

cpsc 313 assignment 3

Ali Akbari

TOTAL POINTS

36 / 50

QUESTION 1

1 question 1 7 / 10

- 0 pts Correct
- 0 pts minor issues
- 1 pts minor step missing
- 1 pts Issues in the \cup / δ^* notation

- 2 pts Not using δ^* (NFAs may process a word in multiple paths)

- 2 pts Sloppiness in the proof
- 3 pts Major Sloppiness
- 3 pts The proof is specific to special cases.

✓ - 3 pts Assuming the shortest word = n

- 6 pts Major errors/Missing too much steps/Incomplete
- 7 pts Something is correct
- 8 pts provided only an example, not a proof.
- 10 pts Incorrect/No answer

- 1 no
- 2 how do you know??
- 3 you are making huge logical jumps. and assuming your claim is true without providing proof
- 4 at most

QUESTION 2

2 question 2 10 / 10

- ✓ - 0 pts Correct
- 1 pts minor sloppiness
- 2 pts Sloppiness in explanation
- 3 pts Machine design mistakes
- 3 pts Explanation mistakes/insufficient
- 4 pts Completely wrong NFA design
- 4 pts Completely wrong/no Explanation

- 7 pts Mostly incorrect
- 10 pts Incorrect/No answer

QUESTION 3

3 question 3 6 / 10

- 0 pts Correct
- 0 pts Using relevant mathematical symbols to write a proof makes the idea clear and rules out ambiguities. For example, $\delta(q_0, 0) = q_1$ means that when the DFA in state q_1 reads a '0' it transitions to state q_1 . If you wish to write the same thing in English, you have to make sure that your sentence is grammatically correct, makes sense, contains all the information same as the mathematical language, and that there are no ambiguities. This is why, using formal language is more efficient.
- ✓ - 1 pts incorrect inductive hypothesis. There exists a positive integer n such that for all $0 \leq k \leq n$, $\delta^*(q_0, 0^k) = q_1$

- 3 pts Part 1: If $w \in L$, then $w \in L(M)$. You have to prove by induction. Refer to the comments
- 3 pts Errors in part 2 of the proof. See comments
- 3 pts Errors in proof of part 1. See comments.
- ✓ - 2 pts Errors in part 2. See comments.
- 2 pts Errors in proof in part 1. See comments.
- ✓ - 1 pts Errors in part 1. See comments
- 1 pts Errors in proof of part 2. See comments
- 0.5 pts Minor errors in part 1 or part 2. See comments.
- 3 pts Major errors in proof of part 1.
- 3 pts Proof for part 2 has major errors.
- 2 pts Error in Base case of induction.
- 2 pts Proof has crucial components but is missing a formal structure missing in the proof. See comments.
- 8 pts Major errors / hand wavy arguments

- **1 pts** Sloppy and hard to read submission
- **3 pts** Errors all over the solution. See comments.
- **10 pts** No solution / completely incorrect solution
- **2 pts** Inductive step incorrect.
- **0.5 pts** Strainful/hard to read submission.
- **9 pts** Your solution is incorrect.
- **5 pts** Proof for part 1 (if $w \in L$, w is accepted) is missing.
- **0 pts** Click here to replace this description.

- 5** L is a set.
- 6** Base case just proves that 0 is accepted by M
- 7** Incorrect use of inductive hypothesis. See solution given.
- 8** please use non-italic font. It is a norm to use italics for statement of Lemma and not the proof itself.
- 9** very sloppy statement. I guess you want to say, $w \in L(M)$ and $w \in L$ complement.
- 10** incoherent proof. How does this pose a contradiction? Your initial assumption is not clear. You can't choose a particular string, say 0, to show a contradiction. The contradiction must hold for every possible string that can be accepted by M.

QUESTION 4

4 question 4 7 / 10

- **0 pts** Correct
- **0 pts** Start/Finish states not labelled
- **1 pts** Minor errors
- **3 pts** Minimizing NFA as a DFA
- **3 pts** Missing proof
- **4 pts** Major Errors
- **3 pts** Jumping to DFA without proof
- ✓ - **3 pts** Did not minimize DFA
- **9 pts** Incorrect
- **10 pts** Missing Solution
- 11** this is not a DFA since you have a state that has two different outputs on the same symbol
- 12** where is the proof that this is minimized?

QUESTION 5

5 question 5 6 / 10

- **0 pts** Correct
- **0 pts** Use the correct notation. ϵ is not a replacement for ϵ
- ✓ - **0 pts** First half is correct
- **1 pts** Minor errors
- **2 pts** Errors
- **3 pts** Not showing the needed contradiction
- ✓ - **4 pts** Major errors
- **5 pts** Proving set equality is a two-way proof. Half of the proof is missing.
- **5 pts** Not a proof.
- **5 pts** Begging the question
- **7 pts** Mostly Incorrect
- **8 pts** Incorrect
- **10 pts** Missing
- **0 pts** check
- 13** $\{0, 1\}^*$ is a subset of all of the states as indicated in the question.
- 14** this direction is for the first half of the proof
- 15** Good. This is the easy direction
- 16** This part is incorrect. If $\delta(q, a)$ returns a set as it does two lines up, then it can't be equal to a state
- 17** not $\{0, 1\}^*$

- 1) Let M be an NFA with n states. Show that if $|L(M)| > 1$ then $\exists w \in L(M)$ with $|w| < n$.
 You may use without proof the fact that $\delta^*(q_0, xy) = \bigcup_{p \in \delta^*(q_0, x)} \delta^*(p, y)$.

From Theorem 1 of the pumping lemma notes in page 1 of pumping-ln.pdf.

The following can be proved by contradiction :

Assume that the NFA does not accept words with a cardinality of less than n ,

let w be the smallest word such that $\forall w \in L(M)$ with $|w| \geq n$.

Intuitively if an NFA has n number of states, the shortest path to visit all states is with $n - 1$ number of transitions.

The number of states that are encountered as w is processed is $|w| \geq n$.

Because there are n total states there must be that one or more states that are visited are visited multiple times (by the pigeonhole principle). This means that there is atleast one cycle. So if we remove the cycles then we get a shorter path, which means a shorter string that will be accepted. So then if we repeatedly remove the cycles we will have $|w|$ 1 n , however since there is no transition to input for the initial state the shortest path to reach all n states it must be $n -$ 2 we get a shorter path and a shorter string that can be accepted. So, $|w|$ 4 $n - 1$. This poses a contradiction to our assumption that the NFA $\forall w \in L(M)$ with $|w| \geq n$ as $|w| = n - 1$ which is $|w| < n$. Thus, we have 3 proven by contradiction that $\exists w \in L(M)$ with $|w| < n$ is true.

1 question 1 7 / 10

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- 0 pts minor issues
- 1 pts minor step missing
- 1 pts Issues in the $\cup \delta^*$ notation
- 2 pts Not using δ^* (NFAs may process a word in multiple paths)
- 2 pts Sloppiness in the proof
- 3 pts Major Sloppiness
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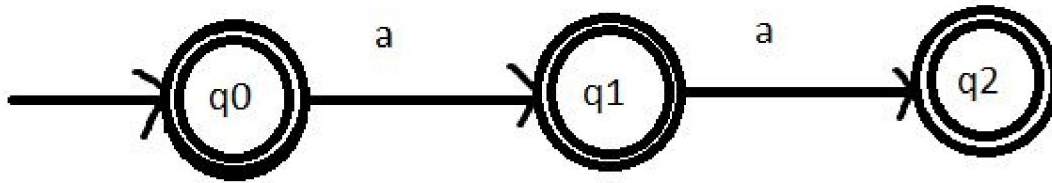
- 2) Let M be an NFA with n states. Show that if $|\overline{L(M)}| > 1$ then there does not necessarily $\exists w \in \overline{L(M)}$ with $|w| < n$.

Consider the following alphabets :

$$\Sigma = \{a\}$$

number of states, $n = 3$

Consider the following NFA denoted as M , it is non deterministic as state q_2 has no transitions defined for the alphabet a .



The following NFA, has a language L , so $L(M) = \{e, a, aa\}$.

Then the complement of the language L is $\overline{L} = \{w \in \Sigma^* : n_a(w) \geq 3\}$.

The language \overline{L} which is also $\overline{L(M)}$, based on the above definition of $\overline{L(M)}$,

has a cardinality $|\overline{L(M)}| \geq 3$, which is greater than 1. This states that the complement language has at least 3 words, $\{a^i : i \geq 3\}$. So the smallest word in the language

\overline{L} is $w = aaa$ which has a cardinality of $|aaa| = 3 = n$, since there are 3 states in the NFA.

Thus, this shows by given example that if $|\overline{L(M)}| > 1$, then $\exists w \in \overline{L(M)}$

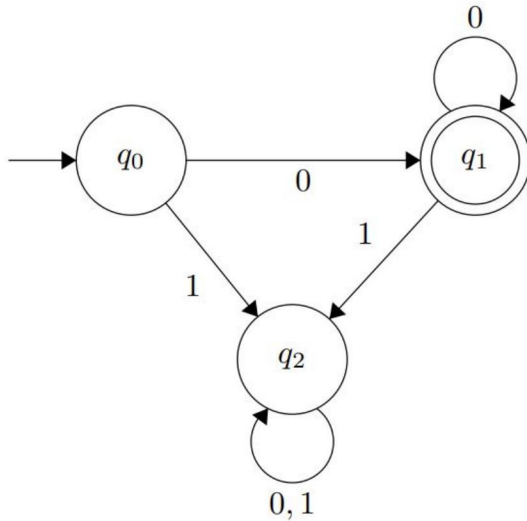
with $|w| \geq n$ holds true.

2 question 2 10 / 10

✓ - 0 pts Correct

- 1 pts minor sloppiness
- 2 pts Sloppiness in explanation
- 3 pts Machine design mistakes
- 3 pts Explanation mistakes/insufficient
- 4 pts Completely wrong NFA design
- 4 pts Completely wrong/no Explanation
- 7 pts Mostly incorrect
- 10 pts Incorrect/No answer

- 3) Prove that the following machine M that accepts the language $L = \{0^i : i > 0\}$. Prove both directions: that all words in L are accepted, and that anything that is accepted is in L . The alphabet $\Sigma = \{0, 1\}$. One way to prove is that L is accepted and \bar{L} is rejected. If you go that way you may use without proof the fact that $\bar{L} = \{\epsilon\} \cup \{w \in \Sigma^* : n_1(w) > 0\}$ and that $\delta^*(q_2, x) = q_2$ without proof.



$$M = (Q, \Sigma, \delta, q_0, A)$$

$$Q = 3$$

$$\Sigma = \{0, 1\}$$

$$q_0 = q_0$$

$$A = \{q_1\}$$

Lemma #1 :

Prove by induction :

That if $L = \{0^i : i > 0\}$, then all words in L are accepted in the above machine M .

Base case :

Let $i = 1$.

Then $L = 0$, then for the machine M , when at q_0 reading a 0 we go to state q_1 ,

$$\delta(q_0, 0) = q_1.$$

Then it holds for the base case that the $L = \{0^1\}$ is accepted in the above machine M .

Induction Step :

IH : Assume for $i > 0$, where $L = \{0^i\}$, then all words in L are accepted by the above machine M .

Prove this holds true for $L = \{0^{i+1} : i > 0\}$.

$$L = 0^{i+1}$$

$$L = 0^i * 0^1$$

$$\delta(q_0, 0) = q_1$$

$$\delta(q_1, 0) = q_1$$

L is a unary language so means that reading 0^1 first or 0^i does not matter since both lead to the same language. Since A only has one element q_1 the only accepting state is q_1 .

Since the base case has been proven to be true using the following transition $\delta(q_0, 0) = q_1$, then 0^1 holds true. Then by using our inductive hypothesis we can also show that we will be in an accepting state after reading 0^i since $\delta(q_1, 0) = q_1$. Also, since the language only contains 0s and no 1s based on the transitions states it only leads to q_1 , thus a stream of 0s is always accepting.

So if (0^1) leads to an accepting state as seen in the base case, and if (0^i) leads to an accepting state as seen in the inductive hypothesis. 0^i leads to q_1 and from the above transition states we know reading any number of 0s at q_1 also lead back to q_1 so it holds for 0^{i+1} .

Then it holds for the induction step that the $L = \{0^i : i > 0\}$ is accepted in the above machine M .

So, in conclusion if a language $L = \{0^i : i > 0\}$, then all words in L are accepted in the above machine M .

Lemma #2 :

Prove by contradiction that anything that is accepted by the machine M is in L .

$$\Sigma = \{0, 1\}$$

$$U = \Sigma^*$$

$$\bar{L} = U/L$$

$$\bar{L} = \{\epsilon\} \cup \{w \in \Sigma^* : n_1(w) > 0\}$$

Suppose that anything that is accepted by the machine M is in the language \bar{L} defined above.

The machine M accepts the word $w = 0$, as shown by the transition state $\delta(q_0, 0) = q_1$.

However based on the definition of \bar{L} the number of 1s in word of the language $\bar{L} > 0$.

This poses a contradiction as the word $w = 0$ does not include any 1s.

Also since in the given fact it shows that q_2 is trap based on the extended transition state, to get to q_2 a single 1 must be read which is part of the definition of the \bar{L} . Thus when a single 1 is read it leads to q_2 the error state and any subsequent string that follows still stays in q_2 which is not accepting. Therefore by contradiction we proved that anything that is accepted by the machine M is in L .

Then using lemma # 1 and lemma # 2 we can conclude the proof that all words in L are accepted by the machine M and that anything that is accepted by M it is in L .

3 question 3 6 / 10

- 0 pts Correct

- 0 pts Using relevant mathematical symbols to write a proof makes the idea clear and rules out ambiguities.

For example, $\delta(q_0, 0) = q_1$ means that when the DFA in state q_1 reads a '0' it transitions to state q_1 . If you wish to write the same thing in English, you have to make sure that your sentence is grammatically correct, makes sense, contains all the information same as the mathematical language, and that there are no ambiguities. This is why, using formal language is more efficient.

✓ - 1 pts **incorrect inductive hypothesis. There exists a positive integer n such that for all $0 \leq k \leq n$, $\delta^*(q_0, 0^k) = q_1$**

- 3 pts Part 1: If $w \in L$, then $w \in L(M)$. You have to prove by induction. Refer to the comments

- 3 pts Errors in part 2 of the proof. See comments

- 3 pts Errors in proof of part 1. See comments.

✓ - 2 pts **Errors in part 2. See comments.**

- 2 pts Errors in proof in part 1. See comments.

✓ - 1 pts **Errors in part 1. See comments**

- 1 pts Errors in proof of part 2. See comments

- 0.5 pts Minor errors in part 1 or part 2. See comments.

- 3 pts Major errors in proof of part 1.

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- 9 pts Your solution is incorrect.

- 5 pts Proof for part 1 (if $w \in L$, w is accepted) is missing.

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10 incoherent proof. How does this pose a contradiction? Your initial assumption is not clear. You can't choose a particular string, say 0, to show a contradiction. The contradiction must hold for every possible string that can be accepted by M.

- 4) Prove that no DFA with three states can accept the language
 $L = \{0^i : i > 0\} \cup \{1^i : i > 0\}$. Hint: there are a variety of ways you can prove this, but it is insufficient to simply show a X-state machine (for some $X > 3$) that is claimed to accept the language. You may apply any construction or algorithm we've covered in class and refer to its proof of correctness as part of this question. Clearly state the construction you are using and show the input and the output. You may also use the machine you provide in the previous question for $\{0^i : i > 0\}$ and without loss of generality $\{1^i : i > 0\}$ without reproving that it accepts the language.

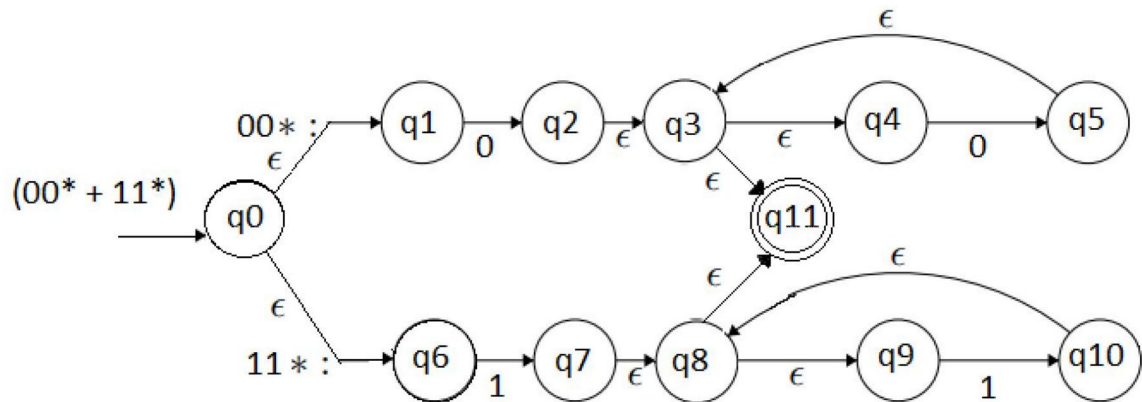
In this question I used the lecture notes and the following sites to make the NFA and the last DFAs:

<http://madebyevan.com/fsm/>

<https://www.youtube.com/watch?v=WSGcmaHNBfM&t=440s>

The regular expression for the language L is $R = 00^* + 11^*$, from this we can use theorem 1 of Kleene's Theorem to get an ϵ -NFA. Used page #3 in kleene - In.pdf notes.

The resulting ϵ -NFA is the following :



* implies accept state

NFA Transition Table :

	0	1
q0	{q2,q3,q4,q11}	{q7,q8,q9,q11}
q1	{q2,q3,q4,q11}	{}
q2	{q3,q4,q5,q11}	{}
q3	{q3,q4,q5,q11}	{}

q4	{q3,q4,q5,q11}	{}
q5	{q3,q4,q5,q11}	{}
q6	{}	{q7,q8,q9,q11}
q7	{}	{q8,q9,q10,q11}
q8	{}	{q8,q9,q10,q11}
q9	{}	{q8,q9,q10,q11}
q10	{}	{q8,q9,q10,q11}
q11*	{}	{}

By sub – set construction we get the following table :

	0	1
q0	{q2,q3,q4,q11}*	{q7,q8,q9,q11}*
q1	{q2,q3,q4,q11}*	{}
{q2,q3,q4,q11}*	{q3,q4,q5,q11}*	{}
{q7,q8,q9,q11}*	{}	{q8,q9,q10,q11}*
{q3,q4,q5,q11}*	{q3,q4,q5,q11}*	{}
{q8,q9,q10,q11}*	{}	{q8,q9,q10,q11}*
{}	{}	{}

Let the following :

q0 = q0

q1 = q1

q2* = {q2,q3,q4,q11}

q3* = {q7,q8,q9,q11}

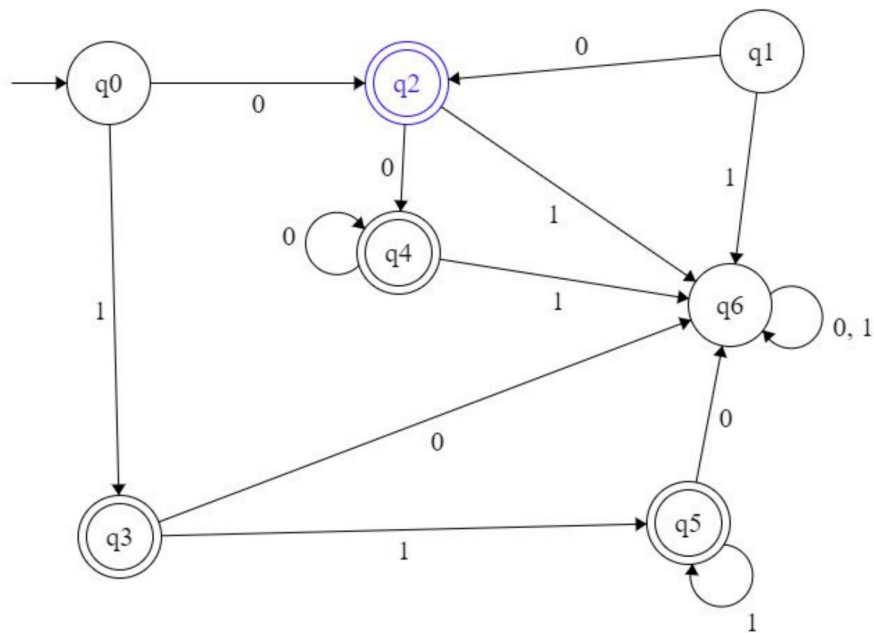
q4* = {q3,q4,q5,q11}

q5* = {q8,q9,q10,q11}

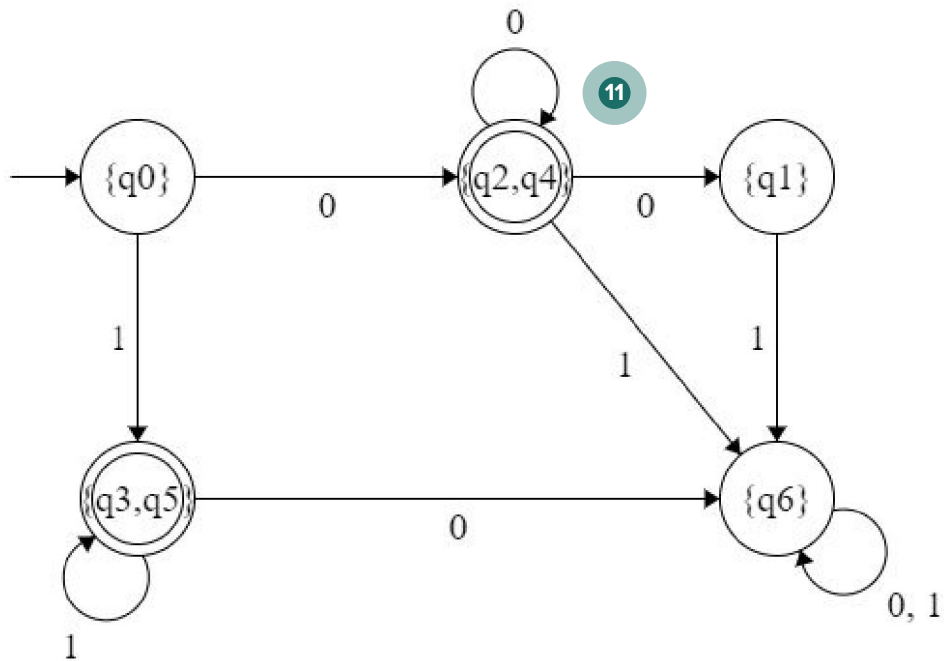
q6 = {} - Fail state

	0	1
q0	q2*	q3*
q1	q2*	q6
q2*	q4	q6

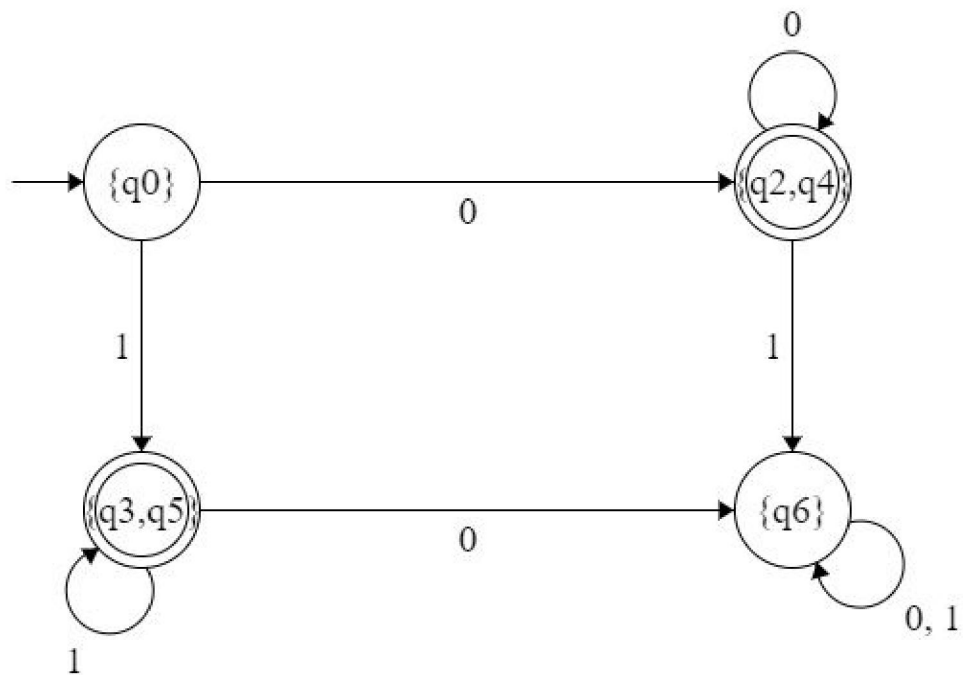
q3*	q6	q5
q4	q4	q6
q5	q6	q5
q6	q6	q6



Color coated states have the same transition so when minimized we get the following DFA :



$q1$ is a dead state that has no incoming transitions, so can be further minimized as :



We see the minimized DFA has 4 states so the following construction shows that the language $L = \{0^i : i > 0\} \cup 1^i : i > 0$ after minimization can only be built with at least 4 states, it is not possible for 3 states to accept the language, thus this is proved false.

4 question 4 7 / 10

- 0 pts Correct
 - 0 pts Start/Finish states not labelled
 - 1 pts Minor errors
 - 3 pts Minimizing NFA as a DFA
 - 3 pts Missing proof
 - 4 pts Major Errors
 - 3 pts Jumping to DFA without proof
 - ✓ - 3 pts **Did not minimize DFA**
 - 9 pts Incorrect
 - 10 pts Missing Solution
- 11 this is not a DFA since you have a state that has two different outputs on the same symbol
- 12 where is the proof that this is minimized?

5) Let $M = (Q, \Sigma, \delta, q_0, A)$ be an ϵ -NFA and let $S \subseteq Q$. Prove that $\epsilon(S) = \epsilon(\epsilon(S))$.

Since S is a set of the starting states and since ϵ is a epsilon transition, $\epsilon(S)$ shows the set of states you can end up on, after any number of transitions of (null or epsilon), starting from S .

From the notes we define the ϵ -closure of S recursively :

(i) $S \subseteq \epsilon(S)$

(ii) $q \in \epsilon(S) \Rightarrow \delta(q, \epsilon) \subseteq \epsilon(S)$

We can prove that $\epsilon(S) = \epsilon(\epsilon(S))$ by set equalities.

First we have to prove that $\epsilon(S) \subseteq \epsilon(\epsilon(S))$:

From the first definition we see that $S \subseteq \epsilon(S)$,

The definition (i), states that a set X is a subset of $\epsilon(X)$ thus if we apply the definition to $\epsilon(S)$ we that $\epsilon(S)$ is a subset of $\epsilon(\epsilon(S))$,

$\epsilon(S) \subseteq \epsilon(\epsilon(S))$ 15

Another definition that we can use :

(iii) If $q \in \epsilon(\epsilon(S)) \Rightarrow \exists q_1, q_2, \dots, q_n$ such that the following hold :

a) $q_1 \in \epsilon(S)$

b) $q = q_n, n \geq 1$

c) $\forall 1 < i \leq q_i \in \delta(q_{i-1}, \epsilon)$

By using the definition (ii) :

$q \in \epsilon(S) \Rightarrow \delta(q, \epsilon) \subseteq \epsilon(S)$

Second we have to prove that $\epsilon(\epsilon(S)) \subseteq \epsilon(S)$:

Choose some arbitrary element q that is in $\epsilon(\epsilon(S))$.

By using the definition (iii), we get that :

a) $q_1 \in \epsilon(S)$

b) $q = q_n, n \geq 1$, our arbitrary q can be further defined as $q_n \in \epsilon(\epsilon(S))$.

c) $\forall 1 < i \leq q_i \in \delta(q_{i-1}, \epsilon) \rightarrow q_i \in \delta(q_{i-1}, \epsilon)$

From the above we get that : $q_2 \in \delta(q_1, \epsilon)$ 17 $q_2 \in \delta(q_1, \epsilon)$

16 $(q_1, \epsilon) = q_2$

This means that from a state q_1 with an epsilon transition you can reach q_2 .

Then it means that q_2 also exists in a) $q_2 \in \epsilon(S)$ as $q_1 \in \epsilon(S)$ but with one more epsilon with get q_2 . Then by using the same logic we can repeatedly follow this procedure and find that eventually we will hit q_n and so we get that $q_n \in \epsilon(S)$ and $q_n \in \epsilon(\epsilon(S))$. Thus this works for any arbitrary state q , then it follows that if any state q is in $\epsilon(S)$ it must also be in $\epsilon(\epsilon(S))$ 14

Then we proved that $\epsilon(\epsilon(S)) \subseteq \epsilon(S)$.

In conclusion we proved by set equalities that if $\epsilon(S) \subseteq \epsilon(\epsilon(S))$ and $\epsilon(\epsilon(S)) \subseteq \epsilon(S)$ then $\epsilon(S) = \epsilon(\epsilon(S))$.

5 question 5 6 / 10

- 0 pts Correct

- 0 pts Use the correct notation. ϵ is not a replacement for ϵ

✓ - 0 pts First half is correct

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17 not $S=S$