Assignment 4

1. Prove that the following language is not regular: $\{a^i b^j a^j b^i : i \leq j\}$.

   solution:

   Prove using the pumping lemma and the adversary argument. Given n, choose $z = b^n a^n$. This is in the language where $i = 0$ and $j = n$.

   Given an arbitrary decomposition $z = uvw$ with $|uv| \leq n$ and $|v| > 0$, we see that $u = b^i$ and $v = b^j$ with $i \geq 0$ and $j > 0$. This means that $w = b^{n-i-j} a^n$.

   We choose $i = 0$ to get $uw = b^i b^{n-i-j} a^n$. Because $j > 0$ we have $n - i - j \neq n$ so uw is not in L.

2. Prove that the following language is not regular: $\{w \in \Sigma^\star : w \neq w^r\}$.

   solutions:

   We prove using closure properties. Assume this language is regular. Then its complement is also regular. The complement of $\{w \in \Sigma^\star : w \neq w^r\}$ is all words $w \in \Sigma^\star$ such that $\neg w \neq w^r$, which is the same as $w \in \Sigma^\star$ such that $w \neq w^r$. This is the language of palindromes, a language that we know is not regular. This is a contradiction, so our assumption that $\{w \in \Sigma^\star : w \neq w^r\}$ is regular is false. Therefore $\{w \in \Sigma^\star : w \neq w^r\}$ is not regular.

3. Consider the following "addition checker" language $\{a^i b^j c^{i+j} : i, j \geq 0\}$ where the number of c's is equal to the sum of the number of a's and b's. Provide a grammar for that language and formally prove that the grammar accepts the language. You may use without proof the fact that the grammar $S \to 0S1|\epsilon$ generates the language $\{0^i 1^i : i \geq 0\}$.

   solution:

   The grammar is $S \to aSc|B$ and $B \to bBc|\epsilon$.

   We note that there are two rules for S and only one of them removes S from the right hand side of the production. Therefore in any derivation that results in a word there must be the use of the production $S \to B$. Thus, if $S \Rightarrow^\star w$ then $S \Rightarrow^\star \alpha S \beta \Rightarrow \alpha B \beta \Rightarrow^\star w$.

   First Prove a lemma that $L(S) = \{a^i w c^i : w \in L(B) \wedge i \geq 0\}$.

   (i) prove $L(S) \subseteq \{a^i w c^i : w \in L(B) \wedge i \geq 0\}$ by induction on length of derivation that if $S \Rightarrow^i \alpha S \beta \Rightarrow \alpha B \beta$ then $\alpha = a^i$ and $\beta = c^i$. Base case $i = 0$ and $S \Rightarrow^0 S = a^0 S c^0 \Rightarrow a^0 B c^0$ as needed. Assume it is true for some particular value i. Prove for $i+1$: $S \to aSc \Rightarrow^i a(a^i S c^i)c \Rightarrow aa^i B c^i c$ as needed. Since $L(B) = \{w : B \Rightarrow^\star w\}$ we have our result.

   (ii) prove that $\{a^i w c^i : w \in L(B) \wedge i \geq 0\} \subseteq L(S)$. Prove by induction on i. Base case $i = 0$: $a^0 w c^0 = w$ where $w \in L(B)$. Then $S \Rightarrow B \Rightarrow^\star w$ is our result. Assume true for some particular i. Prove for $i + 1$: $S \Rightarrow$

$aSc \Rightarrow^\star a(a^iSc^i)c$ (by inductive hypothesis) $\Rightarrow aa^iBc^ic \Rightarrow^\star aa^iwc^ic$ where $w \in L(B)$ and we have our result.

Therefore $L(S) = \left\{a^iwc^i : w \in L(B) \wedge i \geq 0\right\}$. Now we apply the assumption the question allowed us make that $L(B) = \left\{b^jc^j : j \geq 0\right\}$ and get that $L(S) = \left\{a^ib^jc^jc^i : i, j \geq 0\right\} = \left\{a^ib^jc^{i+j}\right\}$ as needed.

4. (Note: this question has two parts that are each worth one fifth of the assignment's marks.) In class we showed that the language $\texttt{prefix}(L)$ is closed for context-free languages, meaning that if L is context-free then $\texttt{prefix}(L)$ is also context free. This does not imply, however, that if L is *not regular* then $\texttt{prefix}(L)$ is *not regular*.

   (a) Give an example non-regular context-free language L_1 and *prove* that $\texttt{prefix}(L_1)$ is *regular*.

   (b) Give an example non-regular context-free language L_2 and *prove* that $\texttt{prefix}(L_2)$ is *not regular*.

   Note that you may specify L_1 or L_2 using math notation and give a grammar *without* doing a proof that the grammar generates the language. If you use a language that we have not shown was non-regular for L_1 or L_2, however, you must prove that the language you use is non-regular (i.e., with pumping lemma, etc.).

   solution for L_1 choose $L_1 = \{xx^r : x \in \Sigma^\star\}$. Then prefix of L_1 is $\Sigma^\star$, which is a known regular language. Proof: if $x \in \Sigma^\star$ then $xx^r \in L_1$ and so x is a prefix of a word in L_1.

   solution for L_2 choose $L_2 = \left\{a^ib^i : i \geq 0\right\}$. Then prefix of L_2 is $\left\{a^ib^j : j \leq i\right\}$, which is a context free language generated by $S \rightarrow aSb|aS|\epsilon$.

   Proof that $\left\{a^ib^j : j \leq i\right\} \subseteq \texttt{prefix}(L_2)$: let a^ib^j be an arbitrary element. Then it is a prefix of $a^ib^jb^{i-j}$ since $j \leq i$, but this is a^ib^i, an element of L_2.

   Proof that $\texttt{prefix}(L_2) \subseteq \left\{a^ib^j : j \leq i\right\}$: let $w = a^ib^i$ be an arbitrary element of L_2 and let x be an arbitrary prefix. Then $w = xy$ and there are two cases: $x = a^j, y = a^{i-j}b^i$ and $x = a^ib^j, y = b^{i-j}$. Case (i) $x = a^j = a^jb^0$ and $j \geq 0$ therefore it is an element of $\texttt{prefix}(L_2)$. Case (ii) $x = a^ib^j$ where $j \leq i$ therefore it is an element of $\texttt{prefix}(L_2)$.

   Proof that the prefix language is context free by adversary argument. Adversary chooses n. We choose a^nb^n. Adversary chooses uvw where $|uv| \leq n$ and $|v| > 0$. Then $u = a^i, v = a^j, w = a^{n-i-j}b^n$. We choose to pump with 0 to get $uw = a^ia^{n-i-j}b^n = a^{n-j}b^n$. Since $j > 0$ we have that $n - j < n$ and so the word is not in L. Therefore L_2 is not regular.