Assignment 3

1. Let M be an NFA with n states. Show that if $|L(M)| > 1$ then $\exists w \in L(M)$ with $|w| < n$. You may use without proof the fact that $\delta^\star(q_0, xy) = \bigcup_{p \in \delta^\star(q_0,x)} \delta^\star(p, y)$.

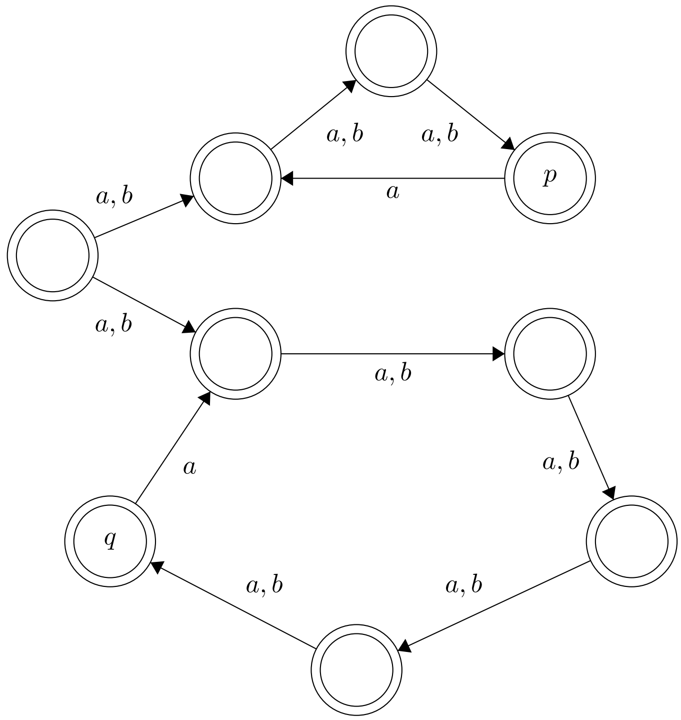
   Let w be the shortest word that M accepts. Suppose to the contrary that w is a word that has length n or more. Let $q_0, \ldots, q_{|w|}$ be the sequence of states visited by M while processing w. Because $w \in L(M)$ we know that $\left\{q_{|w|}\right\} \cap A \neq \emptyset$. Since there are $|w| + 1$ states there is $\geq n + 1$ states on this path, and by the pidgeonhole principle one of these states much repeat. Let q_i be a state that repeats. Then $q_0, \ldots, q_i, \ldots, q_i, \ldots, q_{|w|}$ is the sequence of states. Write $w = xyz$ where $q_i \in \delta^\star(q_0, x)$, $q_i \in \delta^\star(q_0, y)$, and $q_{|w|} \in \delta^\star(q_i, z)$. We have that $y \neq \epsilon$ because it corresponds to the path between two visits to q_i.

   Then $\delta^\star(q_0, xz) = \bigcup_{p \in \delta^\star(q_0,x)} \delta^\star(p, z)$ and so $\delta^\star(q_i, z) \subseteq \bigcup_{p \in \delta^\star(q_0,x)} \delta^\star(p, z)$ and thus $q_{|w|} \in \bigcup_{p \in \delta^\star(q_0,x)} \delta^\star(p, z)$ and so $xz \in L(M)$. Since $|xz| = |xyz| - |y|$ and $|y| > 0$ we have that xz is a shorter word accepted by M than xyz, contradicting xyz being the shortest word.

2. Let M be an NFA with n states. Show that if $|\overline{L(M)}| > 1$ then there does not necessarily $\exists w \in \overline{L(M)}$ with $|w| < n$.

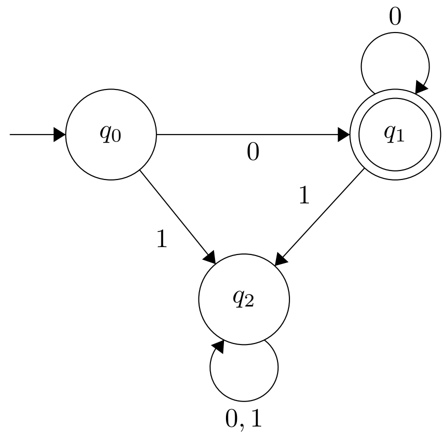
   Solution: There are many examples. The simpliest is a one state machine with no transitions that accepts ϵ. Its complement language is every other word, meaning the shortest word has length 1. As this machine has 1 state, we have that the shortest word matches the number of states. The idea is that if all NFA states are accepting, then the "emptyset" in the subset construction is a state in the DFA that "becomes" the only accepting state.

   Another approach that creates a much larger difference between the shortest word and the number of states.



Here there are two loops, one of length 3 and one of length 5. At the end of each loop, you have to use an a. If you subset construct this, you end up with more than 15 states, and the only way to go to the "no possible moves" state is to be in "both" states p and q, which occurs only after processing 15 letters.

3. Prove that the following machine M that accepts the language $L = \{0^i : i > 0\}$. Prove both directions: that all words in L are accepted, and that anything that is accepted is in L. The alphabet $\Sigma = \{0, 1\}$. One way to prove is that L is accepted and $\overline{L}$ is rejected. If you go that way you may use without proof the fact that $\overline{L} = \{\epsilon\} \cup \{w \in \Sigma^\star : n_1(w) > 0\}$ and that $\delta^\star(q_2, x) = q_2$) without proof.



solution:

Suppose $w \in L$ then $w = 0^i$. Induct on i to show that $\delta^\star(q_0, w) = q_1$.

Base case $i = 1$ then $\delta^\star(q_0, w) = \delta(\delta^\star(q_0, \epsilon), 0) = \delta(q_0, 0) = q_1$.

Inductive hypothesis: assume that there is a k such that when $i = k$ that 0^i is accepted by M.

Inductive step: prove for $i = k + 1$. Write $w = 0^i$ as 0^k0. Then $\delta^\star(q_0, 0^k0) = \delta(\delta^\star(q_0, 0^k), 0)$. By inductive hypothesis, we have that $\delta^\star(q_0, 0^k) = q_1$ so $\delta^\star(q_0, 0^k0) = \delta(q_1, 0) = q_1$.

Because q_1 is in A, we have that all these words are therefore in $L(M)$.

Now prove the reverse direction. Suppose w is a word not in L. There are two cases: $w = \epsilon$ and $n_1(w) > 0$.

Case 1: $\delta^\star(q_0, \epsilon) = q_0 \notin A$ therefore $\epsilon \notin L(M)$.

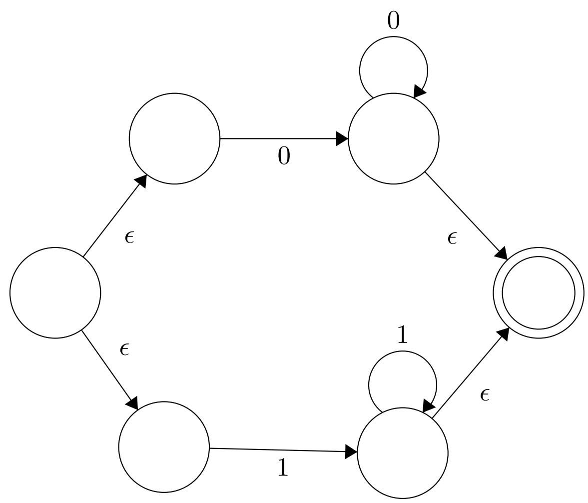
Case 2: write $w = x1y$. Consider $\delta^\star(q_0, x1) = \delta(\delta^\star(q_0, x), 1)$. Since all states q have $\delta(q, 1) = q_2$, we have that $\delta^\star(q_0, x1) = q_2$ regardless of $\delta^\star(q_0, x)$. Similarly, $\delta(q_2, 0) = \delta(q_2, 1) = q_2$ and thus $\delta^\star(q_0, x1y) = q_2 \notin A$ and therefore $w \notin L(M)$.

4. Prove that no DFA with three states can accept the language $L = \left\{0^i : i > 0\right\} \cup \left\{1^i : i > 0\right\}$. Hint: there are a variety of ways you can prove this, but it is insufficient to simply show a X-state machine (for some $X > 3$) that is claimed to accept the language. You may apply any construction or algorithm we've covered in class and refer to its proof of correctness as part of this question. Clearly state the construction you are using and show the input and the output. You may also use the machine you provide in the previous question for $\left\{0^i : i > 0\right\}$ and without loss of generality $\left\{1^i : i > 0\right\}$ without reproving that it accepts the language.

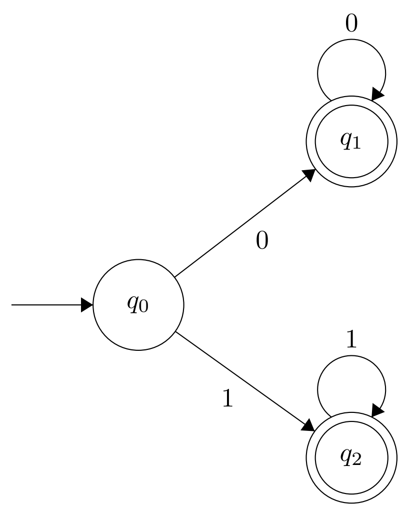
   solutions:

   Step 1: start with the two state machine to accept $\left\{0^i : i > 0\right\}$ and duplicate it with 0 to 1 to make one that accepts $\left\{1^i : i > 0\right\}$

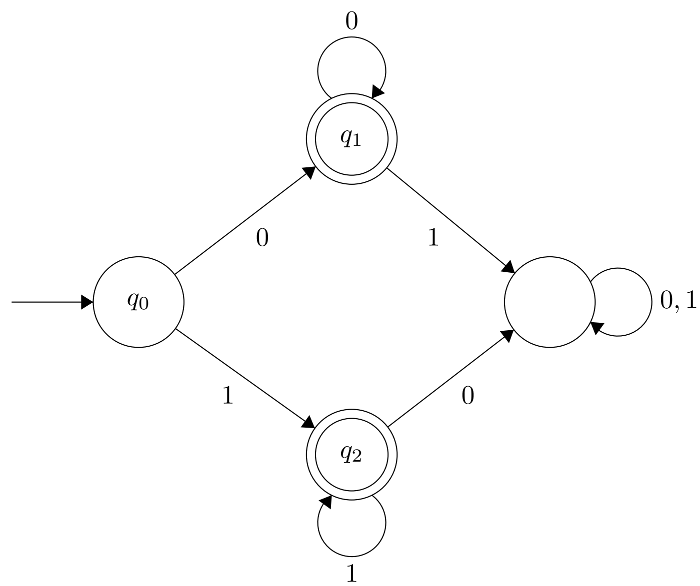
Step 2: use Kleene's theorem to implement the union machine:



Step 3: turn into an NFA



Step 4: subset construct the DFA



Step 5: minimize the DFA using the table filling algorithm:

	$\{q_0\}$	$\{q_1\}$	$\{q_2\}$	$\{\}$
$\{q_0\}$				
$\{q_1\}$				
$\{q_2\}$				
$\{\}$				

base case

	$\{q_0\}$	$\{q_1\}$	$\{q_2\}$	$\{\}$
$\{q_0\}$		$\times$	$\times$	
$\{q_1\}$	$\times$			$\times$
$\{q_2\}$	$\times$			$\times$
$\{\}$		$\times$	$\times$	

after one loop

	$\{q_0\}$	$\{q_1\}$	$\{q_2\}$	$\{\}$
$\{q_0\}$		$\times$	$\times$	$\times$
$\{q_1\}$	$\times$		$\times$	$\times$
$\{q_2\}$	$\times$	$\times$		$\times$
$\{\}$	$\times$	$\times$	$\times$	

At this point, there are no more empty cells to consider, therefore every single state is distinguishable from the others. That means that 4 is the minimum number of states required to accept this language. Therefore there can be no 3 state machine that accepts it.

5. Let $M = (Q, \Sigma, \delta, q_0, A)$ be an ϵ-NFA and let $S \subseteq Q$. Prove that $\epsilon(S) = \epsilon(\epsilon(S))$.

solution:

We must show that $\epsilon(S) \subseteq \epsilon(\epsilon(S))$ and $\epsilon(\epsilon(S)) \subseteq \epsilon(S)$.

The first direction follows directly from the definition of ϵ closure: $T \subseteq \epsilon(T)$, just choose $T = \epsilon(S)$.

For the reverse, assume that some $q \in \epsilon(\epsilon(S))$ but $q \notin \epsilon(S)$. By definition, there is a sequence of states $q_1, q_2, \ldots q_n, q_{n+1}$ such that $q_{n+1} = q$, $q_1 \in \epsilon(S)$ and $q_{i+1} \in \delta(q_i, \epsilon)$ for $1 \leq i \leq n$. Because $q_1 \in \epsilon(S) \wedge q \notin \epsilon(S)$, there must be some state q_i such that $q_i \in \epsilon(S)$ and $q_{i+1} \notin \epsilon(S)$. By definition of $\epsilon(S)$ (recursive part) we have that if $q_i \in \epsilon(S)$ and $q_{i+1} \in \delta(q_i, \epsilon)$, then $q_{i+1} \in \epsilon(S)$, a contradiction.