CPSC 313 Assignment #1 **Ali Akbari**

1) Prove with induction that $\forall n > 0\ \forall w \in \{a, b\}^* : n_a(w^n) = n \cdot n_a(w)$. You may use without proof that $n_a(xy) = n_a(x) + n_a(y)$. Hint: the recursive definition of w^n is $w^0 = \varepsilon$ and $w^{n+1} = w \cdot w^n$.

Base Case:
$n = 1$
$n_a(w^1) = 1 \cdot n_a(w) =$
$n_a(w\ ) = n_a(w)$
$This\ holds\ true\ for\ the\ base\ case,\ now\ prove\ for\ n + 1.$

Induction Step:
$n_a(w^{n+1}) = n_a(w^n) + n_a(w^1)$
$n_a(w^{n+1}) = n_a(w^n) + n_a(w)\ - From\ base\ case$
$Let\ x = n_a(w^n)$ $- we\ know\ that\ w\ is\ the\ same\ word\ concatenated\ with\ self\ n\ time\ so,$
$x = n_a(w) + n_a(w) + n_a(w) + n_a(w) + n_a(w)...\ goes\ on\ for\ n\ times$
$Since\ the\ word\ is\ the\ same\ n_a(w)\ equates\ to\ the\ same\ value\ thus,\ this\ can\ be\ simiplified\ as\ = n \cdot n_a(w)$
$When\ we\ plug\ this\ back\ to\ the\ above\ equation:$
$n_a(w^{n+1}) = n \cdot n_a(w) + n_a(w)$
$And\ with\ some\ simplification\ of\ dividing\ out\ common\ term\ we\ get:$
$n_a(w^{n+1}) = n \cdot n_a(w) + n_a(w)\ ->$
$n_a(w^{n+1}) = (n + 1) \cdot n_a(w)$
$Thus,\ this\ proves\ that\ n_a(w^n) = n \cdot n_a(w)\ hold\ for\ n + 1.$
$QED.$

2) Prove that for any language $L \subseteq \Sigma^*$ that $(L^r)^r = L$. You may use the fact that $(w^r)^r = w$ without proof. Hint: recall languages are sets, so this question relates to set equality.

$Using\ the\ formal\ definition\ of\ set\ equality:$
$\forall L \forall (L^r)^r [\forall w\ (w \in L \Leftrightarrow w \in (L^r)^r \Rightarrow L = (L^r)^r$
$To\ prove\ equivalence\ we\ have\ to\ prove\ that\ each\ is\ a\ subset/(\subseteq)\ of\ each\ other.$
$Let:$
$1)\ L = \{w : w \in L\}$
$2)\ L^r = \{w^r : w \in L\}$
$3)\ (L^r)^r = \{(w^r)^r : w^r \in L^r\}$

$First\ prove\ that\ (L^r)^r \subseteq L:$
$L^r = \{w^r : w \in L\}\ then,$
$(L^r)^r = \{(w^r)^r : w^r \in L^r\}$

$and\ by\ the\ fact\ that\ (w^r)^r\ =\ w,\ we\ can\ use\ this\ operation\ for\ all\ the\ words\ in\ the$
$language\ (L^r\ )^r = \{(w^r)^r\ =\ w,\ (w^r)^r\ =\ w,\ (w^r)^r\ =\ w,\ (w^r)^r\ =\ w,\ (w^r)^r\ =\ w\ ....\}$
$This\ means\ that\ all\ elements\ of\ (L^r)^r\ =\ w\ which\ is\ \{w\ :\ w \in L\ \}.$
$Since\ all\ element\ of\ (L^r\ )^r\ exist\ within\ the\ L\ from\ number\ 1\ this\ proves\ that\ (L^r\ )^r\ \subseteq\ L.$

$Now\ prove\ L \subseteq\ (L^r\ )^r\ :$
$L =\ \{w\ :\ w \in L\ \}$
$L^r\ =\ \{w^r\ :\ w \in L\ \}$ $\qquad - Reverse\ once$
$(L^r\ )^r\ =\ \{(w^r\ )^r\ :\ w^r \in L^r\}$ $\qquad - Reverse\ twice$
$We\ can\ do\ this\ because\ of\ the\ fact\ that\ (w^r)^r\ =\ w,\ we\ can\ use\ this\ operation\ for\ all\ the\ words\ in\ the$
$L,\ and\ this\ operation\ of\ double\ reversing\ should\ not\ change\ the\ words.$
$L\ =\ \{w =\ (w^r)^r,\ w =\ (w^r)^r,\ w =\ (w^r)^r,\ w =\ (w^r)^r\ ,\ w = (w^r)^r.....\}$
$This\ means\ that\ all\ elements\ of\ L\ =\ (w^r)^r\ which\ is\ \{(w^r\ )^r\ :\ w^r \in L^r\}.$
$Since\ all\ element\ of\ L\ exist\ within\ the\ (L^r\ )^r\ from\ number\ 3\ this\ proves\ that\ \ L \subseteq\ (L^r\ )^r.$
$By\ the\ definition\ of\ set\ equality,\ and\ since\ both\ are\ L\ and\ \ (L^r\ )^r\ are\ subsets\ of\ each\ other,$
$L \subseteq\ (L^r\ )^r\ and\ (L^r\ )^r\ \subseteq\ L$
$then\ (L^r\ )^r\ \ =\ L.$

3) Prove $(\forall i \in N\ :\ w^i\ = \varepsilon\ ) \Leftrightarrow w\ = \varepsilon$. Be sure to prove both directions of the implication. You may use without proof the fact that $\forall w \in \Sigma^*\ \forall i \geq 0\ :\ |w^i|\ =\ i \cdot |w|.$

$Using\ Biconditional\ proof\ p \Leftrightarrow q\ :$
$p\ \Rightarrow (\forall i \in N\ :\ w^i\ = \varepsilon\ )$
$q\ \Rightarrow w\ = \varepsilon$

$LHS\ (p\ \rightarrow\ q) :$
$Based\ on\ p\ assume\ (\forall i \in N\ :\ w^i\ = \varepsilon\ ).$
$(\forall i \in N\ :\ w^i\ = \varepsilon\ )\ by\ using\ the\ above\ fact\ that\ |w^i|\ =\ i \cdot |w|$
$No\ matter\ what\ value\ i\ posses\ \forall i \geq 0\ which\ is\ the\ natural\ numbers,\ and\ no$
$matter\ what\ the\ word\ is,\ the\ cardinality\ is,\ \forall w \in \Sigma^*\ :\ |w^i|\ =\ i \cdot |w|.\ In\ this\ case\ the\ word\ is,$
$based\ on\ q,\ w\ = \varepsilon\ ,\ |\varepsilon^i\ |\ =\ i \cdot |\varepsilon|\ \rightarrow$
$|\varepsilon \cdot \varepsilon \cdot \varepsilon \cdot \varepsilon....\ (i\ times)|\ =\ i \cdot |\varepsilon|$
$since\ |\varepsilon|\ is\ =\ 0\ \ \rightarrow$
$0 = i \cdot 0\ =\ 0$
$This\ means\ that\ no\ matter\ what\ value\ of\ i\ is,\ if\ the\ word\ is\ of\ size\ 0\ which\ is$
$epsilon,\ \varepsilon.\ Then\ the\ length\ will\ be\ 0\ for\ any\ i\ (multiple\ concatenation\ of\ epsilon\ =\ epsilon).$
$w^i = w = \varepsilon.$
$Thus\ w\ =\ \varepsilon,\ as\ p\ implies\ q\ (\ p \rightarrow q).$

$This\ can\ be\ proved\ by\ contradiction\ as\ well:$

$Suppose\ (\forall i \in N\ :\ w^i\ != \varepsilon) \Leftrightarrow w\ = \varepsilon$

$Let\ i\ =\ 2,\ then:$

$(\forall i \in N\ :\ w^i\ != \varepsilon) \Leftrightarrow w\ = \varepsilon,\ w\ =\ \varepsilon\ based\ on\ q\ and\ if\ we\ raise\ the\ word\ to\ the\ second\ power$

$w^2\ =\ \varepsilon\ \cdot \varepsilon\ = \varepsilon$

$This\ poses\ a\ contradiction\ on\ the\ above\ assumption\ w^i\ != \varepsilon \Leftrightarrow w\ = \varepsilon.\ As\ w^2\ does\ =\ \varepsilon.$

$Thus,\ (\forall i \in N\ :\ w^i\ = \varepsilon)\ \Leftrightarrow w\ = \varepsilon.$

$RHS\ (q\ \rightarrow\ p):$

$Based\ on\ q\ assume\ w\ = \varepsilon$

$Suppose\ we\ raise\ the\ word\ w\ to\ an\ arbitrary\ i,\ \ \forall i \in N.$

$w^i = \varepsilon \cdot \varepsilon \cdot \varepsilon \cdot \varepsilon....\ i\ times.$

$By\ using\ the\ fact\ that\ \forall w \in \Sigma^*\ \forall i \geq 0\ :\ |w^i|\ =\ i \cdot |w|,$

$we\ get\ that\ |\varepsilon^i|\ =\ i \cdot |\varepsilon|\ -{>}\ |\varepsilon \cdot \varepsilon \cdot \varepsilon \cdot \varepsilon....(i\ times)|\ = i \cdot |\varepsilon|,\ since\ length\ of\ empty\ string\ is\ 0,$

$and\ using\ the\ fact\ that\ \varepsilon \cdot \varepsilon \cdot \varepsilon \cdot \varepsilon....(i\ times)\ =\ \varepsilon.\ The\ cardinality\ equates\ to\ 0\ =\ i \cdot 0\ =\ 0.$

$This\ means\ that\ no\ matter\ what\ value\ of\ i\ is,\ if\ the\ word\ is\ of\ size\ 0\ which\ is$

$epsilon,\ \varepsilon.\ Then\ the\ length\ will\ be\ 0\ for\ any\ i\ (multiple\ concatenation\ of\ epsilon\ =\ epsilon).$

$w = w^i = \varepsilon.$

$Thus\ (\forall i \in N\ :\ w^i\ = \varepsilon),\ as\ q\ implies\ p\ (q \rightarrow p).$

$This\ can\ be\ proved\ by\ contradiction\ as\ well:$

$Suppose\ (\forall i \in N\ :\ w^i = \varepsilon) \Leftrightarrow w!\ = \varepsilon$

$Let\ i\ =\ 3,\ then:$

$(\forall i \in N\ :\ w^i = \varepsilon) \Leftrightarrow w\ != \varepsilon,\ by\ using\ the\ fact\ that\ \forall w \in \Sigma^*\ \forall i \geq 0\ :\ |w^i|\ =\ i \cdot |w|,\ and\ that$

$the\ length\ will\ be\ 0\ for\ any\ i\ (multiple\ concatenation\ of\ epsilon\ =\ epsilon),\ \ w^3 = \varepsilon\ \cdot \varepsilon\ \cdot \varepsilon\ = \varepsilon$

$w\ =\ \varepsilon,\ \ even\ if\ the\ word\ is\ raised\ to\ the\ third\ power$

$This\ poses\ a\ contradiction\ on\ the\ above\ assumption\ w^i = \varepsilon \Leftrightarrow w\ != \varepsilon.\ As\ w^3\ does\ =\ \varepsilon.$

$Thus,\ (\forall i \in N\ :\ w^i\ = \varepsilon)\ \Leftrightarrow w\ = \varepsilon.$

$Since\ LHS\ =\ RHS,\ and\ by\ the\ definition\ of\ biconditional\ proof,\ we\ see\ that\ either$

$p\ \&\ q\ are\ true\ or\ false.$

$QED.$

4) Let L, L' languages over Σ^* with $0 < |L| \leq |L'| < n$ for some $n \in N$. Prove formally using the definition of concatenation that $|L'| \leq |L \cdot L'| \leq |L| \cdot |L'|$. You may use the following lemma without proof: if w, x, y are words and $x \neq y$ then $wx \neq wy$.

Definition of concatention of $L \cdot L' = \{xy \mid x \, \varepsilon \, L \, ; \, y \, \varepsilon \, L'\}$

Prove the lower bound that $|L'| \leq |L \cdot L'|$:
Let L, L' *be languages over* Σ^* *with* $0 < |L| \leq |L'| < n$

There are two cases :
Case 1 : *L consist of one element* $|L'| = |L \cdot L'|$.
$|L| = 1$
The maximum number of words after concatenation of L and L' is
$|L \cdot L'|$, *since L has one word x and it is concatenated once to every word y of L'* ,
this makes a new word in each instance xy. From the definition of concatenation.
Since a word is made for every word in L', the number of new words is equal to the number element in L'. Thus $|L \cdot L'| = |L'|$.
For example :
$L = \{x\}$
$L' = \{y, z\}$
$|L \cdot L'| = |\{xy, xz\}| = 2$ *and* $|L'| = |\{y, z\}| = 2$
$|L'| = |L \cdot L'|$

Case 2 : *L contains multiple elements* $|L'| < |L \cdot L'|$.
$|L| = n$
Intuitively if the cardinality of $|L'| = |L \cdot L'|$ *when* $|L| = 1$ *from the above case then with multiple words in L the cardinality of* $|L'| < |L \cdot L'|$. *From the definition ofconcatenation we see that a word x in L and a word y in L' make a unique word for every word in L, thus the cardinality is* $|L \cdot L'|$ *and that has to be bigger as the cardinality of* $|L'|$
For example :
$L = \{y, yx\}$
$L' = \{y, xy\}$
$|L \cdot L'| = |\{yy, yxy, yxxy\}| = 3$ *and* $|L'| = |\{y, yx\}| = 2 \Rightarrow |L'| < |L \cdot L'|$

Prove the upper bound that $|L \cdot L'| \leq |L| \cdot |L'|$:
Let L, L' *be languages over* Σ^* *with* $0 < |L| \leq |L'| < n$

There are two cases :

Case 1 : *No words are repeated and* $|L{\cdot}L'| = |L|{\cdot}|L'|$.

The maximum number of words with no repeats after concatenation of L and L' is $|L| \cdot |L'|$, *since for every word x of L, it is concatenated once to every word y of L' when the two languages are concatenated. This makes a new word in each instance xy. From the definition of concatenation. Thus* $|L \cdot L'| = |L| \cdot |L'|$, *for example :*

$L = \{x, w\}$

$L' = \{y, z\}$

$|L \cdot L'| = |\{xy, xz, wy, wz\}| = 4$ *and* $|L| \cdot |L'| = |\{x, w\}| \cdot |\{y, z\}| = 2 \cdot 2 = 4$

Case 2 : *Words are repeated and* $|L{\cdot}L'| < |L|{\cdot}|L'|$.

Intuitively if the maximum words of $|L{\cdot}L'|$ *is* $|L|{\cdot}|L'|$ *from the above case then with repeated word it would be that those word(s) can cancel out and the length of* $|L{\cdot}L'|$ *will become shorter.*

So, $|L \cdot L'| < |L| \cdot |L'|$

For example :

$L = \{y, yx\}$

$L' = \{y, xy\}$

$|L \cdot L'| = |\{yy, yxy, yxxy\}| = 3$ *and* $|L| \cdot |L'| = |\{y, yx\}| \cdot |\{y, xy\}| = 2 \cdot 2 = 4$

5) Prove that $\forall L \subseteq \{0\}^* \Rightarrow L = L^r$. You may use without prove the fact that $|w| = |w^r|$.

Intuitively :

Assume w is an arbituary element from the unary set $\{0\}^*$, *then* $w = 0^i$, $\forall\, i \,\varepsilon\, N$.

And from the given fact that w^r *must equal the same length/ cardinality as w, and since it is a unary set and the form of the word and length of the word is the same reversed, then it must mean that for the languages containing those words, it is also the same in reverse. Since w &* w^r, *are element of L L is a Palindrome language, which mean* $L = L^r$.

First using Induction I will prove that $\forall w, x$ *in L where* $\forall L \subseteq \{0\}^*$ *if* $|w| = |x|$ *then* $w = x$, *where* $n = n_0(w)$, *as lemma.*

Base Case (n = 1):

Let $n = n_0(w)$, *the number unary unit in the word.*

Choose $n = 1$, *so* $w = 0$ *then,*

$|w| = |x| \Rightarrow |0| = |x| \Rightarrow 1 = |x|$, *length of x must equal* 1, *therefore*

$x = 0$, $w = x$, *w & x in unary* $= 0$.

Thus this holds true for the base case.

Induction Step(n + 1):

Suppose one more unary unit is added so $n + 1 = n_0(w)$,

$|w| = |x| \Rightarrow n + 1 = |x| \Rightarrow n + 1 = |x|$, *length of x must equal* $n + 1$, *therefore*

$x = n+1,\ w = x,\ w\ \&\ x\ in\ unary = n+1\ unary\ units.$
$This\ means\ that\ if\ two\ words\ of\ the\ unary\ set\ \{0\}^*\ have\ the\ same\ length,\ they\ must\ be\ the\ same\ word.$
$Thus\ this\ holds\ true\ for\ the\ induction\ step.$

$Then\ by\ the\ given\ fact\ above\ for\ the\ any\ set\ in\ this\ case\ \{0\}^*,\ we\ know\ that\ the\ length\ of$
$|w| = |w^r|,\ then\ w\ must\ equal\ w^r\ based\ on\ the\ above\ lemma.$

$\forall L \subseteq \{0\}^* \Rightarrow L = L^r\ can\ be\ proven\ as\ set\ equality.$
$To\ prove\ equivalence\ we\ have\ to\ prove\ that\ each\ is\ a\ subset/(\subseteq)\ of\ each\ other.$

$1)\ L = \{w : w \in L\}$
$2)\ L^r = \{w^r : w \in L\}$

$First\ prove\ that\ L \subseteq L^r :$
$L = \{w : w \in L\}$
$By\ the\ cardinality\ |w| = |w^r|,\ and\ by\ our\ lemma\ w = w^r\ for\ all\ words\ in\ \forall L \subseteq \{0\}^*.$
$And\ since\ all\ w^r\ words\ are\ elements\ of\ L^r\ from\ number\ 2,\ it\ must\ be\ that\ L \subseteq L^r.$

$Now\ prove\ L^r \subseteq L :$
$L^r = \{w^r : w \in L\}$
$By\ the\ cardinality\ |w^r| = |w|,\ and\ by\ our\ lemma\ w^r = w\ for\ all\ words\ in\ \forall L^r \subseteq \{0\}^*.$
$And\ since\ all\ w\ words\ are\ elements\ of\ L,\ it\ must\ be\ that\ L^r \subseteq L.$
$By\ the\ definition\ of\ set\ equality,\ and\ since\ both\ are\ L\ and\ L^r\ are\ subsets\ of\ each\ other,$
$L \subseteq L^r\ and\ L \subseteq L^r$
$then\ L = L^r.$