# The Reynolds Averaged Navier Stokes Equations RANS

We start off by writing down the classical Navier Stokes equations for incompressible, constant viscosity, Newtonian fluids on the form

$$\frac{D}{Dt}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \frac{\partial}{\partial t}\begin{pmatrix} u \\ v \\ w \end{pmatrix} + (\mathbf{u}\cdot\nabla)\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \frac{-\nabla P}{\rho} + \mathbf{g} + \nu\nabla^2\begin{pmatrix} u \\ v \\ w \end{pmatrix} \tag{1}$$

where this notation highlights the fact that the operators $\frac{\partial}{\partial t}$, $(\mathbf{u}\cdot\nabla)$ and $(\nu\nabla^2)$ must be applied to each component of the velocity vector $\mathbf{u}$. To derive the Reynolds Averaged Navier-Stokes Equations (RANS) we assume that the dependent variables (velocity and pressure) consist of a mean part $\overline{\phi}$ and a random fluctuating turbulent part $\phi'$, such that $\phi = \overline{\phi} + \phi'$. This is called Reynolds decomposition. Before we go on into the algebra, we should note some laws for averaging.

## Laws for averaging

Let the mean of a time signal $\phi(t)$ be defined as $\overline{\phi} = \frac{1}{T}\int_0^T \phi dt$, where the integral is taken over a time period $T$ which is considerably larger than the period of the turbulent fluctuations. From this definition we may define several algebraic rules for manipulation of turbulent signals. Let $f = \overline{f} + f'$ and $g = \overline{g} + g$ be two turbulent signals:

$$(1) : \overline{f'} = 0 \qquad (2) : \overline{\overline{f}} = \overline{f} \qquad (3) : \overline{f\overline{g}} = \overline{f}\overline{g}$$

$$(4) : \overline{f'g} = 0 \qquad (5) : \overline{f+g} = \overline{f} + \overline{g} \qquad (6) : \overline{\overline{f}\overline{g}} = \overline{f}\overline{g}$$

$$(7) : \overline{fg} = \overline{f}\overline{g} + \overline{f'g'} \qquad (8) : \overline{\frac{\partial f}{\partial s}} = \frac{\partial \overline{f}}{\partial s} \qquad (9) : \overline{\int f ds} = \int \overline{f} ds$$

## Deriving RANS

The process of Reynolds averaging is now to insert $\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}'$ and $P = \overline{P} + P'$ into (1), and thereafter take the average of the whole equation term for term, utilizing the laws above. The result is that the transport equations for $\overline{\mathbf{u}}$ involves additional terms called *Reynolds stresses*, that acts as additional unknowns in the equations, and needs proper modelling to find suitable solutions. As we will see, these Reynolds stresses emerge from the non linear convective acceleration term. We will now conduct the Reynolds averaging of the $u-$ momentum equation, and then generalize the result. Inserting $u = \overline{u} + u'$ and $p = \overline{P} + P'$ into the $u-$ momentum, and then averaging the equation term for term gives:

### Time derivative

$$\overline{\frac{\partial(\overline{u} + u')}{\partial t}} = \overline{\frac{\partial \overline{u}}{\partial t} + \frac{\partial u'}{\partial t}} = \overline{\frac{\partial \overline{u}}{\partial t}} + \overline{\frac{\partial u'}{\partial t}} = \frac{\partial \overline{\overline{u}}}{\partial t} + \frac{\partial \overline{u'}}{\partial t} = \frac{\partial \overline{u}}{\partial t}$$

**Pressure gradient**

$$\overline{\frac{\partial(\overline{P}+P')}{\partial x}} = \overline{\frac{\partial \overline{P}}{\partial x} + \frac{\partial P'}{\partial x}} = \frac{\partial \overline{\overline{P}}}{\partial x} + \frac{\partial \overline{P'}}{\partial x} = \frac{\partial \overline{\overline{P}}}{\partial x} + \frac{\partial \overline{P'}}{\partial x} = \frac{\partial \overline{P}}{\partial x}$$

**Viscous Diffusion**

$$\overline{\nabla^2(\overline{u}+u')} = \overline{\nabla^2 \overline{u} + \nabla^2 u'} = \nabla^2 \overline{\overline{u}} + \nabla^2 \overline{u'} = \nabla^2 \overline{u}$$

As seen from the previous calculations, the Reynolds averaging does not produce any additional term in the linear terms.

**Convective acceleration**

Before we calculate the non linear convective acceleration term, we should utilize the following identity

$$(\mathbf{u}\cdot\nabla)\phi = \nabla\cdot(\mathbf{u}\phi) - \phi(\nabla\cdot\mathbf{u})$$

where of course the term $\phi(\nabla\cdot\mathbf{u}) = 0$, for an incompressible fluid. Reynolds averaging gives:

$$\overline{([\overline{\mathbf{u}}+\mathbf{u}']\cdot\nabla)(\overline{u}+u')} = \overline{\nabla\cdot[(\overline{\mathbf{u}}+\mathbf{u}')\ (\overline{u}+u']} =$$

$$\overline{\frac{\partial}{\partial x}\left((\overline{u}+u')(\overline{u}+u')\right)} + \overline{\frac{\partial}{\partial y}\left((\overline{v}+v')(\overline{u}+u')\right)} + \overline{\frac{\partial}{\partial z}\left((\overline{w}+w')(\overline{u}+u')\right)} \quad (2)$$

Let us work out the first of these terms:

$$\overline{\frac{\partial}{\partial x}\left((\overline{u}+u')(\overline{u}+u')\right)} = \overline{\frac{\partial}{\partial x}\left(\overline{u}\ \overline{u}+2\overline{u}u'+u'u'\right)} = \frac{\partial(\overline{u}\ \overline{u})}{\partial x} + 2\frac{\partial\overline{\overline{u}u'}}{\partial x} + \frac{\partial\overline{u'u'}}{\partial x}$$

where the terms $\overline{\overline{u}u'} = 0$ according to the averaging laws, but the terms $\overline{u'u'} \neq 0$, and this is the cause of the Reynolds stresses. We may generalize this results into to yield

$$\overline{\frac{\partial}{\partial x}\left((\overline{u}+u')(\overline{u}+u')\right)} = \frac{\partial \overline{u}^2}{\partial x} + \frac{\partial \overline{u'u'}}{\partial x}$$

$$\overline{\frac{\partial}{\partial y}\left((\overline{v}+v')(\overline{u}+u')\right)} = \frac{\partial(\overline{u}\ \overline{v})}{\partial y} + \frac{\partial \overline{u'v'}}{\partial y}$$

$$\overline{\frac{\partial}{\partial z}\left((\overline{w}+w')(\overline{u}+u')\right)}\frac{\partial(\overline{u}\ \overline{w})}{\partial z} + \frac{\partial \overline{u'w'}}{\partial z}$$

such that

$$\overline{\nabla\cdot[(\overline{\mathbf{u}}+\mathbf{u}')\ (\overline{u}+u']} = \nabla\cdot(\overline{\mathbf{u}}\ u) + \nabla\cdot\overline{\mathbf{u}'u'}$$

We are now in a position to write the full form of the $u-$momentum RANS equation which becomes

$$\frac{\partial\overline{u}}{\partial t} + (\overline{\mathbf{u}}\cdot\nabla)\overline{u} = -\frac{1}{\rho}\frac{\partial\overline{P}}{\partial x} + \overline{\mathbf{g}} + \nu\nabla^2\overline{u} + \nabla\cdot\overline{\mathbf{u}'u'}$$

This results may be generalized in 3D to form the complete RANS equations

$$\frac{D\overline{\mathbf{u}}}{Dt} = \frac{\partial}{\partial t}\begin{pmatrix}\overline{u}\\ \overline{v}\\ \overline{w}\end{pmatrix} + (\overline{\mathbf{u}}\cdot\nabla)\begin{pmatrix}\overline{u}\\ \overline{v}\\ \overline{w}\end{pmatrix} = \frac{-\nabla\overline{P}}{\rho} + \overline{\mathbf{g}} + \nu\nabla^2\begin{pmatrix}\overline{u}\\ \overline{v}\\ \overline{w}\end{pmatrix} + \nabla\cdot\begin{pmatrix}[-\overline{u'u'} & -\overline{u'v'} & -\overline{u'w'}]\\ [-\overline{v'u'} & -\overline{v'v'} & -\overline{v'w'}]\\ [-\overline{w'u'} & -\overline{w'v'} & -\overline{w'w'}]\end{pmatrix} \quad (3)$$

The last term of this equations is the divergence of the Reynolds stress tensor. Written in this form, it illustrates that for each of the momentum equations, the divergence operator should be applied to the *row vector* of the corresponding momentum equation. For example if we were to write out the $v-$momentum, we must apply the divergence operator to the $2^{nd}$ row of the Reynolds stress tensor.

**Continuity**

Performing Reynolds averaging of continuity gives us $\overline{\nabla \cdot (\overline{\mathbf{u}} + \mathbf{u}')} = 0 \rightarrow$

$$\nabla \cdot \overline{\mathbf{u}} = 0, \qquad \nabla \cdot \mathbf{u}' = 0 \tag{4}$$

So we see that both the mean and fluctuation parts of velocity both satisfy continuity individually.

## RANS on tensor notation

The equations in turbulence are notoriously long, tedious and cumbersome to write out in full vector notation. Therefore it is an absolute must to utilize tensor notation (also called index notation). The important thing to remember is to know the Einstein summation convention that says that if a term consist of variables with repeated index, the term must be summed over that index. An example is continuity

$$\nabla \cdot \mathbf{u} = \frac{\partial u_i}{\partial x_i} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

Since the $i$ appears in both $u_i$ and $x_i$, we say that that index is repeated, and therefore the term must by summed over $i = 1, 2, 3$. If a term has both $i$ and $j$ repeated, the term must be summed over both $i$ and $j$ giving a total of 9 terms if $i, j = 1, 2, 3$. Using this convention, we may now write the RANS equations on tensor notation

$$\rho(\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j}) = -\frac{\partial \overline{P}}{\partial x_i} + \rho \overline{g}_i + \mu \frac{\partial^2 \overline{u}_i}{\partial {x_j}^2} - \frac{\partial}{\partial x_j} \rho \overline{u_i' u_j'} \tag{5}$$

Note that the Laplacian term $\frac{\partial^2 u_j}{\partial {x_i}^2} = \frac{\partial^2 u_j}{\partial x_i \partial x_i}$ must be summed over $i$ since the $i$ is repeated. Note also that since the Reynolds stress tensor is symmetric, we have a total of 6 additional unknowns. The question now is how to model the additional unknowns $-\frac{\partial}{\partial x_j}(\rho \overline{u_i' u_j'})$. If we assume 2D flow where $\overline{w} = \frac{\partial}{\partial z} = 0$ but $\overline{w'^2} \neq 0$, we are left with the additional terms $-\rho \overline{u'^2}$, $-\rho \overline{u'v'}$ and $-\rho \overline{v'^2}$

It is the aim of turbulence models to be able to relate the unknown components of the Reynolds stress tensor to the mean flow quantities and thus providing *closeure* to the equations. One line of modelling known as *eddy viscosity models* is to relate the Reynolds stress tensor components to a the mean rate of strain tensor through a turbulent (eddy) viscosity. This is known as the *Boussinesq approximation* for turbulence, and takes the form

$$-\overline{u_i' u_j'} = \nu_t \left( \frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} k \delta_{ij} \tag{6}$$

where $\delta_{ij}$ is the Kronecker delta function $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$. $k$ is the turbulent kinetic energy defined as $k = \dfrac{\overline{u_i u_i}}{2}$. A transport equation of $k$ can be derived by forming the dot product of $u_i$ with the RANS equations (3), but the result involves *many* terms.

### k - $\epsilon$ turbulence model

Now we need to model the turbulent viscosity $\nu_t$. The $k-\epsilon$ model, uses

$$\nu_t = \frac{C_\mu k^2}{\epsilon} \tag{7}$$

where $C_\mu$ is an empirical constant, and $\epsilon$ is the rate of turbulent dissipation. A transport equations for $\epsilon$ may be derived by applying the operator $\nu \dfrac{\partial u_i'}{\partial x_j}\dfrac{\partial}{\partial x_j}$ to the RANS equations (3), but again the result involves many complicated terms. Instead of incorporating all the complexities of the analytically derived $k$ and $\epsilon$ equations, the $k-\epsilon$ equation uses approximated versions of them involving only the most important terms:

$$\frac{Dk}{Dt} = \frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_j}\right) + \nu_t \frac{\partial \overline{u}_i}{\partial x_j}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) - \epsilon \tag{8}$$

$$\frac{D\epsilon}{Dt} = \frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_\epsilon}\frac{\partial \epsilon}{\partial x_j}\right) + C_1 \nu_t \frac{\epsilon}{k}\frac{\partial \overline{u}_i}{\partial x_j}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) - C_2 \frac{\epsilon^2}{k} \tag{9}$$

with empirically determined constants

$$C_{mu} = 0.09 \qquad C_1 = 1.44 \qquad C_2 = 1.92 \qquad \sigma_K = 1.0 \qquad \sigma_\epsilon = 1.3$$

These constants should ideally be tuned to the particular flow type at and. Let us now expand the $k$ equation to get a feeling of their complexity. Expanding the $k$ equation term for term gives:

1. $\dfrac{\partial}{\partial x_j}\left(\dfrac{\nu_t}{\sigma}\dfrac{\partial k}{\partial x_j}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\nu_t}{\sigma}\dfrac{\partial k}{\partial x}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{\nu_t}{\sigma}\dfrac{\partial k}{\partial y}\right) + \dfrac{\partial}{\partial z}\left(\dfrac{\nu_t}{\sigma}\dfrac{\partial k}{\partial x}\right) = \nabla \cdot \left(\dfrac{\nu_t}{\sigma}\nabla k\right)$
2. $\dfrac{\partial \overline{u}_i}{\partial x_j}\dfrac{\partial \overline{u}_i}{\partial x_j} = \dfrac{\partial \overline{u}}{\partial x_j}\dfrac{\partial \overline{u}}{\partial x_j} + \dfrac{\partial \overline{v}}{\partial x_j}\dfrac{\partial \overline{v}}{\partial x_j} + \dfrac{\partial \overline{w}}{\partial x_j}\dfrac{\partial \overline{w}}{\partial x_j} = [\nabla\overline{u}\cdot\nabla\overline{u}] + [\nabla\overline{v}\cdot\nabla\overline{v}] + [\nabla\overline{w}\cdot\nabla\overline{w}]$
3. $\dfrac{\partial \overline{u}_i}{\partial x_j}\dfrac{\partial \overline{u}_j}{\partial x_i} = \dfrac{\partial \overline{u}}{\partial x_j}\dfrac{\partial \overline{u}_j}{\partial x} + \dfrac{\partial \overline{v}}{\partial x_j}\dfrac{\partial \overline{u}_j}{\partial y} + \dfrac{\partial \overline{w}}{\partial x_j}\dfrac{\partial \overline{u}_j}{\partial z} = [\nabla\overline{u}\cdot\dfrac{\partial \overline{\mathbf{u}}}{\partial x}] + [\nabla\overline{v}\cdot\dfrac{\partial \overline{\mathbf{u}}}{\partial y}] + [\nabla\overline{w}\cdot\dfrac{\partial \overline{\mathbf{u}}}{\partial z}]$

In 3 dimensions (1) consists of 3 terms, (2) consists of 9 terms, and (3) consists of 9 terms, so the RHS of equations (8) consists of 22 terms including the additional $\epsilon$ term. In 2D this is reduced to a total of 10 terms. We also see that the $\epsilon$ equation contains exactly the same tensor terms.

### Boundary treatment

The $k-\epsilon$ equations written above are only valid for fully turbulent high Reynolds number flow, and are thus not valid near the wall. Therefore these equations are not integrated all the way to the walls. Instead we use one of two options: 1) patch a turbulent wall-law (a logarithmic velocity profile) onto the solution near the walls, and 2) add *damping functions* to the equations

**wall law treatment**

let $\overline{u}_p$ be a velocity component at a node $p$ closest to the wall. We then impose that the velocity at this node should follow the logarithmic wall law according to

$$\frac{\overline{u}_p}{v^*} = \frac{1}{\kappa} ln\left(\frac{v^* y_p}{\nu}\right) + B \tag{10}$$

$$K_p = \frac{{v^*}^2}{\sqrt{C_\mu}} \tag{11}$$

$$\epsilon_p = \frac{{v^*}^3}{\kappa y_p} \tag{12}$$

where $v^*$ is the wall friction velocity $v^* = \left(\frac{\tau_w}{\rho}\right)^{1/2}$, $\kappa \approx 0.40$ is the Von Karman constant

# Implementation of the k - $\epsilon$ model in openFoam

## Initial and Boundary Condition

To set initial and boundary data for turbulence we need to specify the *turbulence intensity* $I$ which is the ratio of the magnitude of the turbulent fluctuations to the magnitude of the characteristic mean velocity.

$$I = \frac{\sqrt{\frac{1}{3}(\overline{u'^2} + \overline{v'^2} + \overline{w'^2})}}{U_{ref}} = \frac{\sqrt{\frac{2}{3}k}}{U_{ref}} \rightarrow k = \frac{3}{2}(U_{ref} I)^2$$

$I$ is a dimensionless quantity and its value typically has a value ranging from $[0, 20]$. $I = [0, 1]$ is considered low intensity, $[1, 5]$ is medium intensity, and $[5, 20]$ is considered high turbulent intensity. Note also that if one has access to fluid dynamical experimental equipments, one may measure the turbulent fluctuations quite accurately. For fully developed pipe flow we may use the relation

$$I = 0.16 Re^{-\frac{1}{8}}$$

As boundary condition on $\epsilon$ we may use

$$\epsilon = \frac{0.164 k^{1.5}}{0.07 L}$$