Bundles of metric structures as left ultrafunctors

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    1. For each ultrafilter μ on a set S a functor $\int_S \bullet d\mu\colon\ \mathcal{M}^S \to \mathcal{M}$
    2. for every family of objects $(M_s)_{s\in S}$ and every principal ultrafilter δ_{s_0} a natural isomorphism ϵ_{S,s_0} between $\int_S M_s d\delta_{s_0}$ and M_{s_0}.

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  3. Suppose we have a family of ultrafilters $(\nu_s)_{s\in S}$ on T and a family of objects of $\mathcal{M}$ $(M_t)_{t\in T}$, and an ultrafilter μ on S we have a map called the categorical Fubini transform $\Delta_{\mu,\nu_\bullet}$ from $\int_T M_t d(\int_S \nu_s d\mu)$ to $\int_S(\int_T M_t d\nu_s)d\mu$, which is required to be natural in the family $(M_t)_{t\in T}$.Here $\int_S \nu_s d\mu$ is the ultrafilter defined by $U \in \int_S \nu_s d\mu$ iff $\{s \in S|\ U \in \nu_s\} \in \mu$

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- These data are required to satisfy certain compatibility axioms.

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2-categorical aspect of ultracategories

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- This concept is too restrictive for our applications thus we look for a weakening.

Leftultrafunctors

- The important notion of morphism between ultracategories that we are going to deal with is that of left ultrafunctor.
- Let $\mathcal{M}$ and $\mathcal{N}$ be ultracategories, we call a leftultrafunctor F from $\mathcal{M}$ to $\mathcal{N}$ a functor equipped with a left ultrastructure, which consists of the following data:

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- Let $\mathcal{M}$ and $\mathcal{N}$ be ultracategories, we call a leftultrafunctor F from $\mathcal{M}$ to $\mathcal{N}$ a functor equipped with a left ultrastructure, which consists of the following data:
- For every collection of objects of $\mathcal{M}$, $(M_s)_{s \in S}$ and every ultrafilter μ on S a map $\sigma_\mu : \ F(\int_S M_s d\mu) \to \int_S F(M_s) d\mu$.
- These families of morphisms are required to satisfy compatibility axioms, with respect to the data of the ultracategory structure, on both categories.

Natural transformations of left ultrafunctors

- Suppose F and G are two left ultrafunctors a natural transformation $u : F \to G$ is called a natural transformation of left ultrafunctors if for every family of objects $(M_s)_{s \in S}$ and every ultrafilter μ on S the following diagram commutes:

$$\begin{tikzcd}
F(\int_S M_s d\mu) \arrow[r, "\sigma_\mu"] \arrow[d, "u_{\int_S M_s d\mu}"] & \int_S F(M_s) d\mu \arrow[d, "\int_S u_{M_s} d\mu"] \\
G(\int_S M_s d\mu) \arrow[r, "\sigma_\mu"] & \int_S G(M_s) d\mu
\end{tikzcd}$$

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Examples of ultracategories

The category of sets

- The most basic example of a category with ultraproduct is the category of sets.
- Let $(A_i)_{i \in I}$ be a family of non-empty sets then if we define, and let μ be an ultrafilter on I, then the ultraproduct of (A_i) with respect to μ is defined as follows

$$\int_I A_i d\mu = \prod_{i \in I} A_i / \sim$$

- Here $\sim$ is the equivalence relation that identifies two tuples if they agree on some set $J \in \mu$.

Examples of ultracategories

Category of models of first order theories

- An example of the construction above is the category of models of a first order theory.
- Let $\mathcal{L} = \langle \mathfrak{S}, \mathfrak{F}, \mathfrak{R} \rangle$ be a first order signature, and let $\mathbb{M}$ be family of sentences in that language.
- suppose that we have a family $(M_i)_{\in I}$ of models of $\mathcal{M}$, then for every sort $S \in \mathfrak{S}$, we can form the ultraproduct of sets $\int_I M_i^S d\mu$.

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- Los theorem tells us that the ultraproduct of $(M_i)_{\in I}$ constructed sortwise is also a model of $\mathbb{M}$.

Examples of ultracategories

The category of complete bounded metric spaces

- The building block of categories of metric structures, is the category complete metric spaces bounded by k, with contractions as morphisms, which we denote by k-CompMet

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$$\int_I A_i d\mu = \prod_{i \in I} A_i / \sim$$

- Here $\sim$ is the equivalence relation defined by $(a_i)_{i \in I} \sim (b_i)_{i \in I}$ iff for any $\epsilon > 0$ the set $\{i \in I \mid d_i(a_i, b_i) < \epsilon\} \in \mu$

Examples of ultracategories

Ultrasets

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- A theorem by Lurie shows an equivalence between ultrasets and compact Hausdorff spaces.
- Let I be a set, let $(x_i)_{i \in I}$ be a family of points of X indexed by I, regarded as a function from I to X $x : i \mapsto x_i$,, then $\int_I x_i d\mu$ is just the limit of the ultrafilter $x\mu$.

An important theorem regarding ultracategories

Theorem (Lurie)
Let X be a compact Hausdorff space, then there exists an equivalence of categories between the category of sheaves on X, $Sh(X)$, and the category of left ultrafunctors from X to Set

- Since there exists an equivalence of categories between sheaves of sets over X and etale spaces over X, then this shows that etale spaces over X are equivalent to left ultrafunctors from X to Set.
- Our aim was to find the correct notion of bundle which we get by replacing Set with a category of metric structures.

Bundles of complete bounded metric spaces

Definition (H.)
Let k be a positive real, A triple (E, X, π) with $\pi : E \to X$ a surjection such that for every $x \in X$, $\pi^{-1}(x)$ is a complete metric space bounded by k, is said to define a bundle of k bounded metric spaces if it satisfies the following conditions:

1. The topology of E is such that the distance function is upper semi-continuous.
2. π is continuous and open.
3. for every open set W and every $f \in W$ there exist an open nhood V of f and $\epsilon > 0$ such that $V \subseteq_\epsilon W$.

Here $V \subseteq_\epsilon W$ means that $V \subseteq V_\epsilon \subseteq W$ where $V_\epsilon = \{x \in E\ \exists y \in V\ \pi(x) = \pi(y) \text{ and } d_{\pi(x)}(x, y) < \epsilon\}$

Bundles of complete metric spaces(Ctd)

Theorem (H.)

*Let X be a compact Hausdorff space, then there exists an equivalence of categories between bundles of k bounded metric spaces over X, and left ultrafunctors from X to the category k-*CompMet*.*

- Now we want to extend the theorem above to more metric structures, the adequate framework turned out to be continuous model theory.
- Continuous model theory, is an extension of first order model theory that allows the axiomatisation of various metric structures.
- In continuous models theory, sorts are interpreted as bounded metric spaces, function symbols are interpreted as uniformly continuous functions, and relation symbols takes value in compact intervals of $\mathbb{R}$, quantifiers are $\mathrm{Inf}_{x \in S}$ and $\mathrm{Sup}_{x \in S}$, Connectives are continuous functions from $S_1 \times \ldots S_n$ to $\mathbb{R}$.
- Examples of structures axiomatisable in continuous model theory: Banach spaces, Hilbert spaces, C^* algebras, etc.

Ultracategories of continuous models

- Let $\mathcal{L} = \langle \mathfrak{S}, \mathfrak{F}, \mathfrak{R} \rangle$ be a continuous first order signature, and let $\mathbb{M}$ be family of sentences in that language.
- Suppose that we have a family $(M_i)_{in \in I}$ of models of $\mathcal{M}$, then for every sort $S \in \mathfrak{S}$, we can form the ultraproduct of sets $\int_I M_i^S d\mu$ (as bounded metric spaces).

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Bundles of models of continuous theories

Let $\mathcal{L} = \langle \mathfrak{S}, \mathfrak{F}, \mathfrak{R} \rangle$ be a continuous first order signature. we define a bundle of structure of $\mathcal{L}$ with base space X as follows:

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- For each sort $S \in \mathfrak{S}$ we define a require a bundle E^S of metric spaces bounded by k_S.
- For each relation symbol with formal domain $S_1 \times \cdots \times S_n$ we require the global relation defined from $E^{S_1} \times_X \cdots \times_X E^{S_n}$ to some compact subset of $\mathbb{R}$ to be upper semi-continuous.
- For each function symbol with formal domain $S_1 \times \cdots \times S_n$ and formal range S we require the global function defined from $E^{S_1} \times_X \cdots \times_X E^{S_n}$ to E^S to be continuous.

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- For each function symbol with formal domain $S_1 \times \cdots \times S_n$ and formal range S we require the global function defined from $E^{S_1} \times_X \cdots \times_X E^{S_n}$ to E^S to be continuous.
- let $\mathbb{M}$ be a family of sentences in the language $\mathcal{L}$. We define a bundle of models of $\mathbb{M}$ to be a bundle of structures such that every fibre is a model.

Bundles of models of continuous theories

Let $\mathcal{L} = \langle \mathfrak{S}, \mathfrak{F}, \mathfrak{R} \rangle$ be a continuous first order signature, and $\mathbb{M}$ be a family of sentences in the language

Theorem (H.)
There exists an equivalence of categories between the category of bundles of models of the continuous theory $\mathbb{M}$ with base space a compact Hausdorff space X, and the category of left ultrafunctors from X to the category of $\mathbb{M}$-models.

Examples

Banach bundles (Fell, Hofmann, ...)

A (semi-continuous) Banach bundle is defined to be a triple (E, π, X) such that the following conditions are satisfied

- For every x, $\pi^{-1}(x)$ is a Banach space.
- π is continuous and open.
- scalar multiplication from $\mathbb{K} \times E$ to E, addition from $E \times_X E$ to E are continuous.
- norm $\| \ \|$ from E to $[0, \infty)$ is upper semi-continuous (it is not hard to see that in the presence of the other axioms, this is equivalent to saying that the distance from $E \times_X E$ to $[0, \infty)$ is upper semi-continuous)
- for any $x \in X$, if we call $\mathcal{N}_x$ the set of all open neighbourhoods of x, then $(\coprod_{y \in U} B(0_y, r))_{r>0, U \in \mathcal{N}_x}$ is neighbourhood basis at 0_x

Banach bundles

Theorem (H.)
Let X be a compact Hausdorff There exists an equivalence between Semi-continuous Banach bundles over X, and left ultrafunctors from X to the category of Banach spaces with contractions as morphisms.

Banach bundles

- What if we want a continuous norm instead of a semi-continuous norm in the definition of Banach bundles instead of semi-continuous norms?

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Banach bundles

- What if we want a continuous norm instead of a semi-continuous norm in the definition of Banach bundles instead of semi-continuous norms?
- There is an equivalence of categories between continuous Banach bundles and left ultrafunctors from X to the category of Banach spaces with isometries as morphism.

Examples

Hilbert bundles (Fell, ...)

- Hilbert bundles are continuous Banach bundles for which every fibre is a Hilbert space.

Theorem (H.)
Let X be a compact Hausdorff There exists an equivalence between Hilbert bundles over X, and left ultrafunctors from X to the category of Hilbert spaces.

Other examples

- Semi-continuous and continuous C^* bundles (Fell, $\ldots$) which turn out to be equivalent to bundles of the continuous theory of C^* algebra with $*$ homomorphisms, and to the continuous theory of C^* algebra respectively with injective $*$-homomorphism, respectively.
- W^* bundles (Ozawa), which are equivalent to bundles of the continuous theory of tracial von Neumann algebras .