Markowitz Optimization

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1 Introduction

Markowitz Optimization to investors it mostly known as Market Portfolio a term from portfolio theory that refers to the whole set of investible risky assets, like stocks, bonds, real estate, collectibles and human capital. People often use an index of stocks as a substitute. Portfolio weighs are based on market value, or capitalization.
By choosing securities that do not move exactly together, the Markowitz model shows investors how to reduce their risk. The Markowitz model is also called mean-variance model due to the fact that it is based on expected returns (mean) and the standard deviation (variance) of the various portfolios.
The code for this paper is posted on the following can be accessed through this link:
`PorfolioOptimizationUsingMarkowitzOptimization`

1.1 Investopedia Definition:

Modern portfolio theory (MPT) is a theory on how risk-averse investors can construct portfolios to maximize expected return based on a given level of market risk.

2 Assumptions to the Markowitz's Modern Portfolio Theory (MPT):

2.1 Investors:

1. Are interested in maximizing mean return for a portfolio at a given variance
2. Aim to maximize economic utility
3. Are rational and risk-averse
4. Have access to the same information at the same time
5. Are price takers
6. Can divide weights infinitely small position sizes
7. Asset returns are jointly normally distributed random variables
8. Asset correlations are fixed and constant
9. Expectations match the asset's distribution of returns
10. There are no taxes or transactions costs
11. Asset risk is known in advance and is constant
12. Market weights are based on market value

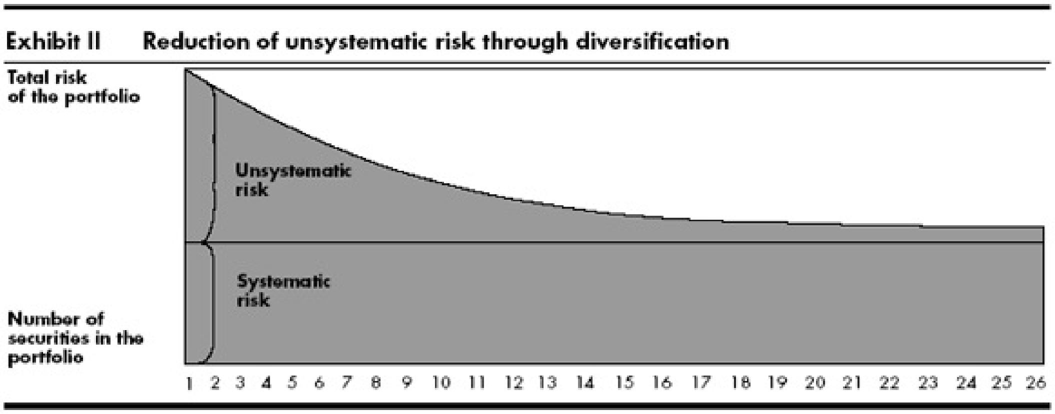

Figure 1: The Risk against number of stocks in the Portfolio

2.2 Estimation - Individual Assets

$$Expected\ Return = \Sigma R_i * P_i$$

$$Variance\ \sigma^2 = \frac{\Sigma(X_i - \mu)^2}{N}$$

2.3 Estimation - Portfolio

$$R_P = I_{RF} + (R_M - I_{RF})\frac{\sigma_P}{\sigma_M}$$

$\mathrm{R}_P = expected\ return\ of\ portfolio$
$R_M = return\ on\ the\ market\ portfolio$
$I_{RF} = risk - free\ rate\ of\ interest$
$\sigma_M = standard\ deviation\ of\ the\ market\ portfolio$
$\sigma_P = standard\ deviation\ of\ portfolio$

3 Diversification

3.1 Types of Risks

- Unsystematic risk

  Is the risk that is inherent in a specific company or industry.

- Systematic risk

  Refers to the risk inherent to the entire market or market segment

3.2 How to Diversify?

Diversification is done through Covariance and Correlation

Covariance

$$\widehat{\sigma_{i,j}} = \widehat{Cov_{i,j}} = \frac{\sum_{t=1}^{T}(r_{it} - \widehat{r_i})(r_{jt} - \widehat{r_j})}{N-1}$$

Correlation

$$\widehat{\rho_{i,j}} = \widehat{Corr}_{i,j} = \frac{\widehat{\sigma_{i,j}}}{\widehat{\sigma_i}\widehat{\sigma_j}}$$

$\widehat{\sigma_{i,j}} = Covariance\ between\ two\ assets$
$\widehat{\rho_{i,j}} = Correlation\ between\ two\ assets$
$r_{it} = return\ for\ each\ timestamp\ t\ for\ stocki$
$\widehat{r_i} = mean\ return\ for\ each\ stock\ i$
$r_{jt} = return\ for\ each\ timestamp\ t\ for\ stock\ j$
$\widehat{r_j} = mean\ return\ for\ each\ stock\ j$

3.3 Optimization Problem

Given a number of assets with estimated returns, variance and covariance.
In my portfolio, I want to choose my weightings in each asset as to maximise my return, while minimising my variance.
Solve the quadratic programming problem using Lagrange Multipliers.

Portfolio Variance

$$min\ \frac{1}{2} w^T \Sigma w$$

[Jan20]

Portfolio Return above μ_b

$$s.t.\ m^T\ w\ \geq\ \mu_b$$
$$e^T w\ =\ 1$$

w = vector of portfolio weightings
$\Sigma =\ covariance\ matrix$
$m = vector\ asset\ returns$
$e\ = vectors\ of\ 1s$
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3.4 Implementation

Building our stocks Universe

The calculations were performed on stocks data of stocks data of companies listed on US stock exchanges atleast before $1^{st}\ jan\ 2000\ and\ were\ among\ the\ top\ 100\ companies\ by\ Market\ Capitalization.$ $42\ Companies\ were\ filtered\ through\ this\ criteria.$

Building Portfolios from these stocks

For optimization purpose only 2 stocks portfolio were built. As for addition of each stock in our Stocks Universe the computation power required rises by power of 1. For our 42 stock universe the computation power required will be $O(2^{41})$. The time complexity for algorithm is $O(2^{N-1})$.
N = Number of stocks in the universe

3.5 Sharpe Ratio

The Sharpe ratio compares the return of an investment with its risk. It's a mathematical expression of the insight that excess returns over a period of time may signify more volatility and risk

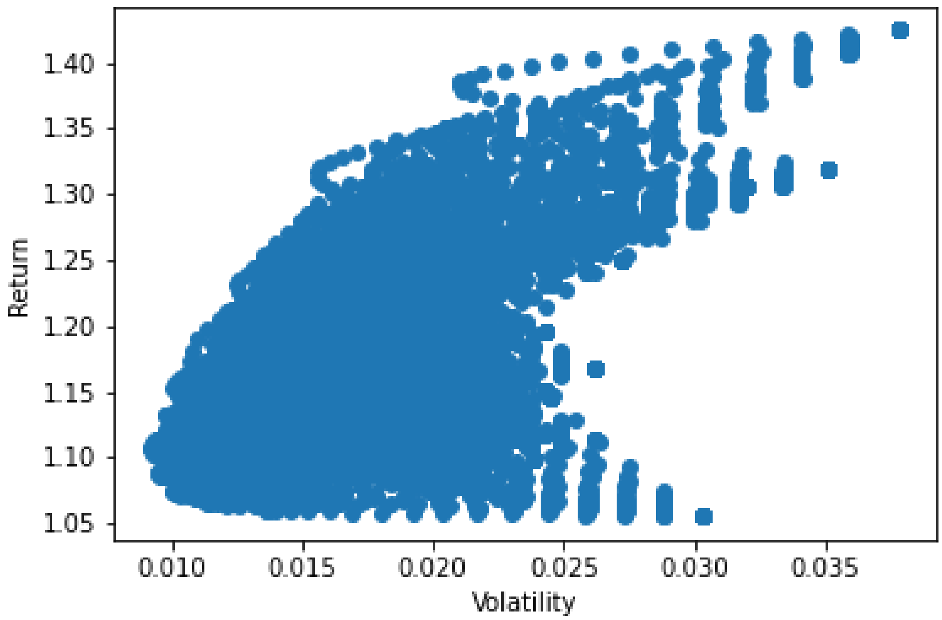

Figure 2: All Portfolios Risk-Reward plot

Using Sharpe Ratio

$$SharpeRatio = \frac{R_p - R_f}{\sigma_p}$$

$\mathrm{R}_p = Return\ of\ Portfolio$
$P_f = Risk - free\ rate$
$\sigma_p = standard\ deviation\ of\ the\ portfolios\ excess\ return$

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we selected highest returns For each risk level and By using Sharpe ratio we selected the portfolios having greater the average Sharpe Ratios. After it 98 portfolios were left.
Figure 3 Represents there risk-reward ratios

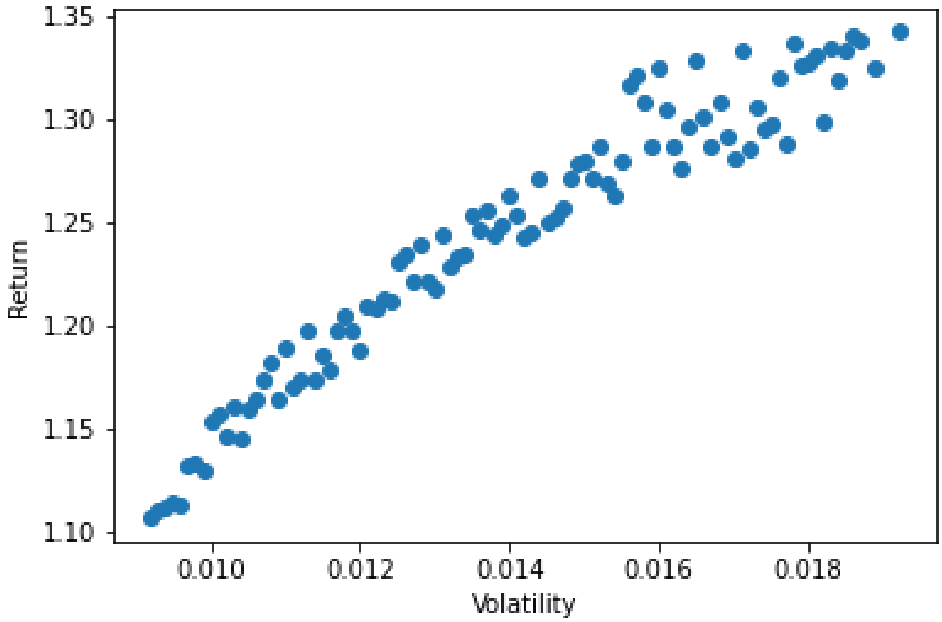

Figure 3: The portfolios having greater than average Sharpe Ratios

References

[Jan20] S. Jansen. *Machine Learning for Algorithmic Trading: Predictive Models to Extract Signals from Market and Alternative Data for Systematic Trading Strategies with Python.* Packt Publishing, 2020.