Ultracategories as colax algebras for the ultracompletion pseudo-monad

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- Makkai's ultracategories were used to show conceptual completeness, a result about constructing a pretopos (completion of the syntactic category of a coherent first-order theory), from the category of models + ultraproduct.
- Lurie ultracategories which are different than Makkai's, are used to show the same result as Makkai, and some variations.

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 - 3. Suppose we have a family of ultrafilters $(v_s)_{s\in S}$ on T and a family of objects of \mathcal{M} $(M_t)_{t\in T}$, and an ultrafilter μ on S we have a map called the categorical Fubini transform $\Delta_{\mu,\nu_{ullet}}$ from $\int_{T} M_t d(\int_{S} v_s d\mu)$ to $\int_{S} (\int_{T} M_t dv_s) d\mu$, which is required to be natural in the family $(M_t)_{t\in T}$. Here $\int_{S} v_s d\mu$ is the ultrafilter defined by $U\in \int_{S} v_s d\mu$ iff $\{s\in S|\ U\in v_s\}\in \mu$.

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- These data are required to satisfy certain compatibility axioms.

- An important class of mappings in the theory of ultracategories is the ultraproduct diagonal map.
- Suppose that we have map of set $I \xrightarrow{f} J$, an ultrafilter μ on a family of objects $(M_j)_{j \in J}$, then we can define a map $\Delta_{\mu,f}$ from $\int_J M_j d\mu$ to $\int_I M_i df \mu$ as follows:

$$\int_{J} M_{j} df \mu = \int_{J} M_{j} d \int_{I} \delta_{f(i)} d\mu \xrightarrow{\Delta_{\mu, \delta_{f \bullet}}} \int_{I} \int_{J} M_{j} d\delta_{f(i)} d\mu \xrightarrow{\int_{I} \varepsilon_{J, f(i)} d\mu} \int_{I} M_{f(i)} d\mu$$

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- Ultraproduct diagonal maps satisfy nice properties, for example $\Delta_{\mu,g\circ f} = \Delta_{\mu,f} \circ \Delta_{f\mu,g} \text{, and } \Delta_{\mu,\mathrm{Id}_S} = \mathrm{Id}_{\int_S M_S d\mu}.$

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- An interesting question would be to whether it is possible to rewrite the ultracategory axioms in terms of the ultraproduct diagonal map.

• An ultrafunctor between ultracategories is an ultraproduct respecting functor, that means this is a functor together with the data of isomorphism between $F(\int_I M_i d\mu)$ and $\int_I F(M_i) d\mu$, which behaves well with respect to the compatibility conditions imposed in the definition of an ultracategory.

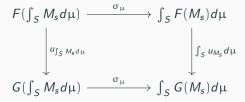
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- We may weaken these isomorphisms in both directions which gives rise to definition of left and right ultrafunctors, more formally:
- Let $\mathcal M$ and $\mathcal N$ be ultracategories, we call a leftultrafunctor F from $\mathcal M$ to $\mathcal N$ a functor equipped with a left ultrastructure, which consists of the following data:

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- These families of morphisms are required to satisfy compatibility axioms, with respect to the data of the ultracategory structure, on both categories.

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- Requiring the morphism to go in the other direction leads to the definition of right ultrafunctors.
- This allows the definition of various 2-categories of ultracategories

Natural transformations of left ultrafunctors

• Suppose F and G are two left ultrafunctors a natural transformation $u: F \to G$ is called a natural transformation of left ultrafunctors if for every family of objects $(M_s)_{s \in S}$ and every ultrafilter μ on S the following diagram commutes:



 Similarly we can define natural transformation of right ultrafunctors or ultrafunctors.

Examples

- The category of points of compact Hausdorff spaces, is an ultracategory where the ultrastructure encodes the convergence of ultrafilters.
- This observance leads to a much better result: There is an equivalence of categories between ultrasets (small ultracategories with no non-identity morphism) and compact Hausdorff spaces.

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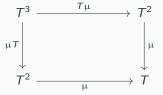
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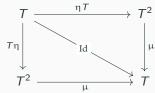
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- For any first-order theory, the category of models has a natural ultrastructure, where the ultraproduct is the usual ultraproduct.
- Many categories of metric structures have natural ultraproduct,
 examples: Bounded metric spaces, Banach spaces, C* algebras, etc.

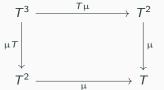
- $T: A \rightarrow A$ is an endofunctor on a category A.
- η is a natural transformation from 1_A to \tilde{T} .
- μ is a natural transformation from T^2 to T.

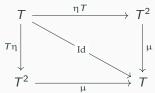
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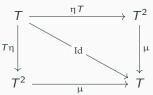




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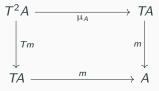


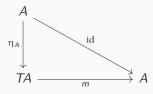
- We want to generalise the definition to 2-categories.
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- We can weaken this further, still requiring T to be strict, but asking the diagrams above to commute up to isomorphism, this is the notion of pseudo-monad.
- Further weakening is possible (requiring T to be a pseudo or lax functor for example).

Monads and 2-monads (Ctd.)

The data of an algebra for an monad is given by:

• An object $C \in C$ together with a morphism m from TC to C, such that following diagrams commute:

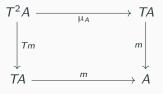


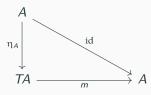


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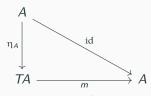
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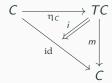
Examples:

- Every algebraic theory is monadic over Set.
- Compact Hausdorff spaces are algebras for the ultrafilter monad.

Categories of normal colax algebras for a 2-monad

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- 2-isomorphism *i*:

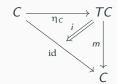


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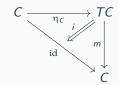
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This data is required this data to satisfy compatibility conditions.

• Given two colax algebras A and B for a 2-monad M with colax algebras m and m' respectively, the data of a lax (left) algebra morphism is given by a morphisms f between A and B, together with a 2-morphism $\alpha: f \circ m \longrightarrow m' \circ Tf$, and certain compatibility conditions.

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- Finally it is possible to define 2-morphisms between lax(colax,pseudo)-morphisms of algebras.
- This allows for the definition of various 2-categories of colax algebras for a pseudo-monad T.

The ultracompletion pseudo-monad

- We define the following pseudo-monad on the category of locally small categories, and we denote it by T as follows:
- Let E be a locally small category then the objects of T, are given by $(I, \mu, (A_i)_{i \in I})$, where I is set μ is an ultrafilter on this set and $(A_i)_{i \in I}$ is a collection of objects in E.

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- We define a morphism in the category TA, between $(I, \mu, (A_i)_{i \in I})$ and $(J, \nu, (B_i)_{i \in I})$ to be a pair $(f, (g_j))$, where f is a map from $\eta \ni J^{'} \subseteq J$ to I and (g_j) is a family of morphism $g_i A_{f(j)} \to B_j$, subject to the equivalence relation that identifies $(f, (g_j))$ and $(f^{'}, (g_j^{'}))$ if there exists some set $J^{'} \in \nu$, such that $f|_{J^{'}} = g|_{J^{'}}$, and such that for every $j \in J^{'}$ $g_j = f_j$.

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- Similarly we may easily define what T does to functors and natural transformations.
- T is a strict 2-endofunctor on the 2-category of locally small categories

Multiplication and unit of T

- We want to give T the necessary data to define a pseud-monad.
- We define the monad multiplication μ as follows let A be a category, we define $\mu_E: T^2E \to TE$ by $(I, \nu, (J_i, \lambda_i, (M_{(i,j)})) \mapsto (\coprod_{i \in I} J_i, \int_I \iota_i \lambda_i d\mu, (M_{(i,j)})_{(i,j) \in \coprod_{i \in I} J_i}).$
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- this data is not the data of a strict 2-moand on the category of small categories but rather that of a pseudo-monad, the problem is that $(\coprod_{i}I,\mu,(M_{i}))$ is isomorphic and not equal to $(\coprod_{i}*,\mu,(M_{i}))$, similarly we have $(\coprod_{x\in X}\coprod_{y\in Y_{x}}Z_{(x,y)},\int_{X}\iota_{x}\int_{Y_{x}}\iota_{y}\xi_{(x,y)}d\lambda_{x}d\mu,M_{(x,y,z)}) \text{ is only isomorphic to}\\ (\coprod_{(x,y)\in\coprod_{x\in X}Y_{x}}Z_{(x,y)},\int_{\coprod_{x\in X}Y_{x}}\iota_{(x,y)}d\int_{X}\lambda_{x}d\mu,M_{(x,y,z)}).$

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- So the actual definition of the pseudo-monad includes in fact some invertible 3-cells.

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- In that case we identify
 \(\limit_{s \in S} \) T_s with the unique ordinal order isomorphic to
 \(\limit_{s \in S} \) T_s as a set equipped with lexicographic order.

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- The only difference is that in T' we restrict our attention to ordinals so T'A has objects $(I, \mu, (A_i)_{i \in I})$ where I is an ordinals.
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 \(\limit_{s \in S} \) T_s with the unique ordinal order isomorphic to
 \(\limit_{s \in S} \) T_s as a set equipped with lexicographic order.
- And hence $I = \coprod_* I = \coprod_I *$, and $\coprod_{x \in X} \coprod_{y \in Y_x} Z_{(x,y)} = \coprod_{(x,y) \in \coprod_{x \in X} Y_x} Z_{(x,y)}.$
- Morphisms remains the same.
- Since T and T' are equivalent, their respective categories of algebras are equivalent.

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Theorem

Let f and g be two (left,right) ultrafunctors then there is a bijection between natural of left ultrafunctors between f and g, and the corresponding 2-morphisms in the 2-category of normal colax algebras for the pseudo-monad T.

From ultracategories to colax algebra

Now suppose that we are given an ultracategory A, we want to define the normal colax algebra structure on A:

- The data of the algebra functor comes from that of the ultraproduct functors, i.e we send $(I, \mu(M_i)_{i \in I})$ to $\int_I M_i d\mu$.
- Now we want to define the colax associator: Suppose that we have a family of sets (T_s) together with an ultrafilter λ_s on each T_s and an ultrafilter $s \in S$, then we want to define a map from $\int_{\coprod_{s \in S} T_s} M_{(s,t)} d \int_S \iota_s \lambda_s d\mu \text{ to } \int_S \int_{T_s} M_{(s,t)} d\lambda_s d\mu.$

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- We may think of using the categorical Fubini transform which would give us a map from $\int_{\coprod_{s\in S} T_s} M_{(s,t)} d\int_S \iota_s \lambda_s d\mu$ to $\int_S \int_{\coprod_{s\in S} T_s} M_{(s,t)} d\iota_s \lambda_s d\mu$.

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- Now, we can notice that for each s we have an ultraproduct diagonal map $\Delta_{\lambda_s,\iota_s}$ from $\int_{\coprod_S T_s} M(s,t) d\iota \lambda_s$ to $\int_S M_{(s,t)} d\lambda_s$.
- So we define the associator by the following composition: $\int_{\coprod_{s \in S} T_s} M_{(s,t)} d \int_{\mathcal{S}} \iota_s \lambda_s d\mu \xrightarrow{\Delta_{\mu,\iota_\bullet} \lambda_\bullet} \int_{\mathcal{S}} \int_{\coprod_{s \in S} T_s} M_{(s,t)} d \iota_s \lambda_s d\mu \xrightarrow{\int_{\mathcal{S}} \Delta_{\lambda_s,\iota_s} d\mu} \int_{\mathcal{S}} \int_{T} M_{(s,t)} d \lambda_s d\mu$
- The unitor definition is simple, we may define it to be $\epsilon_{*,*}: \int_* Md* \to M$.

From colax algebras to ultrastructures

- Let A be a T-normal colax algebra, with algebra functor
 m: TA → A, with colax associator a: m ∘ T_m → m ∘ μ_A, we can
 define an ultrastructure on this category as follows:
- The data of the ultraproduct functor(s) is supplied by the algebra functor m.
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- Now we want to define the categorical Fubini transform. Suppose that we have a family of objects of this category given by $(M_t)_{t\in T}$, a set S, a family of ultrafilters $(\gamma_s)_{s\in S}$ and an ultrafilter μ on S, we want to define a map from $\int_T M_t d\int_S \gamma_s d\mu$ to $\int_S \int_T M_t d\gamma_s d\mu$.

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- "Unfortunately" the colax associator cannot supply directly such maps, but notice that we can regard $S \times T$ as the coproduct of |S| copies of T, and that we have a projection map from $S \times T$ to S allowing us to define the following morphism in TA $\pi_T: (S \times T, (M_{(s,t)}^{'})_{(s,t) \in S \times T}) \leftarrow (T, (M_t))$, here $(M_{(s,t)}^{'})$ is defined by $M_{(s,t)}^{'} = M_t$, for every $s \in S$.

From colax algebras to ultrastructure (Ctd.)

• Hence we may define $\Delta_{\mu,\gamma_{\bullet}}$ as follows:

$$\int_{T} M_{t} d(\int_{S} \gamma_{s} d\mu) \xrightarrow{m_{\pi_{T}}} \int_{S \times T} M_{(t,s)}' d\int_{S} i_{s} \gamma_{s} d\mu \xrightarrow{a_{S,\mu,(T_{s}'),(\gamma_{s}),(M_{(s,t)}')}'} \int_{S} \int_{T} M_{t} d\gamma_{s} d\mu$$

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• In the definition of unitor we want an isomorphism from $\int_I M_i d\delta_{i_0}$ to M_{i_0} , the unitor supplies us with an isomorphism from $\int_* M_{i_0} d*$ to M_{i_0} , but notice that in the category TA we have an isomorphism between $(I, \delta_{i_0}, (M_i))$ and $(*, *, M_{i_0})$ hence we define the natural map ϵ_{I, i_0} :

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We can verify that this supplies the data of an ultracategory.

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- Conversely, suppose that we are given a normal colax algebra A for the pseudo-monad T.
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- The hard part is showing that the associators of A and $\mathcal{F}(\mathcal{G})(A)$ agree.
- So the difficulty in general is relating the categorical Fubini transform $\Delta_{\mu,\lambda_{\bullet}}:\int_{I}M_{i}d\int_{j}\lambda_{j}d\mu \rightarrow \int_{J}\int_{I}M_{i}d\lambda_{j}d\mu$, and the associator $a:\int_{\prod_{i}X_{i}}M_{(i,x)}d\int_{I}\iota_{i}\lambda_{i}d\mu \rightarrow \int_{I}\int_{X_{i}}M_{(i,x)}d\lambda_{i}d\mu$

Equivalence of 2-categories

- As we stated before, given two ultracategories, we can find a bijection between (lax,colax)-pseudo algebra morphisms and (left-right) ultrafunctors between them.
- Moreover given two such functors, we can find a bijection between natural transformation of (left,right) ultrafunctors and the corresponding 2-morphisms of (lax,colax) pseudo-morphisms of normal colax algebras

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- Moreover given two such functors, we can find a bijection between natural transformation of (left,right) ultrafunctors and the corresponding 2-morphisms of (lax,colax) pseudo-morphisms of normal colax algebras
- All of this allows us to say that there is an equivalence of 2-categories between the category of locally small ultracataegories with (left,right) ultrafunctors, and the category of normal colax algebras for the pseudo-monad with (lax,colax) pseudo-algebra morphisms and their corresponding 2-morphisms.

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- More formally there is an equivalence of categories between ultracategories for which the above composition is invertible and pseud-algebras for T.
- An interesting observation is that all the examples we gave are in fact pseudo-algebras.