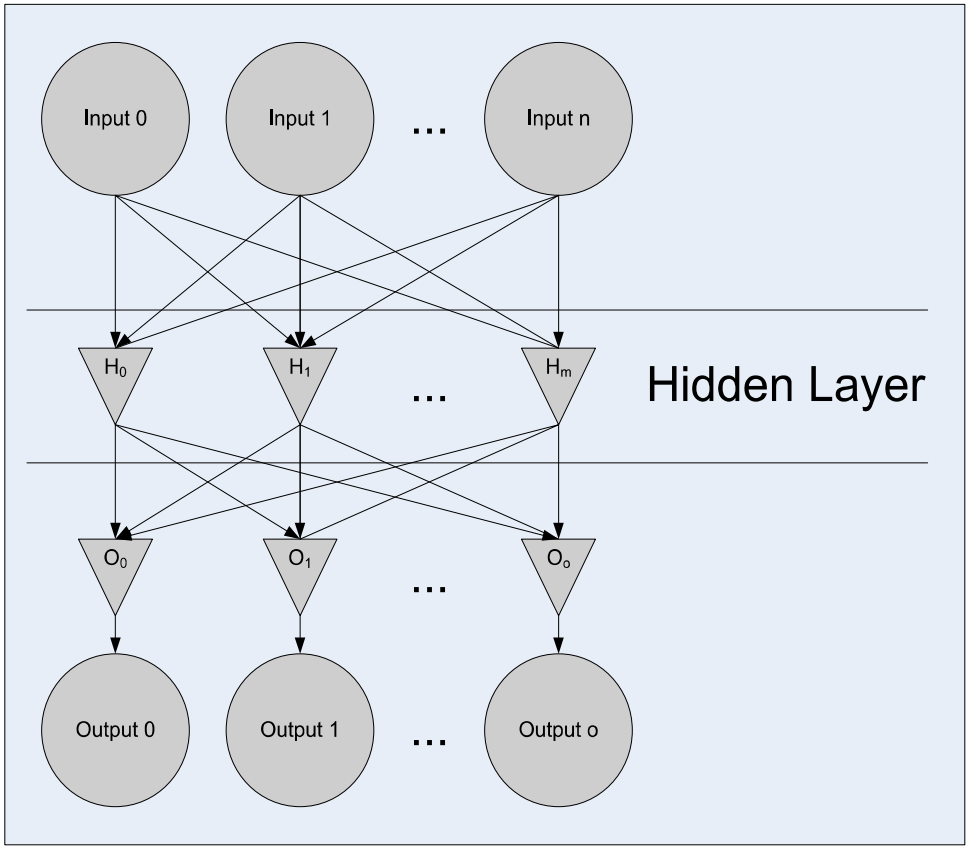
شبکه عصبی چیست؟

- روشی برای محاسبه است که بر پایه اتصال و به هم پیوسته چندین واحد پردازشی ساخته می‌شود.
- شبکه از تعداد دلخواهی سلول (گره، واحد یا نورون) تشکیل می‌شود که مجموعه ورودی را به خروجی ربط می‌دهند.



شبکه عصبی چیست؟

- شبکه عصبی مصنوعی روشی عملی برای یادگیری توابع گوناگون نظیر توابع با مقادیر حقیقی، توابع با مقادیر گسسته و ... می‌باشد.

- یادگیری شبکه عصبی تا حد بالایی در برابر خطاهای داده‌های آموزشی مصون بوده و اینگونه شبکه‌ها با موفقیت در حل مسائلی نظیر شناسایی گفتار، شناسایی و تعبیر تصاویر، و یادگیری ربات مورد استفاده قرار گرفته‌اند.

شبکه عصبی چه قابلیت‌هایی دارد؟

- محاسبه یک تابع معلوم
- تقریب یک تابع ناشناخته
- شناسایی الگو
- پردازش سیگنال
- یادگیری

مسائل مناسب برای یادگیری شبکه‌های عصبی

- خطا در داده‌های آموزشی وجود داشته باشد، مثل مسائلی که داده‌های آموزشی دارای نویز حاصل از دادهای سنسورها نظیر دوربین و میکروفن‌ها هستند.
- مواردی که نمونه‌ها توسط مقادیر زیادی زوج **(ویژگی ، مقدار)** نشان داده شده باشند، نظیر داده‌های حاصل از یک آزمایش پزشکی.
- تابع هدف دارای مقادیر پیوسته باشد.

مسائل مناسب برای یادگیری شبکه‌های عصبی

- زمان کافی برای یادگیری وجود داشته باشد. این روش در مقایسه با روش‌های دیگر نظیر درخت تصمیم نیاز به زمان بیشتری برای یادگیری دارد.
- نیازی به تعبیر تابع هدف نباشد، زیرا به سختی می‌توان وزن‌های یادگرفته شده توسط شبکه را تعبیر نمود.

Early learning algorithms

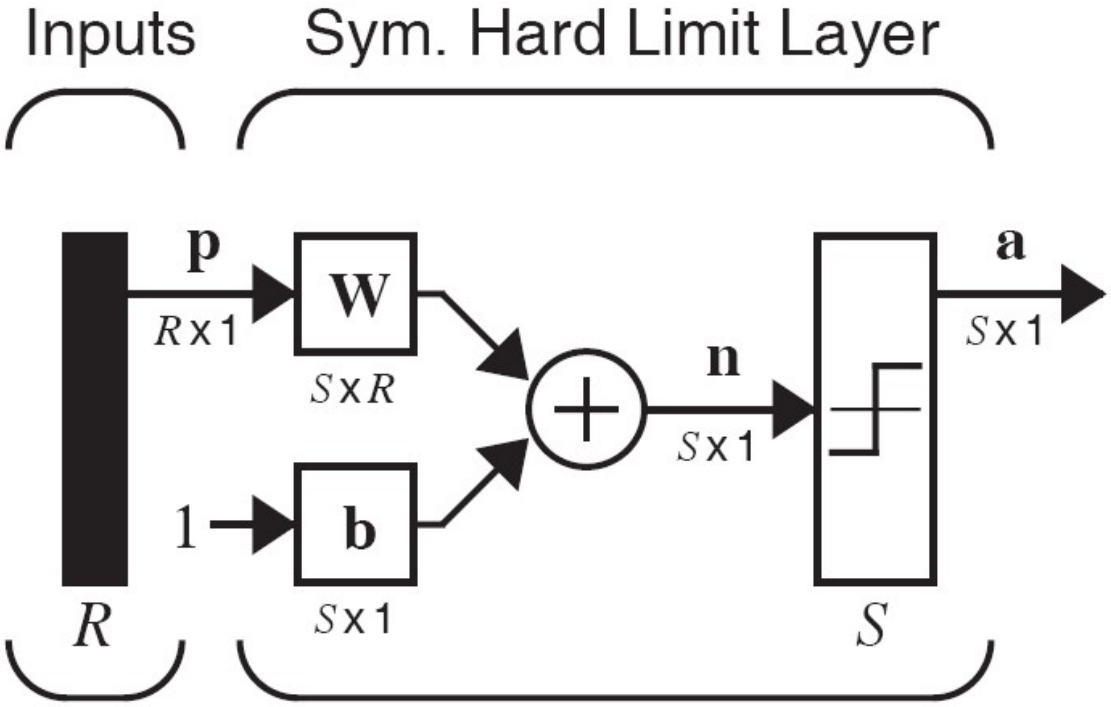
- Designed for single layer neural networks
- Generally more limited in their applicability
- Some of them are
  - Perceptron learning
  - LMS or Widrow- Hoff learning
  - Grossberg learning

پرسپترون

نوعی از شبکه عصبی برمبنای یک واحد محاسباتی به نام پرسپترون ساخته می‌شود. یک پرسپترون برداری از ورودی‌های با مقادیر حقیقی را گرفته و یک ترکیب خطی از این ورودی‌ها را محاسبه می‌کند.

اگر نتیجه حاصل از یک مقدار آستانه بیشتر بود خروجی پرسپترون برابر با 1 و در غیر این‌صورت معادل 1- (یا صفر) خواهد بود.

پرسپترون



$$\mathbf{a} = \mathbf{hardlims}(\mathbf{Wp} + \mathbf{b})$$

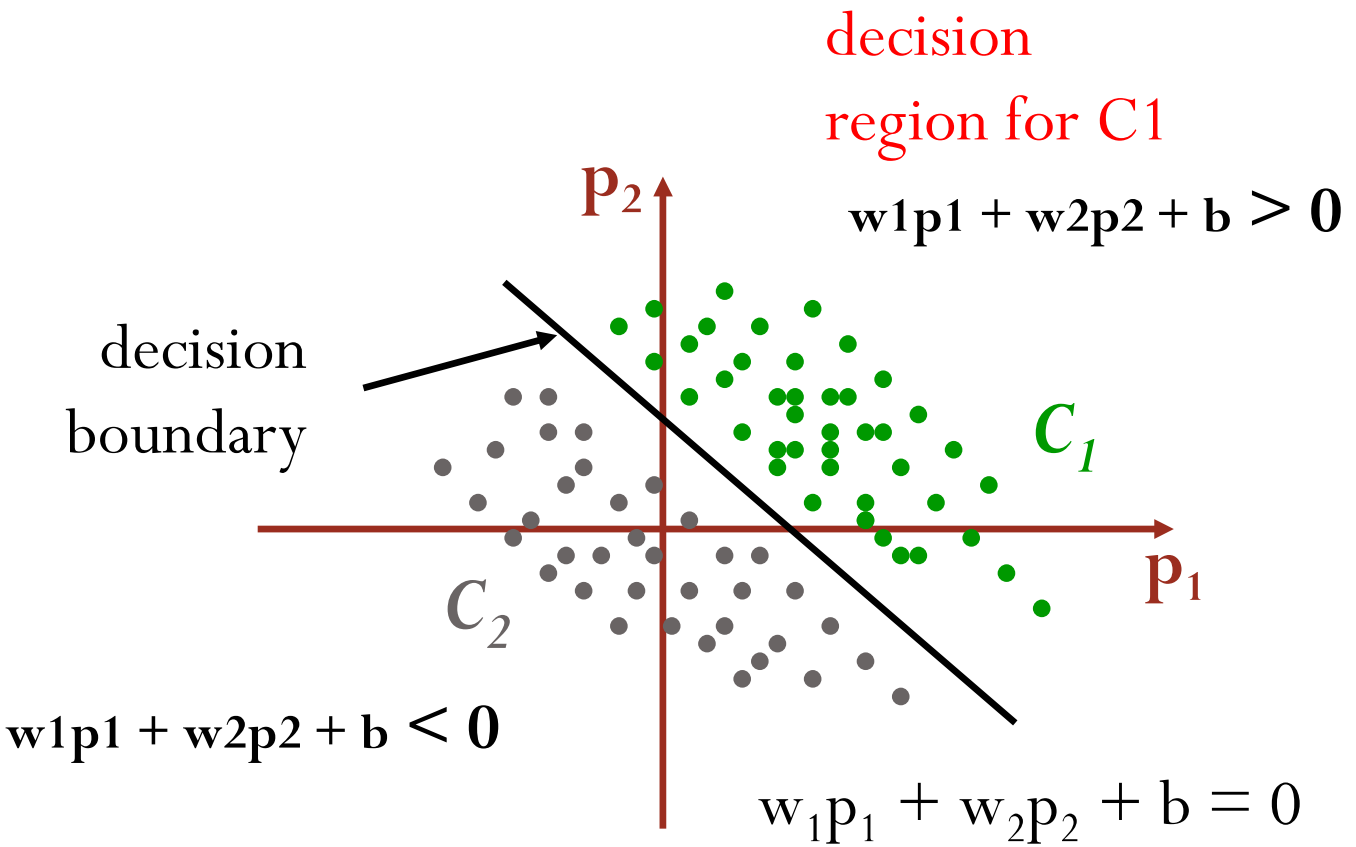
اضافه کردن بایاس

- افزودن بایاس موجب می‌شود تا استفاده از شبکه پرسپترون با سهولت بیشتری مقدور باشد.

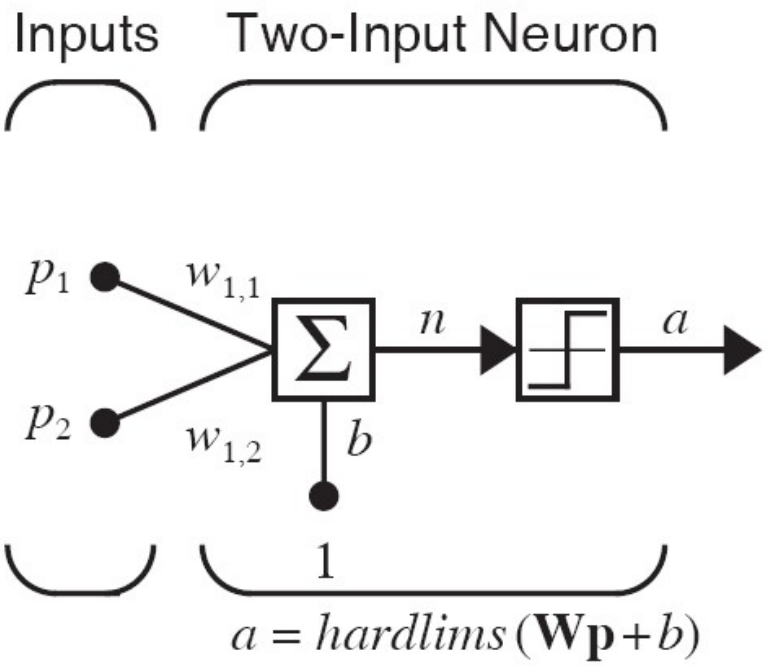
- برای اینکه برای یادگیری بایاس نیازی به استفاده از قانون دیگری نداشته باشیم، بایاس را به صورت یک ورودی با **مقدار ثابت ۱** در نظر گرفته و وزن b را به آن اختصاص می‌دهیم.

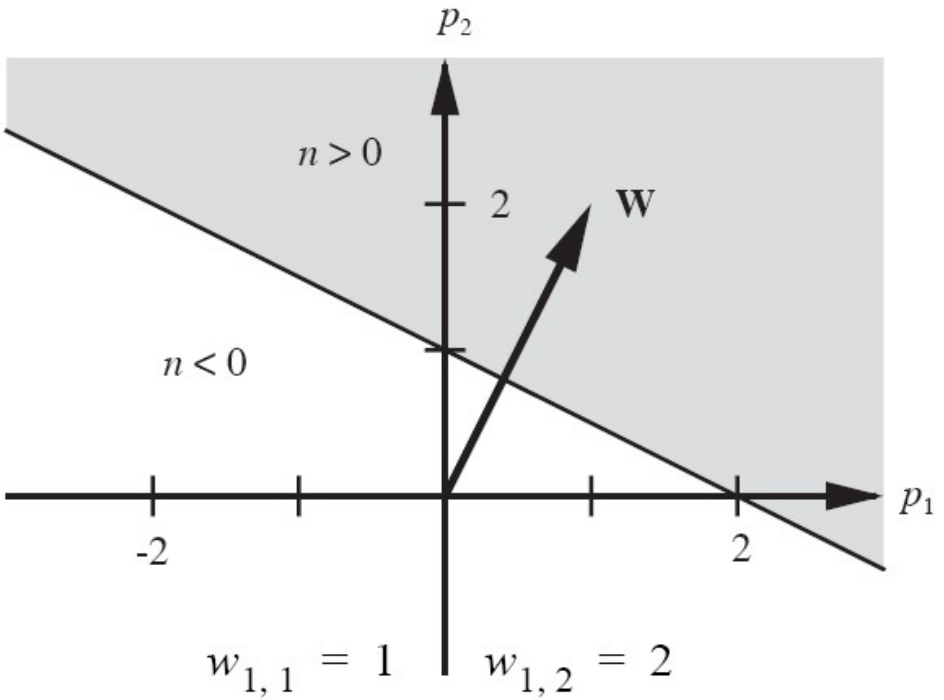
Perceptron Geometric View

The equation below describes a (hyper-)plane in the input space consisting of real valued m-dimensional vectors. The plane splits the input space into two regions, each of them describing one class.



Two-Input Case



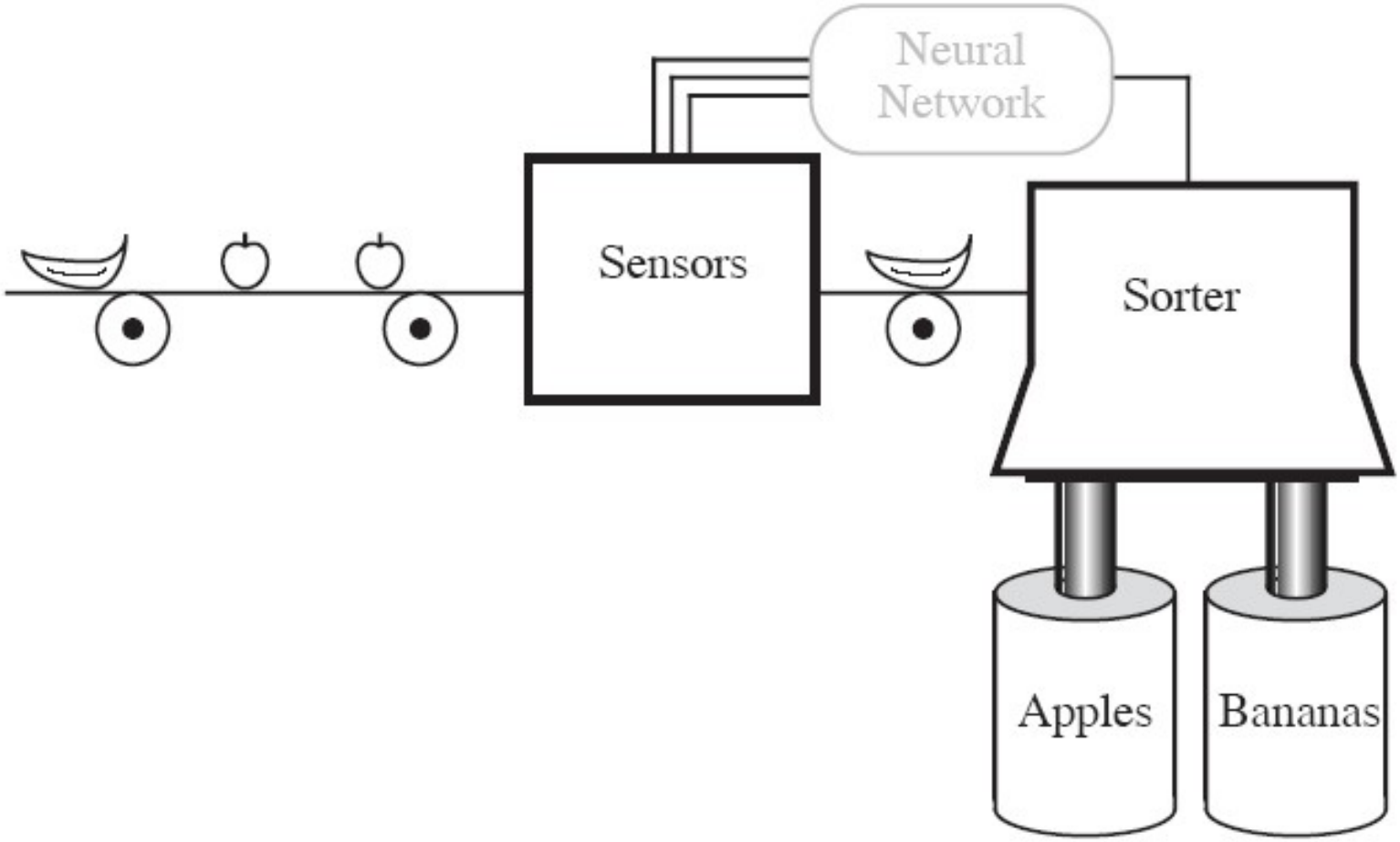


$$a = hardlims(n) = hardlims(\begin{bmatrix}1 & 2\end{bmatrix}\mathbf{p} + (-2))$$

Decision Boundary

$$\mathbf{W}\mathbf{p} + b = 0 \qquad \begin{bmatrix}1 & 2\end{bmatrix}\mathbf{p} + (-2) = 0$$

Apple/Banana Sorter



Class Representation

Prototype Banana

$$p_B = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

Prototype Apple

$$p_A = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Shape : 1 = Circular -1 = Elliptical

Texture : 1= Smooth -1 = Rough

Weight : 1 > 1 lbs. -1 < 1 lbs.

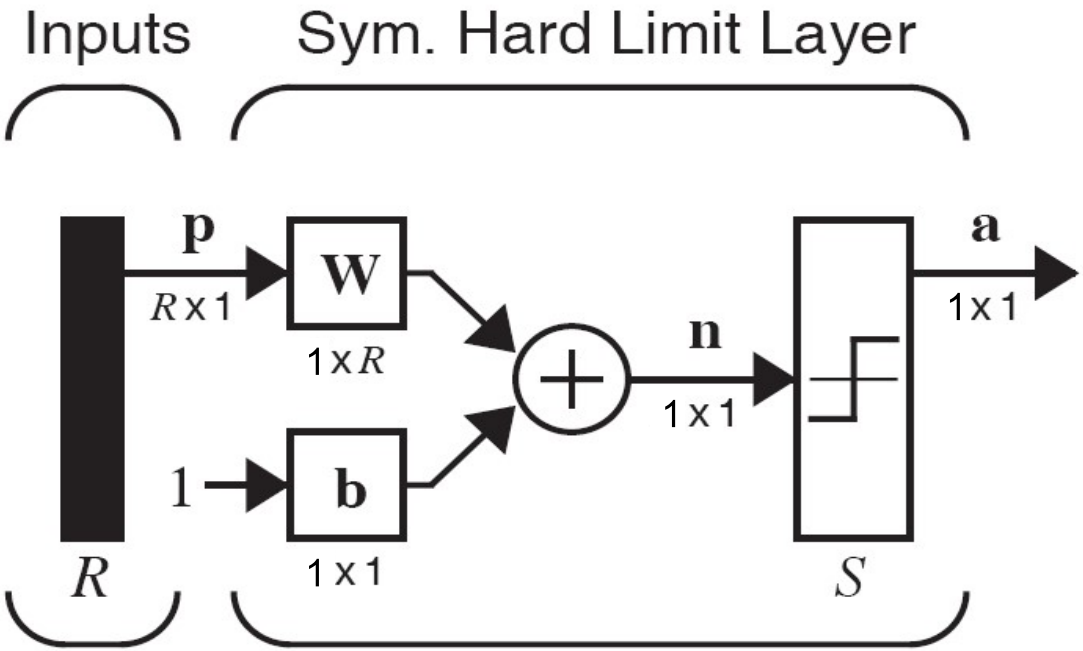
Case #1: 1 -1 → **Perceptron Network**

Case #2: $\begin{bmatrix} a \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ a \end{bmatrix}$ → **Hamming Network**

Case #3: $\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ → **Hopfield Network**

McCulloch-Pitts Perceptron

	Banana	Apple
Case #1:	1	-1



$$\mathbf{a} = \mathbf{hardlims}(\mathbf{Wp} + \mathbf{b})$$

Apple/Banana Example

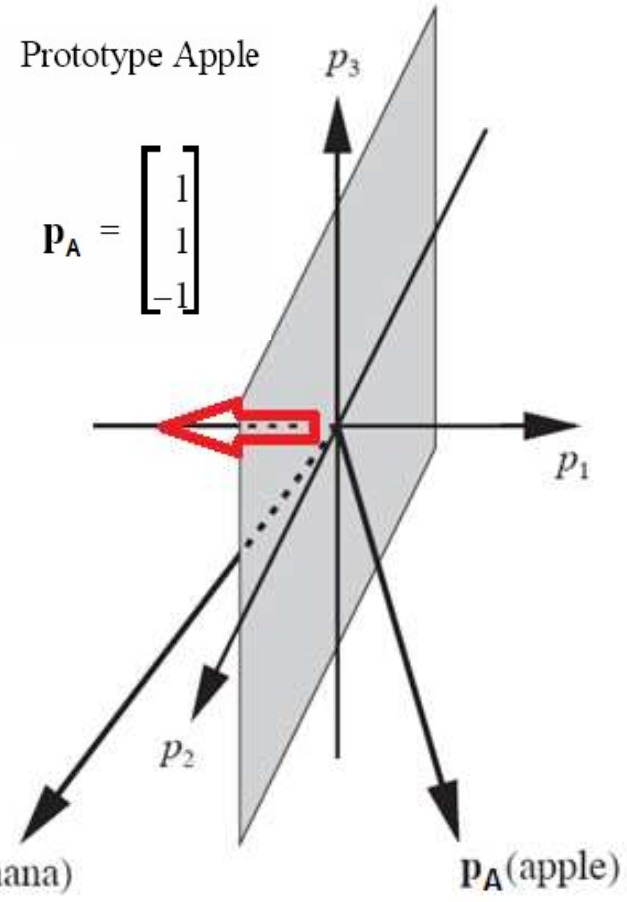
$$a = hardlims\left(\begin{bmatrix} w_{1,1} & w_{1,2} & w_{1,3} \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + b\right)$$

Prototype Banana

$$\mathbf{p_B} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

Prototype Apple

$$\mathbf{p_A} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$



The decision boundary should separate the prototype vectors.

$$p_1 = 0$$

The weight vector should be orthogonal to the decision boundary, and should point in the direction of the vector which should produce an output of 1. The bias determines the position of the boundary

$$\begin{bmatrix} -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + 0 = 0$$

Testing the Network

Banana:

$$a = hardlims\left(\begin{bmatrix}-1 & 0 & 0\end{bmatrix}\begin{bmatrix}-1\\1\\-1\end{bmatrix} + 0\right) = 1 \text{(banana)}$$

Apple:

$$a = hardlims\left(\begin{bmatrix}-1 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\1\\-1\end{bmatrix} + 0\right) = -1 \text{ (apple)}$$

Prototype Banana

$$\mathbf{p}_B = \begin{bmatrix}-1\\1\\-1\end{bmatrix}$$

Prototype Apple

$$\mathbf{p}_A = \begin{bmatrix}1\\1\\-1\end{bmatrix}$$

"Rough" Banana:

$$a = hardlims\left(\begin{bmatrix}-1 & 0 & 0\end{bmatrix}\begin{bmatrix}-1\\-1\\-1\end{bmatrix} + 0\right) = 1 \text{(banana)}$$

XOR problem

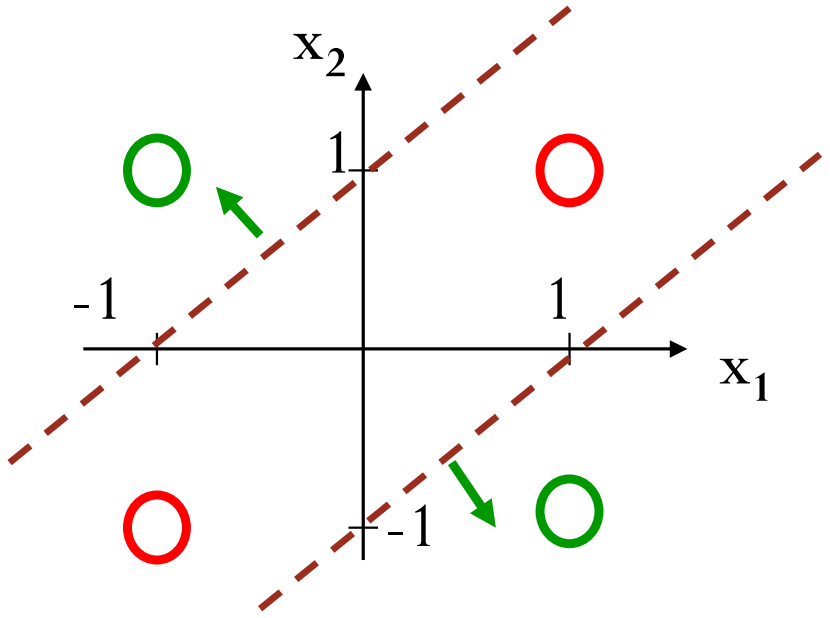
A typical example of non-linearly separable function is the **XOR**. This function takes two input arguments with values in {-1,1} and returns one output in {-1,1}, as specified in the following table:

x_1	x_2	$x_1 \otimes x_2$
-1	-1	**-1**
-1	1	**1**
1	-1	**1**
1	1	**-1**

If we think at -1 and 1 as encoding of the truth values **False** and **True**, respectively, then XOR computes the logical **Exclusive or**, which yields **True** if and only if the two inputs have different truth values.

XOR problem

In this graph of the XOR, input pairs giving output equal to 1 and -1 are depicted with green and red circles, respectively. These two classes (green and red) cannot be separated using a line. We have to use two lines, like those depicted in blue. The following NN with two hidden nodes realizes this non-linear separation, where each hidden node describes one of the two blue lines.

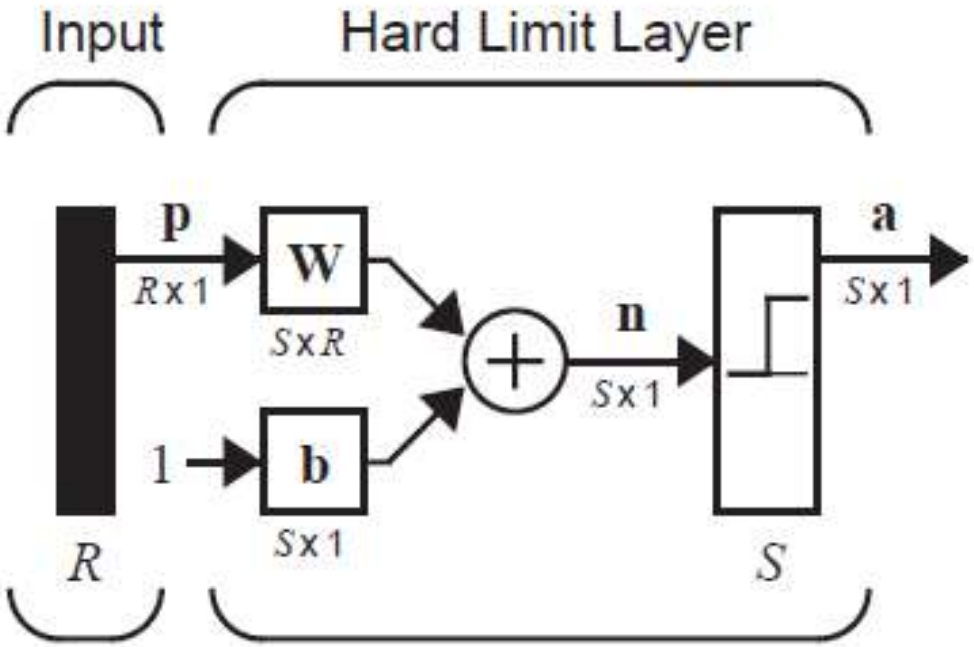


Supervised Learning

Network is provided with a set of examples of proper network behavior **(inputs/targets)**

$$\{\mathbf{p}_1, \mathbf{t}_1\}, \{\mathbf{p}_2, \mathbf{t}_2\}, \ldots, \{\mathbf{p}_Q, \mathbf{t}_Q\}$$

Perceptron Architecture AGAIN



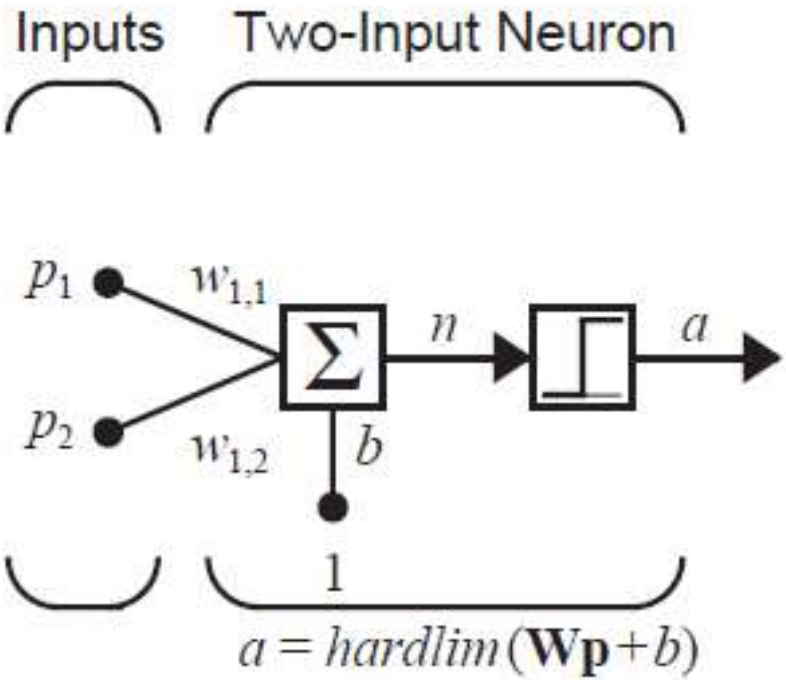
$$\mathbf{a} = \mathbf{hardlim}(\mathbf{Wp}+\mathbf{b})$$

$$\mathbf{W} = \begin{bmatrix} w_{1,1} & w_{1,2} & \cdots & w_{1,R} \\ w_{2,1} & w_{2,2} & \cdots & w_{2,R} \\ \vdots & \vdots & & \vdots \\ w_{S,1} & w_{S,2} & \cdots & w_{S,R} \end{bmatrix}$$

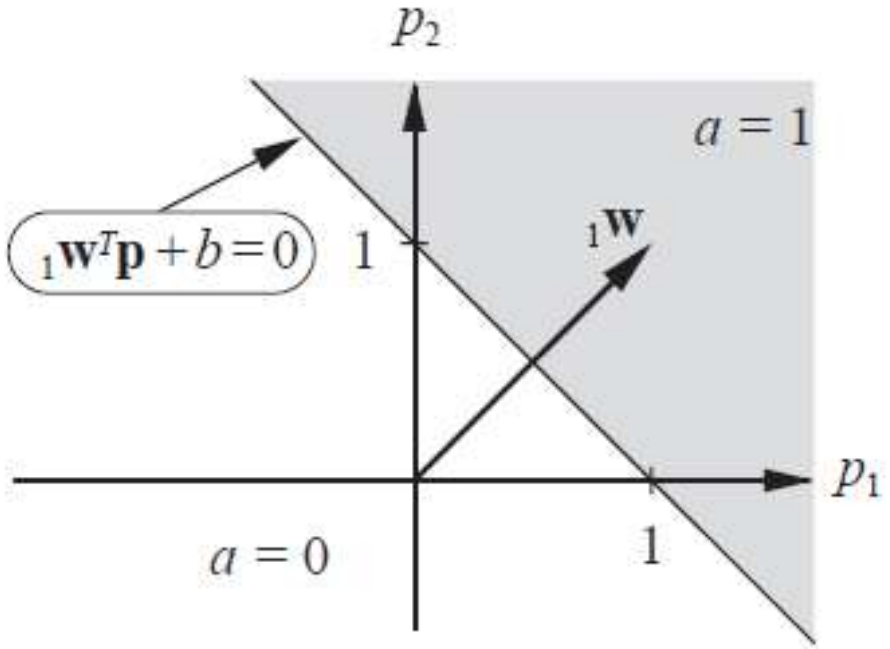
$${}_i\mathbf{w} = \begin{bmatrix} w_{i,1} \\ w_{i,2} \\ \vdots \\ w_{i,R} \end{bmatrix} \qquad \mathbf{W} = \begin{bmatrix} {}_1\mathbf{w}^T \\ {}_2\mathbf{w}^T \\ \vdots \\ {}_S\mathbf{w}^T \end{bmatrix}$$

$$a_i = hardlim(n_i) = hardlim({}_i\mathbf{w}^T\mathbf{p} + b_i)$$

Single-Neuron Perceptron



$w_{1,1} = 1 \qquad w_{1,2} = 1 \qquad b = -1$

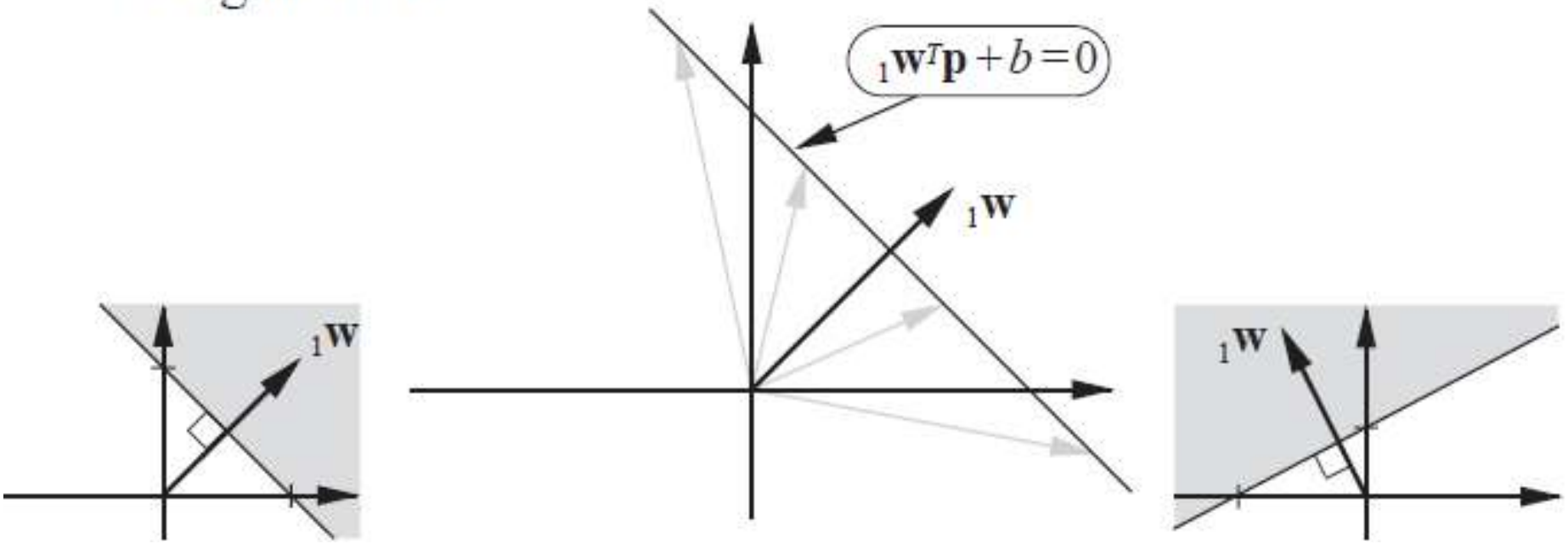


$$a = hardlim(_1\mathbf{w}^{\mathsf{T}}\mathbf{p} + b) = hardlim(w_{1,1}p_1 + w_{1,2}p_2 + b)$$

Decision Boundary

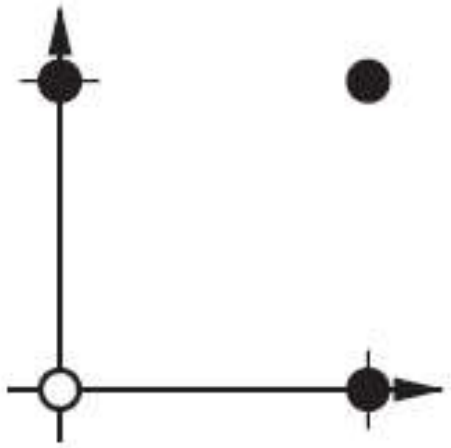
$$ {}_1\mathbf{w}^T\mathbf{p} + b = 0 \qquad {}_1\mathbf{w}^T\mathbf{p} = -b $$

- All points on the decision boundary have the same inner product with the weight vector.
- Therefore they have the same projection onto the weight vector, and they must lie on a line orthogonal to the weight vector

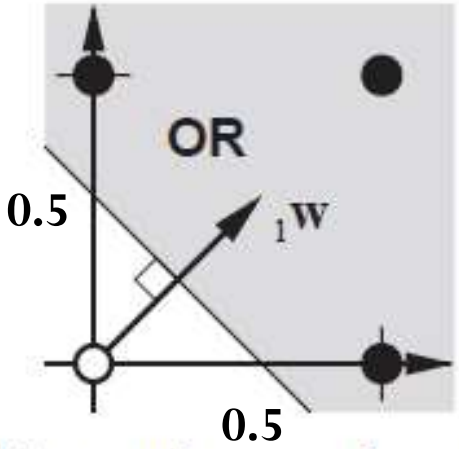


Example - OR

$$\left\{\mathbf{p}_1 = \begin{bmatrix}0\\0\end{bmatrix}, t_1 = 0\right\} \quad \left\{\mathbf{p}_2 = \begin{bmatrix}0\\1\end{bmatrix}, t_2 = 1\right\} \quad \left\{\mathbf{p}_3 = \begin{bmatrix}1\\0\end{bmatrix}, t_3 = 1\right\} \quad \left\{\mathbf{p}_4 = \begin{bmatrix}1\\1\end{bmatrix}, t_4 = 1\right\}$$



OR Solution



Weight vector should be orthogonal to the decision boundary.

$$_1\mathbf{w} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

Pick a point on the decision boundary to find the bias.

$$_1\mathbf{w}^T\mathbf{p} + b = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} + b = 0.25 + \mathrm{b} = 0 \quad \Rightarrow \quad b = -0.25$$

Multiple-Neuron Perceptron

Each neuron will have its own decision boundary.

$$_i\mathbf{w}^T\mathbf{p} + b_i = 0$$

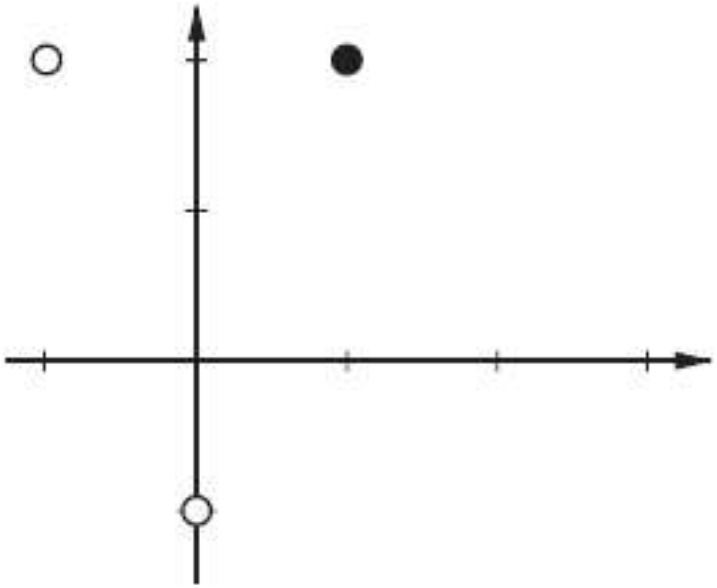
A single neuron can classify input vectors into two categories.

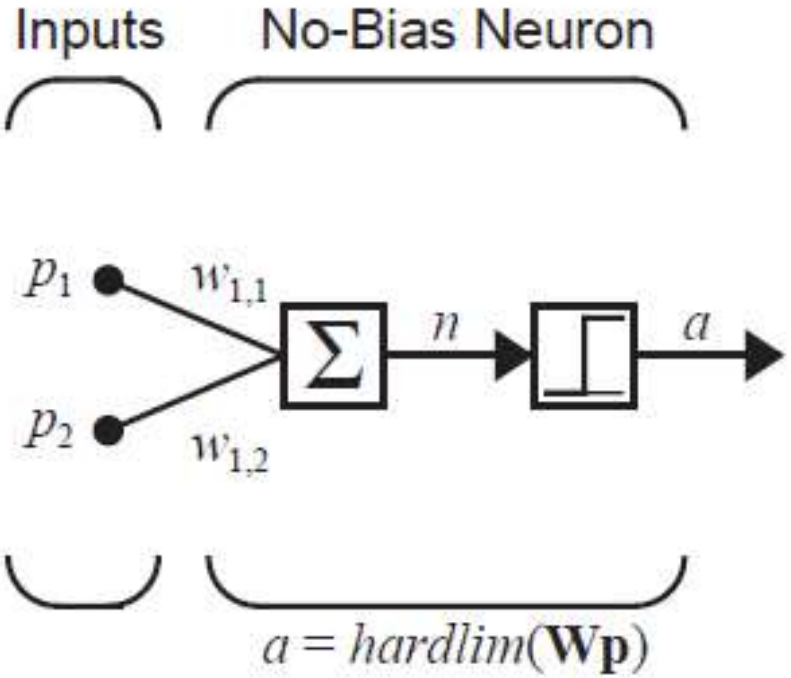
A multi-neuron perceptron can classify input vectors into 2^S categories.

Learning Rule Test Problem

$$\{\mathbf{p}_1, \mathbf{t}_1\}, \{\mathbf{p}_2, \mathbf{t}_2\}, \ldots, \{\mathbf{p}_Q, \mathbf{t}_Q\}$$

$$\left\{\mathbf{p}_1 = \begin{bmatrix}1\\2\end{bmatrix}, t_1 = 1\right\} \quad \left\{\mathbf{p}_2 = \begin{bmatrix}-1\\2\end{bmatrix}, t_2 = 0\right\} \quad \left\{\mathbf{p}_3 = \begin{bmatrix}0\\-1\end{bmatrix}, t_3 = 0\right\}$$

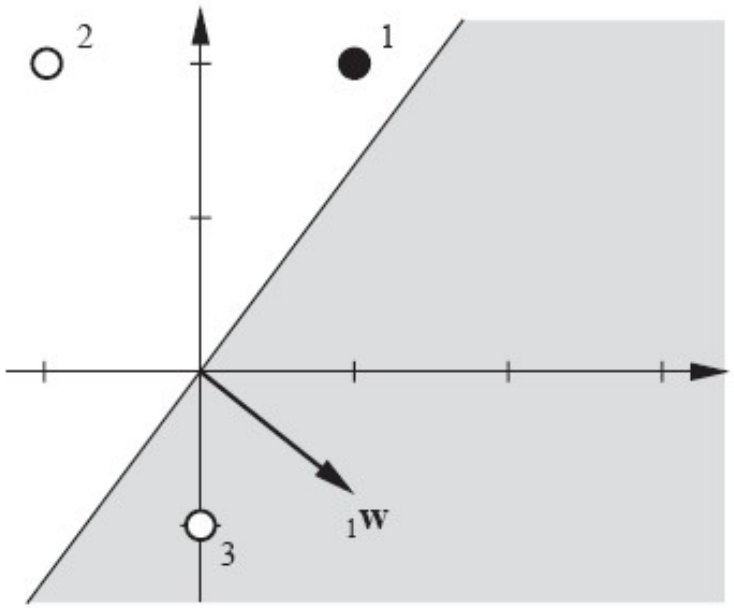




Starting Point

Random initial weight:

$$ {}_1\mathbf{w} = \begin{bmatrix} 1.0 \\ -0.8 \end{bmatrix} $$



Present $\mathbf{p}_1$ to the network:

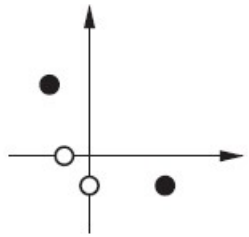
$$ a = hardlim({}_1\mathbf{w}^T\mathbf{p}_1) = hardlim\left(\begin{bmatrix} 1.0 & -0.8 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) $$

$$ a = hardlim(-0.6) = 0 $$

Incorrect Classification.

Tentative Learning Rule

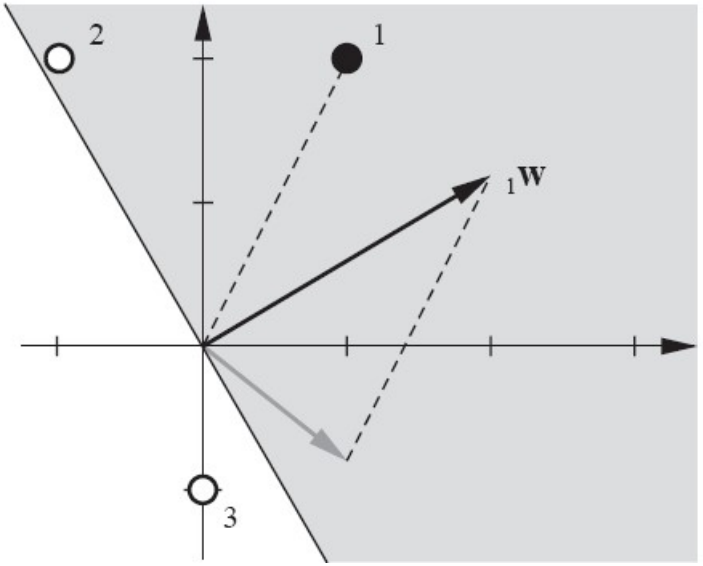
- Set ${}_1\mathbf{w}$ to $\mathbf{p}_1$
  - Not stable ×



- Add $\mathbf{p}_1$ to ${}_1\mathbf{w}$ ✓

Tentative Rule: If $t = 1$ and $a = 0$, then ${}_1\mathbf{w}^{new} = {}_1\mathbf{w}^{old} + \mathbf{p}$

$${}_1\mathbf{w}^{new} = {}_1\mathbf{w}^{old} + \mathbf{p}_1 = \begin{bmatrix} 1.0 \\ -0.8 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2.0 \\ 1.2 \end{bmatrix}$$



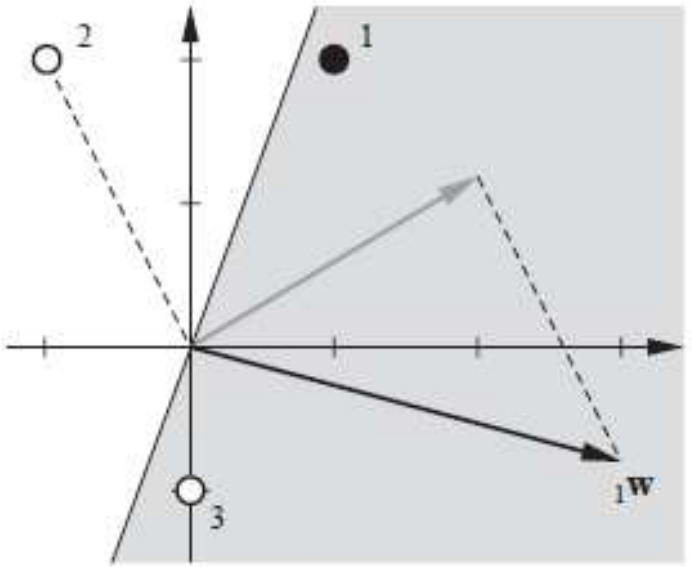
Second Input Vector

$$a = hardlim({}_1\mathbf{w}^T\mathbf{p}_2) = hardlim\left(\begin{bmatrix}2.0 & 1.2\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}\right)$$

$$a = hardlim(0.4) = 1 \quad \text{(Incorrect Classification)}$$

Modification to Rule: If $t = 0$ and $a = 1$, then ${}_1\mathbf{w}^{new} = {}_1\mathbf{w}^{old} - \mathbf{p}$

$${}_1\mathbf{w}^{new} = {}_1\mathbf{w}^{old} - \mathbf{p}_2 = \begin{bmatrix}2.0\\1.2\end{bmatrix} - \begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}3.0\\-0.8\end{bmatrix}$$

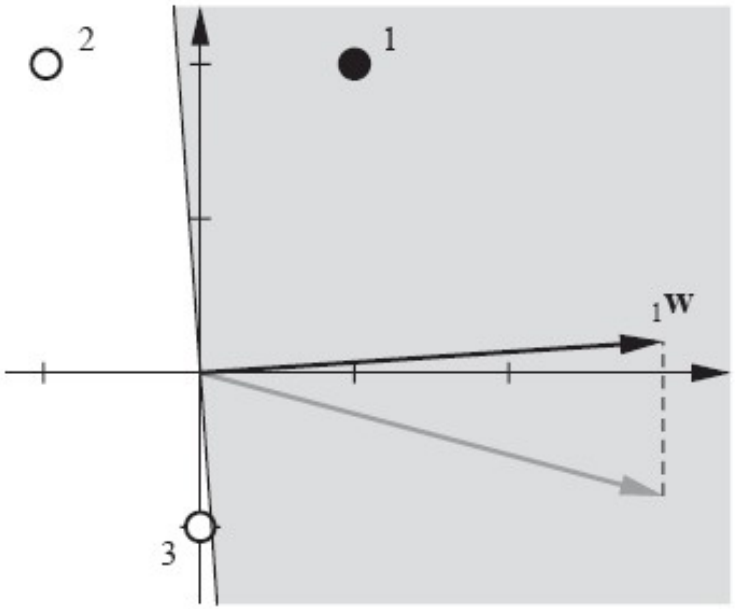


Third Input Vector

$$a = hardlim({}_1\mathbf{w}^T \mathbf{p}_3) = hardlim\left(\begin{bmatrix}3.0 & -0.8\end{bmatrix}\begin{bmatrix}0\\-1\end{bmatrix}\right)$$

$$a = \text{hardlim}(0.8) = 1 \qquad \text{(Incorrect Classification)}$$

$${}_1\mathbf{w}^{new} = {}_1\mathbf{w}^{old} - \mathbf{p}_3 = \begin{bmatrix}3.0\\-0.8\end{bmatrix} - \begin{bmatrix}0\\-1\end{bmatrix} = \begin{bmatrix}3.0\\0.2\end{bmatrix}$$



Patterns are now correctly classified.

If $t = a$, then ${}_1\mathbf{w}^{new} = {}_1\mathbf{w}^{old}$.

Unified Learning Rule

If $t = 1$ and $a = 0$, then ${}_1\mathbf{w}^{new} = {}_1\mathbf{w}^{old} + \mathbf{p}$

If $t = 0$ and $a = 1$, then ${}_1\mathbf{w}^{new} = {}_1\mathbf{w}^{old} - \mathbf{p}$

If $t = a$, then ${}_1\mathbf{w}^{new} = {}_1\mathbf{w}^{old}$

$$e = t - a$$

If $e = 1$, then ${}_1\mathbf{w}^{new} = {}_1\mathbf{w}^{old} + \mathbf{p}$

If $e = -1$, then ${}_1\mathbf{w}^{new} = {}_1\mathbf{w}^{old} - \mathbf{p}$

If $e = 0$, then ${}_1\mathbf{w}^{new} = {}_1\mathbf{w}^{old}$

$${}_1\mathbf{w}^{new} = {}_1\mathbf{w}^{old} + e\mathbf{p} = {}_1\mathbf{w}^{old} + (t-a)\mathbf{p}$$

$$b^{new} = b^{old} + e$$

A bias is a weight with an input of 1.

Multiple-Neuron Perceptrons

To update the ith row of the weight matrix:

$$_i\mathbf{w}^{new} = {}_i\mathbf{w}^{old} + e_i\mathbf{p}$$

$$b_i^{new} = b_i^{old} + e_i$$

Matrix form:

$$\mathbf{W}^{new} = \mathbf{W}^{old} + \mathbf{e}\mathbf{p}^T$$

$$\mathbf{b}^{new} = \mathbf{b}^{old} + \mathbf{e}$$

Apple/Banana Example

Training Set

$$\left\{\mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, t_1 = \begin{bmatrix} 1 \end{bmatrix}\right\} \qquad \left\{\mathbf{p}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, t_2 = \begin{bmatrix} 0 \end{bmatrix}\right\}$$

Initial Weights

$$\mathbf{W} = \begin{bmatrix} 0.5 & -1 & -0.5 \end{bmatrix} \qquad b = 0.5$$

First Iteration

$$a = hardlim(\mathbf{W}\mathbf{p}_1 + b) = hardlim\left(\begin{bmatrix} 0.5 & -1 & -0.5 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + 0.5\right)$$

$$a = hardlim(-0.5) = 0 \qquad e = t_1 - a = 1 - 0 = 1$$

$$\mathbf{W}^{new} = \mathbf{W}^{old} + e\mathbf{p}^T = \begin{bmatrix} 0.5 & -1 & -0.5 \end{bmatrix} + (1)\begin{bmatrix} -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -0.5 & 0 & -1.5 \end{bmatrix}$$

$$b^{new} = b^{old} + e = 0.5 + (1) = 1.5$$

Second Iteration

$$a = hardlim\left(\mathbf{W}\mathbf{p}_2 + b\right) = hardlim\left(\begin{bmatrix}-0.5 & 0 & -1.5\end{bmatrix}\begin{bmatrix}1\\1\\-1\end{bmatrix} + (1.5)\right)$$

$$a = hardlim\,(2.5) = 1$$

$$e = t_2 - a = 0 - 1 = -1$$

$$\mathbf{W}^{new} = \mathbf{W}^{old} + e\mathbf{p}^T = \begin{bmatrix}-0.5 & 0 & -1.5\end{bmatrix} + (-1)\begin{bmatrix}1 & 1 & -1\end{bmatrix} = \begin{bmatrix}-1.5 & -1 & -0.5\end{bmatrix}$$

$$b^{new} = b^{old} + e = 1.5 + (-1) = 0.5$$

Check

$$a = hardlim\,(\mathbf{W}\mathbf{p}_1 + b) = hardlim\left(\begin{bmatrix}-1.5 & -1 & -0.5\end{bmatrix}\begin{bmatrix}-1\\1\\-1\end{bmatrix} + 0.5\right)$$

$$a = hardlim\,(1.5) = 1 = t_1$$

$$a = hardlim\,(\mathbf{W}\mathbf{p}_2 + b) = hardlim\left(\begin{bmatrix}-1.5 & -1 & -0.5\end{bmatrix}\begin{bmatrix}1\\1\\-1\end{bmatrix} + 0.5\right)$$

$$a = hardlim\,(-1.5) = 0 = t_2$$

Perceptron Rule Capability

The Perceptron rule will always converge to weights which accomplish the desired classification, assuming that such weights exist.

Rosenblatt's single layer perceptron is trained as follow:

1. Randomly initialize all the networks weights.
2. Apply inputs and find outputs (**Feedforward**).
3. Compute the errors.
4. Update each weight as

$$w_{ij}(k+1) = w_{ij}(k) + \eta\ p_i(k)\, e_j(k)$$

5. Repeat steps 2 to 4 until the errors reach the satisfactory level.

What is η?

- Name : Learning rate.
- Where is living: usually between 0 and 1.
- It can change it's value during learning.
- Can define separately for each parameters.

Example 1

We have a classification problem with four classes of input vector. The four classes are

$$\text{class 1:}\left\{\mathbf{p}_1 = \begin{bmatrix}1\\1\end{bmatrix}, \mathbf{p}_2 = \begin{bmatrix}1\\2\end{bmatrix}\right\}, \text{class 2:}\left\{\mathbf{p}_3 = \begin{bmatrix}2\\-1\end{bmatrix}, \mathbf{p}_4 = \begin{bmatrix}2\\0\end{bmatrix}\right\},$$

$$\text{class 3:}\left\{\mathbf{p}_5 = \begin{bmatrix}-1\\2\end{bmatrix}, \mathbf{p}_6 = \begin{bmatrix}-2\\1\end{bmatrix}\right\}, \text{class 4:}\left\{\mathbf{p}_7 = \begin{bmatrix}-1\\-1\end{bmatrix}, \mathbf{p}_8 = \begin{bmatrix}-2\\-2\end{bmatrix}\right\}.$$

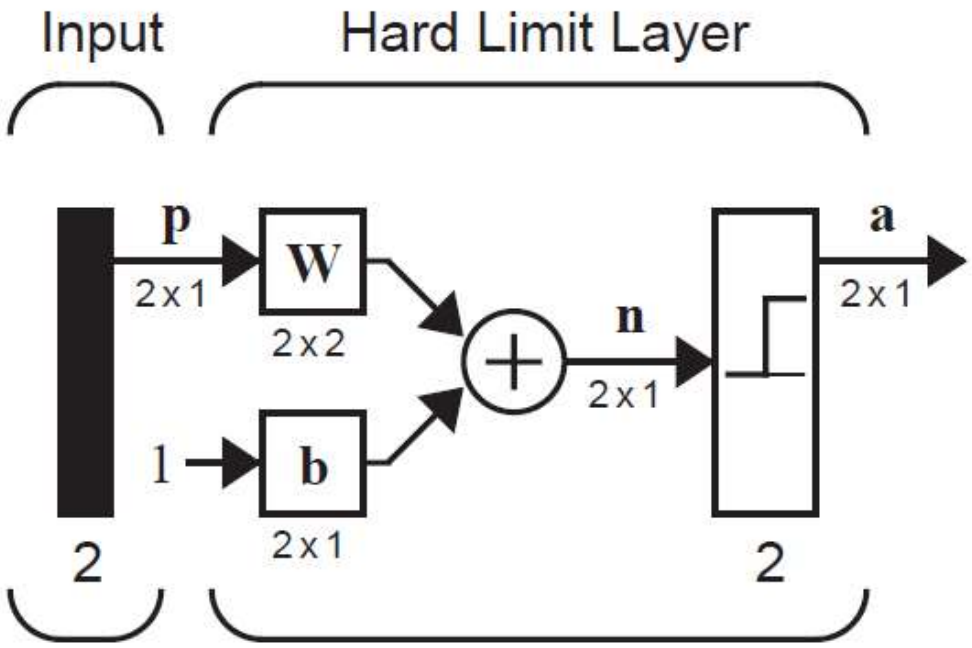
Design a perceptron network to solve this problem.

$$\text{class 1:}\left\{\mathbf{t}_1 = \begin{bmatrix}0\\0\end{bmatrix}, \mathbf{t}_2 = \begin{bmatrix}0\\0\end{bmatrix}\right\}, \text{class 2:}\left\{\mathbf{t}_3 = \begin{bmatrix}0\\1\end{bmatrix}, \mathbf{t}_4 = \begin{bmatrix}0\\1\end{bmatrix}\right\},$$

$$\text{class 3:}\left\{\mathbf{t}_5 = \begin{bmatrix}1\\0\end{bmatrix}, \mathbf{t}_6 = \begin{bmatrix}1\\0\end{bmatrix}\right\}, \text{class 4:}\left\{\mathbf{t}_7 = \begin{bmatrix}1\\1\end{bmatrix}, \mathbf{t}_8 = \begin{bmatrix}1\\1\end{bmatrix}\right\}.$$

Example 1

To solve a problem with four classes of input vector we will need a perceptron with at least two neurons, since an S-neuron perceptron can categorize 2^S classes. The two-neuron perceptron is shown in Figure P4.2.



$$\mathbf{a} = \mathbf{hardlim}(\mathbf{Wp}+\mathbf{b})$$

Example 1

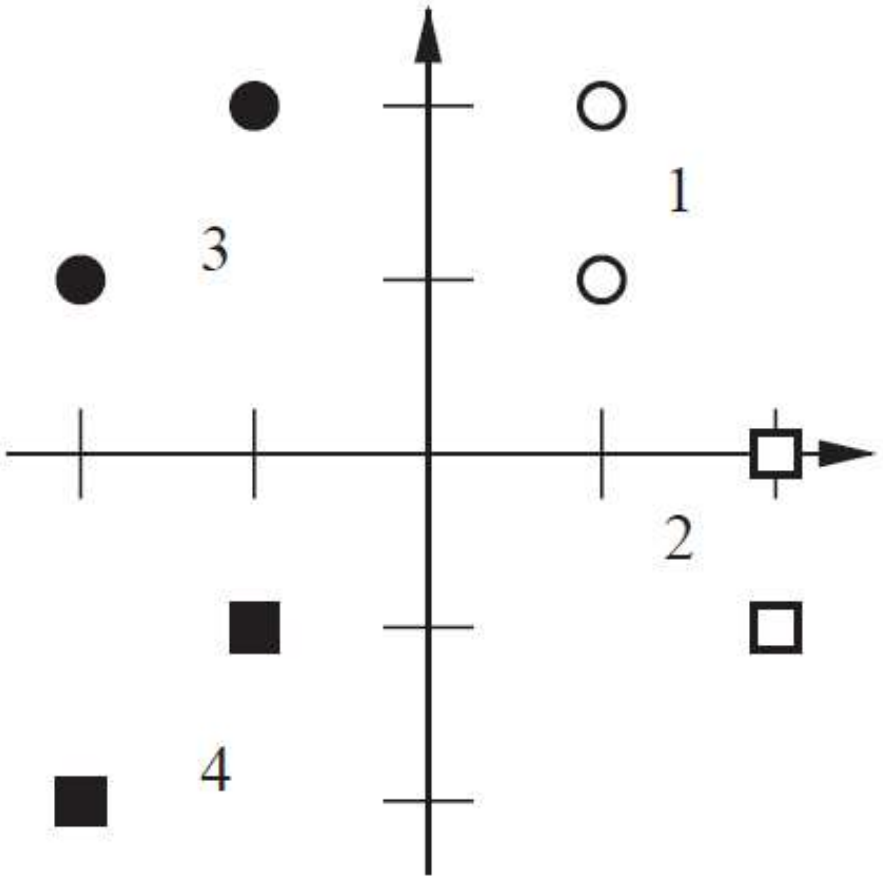


Figure P4.3 Input Vectors for Problem P4.3

Example 1

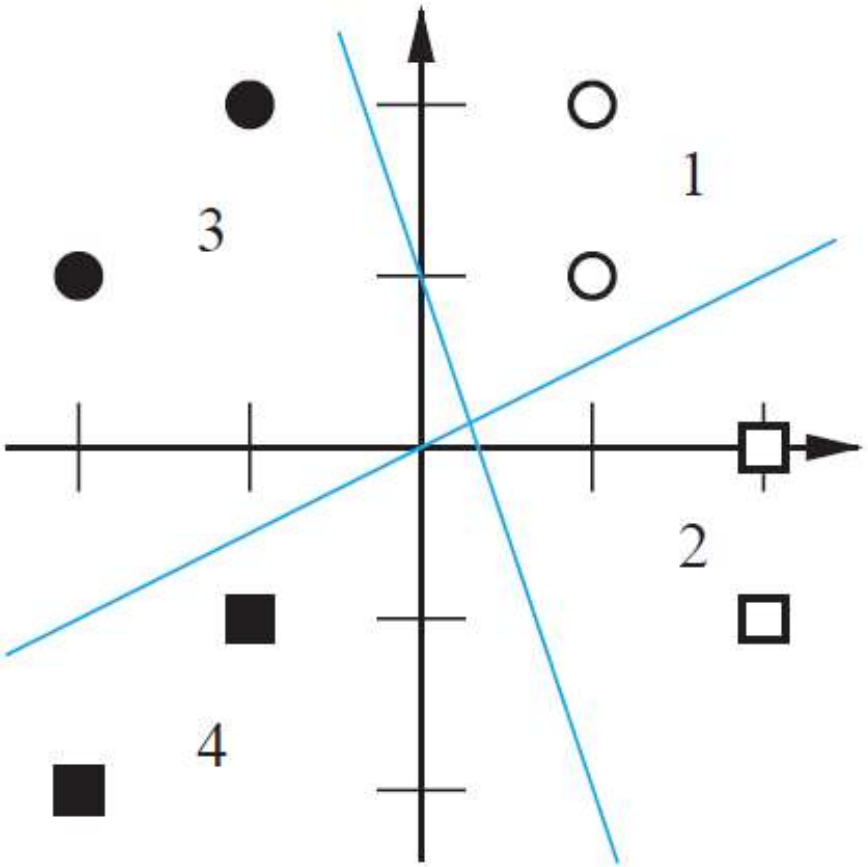
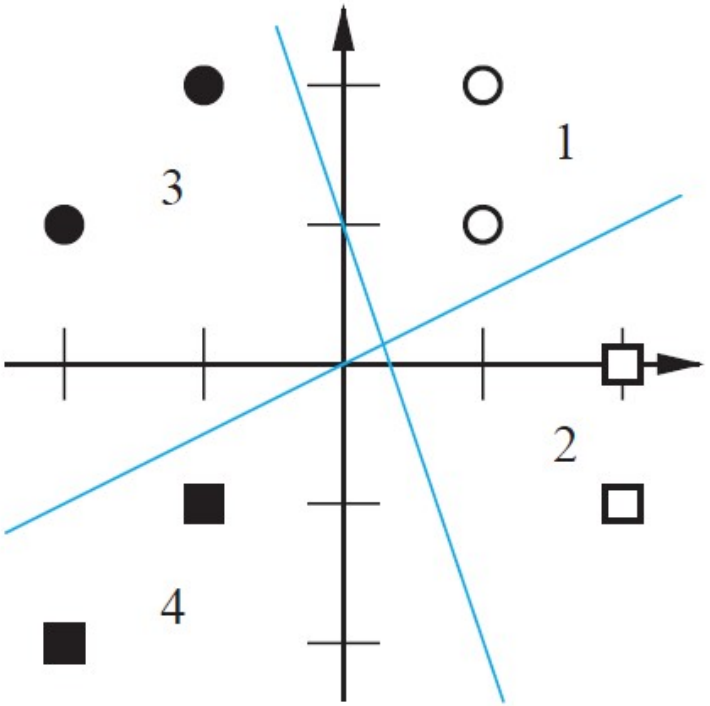


Figure P4.4 Tentative Decision Boundaries for Problem P4.3

Example 1



$$\text{class 1:}\left\{\mathbf{t}_1 = \begin{bmatrix}0\\0\end{bmatrix}, \mathbf{t}_2 = \begin{bmatrix}0\\0\end{bmatrix}\right\}, \text{class 2:}\left\{\mathbf{t}_3 = \begin{bmatrix}0\\1\end{bmatrix}, \mathbf{t}_4 = \begin{bmatrix}0\\1\end{bmatrix}\right\},$$

$$\text{class 3:}\left\{\mathbf{t}_5 = \begin{bmatrix}1\\0\end{bmatrix}, \mathbf{t}_6 = \begin{bmatrix}1\\0\end{bmatrix}\right\}, \text{class 4:}\left\{\mathbf{t}_7 = \begin{bmatrix}1\\1\end{bmatrix}, \mathbf{t}_8 = \begin{bmatrix}1\\1\end{bmatrix}\right\}.$$

Example 1

$$ {}_1\mathbf{w} = \begin{bmatrix} -3 \\ -1 \end{bmatrix} \text{ and } {}_2\mathbf{w} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}. $$

Note that the lengths of the weight vectors is not important, only their directions. They must be orthogonal to the decision boundaries. Now we can calculate the bias by picking a point on a boundary and satisfying Eq. (4.15):

$$ b_1 = -{}_1\mathbf{w}^T\mathbf{p} = -\begin{bmatrix} -3 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1, $$

$$ b_2 = -{}_2\mathbf{w}^T\mathbf{p} = -\begin{bmatrix} 1 & -2 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0. $$

Example 1

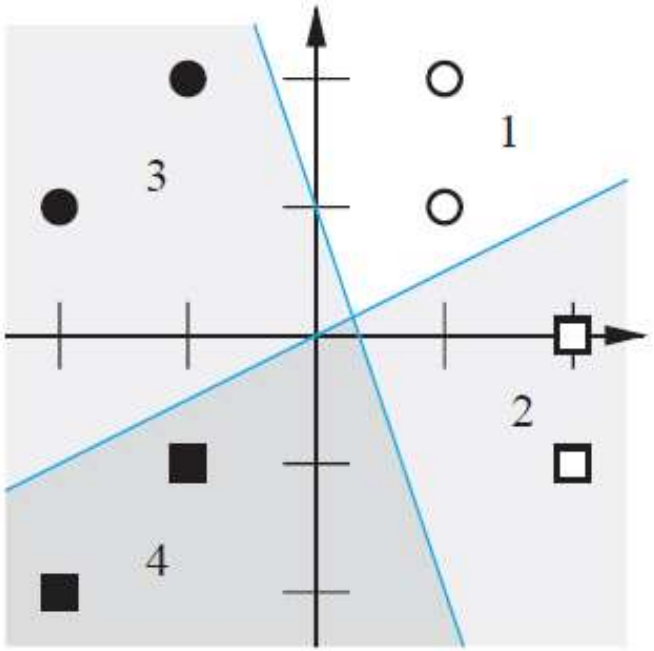


Figure P4.5 Decision Regions for Problem P4.3

In matrix form we have

$$\mathbf{W} = \begin{bmatrix} {}_1\mathbf{w}^T \\ {}_2\mathbf{w}^T \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 1 & -2 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

which completes our design.

Example 2

Solve the following classification problem with the perceptron rule. Apply each input vector in order, for as many repetitions as it takes to ensure that the problem is solved. Draw a graph of the problem only after you have found a solution.

$$\left\{\mathbf{p}_1 = \begin{bmatrix}2\\2\end{bmatrix}, t_1 = 0\right\}\left\{\mathbf{p}_2 = \begin{bmatrix}1\\-2\end{bmatrix}, t_2 = 1\right\}\left\{\mathbf{p}_3 = \begin{bmatrix}-2\\2\end{bmatrix}, t_3 = 0\right\}\left\{\mathbf{p}_4 = \begin{bmatrix}-1\\1\end{bmatrix}, t_4 = 1\right\}$$

Use the initial weights and bias:

$$\mathbf{W}(0) = \begin{bmatrix}0 & 0\end{bmatrix} \qquad b(0) = 0\,.$$

Example 2

We start by calculating the perceptron's output a for the first input vector $\mathbf{p}_1$, using the initial weights and bias.

$$a = hardlim(\mathbf{W}(0)\mathbf{p}_1 + b(0))$$

$$= hardlim\left(\begin{bmatrix}0 & 0\end{bmatrix}\begin{bmatrix}2\\2\end{bmatrix} + 0\right) = hardlim(0) = 1$$

The output a does not equal the target value t_1, so we use the perceptron rule to find new weights and biases based on the error.

$$e = t_1 - a = 0 - 1 = -1$$

$$\mathbf{W}(1) = \mathbf{W}(0) + e\mathbf{p}_1^T = \begin{bmatrix}0 & 0\end{bmatrix} + (-1)\begin{bmatrix}2 & 2\end{bmatrix} = \begin{bmatrix}-2 & -2\end{bmatrix}$$

$$b(1) = b(0) + e = 0 + (-1) = -1$$

Example 2

We now apply the second input vector $\mathbf{p}_2$, using the updated weights and bias.

$$a = hardlim(\mathbf{W}(1)\mathbf{p}_2 + b(1))$$

$$= hardlim\left(\begin{bmatrix} -2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} - 1 \right) = hardlim(1) = 1$$

This time the output a is equal to the target t_2. Application of the perceptron rule will not result in any changes.

$$\mathbf{W}(2) = \mathbf{W}(1)$$

$$b(2) = b(1)$$

We now apply the third input vector.

Example 2

$$a = hardlim(\mathbf{W}(2)\mathbf{p}_3 + b(2))$$

$$= hardlim\left(\begin{bmatrix}-2 & -2\end{bmatrix}\begin{bmatrix}-2\\2\end{bmatrix} - 1\right) = hardlim(-1) = 0$$

The output in response to input vector $\mathbf{p}_3$ is equal to the target t_3, so there will be no changes.

$$\mathbf{W}(3) = \mathbf{W}(2)$$

$$b(3) = b(2)$$

We now move on to the last input vector $\mathbf{p}_4$.

$$a = hardlim(\mathbf{W}(3)\mathbf{p}_4 + b(3))$$

$$= hardlim\left(\begin{bmatrix}-2 & -2\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix} - 1\right) = hardlim(-1) = 0$$

Example 2

This time the output a does not equal the appropriate target t_4. The perceptron rule will result in a new set of values for $\mathbf{W}$ and b.

$$e = t_4 - a = 1 - 0 = 1$$

$$\mathbf{W}(4) = \mathbf{W}(3) + e\mathbf{p}_4^T = \begin{bmatrix} -2 & -2 \end{bmatrix} + (1)\begin{bmatrix} -1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -1 \end{bmatrix}$$

$$b(4) = b(3) + e = -1 + 1 = 0$$

We now must check the first vector $\mathbf{p}_1$ again. This time the output a is equal to the associated target t_1.

$$a = hardlim(\mathbf{W}(4)\mathbf{p}_1 + b(4))$$

$$= hardlim\left(\begin{bmatrix} -3 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} + 0\right) = hardlim(-8) = 0$$

Example 2

Therefore there are no changes.

$$\mathbf{W}(5) = \mathbf{W}(4)$$
$$b(5) = b(4)$$

The second presentation of $\mathbf{p}_2$ results in an error and therefore a new set of weight and bias values.

$$a = hardlim(\mathbf{W}(5)\mathbf{p}_2 + b(5))$$

$$= hardlim\left(\begin{bmatrix}-3 & -1\end{bmatrix}\begin{bmatrix}1\\-2\end{bmatrix} + 0\right) = hardlim(-1) = 0$$

Therefore the algorithm has converged. The final solution is:

$$\mathbf{W} = \begin{bmatrix}-2 & -3\end{bmatrix} \qquad b = 1\,.$$

Perceptron Limitations

Linear Decision Boundary

$$_1\mathbf{w}^T\mathbf{p} + b = 0$$

Linearly Inseparable Problems