Bundles of bounded complete metric spaces as left-ultrafunctors and generalisation to other metric structures

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- An ultrafilter is a maximal element among filters
- Ultrafilter Lemma(Consequence of Zorn's lemma): every filter is contained(subset of) an ultrafilter
- Lemma: A filter μ on X is an ultrafilter iff for every $A \subseteq X$ either $A \in \mu$ or $A \notin \mu$.
- Alternative definition of ultrafilters: An ultrafilter on X is a morphism of Boolean algebras from $P(X)$ to $\{0, 1\}$

Ultrafilters

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- We construct this functor as follows :for any set X we equip the set βX of all ultrafilters with the topology having as basis sets of the form $(\beta S)_{S \subseteq X}$(here βS is the set of all ultrafilters on S identified with the set of all ultrafilters on X that contains S).
- Alternatively this is the unique topology on βX such that for every $\eta \in \beta\beta X$, η to converges to $\mu(\eta)$, where $\mu(\eta)$ is defined by $S \in \mu(\eta)$ iff $\beta S \in \eta$

Some topological properties stated in term of filters ultrafilters

- A space is compact iff every ultrafilter converges to some point
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Theorem (Wyler)

A topology on a set X is a relation R(called the convergence relation) between βX and X such that The following two conditions are satisfied :

1. *for each $x \in X$ $(\delta_x, x) \in R$*
2. *if μ is an ultrafilter on S and $(\gamma_s)_{s \in S}$ is a family of ultrafilters on X indexed by S such that each γ_s $(\gamma_s, f(s)) \in R$ for some $f(s)$ then : if $(f\mu, x) \in R$ then $(\int_S \gamma_s d\mu, x) \in R$*

- Here $\int_S \nu_s d\mu$ is the ultrafilter defined by $U \in \int_S \nu_s d\mu$ iff $\{s \in S |\ U \in \nu_s\} \in \mu$.

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First construction of an ultraproduct

- Let $(M_i)_{i\in I}$ be a family of sets and let μ be an ultrafilter on I then in this case we construct the ultraproduct of the family M_i and we denote it by $\int_I M_i d\mu = \varinjlim_{U\in\mu}(\prod_{i\in U} M_i)$.

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- In the case where all the M_i are non-empty we have an easier description of the ultraproduct as $\prod_{i \in I} M_i / \sim$ where $\sim$ is defined by $(a_i)_{i \in I} \sim (b_i)_{i \in I}$ iff there exists some $U \in \mu$ such that $(a_i)_{i \in U} = (b_i)_{i \in U}$.

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- We can generalise this construction to the category of models of any first order theory using Łos theorem.
- Łos theorem:Let ϕ be a first order sentence in some language $\mathcal{L}$ and let $(M_i)_{i\in I}$ be a family of $\mathcal{L}$-structures such that for each $i \in I$ we have $M_i \models \phi$ then $\int_I M_i d\mu \models \phi$.

Ultracategories

- Ultracategories(introduced by Makkai and then by Lurie) are a categorical axiomatization of the idea a category with an ultraproduct.
- An ultracategory is a category $\mathcal{M}$ with an ultrastructure.
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  2. for every family of objects $\{M_s\}_{s\in S}$ and every principal ultrafilter δ_{s_0} a natural isomorphism ϵ_{S,s_0} from $\int_S M_s d\delta_{s_0}$ to M_{s_0}.

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  3. Suppose we have a family of ultrafilters $\{\nu_s\}_{s\in S}$ on T and a family of objects of $\mathcal{M}$ $\{M_t\}$ and an ultrafilter μ on S we have a map called the categorical Fubini transform $\Delta_{\nu_\bullet,\mu}$from $\int_T M_t d(\int \nu_s \mu)$ to $\int_S(\int_T M_t d\nu_s)d\mu$, which is required to be natural in the family $\{M_t\}_{t\in T}$.

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- these data are required to satisfy certain compatibility axioms.

Important constructions

We highlight some important ways of constructing ultracategories:

- Let $\mathcal{M}$ be a category having both directed colimits and products, then it has an ultrastructure.
- We highlight this ultrastructure construction:Let T be a set and μ an ultrafilter on T,and $\{M_t\}_{t\in T}$ a family of objects of $\mathcal{M}$,Then the ultraproduct functor is defined by : $\{M_t\}_{t\in T} \mapsto \varinjlim_{U\in\mu} \prod_{t\in U} M_t$.

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- Let $\mathcal{M}$ be a full subcategory of some category of $\mathcal{M}^+$ such that $\mathcal{M}^+$ have both products and directed colimits and such that $\mathcal{M}$ is closed under the previous construction,then $\mathcal{M}$ has an ultrastructure given by $\{M_t\}_{t\in T} \mapsto \varinjlim_{U\in\mu} \prod_{t\in U} M_t$.

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Important constructions

The previous ultraproduct construction works for the category of k-bounded metric spaces with contractions(as morphisms).(here k is a non-negative real).

Theorem

let $(A_x)_{x \in X}$ be family of metric spaces all and let μ be an ultrafilter on X then the space $\prod_{x \in X} A_x / \sim$,(where $\sim$ is the equivalence relation given by $(b_x)_{x \in X} \sim (c_x)_{x \in X}$ iff $\forall \epsilon > 0 \; \exists U \in \mu$ such that $\sup_{x \in U} d(b_x, c_x) < \epsilon$) equipped with the distance $d((b_x), (c_x)) = \inf_{U \in \mu} sup_{x \in U} d_x(b_x, c_x)$ defines an ultraproduct functor on the category of k-bounded metric spaces for any $k \in [0, \infty)$.

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Lemma

If every space of $\{A_x\}_{x\in X}$ is complete then $\prod_{x\in X} A_x/\sim$ is also and this construction is the categorical ultraproduct in the category of complete k-bounded metric spaces for any $k \in [0,\infty)$.

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Note

This construction is usually denoted in the literature by ultralimit when the indexing family is $\mathbb{N}$(it is usually done for families of pointed metric spaces and with respect to a non-principal ultrafilter), and in that case the resulting space would be complete for any family of pointed metric spaces.

Another example of ultracategories

- Another important ultracategory construction is an ultracategory with no non-identity morphism,following Lurie we call these ultrasets.
- Theorem: There is a bijection between ultrastructures on a small category with no non-identity morphisms and compact Hausdorff topology on the underlying set of such category.

Functors between ultracategories

- The first idea of a functor between ultracategories is that of a ultraproduct respecting functor, well behaving with respect to the compatibility conditions imposed in the definition of an ultracategory.
- Such functor is called an ultrafunctor.

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- Unfortunately such concept is too restrictive thus we look for a weakening.

Leftultrafunctors

- The important notion of morphism between ulracategories that we are going to deal with is that of Leftultrafunctor.
- let $\mathcal{M}$ and $\mathcal{N}$ be ultracategories ,we call a leftultrafunctor F from $\mathcal{M}$ to $\mathcal{N}$ a functor equipped with a leftultrastructure.

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- let $\mathcal{M}$ and $\mathcal{N}$ be ultracategories ,we call a leftultrafunctor F from $\mathcal{M}$ to $\mathcal{N}$ a functor equipped with a leftultrastructure.
- a left-ultrastructure consists of the following data:
- for every collection of objects of $\mathcal{M}$ $\{M_s\}_{s\in S}$ and every ultrafilter μ on S a map $\sigma_\mu :\ F(\int_S M_s d\mu) \to \int_S F(M_s)d\mu$.

Leftultrafunctors(contd)

- These maps are required to satisfy certain axioms :
  1. The assignment σ_μ is natural in $\{M_s\}$
  2. for every collection of objects of $\mathcal{M}$ $\{M_s\}_{s\in S}$,and every principal ultrafilter δ_{s_0} the following diagram commutes:

$$\begin{tikzcd} F(\int_S M_s d\delta_{s_0}) \arrow[rr, "\sigma_{\delta_{s_0}}"] \arrow[dr, "F(\epsilon_{S,s_0})"'] & & \int_S F(M_S) d\delta_{s_0} \arrow[dl, "\epsilon_{S,s_0}"] \\ & F(M_{s_0}) & \end{tikzcd}$$

  3. For every collection $\{M_t\}_{t\in T}$ of objects of $\mathcal{M}$ indexed by a set T ,and for every collection $\alpha_\bullet = \{\alpha_s\}_{s\in S}$ of ultrafilters on T indexed by a set S, and every ultrafilter μ on S, the following diagram commutes :

$$\begin{tikzcd} F(\int_T M_t d(\int_S d\alpha_s) d\mu) \int_T M_t d(\int_S d(\alpha_s) d\mu) \arrow[rr, "\sigma_{\int_S \alpha_s d\mu}"] \arrow[d, "F(\Delta_{\mu,\alpha_\bullet})"] & & \int_T F(M_t) d(\int_S \alpha_s d\mu) \arrow[d, "\Delta_{\mu,\alpha_\bullet}"] \\ F(\int_S (\int_T M_t d\alpha_s) d\mu) \arrow[r, "\sigma_\mu"] & \int_S F(\int_T M_t d(\alpha_s)) d\mu \arrow[r, "\int_S \sigma_{\alpha_s} d\mu"] & \int_S \int_T F(M_t d(\alpha_s)) d\mu \end{tikzcd}$$

Left-ultrafunctors(contd)

- Suppose F and G are two left ultrafunctors a natural transformation $u: F \to G$ is called a natural transformation of left ultrafunctors if for every family of objects $\{M_s\}_{s\in S}$ and every ultrafilter μ on S the following diagram commutes:

$$\begin{tikzcd}
F(\int_S M_s d\mu) \arrow[r, "\sigma_\mu"] \arrow[d, "u_{\int_S M_s d\mu}"] & \int_S F(M_s) d\mu \arrow[d, "\int_S u_{M_s} d\mu"] \\
G(\int_S M_s d\mu) \arrow[r, "\sigma_\mu"] & \int_S G(M_s) d\mu
\end{tikzcd}$$

Equivalence of left-ultrafunctors and sheaves of sets

The following result indicates that left-ultrafunctors provides information necessary to reconstruct a sheaf of sets from it stalks.

Theorem (Lurie)

Let X be an ultraset, then there exists an equivalence between left-ultrafunctors from X to Set *and sheaves of sets on the compact Hausdorff topology of X.*

Thus we have three description of sheaves of sets over X

Sheaves of sets	Etale Bundles	Left ultrafunctors
functor from the category of open sets	Stalks+extra information(topology of the bundle)	Stalks + extra information(coming from the left-ultrastructure)

Bundle of Banach spaces

Definition (Hofmann)

A pre-bundle of Banach spaces is a triple (E, X, π) where π is a function form E to X such that for any $x \in X$ $\pi^{-1}(x)$ is a Banach space,that satisfies the following axioms:

- The set E carries a topology such that :
  1. The subspace topology on each fiber agrees with its Banach space topology.
  2. Addition $E \times_X E \to E$ is continuous
  3. Scalar multiplication $\mathbb{K} \times E \to E$ is continuous
  4. The norm $|| \; || : E \to [0, \infty)$ is upper semi-continuous
- The set X carries a topology such that :
  1. the map $\pi : E \to X$ is open and continuous
  2. the 0 selection is continuous
- For every $b \in X$ the set $\{\coprod_{x \in U} B(0, r)$ U nhood of b and $r > 0\}$ is a filter basis for nhood filter of 0_b.

Bundles of Banach spaces(contd)

Definition

We say that a pre-bundle (E, X, π) is a full bundle if for every $f \in E$ there exists a global section s such that $s(\pi(f)) = f$

Theorem (Douady,Dal Soglio-Hérault)

If X is compact Hausdorff then every pre-bundle is a full bundle.

Bundles of complete metric spaces

We wanted to find an adequate extension of the notion of Bundle of Banach spaces to complete metric spaces all bounded by a certain constant k so we have the following definition:

Definition (H.)

Let k be a positive real, A triple (E, X, π) with $\pi\ : E \to X$ a surjection such that for every $x \in X$ $\pi^{-1}(x)$ is a complete metric space bounded by k, is said to define a bundle of k bounded metric spaces if it satisfies the following conditions:

1. The topology of E is such that the distance function is upper semi-continuous and the subspace topology agrees with the metric space topology for each fiber.
2. π is continuous and open.
3. for every open set W and every $f \in W$ there exist an open nhood V of f and $\epsilon > 0$ such that $V \subseteq_\epsilon W$.

Here $V \subseteq_\epsilon W$ means that $V \subseteq V_\epsilon \subseteq W$ where $V_\epsilon = \{x \in E\ \exists y \in V\ \pi(x) = \pi(y)$ and $d_{\pi(x)}(x, y) < \epsilon\}$ is an open set.

Bundles of complete metric spaces(contd)

Definition

If (E, X, π) and $(E^{'}, X, \pi^{'})$ are two bundles with base space X we define a map of bundles σ to be a continuous map from E to $E^{'}$

such that the following diagram commutes:

$$\begin{tikzcd} E \arrow[rr, "\sigma"] \arrow[dr, "\pi"'] & & E^{'} \arrow[dl, "\pi^{'}"] \\ & X & \end{tikzcd}$$

and such that for each $x \in X$ the map $\psi|_{\pi^{-1}(x)}$ is a contraction .

Main theorem

Let k-CompMet denote the category of complete metric spaces bounded by a certain k

Theorem (H.)

*Let X be an ultraset, there exists an equivalence of categories between the category of left-ultrafunctors from X to k-*CompMet*,and the category of bundles of complete metric spaces bounded by k with X as base space.*

- In the next few slides we are going to give the construction of two functors $\mathcal{L}$ and $\mathcal{R}$ inverses(up to iso) that are going to realise the equivalence of these categories.
- from now on the compact Hausdorff space X is going to be fixed.

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The functor $\mathcal{L}$

We give a construction of the two functors $\mathcal{L}$ and $\mathcal{R}$ that should realise the equivalence of categories

- The functor $\mathcal{L}$ gives a bundle of complete metric spaces for each left ultrafunctor F from X to k-CompMet.
- As a set the bundle $\mathcal{L}(F)$ is just $\coprod_{x \in X} F(x)$

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Theorem

There exists a well defined topology on $\coprod_{x \in X} F(x)$ that makes into a bundle of complete metric spaces .

- This topology is defined in terms of convergence of ultrafilters : an ultrafilter μ on $\coprod_{x \in X} F(x)$ converges to f iff following conditions are satisfied:
  1. $\pi\mu$ converges to $\pi(f)$
  2. if $\sigma_{\pi\mu}(f) = (b_x)_{x \in X}$ then the following holds: $\forall \epsilon > 0$ $\coprod_{x \in X} B(b_x, \epsilon) \in \mu$
- So we define the functor $\mathcal{L}(F)$ by sending F to the space $\coprod_{x \in X} F(x)$ equipped with this topology.

The functor $\mathcal{R}$

Let (E, X, π) be a bundle of complete metric spaces,we want to construct leftultrafunctor from X(as an ultraset) to k-CompMet.

- Clearly $\mathcal{R}(E)$ as a functor needs to send x to $\pi^{-1}(x)$,but how do we get a left ultrastructure?

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- For every ultrafilter μ converging to some $y \in X$ we need to construct a map σ_μ from $\mathcal{R}(E)(y) = \pi^{-1}(y)$ to $\int_X \pi^{-1}(x) d\mu$

The functor $\mathcal{R}$ (contd)

- In the case every f has a local section s that hits f, we can easily define σ_μ by taking the equivalence class of $(s(x))_{x \in U}$.
- if such local sections are not guaranteed to exist, the construction of σ_μ is more technical, and it involves the construction of certain convergent filter in the ultraproduct of the fibers.

The functor $\mathcal{R}$ (contd)

Lemma

For any bundle E and any left ultrafunctor F $\mathrm{Hom}(\mathcal{L}(F), E) \simeq \mathrm{Hom}(F, \mathcal{R}(E))$ and this bijection is natural in F.

Lemma

This isomorphism induces a functor structure on $\mathcal{R}$ such that $\mathcal{R}$ and $\mathcal{L}$ are adjoints.

Back to main result

Theorem

Both the unit and counit of adjunction are isomorphisms.

Introduction to continuous model theory

We are going to use *Model theory of* C^* *algebras* Farah et al. (2021) as primary source for continuous model theory :
A language $\mathfrak{L}$ in continuous model theory consists of the following triplet $< \mathfrak{S}, \mathfrak{F}, \mathfrak{R} >$:

1. the sorts $\mathfrak{S}$ consists of a set of sorts symbols such that each symbol comes equipped with a symbol d_S (should be interpreted as the distance function) and a constant k_S (actual constant not just a symbol) (which should be interpreted as an upper bound for the distance function).

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1. the sorts $\mathfrak{S}$ consists of a set of sorts symbols such that each symbol comes equipped with a symbol d_S (should be interpreted as the distance function) and a constant k_S (actual constant not just a symbol) (which should be interpreted as an upper bound for the distance function).
2. the function symbols sets $\mathfrak{F}$ which consists on family of symbols,and for each symbol f we specify a formal domain $\mathrm{dom}(f) = (S_1, ..., S_n)$ and a formal range $\mathrm{rng}(f) = S^{'}$ and a function δ_f which should be interpreted as the uniform continuity modulus of f.

Introduction to continuous model theory

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3. the relation symbols set $\mathfrak{R}$ which consist of relation symbols,each equipped with a compact subset of $\mathbb{R}$(which should be interpreted as the range of these relations) as well as a uniform continuity modulus δ_ϕ for every $\phi \in \mathfrak{R}$.

Note

In our work We are going to treat the distance symbol as a relation symbol(with additional requirements).

Introduction to continuous model theory(Ctd)

- connectives are continuous functions from S^n to R where n is non-negative integer.
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- A sentence is a formula with no free variables.
- given M and N $\mathcal{L}$-structures we define a morphism f of $\mathcal{L}$ structures to be a set of functions $(f^S)_{S\in\mathfrak{S}}$ such that for any atomic formula ϕ with free variables $(x_{S_1}, ..., x_{S_n})$ and any $(a_1, ...a_n) \in M(S_1) \times \ldots M(S_n)$ we have $\phi^N(f^{S_1}(a_1), \ldots, f^{S_n}(a_n)) \leq \phi^M(a_1, \ldots, a_n)$.
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Introduction to continuous model theory(Ctd)

- To axiomatise structures where the distance are not bounded like Banach space, we define a sort for each natural number n which is intended to be interpreted as the unit ball with radius n.

Bundle of structures

Suppose that $\mathcal{L}$ is a language, we define a bundle of structure of $\mathcal{L}$ with base space X as follows :

- For each sort $S \in \mathfrak{S}$ we define a require the existence of a bundle E^S of metric spaces bounded by k_S.

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- For each sort $S \in \mathfrak{S}$ we define a require the existence of a bundle E^S of metric spaces bounded by k_S.
- For each relation symbol with formal domain $S_1 \times \cdots \times S_n$ we require the global relation defined from $E^{S_1} \times_X \cdots \times_X E^{S_n}$ to some compact subset of $\mathbb{R}$ to be upper semi continuous.
- For each function symbol with formal domain $S_1 \times \cdots \times S_n$ and formal range S we require the global function defined from $E^{S_1} \times_X \cdots \times_X E^{S_n}$ to E^S to be continuous.

Theorem (H.)

There exists an equivalence of categories between the category of bundles of structures of some language $\mathcal{L}$ with bases a space a compact Hausdorff space X, and the category of left ultrafunctors from X to the category of $\mathcal{L}$ structures.

Bundle of models

Definition

Let $\mathcal{L}$ be a language, and let T be a set of sentences in that language, we simply define a category of bundles of T models over some X to be the full subcategory of $\mathcal{L}$-structures for which every fiber is a T model.

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- We extend the k-CompMet equivalence of the category from $\mathcal{L}$-structures to T-models by noticing that the category of left-ultrafunctors from X to the category of models is full subcategory of the category of left ultrafunctors to the category of structures.
- We can show by this equivalence that bundles of Banach spaces over a compact Hausdorff space X corresponds to left-ultrafunctors from X to the ultracategory of Banach spaces with contractions.
- Also we can extend this equivalence to C^* algebras and Hilbert spaces.

Further results

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- Informally speaking, if we axiomatise a theory without much caring about morphisms, then we usually get structure preserving contractions as morphisms.
- If we want to get isometries instead of contraction we should add a new relation symbol χ_S with formal domain $S \times S$ for each sort S and an additional set of axioms of form : $\sup_{(x,y)} |(\chi_S(x,y) - k_S + d_S(x,y))|$.
- Bundles of such models are going to be bundles of the original theory with distance function being continuous instead of only being upper semi-continuous.