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INTERPOLATED BOUNDARY CONDITIONS IN THE FINITE ELEMENT METHOD*

RIDGWAY SCOTT†

Abstract. This paper shows that the technique introduced in Berger, Scott and Strang [2] can achieve optimal accuracy if the approximating functions interpolate boundary conditions at the Lobatto quadrature points for each element edge on the boundary. No modification of the energy form is required. Estimates are derived in lower norms as well as in the energy norm. A numerical integration scheme is presented that yields optimal accuracy for piecewise quadratics.

Introduction. In this paper, we continue the analysis begun in [2] with Berger and Strang of a finite element technique that (as we shall prove here) treats curved boundaries accurately. The idea is to approximate the solution of a variational problem using a finite-dimensional space of piecewise polynomials that *interpolate* homogeneous boundary conditions at boundary nodes. Thus the boundary conditions are satisfied only at a finite set of points on the boundary, and the rules of the ordinary Ritz–Galerkin theory are violated. The motive for interpolating the boundary conditions is that when the boundary is curved, a smooth function satisfying homogeneous boundary conditions cannot be approximated closely by piecewise polynomials that satisfy the boundary conditions exactly [1], [12], [16]. Our finite element technique does not compensate for the nonsatisfaction of boundary conditions by modifying the energy inner-product (i.e., using penalty functions, Lagrangian multipliers, etc.). Rather, we seek the right spacing of boundary interpolation nodes that makes modification of the energy inner product unnecessary. In [2], we derived the basic error estimates for the interpolated boundary condition technique, but failed to see the right choice of boundary interpolation nodes that gives higher accuracy. Our goal here is to describe this choice.

Optimal accuracy for the special case of piecewise quadratics was first verified by Alan Berger in his thesis (MIT, 1972). His results were announced in [3], which also contains a summary of some computations using the interpolated boundary condition technique. J. Nitsche [10] has also studied subspaces that interpolate boundary conditions. In [10], the basic energy inner product is modified to obtain optimal accuracy with subspaces (e.g., the one studied in [2]) that do not interpolate boundary conditions accurately (in the sense of our Lemma 1).

In this paper, we treat second order elliptic problems. In a later paper, we shall discuss higher order elliptic problems, such as the (4th order) plate-bending problem. In both papers, we deal only with a two-dimensional domain. We remark that our introduction to the interpolated boundary condition technique was via the work of a group of Canadian engineers [6], who used it to treat the plate-bending problem.

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1. Notation. Let Ω be a bounded domain in $\mathbb{R}^2$, with smooth boundary $\partial\Omega$. We denote by $\|\cdot\|_m$ ($m \geqq 0$) the norm for the Sobolev space $H^m(\Omega)$:

$$\|v\|_m^2 = \sum_{|\alpha| \leqq m} \int_\Omega |D^\alpha v|^2, \qquad m \geqq 0.$$

When m is negative, we mean the dual norm:

$$\|v\|_m = \sup_{\varphi \in H^{-m}(\Omega)} \frac{|(v, \varphi)|}{\|\varphi\|_{-m}}, \qquad m \leqq 0,$$

where $(\cdot, \cdot)$ is the $L^2(\Omega)$-inner product. Similarly, we denote by $|\cdot|_m$ the Sobolev norm for $H^m(\partial\Omega)$: if σ denotes arc length for $\partial\Omega$, then

$$|v|_m^2 = \sum_{j=0}^{m} \int_{\partial\Omega} \left|\frac{\partial^j v}{\partial \sigma^j}\right|^2 d\sigma,$$

$$|v|_{-m} = \sup_{\varphi \in H^m(\partial\Omega)} \frac{|\int_{\partial\Omega} \varphi v|}{|\varphi|_m}, \qquad m \geqq 0.$$

We denote by $\mathring{H}^1(\Omega)$ the space of functions in $H^1(\Omega)$ that are zero on $\partial\Omega$. Our third and final norm is the $C^m(\Omega)$-norm:

$$[v]_m = \sup_{\substack{\Omega \\ |\alpha| \leqq m}} |D^\alpha v|.$$

We shall also use restrictions of these norms. Given any subset ω of $\mathbb{R}^2$, we denote by $\|\cdot\|_{\omega,m}$ and $[\cdot]_{\omega,m}$ the $H^m(\omega)$- and $C^m(\omega)$-norms. If $\omega \subset \partial\Omega$, then $|\cdot|_{\omega,m}$ will mean the boundary norm restricted to ω. These restrictions will be used only when $m \geqq 0$, and when the subscript is omitted, $\omega = \Omega$ (or $\partial\Omega$) is implied.

2. Description of the interpolated boundary condition technique. Suppose we want to approximate the solution u of a variational problem of the following form: Given f in $L^2(\Omega)$, find u in $\mathring{H}^1(\Omega)$ such that

$$a(u, v) = (f, v) \quad \text{for all } v \text{ in } \mathring{H}^1(\Omega), \tag{2.1}$$

where

$$a(u, v) \equiv \int_\Omega \sum_{|\alpha|, |\beta| \leqq 1} c_{\alpha\beta} D^\alpha u D^\beta v. \tag{2.2}$$

This corresponds to solving the boundary value problem

$$\begin{aligned} Au \equiv \sum_{|\alpha|, |\beta| \leqq 1} (-1)^{|\beta|} D^\beta c_{\alpha\beta} D^\alpha u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega.^1 \end{aligned} \tag{2.3}$$

We shall assume that A is elliptic [9], that the coefficients $c_{\alpha\beta}$ are smooth in $\overline{\Omega}$, and that $a(\cdot, \cdot)$ is coercive:[2]

$$\gamma_0 \|v\|_1^2 \leqq |a(v, v)| \quad \text{for all } v \text{ in } \mathring{H}^1(\Omega). \tag{2.4}$$

[1] Inhomogeneous boundary data is treated in §5.

[2] It may be possible to weaken this requirement by using an idea due to Schatz [11].

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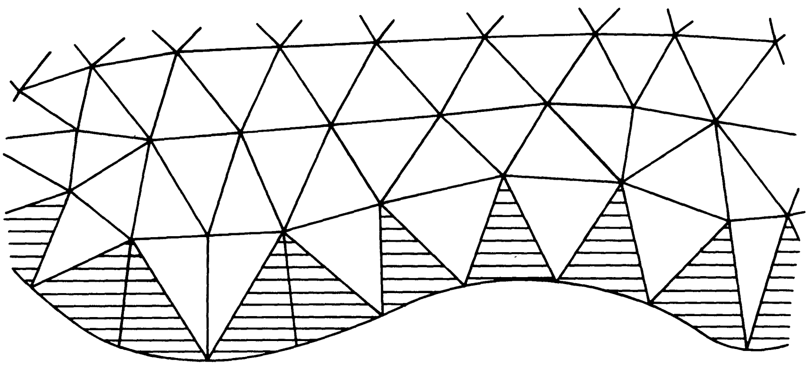

FIG. 1. *Pie-shaped boundary elements are shaded*

The approximation u^h to u is defined by a variational equation similar to (2.1), but with the variations over a finite-dimensional space $S^h \subset H^1(\Omega)$. We define u^h in S^h by

$$a(u^h, v) = (f, v) \quad \text{for all } v \text{ in } S^h. \tag{2.5}$$

To insure that u^h is well-defined, we must assume that the bilinear form $a(\cdot, \cdot)$ is nondegenerate on S^h (since $S^h \not\subset \mathring{H}^1(\Omega)$), namely, we assume that

$$v \text{ in } S^h, a(v, v) = 0 \quad \text{implies} \quad v = 0. \tag{2.6}$$

We now describe the spaces S^h studied in this paper. The construction of S^h begins with the "finite element" idea of dividing Ω into a collection of (non-overlapping) elements e_j: $\bar{\Omega} = \bigcup_j e_j$. The elements must be of two types because $\partial\Omega$ is curved. In the interior, we simply use triangles, but at the boundary, we use triangles with one edge replaced by a segment of the boundary. We refer to these boundary elements as *pie-shaped* because where the boundary is convex, they resemble a slice of pie. (See Fig. 1). We demand that the division of Ω into elements satisfy the usual rules of a *triangulation*, and we shall refer to it as such. As a technical device, we need the following definition.

DEFINITION 1. We say that a family of triangulations of Ω (depending on a parameter h) is *nondegenerate* if the ratio of the radii r_1 and r_2 of the circumscribed and inscribed circles of each element is bounded:

$$r_1/r_2 \leqq K \text{ for all elements and all } h.$$

(For a boundary element e, the circumscribed circle $\bar{e}$ is the smallest one containing e, and the inscribed circle $\underline{e}$ is the largest one contained both in e and in the triangle e_0 formed from e by joining its boundary vertices with a straight line.)

The reason we need this definition is that it allows us to relate the L^2 and L^∞ norms of polynomials in each element, namely, for all polynomials P of degree $k - 1$, we have

$$[P]_{\bar{e},s} \leqq c r_1^{m-s-1} \|P\|_{\underline{e},m}, \qquad 0 \leqq m \leqq s, \tag{2.7}$$

where c depends only on k and K. This is proved by homogeneity: Expand coordinates so that $\bar{e}$ is a disc of radius 1, use the equivalence of norms for polynomials

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of degree $k - 1$, and then contract, keeping track of the powers of r_1.

From now on, we consider a nondegenerate family of triangulations of Ω, parametrized by h = maximum radius (r_1 in Definition 1) of the elements in the triangulation.[3] For each h, we denote by $\bar{S}^h$ the space of continuous piecewise polynomials of degree $k - 1$ on the corresponding triangulation; i.e., the subspace of $C^0(\Omega)$ that consists of functions whose restriction to each element of the given triangulation is a polynomial of degree $\leqq k - 1$.[4] We emphasize that the "displacement" functions in $\bar{S}^h$ are polynomials even in the boundary elements. This differs from the *isoparametric* technique [7], [17] where the displacement functions in the (curved) boundary elements are polynomials not in the Cartesian coordinates but in the isoparametric coordinates. When h is sufficiently small relative to the radius of curvature of $\partial\Omega$ (as we now assume it is), the functions in $\bar{S}^h$ can be determined by nodal values, in this case, their values at

(2.8)
(a) the vertices in the triangulation,
(b) $k - 2$ points in the interior of each edge (including the curved boundary edges), and
(c) $\frac{1}{2}(k - 3)(k - 2)$ points in the interior of each element, chosen so that if a polynomial of degree $k - 4$ vanishes at them, it vanishes identically.

(See Fig. 2 for an illustration of the $k = 3$ case. Such nodal values give rise to a nodal basis of functions that are 1 at one node, zero at all others. This basis would usually be used to turn (2.5) into a matrix equation).

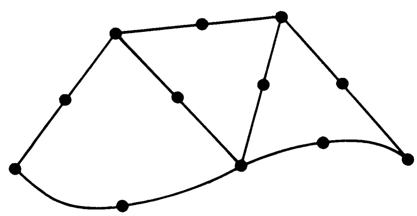

FIG. 2. *Nodal configuration for piecewise quadratics*

It turns out that it is only the boundary edge nodal placement that affects the accuracy of u^h, and we describe that now. Let e be a boundary element, and choose Cartesian coordinates so that the boundary vertices of e lie on the x-axis, with one of them at the origin $x = 0$. We now assume that the mesh size h is small enough so that $\partial e = e \cap \partial\Omega$ can be represented as a graph:

$$\partial e = \{(x, \rho(x)) : 0 \leqq x \leqq x_0\}.$$

(See Fig. 3.) Let p be a polynomial of degree $k - 1$ that approximates ρ accurately,

$$\sup_{0 \leqq x \leqq x_0} |\rho(x) - p(x)| \leqq c x_0^k, \tag{2.9}$$

[3] We assume that h takes on a sequence of values tending to zero. Strictly speaking, the constants in Theorems 1–5 depend on the largest h in addition to the quantities metioned there.

[4] We assume $k \geqq 3$ as the piecewise linear case is well studied.

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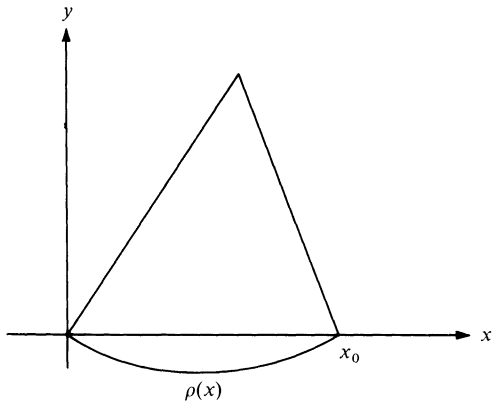

FIG. 3

and such that $p(0) = p(x_0) = 0$. (Thus, p could be an interpolate of ρ, but we leave flexible the type of interpolation used.) We place boundary edge interpolation nodes at the points

$$(\xi_i x_0, p(\xi_i x_0)), \qquad i = 1, 2, \cdots, k-2, \tag{2.10}$$

where the points $0 < \xi_1 < \xi_2 < \cdots < \xi_{k-2} < 1$ will be fixed.[5] (We make the same choice for all boundary elements e and for all h.) To interpolate the homogeneous boundary data, we let S^h be those functions in $\bar{S}^h$ that are zero at the boundary vertices and the boundary edge nodes described in (2.10). The choice of the points ξ_i in [0, 1] determines the rate of convergence of u^h to u, and the optimal choice is based on the Lobatto quadrature rule [8], [15]. We recall that the quadrature points for the Lobatto rule for [0, 1] are the k roots of

$$x(1-x)P'_{k-1}(x) = c_k\left(\frac{d}{dx}\right)^{k-2}[x(1-x)]^{k-1},$$

where P_n is the nth Legendre (orthogonal) polynomial. We choose the ξ_i as the $k-2$ roots in]0, 1[. Thus each function in S^h vanishes at the Lobatto quadrature points for ∂e for each boundary element e.

THEOREM 1. *Let S^h be the space (described above) of continuous piecewise polynomials of degree $k-1$ that are zero at the Lobatto quadrature points for ∂e for all boundary elements e. Let u and u^h solve* (2.1) *and* (2.5), *respectively. Then*

$$\|u - u^h\|_1 \leqq ch^{k-1}\|u\|_k,$$

where c depends only on k, Ω, γ_0 (see (2.4)), K (Definition 1), *the $C^{k-2}(\Omega)$-norm of the coefficients $c_{\alpha\beta}$, and the constant in estimate* (2.9).

We postpone the proof of Theorem 1 (the main result of the paper) until the next section, concluding this section with some remarks about Theorem 1.

[5] The boundary nodes are allowed to be slightly off $\partial\Omega$ to make the method computationally simple. We could also have parametrized ∂e by arc length $\sigma_1 \leqq \sigma \leqq \sigma_2$ and placed the nodes at $\sigma_1 + \xi_i(\sigma_2 - \sigma_1)$. The analysis is simpler in this case (with the same optimal results), but we feel the nodes in (2.10) are easier to locate in practice.

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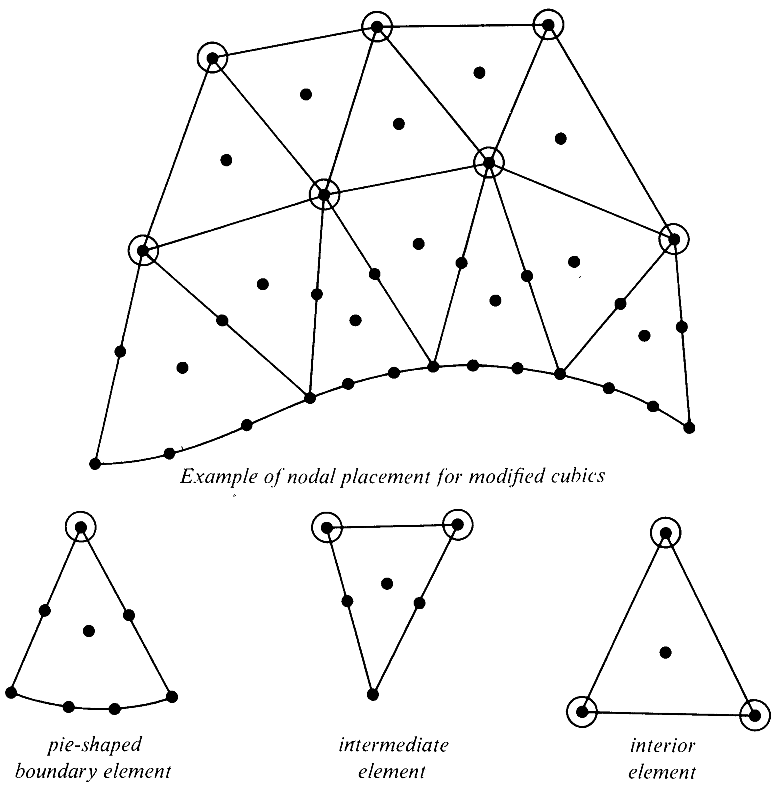

● means value of function is interpolated
○ means first derivatives are interpolated

FIG. 4

Remark 1. The placement of the boundary edge nodes is crucial to obtaining optimal accuracy. In [2], we proved that (for generic u) $\|u - u^h\|_1 \geqq ch^{3/2}$ for a piecewise cubic space with different boundary nodes.

Remark 2. Theorem 1 is also true (with the same proof) if we start with $\bar{S}^h$ as the (Hermite) space of continuous piecewise polynomials of degree $k - 1$ that are C^1 at the interior vertices ($k \geqq 4$). In terms of the nodal representation of $\bar{S}^h$, this corresponds to sliding an edge node on each interior edge to the adjacent interior vertex, with the boundary edge nodes remaining the same. (When nodes coalesce in this fashion, they become derivative nodes.) An example of the nodal placement for piecewise cubics ($k = 4$) is shown in Fig. 4.

Remark 3. For $k = 3$, the interior Lobatto point corresponds to the midpoint of ∂e ($\xi_1 = \frac{1}{2}$). For $k = 4$, we have $\xi_1, \xi_2 = \frac{1}{2} \pm \sqrt{1/20}$. See [15] for values of the Lobatto quadrature points, accurate to 32 digits, for $k = 3, 4, \cdots, 32, 36, \cdots, 64$.

3. Proof of Theorem 1. Let $C < \infty$ and $\gamma = \gamma(h) > 0$ be numbers such that

(3.1) $$|a(u, v)| \leqq C\|u\|_1\|v\|_1 \quad \text{for all } u, v \text{ in } H^1(\Omega),$$

and

(3.2) $$\gamma\|v\|_1^2 \leqq |a(v, v)| \quad \text{for all } v \text{ in } S^h.$$

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($\gamma > 0$ exists because of (2.6). We show later in Lemma 2 that γ can be chosen independently of h.) Our first step is to apply the basic error estimate of [2] (generalized to variable $c_{\alpha\beta}$—see § A.1 of the Appendix for a proof). This says that for any integers $\geqq 0$, the error $u - u^h$ satisfies

$$\|u - u^h\|_1 \leqq \left(1 + \frac{C}{\gamma}\right) \inf_{v \in S^h} \|u - v\|_1 + \frac{c_s}{\gamma} \|u\|_{s+2} \sup_{v \in S^h} \frac{|v|_{-s}}{\|v\|_1}. \tag{3.3}$$

The constant c_s depends on s and Ω (via the trace theorem) and the $C^s(\Omega)$-norm of the coefficients $c_{\alpha\beta}$. By the approximation theorem in § A.3 of the Appendix, we have $\inf_{v \in S^h} \|u - v\|_1 \leqq ch^{k-1}\|u\|_k$, so we are done if we prove the following lemma.

LEMMA 1. *Let S^h be as in Theorem 1. Then for* $-1 \leqq s \leqq k - 2$, *we have*

$$\sup_{v \in S^h} \frac{|v|_{-s}}{\|v\|_1} \leqq ch^{s+3/2},$$

where c depends only on Ω, K, *and the constant in* (2.9).

Remark 1. Notice that the boundary condition error is smaller by a factor $h^{1/2}$ than the approximation error.

Remark 2. Suppose we chose S^h to be those functions v in $\bar{S}^h$ that are zero at the boundary vertices and, for each boundary element e, instead of being zero at the interior Lobatto points for ∂e, satisfy

$$\int_{\partial e} Pv = 0 \quad \text{for all polynomials (in arc length) } P \text{ with } \deg P \leqq k - 3. \tag{3.4}$$

The functions in this S^h satisfy an extended Poincaré inequality,

$$|v|_{-s} \leqq ch^s |v|_0 \quad \text{for } v \text{ in } S^h, \quad 0 \leqq s \leqq k - 2,$$

and we can show (similar to (3.7)) that

$$|v|_0 \leqq c' h^{3/2} \|v\|_1 \quad \text{for } v \text{ in } S^h.$$

The point of this paper is that the more complicated condition (3.4) need not hold exactly; that is, if v is zero at the Lobatto points for ∂e, then $\int_{\partial e} Pv$ is appropriately small. We note that condition (3.4) indicates a relationship between estimate (3.3) and what is known in solid mechanics as Saint-Venant's principle.

Proof of Lemma 1. Let e be a boundary element, choose coordinates as in Fig. 3, and let ρ and p be the functions used to determine the boundary edge nodes. For v in S^h and φ in $H^s(\partial\Omega)$, $0 \leqq s \leqq k - 2$, we shall prove that [6]

$$\begin{aligned} \left|\int_{\partial e} v\varphi\right| &= \left|\int_0^{x_0} v(x, \rho(x))\varphi(\sigma(x)) \frac{d\sigma}{dx}\, dx\right| \\ &\leqq ch^{s+3/2} \|v\|_{e,1} |\varphi|_{\partial e, s}. \end{aligned} \tag{3.5}$$

[6] $\sigma(x) = \int_0^x \sqrt{1 + \dot{\rho}^2}$ is the arc length of ∂e.

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Summing this estimate over all boundary elements e proves the lemma for $0 \leqq s \leqq k-2$. The case $s=-1$ follows the proof of (3.5). To prove (3.5), we approximate $\varphi(d\sigma/dx)$ by a polynomial ψ of degree $s-1$ ($\psi=0$ if $s=0$):

$$\int_0^{x_0} \left|\varphi(\sigma(x))\frac{d\sigma}{dx} - \psi(x)\right|^2 dx \leqq cx_0^s|\varphi|_{\partial e,s}.$$

(The notation c is used here and in the rest of the paper for various constants depending only on the specified quantities in the respective theorem or lemma. Here c depends on s and the derivatives of ρ through order $s+1$).
Therefore

$$\begin{aligned}\left|\int_{\partial e} v\varphi\right| &\leqq \left|\int_0^{x_0} (v(x,\rho(x)) - v(x,p(x)))\varphi(\sigma(x))\frac{d\sigma}{dx}\,dx\right| \\ &\quad + \left|\int_0^{x_0} v(x,p(x))\psi(x)\,dx\right| \\ &\quad + \left|\int_0^{x_0} v(x,p(x))\left(\varphi(\sigma(x))\frac{d\sigma}{dx} - \psi(x)\right)dx\right| \\ &\leqq cx_0^{s+3/2}\|v\|_{e,1}|\varphi|_{\partial e,0} + \left|\int_0^{x_0} v(x,p(x))\psi(x)\,dx\right| \\ &\quad + \left(\int_0^{x_0} v(x,p(x))^2\,dx\right)^{1/2} cx_0^s|\varphi|_{\partial e,s}.\end{aligned} \tag{3.6}$$

For the first term, we used the mean value theorem and (2.7) to estimate v_y. We have reduced the problem to studying polynomials in x, $v(x,p(x))$ and $v(x,p(x))\psi(x)$. For the former, we have

$$\sup_{0\leqq x\leqq x_0} |v(x,p(x))| \leqq cx_0\|v\|_{e,1}. \tag{3.7}$$

To prove this, we define, for a function $g(x)$, an interpolate g_I as the polynomial of degree $k-1$ such that $(g-g_I)(x)=0$ at the k Lobatto points for $[0,x_0]$. Thus

$$\sup_{0\leqq x\leqq x_0} |(g-g_I)(x)| \leqq cx_0^k \sup_{0\leqq x\leqq x_0} |g^{(k)}(x)|.$$

Applying this with $g(x)=v(x,p(x))$, we have

$$\sup_{0\leqq x\leqq x_0} |v(x,p(x))| \leqq cx_0^k \sup_{0\leqq x\leqq x_0} \left|\left(\frac{d}{dx}\right)^k v(x,p(x))\right|,$$

since the interpolate is $=0$. Applying (2.7) yields (3.7), where the constant c depends on the derivatives of p. To estimate $\int_0^{x_0} v(x,p(x))\psi(x)\,dx$, we simply recall [8] the form of the error term in Lobatto quadrature, since the integrand vanishes at the Lobatto points. Thus we have (using (2.7))

$$\begin{aligned}\left|\int_0^{x_0} v(x,p(x))\psi(x)\,dx\right| &\leqq cx_0^{2k-1} \sup_{0\leqq x\leqq x_0}\left|\left(\frac{d}{dx}\right)^{2k-2} v(x,p(x))\psi(x)\right| \\ &\leqq cx_0^{k-1/2}\|v\|_{e,1}|\varphi|_{\partial e,s}.\end{aligned} \tag{3.8}$$

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This completes the proof of (3.5), since $x_0 \leqq 2h$.

We now prove the lemma for $s = -1$. From (3.7) and the one-dimensional version of (2.7), we have

$$\sup_{0 \leqq x \leqq x_0} \left| \frac{d}{dx} v(x, p(x)) \right| \leqq c \|v\|_{e,1}.$$

Applying the mean value theorem, (2.9) and (2.7), we thus have

$$\sup_{0 \leqq x \leqq x_0} \left| \frac{d}{dx} v(x, \rho(x)) \right| \leqq c \|v\|_{e,1}.$$

We want to bound $\partial v/\partial\sigma$ on ∂e, and this may be written as

$$\frac{\partial}{\partial\sigma} v(x, \rho(s)) = \frac{dx}{d\sigma} \frac{d}{dx} v(x, \rho(x)).$$

We have $dx/d\sigma = (1 + \dot{\rho}(x)^2)^{-1/2} \leqq 1$, so we find that

$$\sup_{\partial e} \left| \frac{\partial v}{\partial\sigma} \right| \leqq c \|v\|_{e,1} \quad \text{for } v \text{ in } S^h.$$

Squaring and integrating over ∂e, then summing over all boundary elements e, we complete the lemma.

To complete the proof of Theorem 1, we must show that the constant γ in (3.2) can be chosen independently of h. To do so, we introduce nonintegral Sobolev boundary norms, which are defined by interpolation [9].

LEMMA 2. *Suppose that* $a(\cdot, \cdot)$ *satisfies* (2.4) *and* (3.1). *Then there is a constant* $c = c(\Omega, \gamma_0, C)$ *such that*

$$|a(v, v)| \geqq \gamma_0 \left(1 - c \frac{|v|_{1/2}}{\|v\|_1} \right) \|v\|_1^2,$$

for all v *in* $H^1(\Omega)$.

Proof. Let P be the orthogonal projection of $H^1(\Omega)$ onto $\mathring{H}^1(\Omega)$ with respect to the natural inner product on $H^1(\Omega)$. Then we have

$$\|v - Pv\|_1 \leqq c_1 |v|_{1/2}, \tag{3.9}$$

where c_1 depends only on Ω. Writing $v = Pv + (v - Pv)$ and expanding, we find

$$|a(v, v) - a(Pv, Pv)| \leqq 2C \|v - Pv\|_1 \|v\|_1.$$

Since Pv is in $\mathring{H}^1(\Omega)$, we have

$$|a(v, v)| \geqq \gamma_0 \|Pv\|_1^2 - 2Cc_1 |v|_{1/2} \|v\|_1.$$

Writing $\|Pv\|_1^2 = \|v\|_1^2 - \|v - Pv\|_1^2$ and applying (3.9) and the trace theorem [9], we complete the lemma.

In view of Lemma 1 and the interpolation norm inequalities [9], we have

$$\sup_{v \in S^h} \frac{|v|_{1/2}}{\|v\|_1} \leqq ch.$$

Thus Lemma 2 shows that γ can be chosen independently of h (we are assuming that the h's form a sequence tending to zero, so that $h \geqq h_1$ corresponds to only a finite number of h's for any $h_1 > 0$).

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4. A quadrature rule for piecewise quadratics. In Theorem 1, we assumed that the integrals involved in the definition (2.5) of u^h were computed exactly. In practice, the inner products $a(u^h, v)$ and (f, v) are computed using numerical quadrature, and in this section we describe a quadrature scheme that yields the same accuracy as in Theorem 1 for piecewise quadratics ($k = 3$). Numerical quadrature for triangles is well understood, so our contribution is a quadrature rule for pie-shaped boundary elements. Another approach to computing integrals over such elements was studied in [6]. Their method was to approximate the curved side by a polynomial, and then integrate exactly on the approximate element. This presupposes either that the coefficients $c_{\alpha\beta}$ in $a(\cdot, \cdot)$ are polynomials or that they have been approximated by polynomials. In the constant coefficient case, the resulting error was analyzed in [2].[7] Formulas for the integral of polynomials on a triangle with a quadratic side and a quartic side can be found in [6].

We now begin the construction of our quadrature rule for pie-shaped elements. It begins with the formula [16] for triangles that has a quadrature point at the midpoint of each edge getting a weight equal to 1/3 of the area of the triangle. (This quadrature scheme integrates quadratic polynomials exactly on triangles.) For a pie-shaped element e, we use this quadrature formula for the triangle e_0 formed by joining the two boundary vertices with a straight line, and then we simply add something for the region between this line and the boundary. If we ignored this region completely, it is known [2], [12], [16] that the resulting error is $O(h^{3/2})$, so it seems reasonable that we would only need to integrate constants exactly in this region to improve to the optimal $O(h^2)$. However, this is not the case—the term of order $h^{3/2}$ is not constant in this region. We show later that it is constant in the direction normal to the boundary, but it is *linear* in the tangential direction. This presents no real problem though, because by placing a quadrature point at the midpoint of the line joining the boundary vertices, with weight equal to the area of e minus the area of e_0, we can integrate such functions accurately. With a quadrature point so placed, we actually have only three quadrature points, the midpoint of the line joining the boundary vertices getting a different weight from the other two points (see Fig. 5). Notice that in case the boundary is straight, the quadrature rule agrees with the original equal weight rule, which we use for the elements in the interior. To make the quadrature well-defined in boundary elements with a nonconvex boundary edge, we must extend the functions involved so that they are defined at the quadrature points. So we suppose that f and the coefficients $c_{\alpha\beta}$ in $a(\cdot, \cdot)$ are extended so that

$$\|f\|_{\mathbb{R}^2,2} \leqq c\|f\|_{\Omega,2} \quad \text{and} \quad [c_{\alpha\beta}]_{\mathbb{R}^2,2} \leqq c[c_{\alpha\beta}]_{\Omega,2}. \tag{4.1}$$

The functions in S^h we extend locally as polynomials. (If Ω is convex, these extensions are unnecessary.) Denote by $a^*(u^h, v)$ and $(f, v)^*$ the corresponding inner products where integration has been replaced by quadrature. We let u_*^h be the approximate solution determined by using quadrature:

$$a^*(u_*^h, v) = (f, v)^* \quad \text{for all } v \text{ in } S^h. \tag{4.2}$$

[7] See also [17].

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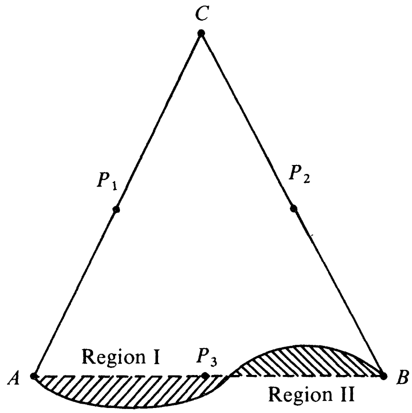

$$\int_e^x g \sim \sum_{i=1}^{3} w_i g(P_i), \text{ where } w_1 = w_2 = \tfrac{1}{3} \text{ area of triangle ABC and}$$

$$w_3 = \tfrac{1}{3} \text{ area of triangle ABC} + \text{area of region I} - \text{area of region II}$$

FIG. 5. *Quadrature formula with optimal accuracy for piecewise quadratics*

THEOREM 2. *Let S^h be as in Theorem* 1 *with $k = 3$ (piecewise quadratics). Then the solution u_*^h to* (4.1) *computed using quadrature is well-defined for h sufficiently small and satisfies*

$$\|u - u_*^h\|_1 \leqq ch^2(\|u\|_3 + \|f\|_2),$$

where the constant c depends on Ω, γ_0, K, the $C^2(\Omega)$ norm of the coefficients $c_{\alpha\beta}$, and the constants in (2.9) *and* (4.1).

Remark 1. The requirement that h be sufficiently small comes from our proof of coerciveness of $a^*(\cdot, \cdot)$ for h sufficiently small (Lemma 5). In fact, the theorem is true for all h such that $a^*(\cdot, \cdot)$ is coercive over S^h, if we let c depend on the coerciveness constant for $a^*(\cdot, \cdot)$.

Remark 2. The inclusion of $\|f\|_2$ in the error estimate is required for the quadrature $(f, v)^*$ to be accurate.

Remark 3. In the quadrature rule of Fig. 5, we need to have w_3 accurate only to order h^3, as an inspection of our proof shows (see (4.8) and its proof). Writing $w_3 = \frac{1}{3}$ area of $ABC + w_0$, we may take w_0 to be either $\frac{2}{3}ab$ or $\frac{1}{12}a^3/R$, where a is the length of the line AB, b is the distance from the midpoint of AB to ∂e, and R is the radius of curvature of $\partial\Omega$ at some point on ∂e.

Proof of Theorem 2. In view of Theorem 1 and the triangle inequality, we need only show that

$$\|u^h - u_*^h\|_1 \leqq ch^2(\|u\|_3 + \|f\|_2). \tag{4.3}$$

Our first step in proving this is to apply Theorem 4.1 of [14] as follows.

LEMMA 3. *Suppose that $a^*(\cdot, \cdot)$ is coercive over S^h, i.e., $|a^*(v, v)| \geqq \theta\|v\|_1^2$ for all v in S^h, and that*

$$|(a^* - a)(u^h, v)| + |(f, v)^* - (f, v)| \leqq \kappa h^2\|v\|_1. \tag{4.4}$$

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Then

$$\|u_*^h - u^h\|_1 \leqq \kappa\theta^{-1}h^2.$$

In view of Lemma 3, we just have to verify the coerciveness of $a^*(\cdot, \cdot)$ uniformly in h and prove (4.4) with κ satisfying

$$\kappa \leqq c(\|u\|_3 + \|f\|_2), \tag{4.5}$$

where the constant is as specified in the theorem. Postponing the proof of coerciveness for the moment (see Lemma 5), we concentrate on the proof of (4.4), which would follow directly from the analysis in [7] or [13], [14] if it were not for the pie-shaped boundary elements. Thus if e is a boundary element, we introduce the triangle e_0 formed by joining the boundary vertices of e with a straight line, and we let Ω_0 be the polygonal domain obtained by replacing each boundary element e by e_0. We use the following notation for the inner products transferréd to the domain Ω_0:

$$a_0(u, v) = \int_{\Omega_0} \sum_{\alpha,\beta} c_{\alpha\beta} D^\alpha u D^\alpha v, \qquad (u, v)_0 = \int_{\Omega_0} uv.$$

We denote by $a_0^*(\cdot, \cdot)$ and $(\cdot, \cdot)_0^*$ the quadrature inner products for Ω_0, where quadrature for both the interior and boundary triangles is given by the equal weight rule (i.e., Fig. 5, with $w_1 = w_2 = w_3 = \frac{1}{3}$ area of triangle). The following lemma is a simple generalization (to allow $c_{\alpha\beta} \neq 0$ for $|\alpha| + |\beta| = 1$) of the theory in [7].

LEMMA 4. *If v is piecewise quadratic, and if u^h is as in Theorem 1 for $k = 3$, then*

$$|a_0(u^h, v) - a_0^*(u^h, v)| \leqq \text{const.}\left(\max_{\alpha,\beta} [c_{\alpha\beta}]_2\right) h^2\|u\|_3\|v\|_1, \tag{4.6a}$$

$$|(f, v)_0 - (f, v)_0^*| \leqq \text{const. } h^2\|f\|_2\|v\|_1. \tag{4.6b}$$

The first constant depends on the constant c in Theorem 1 as well as Ω and K, and the second constant depends only on Ω and K.

Now we must estimate the difference between the terms in (4.4) and (4.6). For any function g and any boundary element e, define

$$\tilde{\int}_{e\Delta e_0} g \equiv \int_{e-e_0} g - \int_{e_0-e} g.$$

and

$$E(g) = \left| \tilde{\int}_{e\Delta e_0} g - w_0 g(P_3)\right|,$$

where P_3 is the midpoint of the line joining the boundary vertices of e (see Fig. 5), and $w_0 = \tilde{\int}_{e\Delta e_0} 1$. Then the difference between (4.4) and (4.6) is a sum of terms of the form

$$E(c_{\alpha\beta} D^\alpha u^h D^\beta v), \tag{4.7a}$$

$$E(fv). \tag{4.7b}$$

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We shall show that the quadrature is accurate on a linear polynomial p:

$$E(p) \leqq ch_0^5[p]_{e,1}, \tag{4.8}$$

where $h_0 \leqq h$ is the radius of the smallest disc $\bar{e}$ containing e. For any function g on $\bar{e}$, we can choose a linear polynomial p (using the Bramble–Hilbert lemma [5]) such that

$$[g - p]_{\bar{e},0} + h_0\|p\|_{\bar{e},1} \leqq ch_0\|g\|_{\bar{e},2}.$$

Thus by (4.8) and (2.7) we have

$$E(g) \leqq E(g - p) + E(p) \leqq ch_0^4\|g\|_{\bar{e},2}. \tag{4.9}$$

Applying this with $g = c_{\alpha\beta}D^\alpha u^h D^\beta v$ yields the following in view of (2.7) and (4.1):

$$E(c_{\alpha\beta}D^\alpha u^h D^\beta v) \leqq \text{const. } h_0^2[c_{\alpha\beta}]_2\|u^h\|_{e,2}\|v\|_{e,1}. \tag{4.10}$$

To estimate $\|u^h\|_{e,2}$, we introduce the interpolate u_I of u:[8]

$$\begin{aligned} \|u^h - u_I\|_{e,2} &\leqq ch_0^{-1}\|u^h - u_I\|_{e,1} \quad \text{(by (2.7))} \\ &\leqq ch_0^{-1}(\|u^h - u\|_{\Omega,1} + \|u - u_I\|_{\Omega,1}) \\ &\leqq ch_0^{-1}h^2\|u\|_{\Omega,3}. \end{aligned} \tag{4.11}$$

Therefore, the triangle inequality yields

$$\|u^h\|_{e,2} \leqq \|u_I\|_{e,2} + ch_0^{-1}h^2\|u\|_{\Omega,3}, \tag{4.12}$$

Combining (4.12) and (4.10), we have

$$E(c_{\alpha\beta}D^\alpha u^h D^\beta v) \leqq \text{const. } h^2[c_{\alpha\beta}]_2(\|u_I\|_{e,2} + h_0\|u\|_3)\|v\|_{e,1}. \tag{4.13}$$

Summing this estimate over all α, β and e, and applying the Schwarz inequality for sums, we find by estimate (A.8) in the Appendix that

$$|(a - a_0)(u^h, v) - (a^* - a_0^*)(u^h, v)| \leqq \text{const. } h^{5/2}\left(\max_{\alpha,\beta}[c_{\alpha\beta}]_2\right)\|u\|_3\|v\|_1.$$

This combines with Lemma 4 to prove the first half of (4.4).

To estimate $E(fv)$, we simply note that

$$[v]_{e\Delta e_0,0} \leqq ch_0\|v\|_{e,1} \quad \text{for } v \text{ in } S^h. \tag{4.14}$$

To prove this, recall that (3.7) says that v is $\leqq ch_0\|v\|_{e,1}$ on ∂e. Since the points of $e\Delta e_0$ are within $O(h^2)$ of ∂e, (2.7) plus the mean value theorem implies (4.14). Therefore by (4.14) and Sobolev's inequality, we have

$$\begin{aligned} E(fv) &\leqq ch_0^3[fv]_{e\Delta e_0,0} \\ &\leqq ch_0^4\|f\|_{\mathbb{R}^2,2}\|v\|_{e,1} \\ &\leqq ch_0^4\|f\|_{\Omega,2}\|v\|_{e,1}. \end{aligned}$$

Summing over all e, we get

$$|(f, v) - (f, v)_0 - ((f, v)^* - (f, v)_0^*)| \leqq ch^{7/2}\|f\|_2\|v\|_1. \tag{4.15}$$

[8] See §A.3 of the Appendix.

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In view of Lemma 4, this completes the proof of (4.4), with κ as specified in (4.5).

For the proof of (4.8), we introduce special coordinates for e: Let the line joining the boundary vertices of e be the x-axis, and let $P_3 = (0, 0)$. Then we have

$$E(a_0 + a_1x + a_2y) = \left|\tilde{\int}_{e\Delta e_0} a_1x + a_2y\right|$$

because $w_0 = \tilde{\int}_{e\Delta e_0} 1$. Since the y-coordinate of a point in $e\Delta e_0$ is $O(h_0^2)$, we have

$$E(a_0 + a_1x + a_2y) \leqq |a_1|\left|\tilde{\int}_{e\Delta e_0} x\right| + ch_0^5|a_2|.$$

Thus we must prove that

$$\left|\tilde{\int}_{e\Delta e_0} x\right| = O(h_0^5).$$

We let $\rho(x)$ parametrize ∂e, i.e., $\partial e = \{(x, \rho(x)) \colon -\alpha \leqq x \leqq \alpha\}$, and we let $p(x)$ be the quadratic that interpolates ρ at $x = -\alpha, 0, \alpha$. Using this parametrization, we compute

$$\begin{aligned}\left|\tilde{\int}_{e\Delta e_0} x\right| &= \left|\int_{-\alpha}^{\alpha}\int_0^{\rho(x)} x\,dy\,dx\right| = \left|\int_{-\alpha}^{\alpha} x\rho(x)\,dx\right| \\ &\leqq \left|\int_{-\alpha}^{\alpha} x(\rho - p)(x)\,dx\right| + \left|\int_{-\alpha}^{\alpha} xp(x)\,dx\right| \\ &\leqq ch_0^5,\end{aligned}$$

because $|\rho - p| \leqq c\alpha^3$ and p is even. This proves (4.8). We now show that $a^*(\cdot, \cdot)$ is coercive over S^h for h sufficiently small.

LEMMA 5. *Suppose* (3.2) *holds for* $\gamma > 0$ *independent of* h. *Then for* $h < h_1$, *we have*

$$\tfrac{1}{2}\gamma\|v\|_1^2 \leqq |a^*(v, v)| \quad \textit{for all } v \textit{ in } S^h,$$

where $h_1 > 0$ *depends on* γ, Ω, K *and the* $C^1(\Omega)$-*norm of the* $c_{\alpha\beta}$'s.

Proof. What we must prove is that

$$|a(v, v) - a^*(v, v)| \leqq \text{const.}\left(\max_{\alpha,\beta}\,[c_{\alpha\beta}]_1\right)h\|v\|_1^2. \tag{4.16}$$

From this and (3.2), it follows that

$$|a^*(v, v)| \geqq |a(v, v)| - |a^*(v, v) - a(v, v)| \geqq (\gamma - ch)\|v\|_1^2.$$

Thus $a^*(\cdot, \cdot)$ will be coercive for $h < \gamma/c$, and we choose $h_1 = \gamma/2c$.

To prove (4.16), we recall from the proof of Lemma 3 the reduction to the polygonal domain Ω_0: We shall show that

$$|a_0(v, v) - a_0^*(v, v)| \leqq \text{const.}\left(\max_{\alpha,\beta}\,[c_{\alpha\beta}]_{\Omega_0,1}\right)h\|v\|_1^2. \tag{4.17}$$

The difference between this estimate and (4.16) is again a sum of terms of the form (4.7a), but with u^h replaced by v. Because the area of $e\Delta e_0$ and w_0 are $O(h_0^3)$ (see

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after (4.7) for notation), we find that each of these terms may be bounded using (2.7):

$$E(c_{\alpha\beta}D^\alpha vD^\beta v) \leqq \left| \tilde{\int}_{e\Delta e_0} c_{\alpha\beta}D^\alpha vD^\beta v \right| + |w_0(c_{\alpha\beta}D^\alpha vD^\beta v)(P_0)|$$
$$\leqq \text{const. } [c_{\alpha\beta}]_0 h_0 \|v\|_{e,1}^2.$$

Summing this estimate over all α, β and e, we get

$$|(a - a_0)(v, v) - (a^* - a_0^*)(v, v)| \leqq \text{const.} \left(\max_{\alpha,\beta} [c_{\alpha\beta}]_0 \right) h \|v\|_1^2.$$

The difference in (4.17) is a sum, over all α, β and all the triangles T that make up Ω_0, of terms of the form $E(c_{\alpha\beta}D^\alpha vD^\beta v)$ where now

$$E(g) = \left| \int_T g - \sum_{i=1}^{3} w_i g(P_i) \right|, \tag{4.18}$$

where P_1, P_2, P_3 are the midpoints of the sides of T, and $w_i = \frac{1}{3}$ area of T. If $|\alpha| = |\beta| = 1$, then $D^\alpha vD^\beta v$ is a quadratic polynomial, and the quadrature is exact:

$$E(D^\alpha vD^\beta v) = 0.$$

Therefore, for any constant c, we have

$$E(c_{\alpha\beta}D^\alpha vD^\beta v) = E((c_{\alpha\beta} - c)D^\alpha vD^\beta v).$$

Choosing c close to $c_{\alpha\beta}$ on T, we have

$$E(c_{\alpha\beta}D^\alpha vD^\beta v) \leqq \text{const. } h_0^3[c_{\alpha\beta}]_{T,1}[v]_{T,1}^2 \leqq \text{const. } h_0[c_{\alpha\beta}]_{T,1}\|v\|_{T,1}^2 \quad \text{(by (2.7))},$$

where h_0 is the radius of the circle circumscribing T. Now suppose that $|\alpha| = 0$. Then we write

$$\begin{aligned} E(c_{\alpha\beta}vD^\beta v) &= E((c_{\alpha\beta}v - c)D^\beta v) \\ &\leqq \text{const. } h_0^3[c_{\alpha\beta}]_{T,1}[v]_{T,1}^2 \\ &\leqq \text{const. } h_0[c_{\alpha\beta}]_{T,1}\|v\|_{T,1}^2 \quad \text{(by (2.7))}, \end{aligned}$$

where we chose c close to $c_{\alpha\beta}v$ on T. The argument for $|\beta| = 0$ is similar, so we have

$$E(c_{\alpha\beta}D^\alpha vD^\beta v) \leqq \text{const. } h_0[c_{\alpha\beta}]_{T,1}\|v\|_{T,1}^2 \quad \text{for all } \alpha, \beta.$$

Summing this over all α, β and T proves (4.17).

5. Negative norm estimates. In this section, we extend Theorem 1 to prove that the error $u - u^h$ is of the optimal order in negative norms:

$$\|u - u^h\|_{-s} \leqq ch^{k+s}\|u\|_k \quad \text{for } 0 \leqq s \leqq k - 2. \tag{5.1}$$

We want to include a description of the error when the boundary data is inhomogeneous to indicate the smoothness required of the boundary data. So let the variational problem (2.1) be generalized to:

$$\begin{gathered} \text{Find } u \text{ in } H^1(\Omega) \\ \text{such that } u = g \quad \text{on } \partial\Omega \quad \text{and} \quad a(u, v) = (f, v) \quad \text{for all } v \text{ in } \mathring{H}^1(\Omega). \end{gathered} \tag{5.2}$$

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To achieve higher accuracy in any form, we expect that our boundary interpolation nodes for $\bar{S}^h$ will have to be chosen more accurately than the $O(h^k)$ approximation in (2.9). So let us suppose that the boundary edge nodes for each boundary element e are given by $(x_0\xi_i, y_i)$, $i = 1, 2, \cdots, k-2$, in the coordinates of Fig. 3, and suppose they satisfy

$$(5.3) \qquad |y_i - \rho(x_0\xi_i)| \leqq c x_0^{k+s},$$

(the ξ_i's are the interior Lobatto points for $[0, 1]$—see § 2). Let us define the following subspaces of $\bar{S}^h$:

S_g^h (resp. S_0^h) = the set of v in $\bar{S}^h$ that equal g (resp. 0) at the boundary vertices and, at the boundary edge nodes corresponding to the points $(x_0\xi_i, y_i)$, equal $g(x_0\xi_i, \rho(x_0\xi_i))$ (resp. 0).

The variational definition of u^h becomes

$$(5.4) \qquad \text{Find } u^h \text{ in } S_g^h \quad \text{such that} \quad a(u^h, v) = (f, v) \quad \text{for all } v \text{ in } S_0^h.$$

The following theorem generalizes the L^2 estimates given by Alan Berger [4]. The techniques we use in the proof are derived from [4] and [10].

THEOREM 3. *Suppose that the boundary interpolation nodes satisfy* (5.3) *for s an integer,* $0 \leqq s \leqq k-2$. *If u and* u^h *are defined by* (5.2) *and* (5.4), *respectively, then*

$$\|u - u^h\|_1 \leqq ch^{k-1}\|u\|_k.$$

If the operator A in (2.3) *is properly elliptic* [9], *then*

$$\|u - u^h\|_{-s} \leqq c'h^{k+s}(\|u\|_k + |g|_{k+s}).$$

The constants c and c′ depend on the same quantities as in Theorem 1 ((5.3) *replaces* (2.9)), *and in addition, c′ depends on the norm of* A^{-1} *as an operator from* $H^s(\Omega)$ *onto* $H^{s+2}(\Omega) \cap \mathring{H}^1(\Omega)$.

Proof. When inhomogeneous data is present, the basic error estimate becomes (see the Appendix, § A.2)

$$(5.5) \qquad \|u - u^h\|_1 \leqq \left(1 + \frac{c}{\gamma}\right) \inf_{v \in S_g^h} \|u - v\|_1 + \frac{c_\sigma}{\gamma} \sup_{v \in S_0^h} \frac{|v|_{-\sigma}}{\|v\|_1} \|u\|_{\sigma+2}.$$

By the approximation theorem in § A.3 of the Appendix, the first term is bounded by $ch^{k-1}\|u\|_k$. Thus, we must show that the boundary term is small:

$$\sup_{v \in S_0^h} \frac{|v|_{2-k}}{\|v\|_1} = O(h^{k-1/2}).$$

But this is just Lemma 1: in each boundary element, we let p be the polynomial of degree $k-1$ that satisfies

$$p(0) = p(x_0) = 0, \qquad p(x_0\xi_i) = y_i, \quad i = 1, 2, \cdots, k-2,$$

and then condition (5.3) implies (2.9). Therefore,

$$(5.6) \qquad \|u - u^h\|_1 \leqq ch^{k-1}\|u\|_k.$$

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We shall now use this estimate to derive negative norm estimates by the standard duality argument.

We first solve an adjoint problem to (2.3): Given $\varphi \in H^s(\Omega)$, let z solve

$$\begin{aligned} A^* z &= \varphi \quad \text{in } \Omega, \\ z &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Integrating by parts, we obtain the following:

$$(u - u^h, \varphi) = a(u - u^h, z - v) + \int_{\partial\Omega} [(g - u^h)N^* z + vNu] \quad \text{for } v \text{ in } S_0^h.$$

(N and N^* are of the form $a\partial_n + b\partial_t + c$ with $a \neq 0$). Applying Schwarz's inequality and the trace theorem [9] yields

(5.7) $|(u - u^h, \varphi)| \leqq C\|u - u^h\|_1 \|z - v\|_1 + |g - u^h|_{-s}\|z\|_{s+2} + |v|_{2-k}\|u\|_k.$

To bound the last two terms, we must introduce the interpolate u_I of a continuous function u. This is the function in $S^h_{u|\partial\Omega}$ that equals u at the interior nodes (see (2.8) and §A.3 of the Appendix). The following lemma is a generalization of a result due to J. Nitsche [10].

LEMMA 6. *Let u be a continuous function, and let $u_I \in \bar{S}^h$ be its interpolate. If* (5.3) *holds for* $0 \leqq s \leqq k - 2$, *then*

$$|u - u_I|_{-r} \leqq ch^{k+s}(\|u\|_{k+s-r} + |u|_{k+s}),$$

for $s \leqq r \leqq k - 2$, where c depends only on Ω, K, k, and the constant in (5.3).

We use the lemma (with $r = s$) to estimate $|g - u^h|_{-s}$:

$$\begin{aligned} |g - u^h|_{-s} &\leqq |u - u_I|_{-s} + |u^h - u_I|_{-s} \\ &\leqq ch^{k+s}(\|u\|_k + |u|_{k+s}) + ch^{s+3/2}\|u^h - u_I\|_1. \end{aligned}$$

The estimate of $|u^h - u_I|_{-s}$ was derived from Lemma 1. In view of (5.6) and estimate (A.6) in the Appendix, we have

$$\|u^h - u_I\|_1 \leqq \|u - u^h\|_1 + \|u - u_I\|_1 \leqq ch^{k-1}\|u\|_k,$$

so that our estimate becomes

$$|g - u^h|_{-s} \leqq ch^{k+s}(\|u\|_k + |g|_{k+s}). \tag{5.8}$$

We now set v in (5.7) equal to z_I, and apply Lemma 6 (with $r = k - 2$) plus estimate (A.6) in the Appendix to get (recall $z = 0$ on $\partial\Omega$)

$$h^{k-1}\|z - z_I\|_1 + |z_I|_{2-k} \leqq ch^{k+s}\|z\|_{s+2}. \tag{5.9}$$

Using (5.6), (5.8) and (5.9) to bound the right side of (5.7), we get

$$|(u - u^h, \varphi)| \leqq ch^{k+s}\|z\|_{s+2}(\|u\|_k + |g|_{k+s}). \tag{5.10}$$

Because A (and hence A^*) is properly elliptic, we have [9]

$$\|z\|_{s+2} \leqq c\|\varphi\|_s.$$

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Thus (5.10) becomes

$$\frac{|(u - u^h, \varphi)|}{\|\varphi\|_s} \leqq ch^{k+s}(\|u\|_k + |g|_{k+s}).$$

This completes the proof of the theorem.

Proof of Lemma 6. Let e be a boundary element, with $\partial e = \{(x, \rho(x)) : 0 \leqq x \leqq x_0\}$ as in Fig. 3. We let p be the polynomial of degree $k + s - 1$ such that

$$\begin{aligned} &p(0) = p(x_0) = 0, \qquad p(x_0\xi_i) = y_i, \quad i = 1, \cdots, k-2, \\ &(p - \rho)(x_0(\xi_i + \xi_{i+1})/2) = 0, \qquad i = 1, \cdots, s, \end{aligned} \tag{5.11}$$

where $\xi_{k-1} \equiv 1$. Then by assumption (5.3), we have

$$\sup_{0 \leqq x \leqq x_0} |p(x) - \rho(x)| \leqq cx_0^{k+s}. \tag{5.12}$$

Further, we let $\tilde{u}(x)$ be the polynomial of degree $k + s - 1$ that equals $u(x, p(x))$ at $x = 0, x_0, x_0\xi_i$ $(i = 1, \cdots, k-2)$, $x_0(\xi_i + \xi_{i+1})/2$ $(i = 1, \cdots, s)$. For $\varphi \in H^r(\partial\Omega)$, we let $\psi(x)$ be a polynomial of degree $r - 1$ approximating $\varphi\sqrt{1 + \dot{\rho}^2}$. Expanding, we obtain the following analogue of (3.6):

$$\begin{aligned} \left|\int_{\partial e} (u - u_I)\varphi\right| \leqq{}& cx_0^{k+s}(|u|_{\partial e,k+s} + h^{-1/2}\|u_I\|_{e,1})|\varphi|_{\partial e,0} \\ &+ cx_0^r|\varphi|_{\partial e,r}\left(\int_0^{x_0} (\tilde{u}(x) - u_I(x, p(x)))^2\, dx\right)^{1/2} \\ &+ \left|\int_0^{x_0} (\tilde{u}(x) - u_I(x, p(x)))\psi(x)\, dx\right|. \end{aligned} \tag{5.13}$$

Since the integrand in the last term is zero at the Lobatto points, we have (see (3.8))

$$\begin{aligned} &\left|\int_0^{x_0} (\tilde{u}(x) - u_I(x, p(x)))\psi(x)\, dx\right| \\ &\leqq cx_0^{2k-1} \sup_{0 \leqq x \leqq x_0} \left|\left(\frac{d}{dx}\right)^{2k-2} (\tilde{u}(x) - u_I(x, p(x)))\psi(x)\right| \\ &\leqq cx_0^{2k-2}|\varphi|_{\partial e,r}(|u|_{\partial e,k+s} + h^{s-r-1/2}\|u_I\|_{e,k+s-r-1}). \end{aligned} \tag{5.14}$$

To estimate the remaining term in (5.13), we have, as in the proof of (3.7),

$$\begin{aligned} &\sup_{0 \leqq x \leqq x_0} |\tilde{u}(x) - u_I(x, p(x))| \\ &\qquad \leqq cx_0^k \sup_{0 \leqq x \leqq x_0} \left|\left(\frac{d}{dx}\right)^k (\tilde{u}(x) - u_I(x, p(x)))\right| \\ &\qquad \leqq cx_0^{k-1/2}|u|_{\partial e,k+s} + c'x_0^{k+s-r-1}\|u_I\|_{e,k+s-r-1}. \end{aligned}$$

Therefore,

$$\left(\int_0^{x_0} (\tilde{u}(x) - u_I(x, p(x)))^2\, dx\right)^{1/2} \leqq cx_0^{k+s-r}(|u|_{\partial e,k+s} + x_0^{-1/2}\|u_I\|_{e,k+s-r-1}). \tag{5.15}$$

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Using (5.14) and (5.15) in (5.13) yields

$$\left|\int_{\partial e}(u - u_I)\varphi\right| \leqq ch^{k+s}|\varphi|_{\partial e,r}(|u|_{\partial e,k+s} + h^{-1/2}\|u_I\|_{e,k+s-r-1}).$$

Summing this estimate over all boundary elements e (the ones such that $e \cap \partial\Omega \neq \varnothing$), we have

$$\left|\int_{\partial\Omega}(u - u_I)\varphi\right| \leqq ch^{k+s}|\varphi|_r\left(|u|_{k+s} + h^{-1/2}\left(\sum_{e \cap \partial\Omega \neq \varnothing}\|u_I\|^2_{e,k+s-r-1}\right)^{1/2}\right).$$

But by (A.8) in the Appendix, we have

$$\sum_{e \cap \partial\Omega \neq \varnothing}\|u_I\|^2_{e,k+s-r-1} \leqq ch\|u\|^2_{k+s-r}.$$

This completes the proof of Lemma 6.

We can also prove an optimal $L^2(\Omega)$-estimate for the piecewise quadratic approximation u^h_* determined by quadrature (§4). For simplicity, we let the boundary data be homogeneous, as the case of inhomogeneous data can be treated by combining the ideas of Theorem 3 with the following.

THEOREM 4. *Under the same assumptions as Theorem* 2, *we have*

$$\|u - u^h_*\|_0 \leqq ch^3(\|u\|_3 + \|f\|_2),$$

if the operator A in (2.3) *is properly elliptic, where c depends on the norm of A^{-1} as an operator from $L^2(\Omega)$ onto $H^2(\Omega) \cap \mathring{H}^1(\Omega)$, the $C^3(\Omega)$-norm of the $c_{\alpha\beta}$'s, and Ω, γ and K.*

Remark. The accuracy of u^h_* does not appear to improve in negative norms because the quadrature rule used to define u^h_* is only accurate to order h^3.

Proof. As in the proof of Theorem 3, we solve the adjoint problem

$$A^*z = u^h - u^h_* \text{ in } \Omega,$$
$$z = 0 \text{ on } \partial\Omega.$$

Integrating by parts, we find that

$$\|u^h - u^h_*\|^2_0 = a(u^h - u^h_*, z) + \int_{\partial\Omega} Nz(u^h - u^h_*).$$

As in Theorem 3, we have

$$\left|\int_{\partial\Omega} Nz(u^h - u^h_*)\right| \leqq ch\|z\|_2\|u^h - u^h_*\|_1 \leqq ch^3\|z\|_2\|u\|_3.$$

For the other term, we have

$$a(u^h - u^h_*, z) = a(u^h - u^h_*, z - v) + ((f, v) - (f, v)^*) - (a(u^h_*, v) - a^*(u^h_*, v)).$$

Setting $v = z_I$, we shall show that

(5.16) $$|a(u^h_*, z_I) - a^*(u^h_*, z_I)| \leqq ch^3\|u\|_3\|z\|_2,$$

(5.17) $$|(f, z_I) - (f, z_I)^*| \leqq ch^3\|f\|_2\|z\|_2.$$

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Thus we have

$$\|u^h - u_*^h\|_0^2 \leqq ch^3(\|u\|_3 + \|f\|_2)\|z\|_2.$$

Because A^* is properly elliptic, we have

$$\|z\|_2 \leqq c\|u^h - u_*^h\|_0.$$

Dividing by $\|u^h - u_*^h\|_0$ thus yields

$$\|u^h - u_*^h\|_0 \leqq ch^3(\|u\|_3 + \|f\|_2).$$

In view of the triangle inequality and Theorem 3, this proves the theorem.

To prove (5.16) and (5.17), we reduce to the polygonal domain Ω_0 as in the proof of Theorem 2. The following is a simple generalization of the results in [7].

LEMMA 7. *Under the conditions of Theorem* 4, *we have*

$$|a_0(u_*^h, z_I) - a_0^*(u_*^h, z_I)| \leqq ch^3\|u\|_3\|z\|_2, \tag{5.18}$$

$$|(f, z_I)_0 - (f, z_I)_0^*| \leqq ch^3\|f\|_2\|z\|_2. \tag{5.19}$$

To prove (5.16) from (5.18), we must estimate terms of the form (4.7a). In fact, it follows from (4.9) and (2.7) that

$$E(c_{\alpha\beta}D^\alpha u_*^h D^\beta z_I) \leqq \text{const. } h_0^2[c_{\alpha\beta}]_2\|u_*^h\|_{e,2}\|z_I\|_{e,1}.$$

Summing this estimate over all α, β and e yields by (A.8)

$$|(a - a_0)(u_*^h, z_I) - (a^* - a_0^*)(u_*^h, z_I)| \leqq \text{const. } h^3[c_{\alpha\beta}]_2\|u\|_3\|z\|_2,$$

(we have $\|u_*^h\|_{e,2} \leqq \|u_I\|_{e,2} + ch_0^{-1}h^2\|u\|_{\Omega,3}$ by an argument similar to the proof of (4.12)). This proves (5.16). To prove (5.17), we simply combine (4.15) (with $v = z_I$) and (5.19).

Appendix.

A.1. Proof of (3.3). The first step is to show that for any u in $H^1(\Omega)$ and any u^h in S^h, the following inequality holds:

$$\|u - u^h\|_1 \leqq \left(1 + \frac{C}{\gamma}\right) \inf_{v \in S^h} \|u - v\|_1 + \frac{1}{\gamma} \sup_{v \in S^h} \frac{|a(u - u^h, v)|}{\|v\|_1}. \tag{A.1}$$

To prove this, let v be any function in S^h. Then

$$\gamma\|u^h - v\|_1^2 \leqq |a(u^h - v, u^h - v)| \leqq |a(u - v, u^h - v)| + |a(u - u^h, u^h - v)|.$$

Dividing by $\gamma\|u^h - v\|_1$ and applying (3.1), we obtain

$$\|u^h - v\|_1 \leqq \frac{C}{\gamma}\|u - v\|_1 + \frac{1}{\gamma}\frac{|a(u - u^h, u^h - v)|}{\|u^h - v\|_1}.$$

Applying the triangle inequality, we find

$$\begin{aligned}\|u - u^h\|_1 &\leqq \|u - v\|_1 + \|u^h - v\|_1 \\ &\leqq \left(1 + \frac{C}{\gamma}\right)\|u - v\|_1 + \frac{1}{\gamma}\frac{|a(u - u^h, u^h - v)|}{\|u^h - v\|_1},\end{aligned}$$

and this proves (A.1).

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If u and u^h are the solutions to (2.1) and (2.5), then an integration by parts yields

$$a(u, v) = (Au, v) + \int_{\partial\Omega} vNu, \qquad v \text{ in } H^1(\Omega), \tag{A.2}$$

where $N = a\partial_n + b\partial_t + c$, with $a \neq 0$. Since $Au = f$ and $(f, v) = a(u^h, v)$ for v in S^h, (A.2) becomes

$$a(u - u^h, v) = \int_{\partial\Omega} vNu, \qquad v \text{ in } S^h. \tag{A.3}$$

By the generalized Schwarz inequality and the trace theorem [9], we have

$$\begin{aligned} |a(u - u^h, v)| &= \left|\int_{\partial\Omega} vNu\right| \\ &\leqq |v|_{-s}|Nu|_{+s} \leqq c_s|v|_{-s}\|u\|_{s+2}. \end{aligned} \tag{A.4}$$

Thus c_s is the norm of the map $u \to Nu$ of $H^{s+2}(\Omega) \to H^s(\partial\Omega)$ and is bounded by the $C^s(\Omega)$ norm of the $c_{\alpha\beta}$'s. A more precise estimate than (A.4), using nonintegral s, can be made [2], but it is not necessary for this paper.

A.2. Proof of (5.5)—inhomogeneous boundary data. By an argument similar to the proof of (A.1), we have for u in $H^1(\Omega)$ and u^h in S_g^h the following:

$$\|u - u^h\|_1 \leqq \left(1 + \frac{C}{\gamma}\right) \inf_{v \in S_g^h} \|u - v\|_1 + \frac{1}{\gamma} \sup_{v \in S_0^h} \frac{|a(u - u^h, v)|}{\|v\|_1}. \tag{A.5}$$

(This follows because $u^h - v$ is in S_0^h for all v in S_g^h). Equation (A.3) remains true for v in S_0^h, because $(f, v) = a(u^h, v)$. Therefore, (A.4) and (A.5) combine to prove (5.5).

A.3. An approximation theorem. Given a triangulation of Ω, let $\bar{S}^h$ be the space of continuous piecewise polynomials of degree $k - 1$ (see §2). For a continuous function u, we define its *interpolate* u_I in $\bar{S}^h$ by the requirement that $u_I = u$ at the interior nodes (see (2.8)) and the boundary vertices, and at the boundary edge nodes for a boundary element e, we have

$$u_I(x_0\xi_i, p(x_0\xi_i)) = u(x_0\xi_i, \rho(x_0\xi_i))$$

(see Fig. 3 and (2.9)–(2.10) for notation).

THEOREM. *Let u be any function in $H^s(\Omega)$, $2 \leqq s \leqq k$. Then*

$$\|u - u_I\|_1 \leqq ch^{s-1}\|u\|_s, \tag{A.6}$$

$$\|u_I\|_{e,s} \leqq c\|u\|_{e,s} \quad \textit{if } e \textit{ is a triangle}, \tag{A.7}$$

$$\sum_{e \cap \partial\Omega \neq \varnothing} \|u_I\|_{e,s-1}^2 \leqq ch\|u\|_s^2, \tag{A.8}$$

where c depends only on Ω, the constant K in Definition 1 *and k.*

Remark. Estimate (A.6) and its proof, which generalize the approximation theorem in [14], are due to G. Strang (verbal communication).

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Proof. First, we extend u to $\mathbb{R}^2$ so that $\|u\|_{\mathbb{R}^2,s} \leqq c\|u\|_{\Omega,s}$.[9] As a type of reference element for the boundary elements, we introduce

$$E_\kappa = \{(\xi,\eta): 0 \leqq \xi \leqq 1, -\kappa\xi(1-\xi) \leqq \eta \leqq 1-\xi\}.$$

Given a boundary element e, let $\tilde{e}$ be the image of E_κ under the affine identification of the straight sides of e with those of E_κ. We choose $\kappa > 0$ (depending on K and the curvature of $\partial\Omega$) so that the $\tilde{e}$'s do not overlap for h sufficiently small. For any choice of $\kappa > 0$, we have $e \subset \tilde{e}$ for h sufficiently small since the curvature of the boundary edge of $\tilde{e}$ is $O(h^{-1})$, and from now on, we suppose that h is small enough to insure that the $\tilde{e}$'s do not overlap and that $e \subset \tilde{e}$. Pick a particular boundary element e. As a consequence of the Bramble–Hilbert lemma [5], there is a polynomial P of degree $s - 1$ such that

$$\|u - P\|_{\tilde{e},1} + [u - P]_{\tilde{e},0} \leqq ch_0^{s-1}\|u\|_{\tilde{e},s}, \tag{A.9}$$

where h_0 is the radius of the smallest disc containing e. One unusual feature of the above interpolation is that $P_I \neq P$ in e. However, the difference is small, and we find

$$\begin{aligned} \|u - u_I\|_{e,1} &\leqq \|u - P\|_{e,1} + \|(u - P)_I\|_{e,1} + \|P - P_I\|_{e,1} \\ &\leqq c\left(1 + \max_{z_j \in e} \|\varphi_j\|_{e,1}\right) h_0^{s-1}\|u\|_{\tilde{e},s}, \end{aligned} \tag{A.10}$$

where $\{z_j\}$ is the set of all nodes in (2.8) and $\{\varphi_j\}$ is the corresponding nodal basis for $\bar{S}^h$ defined by $\varphi_i(z_j) = \delta_{ij}$. Thus we must show that $\{\varphi_j\}$ is a *stable basis*:

$$[\varphi_j]_{e,m} \leqq ch_0^{-m} \quad \text{for all elements } e \quad (h_0 = \text{diameter of } e). \tag{A.11}$$

Estimate (A.11) implies that $\|\varphi_j\|_{e,1} \leqq c$, and will be proved shortly. Thus (A.10) becomes

$$\|u - u_I\|_{e,1} \leqq ch^{s-1}\|u\|_{\tilde{e},s}.$$

When e is a triangle, we obtain

$$\|u - u_I\|_{e,1} \leqq ch^{s-1}\|u\|_{e,s}$$

by the above argument with e replacing $\tilde{e}$. Squaring and summing over all the elements, we find

$$\|u - u_I\|_{\Omega,1} \leqq ch^{s-1}\|u\|_{\mathbb{R}^2,s} \leqq c'h^{s-1}\|u\|_{\Omega,s}.$$

To prove (A.7), let e be a triangle and (using the Bramble–Hilbert lemma [5] again) let P be a polynomial of degree $s - 1$ such that

$$\|P\|_{e,s} \leqq \|u\|_{e,s} \quad \text{and} \quad [u - P]_{e,0} \leqq ch_0^{s-1}\|u\|_{e,s}.$$

Using (A.11) and the fact that $P = P_I$ on a triangle, we have

$$\begin{aligned} \|u_I\|_{e,s} &\leqq \|(u - P)_I\|_{e,s} + \|P\|_{e,s} \\ &\leqq c\|u\|_{e,s}. \end{aligned}$$

[9] c will denote various constants depending only on Ω, K and k.

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To prove (A.8), we must distinguish the cases $s = 2$ and $s > 2$. For $s = 2$, we have

$$\frac{1}{2} \sum_{e \cap \partial\Omega \neq \varnothing} \|u_I\|_{e,1}^2 \leqq \|u - u_I\|_{\Omega,1}^2 + \sum_{e \cap \partial\Omega \neq \varnothing} \|u\|_{e,1}^2 .$$

To estimate the first term, we apply (A.6):

$$\|u - u_I\|_1 \leqq ch\|u\|_2 .$$

For the second, we introduce the set $\Omega_h = \{(x, y) \in \mathbb{R}^2 : \operatorname{dist}((x, y), \partial\Omega) \leqq h\}$ and use the following:

(A.12) $$\|u\|_{\Omega_h, s-1} \leqq ch^{1/2}\|u\|_{\mathbb{R}^2, s} .$$

This is essentially the trace theorem applied to translates of $\partial\Omega$, plus an integration and can be proved simply by flattening the boundary or introducing a tubular neighborhood of the boundary. Applying (A.12) to the above proves (A.8) for $s = 2$. When $s > 2$, we make an argument similar to the proof of (A.7) to obtain

$$\|u_I\|_{\tilde{e}, s-1} \leqq c\|u\|_{\tilde{e}, s-1} ,$$

where $\tilde{e}$ was described earlier (this is false for $s = 2$). We now square this and sum over all the boundary elements:

$$\sum_{e \cap \partial\Omega \neq \varnothing} \|u_I\|_{e, s-1}^2 \leqq c\|u\|_{\Omega_h, s-1}^2 .$$

Applying (A.12) again completes the proof of (A.8).

Proof of (A.11). When e is a triangle, this is easy: We map to a fixed triangle of size 1 and the change of variables brings in the powers of h. When e is a boundary element, we let e_0 be the triangle formed from e by joining its boundary vertices with a straight line and we let $\{\psi_j\}$ be the basis functions for e_0: $\psi_i(w_j) = \delta_{ij}$, where $w_j = z_j$ except on the boundary edge where (in the notation of Fig. 3) the nodes are changed from $(x_0\xi_i, p(x_0\xi_i))$ to $(x_0\xi_i, 0)$. Because of our nondegeneracy assumption,

(A.13) $$[\psi_j]_{e_0, m} \leqq ch_0^{-m}$$

by the argument above for triangles. Let us write

(A.14) $$\varphi_j = \sum a_{jl}\psi_l .$$

We claim that the coefficients a_{jl} are bounded. The inverse of (a_{ij}) can be determined from

$$\delta_{ij} = \varphi_i(z_j) = \sum a_{il}\psi_l(z_j),$$

so that $(\psi_i(z_j))$ is the inverse of (a_{ij}). But since $|z_j - w_j| = O(h^2)$, we have

$$|\psi_i(z_j) - \delta_{ij}| = |\psi_i(z_j) - \psi_i(w_j)| = O(h).$$

Therefore (a_{ij}) = identity + $O(h)$. Thus (A.11) follows from (A.13) and (A.14) (plus an application of (2.7)).

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