Ultracategories as colax algebras for the ultracompletion pseudo-monad

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- Makkai's ultracategories were used to show conceptual completeness, a result about constructing a pretopos (completion of the syntactic category of a coherent first-order theory), from the category of models + ultraproduct.
- Lurie ultracategories which are different than Makkai's, are used to show the same result as Makkai, and some variations.

Ultrastructure

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    2. for every family of objects $(M_i)_{i\in I}$ and every principal ultrafilter $\delta_{\mathbf{i_0}}$ a natural isomorphism $\epsilon_{I,\mathbf{i_0}}$ between $\int_I M_i d\delta_{\mathbf{i_0}}$ and $M_{\mathbf{i_0}}$.

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    3. Suppose we have a family of ultrafilters $(\nu_s)_{s\in S}$ on T and a family of objects of $\mathcal{M}$ $(M_t)_{t\in T}$, and an ultrafilter μ on S we have a map called the categorical Fubini transform $\Delta_{\mu,\nu_\bullet}$from $\int_T M_t d(\int_S \nu_s d\mu)$ to $\int_S(\int_T M_t d\nu_s)d\mu$, which is required to be natural in the family $(M_t)_{t\in T}$.Here $\int_S \nu_s d\mu$ is the ultrafilter defined by $U \in \int_S \nu_s d\mu$ iff $\{s\in S|\ U\in\nu_s\}\in\mu$.

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- These data are required to satisfy certain compatibility axioms.

Ultraproduct diagonal map

- An important class of mappings in the theory of ultracategories is the ultraproduct diagonal map.
- Suppose that we have map of set $I \xrightarrow{f} J$, an ultrafilter μ on a family of objects $(M_j)_{j\in J}$, then we can define a map $\Delta_{\mu,f}$ from $\int_J M_j d\mu$ to $\int_I M_i df\mu$ as follows:

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- Ultraproduct diagonal maps satisfy nice properties, for example $\Delta_{\mu,g\circ f} = \Delta_{\mu,f} \circ \Delta_{f\mu,g}$, and $\Delta_{\mu,\mathrm{Id}_S} = \mathrm{Id}_{\int_S M_s d\mu}$.

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- An interesting question would be to whether it is possible to rewrite the ultracategory axioms in terms of the ultraproduct diagonal map.

- An ultrafunctor between ultracategories is an ultraproduct respecting functor,that means this is a functor together with the data of isomorphism between $F(\int_I M_i d\mu)$ and $\int_I F(M_i) d\mu$, which behaves well with respect to the compatibility conditions imposed in the definition of an ultracategory.

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- We may weaken these isomorphisms in both directions which gives rise to definition of left and right ultrafunctors, more formally:
- Let $\mathcal{M}$ and $\mathcal{N}$ be ultracategories, we call a leftultrafunctor F from $\mathcal{M}$ to $\mathcal{N}$ a functor equipped with a left ultrastructure, which consists of the following data:

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- These families of morphisms are required to satisfy compatibility axioms, with respect to the data of the ultracategory structure, on both categories.
- Requiring the morphism to go in the other direction leads to the definition of right ultrafunctors.
- This allows the definition of various 2-categories of ultracategories

Natural transformations of left ultrafunctors

- Suppose F and G are two left ultrafunctors a natural transformation $u: F \to G$ is called a natural transformation of left ultrafunctors if for every family of objects $(M_s)_{s \in S}$ and every ultrafilter μ on S the following diagram commutes:

$$\begin{tikzcd}
F(\int_S M_s d\mu) \arrow[r, "\sigma_\mu"] \arrow[d, "u_{\int_S M_s d\mu}"] & \int_S F(M_s) d\mu \arrow[d, "\int_S u_{M_s} d\mu"] \\
G(\int_S M_s d\mu) \arrow[r, "\sigma_\mu"] & \int_S G(M_s) d\mu
\end{tikzcd}$$

- Similarly we can define natural transformation of right ultrafunctors or ultrafunctors.

Examples

- The category of points of compact Hausdorff spaces, is an ultracategory where the ultrastructure encodes the convergence of ultrafilters.
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- This justifies the notaion $\int_I \lambda_i d\mu$, for ultrafilters, this is just an ultraproduct in the category of points of βI (the Stone-Cech compactification of I).
- For any first-order theory, the category of models has a natural ultrastructure, where the ultraproduct is the usual ultraproduct.
- Many categories of metric structures have natural ultraproduct, examples: Bounded metric spaces, Banach spaces, C^* algebras, etc.

Monads and 2-monads

A monad is a triple (T, η, μ) where:

- $T : A \to A$ is an endofunctor on a category A.
- η is a natural transformation from 1_A to T.
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  Such that the following conditions are satisfied:

$$\begin{tikzcd}
T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"'] & T
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\begin{tikzcd}
T \arrow[r, "\eta T"] \arrow[d, "T\eta"'] \arrow[dr, "\mathrm{Id}" description] & T^2 \arrow[d, "\mu"] \\
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- We want to generalise the definition to 2-categories.
- A straight foreword generalisation is the notion of strict 2-monad where T is a strict 2-functor, and the diagrams above commute to the point.

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- We can weaken this further, still requiring T to be strict, but asking the diagrams above to commute up to isomorphism, this is the notion of pseudo-monad.
- Further weakening is possible (requiring T to be a pseudo or lax functor for example).

Monads and 2-monads (Ctd.)

The data of an algebra for an monad is given by:

- An object $C \in \mathcal{C}$ together with a morphism m from TC to C, such that following diagrams commute:

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Examples:

- Every algebraic theory is monadic over Set.
- Compact Hausdorff spaces are algebras for the ultrafilter monad.

Categories of normal colax algebras for a 2-monad

Suppose that we have 2-monad (T, η, μ) (strict 2-monad) over some 2-category $\mathcal{C}$ then a normal colax algebra is an object $C \in \mathcal{C}$ equipped with:

- 1-morphism m: $TC \xrightarrow{m} C$ which is called the algebra morphism (in our case it's called the algebra functor).
- 2-isomorphism i:

$$\begin{tikzcd} C \arrow[r, "\eta_C"'] \arrow[dr, "\mathrm{id}"'] & TC \arrow[d, "m"] \arrow[dl, phantom, "\Swarrow i"] \\ & C \end{tikzcd}$$

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- This data is required this data to satisfy compatibility conditions.

Colax algebra morphisms

- Given two colax algebras A and B for a 2-monad M with colax algebras m and $m^{'}$ respectively, the data of a lax (left) algebra morphism is given by a morphisms f between A and B, together with a 2-morphism $\alpha : f \circ m \longrightarrow m^{'} \circ Tf$, and certain compatibility conditions.

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- By changing the direction of this two cell we may define a colax (right) algebra morphism.
- By requiring this 2-morphism to be invertible, then we get pseudo-morphisms.
- Finally it is possible to define 2-morphisms between lax(colax,pseudo)-morphisms of algebras.
- This allows for the definition of various 2-categories of colax algebras for a pseudo-monad T.

The ultracompletion pseudo-monad

- We define the following pseudo-monad on the category of locally small categories, and we denote it by T as follows:
- Let E be a locally small category then the objects of T, are given by $(I, \mu, (A_i)_{i \in I})$, where I is set μ is an ultrafilter on this set and $(A_i)_{i \in I}$ is a collection of objects in E.

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- Similarly we may easily define what T does to functors and natural transformations.
- T is a strict 2-endofunctor on the 2-category of locally small categories

Multiplication and unit of T

- We want to give T the necessary data to define a pseud-monad.
- We define the monad multiplication μ as follows let A be a category, we define $\mu_E : T^2E \to TE$ by $(I, \nu, (J_i, \lambda_i, (M_{(i,j)})) \mapsto (\coprod_{i \in I} J_i, \int_I \iota_i \lambda_i d\mu, (M_{(i,j)})_{(i,j) \in \coprod_{i \in I} J_i})$.
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- We define the monad unit by sending η by $\eta_E : E \to TE$ by $A \mapsto (A, *, *)$.
- this data is not the data of a strict 2-moand on the category of small categories but rather that of a pseudo-monad, the problem is that $(\coprod_* I, \mu, (M_i))$ is isomorphic and not equal to $(\coprod_I *, \mu, (M_i))$, similarly we have $(\coprod_{x\in X}\coprod_{y\in Y_x} Z_{(x,y)}, \int_X \iota_x \int_{Y_x} \iota_y \xi_{(x,y)} d\lambda_x d\mu, M_{(x,y,z)})$ is only isomorphic to $(\coprod_{(x,y)\in\coprod_{x\in X} Y_x} Z_{(x,y)}, \int_{\coprod_{x\in X} Y_x} \iota_{(x,y)} d \int_X \lambda_x d\mu, M_{(x,y,z)})$.

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- We define the monad multiplication μ as follows let A be a category, we define $\mu_E : T^2E \to TE$ by $(I, \nu, (J_i, \lambda_i, (M_{(i,j)})) \mapsto (\coprod_{i\in I} J_i, \int_I \iota_i\lambda_i d\mu, (M_{(i,j)})_{(i,j)\in\coprod_{i\in I} J_i})$.
- We define the monad unit by sending η by $\eta_E : E \to TE$ by $A \mapsto (A, *, *)$.
- this data is not the data of a strict 2-moand on the category of small categories but rather that of a pseudo-monad, the problem is that $(\coprod_* I, \mu, (M_i))$ is isomorphic and not equal to $(\coprod_I *, \mu, (M_i))$, similarly we have $(\coprod_{x\in X}\coprod_{y\in Y_x} Z_{(x,y)}, \int_X \iota_x \int_{Y_x} \iota_y \xi_{(x,y)} d\lambda_x d\mu, M_{(x,y,z)})$ is only isomorphic to $(\coprod_{(x,y)\in\coprod_{x\in X} Y_x} Z_{(x,y)}, \int_{\coprod_{x\in X} Y_x} \iota_{(x,y)} d\int_X \lambda_x d\mu, M_{(x,y,z)})$.
- So the actual definition of the pseudo-monad includes in fact some invertible 3-cells.

Multiplication an unit of T (Ctd.)

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- And hence $I = \coprod_* I = \coprod_I *$, and $\coprod_{x \in X} \coprod_{y \in Y_x} Z_{(x,y)} = \coprod_{(x,y) \in \coprod_{x \in X} Y_x} Z_{(x,y)}$.
- Morphisms remains the same.
- Since T and $T^{'}$ are equivalent, their respective categories of algebras are equivalent.

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Theorem
Let f and g be two (left,right) ultrafunctors then there is a bijection between natural of left ultrafunctors between f and g, and the corresponding 2-morphisms in the 2-category of normal colax algebras for the pseudo-monad T.

From ultracategories to colax algebra

Now suppose that we are given an ultracategory A, we want to define the normal colax algebra structure on A:

- The data of the algebra functor comes from that of the ultraproduct functors, i.e we send $(I, \mu(M_i)_{i\in I})$ to $\int_I M_i d\mu$.
- Now we want to define the colax associator: Suppose that we have a family of sets (T_s) together with an ultrafilter λ_s on each T_s and an ultrafilter $s \in S$, then we want to define a map from $\int_{\coprod_{s\in S} T_s} M_{(s,t)} d \int_S \iota_s \lambda_s d\mu$ to $\int_S \int_{T_s} M_{(s,t)} d\lambda_s d\mu$.

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- We may think of using the categorical Fubini transform which would give us a map from $\int_{\coprod_{s \in S} T_s} M_{(s,t)} d \int_S \iota_s \lambda_s d\mu$ to $\int_S \int_{\coprod_{S \in S T_S}} M_{(s,t)} d\iota_s \lambda_s d\mu$.

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- Now, we can notice that for each s we have an ultraproduct diagonal map $\Delta_{\lambda_s, \iota_s}$ from $\int_{\coprod_S T_s} M_(s,t) d\iota\lambda_s$ to $\int_S M_{(s,t)} d\lambda_s$.
- So we define the associator by the following composition:

$$\int_{\coprod_{s\in S} T_s} M_{(s,t)} d \int_S \iota_s \lambda_s d\mu \xrightarrow{\Delta_{\mu,\iota_\bullet\lambda_\bullet}} \int_S \int_{\coprod_{s\in S} T_s} M_{(s,t)} d\iota_s\lambda_s d\mu \xrightarrow{\int_S \Delta_{\lambda_s,\iota_s} d\mu} \int_S \int_T M_{(s,t)} d\lambda_s d\mu$$

- The unitor definition is simple, we may define it to be $\epsilon_{*,*} : \int_* M d* \to M$.

From colax algebras to ultrastructures

- Let A be a T-normal colax algebra, with algebra functor $m : TA \to A$, with colax associator $a : m \circ T_m \longrightarrow m \circ \mu_A$, we can define an ultrastructure on this category as follows:
- The data of the ultraproduct functor(s) is supplied by the algebra functor m.
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- Now we want to define the categorical Fubini transform. Suppose that we have a family of objects of this category given by $(M_t)_{t \in T}$, a set S, a family of ultrafilters $(\gamma_s)_{s \in S}$ and an ultrafilter μ on S, we want to define a map from $\int_T M_t d \int_S \gamma_s d\mu$ to $\int_S \int_T M_t d\gamma_s d\mu$.

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- "Unfortunately" the colax associator cannot supply directly such maps, but notice that we can regard $S \times T$ as the coproduct of $|S|$ copies of T, and that we have a projection map from $S \times T$ to S allowing us to define the following morphism in TA $\pi_T : (S \times T, (M^{'}_{(s,t)})_{(s,t) \in S \times T}) \leftarrow (T, (M_t))$, here $(M^{'}_{(s,t})$ is defined by $M^{'}_{(s,t)} = M_t$, for every $s \in S$.

- Hence we may define $\Delta_{\mu,\gamma_\bullet}$ as follows:

$$\int_T M_t d(\int_S \gamma_s d\mu) \xrightarrow{m_{\pi_T}} \int_{S\times T} M'_{(t,s)} d\int_S i_s\gamma_s d\mu \xrightarrow{a_{S,\mu,(T'_s),(\gamma_s),(M'_{(s,t)})}} \int_S \int_T M_t d\gamma_s d\mu$$

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- We can verify that this supplies the data of an ultracategory.

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- The hard part is showing that the associators of A and $\mathcal{F}(\mathcal{G})(A)$ agree.
- So the difficulty in general is relating the categorical Fubini transform $\Delta_{\mu,\lambda_\bullet} : \int_I M_i d \int_j \lambda_j d\mu \to \int_J \int_I M_i d\lambda_j d\mu$, and the associator $a : \int_{\coprod_I X_i} M_{(i,x) d \int_I \iota_i \lambda_i d\mu} \to \int_I \int_{X_i} M_{(i,x)} d\lambda_i d\mu$

Equivalence of 2-categories

- As we stated before, given two ultracategories, we can find a bijection between (lax,colax)-pseudo algebra morphisms and (left-right) ultrafunctors between them.
- Moreover given two such functors, we can find a bijection between natural transformation of (left,right) ultrafunctors and the corresponding 2-morphisms of (lax,colax) pseudo-morphisms of normal colax algebras

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- Moreover given two such functors, we can find a bijection between natural transformation of (left,right) ultrafunctors and the corresponding 2-morphisms of (lax,colax) pseudo-morphisms of normal colax algebras
- All of this allows us to say that there is an equivalence of 2-categories between the category of locally small ultracataegories with (left,right) ultrafunctors, and the category of normal colax algebras for the pseudo-monad with (lax,colax) pseudo-algebra morphisms and their corresponding 2-morphisms.

Pseudo-algebras for the ultracompletion pseudo-monad

- We have showed that normal colax algebras for T are equivalent to ultracategories.
- Requiring the colax associator a
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  to be invertible would allows to get pseudo-algebras.
- More formally there is an equivalence of categories between ultracategories for which the above composition is invertible and pseud-algebras for T.
- An interesting observation is that all the examples we gave are in fact pseudo-algebras.