Assignment 1 Solutions

1. Prove with induction that $\forall n > 0 \forall w \in \{a, b\}^\star : n_a(w^n) = n \cdot n_a(w)$.

   Proof. Note that the recursive definition of w^n is as follows: $w^0 = \epsilon$ and $w^{n+1} = w \cdot w^n$.

   Prove by induction on n. Base case: $n = 0$. Then $w^n = w^0 = \epsilon$ (recursive definition base case) and $n_a(w) = n_a(\epsilon) = 0$. So $n_a(w^n) = 0 = 0 \cdot n_a(w)$, as needed.

   Inductive hypothesis: assume there exists a $k \geq 0$ such that $n_a(w^k) = k \cdot n_a(w)$.

   Now prove for $n = k + 1$ that $n_a(w^{k+1}) = (k + 1) \cdot n_a(w)$. The recursive definition (recursive case) tells us $w^{k+1} = w \cdot w^k$, so $n_a(w^{k+1}) = n_a(w \cdot w^k) = n_a(w) + n_a(w^k)$. Inductive case tells us that $n_a(w^k) = k \cdot n_a(w)$ so $n_a(w) + n_a(w^k) = n_a(w) + k \cdot n_a(w) = (k + 1) \cdot n_a(w)$ as needed. $\square$

2. Prove that for any language $L \subseteq \Sigma^\star$ that $(L^r)^r = L$. Hint: recall languages are sets, so this question relates to set equality.

   Proof. First, note that $|L^r| \leq |L|$. (In fact they are equal but that would require a proof that $w \neq y \Rightarrow w^r \neq y^r$ to show, whereas the $\leq$ relationship follows immediately from the definition of $L^r = \{w^r : w \in L\}$.) This means further that $|(L^r)^r| \leq |L^r|$ by applying this again and thus that $|(L^r)^r| \leq |L|$.

   Thus, we need only show that $\forall x \in L : x \in (L^r)^r$ to prove equality. Let $x \in L$ be an arbitrary element. By definition of L^r, $x \in L \Rightarrow x^r \in L^r$. If we make $x' = x^r$ and $L' = L^r$ it is by definition that $x' \in L' \Rightarrow (x')^r \in (L')^r$, which is the same as $(x^r)^r \in (L^r)^r$, as needed. (Note this last step with the prime is only to help show how the defintion applies again, it would not be necessary for the proof.) $\square$

3. Prove $(\forall i \in \mathbb{N} : w^i = \epsilon) \Leftrightarrow w = \epsilon$.

   Proof. First prove $(\forall i \geq 0 : w^i = \epsilon) \Rightarrow w = \epsilon$. This is vacuously true: if $w^i = \epsilon$ and 1 is a valid choice for i then $w = w^1 = \epsilon$.

   Now prove $w = \epsilon \Rightarrow \forall i \geq 0 : w^i = \epsilon$. Base cases: $i = 0$ then $w^0 = \epsilon$ by definition. $i = 1$ then $w = \epsilon$ by assumption. Inductive hypothesis: assume that there exists an n such that $w^n = \epsilon$. Inductive step: prove that $w^{n+1} = \epsilon$. By definition $w^{n+1} = ww^n = \epsilon w^n = \epsilon\epsilon = \epsilon$ as needed. $\square$

4. Let L, L' languages over $\Sigma^\star$ with $0 < |L| \leq |L'| < n$ for some $n \in \mathbb{N}$. Prove formally using the definition of concatenation that $|L'| \leq |L \cdot L'| \leq |L| \cdot |L'|$. You may use the following lemma without proof: if w, x, y are words and $x \neq y$ then $wx \neq wy$.

Proof. First let's do the upper bound. By definition of concatenation, $L \cdot L' = \{x \cdot y : x \in L \wedge y \in L'\}$. This means that if $w \in L \cdot L'$ then $w = xy$ with $x \in L$ and $y \in L'$. There are then $|L|$ choices for x and $|L'|$ choices for y, yielding a total of $|L| \cdot |L'|$ possible choices for w, giving the upper bound on the set.

Now let's do the lower bound. We know that both sets are finite and non-empty, so let $w \in L$ be an arbitrary word in L. By definition of concatenation, we have that $\{w\} \cdot L' \subseteq L \cdot L'$. For any two $x, y \in L$ with $x \neq y$, we have that $wx \neq wy$, and so $|\,\{w\} \cdot L'| = |\,\{wx : x \in L'\}\,| = |L'|$. Therefore, there are at least $|L'|$ unique elements in $L \cdot L'$.

□

5. Prove that $\forall L \subseteq \{0\}^\star \Rightarrow L = L^r$.

Proof. First we prove a lemma: if $w \in \{0\}^\star \Rightarrow w = w^r$. If $w \in \{0\}$ then $w^r \in \{0\}$ as well. Let $w = 0^i$ and $w^r = 0^j$ be the general form of words in the unary alphabet. But $|w| = |w^r| \Rightarrow i = j$. Thus $0^i = 0^j \Rightarrow w = w^r$.

Now consider $L^r = \{w^r : w \in L\}$.

But $w = w^r$ so $\{w^r : w \in L\} = \{w : w \in L\} = L$.

□