Ali Akbari **30010402**

1) Let M be an NFA with n states. Show that if $|L(M)| > 1$ then $\exists w \in L(M)$ with $|w| < n$.
You may use without proof the fact that $\delta^*(q_0, xy) = \cup_{p \in \delta^*(q_0,x)} \delta^*(p, y)$.
From Theorem 1 of the pumping lemma notes in page 1 of pumping-ln.pdf.

The following can be proved by contradiction :
Assume that the NFA does not accept words with a cardinality of less than n,
let w be the smallest word such that $\forall w \in L(M)$ *with* $|w| \geq n$.
Intuitively if an NFA has n number of states, the shortest path to visit all states is with $n - 1$ *number of transitions.*

The number of states that are encountered as w is proccessed is $|w| \geq n$.
Because there are n total states there must be that one or more states that are visited are visited multiple times(by the pigeonhole principle). This means that there is atleast one cycle. So if we remove the cycles then we get a shorter path, which means a shorter string that will be accepted. So then if we repeated remove the cycles we will have $|w| = n$, *however since there is no transiton to input for the initial state the shortest path to reach all n states it must be* $n - 1$, *we get a shorter path and a shorter string that can be accepted. So,* $|w| = n - 1$. *This poses a contradiction to our assumption that the NFA* $\forall w \in L(M)$ *with* $|w| \geq n$ *as as* $|w| = n - 1$ *which is* $|w| < n$. *Thus, we have proven by contradiction that* $\exists w \in L(M)$ *with* $|w| < n$ *is true.*

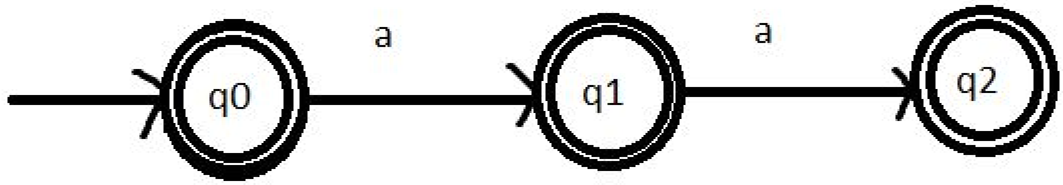
2) Let M be an NFA with n states. Show that if $|\overline{L(M)}| > 1$ then there does not necessarily $\exists w \in \overline{L(M)}$ with $|w| < n$.

Consider the following alphabets :

$\Sigma = \{a\}$

number of states, $n = 3$

Consider the following NFA denoted as M, it is non deterministic as state q2 has no transitions defined for the alphabet a.



$The\ following\ NFA,\ has\ a\ language\ L,\ so\ L(M) = \{e,\ a,\ aa\}.$

$Then\ the\ complement\ of\ the\ language\ L\ is\ \overline{L} = \{w \in \Sigma^* : n_a(w) >= 3\}.$

$The\ language\ \overline{L}\ which\ is\ also\ \overline{L(M)},\ based\ on\ the\ above\ definition\ of\ \overline{L(M)},$

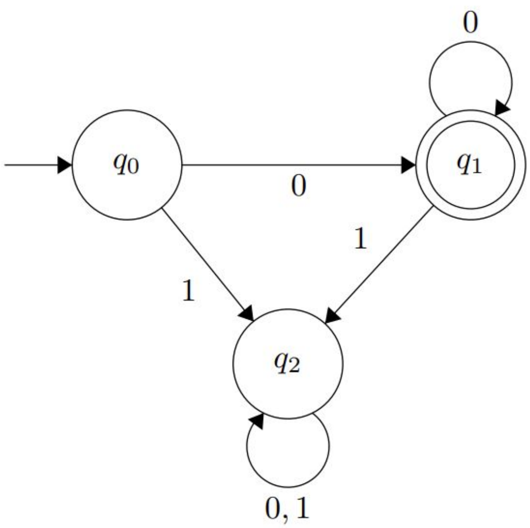
$has\ a\ cardinality\ |\overline{L(M)}| >= 3,\ which\ is\ greater\ than\ 1.\ This\ states\ that\ the\ complement$
$language\ has\ atleast\ 3\ a(s),\ \{a^i : i >= 3\}.\ So\ the\ smallest\ word\ in\ the\ language$

$\overline{L}\ is\ w = aaa\ which\ has\ a\ cardinality\ of\ |aaa| = 3 = n,\ since\ there\ are\ 3\ states\ in\ the\ NFA.$

$Thus,\ this\ shows\ by\ given\ example\ that\ if\ |\overline{L(M)}| > 1,\ then\ \exists w \in \overline{L(M)}$

$with\ |w| \geq n\ holds\ true.$

3) Prove that the following machine M that accepts the language $L = \{0^i : i > 0\}$. Prove both directions: that all words in L are accepted, and that anything that is accepted is in L. The alphabet $\Sigma = \{0, 1\}$. One way to prove is that L is accepted and L is rejected. If you go that way you may use without proof the fact that
$\overline{L} = \{\varepsilon\} \cup \{w \in \Sigma^* : n_1(w) > 0\}$ and that $\delta^*(q_2, x) = q_2$ without proof.



$M = (Q, \Sigma, \delta, q_0, A)$

$Q = 3$

$\Sigma = \{0, 1\}$

$q_0 = q_0$

$A = \{q_1\}$

Lemma #1 :

Prove by induction :

That if $L = \{0^i : i > 0\}$, *then all words in L are accepted in the above machine* M.

Base case :

Let $i = 1$.

Then $L = 0$, *then for the machine* M, *when at* q_0 *reading a* 0 *we go to state* q_1,

$\delta(q_0, 0) = q_1$.

Then it holds for the base case that the $L = \{0^i : i > 0\}$ *is accepted in the above machine* M.

Induction Step :

IH : Assume for $i > 0$, *where* $L = \{0^i\}$, *then all words in L are accepted by the above machine* M.

Prove this holds true for $L = \{0^{i+1} : i > 0\}$.

$L = 0^{i+1}$

$L = 0^i * 0^1$

$\delta(q_0, 0) = q_1$

$\delta(q_1, 0) = q_1$

L is a unary language so means that reading 0^1 *first or* 0^i *does not matter since both lead to the same language. Since A only has one element* q_1 *the only accepting state is* q_1.

Since the base case has been proven to be true using the following transition $\delta(q_0, 0) = q_1$, *then* 0^1 *holds true. Then by using our inductive hypothesis we can also show that we will be in an accepting state after reading* 0^i *since* $\delta(q_1, 0) = q_1$. *Also, since the language only contains* $0s$ *and no* $1s$ *based on the transitions states it only leads to* q_1, *thus a stream of* $0s$ *is always accepting.*

So if (0^1) *leads to an accepting state as seen in the base case, and if* (0^i) *leads to an accepting state as seen in the inductive hypothesis.* 0^i *leads to* q_1 *and from the above transition states we know reading any number of* $0s$ *at* q_1 *also lead back to* q_1 *so it holds for* 0^{i+1}.

Then it holds for the induction step that the $L = \{0^i : i > 0\}$ *is accepted in the above machine* M.

So, in conclusion if a language $L = \{0^i : i > 0\}$, *then all words in L are accepted in the above machine* M.

Lemma #2 :

Prove by contradiction that anything that is accepted by the machine M is in L.

$\Sigma = \{0, 1\}$

$U = \Sigma^*$

$\overline{L} = U/L$

$\overline{L} = \{\varepsilon\} \cup \{w \in \Sigma^* : n_1(w) > 0\}$

Suppose that anything that is accepted by the machine M is in the language $\overline{L}$ *defined above.*

The machine M accepts the word $w = 0$, *as shown by the transition state* $\delta(q_0, 0) = q_1$.

However based on the definition of $\overline{L}$ *the number of* $1s$ *in word of the language* $\overline{L} > 0$.

This poses a contradiction as the word $w = 0$ *does not include any* $1s$.

Also since in the given fact it shows that q_2 *is trap based on the extended transition state, to get to* q_2 *a single* 1 *must be read which is part of the definition of the* $\overline{L}$. *Thus when a single* 1 *is read it leads to* q_2 *the error state and any subsequent string that follows still stays in* q_2 *which is not accepting.*

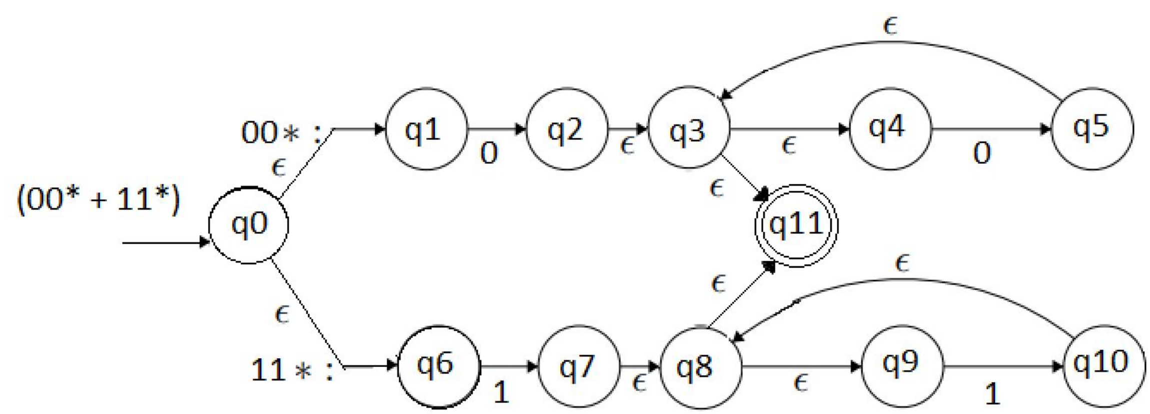
Therefore by contradiction we proved that anything that is accepted by the machine M is in L.

Then using lemma # 1 and lemma # 2 we can conclude the proof that all words in L are accepted by the machine M and that anything that is accepted by M it is in L.

4) Prove that no DFA with three states can accept the language $L = \{0^i : i > 0\} \cup 1^i : i > 0.$ Hint: there are a variety of ways you can prove this, but it is insufficient to simply show a X-state machine (for some X > 3) that is claimed to accept the language. You may apply any construction or algorithm we've covered in class and refer to its proof of correctness as part of this question. Clearly state the construction you are using and show the input and the output. You may also use the machine you provide in the previous question for $\{0^i : i > 0\}$ and without loss of generality $\{1^i : i > 0\}$ without reproving that it accepts the language.

In this question I used the lecture notes and the following sites to make the NFA and the last DFAs:
http://madebyevan.com/fsm/
https://www.youtube.com/watch?v=WSGcmaHNBFM&t=440s

The regular expression for the language L is $R = 00^* + 11^*$, *from this we can use theorem 1 of Kleene's Theorem to get an* $e-NFA$. *Used page #3 in kleene − In.pdf notes.*
The resulting $e-NFA$ *is the following* :



$*$ *implies accept state*

NFA Transition Table :

	0	1
q0	{q2,q3,q4,q11}	{q7,q8,q9,q11}
q1	{q2,q3,q4,q11}	{}
q2	{q3,q4,q5,q11}	{}
q3	{q3,q4,q5,q11}	{}

q4	{q3,q4,q5,q11}	{}
q5	{q3,q4,q5,q11}	{}
q6	{}	{q7,q8,q9,q11}
q7	{}	{q8,q9,q10,11}
q8	{}	{q8,q9,q10,q11}
q9	{}	{q8,q9,q10,q11}
q10	{}	{q8,q9,q10,q11}
q11*	{}	{}

By sub − set construction we get the following table :

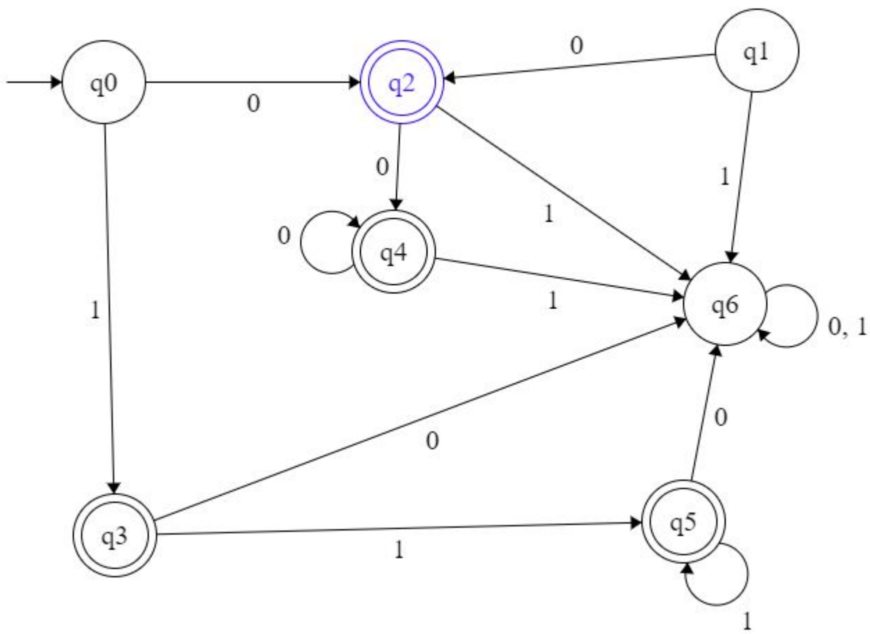
	0	1
q0	{q2,q3,q4,q11}*	{q7,q8,q9,q11}*
q1	{q2,q3,q4,q11}*	{}
{q2,q3,q4,q11}*	{q3,q4,q5,q11}*	{}
{q7,q8,q9,q11}*	{}	{q8,q9,q10,q11}*
{q3,q4,q5,q11}*	{q3,q4,q5,q11}*	{}
{q8,q9,q10,q11}*	{}	{q8,q9,q10,q11}*
{}	{}	{}

Let the following :

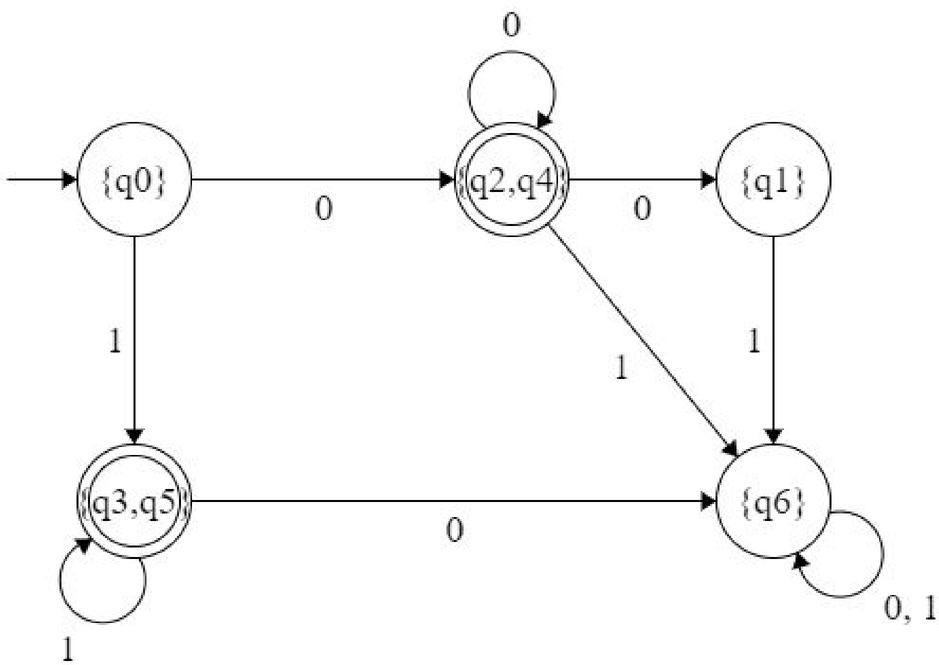
q0 = q0
q1 = q1
q2* = {q2,q3,q4,q11}
q3* = {q7,q8,q9,q11}
q4* = {q3,q4,q5,q11}
q5* = {q8,q9,q10,q11}
q6 = {} - Fail state

	0	1
q0	q2*	q3*
q1	q2*	q6
q2*	q4	q6

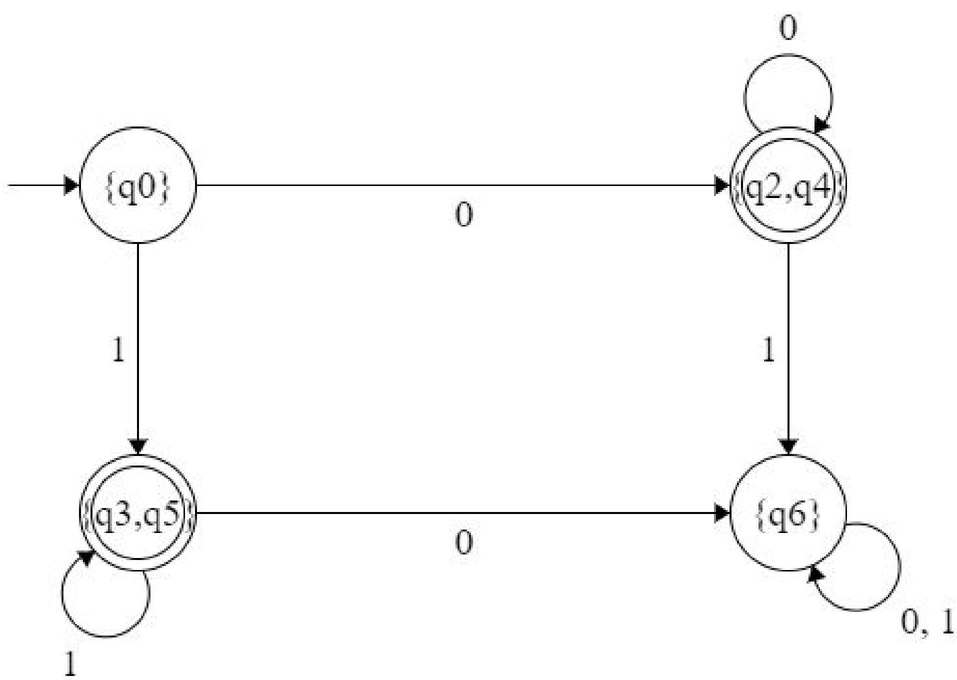
q3*	q6	q5
q4	q4	q6
q5	q6	q5
q6	q6	q6



Color coated states have the same transition so when minimized we get the following DFA :



$q1$ is a dead state that has no incoming transitions, so can be further minimized as :



We see the minimized DFA has 4 states so the following construction shows that the language $L = \{0^i : i > 0\} \cup 1^i : i > 0$ after minimization can only be built with atleast 4 states, it is not possible for 3 states to accept the language, thus this is proved false.

5) Let $M = (Q, \Sigma, \delta, q_0, A)$ be an ε-NFA and let $S \subseteq Q$. Prove that $\varepsilon(S) = \varepsilon(\varepsilon(S))$.

$Since\ S\ is\ a\ set\ of\ the\ starting\ states\ and\ since\ \varepsilon\ is\ a\ epsilon\ transtion,\ \varepsilon(S)\ shows\ the\ set\ of$
$states\ you\ can\ end\ up\ on,\ after\ any\ number\ of\ transitions\ of\ (null\ or\ epsilon),\ starting$
$from\ S.$
$From\ the\ notes\ we\ define\ the\ e - eclosure\ of\ S\ recursively:$
$(i)\ S \subseteq \varepsilon\,(S)$
$(ii)\ q \in \varepsilon\,(S) \Rightarrow \delta(q,\ \varepsilon) \subseteq \varepsilon(S)$

$We\ can\ prove\ that\ \varepsilon(S) = \varepsilon(\varepsilon(S))\ by\ set\ equalities.$
$First\ we\ have\ to\ prove\ that\ \varepsilon(S) \subseteq \varepsilon(\varepsilon(S)):$
$From\ the\ first\ definition\ we\ see\ that\ S \subseteq \varepsilon\,(S),$
$The\ definition\ (i),\ states\ that\ a\ set\ X\ is\ a\ subset\ of\ \varepsilon\,(X)\ thus\ if\ we$
$apply\ the\ definition\ to\ \varepsilon(S)\ we\ that\ \varepsilon(S)\ is\ a\ subset\ of, \varepsilon(\varepsilon(S)),$
$\varepsilon(S) \subseteq \varepsilon(\varepsilon(S)).$

$Another\ definition\ that\ we\ can\ use:$
$(iii)\ If\ q \in \varepsilon(\varepsilon(S)) \Rightarrow \exists\ q_1,\ q_2,\ ....\ q_n\ such\ that\ the\ following\ hold:$
$a)\ q_1 \in \varepsilon(S)$
$b)\ q = q_n, n \geq 1$
$c)\ \forall\ 1 < i \leq q_i\ \varepsilon\ \delta(q_{i-1},\ \varepsilon)$
$By\ using\ the\ definition\ (ii):$
$q \in \varepsilon\,(S) \Rightarrow \delta(q,\ \varepsilon) \subseteq \varepsilon(S)$

$Second\ we\ have\ to\ prove\ that\ \varepsilon(\varepsilon(S)) \subseteq \varepsilon(S):$
$Choose\ some\ arbitrary\ element\ q\ that\ is\ in\ \varepsilon(\varepsilon(S)).$
$By\ using\ the\ definition\ (iii),\ we\ get\ that:$
$a)\ q_1 \in \varepsilon(S)$
$b)\ q = q_n, n \geq 1,\ our\ arbitrary\ q\ can\ be\ further\ defined\ as\ q_n \in \varepsilon(\varepsilon(S)).$
$c)\ \forall\ 1 < i \leq q_i \in \delta(q_{i-1},\ \varepsilon) \rightarrow q_i \in \delta(q_{i-1},\ \varepsilon)$
$From\ the\ above\ we\ get\ that: q_2\ \varepsilon\ \delta(q_{2-1},\ \varepsilon) = q_2\ \varepsilon\ \delta(q_1,\ \varepsilon)$
$\delta(q_1,\ \varepsilon) = q_2$
$This\ means\ that\ from\ a\ state\ q_1 with\ an\ epsilon\ transition\ you\ can\ reach\ q_2.$
$Then\ it\ means\ that\ q_2\ also\ exists\ in\ a)\ q_2\ \varepsilon\ \varepsilon(S)\ as\ q_1\ \varepsilon\ \varepsilon(S)\ but\ with\ one\ more\ epsilon\ with\ get$
$q_2.\ Then\ by\ using\ the\ same\ logic\ we\ can\ repeatedly\ follow\ this\ procedure\ and\ find\ that$
$eventually\ we\ will\ hit\ q_n\ and\ so\ we\ get\ that\ q_n\ \varepsilon\ \varepsilon(S)\ and\ q_n\ \varepsilon\ \varepsilon(\varepsilon(S)).\ Thus\ this\ works\ for\ any$
$arbitrary\ state\ q,\ then\ it\ follows\ that\ if\ any\ state\ q\ is\ in\ \varepsilon(S)\ it\ must\ also\ be\ in\ \varepsilon(\varepsilon(S)).$
$Then\ we\ proved\ that\ \varepsilon(\varepsilon(S)) \subseteq \varepsilon(S).$

$In\ conclusionwe\ proved\ by\ set\ equalities\ that\ if\ \varepsilon(S) \subseteq \varepsilon(\varepsilon(S))\ and\ \varepsilon(\varepsilon(S)) \subseteq \varepsilon(S)$
$then\ \varepsilon(S) = \varepsilon(\varepsilon(S)).$