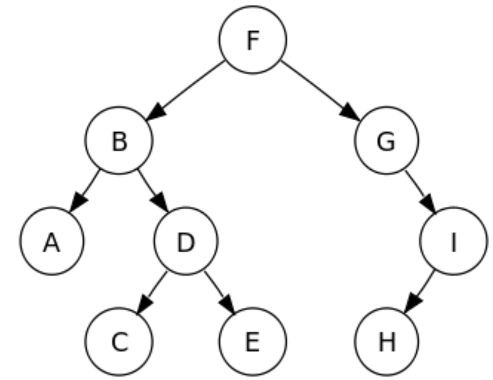
# Lecture 7

Binary Search Trees and Red-Black Trees

# Announcements

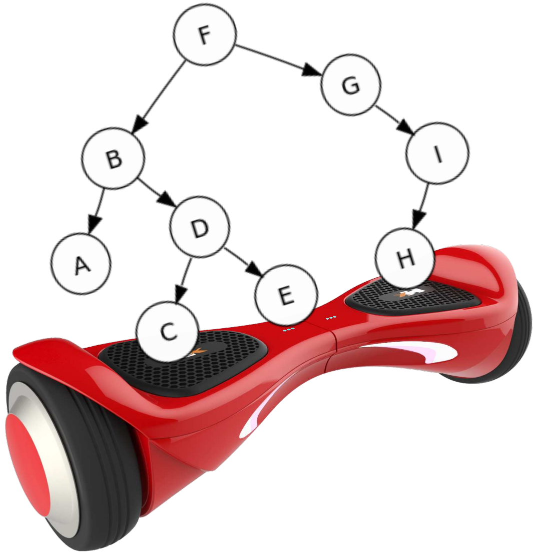
- HW 3 released!  (Due Friday)

# Today: binary search trees




- Brief foray into data structures!
  - See CS 166 for more!
- What are binary search trees?
  - You may remember these from CS 106B
  - Why are they good?
  - Why are they bad?

this will lead us to…




- Self-Balancing Binary Search Trees
  - **Red-Black** trees.

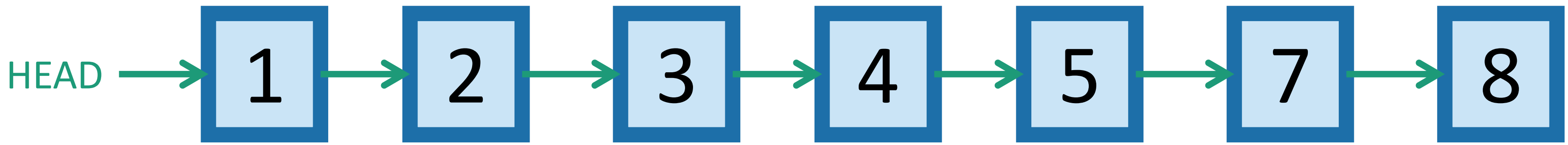
# Why are we studying self-balancing BSTs?

1. The punchline is **important**:
   - A data structure with O(log(n)) INSERT/DELETE/SEARCH

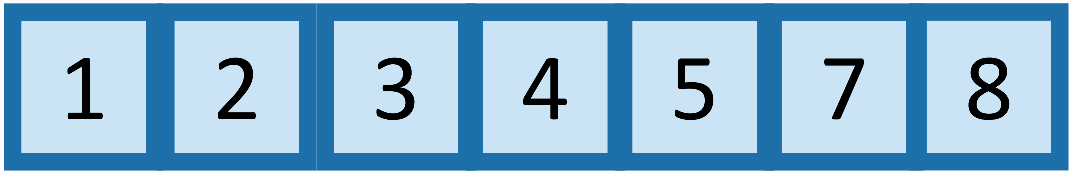
2. The idea behind **Red-Black Trees** is clever
   - It's good to be exposed to clever ideas.
   - Also it's just aesthetically pleasing.

# Motivation for binary search trees

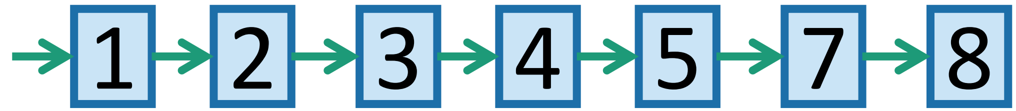
- We've been assuming that we have access to some basic data structures:
  - (Sorted) linked lists



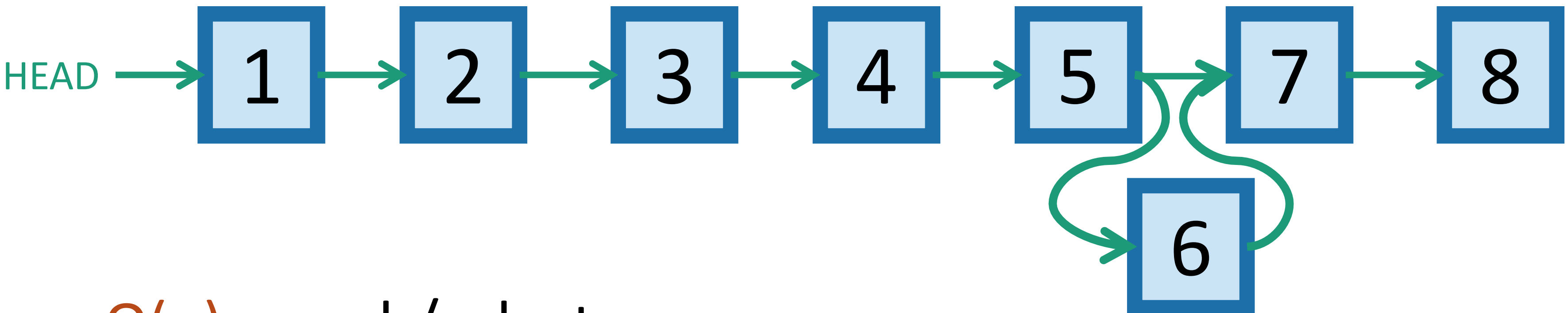

  - (Sorted) arrays



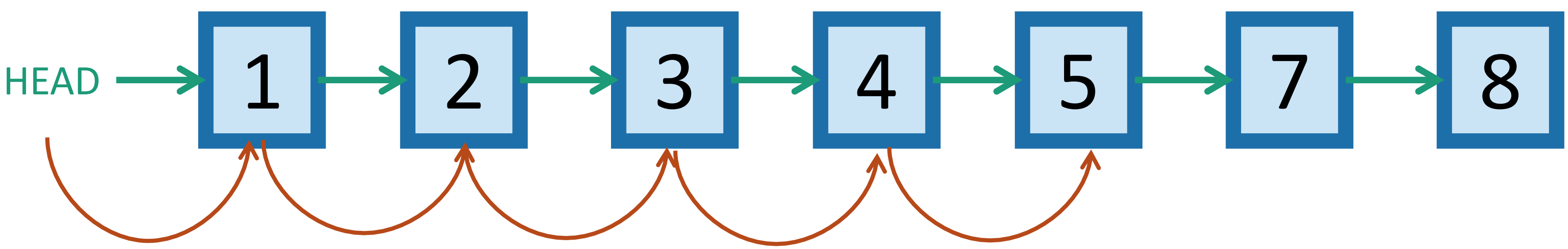
# Sorted linked lists



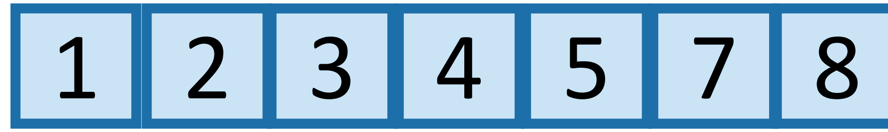

- O(1) insert/delete (assuming we have a pointer to the location of the insert/delete):



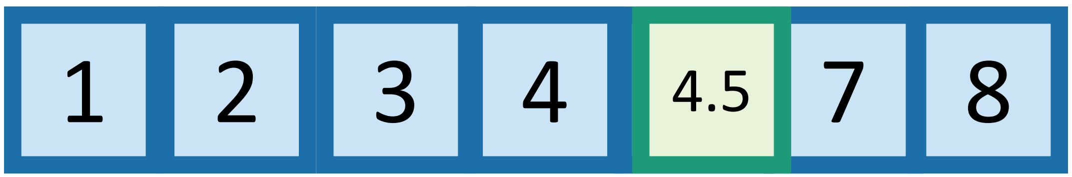

- O(n) search/select:



# Sorted Arrays




- O(n) insert/delete:




- O(log(n)) search, O(1) select:


Search: Binary search to see if 3 is in A.

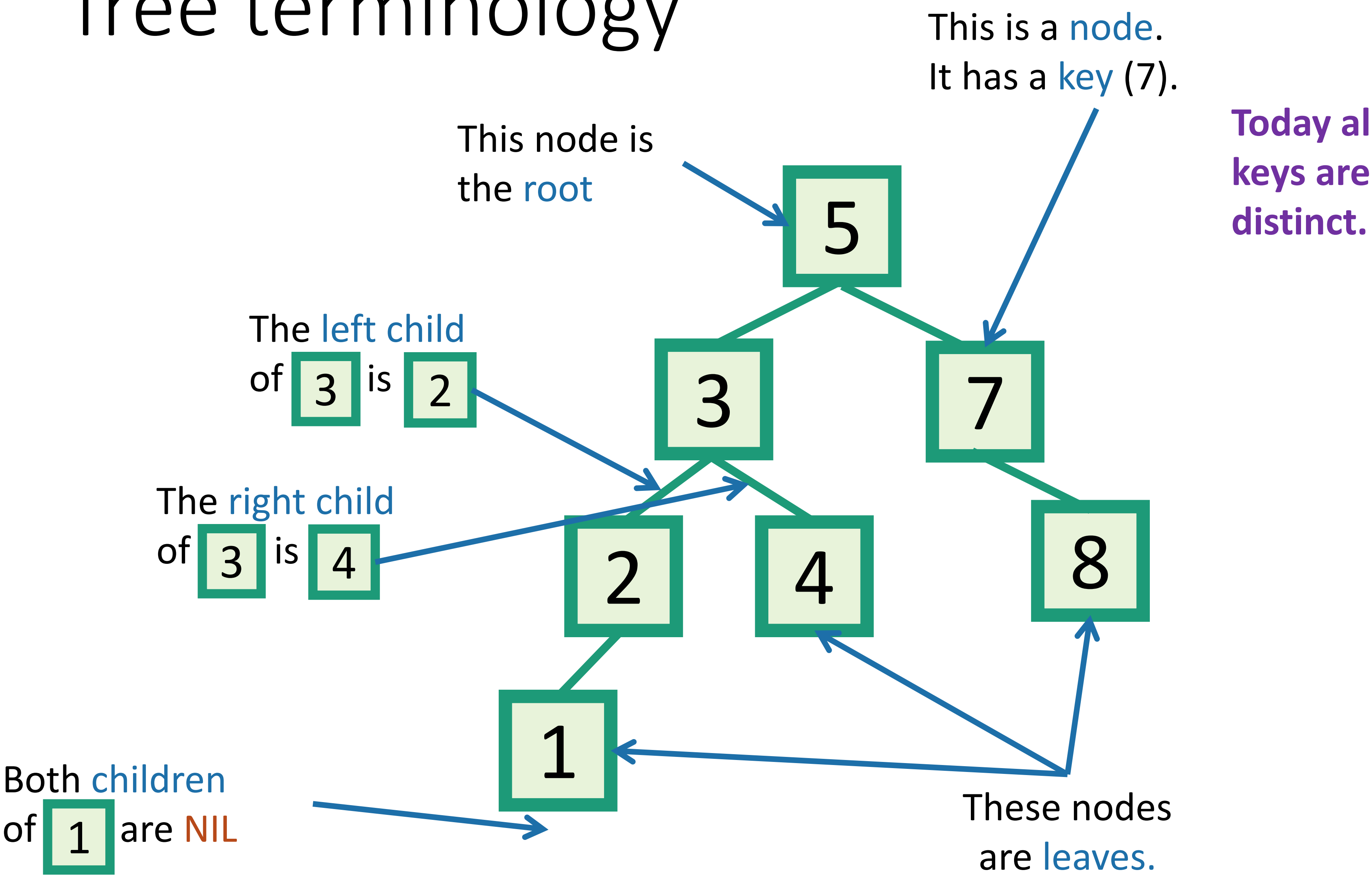
Select: Third smallest is A[3].

# The best of both worlds

TODAY!

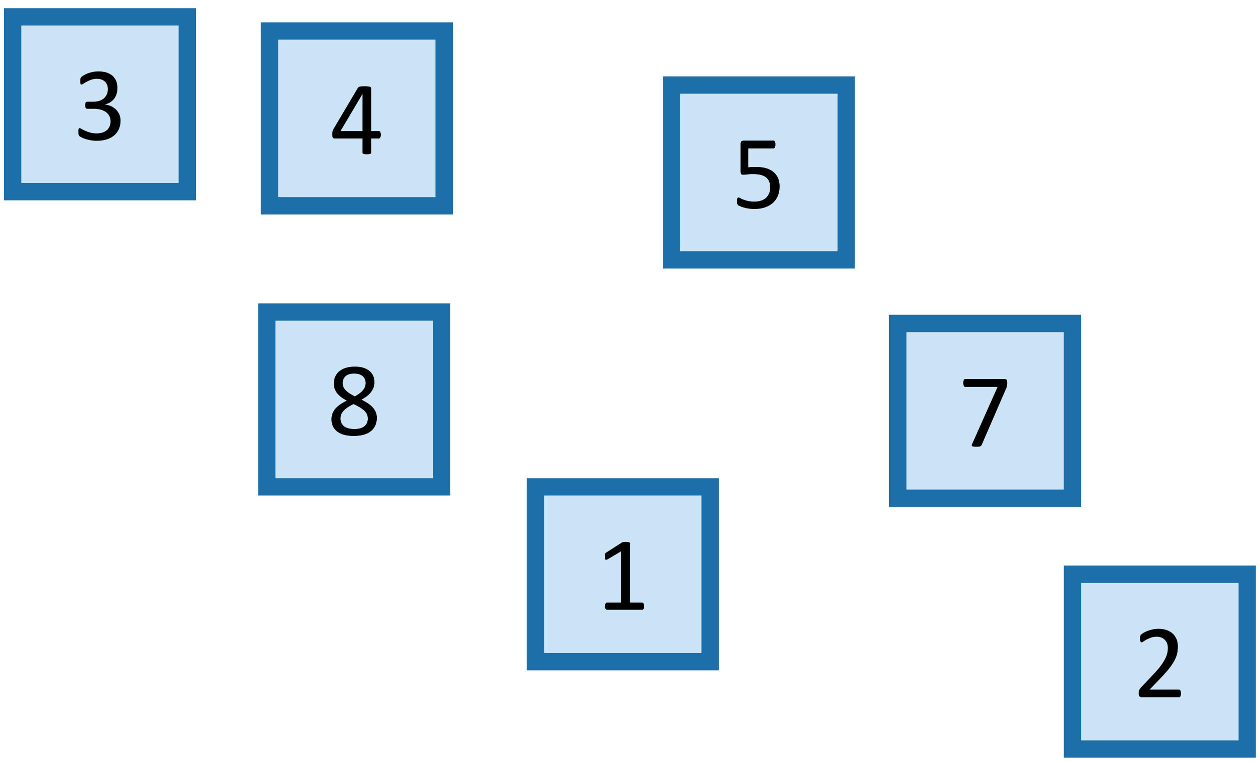
| | Sorted Arrays | Linked Lists | Binary Search Trees* |
|---|---|---|---|
| Search | O(log(n)) 😃 | O(n) ☹ | O(log(n)) 😃 |
| Insert/Delete | O(n) ☹ | O(1) 😃 | O(log(n)) 😃 |

# Tree terminology



# Binary Search Trees

- It's a binary tree so that:
  - Every LEFT descendent of a node has key less than that node.
  - Every RIGHT descendent of a node has key larger than that node.
- Example of building a binary search tree:



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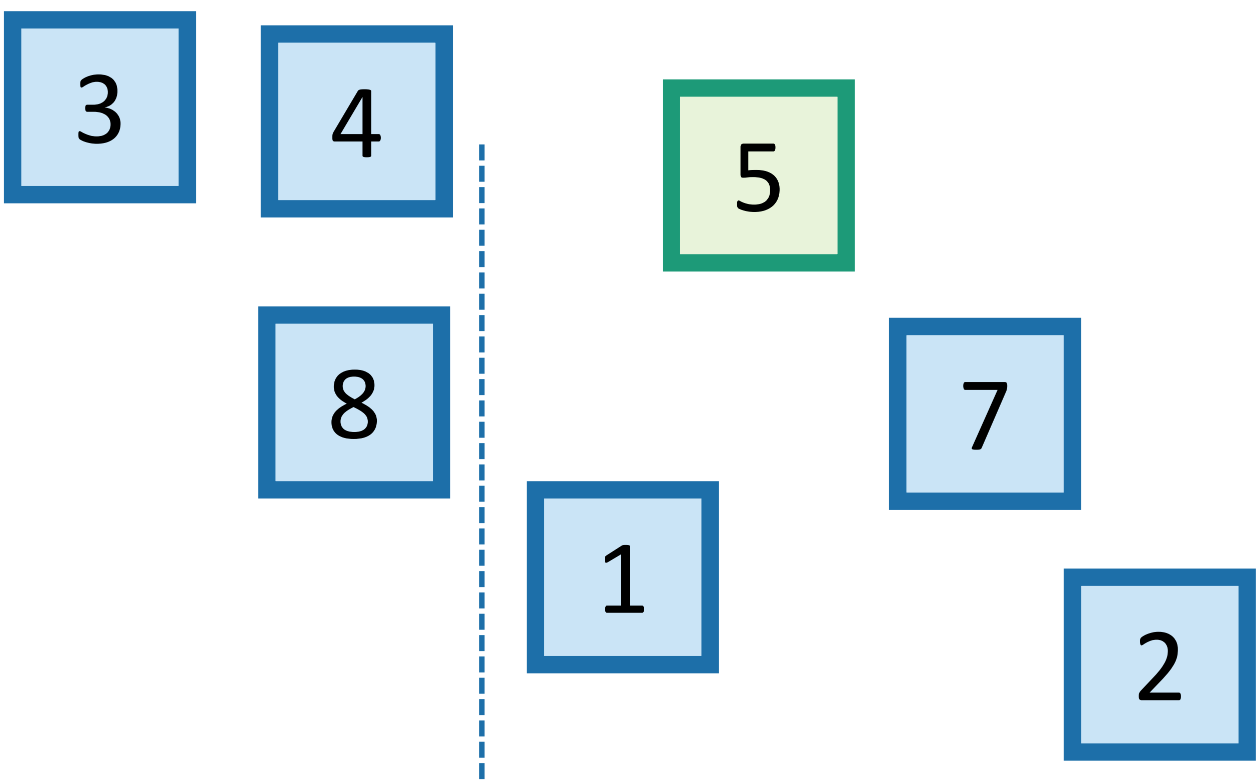

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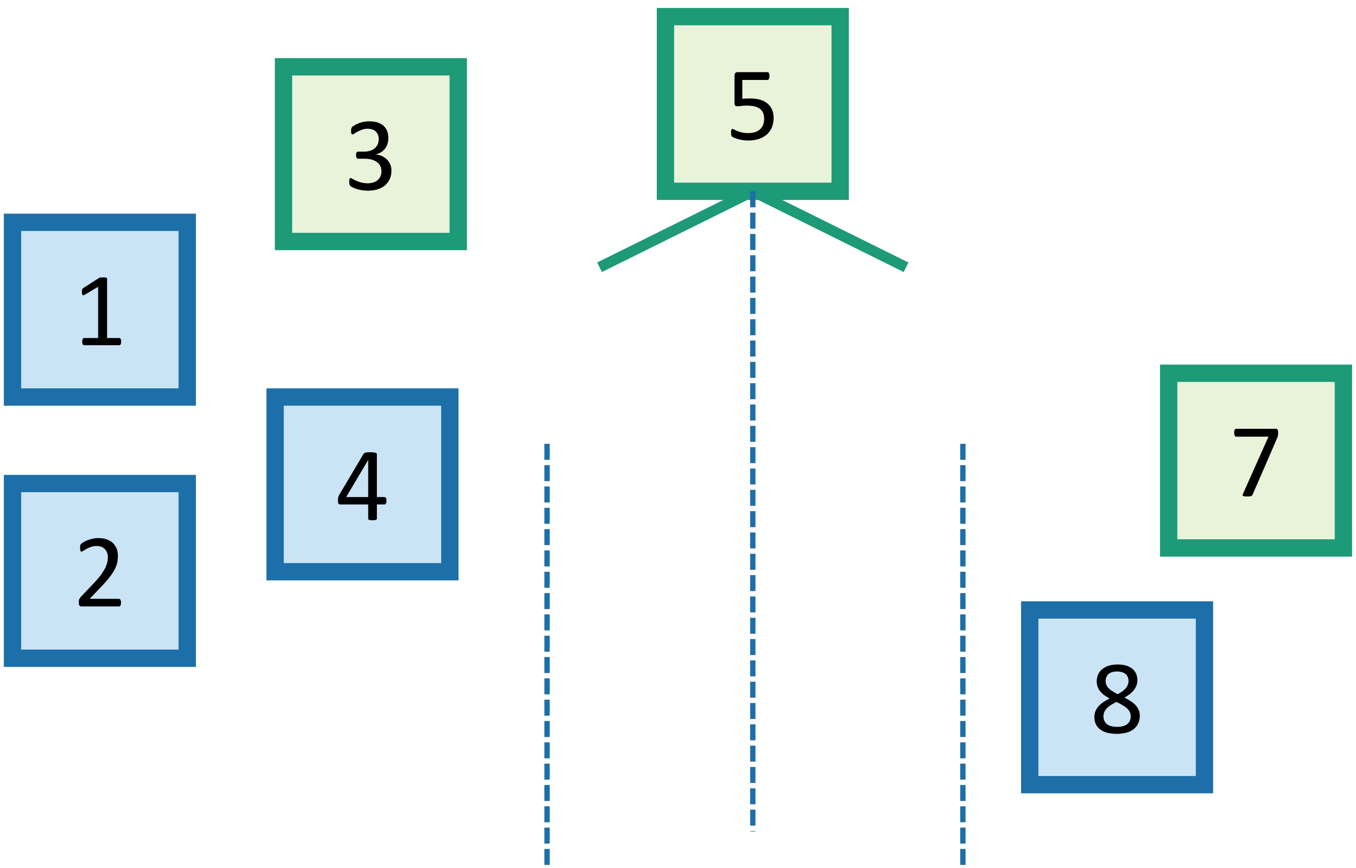

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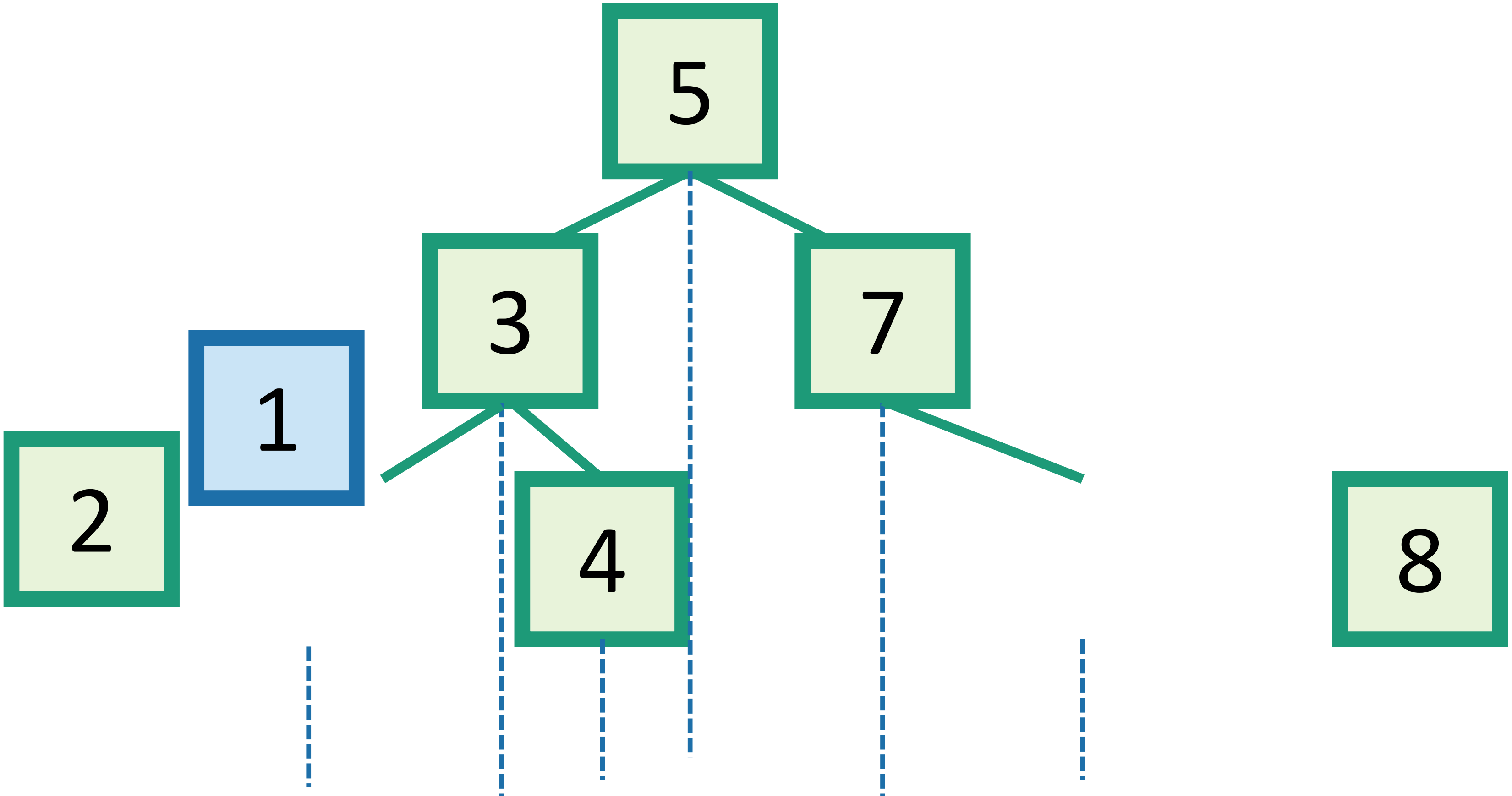

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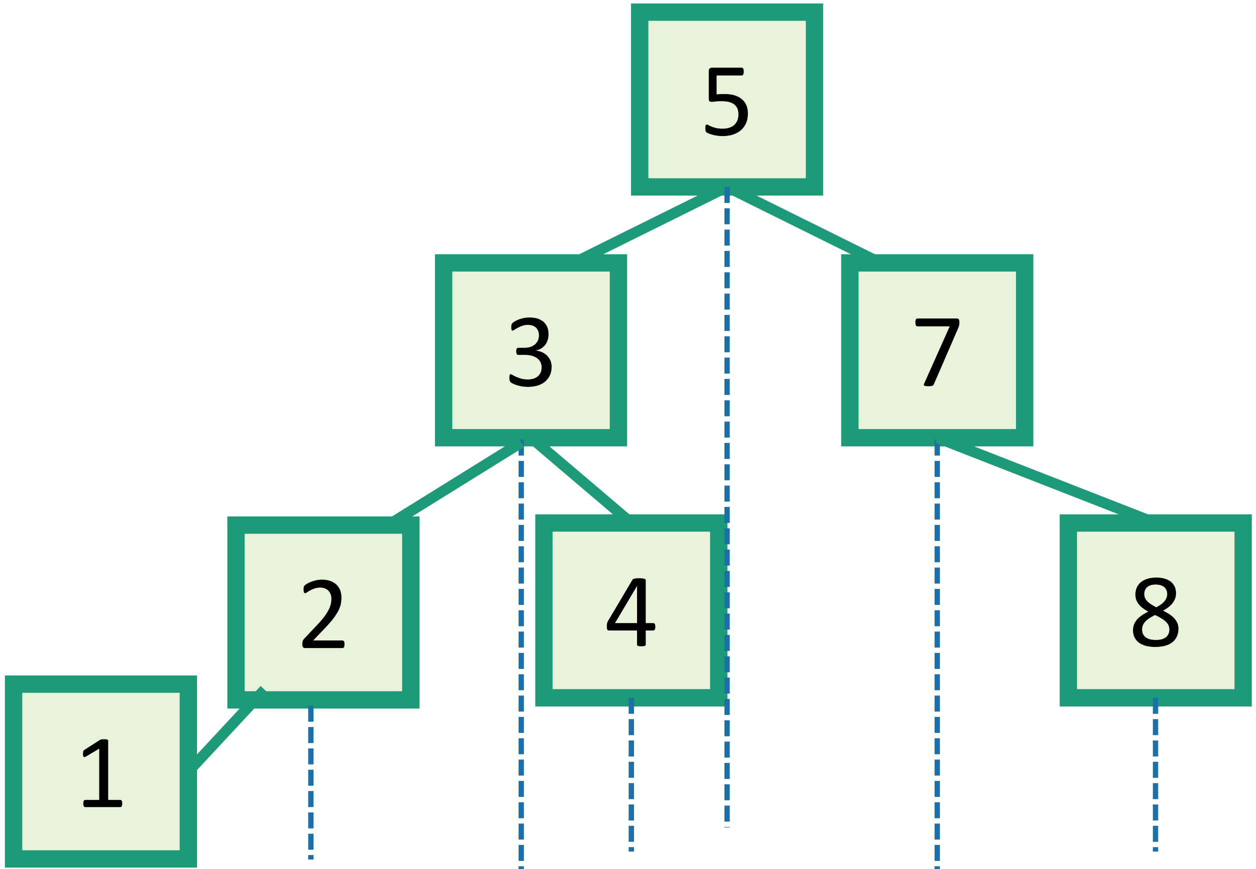
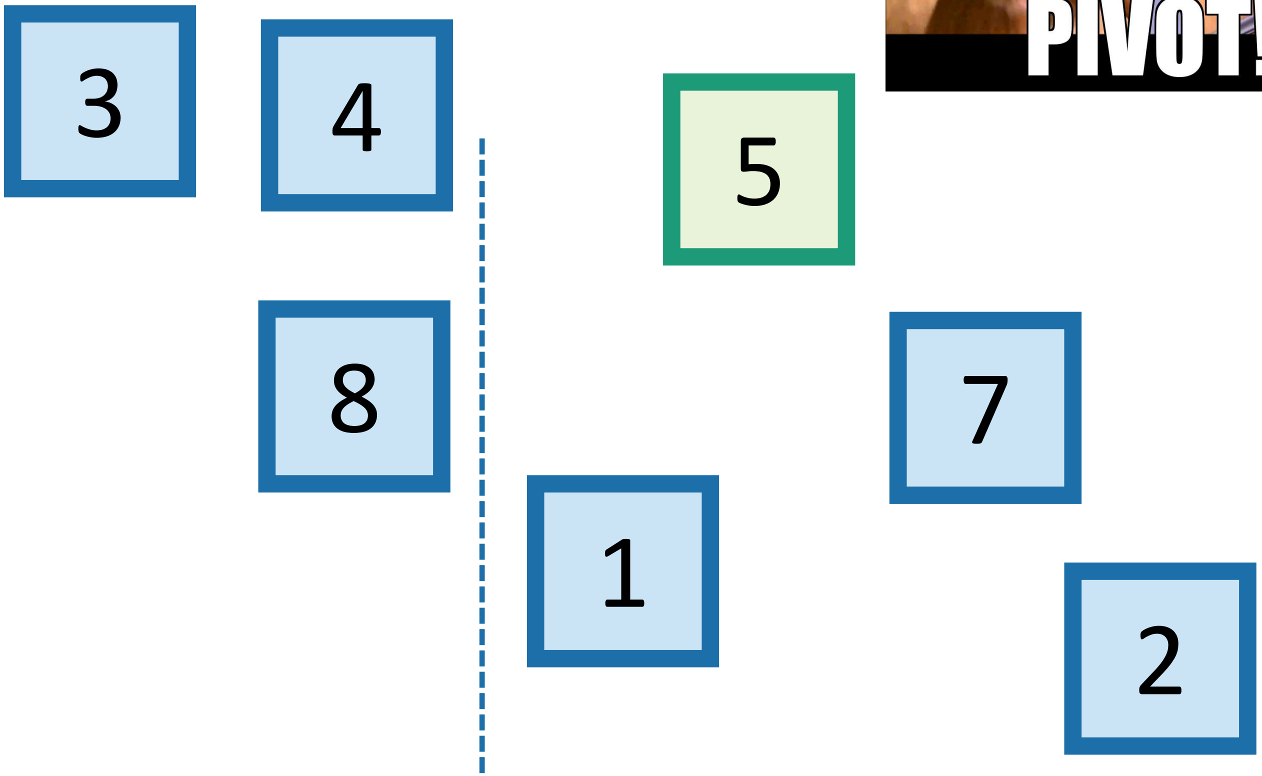


# Aside: this should look familiar

kinda like QuickSort

# Binary Search Trees

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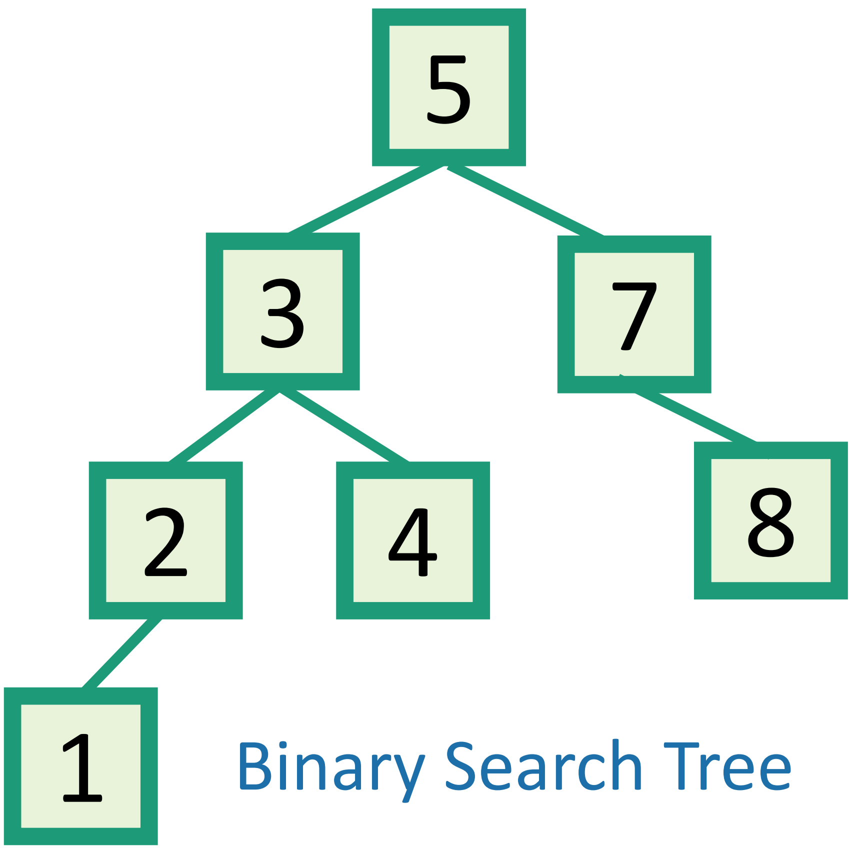
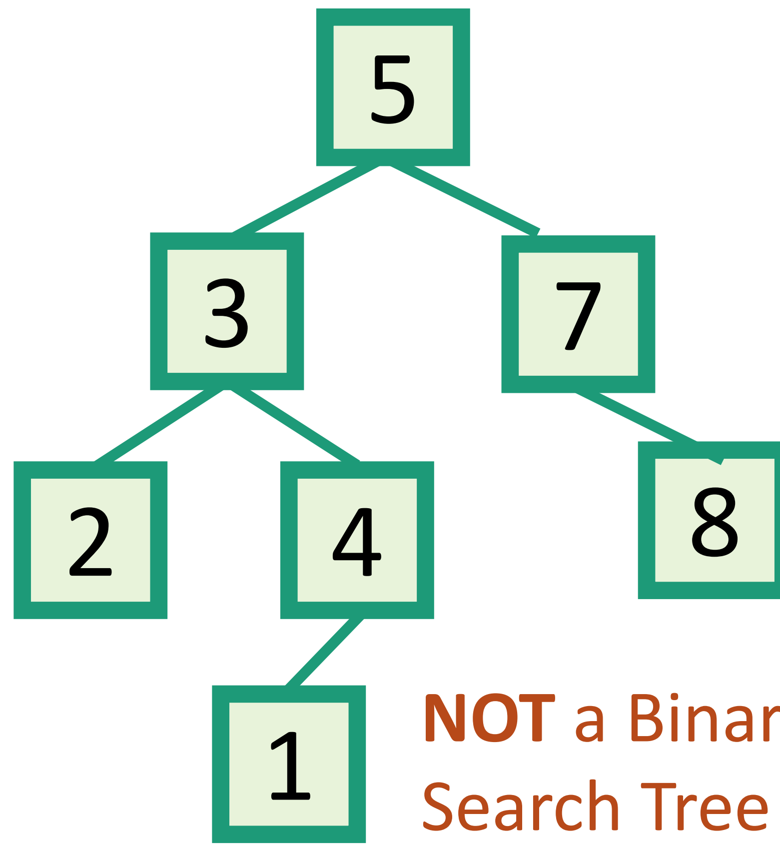
  - **Every LEFT descendent** of a node has key less than that node.
  - **Every RIGHT descendent** of a node has key larger than that node.



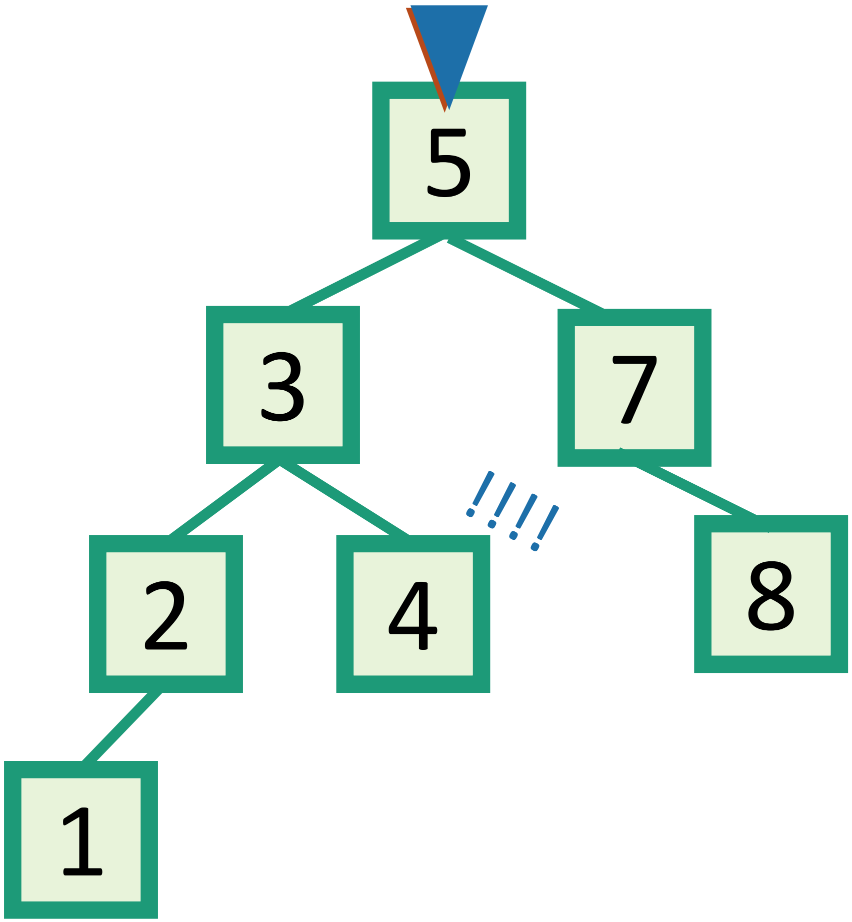

Binary Search Tree




**NOT** a Binary Search Tree

# SEARCH in a Binary Search Tree

## definition by example

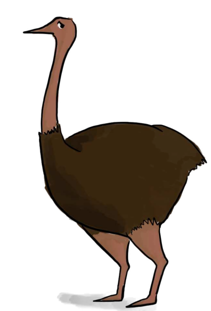



**EXAMPLE:** Search for 4.

**EXAMPLE:** Search for 4.5

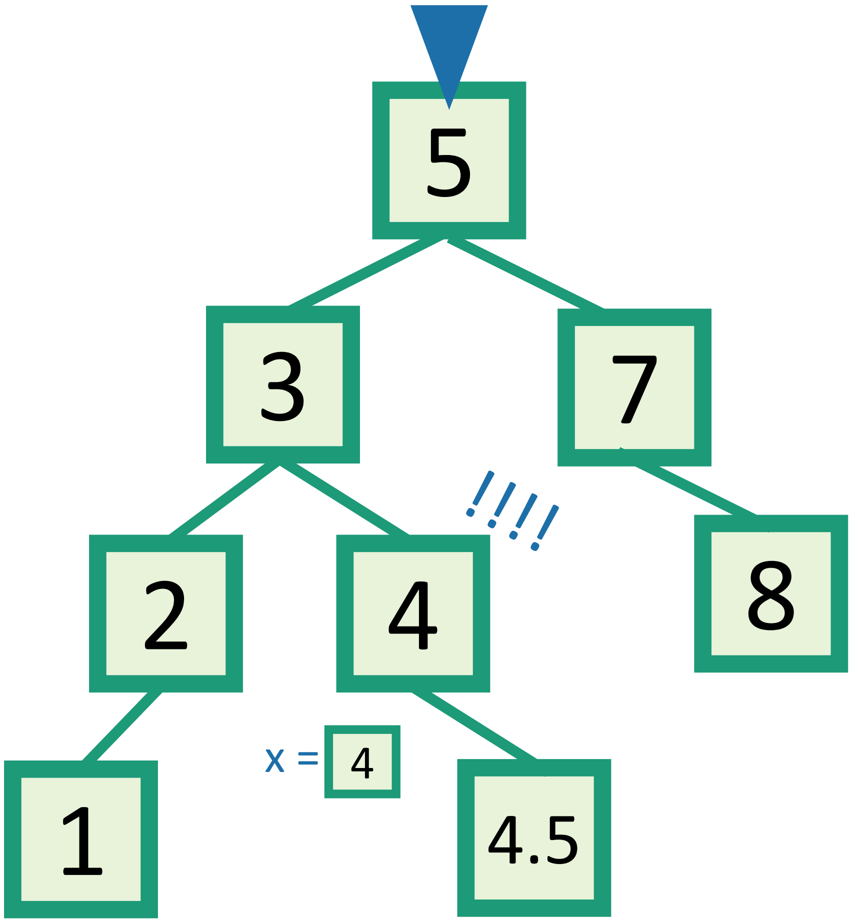
- It turns out it will be convenient to **return 4** in this case
- (that is, **return** the last node before we went off the tree)

Write pseudocode (or actual code) to implement this!



Ollie the over-achieving ostrich

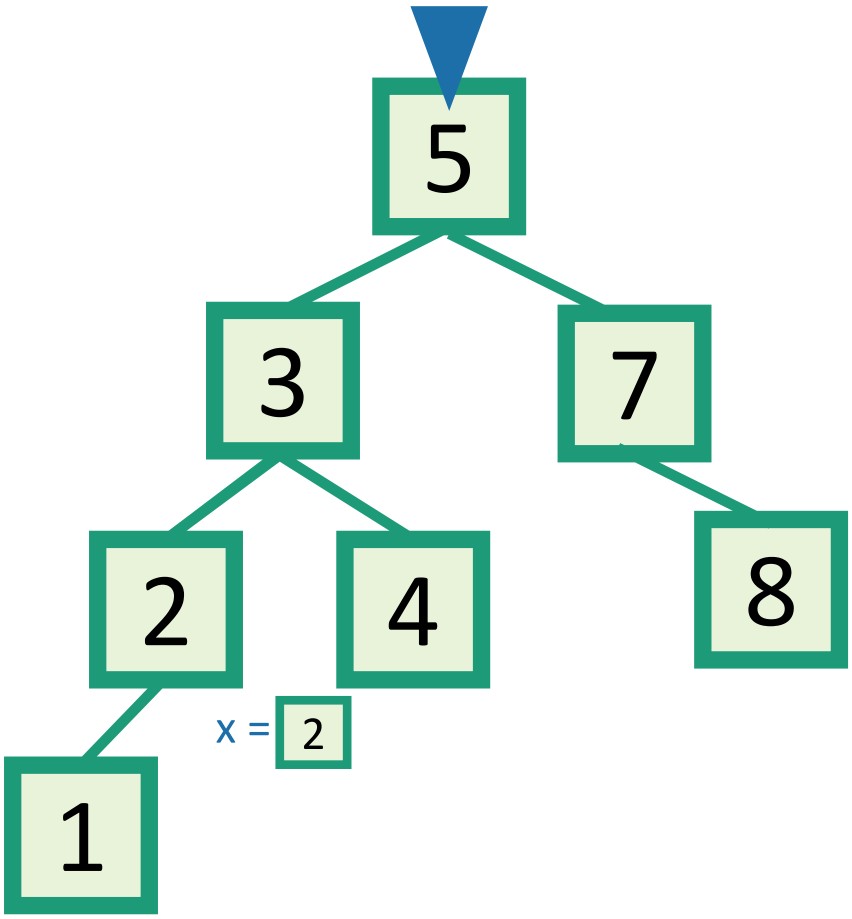
# INSERT in a Binary Search Tree




**EXAMPLE:** Insert 4.5

- INSERT(key):
  - x = SEARCH(key)
  - **if** key > x.key:
    - Make a new node with the correct key, and put it as the right child of x.
  - **if** key < x.key:
    - Make a new node with the correct key, and put it as the left child of x.
  - **if** x.key == key:
    - **return**

# DELETE in a Binary Search Tree




**EXAMPLE:** Delete 2

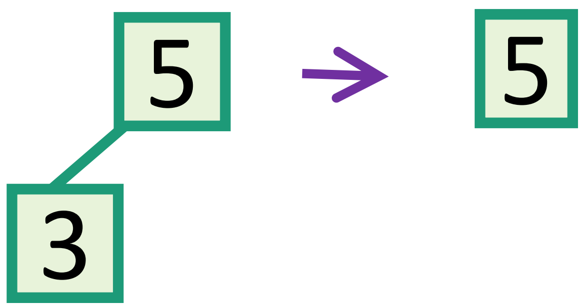
- DELETE(key):
  - x = SEARCH(key)
  - **if** x.key == key:
    - ….delete x….

This is a bit more complicated…

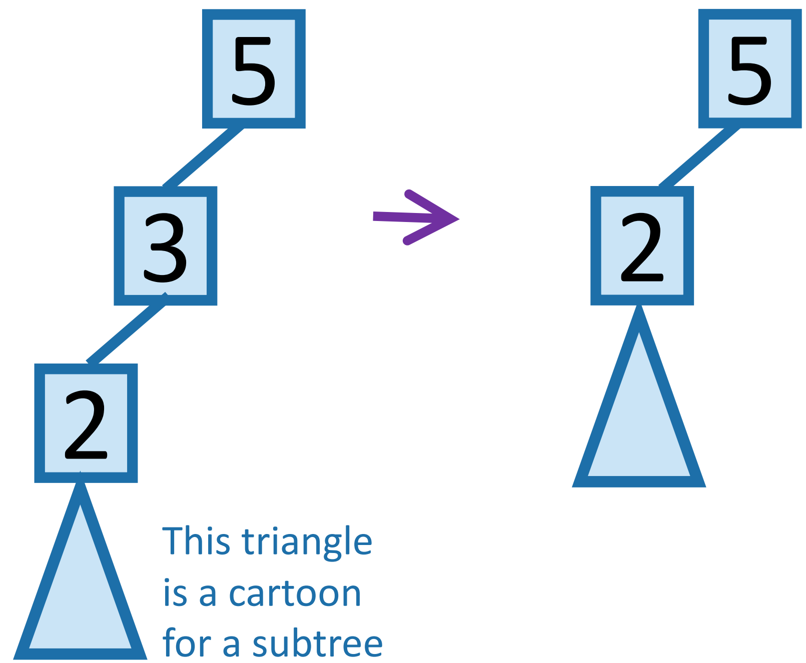
# DELETE in a Binary Search Tree

## several cases (by example)

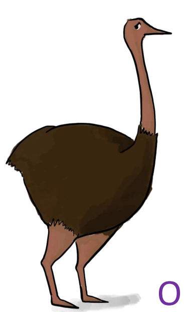
## say we want to delete 3




**Case 1**: if 3 is a leaf, just delete it.




**Case 2:** if 3 has just one child, move that up.



We won't write down pseudocode for this – try to do it yourself!

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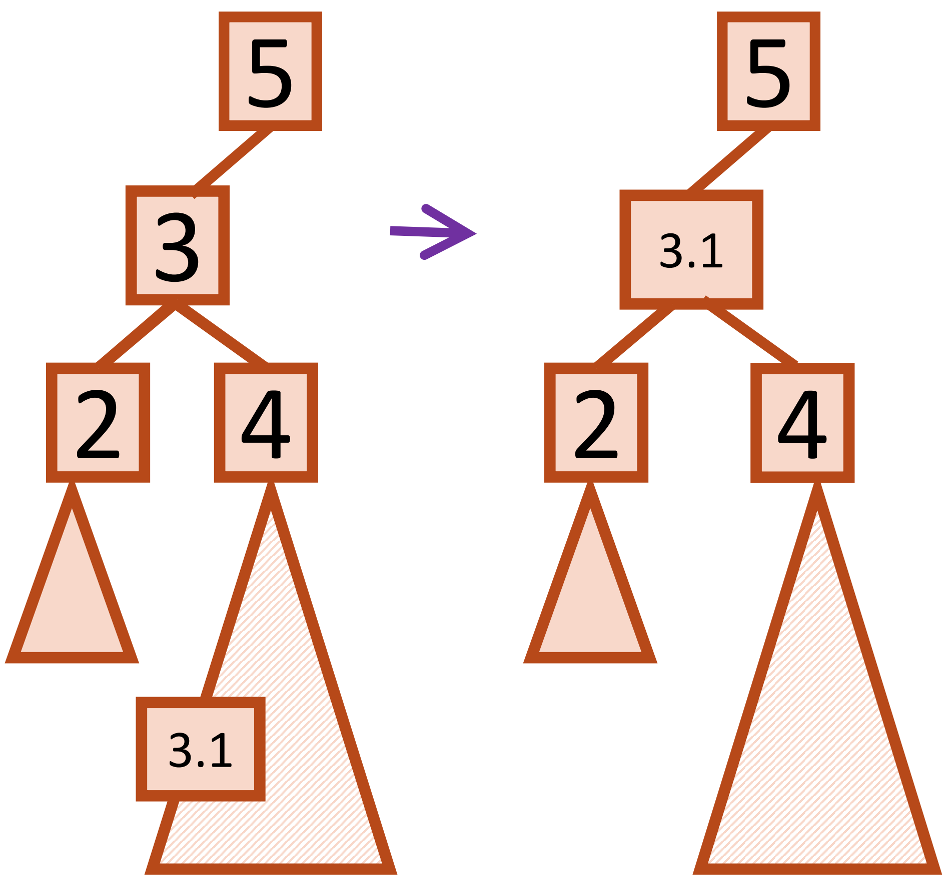
# DELETE in a Binary Search Tree ctd.

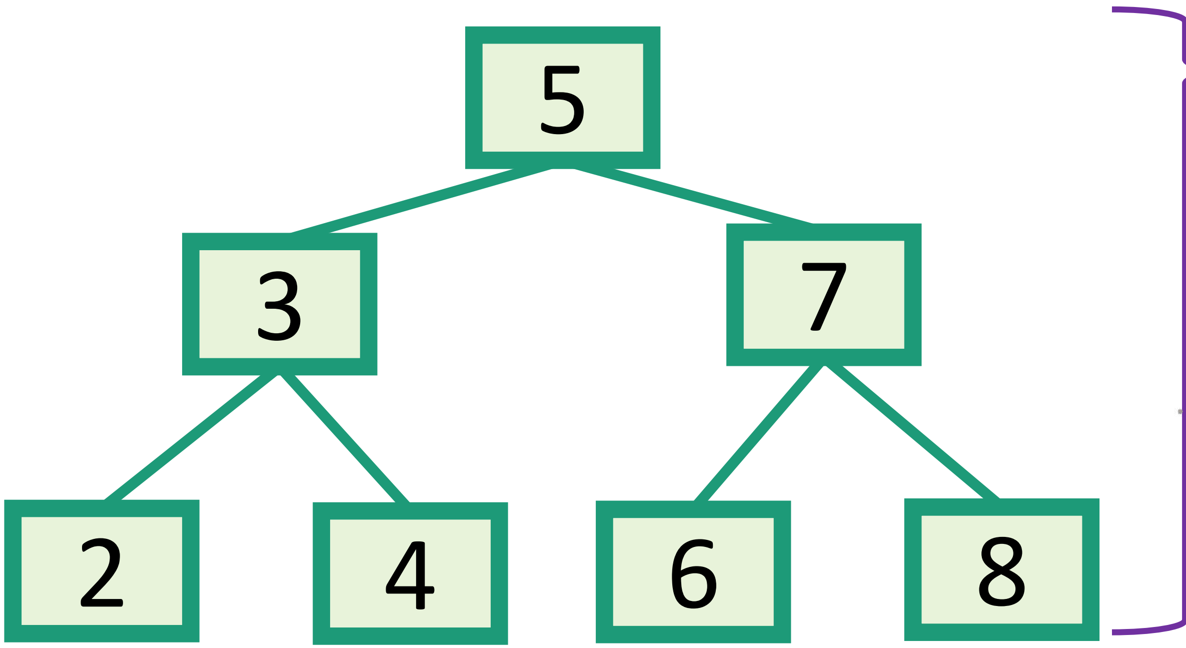
**Case 3**: if 3 has two children,
replace 3 with it's immediate successor.
(aka, next biggest thing after 3)




- How do we find the immediate successor?
  - SEARCH(3.right, 3)
- How do we remove it when we find it?
  - Run DELETE for one of the previous two cases.
- Wait, what if it's **THIS** case? (Case 3).
  - It's not.

# How long do these operations take?

- SEARCH is the big one.
- Everything else just calls SEARCH and then does some small O(1)-time operation.

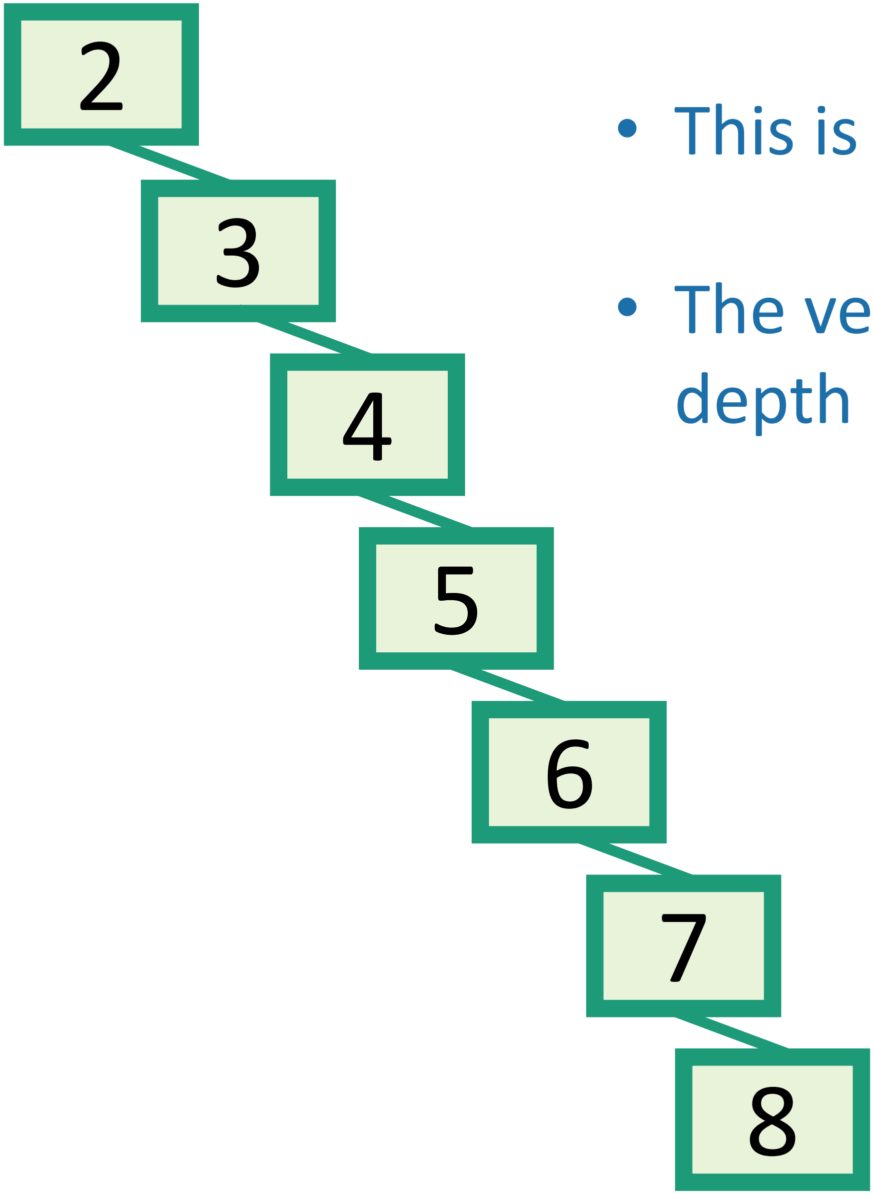



Time = O(depth of tree)

Trees have depth O(log(n)). **Done!**

Lucky the lackadaisical lemur.

# Wait…



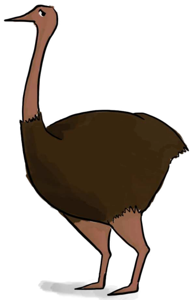

- This is a valid binary search tree.
- The version with n nodes has depth n, not O(log(n)).

**Could such a tree show up?**
In what order would I have to insert the nodes?

Inserting in the order 2,3,4,5,6,7,8 would do it.

So this could happen.

# What to do?



How often is "every so often" in the worst case? It's actually pretty often!

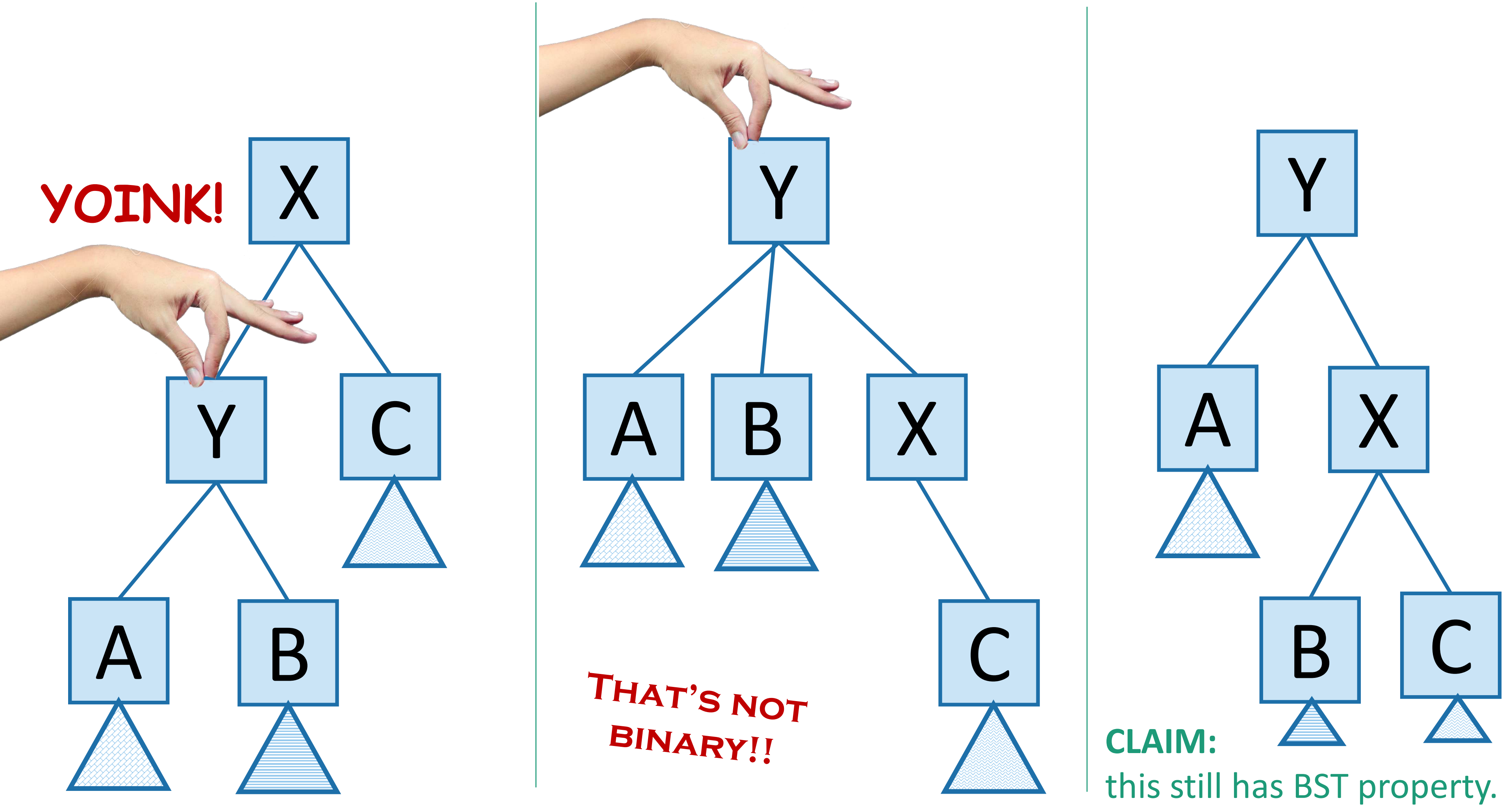
Ollie the over-achieving ostrich

- Keep track of how deep the tree is getting.
- If it gets too tall, re-do everything from scratch.
  - At least $\Omega(n)$ every so often….
- Other ideas?

# Idea 1: Rotations

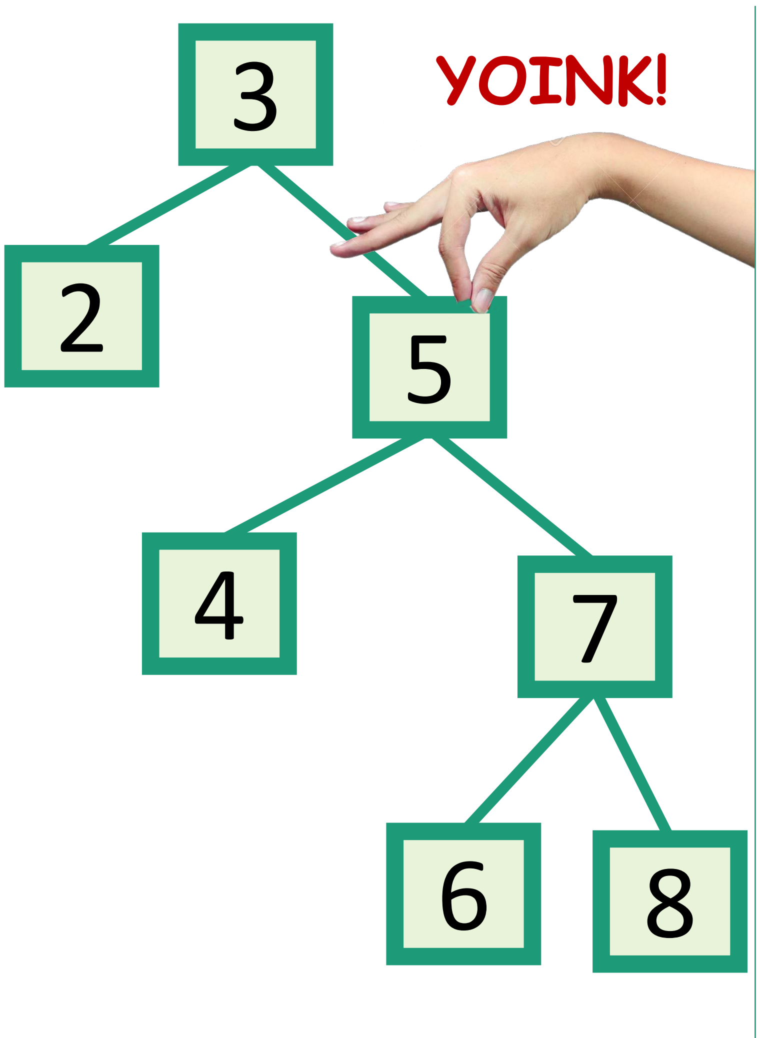
No matter what lives underneath A,B,C,
**this takes time O(1).** (Why?)

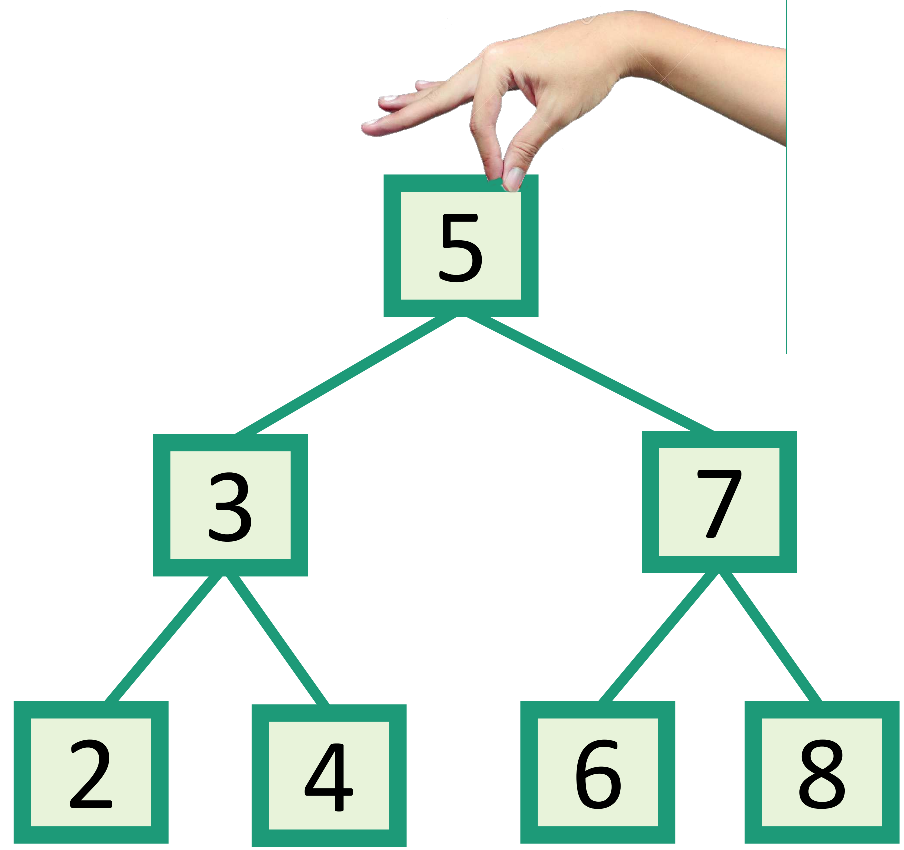
- Maintain Binary Search Tree (BST) property, while moving stuff around.




**CLAIM:**
this still has BST property.

# This seems helpful



# Does this work?

- Whenever something seems unbalanced, do rotations until it's okay again.

Lucky the Lackadaisical Lemur

Even for me this is pretty vague. What do we mean by "seems unbalanced"? What's "okay"?

# Idea 2: have some proxy for balance

- Maintaining perfect balance is too hard.
- Instead, come up with some proxy for balance:
  - If the tree satisfies [SOME PROPERTY], then it's pretty balanced.
  - We can maintain [SOME PROPERTY] using rotations.

There are actually several ways to do this, but today we'll see…

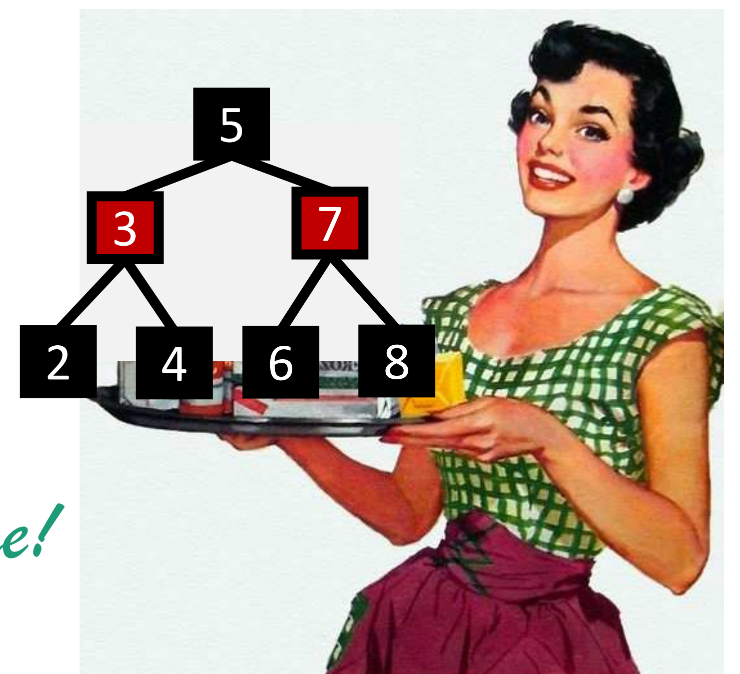
# Red-Black Trees

- A Binary Search Tree that balances itself!
- No more time-consuming by-hand balancing!
- Be the envy of your friends and neighbors with the time-saving...

**Red-Black tree!**

*Maintain balance* by stipulating that **black nodes** are balanced, and that there *aren't too many* **red nodes**.

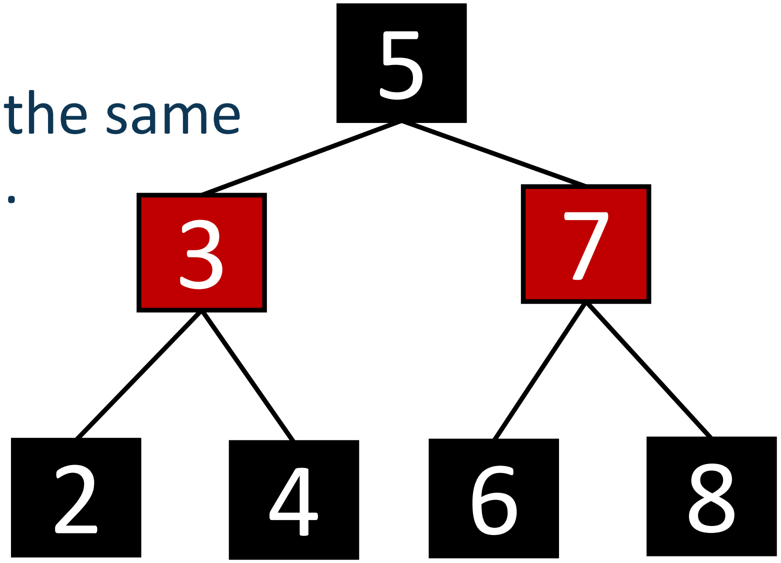
*It's just good sense!*



# Red-Black Trees

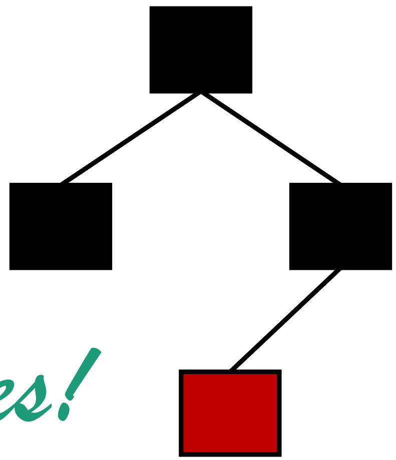
these rules are the proxy for balance

- Every node is colored **red** or **black.**
- The root node is a **black node**.
- NIL children count as **black nodes**.
- Children of a **red node** are **black nodes**.
- For all nodes x:
  - all paths from x to NONE's have the same number of black nodes on them.



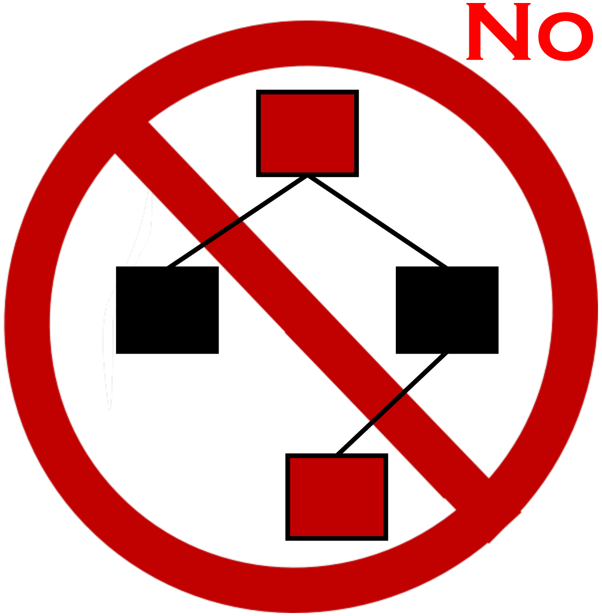
# Examples(?)

- Every node is colored **red** or **black.**
- The root node is a **black node**.
- NONE children count as **black nodes**.
- Children of a **red node** are **black nodes**.
- For all nodes x:
  - all paths from x to NONE's have the same number of black nodes on them.



Yes!

No!



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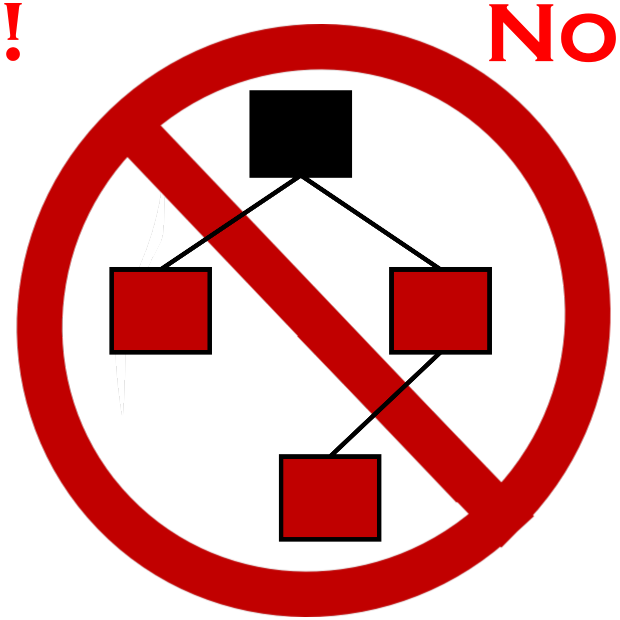

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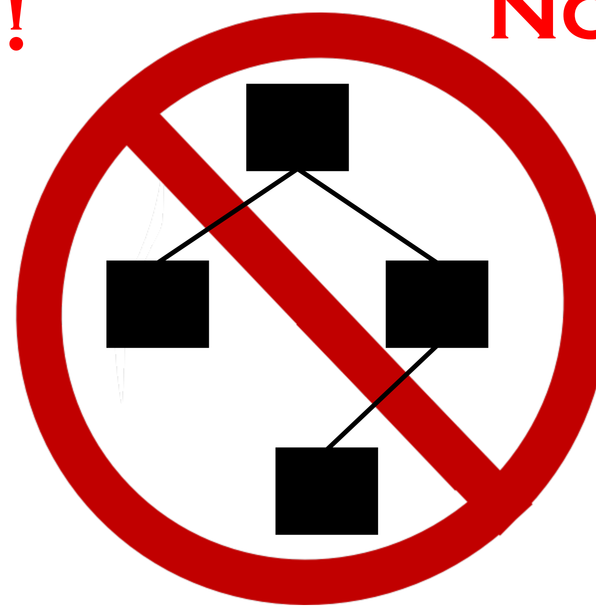
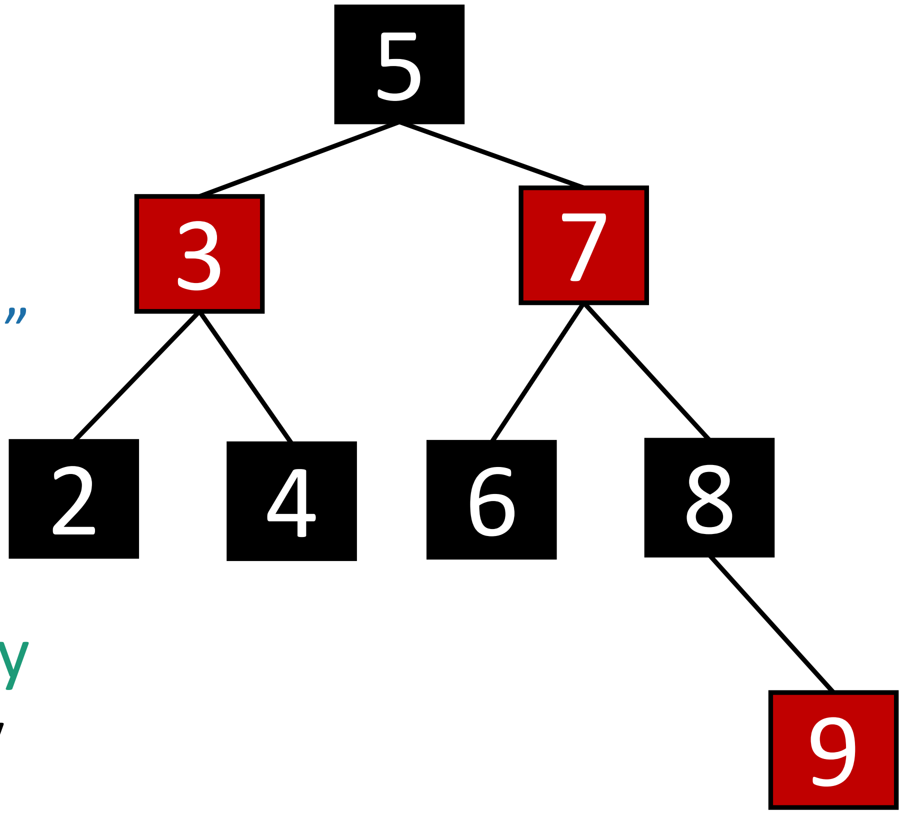


# Why???????

- This is pretty balanced.
  - The **black nodes** are balanced
  - The **red nodes** are “spread out” so they don’t mess things up too much.



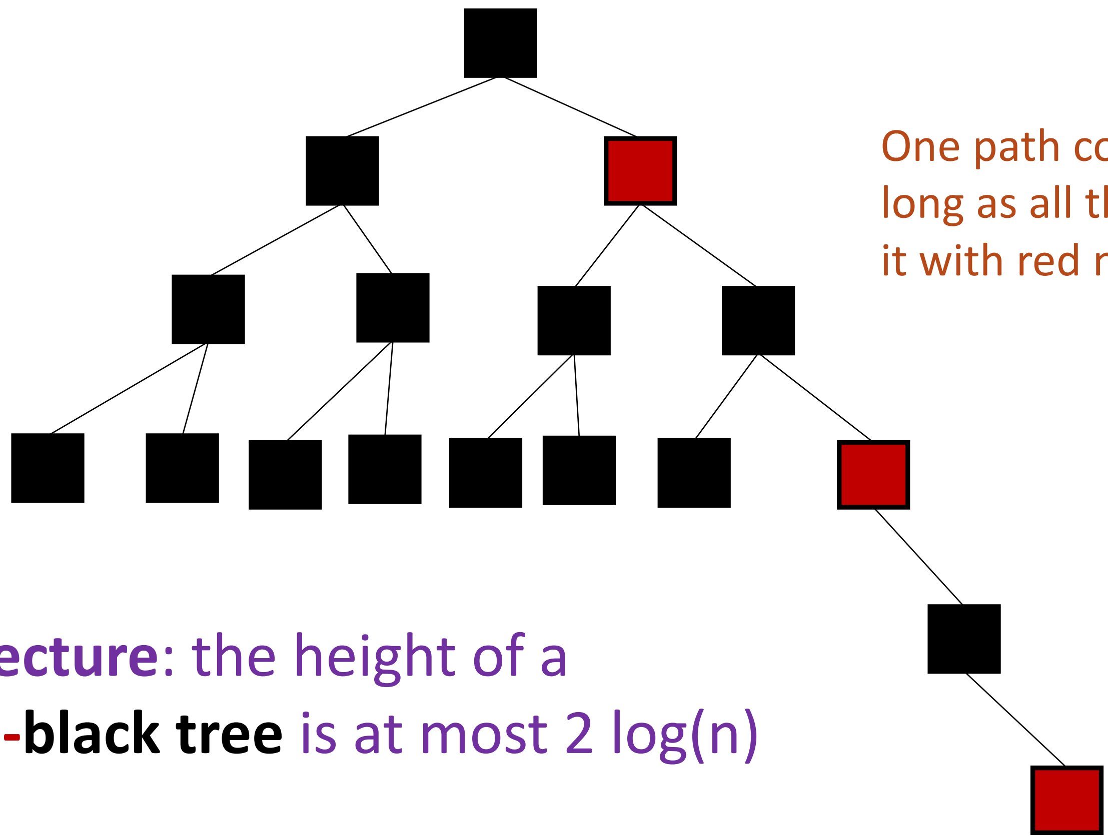

- We can maintain this property as we insert/delete nodes, by using rotations.

Let’s build some intuition!

Lucky the lackadaisical lemur

# This is “pretty balanced”

- To see why, intuitively, let’s try to build a Red-Black Tree that’s unbalanced.

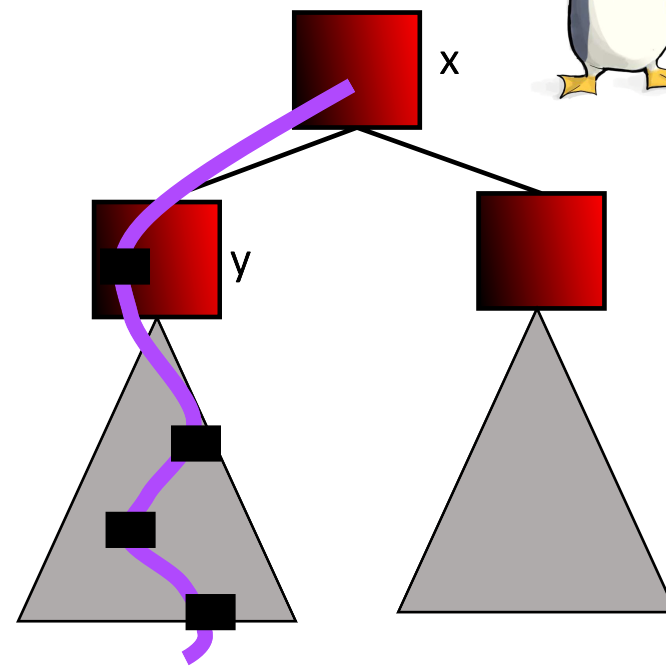


One path could be twice as long as all the others if we pad it with red nodes.

**Conjecture**: the height of a **red-black tree** is at most 2 log(n)

# That turns out the be basically right.

[proof sketch]




- Say there are b(x) black nodes in any path from x to NONE.
  - (including x).
- **Claim:**
  - Then there are at least $2^{b(x)} - 1$ nodes in the subtree underneath x.
- [Proof by induction – on board if time]

Then:

$$n \geq 2^{b(root)} - 1 \qquad \text{using the } \textbf{Claim}$$

$$\geq 2^{height/2} - 1 \qquad \text{b(root) >= height/2 because of RBTree rules.}$$

Rearranging:

$$n + 1 \geq 2^{height/2} \Rightarrow height \leq 2\log(n+1)$$

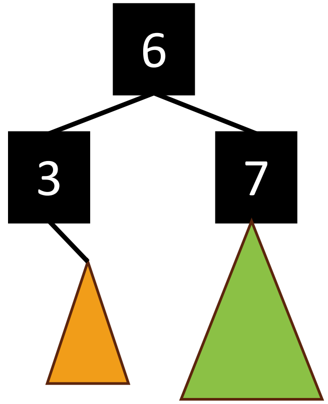
# Okay, so it's balanced…
# …but can we maintain it?

- Yes!

- For the rest of lecture:
  - sketch of how we'd do this.

- See CLRS for more details.

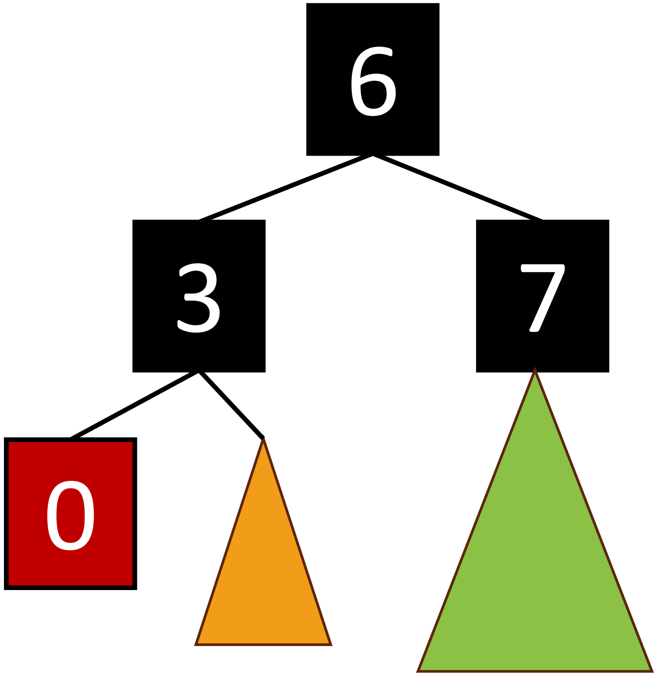
# Inserting into a Red-Black Tree



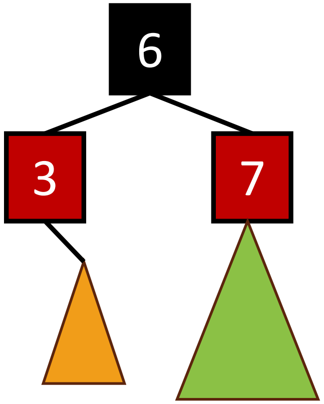

What if it looks like this?

- Make a new **red node**.
- Insert it as you would normally.

Example: insert 0

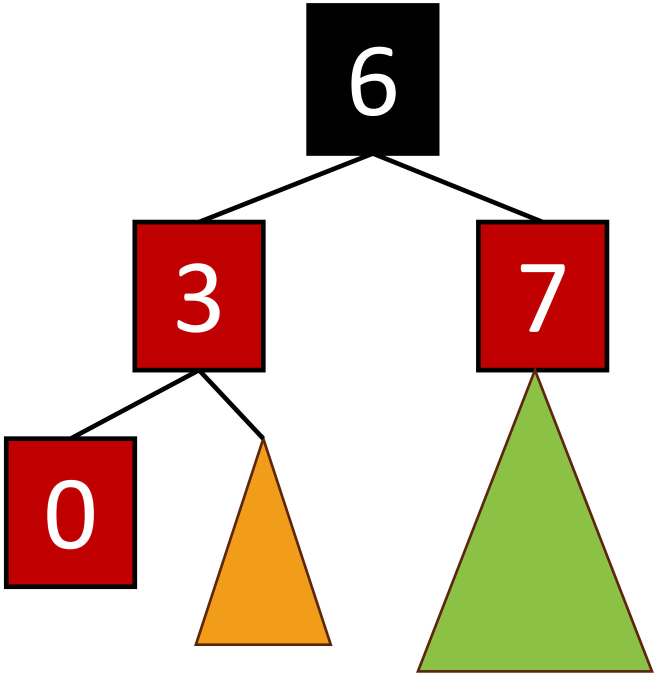


# Inserting into a Red-Black Tree



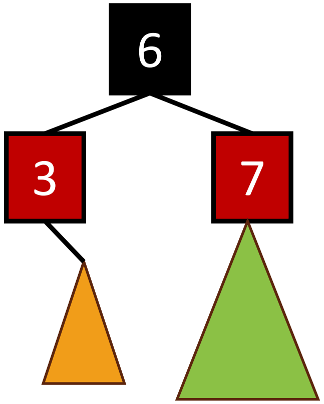

What if it looks like this?

- Make a new **red node.**
- Insert it as you would normally.
- Fix things up if needed.




Example: insert 0

# Inserting into a Red-Black Tree

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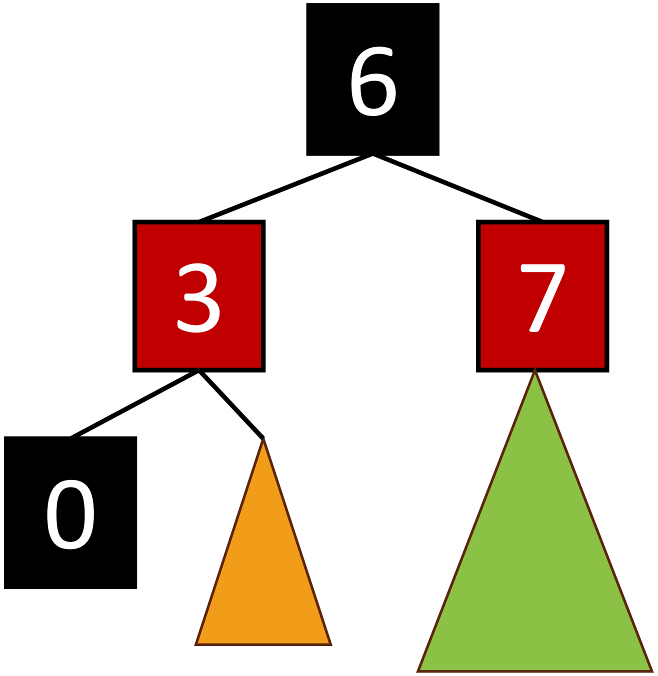
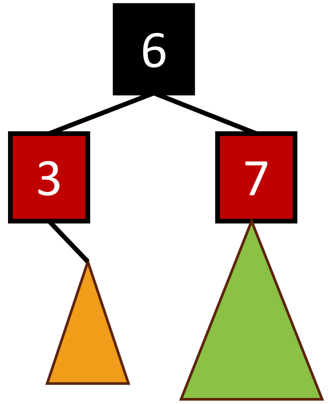


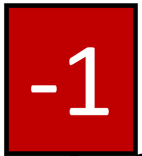

Example: insert 0

Can’t we just insert 0 as a **black node?**

# Inserting into a Red-Black Tree




What if it looks like this?




- Make a new **red node.**
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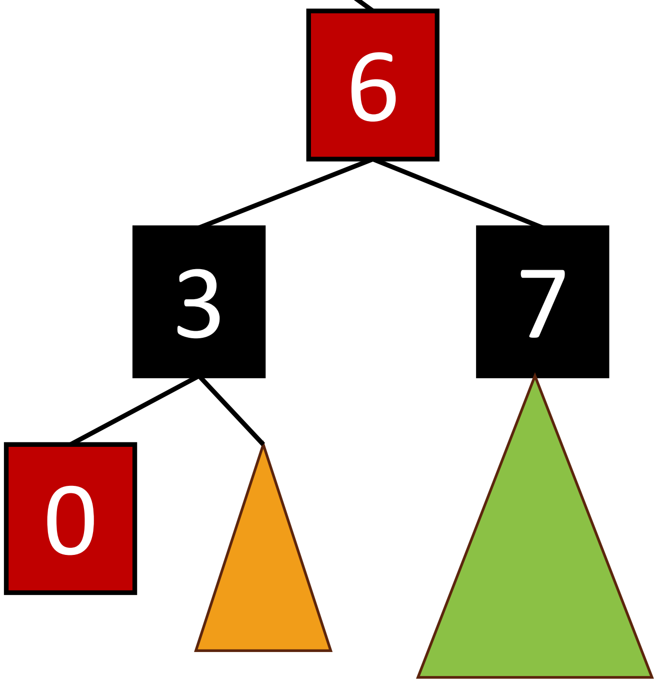
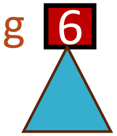
Example: insert 0

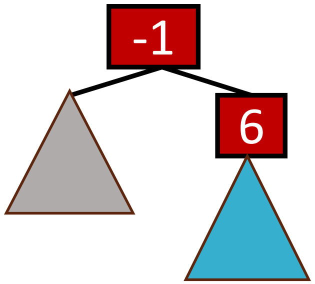



- Instead recolor like this.
- Need to argue:
  - RB-Tree properties still hold.

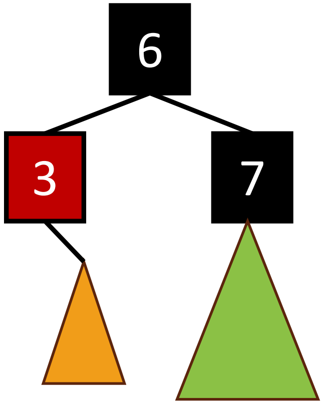

- What about the red root?
  - if 6 is actually the root, color it black.
  - Else, recursively re-color up the tree.

Now the problem looks like this, where I'm inserting



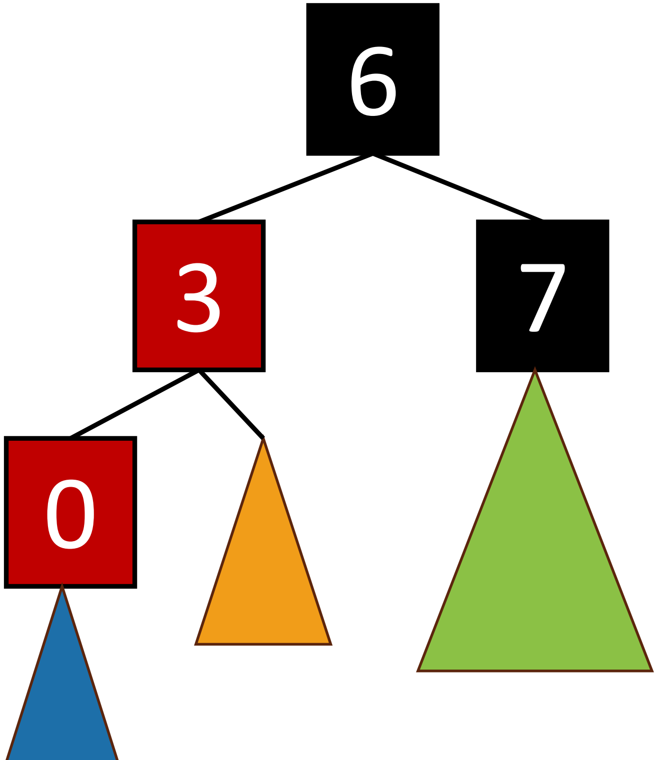



# Inserting into a Red-Black Tree




What if it looks like this?

- Make a new **red node**.
- Insert it as you would normally.
- Fix things up if needed.

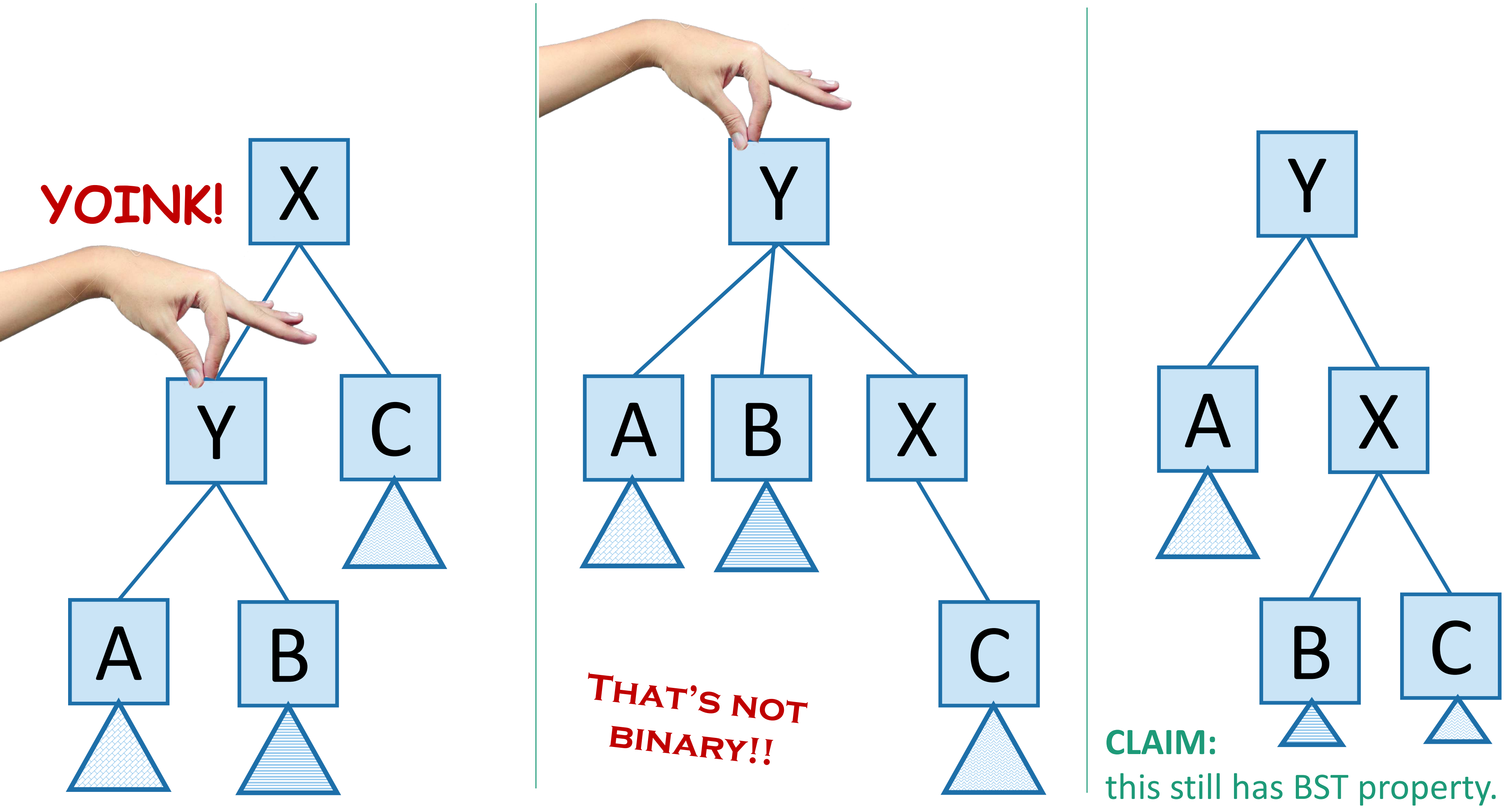



Example: Insert 0.

- Actually, **this can't happen?**
- It might happen that we just turned 0 red from the previous step.
- Or it could happen if 7 is actually NIL.

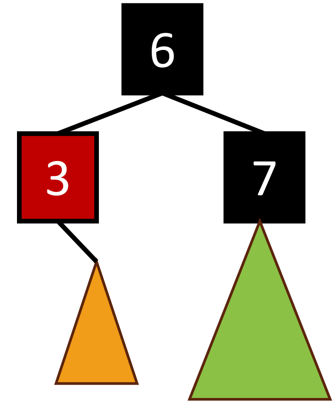
# Recall Rotations

- Maintain Binary Search Tree (BST) property, while moving stuff around.

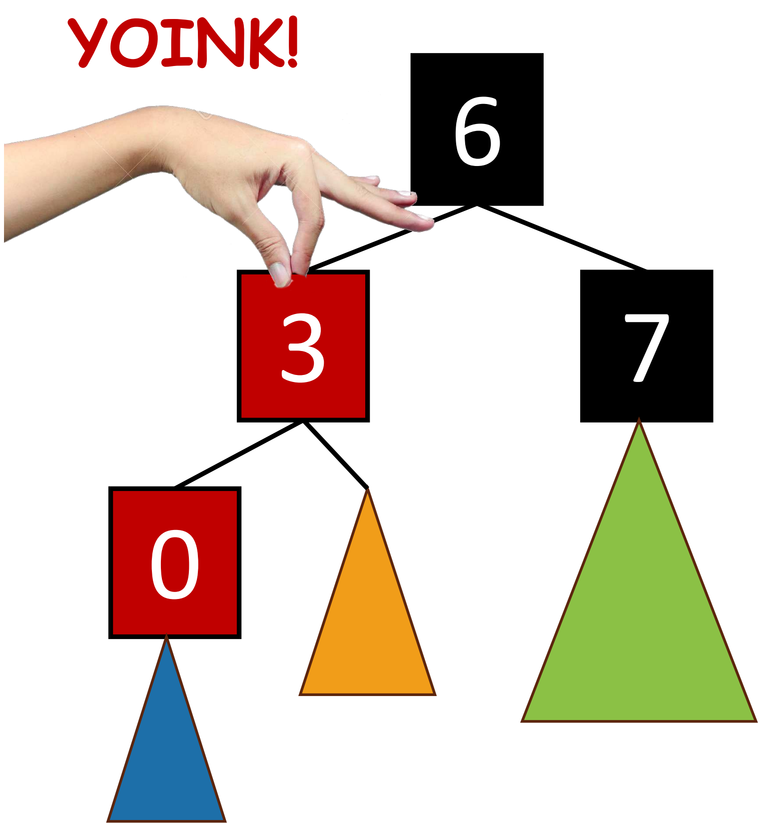


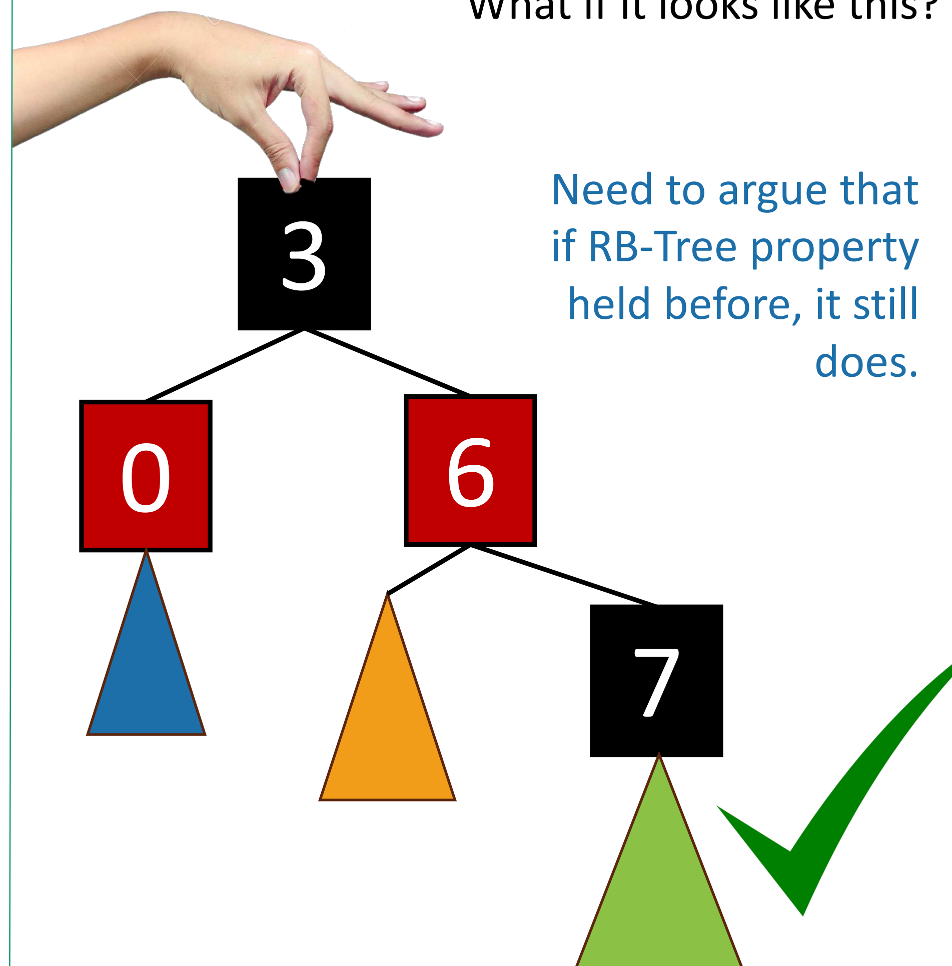
# Inserting into a Red-Black Tree

- Make a new **red node**.
- Insert it as you would normally.
- Fix things up if needed.




What if it looks like this?




Need to argue that if RB-Tree property held before, it still does.

# That’s basically it

- A few things still left to check for INSERT!
  - Anything else that might happen looks basically like what we just did.
  - Formally dealing with the recursion.
  - You check these! (or see CLRS)
- DELETE is similar.

Plucky the pedantic penguin

# The punchline:

- Red-Black Trees always have height at most 2log(n+1).
- As with general Binary Search Trees, all operations are O(height)
- So all operations are O(log(n)).

# Conclusion: The best of both worlds

| | Sorted Arrays | Linked Lists | Balanced Binary Search Trees |
|---|---|---|---|
| **Search** | O(log(n)) | O(n) | O(log(n)) |
| **Insert/Delete** | O(n) | O(1) | O(log(n)) |

# Recap

- Balanced binary trees are the best of both worlds!
- But we need to keep them balanced.
- **Red-Black Trees** do that for us.
  - We get O(log(n))-time INSERT/DELETE/SEARCH
  - Clever idea: have a proxy for balance

# Next time

- Hashing!