Modeling Statistical phenomena
Assignment 1

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1 Introduction

This assignment will examine the SIR system of differential equations 3 and its properties, such as the reproductive ratio and the system's different variables' long-term behavior.

$$\frac{dS}{dt} = -\beta SI, \tag{1}$$

$$\frac{dI}{dt} = \beta SI - \gamma I, \tag{2}$$

$$\frac{dR}{dt} = \gamma I. \tag{3}$$

2 SIR Model

In this section, we solve the SIR differential system of equations 3 numerically using scipy.integrate module. In the jupyter notebook provided in the zip file, we have defined a function f which contains the right-hand side of the SIR equations. We feed this function to solve-ivp function and solve this system numerically in time between tstart=0 and tend=350. The default algorithm of solve ivp is RK45 which is a Runge-Kutta fourth-order. For the entirety of this assignment, we work with this algorithm to solve the equations. For this part, we use 4 as our initial conditions.

$$\begin{aligned} &S(0) = 1 - 10^{-6} - 10^{-4} \\ &I(0) = 10^{-6} \\ &R(0) = 10^{-4} \end{aligned} \tag{4}$$

For the parameters, we set $\beta = 1.5, \gamma = 0.3$. In the figure you see the plot of S(t), I(t), and R(t). To ensure that our numerical solutions are correct, we plot I versus R and S versus R for an analytical solution and numerical solution. To

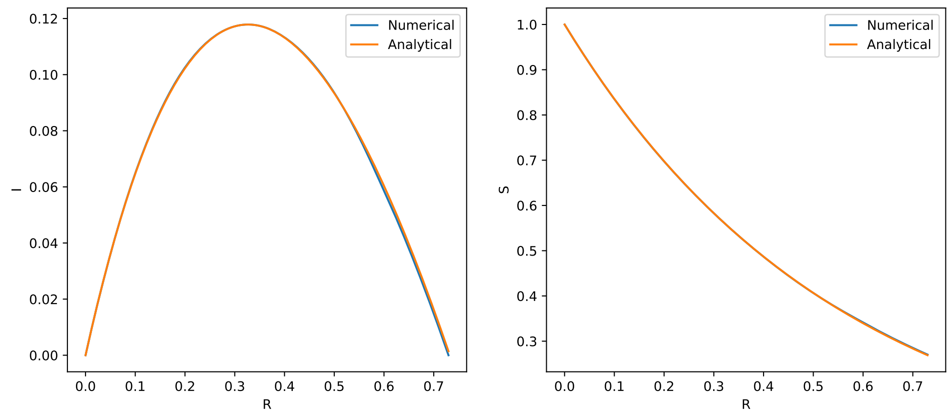


Figure 1: comparing analytical solution and numerical solution for SIR equations 3. We plot S vs. R and I vs. R for both analytical and numerical solutions.

obtain the analytical solution, we should eliminate t from equations. To this end, we divide the first equation by the third one 3 and obtain an equation for S in terms of R (5).

$$\frac{dS}{dR} = -\frac{\beta}{\gamma}S \tag{5}$$

this equation has a an exponential solution of the form $e^{R(0)R_0}S(0)exp(-R_0R)$. Furthermore, we could divide equation 2 by 3 to find I in terms of R by substituting the S(R) in the differential equation 6. The solution of this equation has the form $C_0exp(-RR_0) - R + C_1$ which C_0 and C_1 are constants that depend on the initial values and the reproductive ratio (R_0).

$$\frac{dI}{dR} = \frac{\beta}{\gamma}S(R) - 1 \tag{6}$$

As you can see in figure 1, the numerical and analytical solutions almost overlap. To see the difference between these two solutions, we have plotted the absolute value of the difference in figure 2.As you can see the difference between these two values is of the order 10^{-4}.

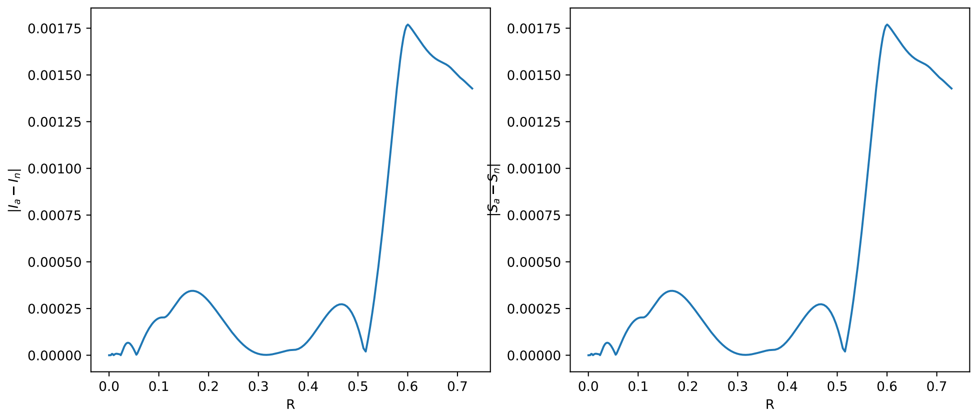
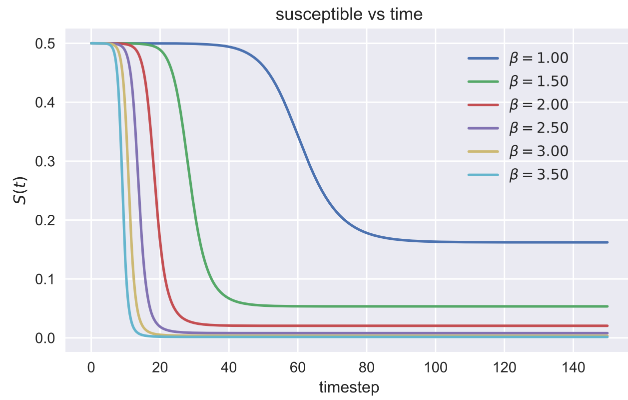


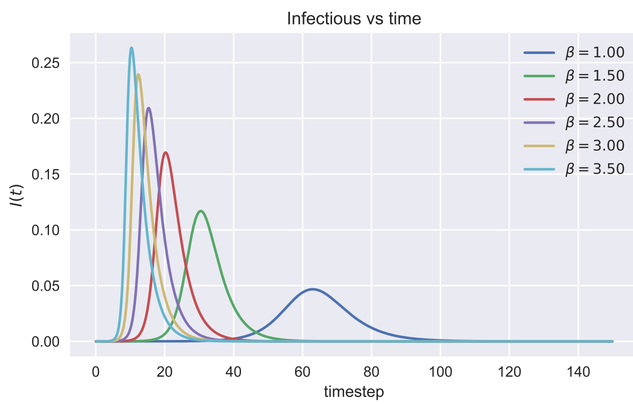
Figure 2: the absolute value difference between analytical and numerical solutions

3 changing the value of β

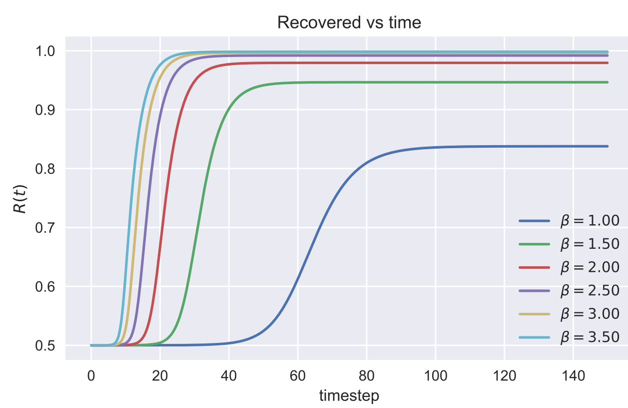
in this section we investigate the behaviour of the system as we varying the parameter β. The initial conditions here are $S(0) = 1 - 10^{-6} - 0.5, I(0) = 10^{-6}$ and $R(0) = 0.5$. You could see the results in figure 3(a) to figure 3(c).



(a) susceptible vs time for various values of β



(b) infections vs time for various values of β

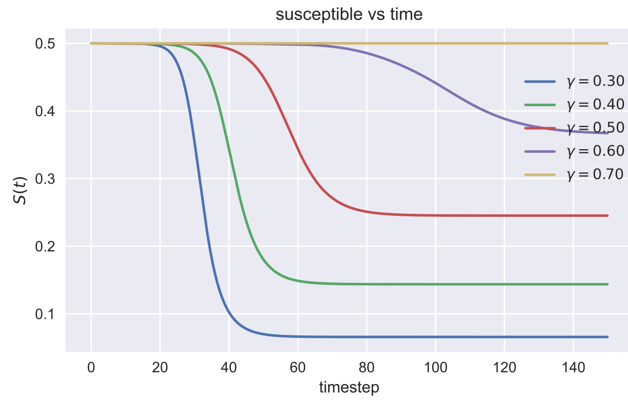


(c) recovered vs time for various values of β

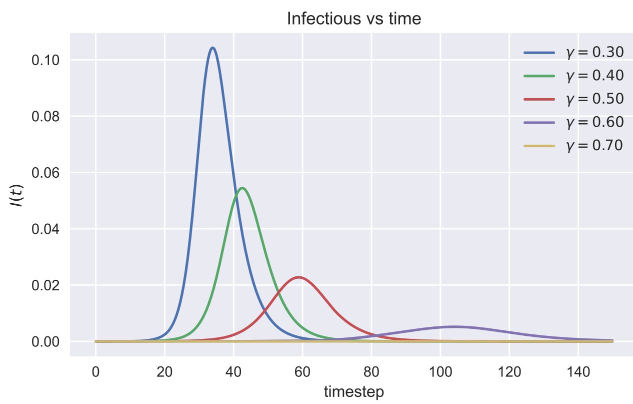
As you can see from figures the by increasing the value of β, the slope of changes in the number of the Susceptible and the plot of recovered shrinks, and it peaks at a higher value which seems natural since by increasing the value of β the disease spreads at faster pace.

4 changing γ

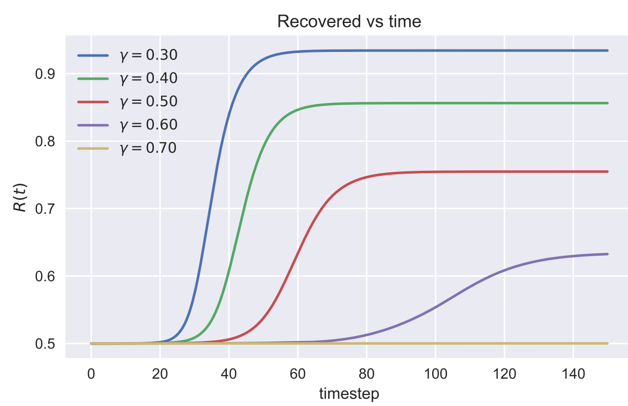
In this section, we vary the value of γ and examine the behavior of the system. The initial values of this section are as same as 3. You could see the results in figure 3(d) to figure 3(f).



(d) Susceptible vs time for various values of γ



(e) infections vs time for various values of γ

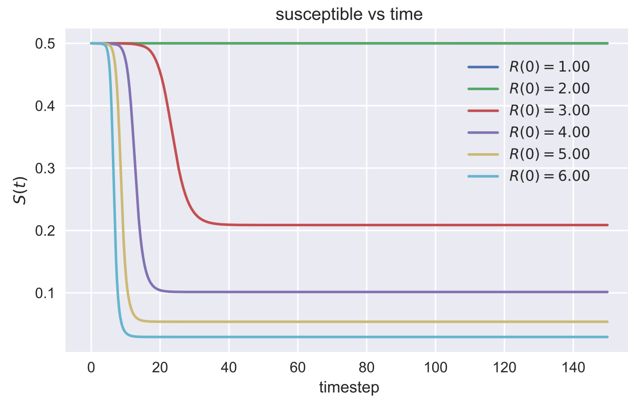


(f) recovered vs time for various values of γ

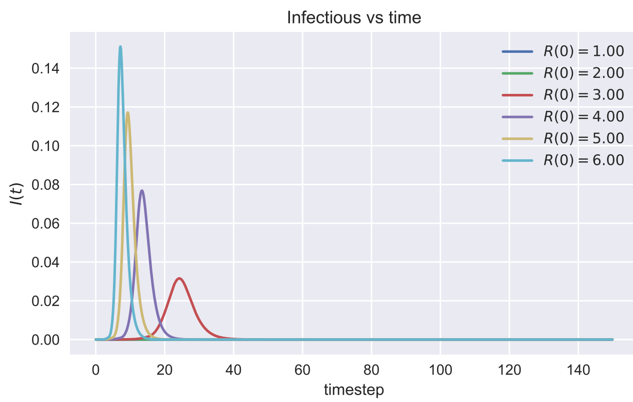
As you can see from the figures, by increasing the value of γ, the slope of S and R plots decreases, and the equilibrium values for S becomes larger, and the opposite happens for R. Moreover, by increasing γ, the plot of I peaks at a smaller values Which is reasonable becausw larger values of γ means the faster rate of recovery, which leads to fewer infections and fewer recovered cases.

5 changing R_0

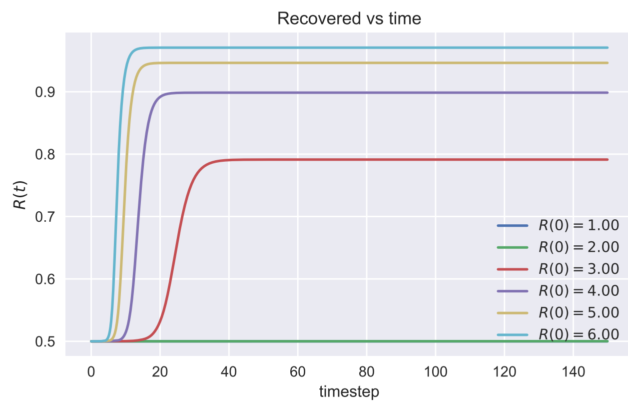
I this section, we vary the value of R_0 and see the results. By definition, $R_0 = \frac{\beta}{\gamma}$. Here we change the value of R_0 between 1 and 7. You see the results in figures 3(g) to 3(i).



(g) susceptible vs time for various values of R_0



(h) infections vs time for various values of R_0



(i) recovered vs time for various values of R_0

As you could see from the figure for larger values of R_0, we would have infections and recovered cases and fewer susceptible which is what we expect to see considering the definition of R_0.

6 $R(\infty)$ vs R_0 for various values of $R(0)$

In this section, we solve 3 different values of R_0 (between 1 and 6) and plot the final state of $R(t)$ for these R_0 values. We then repeat this procedure for different initial values of R between 0 and 0.5. You could see the results in figure 3.

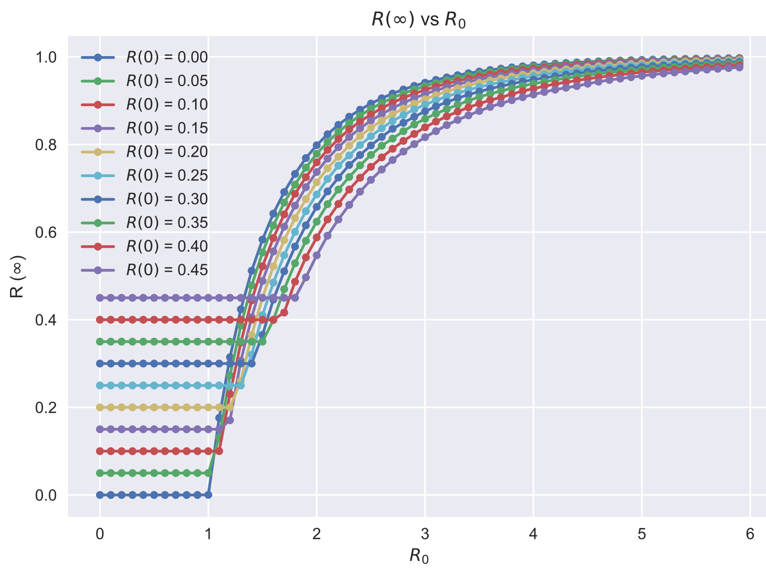


Figure 3: R_∞ vs R_0 for different values of R(0)

As you can see, by increasing the R(0), the value of R_0 in which the phase transition occurs increases. This happens because by increasing the value of R(0) and keep the I(0) as same as before, to satisfy the $S + I + R = 1$ condition what we are doing in practice is to decrease the value of S. By decreasing the value of S there is less chance for disease to spread and therefore for infection to spread out we should have bigger values of R_0. Also, for bigger R(0) values, we see that the system reaches its equilibrium at a slower rate, which is also because of the smaller number of susceptible.