k - ϵ turbulence model

Turbulent kinetic energy and dissipation rate

- Turbulent Kinetic Energy (scalar) $k = \frac{1}{2}\overline{u_i' u_i'} = \frac{1}{2}(\overline{\mathbf{u'} \cdot \mathbf{u'}}) = \frac{1}{2}\overline{(u'^2 + v'^2 + w'^2)}$ in units $\frac{m^2}{s^2} = \frac{J}{kg}$
- Turbulent Dissipation Rate (vector) $\epsilon = \nu \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_j}}$ in units $\frac{m^2}{s^3} = \frac{J/kg}{s}$ where

$$\epsilon_x = \nu \overline{\left[\left(\frac{\partial u'}{\partial x} \right)^2 + \left(\frac{\partial u'}{\partial y} \right)^2 + \left(\frac{\partial u'}{\partial z} \right)^2 \right]}$$

$$\epsilon_y = \nu \overline{\left[\left(\frac{\partial v'}{\partial x} \right)^2 + \left(\frac{\partial v'}{\partial y} \right)^2 + \left(\frac{\partial v'}{\partial z} \right)^2 \right]}$$

$$\epsilon_z = \nu \overline{\left[\left(\frac{\partial w'}{\partial x} \right)^2 + \left(\frac{\partial w'}{\partial y} \right)^2 + \left(\frac{\partial w'}{\partial z} \right)^2 \right]}$$

$$\epsilon = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \end{bmatrix}$$

$$k = \frac{3}{2}(UI)^2$$

where U is a characteristic velocity scale, and I is the *turbulent intensity* given as

$$I = \frac{\mathbf{u'}}{\overline{\mathbf{u}}} = \frac{\sqrt{\overline{u'^2} + \overline{v'^2} + \overline{w'^2}}}{\sqrt{\overline{u}^2 + \overline{v}^2 + \overline{w}^2}} = \frac{\sqrt{\overline{u'^2} + \overline{v'^2} + \overline{w'^2}}}{U_\infty}$$

Note that I is a dimensionless quantity. I typically has a value ranging from $[0, 20]\%$, and $[0, 1]\%$ is considered low intensity, $[1, 5]\%$ is medium intensity, and $[5, 20]\%$ is considered high turbulent intensity. Note also that if one has access to fluid dynamical experimental equipments, one may measure the turbulent fluctuations quite accurately.
For fully developed pipe flow we may use the relation

$$I = 0.16 Re^{-\frac{1}{8}}$$

If we assume *isotropic* turbulence $\overline{u'^2} = \overline{v'^2} = \overline{w'^2}$ we have

$$k = \frac{1}{2}\overline{(u'^2 + v'^2 + w'^2)} = \frac{3}{2}\overline{u'^2} \to u' = \sqrt{\frac{2}{3}k}$$

Isotropic turbulence assumption is valid in core regions far from walls where wall effects are negligible. This gives

$$k = \frac{3}{2}(U_\infty I)^2$$

For $Re = 10000$:

$$I = 0.16 \cdot 10000^{-\frac{1}{8}} = 0.0506$$

k

$$k = \frac{3}{2}(U_\infty I)^2 = \frac{3}{2}(1m/s \cdot 0.0506)^2 = 0.00384$$

epsilon

ϵ

$$\epsilon = \frac{0.164 \cdot k^{1.5}}{0.07L} = \frac{0.164 \cdot \left(0.00384\frac{m^2}{s^2}\right)^{1.5}}{0.07 \cdot 1m} = 5.57 \cdot 10^{-4}\frac{m^2}{s^3}$$

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