$$X \sim N(0,1) \longrightarrow f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$Y \sim N(0,1) \qquad f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

الف)

$$f_{UV}(u,v) = \left| \text{Jacobian}(u,v) \right| f_{XY}(x,y)$$

$$U = X \longrightarrow u = x$$

$$V = \frac{X}{Y} \longrightarrow Y = \frac{X}{V} \longrightarrow y = \frac{u}{v}$$

$$\text{Jacobian}(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = \frac{-u}{v^2}$$

$$\xrightarrow{\text{پس}} f_{UV}(u,v) = \left| \frac{-u}{v^2} \right| f_{XY}(x,y)$$

$$\xrightarrow{\text{X و Y مستقل}} f_{UV}(u,v) = \frac{|u|}{v^2} f_X(x) f_Y(y)$$

$$= \frac{|u|}{v^2} \left(\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \times \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} \right)$$

$$\xrightarrow{u = x \text{ و } y = \frac{u}{v}} \boxed{= \frac{|u|}{v^2} \times \frac{e^{-\frac{1}{2}\left(u^2 + \frac{u^2}{v^2}\right)}}{2\pi}}$$

ب)

$$f_V(v) = \int_{-\infty}^{\infty} \frac{|u|}{v^2} \frac{e^{-\frac{1}{2}u^2\left(1+\frac{1}{v^2}\right)}}{2\pi} du$$

$$\xrightarrow[\text{ضرب دو تابع زوج است}]{|u| \text{ و } e^{-\frac{1}{2}u^2\left(1+\frac{1}{v^2}\right)} \text{ توابعی زوج هستند پس}} = \frac{2}{2\pi v^2} \int_0^{\infty} u e^{-\frac{1}{2}u^2\left(1+\frac{1}{v^2}\right)} du$$

$$f_V(v) = \frac{-\left(1+\frac{1}{v^2}\right)}{-\left(1+\frac{1}{v^2}\right)} \times \frac{2}{2\pi v^2} \int_0^{\infty} u e^{-\frac{1}{2}u^2\left(1+\frac{1}{v^2}\right)} du$$

$$z = -\frac{1}{2}u^2\left(1+\frac{1}{v^2}\right)$$

$$dz = -u\left(1+\frac{1}{v^2}\right) du$$

$$\longrightarrow f_V(v) = \frac{-1}{\pi(1+v^2)} \int e^z dz$$

$$= \frac{-1}{\pi(1+v^2)} \times e^{-\frac{1}{2}u^2\left(1+\frac{1}{v^2}\right)} \Big|_0^{\infty}$$

$$= \frac{-1}{\pi(1+v^2)} \times (0-1)$$

$$= \frac{1}{\pi(1+v^2)} \longrightarrow \begin{cases} \gamma = 1 \\ V_0 = 0 \end{cases}$$

$$V \sim \text{Cauchy}(0,1)$$

-۳

$Y_1 \sim exp(\lambda_1)$
$Y_2 \sim exp(\lambda_2)$ $\longrightarrow$ $f_{Y_1}(y_1) = \lambda_1 e^{-\lambda_1 y_1}$ $y_1 > 0$
$f_{Y_2}(y_2) = \lambda_2 e^{-\lambda_2 y_2}$ $y_2 > 0$

الف)

$$f_{U,V}(u,v) = |Jacoubian(v,v)| f_{Y_1 Y_2}(y_1, y_2)$$

$$U = Y_1 \longrightarrow y_1 = u$$

$$V = \frac{Y_1}{Y_2} \longrightarrow Y_2 = \frac{Y_1}{V} \longrightarrow Y_2 = \frac{U}{V} \longrightarrow y_2 = \frac{u}{v}$$

$$Jacoubian(u,v) = \begin{vmatrix} \frac{\partial y_1}{\partial u} & \frac{\partial y_1}{\partial v} \\ \frac{\partial y_2}{\partial u} & \frac{\partial y_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix} = \frac{-u}{v^2}$$

$$\xrightarrow{\text{پس}} f_{uv}(u,v) = \frac{|u|}{v^2} f_{Y_1 Y_2}(y_1, y_2)$$

$$\xrightarrow{\text{مستقل } Y_2 \text{ و } Y_1} f_{uv}(u,v) = \frac{|u|}{v^2} f_{Y_1}(y_1) f_{Y_2}(y_2)$$

$$= \frac{|u|}{v^2} \lambda_1 e^{-\lambda_1 y_1} \times \lambda_2 e^{-\lambda_2 y_2}$$

$$= \frac{|u| \lambda_1 \lambda_2}{v^2} e^{(-\lambda_1 y_1 - \lambda_2 y_2)}$$

$$\xrightarrow{y_1 = u \text{ و } y_2 = \frac{u}{v}} f_{uv}(u,v) = \frac{|u|}{v^2} \lambda_1 \lambda_2 e^{-u(\lambda_1 + \frac{\lambda_2}{v})}$$

$$f_V(v) = \int_{-\infty}^{\infty} \frac{|u|}{v^2} \lambda_1 \lambda_2 e^{-u(\lambda_1 + \frac{\lambda_2}{v})} du$$

$$= \frac{\lambda_1 \lambda_2}{v^2} \int_0^{\infty} u e^{-u(\lambda_1 + \frac{\lambda_2}{v})} du$$

$$f(u) = u \;,\; f'(u) = 1$$

$$g'(u) = e^{-u(\lambda_1 + \frac{\lambda_2}{v})} \;,\; g(x) = \frac{-e^{-u(\lambda_1 + \frac{\lambda_2}{v})}}{\lambda_1 + \frac{\lambda_2}{v}}$$

$$\int f g' = fg - \int f' g$$

$$\int_0^{\infty} u e^{-u(\lambda_1 + \frac{\lambda_2}{v})} du = \frac{-u e^{-u(\lambda_1 + \frac{\lambda_2}{v})}}{\lambda_1 + \frac{\lambda_2}{v}} - \int_0^{\infty} \frac{-e^{-u(\lambda_1 + \frac{\lambda_2}{v})}}{\lambda_1 + \frac{\lambda_2}{v}}$$

روش جزء

$$\lim_{u \to 0} \frac{-u e^{-u(\lambda_1 + \frac{\lambda_2}{v})}}{\lambda_1 + \frac{\lambda_2}{v}} = 0$$

$$\lim_{u \to \infty} \frac{-1}{\lambda_1 + \frac{\lambda_2}{v}} \left(u e^{-u(\lambda_1 + \frac{\lambda_2}{v})} \right) = \frac{-1}{\lambda_1 + \frac{\lambda_2}{v}} \lim_{u \to \infty} \frac{u}{e^{u(\lambda_1 + \frac{\lambda_2}{v})}}$$

$$\overset{Hop}{=} \frac{-1}{\lambda_1 + \frac{\lambda_2}{v}} \lim_{u \to \infty} \frac{1}{(\lambda_1 + \frac{\lambda_2}{v}) e^{u(\lambda_1 + \frac{\lambda_2}{v})}} = \frac{-1}{\infty} = 0$$

$\longleftarrow$ دین

$$\int f g' = \cancel{fg}_{\,0} - \int f' g$$

$$\rightarrow \int_0^{\infty} u e^{-u(\lambda_1 + \frac{\lambda_2}{v})} du = - \int - \frac{e^{-u(\lambda_1 + \frac{\lambda_2}{v})}}{\lambda_1 + \frac{\lambda_2}{v}}$$

$$= \int \frac{e^{-u(\lambda_1 + \frac{\lambda_2}{v})}}{\lambda_1 + \frac{\lambda_2}{v}}$$

$$= - \frac{e^{-u(\lambda_1 + \frac{\lambda_2}{v})}}{(\lambda_1 + \frac{\lambda_2}{v})^2} \Bigg|_0^{\infty}$$

$$= 0 + \frac{1}{(\lambda_1 + \frac{\lambda_2}{v})^2}$$

$$\longrightarrow f_V(v) = \frac{\lambda_1 \lambda_2}{v^2} \times \frac{v^2}{(v\lambda_1 + \lambda_2)^2} = \boxed{\frac{\lambda_1 \lambda_2}{(v\lambda_1 + \lambda_2)^2}}$$

$$F_V(v) = \int_{-\infty}^{v} \frac{\lambda_1 \lambda_2}{(v\lambda_1 + \lambda_2)^2} dv = \int_0^{v} \frac{\lambda_1 \lambda_2}{(v\lambda_1 + \lambda_2)^2} dv$$

$$= \frac{-\lambda_2}{v\lambda_1 + \lambda_2} \Bigg|_0^{v} = \frac{-\lambda_2}{v\lambda_1 + \lambda_2} - \left(\frac{-\lambda_2}{\lambda_2} \right) = \boxed{\frac{v\lambda_1}{v\lambda_1 + \lambda_2}}$$

ب)

$$P(Y_1 < Y_2) = P\left(\frac{Y_1}{Y_2} < 1\right) = \boxed{F_V(1) = \frac{\lambda_1}{\lambda_1 + \lambda_2}}$$

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(۱۴)

$$X = \sum_{i=1}^{n} X_i \longrightarrow \quad X \sim bino(n, p)$$

$$E[X] = np$$

$$Var(X) = npq$$

$$\longrightarrow Z = \frac{X - np}{\sqrt{npq}}$$

$$M_Z(t) = E[e^{tZ}] = \sum_{z} e^{tz} p(z) = \sum_{x=0}^{n} e^{\frac{t(x-np)}{\sqrt{npq}}} p(X=x)$$

$$= \sum_{x=0}^{n} e^{\frac{t(x-np)}{\sqrt{npq}}} \times \binom{n}{x} p^x (q)^{n-x}$$

$$= e^{\frac{-npt}{\sqrt{npq}}} \sum_{x=0}^{n} \binom{n}{x} \times \left(e^{\frac{t}{\sqrt{npq}}}\right)^x p^x q^{n-x}$$

$$= e^{\frac{-npt}{\sqrt{npq}}} \times \left(p e^{\frac{t}{\sqrt{npq}}} + q\right)^n$$

$$M_Z(t) = e^{\frac{-npt}{\sqrt{npq}}} \left(p\left(e^{\frac{t}{\sqrt{npq}}} - 1\right) + 1\right)^n$$

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$E[X] = E[E[X|Y]]$ می دانیم ⑤

$E[XY] = E[XE[Y|X]]$ پس :

از فرض سوال نتایج زیر را خواهیم داشت :

$E[X] = E[A]$

$E[Y] = E[B]$

$E[XY] = E[XB] = E[YA]$

$Cov(X+A, Y+B) - Cov(A,B)$

$= E[XY + XB + AY + AB] - E[X+A]E[Y+B]$

$-(E[AB] - E[A]E[B])$

$= E[XY] + E[XB] + E[AY] + E[AB]$

$-(E[X] + E[A])(E[B] + E[Y]) - E[AB]$

$+ E[A]E[B]$

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$$= E[XY] + E[XB] + E[AY] + \cancel{E[AB]}$$

$$- E[X]E[Y] - E[X]E[B] - E[A]E[Y]$$

$$\cancel{- E[A]E[B]} - \cancel{E[AB]} + \cancel{E[A]E[B]}$$

با استفاده از نتایج اول سوال :

$$Cov(X+A, Y+B) - Cov(A,B) = ۳E[XY] - ۳E[X]E[Y]$$

$$= \boxed{۳\, Cov(X,Y)}$$

(ازکمک جیپیتی کمک گرفتم)

Baran

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۹) الف)

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{0}^{1-x} \frac{1}{2}\,dy + \int_{1-x}^{1} \frac{3}{2}\,dy = \frac{1}{2}(1-x) + \frac{3}{2}(x)$$

$$= \boxed{\frac{1}{2} + x}$$

$$f_X(x) = \begin{cases} 0 & o.w \\ x+1/2 & 0 \leq x \leq 1 \end{cases}$$

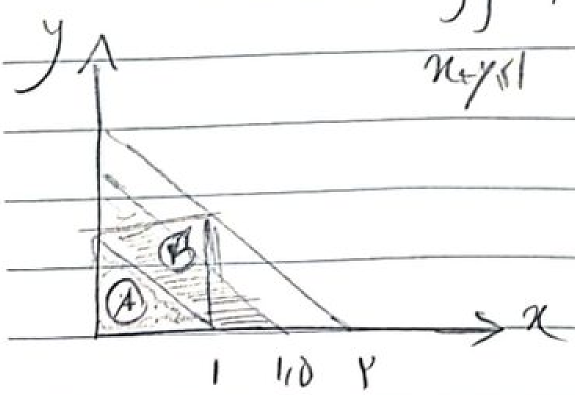
ب)

$$P(X+Y \leq 3/2) = \iint_{x+y \leq 3/2} f_{XY}(x,y)\,dx\,dy$$

$$P(X+Y \leq 3/2) = P(X+Y \leq 1) + P(X+Y \leq 3/2 \mid X+Y \geq 1)$$

$$\iint_{x+y \leq 1} 1/2\,dx\,dy + \iint_{1 \leq x+y \leq 3/2} 3/2\,dx\,dy$$

رسم شکل:



$$P(X+Y \leq 1) = 1/2 \times S_A = 1/4$$

$$P(X+Y \leq 3/2 \mid X+Y \geq 1) = 3/2\, S_B = 9/16$$

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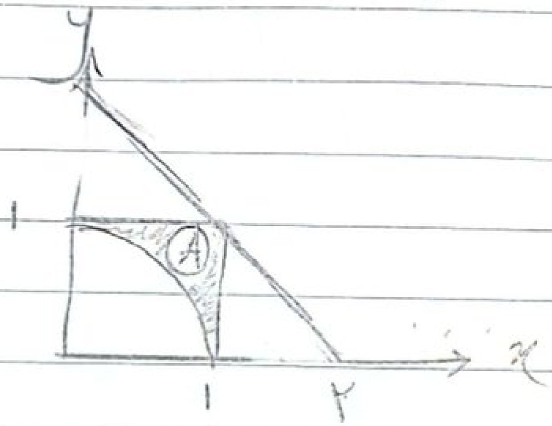
$$\longrightarrow P\left(X+Y \le \tfrac{3}{2}\right) = \frac{1}{4} + \frac{9}{16} = \boxed{\tfrac{13}{16}}$$

و برای $P(x^2+y^2 \geq 1)$ داریم :

$$P(x^2+y^2 \geq 1) = \iint \tfrac{3}{2}\, dx\,dy$$

روش دوم



$$P(x^2+y^2 \geq 1) = \tfrac{3}{2}\, S_A$$

$$= \boxed{\tfrac{3}{2}\left(1 - \tfrac{\pi}{4}\right)}$$

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۷) الف) $Var(\sum_{i=1}^{n} X_i) = Cov(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} X_j)$

$= \sum_{i=1}^{n} \sum_{j=1}^{n} Cov(X_i, X_j)$

$= \sum_{i=1}^{n} \left(Cov(X_i, X_i) + \sum_{j \neq i} Cov(X_i, X_j) \right)$

$= \sum_{i=1}^{n} Var(X_i) + ۲ \sum\sum_{i<j} Cov(X_i, X_j)$

$\longrightarrow Var(\sum_{i=1}^{n} X_i) = n\sigma^{۲} + ۲\binom{n}{۲}\eta$

ب) مانند بخش قبل:

$Y_m = Y_m + Y_{m+1} + Y_{m+۲}$

$Cov(Y_m, Y_{m+j}) = \sum_{i=m}^{m+۲} \sum_{k=m+j}^{m+j+۲} Cov(X_i, X_k)$

حالت‌بندی روی j:

$j=0 \rightarrow Cov(Y_m, Y_{m+j}) = \sum_{i=m}^{m+۲} \sum_{k=m}^{m+۲} Cov(X_i, X_k)$

$= ۳ \underset{i \neq k}{Var}(X_i, X_i) + (۹-۳) \underset{i \neq k}{Cov}(X_i, X_k)$

$= ۳\sigma^{۲} + ۶\eta$

$$j=۱ \rightarrow Cov(Y_m, Y_{m+j}) = \sum_{i=m}^{m+۲} \sum_{K=m+۱}^{m+۳} Cov(X_i, X_K)$$

$$= ۲\, \underset{i=K}{Var(X_i, X_i)} + (۳\times۳-۲)\, \underset{i\neq K}{Cov(X_i, X_K)}$$

$$= ۲\sigma^۲ + ۷\eta$$

$$j=۲ \rightarrow Cov(Y_m, Y_{m+j}) = \sum_{i=m}^{m+۲} \sum_{K=m+۲}^{K=m+۴} Cov(X_i, X_K)$$

$$= ۱\, \underset{i=K}{Var(X_i, X_i)} + (۳\times۳-۱)\, \underset{i\neq K}{Cov(\ldots)}$$

$$= \sigma^۲ + ۸\eta$$

$$j \geq ۳ \rightarrow Cov(Y_m, Y_{m+j}) = \sum_{i=m}^{m+۲} \sum_{K=m+۳}^{m+j+۲} Cov(X_i, X_K)$$

$$= ۰\, \underset{i=K}{Var(X_i, X_i)} + ۳\times۳\, \underset{i\neq K}{Cov(X_i, X_K)}$$

$$= ۰\sigma^۲ + ۹\eta$$

$$Cov(Y_m, Y_{m+j}) = \begin{cases} ۳\sigma^۲ + ۶\eta & j=۰ \\ ۲\sigma^۲ + ۷\eta & j=۱ \\ \sigma^۲ + ۸\eta & j=۲ \\ ۹\eta & j \geq ۳ \end{cases}$$

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$\eta = 0 \longleftarrow$ متغیرها مستقل اند

$$\longrightarrow \operatorname{Cov}(Y_m, Y_{m+j}) = \begin{cases} ۳\sigma^۲ & j=۰ \\ ۲\sigma^۲ & j=۱ \\ ۱\sigma^۲ & j=۲ \\ ۰ & j \geq ۳ \end{cases}$$