discursive strategies (Renn et al., 2011). While most of the time a mixture of these three strategies may be seen as most appropriate, in the context of high stakes and large uncertainties it is sometimes suggested that more weight on precautionary strategies and in the case of interpretative (e.g., being open to different interpretations of specific risk assessments) and normative ambiguity (e.g., being open to different views and concepts of values related to risk) more weight on discursive strategies should be given (SRA, 2018a). Furthermore, the focus should be on reshaping or advancing currently existing risk management and governance strategies instead of developing entirely new governance structures and instruments (Watkisset al., 2015). Risk insurance, while albeit only one of many risk management actions possible (SRA, 2018b), plays an important role in current societies in dealing with extremes and therefore can be seen as one natural starting point for an analysis of how ambiguity may be incorporated to adapt to changing risks. This question is especially important as for an implementable insurance the risk must be quantifiable and in the context of future climate change large ambiguities arise. How to deal with these within a risk-informed approach is the topic of this article. We focus on the demand-side perspective of risk insurance in that regard and neglect supply side considerations (see Charpentier, 2008; Hecht, 2007). Such information can then further be used and embedded within other governance processes (e.g., subsidies of risk insurance and consequences of uptake) including a broader judgment of risk and uncertainties and the involvement of relevant stakeholders (Okada et al., 2018; Scolobig et al., 2015; Webler & Tuler, 2021).

In more detail, we study the question of how ambiguity (i.e., being open to different interpretations of risk assessment) (SRA, 2018b) can affect the uptake of insurance and risk mitigation within a risk-layer approach. The question is especially important in regards to future robust strategies (in the sense that specified goals are achieved despite large information-gaps), see SRA (2018b) compared to the current situation, including what magnitude of risks (e.g., losses) the insured cannot cover anymore through insurance under a given budget and how to make risk insurance affordable (e.g., through subsidies) also in a very uncertain future. From a demand-side perspective, these questions are of primary interest to insurance policyholders as well as governments and can be easily incorporated within iterative approaches that also focus on precautionary or discursive strategies; see, for example, Schinko and Mechler (2017) in the context of climate change. Our contribution to the literature in that regard is twofold. First, we show how possible model ambiguities will affect the uptake of insurance. In doing so, we propose a risk-layer approach (Mechler et al., 2014) where each layer is quantified using distortion risk measures that should reflect the risk aversion of a decisionmaker (DM) toward extreme losses. Importantly, we obtain a closed-form solution for such a problem statement which allows an efficient numerical implementation. Second, we apply this model to a case study of drought risk for Austrian farmers and address the question how ambiguity will affect the uptake of insurance and how subsidies may assist them to deal with such risks. We find that ambiguity as well as subsidies will affect different types of farmers differently, with small-scale farms benefiting more from subsidies compared to larger farms. This also has implications for risk mitigation as subsidies can help especially small-scale farms to invest into risk mitigation and the insurance risk layer. While the suggested approach was applied to drought risk in Austria, the model is flexible enough to be used in any other country and for other hazards as well, given some modest data requirements are met. The suggested methodology and application possibilities should also be beneficial for governance processes that are focusing on how poor communities within a population can be better prepared against current and future changes in risk (Cummins & Mahul, 2009; UNDRR, 2015).

The organization of the article is as follows: in Section 2, we present and discuss risk mitigation, risk insurance, and related ambiguities in some detail to set up the stage for our model description. In doing so, Section 3 presents the formulated optimization problem and the solution with and without the incorporation of ambiguity. Section 4 then applies our method to a case study of Austrian farmers exposed to drought risk, while Section 5 discusses the results within a broader context and ends with a conclusion and an outlook to the future.

2 | METHODOLOGY

To avoid any confusion, we base our discussion on the terminology as suggested by the Society of Risk Analysis (SRA, 2018b) and include additional aspects needed for our discussion where appropriate. Our starting point is in regards to risk management actions including risk acceptance (i.e., risk is judged as acceptable), risk insurance, and risk mitigation (i.e., actions to reduce risk). As will be discussed, these three actions to manage risk can be embedded within a so-called risk-layer approach (Mechler et al., 2014). In our risk-layer approach, we define the term uncertainty as the imperfect information of the occurrence of a loss event. As a measure for representing this uncertainty, we take a probabilities approach, in other words, probabilities of an event or events are assumed to be generally unknown and must be estimated, for example, either through past data or (probability) models. Finally, ambiguity is defined here as the condition to be open to different interpretations of risk assessment input and results (SRA, 2018b), for example, by using different (probability) models to estimate losses and considering all of them as equally valid for decision-making processes. To set up the stage, we start with a detailed discussion of optimal insurance contracts (from a demand-side perspective) and relate these results to risk mitigation and risk acceptance within a risk-layer approach. The extension of such an approach to include ambiguity dimensions explicitly is discussed afterward.

2.1 | Risk insurance

The study of optimal insurance/reinsurance contracts has drawn considerable attention over time, starting from the work of Borch (1960) and Arrow (1963). Borch (1960) showed that the stop-loss reinsurance is optimal when the insurer measures his risk by variance, and the reinsurer uses the expected value premium principle to price the contract. The stop-loss contract was proven to be optimal also when the insurer's objective is to maximize his expected utility (Arrow, 1963). These initial results were further extended to include more advanced risk measures and different constraints of insurance industries. Well-known examples of studied risk measures are value-at-risk (V@R), average value-at-risk (AV@R), or general distortion risk measures. Under the assumption of convex and increasing ceded loss functions, Cai et al. (2008) found that stop-loss reinsurance remains optimal, when the risk is quantified by V@R or AV@R and the premium is priced by the expected value principle. Extensions to the class of ceded functions and insurance premiums are discussed in Cheung et al. (2010), Chi and Tan (2010, 2013), among others. These contributions are further generalized in Cui et al. (2013), where they studied the optimal reinsurance problem for distortion risk measures and general premiums. When both the ceded loss function and the retained loss function are increasing, Cui et al. (2013) present a layered methodology that allows an explicit solution for a reinsurance arrangement. Assa (2015) and Zhuang et al. (2016) offered an alternative approach to the problem in Cui et al. (2013), by exploiting the absolute continuity of the ceded loss function. Note that the majority of these studies derive the structure of optimal insurance contracts under the assumption of complete knowledge of the underlying distribution and we discuss the violation of this assumption within the ambiguity section.

2.2 | Risk mitigation

The question of linking insurance with risk mitigation is a long-standing area of study within the topic of risk management against extremes (Kleindorfer & Kunreuther, 1999; Linnerooth-Bayer & Hochrainer-Stigler, 2015). Past approaches were either based only on empirical studies (Amendola et al., 2012) or on formalized methodologies, that is, optimization techniques (Härdle et al., 2017). In the latter, for a given budget and a given risk measure, different stochastic optimization techniques can be applied to determine the optimal mix between risk mitigation and insurance (Pflug & Romisch, 2007). In reality, several challenges require in-depth consideration (Hanger et al., 2018). Most often, some risk reduction has to be applied already in order to be eligible for insurance coverage. Furthermore, insurance premium reduction is only available if some concrete standards are in place in order to decrease vulnerability to hazards. The lack of knowledge regarding the marginal cost curves presents yet another difficulty for risk management. One last challenge arising in practice is the interaction between insurance and risk mitigation: which tool to apply first and when to switch from one to the other. We discuss this issue in more detail using a risk-layering approach.

2.3 | Risk-layering

Considering the difficulties mentioned above on how to combine risk acceptance, risk mitigation (we also refer to risk reduction or risk retention), and risk insurance, we suggest to first tackle the risk insurance issue and only afterward will ask how risk insurance can be related to risk reduction. To this end, we follow the methodology described in Mechler et al. (2014) and consider a risk-layer approach, were financing instruments are more appropriate for the middle layer risks (respectively, events) and risk reduction or risk acceptance is reserved for more frequent events. In Figure 1, we illustrate the risk-layer approach: losses smaller than a given *attachment point* a are covered by the DM, losses between a and the *exit point* d are insured, while the losses above d are noninsurable and outside assistance may be needed. The noninsurable losses above the exit point can be explained, for example, by the limit in the financial capacity of the insurance sector, facing natural catastrophes. We refer to Doherty and Schlesinger (1991), Cummins et al. (2002) or Cummins and Mahul (2004) for a discussion on the limited coverage in the insurance and reinsurance market. For example, in Cummins and Mahul (2004), the authors investigate the design of optimal insurance contract with an exogenous upper limit that is lower than the maximum loss. In our setting, we do not fix the limit *a priori*, but rather the limit is obtained as the solution of our formulated optimization problem discussed in the next section. Nevertheless, the presence of noninsurable losses is in line with the modeling approach in Cummins and Mahul (2004) or Doherty and Schlesinger (1991), in the context of natural disasters and extremes.

Assuming, as we do here, that insurance protection for the middle-risk layer is available, the corresponding costs can be calculated based on the estimated loss distribution. In this framework, the DM considers splitting a discretionary budget B to purchase insurance and to reduce the risk associated with small losses (see Figure 1). The benefit of doing so is twofold. First, the choice of risk reduction instruments depends on the DM's preferences. Second, rather than assuming a functional relationship between insurance and risk reduction, it allows testing assumptions about the benefits of risk acceptance and mitigation for different risk layers (e.g., assuming larger or smaller attachment points and considering the consequences in regards to risk insurance). Modelingwise this means that we do not assume the presence of a marginal cost curve for risk mitigation measures and the attachment point is preselected based on the DM availability of discretionary budget (see the discussion in Section 3). Note, this also means that this insurance contract is not optimal from the insurance company (supply side) perspective as important dimensions are not incorporated here, such as the reduction of investments in risk mitigation measures after being insured (Shavell, 1979a, 1979b), as well as not incorporating the dependency between risk mitigation and risk insurance (Ehrlich & Becker, 1972).

2.4 | Ambiguity

As indicated, we define ambiguity as being open to different interpretations for risk assessment input and results (SRA, 2018b). Different interpretations of risk assessment can include, for example, the selection of appropriate models to assess risk (e.g., using fundamentally different models for assessing the same kind of risk, for example, drought risk assessed by different climate models) as well as which risk description to use (e.g., expected or tail measures). Since the pioneering work of Borch (1960) and Arrow (1963),

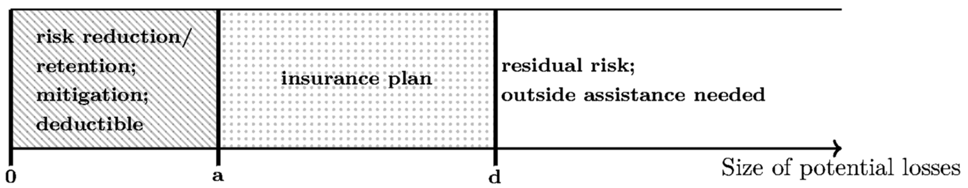

FIGURE 1 The layering approach for risk reduction and risk financing: a is the attachment point and d is the exit point of the contract

a common assumption used in the study of optimal insurance contracts is that the underlying distribution is completely known. In contrast, the notion of *ambiguity* refers to the fact that a DM may also be uncertain about the underlying distribution of the random event itself (Ellsberg, 1961). The impact of ambiguity on the insurance market was analyzed in empirical studies by Hogarth and Kunreuther (1989), Kunreuther et al. (1993), and Kunreuther et al. (1995). Their work showed that individuals prefer to base their decision on known probabilities, rather than unknown ones, paradox postulated already in the work of Ellsberg (1961). For a comprehensive study on the notion of ambiguity, see Machina and Siniscalchi (2014). We focus on ambiguity in regards to alternative probability models as discussed next.

In the classic max–min expected utility preference framework, Gilboa and Schmeidler (1989) described ambiguity in an axiomatic way, by assuming the existence of a set of possible probability models. An alternative approach based on capacities was suggested by Schmeidler (1989). These definitions were further extended in the work of Hansen and Sargent (2011), Maccheroni et al. (2006), and Asano (2012). The modeling of the insurer's belief as a set of nonunique priors can be found in Klibanoff et al. (2005), Alary et al. (2013), and Gollier (2014). A nonaxiomatic approach focuses on building neighborhoods (also known as *ambiguity sets*) around the underlying model and exploring the structure of these sets to obtain a robust contract. Ambiguity sets are defined, for example, based on the Radon–Nikodym derivative for a fixed probability distribution (Balbás et al., 2015), on the Wasserstein distance (Pflug et al., 2017) or on the set of finite number of discrete distributions (Asimit et al., 2017, 2019).

In our work, we will consider only a finite number of possible loss distributions within ambiguity sets. For example, in the case study section loss distributions from fundamentally different regional climate models are used to represent ambiguity. It was already noted that under large uncertainties or ambiguities such as in the case of climate change, precautionary and discursive strategies should be weighted higher compared to risk-informed approaches. However, robust strategies, in the sense that specified goals are achieved despite large information-gaps (SRA, 2018b), which make risk insurance effective (e.g., cover extreme risks, see also the discussion in the case study) and affordable also in a very uncertain future are of primary interest and can be incorporated within iterative approaches that also focus on precautionary or discursive strategies (Schinko & Mechler, 2017).

3 | MODEL DESCRIPTION

Based on the previous discussion, we formalize our suggested approach and start with a short introduction into distortion risk measures and their basic properties, followed by an investigation of optimal insurance contracts and risk-layers under the unambiguous and ambiguity case (defined as the case where the underlying distribution belongs to a specific set of alternative models).

3.1 | Preliminaries

Let L^∞ be the set of all bounded random variables on the probability space $(\Omega, \mathcal{F}, P)$. Each random variable $X \in L^\infty$ represents the accumulated losses over a period $[0, T]$, where at time 0 the contract is written, and at time T, the payments are settled. We denote by $X \sim F$ a random variable X with a distribution function F.

In this article, the risk faced by the DM and the insurance premium are evaluated via distortion risk measures.

Definition 1. Let $g : [0,1] \to [0,1]$ be a nondecreasing, concave function such that $g(0) = 0$ and $g(1) = 1$. The *distortion risk measure* ρ^g of a nonnegative random variable $X \sim F$ with a distortion function g is defined as a Choquet integral:

$$\rho^g(X) = \int_0^\infty g(1 - F(x))\, dx = \mathbb{E}_{F_g}(X), \tag{1}$$

where $F_g(x) := 1 - g(1 - F(x))$ is called the *distorted probability distribution* of X.

According to Theorem 6 in Dhaene et al. (2012), Definition 1 can be equivalently represented as

$$\rho^g(X) = \int_0^1 \mathrm{V@R}_p(X)\, d\tilde{g}(1-p),$$

where $\tilde{g}(p) = 1 - g(1-p)$ and *value-at-risk* is defined as

$$\mathrm{V@R}_p(X) = F^{-1}(p) := \inf\{x \in \mathbb{R} \,|\, P(X \le x) \ge p\}, \text{ for } p \in (0,1). \tag{2}$$

The definition of the distortion risk measure goes back to the axiomatic characterization of insurance premium in Wang et al. (1997), although the concept appeared earlier in Yaari's dual theory of choice (Yaari, 1987). For more reading on the properties of ρ^g and application in portfolio optimization, see Wang et al. (1997), Sereda et al. (2010), Wu and Zhou (2006), and Balbás et al. (2009).

For two random variables $X \sim F$ and $Y \sim G$, the distortion risk measure ρ^g with concave distortion function g satisfies the following properties:

1. Positive homogeneity: $\rho^g(\alpha X) = \alpha\rho^g(X)$, for $\alpha \in \mathbb{R}_+$;
2. Translation equivariance: $\rho^g(X + \alpha) = \rho^g(X) + \alpha$, for $\alpha \in \mathbb{R}$;
3. Monotonicity: $\rho^g(X) \leq \rho^g(Y)$, for $X \leq Y$ a.s.;
4. Comonotone additivity: $\rho^g(X + Y) = \rho^g(X) + \rho^g(Y)$, if X and Y are comonotone[1];
5. Version independence: $\rho^g(X) = \rho^g(Y)$, if $F = G$;
6. Convexity: $\rho^g(\alpha X + (1-\alpha)Y) \leq \alpha\rho^g(X) + (1-\alpha)\rho^g(Y)$, for any $\alpha \in [0, 1]$.

Several observations regarding the class of risk measures are in order. From the representation (1), ρ^g distorts the survival function of the loss variable X according to the weight g, while the magnitude of X is unchanged. Moreover, the translation equivariance property simplifies the optimization problem in the next section, as the insurer's premium is a constant cash flow that can be removed from the objective function. A well-known distortion risk measure used in insurance applications is *average value-at-risk*, also known as *conditional tail expectation* (CTE) or *expected shortfall* (ES).

Definition 2. The AV@R of a random variable $X \sim F$ at confidence level $p \in (0, 1)$ is

$$\mathrm{AV@R}_p(X) := \frac{1}{1-p}\int_p^1 \mathrm{V@R}_t(X)\,dt$$

provided that the integral exists. The corresponding distortion function is

$$g_p(t) = \min\left(\frac{t}{1-p}, 1\right). \tag{3}$$

Observe that the average value-at-risk (as the name suggests) averages over all losses exceeding $\mathrm{V@R}_p(X)$, while neglecting small losses. This behavior is particularly suitable for DM that is risk-averse toward extreme losses, as we will illustrate in Example 1.

For our purpose, we consider the distortion premium principle, introduced by Denneberg (1990) and further developed by Wang et al. (1997).

Definition 3. A *distortion premium principle* of a nonnegative random variable $X \sim F$ is defined as

$$\pi^g(X) = (1+\theta)\rho^g(X),$$

with constant $\theta \geq 0$ called *loading factor* and ρ^g a distortion risk measure.

Here, $\pi^g(X)$ is the annual premium asked by the insurer to fully protect against loss X. To reflect the risk-averse attitude of DM and the insurer, throughout the article, we only focus on risk measures based on the concave distortion function g. Observe that, in this case, the distorted probability distribution F_g assigns higher weights to higher quantiles. Thus, the distorted premium principle gets higher loads for low probability–high consequences events, irrespective of their monetary value. If g is concave, then $\pi^g(X) \geq \mathbb{E}(X)$, which ensures insurer survival.

3.2 | Problem formulation

The quantities that are considered in the optimization problem are the following:

B	the initial budget of the DM;
X	the loss suffered by the DM before compensation;
$\mathcal{P}$	the ambiguity set;
a_1	the attachment point of the contract;
$I(X)$	the insurance contract with fixed attachment $a_1 \geq 0$;
ρ^{g_1}	the distortion risk measure for middle losses ($x \leq a_2$);
ρ^{g_2}	the distortion risk measure for high losses ($x \geq a_2$);
c	the proportion of the budget available for the premium;
β	the proportion of the budget for the premium financed by a government;
$\pi(I(X))$	the insurance premium.

Let X be a nonnegative bounded random variable with continuous distribution function representing losses suffered by a possible policy buyer over a fixed period of time $[0, T]$ and let $[0, M]$ be the support of X. At time T, the DM affected by losses considers to hedge her risk by purchasing insurance. An insurance contract is characterized by a pair $(I(\cdot), \pi(I(\cdot)))$, where $\pi(I(\cdot))$ is the insurance premium and $I(\cdot)$ is the indemnification function/ceded function, that is, $I(x)$ is the payment offered by the insurer, if the damage is x. Typical insurance contracts include proportional contracts with indemnification function $I(x) = c\,x$, $c \in (0, 1)$, stop-loss contracts with $I(x) = \min(x, d)$, $d > 0$ or one-layer contracts $I(x) = \min(\max(x - a, 0), d - a)$, for $0 \leq a \leq d$. See Gollier (2013) and Bernard (2013) for a survey on insurance mechanisms and derivation of optimal insurance contracts.

To add complexity and relevance to our approach, we additionally assume that the government considers to incentivize the insurance purchase by subsidizing the insurance premium, either to make it affordable or under the condition that the potential beneficiary acts to reduce her risk. Let $\beta \in [0, 1]$ be the percentage of the DM's budget that the government decides to subsidize. In order to benefit from subsidies, the policyholder decides to split the available budget B into cB,

[1] Two random variables X and Y are *comonotone* if there exists a random variable $U \sim \mathrm{Uniform}[0, 1]$ such that $X = F^{-1}(U)$ and $Y = G^{-1}(U)$.

$c \in (0,1)$ for the insurance premium and $(1-c)B$ that is put aside to cover small losses $x \le a_1$, where a_1 is the attachment/deductible point. The existence of attachment point a_1 implies that the policyholder acts to accept/mitigate her risk, prior to purchasing insurance. The value of $a_1 \ge 0$ is determined *a priori* by the DM (see Section 4). For fixed $a_1 \ge 0$, the feasible set of insurance contracts with a fixed attachment point a_1 is defined as:

$$\begin{aligned}\mathcal{I}_0 :\ &= \big\{ I : [0,M] \to [0,M] : I(x)\\ &= 0 \text{ for } x \le a_1, |I(x) - I(y)|\\ &\le |x-y| \text{ for } x, y \in [0,M] \big\}.\end{aligned} \tag{4}$$

The 1-Lipschitz constraint for an admissible indemnification function (i.e., $|I(x)-I(y)| \le |x-y|$) in (4) implies that $I(X)$ and $X - I(X)$ are comonotone and bounded by X, and thus *ex post* moral hazard is avoided. See Huberman et al. (1983), Denuit and Vermandele (1998), Young (1999), and Balbás et al. (2015) for an in-depth discussion on the admissible set $\mathcal{I}_0$ in the literature of (re)insurance contract design.

However, at the end of the considered period, there will eventually be some random retained losses $X - I(X)$ that still need to be covered by the insured with possible outside assistance. In the context of natural hazards, possible causes for such losses could be extended periods of drought or intense flooding. Therefore, our objective is to find an optimal insurance contract that will minimize the risk of the retained loss, under the budget constraint of the premium.

As indicated, since catastrophic events cause significant damages, we assume that a DM assigns higher attention to extreme losses, compared to middle ones. To reflect this attitude, two risk measures are used to quantify the middle and extreme losses and thus the overall aim is to reduce the impact of these aggregated risks. We propose to quantify the risk of the retained losses by the following equation:

$$\rho_F^{g_1}(X\mathbf{1}_{A_1} - I_1(X\mathbf{1}_{A_1})) + \delta \rho_F^{g_2}(X\mathbf{1}_{A_2} - I_1(X\mathbf{1}_{A_2})), \tag{5}$$

where $A_1 := [a_1, a_2]$ and $A_2 := [a_2, M]$ are the range of middle and extreme losses, respectively, and ρ^{g_1} and ρ^{g_2} are the corresponding risk measures. The constant $\delta := \frac{g_1(1-F(a_2))}{g_2(1-F(a_2))}$ indicates the DM's different perception toward middle and high risk. In general, $\delta < 1$ suggests that the DM assigns more weight to extreme losses. The value a_2 is chosen as $\mathrm{V@R}_{q,F}(X)$ for q close to 1. In the following, the risk measures will be evaluated with respect to different probability distributions and hence we will use the notation $\rho_F^g(\cdot)$ to specify both the distortion function g and the probability distribution F.

Now we consider the classic case where the policyholder and the insurer share the same probabilistic belief about the loss X and an underlying loss model is available (e.g., estimated using past data).

In this case, we consider the following optimization problem:

Nonambiguous case.

$$\begin{aligned}&\min_{I \in \mathcal{I}_0} \rho_F^{g_1}(X\mathbf{1}_{A_1} - I(X\mathbf{1}_{A_1})) + \delta \rho_F^{g_2}(X\mathbf{1}_{A_2} - I(X\mathbf{1}_{A_2}))\\ &\text{s.t. } \pi_F^g(I(X)) \le cB.\end{aligned} \tag{P$_1$}$$

However, for future risk a considerable amount of ambiguity must be considered which leads to difficulties for determining the optimal insurance contract. As indicated in the previous section our approach is to define a set $\mathcal{P}$ that consists of different loss distributions $F_1, \dots, F_n$ which are stemming from different, but reasonable models of possible future developments (see next section), that is,

$$\mathcal{P} := \{F_1, F_2, \dots, F_n\}.$$

We call $\mathcal{P}$ the *ambiguity set*. Then the optimal insurance contract I^* is obtained by solving the following optimization problem:

Ambiguous case.

$$\begin{aligned}&\min_{I \in \mathcal{I}_0} \max_{F_i \in \mathcal{P}} \rho_{F_i}^{g_1}(X\mathbf{1}_{A_1} - I(X\mathbf{1}_{A_1})) + \delta_i \rho_{F_i}^{g_2}(X\mathbf{1}_{A_2} - I(X\mathbf{1}_{A_2}))\\ &\text{s.t. } \pi_{\hat{F}}^g(I(X)) \le cB,\end{aligned} \tag{P$_2$}$$

where $\delta_i := g_1(1-F_i(a_1))/g_2(1-F_i(a_1)) \in \mathbb{R}_+$, $i = 1, \dots, n$. Observe that the premium for the insurance contract is evaluated with respect to a single distribution $\hat{F}$. The distribution $\hat{F}$ is defined as $\hat{F}(x) := \min(F_1(x), F_2(x), \dots, F_n(x))$ for $x \ge 0$. As $\hat{F}$ dominates F_i in the first stochastic order for all $i = 1 \dots, n$, it implies that

$$\pi_{\hat{F}}^g(I(X)) \ge \pi_{F_i}^g(I(X)) \tag{6}$$

for all nondecreasing distortion functions g. By evaluating the premium with respect to the worst-case distribution in the ambiguity set $\mathcal{P}$, the actuary ensures the contract robustness and thus the protection against model ambiguity. The asymmetric information between participants to the insurance market can be explained by the heterogeneous belief of both parties. On one hand, the insurance company may possess more information about future scenarios or a better probabilistic model to forecast, and thus, there is no ambiguity regarding the underlying model. On the other hand, the policyholders face uncertainty due to model misspecification or lack of information. This uncertainty is thus captured by the ambiguity set $\mathcal{P}$. This set may be interpreted as representing heterogeneous beliefs in the insurance market. The literature of heterogeneous belief dates back to the seminal work of Schmeidler (1989), and was further extended in the work of Marshall (1992) and Huang et al. (2001).

Since, in practice, the insurance market faces limitations on loss coverage, throughout the analysis we impose the following assumption:

Assumption 1. There exists a finite constant $K > 0$ such that $\max(\rho_F^{g_i}(X), \rho_{F_j}^{g_i}(X)) \leq K$, for $i = 1, 2$ and $j = 1, \dots, n$.

Problems (P_1) and (P_2) aim to minimize the risk of the retained loss from a policyholder's perspective, while the premium is computed based on insurer's model. Therefore, the problem is formulated from the demand point of view. In this setting, we assumed that from a supply-side perspective, risk insurance is available. We refer to Meuwissen et al. (2008) and Babcock (2013) or Mahul and Stutley (2010) for a discussion on the availability of risk insurance for drought risk (used in the case study).

Nonambiguous case. By solving the easier problem (P_1), we develop techniques that are also used in the minimax setting of Problem (P_1). The following proposition characterizes the optimal insurance contract in case of a single loss distribution and under a premium principle with concave distortion function g. Observe that in (5), the losses are partitioned based on their magnitude and different risk measures are used to quantify the associated risk. Using a similar technique as in Jin and Zhou (2008), Problem (P_1) can be split into two subproblems, based on the range of $x \in A_1 \dot{\cup} A_2$. Then the optimal solution of the original problem (P_1) is a combination of the optimal solutions of the two subproblems. The splitting technique and the proof of the Proposition 1 are given in Appendix A.

Proposition 1. *Let g, g_1, and g_2 be concave distortion functions as in Definition 1.*

(a) If $\pi_F^g(X) > cB$, then an optimal insurance contract that solves Problem (P_1) is of the form:

$$I^*(x) = I_1^*(x) + I_2^*(x), \tag{7}$$

where

- $I_1^*(x) = \int_{a_1}^x \mathbf{1}_{\{g_1(1-F(z))-\lambda_1^*(1+\theta)g(1-F(z))>0\}} dz + \int_{a_1}^x \eta_1^* \mathbf{1}_{\{g_1(1-F(z))-\lambda_1^*(1+\theta)g(1-F(z))=0\}} dz$;
- $I_2^*(x) = \int_{a_2}^x \mathbf{1}_{\{\delta g_2(1-F(z))-\lambda_2^*(1+\theta)g(1-F(z))>0\}} dz + \int_{a_2}^x \eta_2^* \mathbf{1}_{\{\delta g_2(1-F(z))-\lambda_2^*(1+\theta)g(1-F(z))=0\}} dz$;
- $\lambda_1^*, \lambda_2^* \in \mathbb{R}_+$ *and* $\eta_1^*, \eta_2^* \in [0, 1]$ *such that* $\pi_F^g(I^*(X)) = cB$.

(b) If $\pi_F^g(X) \leq cB$, then $I^*(x) = x$ is the optimal solution of Problem (P_1).

The literature about the optimal insurance problem with 1-Lipschitz constraint frequently obtains bang–bang solutions as in Equation (7) (Assa, 2015; Balbás et al., 2015; Cui et al., 2013; Zhuang et al., 2016). What is important in Proposition 1 is that we are able to explicitly characterize the structure of the indemnity I^*, separately for middle and extreme losses.

In the next example, we apply the results from Proposition 1 to study the optimal indemnity, when the distortion risk measures are $\rho_F^{g_1} = \mathbb{E}_F$ and $\rho_F^{g_2} = \mathrm{AV@R}_{p,F}$, and the premium is computed by power distortion principle. For a concave distortion $g_k : [0, 1] \to [0, 1]$, $g_k(x) = x^k$, for $k \in (0, 1)$, the power premium principle for a positive random variable $X \sim F$ is defined by

$$\pi_F^{g_k}(X) = (1+\theta)\rho_F^{g_k}(X) = (1+\theta)\int_0^\infty (1 - F(x))^k dx.$$

To simplify the analysis, we assume that F is strictly increasing.

Example 1. The DM is risk-neutral with respect to middle losses $x \in A_1$, and chooses $g_1(x) = x$, while she is more risk-averse facing higher losses $x \in A_2$ and selects $g_2 = g_p$, for some $p \in (0, 1)$. The risk-averse coefficient $\delta = 1 - F(a_2)$, if $a_2 < F^{-1}(p)$ and $\delta = 1 - p$, if $a_2 \geq F^{-1}(p)$. According to Proposition 1, the optimal indemnity is of the form $I^*(x) = I_1^*(x) + I_2^*(x)$, where

$$I^*(x) = \begin{cases} \min(x - a_1, d_1 - a_1) + \min\left(x - a_2, d_2^0 - a_2\right), & \text{if } a_2 > \mathrm{V@R}_{p,F}(X), \\ \min(x - a_1, d_1 - a_1) + \min\left(\max\left(x - d_2^1, 0\right), d_2^2 - d_2^1\right), & \text{if } a_2 < \mathrm{V@R}_{p,F}(X), \end{cases}$$

where $d_1 \in [a_1, a_2]$, $d_2^0 \in [a_2, M]$, $d_2^1 \in [a_2, \mathrm{V@R}_{p,F}(X)]$ and $d_2^2 \in [\mathrm{V@R}_{p,F}(X), M]$.

The proof is straightforward and uses the monotonicity of $\frac{g_1(x)}{g_k(x)}$ and $\frac{g_2(x)}{g_k(x)}$. For instance, to obtain the structure of I_1^*, according to Proposition 1, we need to evaluate the monotonicity of $\varphi(x) := (1+\theta)(1 - F(x))^{1-k}$. Since φ is strictly decreasing and continuous over $[a_1, a_2]$, then if $\lambda_1^* \in [\varphi(a_2), \varphi(a_1)]$, there exists a unique $d_1 \in [a_1, a_2]$ such that $\varphi(d_1) = \lambda_1^*$, and thus $I_1^*(x) = \min(x - a_1, d_1 - a_1)$. If $\lambda_1^* < \varphi(a_2)$, then $\varphi(x) - \lambda_1^* > 0$ and $I^*(x) = x - a_1$; while, if $\lambda_1^* > \varphi(a_1)$, then $I_1^*(x) = 0$. The other cases follow in a similar way.

Figure 2 illustrates the shape of the optimal contract in Example 1, when $a_2 > \mathrm{V@R}_{p,F}(X)$ (left) and $a_2 < \mathrm{V@R}_{p,F}(X)$ (right). The optimal indemnity is fully characterized by the pairs of parameters $\{d_1, d_2^0\}$ and $\{d_1, d_2^1, d_2^2\}$, respectively. Hence, we reduced the feasible set $\mathcal{I}_0$ of functionals to a parametric set, with significant complexity reduction on the computation side (see Section 4). The exit point

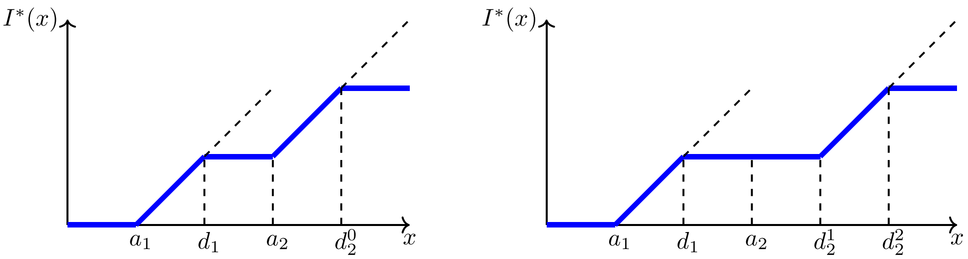

FIGURE 2 Optimal indemnity I^* in Example 1 with fixed attachment point a_1 and exit point $d = d_2^0$ (left) and $d = d_2^2$ (right)

d of the contract I^* is $d = d_2^0$, if $a_2 > \mathrm{V@R}_{p,F}(X)$ and $d = d_2^2$, otherwise.

In Example 1, we assume that the policyholder acts risk-neutral for small losses and risk-averse for large losses, for example, as assumed in expected utility theory (see, for a short summary discussion, Browne et al. (2015) and from a theoretical level, Eeckhoudt et al. (2018)). While not discussed here, it should be noted, however, that in empirical studies a surprising level of risk aversion over modest stakes were found as well (Michel-Kerjan & Kousky, 2010; Sydnor, 2010). Our methodology is flexible enough to include other types of risk preferences, that is, by choosing the appropriate distortions g_1 and g_2 that reflect the behavior of the policyholders.

Ambiguous case. In the previous analysis, we have computed ρ^{g_1}, ρ^{g_2}, and π^g under the assumption that the actuary has full information on the loss distribution F. However, the situation is often not a realistic one. The lack of data or measurement errors is just some of the inherent issues characteristic to the real-world applications. In the extreme case, the dynamics of climate change can increase the difficulty in estimating a single model. Hence, ambiguity cannot be ignored. In order to protect against ambiguity, the actuary will thus determine and price an insurance contract under different loss distributions, say $F_1, F_2, \dots, F_n$. The proof is presented in Appendix B.

Proposition 2. *Let g, g_1, and g_2 be concave distortion functions as in Definition 1.*

(a) If $\pi^g_{\hat{F}}(X) > cB$, then an optimal insurance contract that solves Problem (P_2) is of the form:

$$I^*(x) = \int_{a_1}^{x} \mathbf{1}_{A_1^+}\,dz + \int_{a_1}^{x} \eta_1^* \mathbf{1}_{A_1^0}\,dz + \int_{a_2}^{x} \mathbf{1}_{A_2^+}\,dz + \int_{a_2}^{x} \eta_2^* \mathbf{1}_{A_2^0}\,dz, \tag{8}$$

where

- $A_1^+ := \{z \in A_1 : \sum_{i=1}^n \lambda_{1,i}^* g_1(1 - F_i(z)) - \mu_1^*(1 + \theta)g(1 - \hat{F}(z)) > 0\}$;
- $A_1^0 := \{z \in A_1 : \sum_{i=1}^n \lambda_{1,i}^* g_1(1 - F_i(z)) - \mu_1^*(1 + \theta)g(1 - \hat{F}(z)) = 0\}$;
- $A_2^+ := \{z \in A_2 : \sum_{i=1}^n \delta_i \lambda_{2,i}^* g_2(1 - F_i(z)) - \mu_2^*(1 + \theta)g(1 - \hat{F}(z)) > 0\}$;
- $A_2^0 := \{z \in A_2 : \sum_{i=1}^n \delta_i \lambda_{2,i}^* g_2(1 - F_i(z)) - \mu_2^*(1 + \theta)g(1 - \hat{F}(z)) = 0\}$;
- $\lambda_{j,i}^* \in [0,1]$, *$i = 1, \dots, n$, $j = 1, 2$ such that* $\sum_{i=1}^n \lambda_{j,i}^* = 1$;
- $\mu_1^*, \mu_2^* \geq 0$, $\eta_1^*, \eta_2^* \in [0,1]$ *such that* $\pi_{\hat{F}}(I^*(X)) = cB$.

(b) If $\pi^g_{\hat{F}}(X) \leq cB$, then $I^*(x) = x$ is the optimal solution of Problem (P_2).

Note that the resulting insurance contract is robust under model misspecification since it incorporates all the considered alternative models in $\mathcal{P}$ in the decision process. Moreover, the decomposition of I^* into contracts designed for different types of risks allows an analysis of the impact of ambiguity on each layer of risk. This analysis is exemplified in the next section, where the risk measures for Problems (P_1) and (P_2) are chosen as in Example 1.

4 | APPLICATION

In this section, we apply the results presented in Section 3 to a case study of two representative farms in Austria exposed to current and future drought risk. We focus on an available data set of corn production, for which current and future loss models are estimated. Subsidies provided by the government for risk insurance have already been introduced in Austria due to extreme drought events in the recent past. Our main aim is to quantify the benefits of subsidized insurance for different types of farmers and different future loss scenarios.

We specifically focus on the representative concentration pathway (RCP) 4.5 scenario based on climate projections by Jacob et al. (2014), that were bias-corrected for agriculture-relevant weather variables (e.g., daily temperature, precipitation, and solar radiation) within the FARM and IMPACT2C projects. This scenario served as an input for an agricultural production model called EPIC model (Environmental Policy Integrated Climate model), which subsequently simulated potential crop yields from 1971–2100 for different management scenarios (Hochrainer-Stigler et al., 2019). For our purposes, we used the BAU-ISTA0-RF scenario which assumes business-as-usual fertilization, a rain-fed-based farmer and a dynamic soil profile (Balkovič et al., 2013). In more detail, the simulated crop yields for different time windows, for example, current (1971–2010) as well as the future 2050s

TABLE 1 Farm characteristics used as input for the optimization model: Available budget and losses for different models in Euro/year

Small-sized farm 1 with 30 hectares and available budget of 4847 Euro								
Return period	1000	500	250	100	50	20	10	5
Current	7303	6890	6429	5687	5030	4001	3057	1828
Model 1	11,209	10,474	9579	8294	7210	5542	4050	2256
Model 2	11,670	10,889	9914	8563	7394	5680	4161	2337
Model 3	11,909	10,855	9926	8520	7440	5722	4220	2405
Model 4	16,369	13,014	11,456	9631	8237	6286	4577	2580
Large-sized farm 2 with 100 hectares and available budget of 29,291 Euro								
Current	24,344	22,967	21,429	18,957	16,767	13,336	10,189	6094
Model 1	37,363	34,913	31,929	27,647	24,034	18,472	13,501	7522
Model 2	38,901	36,298	33,045	28,543	24,646	18,934	13,870	7790
Model 3	39,698	36,182	33,086	28,400	24,799	19,073	14,068	8016
Model 4	54,564	43,380	38,186	32,105	27,458	20,954	15,255	8598

(2041–2070), were used to estimate different types of crop distributions. For our analysis, we focused on corn production, since it is one of the most important crops in Austria and will likely be strongly affected by climate change in the future. To account for model ambiguity, we are performing this analysis for four different regional climate models. Hence, the ambiguity set considered in (P_2) is defined as $\mathcal{P} := \{F_1, F_2, F_3, F_4\}$.

For comparison reasons of different DMs, the focus will be on two types of farms, small and large ones, which face the same risk in relative terms; however, different in absolute magnitude. As indicated above, we have four different loss distributions for the future 2050 period due to different modeling approaches. The two representative farmers, called farmer 1 and farmer 2, are selected according to the official statistical definition of such farmers in Austria and are related to available discretionary budget taken from official statistics of the agricultural sector. The input information is shown in Table 1 and displays the losses and the available budget of the two farmers. All the loss distributions are given in return periods, for both the current situation (nonambiguous case) as well as for the future (ambiguous case). Farmer 1 has an available budget of 4847 Euro, that would be split to mitigate the low risk and to purchase insurance. For example, the 20-year event has an estimated loss of 4001 Euro in the current situation. However, in the future, the same 20-year event corresponds to a value of 6286 Euro, according to Model 4. These values imply that even if the farmer's initial budget suffices to cover small losses, a more involved scheme is necessary for future situations.

As observed in Table 1, it is clear that small farms will benefit more from government subsidies, as their allocated budget may not be sufficient to cover extreme losses. Hence, we aim to quantify the risk reduction part when subsidies are introduced and to assess risk insurance opportunities under future model ambiguity.

Figure 3 shows a zoom-in look at the distribution functions for the current case, as well as future losses in the case of farmer 1, according to the estimates in Table 1. We can observe that all the estimated future models dominate the current model in the first stochastic order, already suggesting an increase in future risk. For instance, the 20-year event, corresponding to $P(X \leq x) = 0.95$, corresponds to a loss of 4001 Euro in the current situation, while for future model 1, the loss increases up to 5542 Euro. Moreover, the other future models give even larger estimates for a 20-year event, suggesting a more pessimistic prediction in the context of climate change influence.

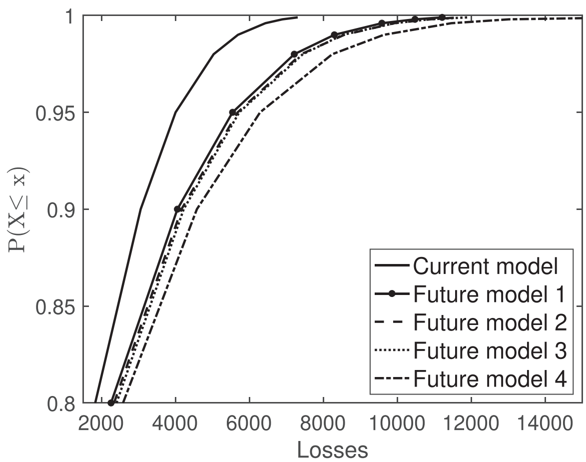

FIGURE 3 Loss distributions for farmer 1, for the current situation, as well as future situation (future model 1, up to model 4)

As already indicated in Section 2, the first question which we need to answer is how to select the attachment point a_1. We suggest a simple method to select a_1 based on the average discretionary budget of the farmer: we choose a_1 to be equal to the maximum available budget for risk mitigation, that is, $a_1 = B(1 - c)$, where c is the proportion of B allocated for premium payment. For example, assuming a budget $B = 4847$ Euro for farmer 1 and a proportion $c = 10\%$ for insurance, then the maximum attachment point $a_1 = 4362.3$

TABLE 2 Optimal parameters d_1^* and $(d_2^0)^*$ of the insurance contract in (P_1) and (P_2) (in Euro)

	a_1	a_2	d_1^*	$(d_2^0)^*$	π^g
Farmer 1 No subsidies					
Current	4362.3	5687	5646.2	5687	484.7
Future	5583.7	11,456	6830	11,456	620.4
Subsidies					
Current	4362.3	5687	5687	6890	969.4
Future	5583.7	11,456	8235	11,635	1240.8
Farmer 2 No subsidies					
Current	26,362	26,362	26,362	26,362	0
Future	33,743	38,186	38,186	52,212	3749.2
Subsidies					
Current	26,362	26,362	26,362	26,362	0
Future	33,743	38,186	38,186	54,564	3421.9

Euro. The underlying heuristic of this approach is that the DM acts risk-neutral for all losses which she could finance with her available budget and starts to behave risk-averse if this is not the case. The argument comes from the resource gap literature (Cardona et al., 2012; Gurenko & Hoeppe, 2015; Mechler et al., 2014), which argues that resource gaps will cause additional long-term or indirect effects which the DM wants to avoid. For our purpose, this approach nicely fits within the risk-layer method as it is usually the frequent events that a risk bearer wants to accept or mitigate and only afterward will start focusing on risk financing instruments (Linnerooth-Bayer & Hochrainer-Stigler, 2015). Further extensions for the choice of both a_1 and c are discussed in Section 5.

We examine again the setting in Example 1: the DM assesses the middle-sized losses by choosing $\rho_F^{g_1}(X) = \mathbb{E}_F(X)$ and extreme losses with $\rho_F^{g_2}(X) = \mathrm{AV@R}_{p,F}$, for $p = 0.9$. Let $q = 0.99$ and the corresponding $a_2 = \mathrm{V@R}_{q,F}(X)$. The risk-aversion coefficient is $\delta = (1-p)$, as $a_2 > \mathrm{V@R}_{p,F}(X)$. For computing the distortion premium, we consider a distortion function $g_k(x) = x^k$ for $k = 0.3$ and the safety loading $\theta = 0.2$. For the future ambiguous situation, we select $a_2 = \mathrm{V@R}_{q,\hat{F}}(X)$, where $\hat{F}$ is the distribution based on which the premium in (P_2) is computed. All the other quantities, that is, δ_i, ρ^{g_1}, ρ^{g_2}, depend on each model in the ambiguity set $\mathcal{P}$. Since the optimal insurance contract is determined by a finite set of parameters, we are able to solve both (P_1) and (P_2) using standard solvers. We used the MATLAB solve `fmincon` for the current situation, and `fminimax` for the future ambiguous situation. The results of the optimization procedures are shown in Table 2. Note, the attachment point a_1 is preselected based on the available budget, a_2 corresponds to $\mathrm{V@R}_{q,F}(X)$, for q chosen close to 1 and F being the distribution used to compute the premium, while the (optimal) exit points d_1^* and $(d_2^0)^*$ as well as the annual premium π^g are calculated through our theoretical model.

As indicated in Figure 1, we are especially interested in risk management strategies according to the risk-layer approach. Based on our suggestion to determine the attachment point, one already sees that they are starting at very different return periods. For example, given the available budget of around 4847 Euro for farmer 1, we select as attachment point $a_1 = 4362.3$ Euro. From Table 1, this value corresponds to a probability $F(a_1) = 0.9605$, which means $1/(1-F(a_1)) \approx 25$-year event for the current situation. We do not assume that the farmer will use his entire available budget for insurance, as this is not realistic, but rather that he would be able to invest a maximum of $c = 10\%$ of his budget for insurance purposes, that is, around 484.7 Euro. The rest of the budget is allocated to cover small losses, giving an attachment point $a_1 = 4362.3$ Euro. In this case, $a_2 = \mathrm{V@R}_{0.99,F}(X) = 5687$ Euro exceeds $\mathrm{V@R}_{0.9,F}(X)$; according to Example 1, the indemnity is a three-layered contract with fixed attachment point a_1: $I(x) = \min(x-a_1, d_1-a_1) + \min(x-a_2, d_2^0-a_2)$, for $d_1 \in [a_1, a_2]$ and $d_2^0 \in [a_2, M]$. Making advantage of our Proposition 1, we may optimize the parameters d_1 and d_2^0 over the interval $[a_1, a_2]$ and $[a_2, M]$, respectively, instead of optimizing the whole indemnity function I. The resulting optimal contract is a one-layer contract with $d_1^* = 5646.2$ and $(d_2^0)^* = a_2$. The optimal exit point $d^* = d_1^*$ suggests an investment up to the 95-year event in insurance for the current situation. Moreover, there is no insurance for extreme losses, as $(d_2^0)^* = a_2$.

For the future ambiguous situation, modeling results show that the risk increases, but the average income would also increase by around 28%; thus, we adjust the available budget accordingly (Hochrainer-Stigler et al., 2017). The insurance premium is computed for the distribution $\hat{F} := F_4$, as it dominates all the others. Similar to the single model case, the attachment point a_1 is computed for $\hat{F}$, giving a value of $a_1 = 5583.7$ Euro, corresponding to a 15-year event. From the *minimax* procedure in MATLAB, the optimal insurance is $I^*(x) = \min(x-a_1, d_1^*-a_1)$, where $\hat{F}(d_1^*) = 0.9584$, or a 24-year event.

As discussed, the government may subsidize insurance premiums by $\beta = 50\%$; hence, we repeat the calculations under this new situation. While the 25-year event still serves as the attachment point for the current situation, the exit point could be increased up to the 500-year event. The future situation is less optimistic as even under subsidized insurance (and increased budget), the exit point would decrease due to ambiguity to the 265-year event or in monetary terms 11,635 Euro. Moreover, in the case of subsidized budget, part of the extreme losses are also covered, meaning that $d^* = (d_2^0)^* = 6890$ Euro for the current situation and $d^* = (d_2^0)^* = 11,635$ Euro for the future ambiguous situation. We also observe that farmer 1 uses all the available budget allocated for premium, with or without subsidies.

Interestingly, for the current risk situation, the large farmer 2 would be able to insure all risk layers due to his large discretionary budget. However, this situation changes if one considers the model ambiguity in regards to future risks: the avail-

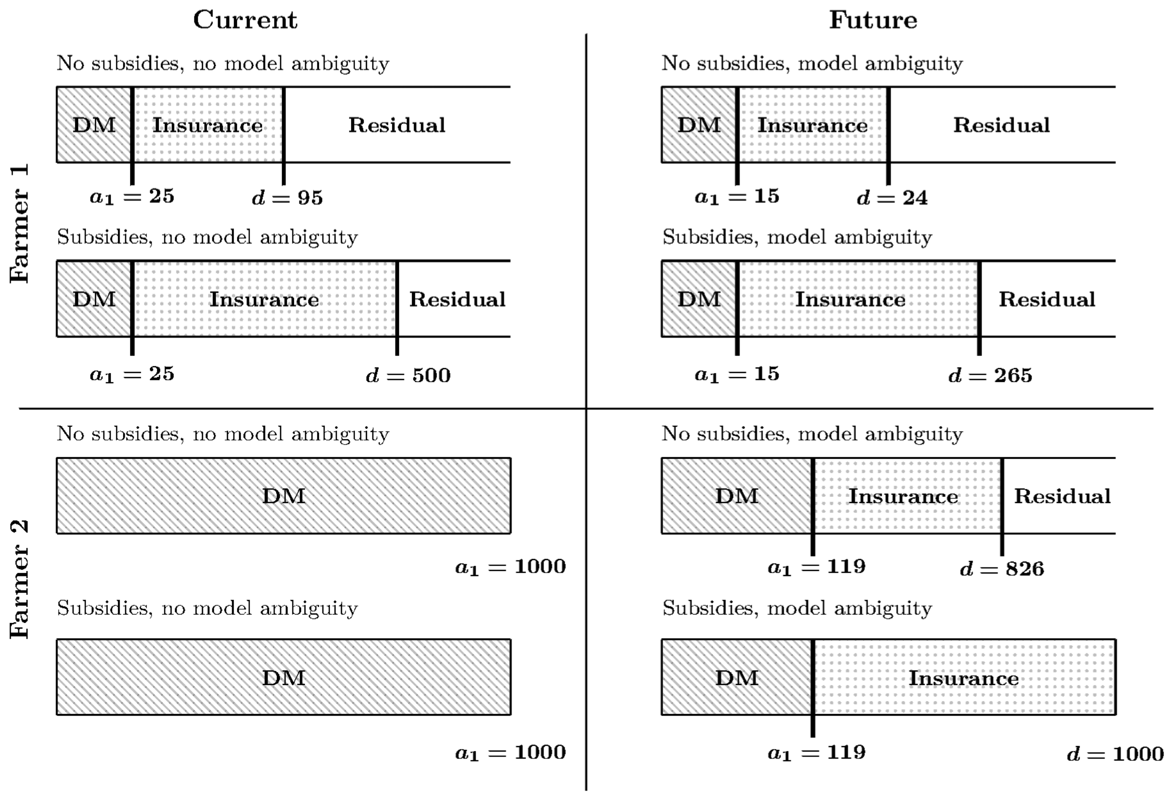

FIGURE 4 Top: optimal insurance for farmer 1 under current situation with or without subsidies and future scenarios. Bottom: optimal insurance for farmer 2 under current situation with or without subsidies and future scenarios

able budget allows the attachment point selected to be around the 826-year event, but the exit point increases considerably when the subsidies are applied, that is, from $d^* = 52,212$ Euro exit point without subsidies to a full coverage offered by insurance. Figure 4 summarizes our results for both farmers.

In Figure 4, we applied the scheme illustrated in Section 2 to our findings: the small losses, up to predetermined attachment point a_1, are covered from DM's budget, while insurance is purchased for middle-sized up to high losses. The extreme losses remain uninsured and outside assistance is likely needed. The impact of subsidies at the insurance level allows a significant increase in the coverage level, from 25 to 500 year event. However, in the future, this beneficial effect is reduced due to model ambiguity, that is, from an exit point of 24- to 265-year event. The mitigation benefits are particularly relevant for large farms, where a higher budget makes even insurance unnecessary (consider the current situation in Figure 4 in the panel). However, as climate change effects tend to intensify in the near future, it is essential to consider a mixture of mitigation and insurance to reduce possible catastrophic risk. Summarizing, the results show a quite drastic change in the future in regards to feasible risk management strategies for both farmer 1 and farmer 2, however, for the small-scale farmer 1 the consequences of increased risk and model ambiguity are much more pronounced and subsidies are especially needed in this case to cover the high-risk layer.

5 | DISCUSSION AND CONCLUSION

Risk analysis of possible climate change impacts can yield important insights into how to tackle future extreme weather events and related losses but also can contribute to the general advancement of knowledge about how to manage risks. Regarding the latter, we formulated and solved an optimization problem regarding the question how to incorporate (model) ambiguity within a risk-layer approach to manage risk (focusing on risk insurance, risk mitigation, and risk acceptance) from a demand-side perspective. The middle and high risk-layers are quantified using different distortion risk measures. The choice of the measure should reflect the DM aversion toward extreme losses. For particular choices of risks and premium, we obtained a closed-form solution, however, there are several limitations of our suggested approach which should be mentioned and are left open for future research. First, the choice of the attachment point was treated as given in the optimization problem but should be chosen based on the costs and benefits of risk mitigation and insurance within the problem formulation setup (P_1) and (P_2). One possible way forward could be the explicit inclusion of marginal cost curves for risk mitigation that would relate the diminishing returns on risk mitigation to risk insurance, for example, at some point further investment in risk mitigation would be too costly and rather risk insurance should be used instead. How to construct such risk mitigation cost curves is a challenging task and is ultimately context (e.g., what type of properties are involved, e.g., houses or infrastructure) as well as hazard-specific (e.g., earthquakes or flooding) (Michel-Kerjan et al., 2013). Second, and related to the first limitation is the fact that the closed-form solution is not optimal from a supply-side perspective, for example, for insurance companies. This is not only related to the insurability question already mentioned above but also to the question of how policyholders react to having insurance, for example, they may reduce investment in risk mitigation or would increase their risk due to moral hazard (Michel-Kerjan, 2010; Wu et al., 2020). Such important considerations were not taken into account by us and are a topic for future research as well. Third, our inclusion of subsidies was related to a demand-side perspective only but can have significant effects on the supply side. While it is true that subsidies can make insurance more affordable we did not look at what an optimal subsidization level for the insurance sector could be, nor did we included any market distortion or possible welfare deterioration effects; for example as there is no incentive for policyholders to optimally invest in mitigation (Jaspersen & Richter, 2015). In our analysis, we included the proportion of the budget c allocated for premium payments and future research could relate this budget with incentives to risk mitigation. Last but not least, a simultaneous analysis of the demand- and supply-side perspective of risk mitigation and risk insurance in the context of ambiguity is yet to be done and should better reflect the necessary tradeoffs needed to be made for a robust strategy on both sides.

Regarding the application aspects in the context of climate change risk we discussed how model ambiguity about future drought risk, stemming from the use of fundamental different climate change models, can inform farmers as well as governments about robust strategies against increasing risk. For the former regarding how to manage future risk compared to the current situation (e.g., through changes in the risk layers for risk mitigation and risk insurance), and for governments to tackle the question of whether and to what extend subsidies have to be expected for very exposed populations. Such anticipated changes have important policy implications, for example, how to finance an increase in future subsidy levels, equity considerations who should benefit and why in that regard, as well as how these changes can be viewed within a broader risk management and governance perspective. Indeed, farmers have a diverse set of risk management instruments at their disposal to cope with agricultural risks. In developed countries, two main policy strategies are available: either by putting emphasis on training, competitiveness, liberalization, and compensation, or relying on (subsidized) insurance mechanisms (Meuwissen et al., 2008). Current debates are concerned about how subsidies can eventually help establish insurance markets as well as how they can distort insurance prices in the long run (Glauber et al., 2002; Kousky et al., 2018). The current Austrian insurance scheme for drought risk is in line with the EU's vision on common agricultural policy which supports a risk management approach. Hence, insurance subsidies are the preferred option compared to compensation. However, the agriculture sector and farmers in Austria (as well as elsewhere) face a complex set of risks caused not only by weather and climate but also by global changes, including political changes, volatile prices, and markets (Hochrainer-Stigler & Hanger-Kopp, 2017). In that regard, small farms as looked at here may even struggle to survive in the future despite generous subsidies. In our analysis, we found that for small farms the attachment point is already at a very low risk-layer level, indicating problems even for very frequent events. The results uphold reality, where subsidies play a vital role. Indeed, especially for small-scale farms, nondrought-related subsidies represent nearly two-third of the farm's income in Austria. Hence, other risk management options may be more feasible under a future changing climate, including niche products and looking for additional sources of income. Future work should extend the risk-layer approach to include as many options as possible to navigate through the large uncertainties farmers are exposed to. This requires collaborative effort and continuous interaction between the relevant stakeholders on all scales, from the farmers up to the country and regional levels.

ACKNOWLEDGMENTS

The authors would like to thank editors for their valuable suggestions and support during the process, and three anonymous reviewers for their comments which have led to the improvement of the present article. The work was partially supported by the Vienna Graduate School on Computational Optimization (VGSCO) (FWF-Project No.W1260- N35) (C. Birghila), the Austrian Climate Research Program (ACRP), project FARM, project number B567169 (S. Hochrainer-Stigler), and the International Institute for Applied Systems Analysis (IIASA) (G. Pflug). The authors declare that no competing interests exist.

ORCID

Stefan Hochrainer-Stigler https://orcid.org/0000-0002-9929-8171

REFERENCES

Alary, D., Gollier, C., & Treich, N. (2013). The effect of ambiguity aversion on insurance and self-protection. *Economic Journal*, *123*(573), 1188–1202.

Amendola, A., Ermolieva, T., Linnerooth-Bayer, J., & Mechler, R. (2012). *Integrated catastrophe risk modeling: Supporting policy processes*. Springer Science & Business Media.

Arrow, K. J. (1963). Uncertainty and the welfare economics of medical care. *American Economic Review*, *53*(5), 941–973.

Asano, T. (2012). Uncertainty aversion and portfolio inertia. *Bulletin of Economic Research*, *64*(3), 334–343.

Asimit, A. V., Bignozzi, V., Cheung, K. C., Hu, J., & Kim, E.-S. (2017). Robust and Pareto optimality of insurance contracts. *European Journal of Operational Research*, *262*(2), 720–732.

Asimit, A. V., Hu, J., & Xie, Y. (2019). Optimal robust insurance with a finite uncertainty set. *Insurance: Mathematics and Economics*, *87*, 67–81.

Assa, H. (2015). On optimal reinsurance policy with distortion risk measures and premiums. *Insurance: Mathematics and Economics*, *61*, 70–75.

Aven, T. (2020). Risk science contributions: Three illustrating examples. *Risk Analysis*, *40*(10), 1889–1899.

Babcock, B. (2013). *Taxpayers, crop insurance, and the drought of 2012*. Environmental Working Group.

Balbás, A., Balbás, B., Balbás, R., & Heras, A. (2015). Optimal reinsurance under risk and uncertainty. *Insurance: Mathematics and Economics*, *60*, 61–74.

Balbás, A., Garrido, J., & Mayoral, S. (2009). Properties of distortion risk measures. *Methodology and Computing in Applied Probability*, *11*(3), 385–399.

Balkovič, J., van der Velde, M., Schmid, E., Skalskỳ, R., Khabarov, N., Obersteiner, M., Stürmer, B., & Xiong, W. (2013). Pan-European crop modelling with EPIC: Implementation, up-scaling and regional crop yield validation. *Agricultural Systems*, *120*, 61–75.

Bernard, C. (2013). Risk sharing and pricing in the reinsurance market. In *Handbook of insurance* (pp. 603–626). Springer.

Borch, K. (1960). An attempt to determine the optimum amount of stop loss reinsurance. In *Transactions of the 16th International Congress of Actuaries*, (Vol. I, pp. 597–610).

Browne, M. J., Knoller, C., & Richter, A. (2015). Behavioral bias and the demand for bicycle and flood insurance. *Journal of Risk and Uncertainty*, *50*(2), 141–160.

Cai, J., Tan, K. S., Weng, C., & Zhang, Y. (2008). Optimal reinsurance under VaR and CTE risk measures. *Insurance: Mathematics and Economics*, *43*, 185–196.

Cardona, O. D., van Aalst, M. K., Birkmann, J., Fordham, M., McGregor, G., Rosa, P., Pulwarty, R. S., Schipper, E. L. F., Sinh, B. T., Décamps, H., Keim, M., Davis, I., Ebi, K. L., Lavell, A., Mechler, R., Murray, V., Pelling, M., Pohl, J., Smith, A. O., & Thomalla, F. (2012). Determinants of risk: Exposure and vulnerability. In *Managing the Risks of Extreme Events and Disasters to Advance Climate Change Adaptation: Special Report of the Intergovernmental Panel on Climate Change* (pp. 65–108). Cambridge University Press.

Charpentier, A. (2008). Insurability of climate risks. *Geneva Papers on Risk and Insurance-Issues and Practice*, *33*(1), 91–109.

Cheung, K. C., Chong, W. F., & Lo, A. (2019). Budget-constrained optimal reinsurance design under coherent risk measures. *Scandinavian Actuarial Journal*, *2019*(9), 729–751.

Cheung, K. C., et al. (2010). Optimal reinsurance revisited—A geometric approach. *Astin Bulletin*, *40*(1), 221–239.

Chi, Y., & Tan, K. S. (2010). Optimal reinsurance under VaR and CVaR risk measures: A simplified approach. *Astin Bulletin*, *41*(2), 487–509.

Chi, Y., & Tan, K. S. (2013). Optimal reinsurance with general premium principles. *Insurance: Mathematics and Economics*, *52*(2), 180–189.

Cui, W., Yang, J., & Wu, L. (2013). Optimal reinsurance minimizing the distortion risk measure under general reinsurance premium principles. *Insurance: Mathematics and Economics*, *53*(1), 74–85.

Cummins, J. D., Lalonde, D., & Phillips, R. D. (2002). Managing risk using index-linked catastrophic loss securities. *Alternative Risk Strategies*, 19–46.

Cummins, J. D., & Mahul, O. (2004). The demand for insurance with an upper limit on coverage. *Journal of Risk and Insurance*, *71*(2), 253–264.

Cummins, J. D., & Mahul, O. (2009). *Catastrophe risk financing in developing countries: Principles for public intervention*. World Bank Publications.

Denneberg, D. (1990). Distorted probabilities and insurance premiums. *Methods of Operations Research*, *63*(3), 3–5.

Denuit, M., & Vermandele, C. (1998). Optimal reinsurance and stop-loss order. *Insurance: Mathematics and Economics*, *22*(3), 229–233.

Dhaene, J., Kukush, A., Linders, D., & Tang, Q. (2012). Remarks on quantiles and distortion risk measures. *European Actuarial Journal*, *2*(2), 319–328.

Doherty, N. A., & Schlesinger, H. (1991). Rational insurance purchasing: Consideration of contract non-performance. In *Managing the Insolvency Risk of Insurance Companies* (pp. 283–294). Springer.

Eeckhoudt, L., Fiori, A. M., & Rosazza Gianin, E. (2018). Risk aversion, loss aversion, and the demand for insurance. *Risks*, *6*(2), 60.

Ehrlich, I., & Becker, G. S. (1972). Market insurance, self-insurance, and self-protection. *Journal of Political Economy*, *80*(4), 623–648.

Ellsberg, D. (1961). Risk, ambiguity, and the Savage axioms. *Quarterly Journal of Economics*, *75*(4), 643–669.

GAR. (2015). *Global Assessment Report on Disaster Risk Reduction 2015: Making Development Sustainable: The Future of Disaster Risk Management* (Vol. 4). UNDRR.

Gilboa, I., & Schmeidler, D. (1989). Maxmin expected utility with a non-unique prior. *Journal of Mathematical Economics*, *97*, 141–153.

Glauber, J. W., Collins, K. J., & Barry, P. J. (2002). Crop insurance, disaster assistance, and the role of the federal government in providing catastrophic risk protection. *Agricultural Finance Review*, *62*(2), 81–101.

Gollier, C. (2013). The economics of optimal insurance design. In *Handbook of Insurance* (pp. 107–122). Springer.

Gollier, C. (2014). Optimal insurance design of ambiguous risks. *Economic Theory*, *57*(3), 555–576.

Gurenko, E., & Hoeppe, P. (2015). Scientific and economic rationales for innovative climate insurance solutions. In *Climate Change and Insurance* (pp. 15–28). Routledge.

Hanger, S., Linnerooth-Bayer, J., Surminski, S., Nenciu-Posner, C., Lorant, A., Ionescu, R., & Patt, A. (2018). Insurance, public assistance, and household flood risk reduction: A comparative study of Austria, England, and Romania. *Risk Analysis*, *38*(4), 680–693.

Hansen, L. P., & Sargent, T. J. (2011). Robustness and ambiguity in continuous time. *Journal of Economic Theory*, *146*(3), 1195–1223.

Härdle, W. K., Chen, C. Y.-H., & Overbeck, L. (2017). *Applied quantitative finance*. Springer.

Hecht, S. B. (2007). Climate change and the transformation of risk: Insurance matters. *UCLA Law Review*, *55*, 1559.

Hochrainer-Stigler, S., Balkovič, J., Silm, K., & Timonina-Farkas, A. (2019). Large scale extreme risk assessment using copulas: An application to drought events under climate change for Austria. *Computational Management Science*, *16*(4), 651–669.

Hochrainer-Stigler, S., & Hanger-Kopp, S. (2017). Subsidized drought insurance in Austria: Recent reforms and future challenges. *Wirtschaftspolitische Blätter*, *6*(4), 599–614.

Hochrainer-Stigler, S., Linnerooth-Bayer, J., & Lorant, A. (2017). The European Union solidarity fund: An assessment of its recent reforms. *Mitigation and Adaptation Strategies for Global Change*, *22*(4), 547–563.

Hogarth, R. M., & Kunreuther, H. (1989). Risk, ambiguity, and insurance. *Journal of Risk and Uncertainty*, *2*(1), 5–35.

Huang, J.-C., Tzeng, L. Y. & Wang, C.-C. (2001). Can an optimal insurance contract violate the principle of indemnity? Working paper.

Huberman, G., Mayers, D., & Smith, C. W., Jr, (1983). Optimal insurance policy indemnity schedules. *Bell Journal of Economics*, *14*(2), 415–426.

IPCC. (2012). Managing the risks of extreme events and disasters to advance climate change adaptation. In C. B. Field, V. Barros, T. F. Stocker, D. Qin, D. J. Dokken, K. L. Ebi, M. D. Mastrandrea, K. J. Mach, G.-K. Plattner, S. K. Allen, M. Tignor, & P. M. Midgley (Eds.), *A Special Report of Working Groups I and II of the Intergovernmental Panel on Climate Change*. Cambridge University Press.

Jacob, D., Petersen, J., Eggert, B., Alias, A., Christensen, O. B., Bouwer, L. M., Braun, A., Colette, A., Déqué, M., Georgievski, G., et al. (2014). EURO-CORDEX: New high-resolution climate change projections for European impact research. *Regional Environmental Change*, *14*(2), 563–578.

Jaspersen, J. G., & Richter, A. (2015). The wealth effects of premium subsidies on moral hazard in insurance markets. *European Economic Review*, *77*, 139–153.

Jin, H., & Zhou, X. Y. (2008). Behavioral portfolio selection in continuous time. *Mathematical Finance*, *18*(3), 385–426.

Kleindorfer, P. R., & Kunreuther, H. (1999). The complementary roles of mitigation and insurance in managing catastrophic risks. *Risk Analysis*, *19*(4), 727–738.

Klibanoff, P., Marinacci, M., & Mukerji, S. (2005). A smooth model of decision making under ambiguity. *Econometrica*, *73*(6), 1849–1892.

Kousky, C., Michel-Kerjan, E. O., & Raschky, P. A. (2018). Does federal disaster assistance crowd out flood insurance? *Journal of Environmental Economics and Management*, *87*, 150–164.

Kunreuther, H., Hogarth, R., & Meszaros, J. (1993). Insurer ambiguity and market failure. *Journal of Risk and Uncertainty*, *7*(1), 71–87.
Kunreuther, H., Meszaros, J., Hogarth, R. M., & Spranca, M. (1995). Ambiguity and underwriter decision processes. *Journal of Economic Behavior & Organization*, *26*(3), 337–352.
Linnerooth-Bayer, J., & Hochrainer-Stigler, S. (2015). Financial instruments for disaster risk management and climate change adaptation. *Climatic Change*, *133*(1), 85–100.
Maccheroni, F., Marinacci, M., & Rustichini, A. (2006). Ambiguity aversion, robustness, and the variational representation of preferences. *Econometrica*, *74*(6), 1447–1498.
Machina, M. J., & Siniscalchi, M. (2014). Ambiguity and ambiguity aversion. In *Handbook of the economics of risk and uncertainty* (pp. 729–807). Elsevier.
Mahul, O., & Stutley, C. J. (2010). *Government support to agricultural insurance: Challenges and options for developing countries*. World Bank Publications.
Marshall, J. M. (1992). Optimum insurance with deviant beliefs. In *Contributions to Insurance Economics* (pp. 255–274). Springer.
Mechler, R., Bouwer, L. M., Linnerooth-Bayer, J., Hochrainer-Stigler, S., Aerts, J. C., Surminski, S., & Williges, K. (2014). Managing unnatural disaster risk from climate extremes. *Nature Climate Change*, *4*(4), 235–237.
Meuwissen, M. P., van Asseldonk, M. A., & Huirne, R. B. (2008). *Income stabilisation in European agriculture: Design and economic impact of risk management tools*. Wageningen Academic Publishers.
Michel-Kerjan, E., Hochrainer-Stigler, S., Kunreuther, H., Linnerooth-Bayer, J., Mechler, R., Muir-Wood, R., Ranger, N., Vaziri, P., & Young, M. (2013). Catastrophe risk models for evaluating disaster risk reduction investments in developing countries. *Risk Analysis*, *33*(6), 984–999.
Michel-Kerjan, E. O. (2010). Catastrophe economics: The national flood insurance program. *Journal of Economic Perspectives*, *24*(4), 165–186.
Michel-Kerjan, E. O., & Kousky, C. (2010). Come rain or shine: Evidence on flood insurance purchases in Florida. *Journal of Risk and Insurance*, *77*(2), 369–397.
Okada, N., Chabay, I., & Renn, O. (2018). Participatory risk governance for reducing disaster and societal risks: Collaborative knowledge production and implementation. *International Journal of Disaster Risk Science*, *9*(4), 429–433.
Pflug, G. C., & Romisch, W. (2007). *Modeling, measuring and managing risk*. World Scientific.
Pflug, G. C., Timonina-Farkas, A., & Hochrainer-Stigler, S. (2017). Incorporating model uncertainty into optimal insurance contract design. *Insurance: Mathematics and Economics*, *73*, 68–74.
Renn, O., Klinke, A., & van Asselt, M. (2011). Coping with complexity, uncertainty and ambiguity in risk governance: A synthesis. *Ambio*, *40*(2), 231–246.
Renn, O., Laubichler, M., Lucas, K., Kröger, W., Schanze, J., Scholz, R. W., & Schweizer, P.-J. (2020). Systemic Risks from Different Perspectives. *Risk Analysis*.
Schinko, T., & Mechler, R. (2017). Applying recent insights from climate risk management to operationalize the loss and damage mechanism. *Ecological Economics*, *136*, 296–298.
Schmeidler, D. (1989). Subjective probability and expected utility without additivity. *Econometrica*, *57*(3), 571–587.
Scolobig, A., Prior, T., Schröter, D., Jörin, J., & Patt, A. (2015). Towards people-centred approaches for effective disaster risk management: Balancing rhetoric with reality. *International Journal of Disaster Risk Reduction*, *12*, 202–212.
Sereda, E. N., Bronshtein, E. M., Rachev, S. T., Fabozzi, F. J., Sun, W., & Stoyanov, S. V. (2010). Distortion risk measures in portfolio optimization. In *Handbook of portfolio construction* (pp. 649–673). Springer.
Shavell, S. (1979a). On moral hazard and insurance. In *Foundations of Insurance Economics* (pp. 280–301). Springer.
Shavell, S. (1979b). Risk sharing and incentives in the principal and agent relationship. *Bell Journal of Economics*, *10*(1), 55–73.
SRA (2018a). Society for Risk Analysis: Fundamental Principles. Retrieved from https://www.sra.org/risk-analysis-introduction/risk-analysis-fundamental-principles/
SRA (2018b). Society for Risk Analysis: Glossary. Retrieved from https://www.sra.org/risk-analysis-introduction/risk-analysis-glossary/
Sydnor, J. (2010). (Over) insuring modest risks. *American Economic Journal: Applied Economics*, *2*(4), 177–199.
UNDRR. (2015). *Sendai Framework for Disaster Risk Reduction 2015-2030* (Vol. 1). United Nations Office for Disaster Risk Reduction.
Wang, S. S., Young, V. R., & Panjer, H. H. (1997). Axiomatic characterization of insurance prices. *Insurance: Mathematics and Economics*, *21*(2), 173–183.
Watkiss, P., et al. (2015). *A review of the economics of adaptation and climate-resilient development*. Grantham Research Institute on Climate Change and the Environment.
Webler, T., & Tuler, S. (2021). Four decades of public participation in risk decision making. *Risk Analysis*, *41*(3), 503–518.
Wu, S., Goodwin, B. K., & Coble, K. (2020). Moral hazard and subsidized crop insurance. *Agricultural Economics*, *51*(1), 131–142.
Wu, X., & Zhou, X. (2006). A new characterization of distortion premiums via countable additivity for comonotonic risks. *Insurance: Mathematics and Economics*, *38*(2), 324–334.
Yaari, M. E. (1987). The dual theory of choice under risk. *Econometrica*, *55*(1), 95–115.
Young, V. R. (1999). Optimal insurance under Wang's premium principle. *Insurance: Mathematics and Economics*, *25*(2), 109–122.
Zhuang, S. C., Weng, C., Tan, K. S., & Assa, H. (2016). Marginal indemnification function formulation for optimal reinsurance. *Insurance: Mathematics and Economics*, *67*, 65–76.

How to cite this article: Birghila, C, Pflug, C. Ch., & Hochrainer-Stigler, S. (2022). Risk-layering and optimal insurance uptake under ambiguity: With an application to farmers exposed to drought risk in Austria. *Risk Analysis*, *42,* 2639–2655. https://doi.org/10.1111/risa.13884

APPENDIX A: PROOF OF PROPOSITION 1

If the budget allocated for insurance premium is large enough, that is, $\pi_F^g(X) \le cB$, then full insurance is optimal for (P_1). Otherwise, assume that $\pi_F^g(X) > cB$. We based the technique to solve Problem (P_1) on the idea in Jin and Zhou (2008); however, the decomposition criterion to split (P_1) into (A_1) and (A_2) below is given by the range of losses $x \in A_1 \dot{\cup} A_2$. Consider the following three problems:

- For given $\Delta \in [0, \min(cB, \pi_F(X\mathbf{1}_{A_1}))]$,

$$\inf_{I_1 \in \mathcal{I}_1} \left\{ \rho_F^{g_1}(X\mathbf{1}_{A_1} - I_1(X\mathbf{1}_{A_1})) : \pi_F^g(I_1(X\mathbf{1}_{A_1})) \le \Delta \right\}, \tag{A_1}$$

where $\mathcal{I}_1 := \{I_1 \in \mathcal{I}_0 : 0 \le I_1(X\mathbf{1}_{A_1}) \le X\mathbf{1}_{A_1}\}$.

- For same Δ as in Problem (A_1),

$$\inf_{I_2\in\mathcal{I}_2}\left\{\delta\rho_F^{g_2}(X\mathbf{1}_{A_2}-I_2(X\mathbf{1}_{A_2})):\ \pi_F^g(I_2(X\mathbf{1}_{A_2}))\right.$$
$$\left.\le\min(cB-\Delta,\pi_F^g(X\mathbf{1}_{A_2}))\right\},\qquad (A_2)$$

  where $\mathcal{I}_2:=\{I_2\in\mathcal{I}_0:\ 0\le I_2(X\mathbf{1}_{A_2})\le X\mathbf{1}_{A_2}\}$.
- For the same Δ, let $\mathrm{val}_1(\Delta)$ and $\mathrm{val}_2(\Delta)$ be the optimal values of Problems (A_1) and (A_2), respectively,

$$\inf_{\Delta\in[0,\min(cB,\pi_F^g(X\mathbf{1}_{A_1}))]}\mathrm{val}_1(\Delta)+\mathrm{val}_2(\Delta).\qquad (A_3)$$

One benefit of this approach is that we can derive separately the structure of I_1^* and I_2^*, corresponding to (A_1) and (A_2). The technique to solve these problems is the same and it is based on standard stochastic optimization tools. We present below the structure of I_1^* for completeness.

Proposition A1. *For each $\Delta\in[0,\min(cB,\pi_F^g(X\mathbf{1}_{A_1}))]$, indemnification function $I_1^*\in\mathcal{I}_1$ that solves Problem (A_1) is of the form*

$$I_1^*(x)=\int_{a_1}^x\mathbf{1}_{\{g_1(1-F(z))-\lambda^*(1+\theta)g(1-F(z))>0\}}dz+\int_{a_1}^x\eta_1^*\mathbf{1}_{\{g_1(1-F(z))-\lambda^*(1+\theta)g(1-F(z))=0\}}dz$$

for $x\in A_1=[a_1,a_2]$, where $\lambda^*\in\mathbb{R}_+$ and $\eta_1^*\in[0,1]$ satisfy $\pi_F^g(I_1^*(X\mathbf{1}_{A_1}))=\Delta$.

Proof. Denoting $R_1(X):=X-I_1(X)$ the retention random variable and using the comonotone additive property of the premium π, Problem (A_1) can be written as

$$\inf_{R_1\in\mathcal{R}}\left\{\rho_F^{g_1}(R_1(X\mathbf{1}_{A_1})):\ \pi_F^g(R_1(X\mathbf{1}_{A_1}))\ge\overline{\Delta}\right\},\qquad ({A_1}')$$

where $\overline{\Delta}:=\pi_F^g(X\mathbf{1}_{A_1})-\Delta\ge 0$, and for $A_1=[a_1,a_2]$, the feasible set is

$$\mathcal{R}:=\left\{R:[0,M]\to[0,M]:\ |R_1(x)-R_1(y)|\le|x-y|,\ \forall x,y\in[0,M]\right\}.$$

Observe that $\mathcal{R}$ is convex and compact with respect to the weak*-topology, see Cheung et al. (2019, Appendix). Moreover, as both risk measures ρ^{g_1} and ρ^g are convex in R_1, Problem (A_1) is a convex optimization problem. As R_1 is 1-Lipschitz function, it is almost everywhere differentiable on A_1. Hence, there exists some Lebesgue integrable function $h:A_1\to[0,1]$ such that $R_1(x)=\int_{a_1}^x h(z)\,dz$, for any $x\in[a_1,a_2]$. Let $\mathcal{H}$ be the feasible set of h. According to Zhuang et al. (2016, Lemma 2.1), Problem (A_1') can be written equivalently as:

$$\left\|\begin{array}{l}\min_{h\in\mathcal{H}}\int_{A_1}g_1(1-F(z))h(z)\,dz\\ \text{s.t. }(1+\theta)\int_{A_1}g(1-F(x))h(z)\ge\overline{\Delta}.\end{array}\right.\qquad ({A_1}'')$$

The corresponding Lagrange function is

$$L(h,\lambda)=\int_{A_1}\left[g_1(1-F(z))-\lambda(1+\theta)g(1-F(z))\right]h(z)\,dz+\lambda\overline{\Delta},$$

where $\lambda\in\mathbb{R}_+$. Strong duality yields that $\min_{h\in\mathcal{H}}\sup_{\lambda\in\mathbb{R}_+}L(h,\lambda)=\sup_{\lambda\in\mathbb{R}_+}\min_{h\in\mathcal{H}}L(h,\lambda)$. For some $\lambda\in\mathbb{R}_+$, a necessary and sufficient condition for a retention function h^* to be optimal is that it pointwise minimizes the integrand for each $z\in A_1$. That is, h^*, depending on λ, is of the form

$$h^*(z)=\begin{cases}0, & \text{if } z\in\{z\in A_1:\ g_1(1-F(z))-\lambda(1+\theta)g(1-F(z))>0\},\\ h(z), & \text{if } z\in\{z\in A_1:\ g_1(1-F(z))-\lambda(1+\theta)g(1-F(z))=0\},\\ 1, & \text{if } z\in\{z\in A_1:\ g_1(1-F(z))-\lambda(1+\theta)g(1-F(z))<0\}\end{cases}$$

for some arbitrary $h\in\mathcal{H}$. The corresponding optimal retention is $R_1^*(x)=\int_{a_1}^x h(z)dz,\ \forall x\in[a_1,a_2]$. In particular, any constant $h(z)=:\eta\in[0,1]$ is still a solution of (A_1''), provided the existence of the Lagrange multiplier. According to Zhuang et al. (2016, Theorem 4.1), there exists $\lambda^*\ge 0$ and $\eta^*\in[0,1]$ such that the constraint in (A_1'') is binding and thus, the corresponding h^* is optimal. □

For the same parameter Δ as in Problem (A_1), the structure of optimal indemnity in (A_2) is given below. The proof is the same as in Proposition A1, and therefore it is omitted.

Proposition A2. *For each $\Delta\in[0,\min(cB,\pi_F^g(X\mathbf{1}_{A_1}))]$, indemnification function $I_2^*\in\mathcal{I}_2$ that solves Problem (A_2) is of the form*

$$I_2^*(x)=\int_{a_2}^x\mathbf{1}_{\{\delta g_2(1-F(z))-\lambda^*(1+\theta)g(1-F(z))>0\}}dz+\int_{a_2}^x\eta_2^*\mathbf{1}_{\{\delta g_2(1-F(z))-\lambda^*(1+\theta)g(1-F(z))=0\}}dz$$

for $x\in A_2=[a_2,M]$, where $\lambda^*\in\mathbb{R}_+$ and $\eta_2^*\in[0,1]$ satisfy $\pi_F^g(I_2^*(X\mathbf{1}_{A_2}))=\min(cB-\Delta,\pi_F^g(X\mathbf{1}_{A_2})))$.

The next result shows that the solutions of Problems (A_1) and (A_2) can be combined to derive the solution of the original Problem (P_1).

Proposition A3. *Let $\Delta^* \in [0, \min(cB, \pi_F^{g_1}(X\mathbf{1}_{A_1}))]$ be the optimal solution of Problem (A_3) and let $I_1^* \in \mathcal{I}_1$ and $I_2^* \in \mathcal{I}_2$ be the optimal solutions of Problems (A_1) and (A_2) for parameter Δ^*, respectively. Then the indemnification function $I^* := I_1^* + I_2^* \in \mathcal{I}_0$ is the optimal solution of Problem (P_1).*

Proof. Since $I_1^* \in \mathcal{I}_1$ is 1-Lipschitz function on $A_1 = [a_1, a_2]$ and $I_2^* \in \mathcal{I}_2$ is 1-Lipschitz function on $A_2 = [a_2, M]$, then $I^* = I_1^* + I_2^*$ is 1-Lipschitz on $A_1 \cup A_2$. Hence, I^* is feasible for (P_1). Let $\tilde{I}$ be a feasible solution of (P_1) that can be decomposed into $\tilde{I} = \tilde{I}\mathbf{1}_{A_1} + \tilde{I}\mathbf{1}_{A_2}$, and consider $\tilde{\Delta} := \pi_F^g(\tilde{I}(X\mathbf{1}_{A_1})) \geq 0$. By definition, $\tilde{\Delta} \leq \min(cB, \pi_F^g(X\mathbf{1}_{A_1})$, hence $\tilde{\Delta}$ is feasible for (A_3). Furthermore, $\tilde{I}\mathbf{1}_{A_1}$ and $\tilde{I}\mathbf{1}_{A_2}$ are optimal solution for (A_1) and (A_2), respectively, for the same $\tilde{\Delta}$. Since Δ^* is the optimal solution for (A_3), it follows that

$$\begin{aligned}\mathrm{val}_1(\Delta^*) \quad +\mathrm{val}_2(\Delta^*) &\leq \rho_F^{g_1}(X\mathbf{1}_{A_1} - \tilde{I}(X\mathbf{1}_{A_1})) \\ &+\delta\rho_F^{g_2}(X\mathbf{1}_{A_2} - \tilde{I}(X\mathbf{1}_{A_2}))\end{aligned}$$

for any $\tilde{I} \in \mathcal{I}$; thus I^* is optimal for (P_1). $\square$

APPENDIX B: PROOF OF PROPOSITION 2

If $\rho_{\hat{F}}^g(X) \leq cB$, then $I^*(x) = x$ is obviously an optimal solution to Problem (P_2). In the subsequent analysis, we assume $\rho_{\hat{F}}^g(X) > cB$ and we use a similar procedure as in (P_1) to split the Problem (P_2).

- For given $\Delta \in [0, \min(cB, \pi_{\hat{F}}^g(X\mathbf{1}_{A_1}))]$,

$$\inf_{I_1 \in \mathcal{I}_1} \sup_{F_i \in \mathcal{P}} \left\{ \rho_{F_i}^{g_1}(X\mathbf{1}_{A_1} - I_1(X\mathbf{1}_{A_1})) : \pi_{\hat{F}}^g(I_1(X\mathbf{1}_{A_1})) \leq \Delta \right\}, \tag{B_1}$$

  where $\mathcal{I}_1 := \{I_1 \in \mathcal{I}_0 : 0 \leq I_1(X\mathbf{1}_{A_1}) \leq X\mathbf{1}_{A_1}\}$.
- For same Δ as in Problem (B_1),

$$\begin{aligned}\inf_{I_2 \in \mathcal{I}_2} \sup_{F_i \in \mathcal{P}} \Big\{ \theta_i \rho_{F_i}^{g_2}(X\mathbf{1}_{A_2} - I_2(X\mathbf{1}_{A_2})) &: \pi_{\hat{F}}^g(I_2(X\mathbf{1}_{A_2})) \\ &\leq \min(cB - \Delta, \pi_{\hat{F}}^g(X\mathbf{1}_{A_2})) \Big\},\end{aligned} \tag{B_2}$$

  where $\mathcal{I}_2 := \{I_2 \in \mathcal{I}_0 : 0 \leq I_2(X\mathbf{1}_{A_2}) \leq X\mathbf{1}_{A_2}\}$.
- For the same Δ, let $\mathrm{val}_1(\Delta)$ and $\mathrm{val}_2(\Delta)$ be the optimal values of Problems (B_1) and (B_2), respectively,

$$\inf_{\Delta \in [0, \min(cB, \pi_{\hat{F}}^g(X\mathbf{1}_{A_1}))]} \mathrm{val}_1(\Delta) + \mathrm{val}_2(\Delta). \tag{B_3}$$

Proposition B1. *For each $\Delta \in [0, \min(cB, \pi_{\hat{F}}^g(X\mathbf{1}_{A_1}))]$, indemnification function $I_1^* \in \mathcal{I}_1$ that solves Problem (B_1) is of the form*

$$I_1^*(x) = \int_{a_1}^x \mathbf{1}_{A_1^+}\, dz + \int_{a_1}^x \eta_1^* \mathbf{1}_{A_1^0}\, dz,$$

where

(i) $A_1^+ := \{z \in A_1 : \sum_{i=1}^n \lambda_{1,i}^* g_1(1 - F_i(z)) - \mu_1^*(1 + \theta)g(1 - \hat{F}(z)) > 0\}$;
(ii) $A_1^0 := \{z \in A_1 : \sum_{i=1}^n \lambda_{1,i}^* g_1(1 - F_i(z)) - \mu_1^*(1 + \theta)g(1 - \hat{F}(z)) = 0\}$;
(iii) $\lambda_{1,i}^* \in [0, 1]$, $i = 1, \ldots, n$ *such that* $\sum_{i=1}^n \lambda_{1,i}^* = 1$;
(iv) $\mu_1^* \geq 0$, $\eta_1^* \in [0, 1]$ *such that* $\pi_{\hat{F}}(I_1^*(X\mathbf{1}_{A_1})) = \Delta$.

Proof. Using the transformation $R_1 := X - I_1(X)$, for the feasible set $\mathcal{R} := \{R_1 : A_1 \to [0, M] : |R_1(x) - R_1(y)| \leq |x - y|, \forall x, y \in [0, M]\}$, Problem (B_1) is equivalent to

$$\inf_{R_1 \in \mathcal{R}} \sup_{F_i \in \mathcal{P}} \left\{ \rho_{F_i}^{g_1}(R_1(X\mathbf{1}_{A_1}) : \pi_{\hat{F}}^g(R_1(X\mathbf{1}_{A_1}) \geq \overline{\Delta} \right\}, \tag{B1}$$

with $\overline{\Delta} := \pi_{\hat{F}}^g(X\mathbf{1}_{A_1}) - \Delta \geq 0$. Similar to Proposition A1, R_1 can be characterized by its marginal indemnification function $h \in \mathcal{H}$, where $\mathcal{H} := \{h : A_1 \to [0, 1] \text{ Lebesgue integrable}\}$. Problem (B_1) becomes:

$$\min_{h \in \mathcal{H}} \max_{F_i \in \mathcal{P}} \left\{ \int_{A_1} g_1(1 - F_i(z))h(z)\, dz : \int_{A_1} g(1 - \hat{F}(z))h(z) \geq \overline{\Delta} \right\}. \tag{$B_1{}'$}$$

A standard approach for min–max problems is to introduce a new variable $\kappa \in [0, K]$, where $K := \max_{F_i \in \mathcal{P}} \rho_{F_i}^{g_1}(X\mathbf{1}_{A_1})$ is finite by Assumption 1. The epigraph representation of Problem (B_1') is given by

$$\left\| \begin{array}{l} \min_{\substack{h \in \mathcal{H},\\ \kappa \in [0,K]}} \kappa \\ \text{s.t. } \int_{A_1} g_1(1 - F_i(z))h(z)\, dz \leq \kappa, \quad F_i \in \mathcal{P}, \\ \int_{A_1} g(1 - \hat{F}(z))h(z)\, dz \geq \overline{\Delta}. \end{array} \right. \tag{$B_1{}''$}$$

Problem (B_1'') is a convex optimization problem, as the objective function is linear in κ and constant in h, while the constraints are linear in κ and h. Let $\breve{}_{\mathbf{1}} := [\lambda_{1,1}, \ldots, \lambda_{1,n}] \in \mathbb{R}_+^n$ and $\mu_1 \in \mathbb{R}_+$ be the $n + 1$ Lagrange multipliers associated with the constraints in (B_1''). Then, the Lagrange function of Problem (B_1'') is

$$\begin{aligned}L(h, \kappa, \breve{}_{\mathbf{1}}, \mu_1) = \kappa\left(1 - \sum_{i=1}^n \lambda_{1,i}\right) + \int_{A_1} \Bigg(\sum_{i=1}^n \lambda_{1,i}\, g_1(1 - F_i(x)) \\ - \mu_1(1 + \theta)g(1 - \hat{F}(z))\Bigg) h(z)dz + \mu_1\overline{\Delta}.\end{aligned}$$

Since the strong duality holds, the minimization of L with respect to κ implies the simplex constraint $\sum_{i=1}^{n} \lambda_{1,i} = 1$. Similar to the proof in Proposition A1, the pointwise minimization of the Lagrange function with respect to z yields that the optimal retention $R_1^*(x) = \int_{a_1}^{x} h^*(z)\,dz$, where h^* of the form:

$$h^*(z) = \begin{cases} 0, & \text{if } z \in \left\{ z \in A_1 : \sum_{i=1}^{n} \lambda_{1,i}\, g_1(1 - F_i(z)) - \mu_1(1+\theta) g(1 - \hat{F}(z)) > 0 \right\}, \\ \eta, & \text{if } z \in \left\{ z \in A_1 : \sum_{i=1}^{n} \lambda_{1,i}\, g_1(1 - F_i(z)) - \mu_1(1+\theta) g(1 - \hat{F}(z)) = 0 \right\}, \\ 1, & \text{if } z \in \left\{ z \in A_1 : \sum_{i=1}^{n} \lambda_{1,i}\, g_1(1 - F_i(z)) - \mu_1(1+\theta) g(1 - \hat{F}(z)) < 0 \right\} \end{cases}$$

for some constant $\eta \in [0,1]$. Since the premium in (B_1) depends only on a single model $\hat{F}$, the existence of Lagrange multiplier μ_1^* and constant η^* such that $\pi_{\hat{F}}^{g}(R_1^*(X\mathbf{1}_{A_1})) = \overline{\Delta}$ follows similar to Proposition (A_1). Moreover, since the feasible set of $\breve{\ }_{\mathbf{1}}$ is a simplex, hence compact and convex, then there exists some $\breve{\ }_{\mathbf{1}}^{*} = [\lambda_{1,1}^*, \dots, \lambda_{1,n}^*]^\top \geq 0$ such that $\lambda_{1,i} \in [0,1]$, $i = 1, \dots, n$, $\sum_{i=1}^{n} \lambda_{1,i} = 1$ and $\kappa = \sum_{i=1}^{n} \lambda_{1,i}^* \rho_{F_i}^{g_1}(R_1^*(X\mathbf{1}_{A_1}))$. □

Remark B1.

- Using a similar proof as in Proposition B1, the optimal indemnity function $I_2^* \in \mathcal{I}_2$ of Problem (B_2) is of the form

$$I_2^*(x) = \int_{a_2}^{x} \mathbf{1}_{A_2^+}\,dz + \int_{a_2}^{x} \eta_2^* \mathbf{1}_{A_2^0}\,dz,$$

  where the sets $A_2^+ := \{z \in A_2 : \sum_{i=1}^{n} \delta_i \lambda_{2,i}^* g_2(1 - F_i(z)) - \mu_2^*(1+\theta) g(1 - \hat{F}(z)) > 0\}$, $A_2^0 := \{z \in A_2 : \sum_{i=1}^{n} \delta_i \lambda_{2,i}^* g_2(1 - F_i(z)) - \mu_2^*(1+\theta) g(1 - \hat{F}(z)) = 0\}$, $\lambda_{2,i}^*$ satisfy the same conditions as in Proposition B1, (iii), and $\mu_2^* \in \mathbb{R}_+$ and $\eta_2^* \in [0,1]$ are such that $\pi_{\hat{F}}(I_2^*(X\mathbf{1}_{A_2})) = \min(cB - \Delta, \pi_{\hat{F}}^{g}(X\mathbf{1}_{A_2})))$.
- According to Proposition A3, the optimal solution of Problem (B_3) is obtained as combination of the optimal solution of Problems (B_1) and (B_2), respectively, for some optimal $\Delta^* \in [0, \min(cB, \pi_{\hat{F}}^{g}(X\mathbf{1}_{A_1}))]$.