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1) $Prove\ that\ the\ following\ language\ is\ not\ regular : \{a^i\ b^j\ a^j\ b^i\ :\ i \le j\}.$

$From\ the\ pumping\ lemma\ examples :$

$L\ = \{a^n\ b^n\ a^n\ b^n\ :\ n\ \ge 0\}.$ *We will use the pumping lemma to show that* L *is not regular.*

Proof by contradiction. Assume that L *is regular. Therefore there exists some pumping length* n.
While we don't know what n *is exactly, we are free to choose any word in the process of proving the language is not regular. So choose* $z\ =\ a^n\ b^n\ a^n\ b^n$. *This word is in* L *and it is also longer than* $n :\ |z|\ =\ n\ +\ n\ +\ n\ +\ n =\ 4n$. *Now consider all possible decompositions of* $z\ =\ uvw$ *where* $|uv| \le n$ *and* $|v| \ge 1$. *Because our word is* $a^n\ b^n\ a^n\ b^n$ *we must have that* $uv\ =\ a^{i+j}$. *That is, it can only consist of* a. *Let* $v = a^j$ *and* $u\ =\ a^i$. *Then* $w\ =\ a^{n-i-j}\ b^n\ a^n b^n$. *Now consider the word* $z^0\ =\ uv^0w$.
This is equal to $a^i a^{n-i-j}\ b^n\ a^n b^n$ *where* $j \ge 1$. *But* $i\ +\ n - i - j\ =\ n - j\ \ne\ n$.
Thus the word is not of the form $a^n\ b^n\ a^n\ b^n$ *and is not an element of* L. *We conclude that our assumption that the language is regular is wrong. Therefore the language is not regular.*

$\square$

2) $Prove\ that\ the\ following\ language\ is\ not\ regular:\ \{w \in \Sigma^*\ :\ w \neq w^r\}.$

$From\ the\ pumping\ lemma\ example\ \#3,\ complement\ closure\ of\ regularity,\ definition\ of\ palindromes$ $and\ palindromes\ not\ being\ regular\ from\ the\ lecture\ notes:$

$L = \{w \in \Sigma^*\ :\ w \neq w^r\}$ $and\ by\ taking\ the\ complement\ of\ the\ language\ we\ preserve\ the\ regularity$ $of\ the\ langauge,\ the\ complement\ is:\ \overline{L} = \{w \in \Sigma^*\ :\ w = w^r\}.$ $We\ see\ that\ this\ is\ a\ language\ of$ $palindromes,\ based\ on\ the\ lecture\ notes\ of\ pumping\ lemma\ example\ \#3\ we\ know\ that\ palindromes\ are$ $not\ regular\ and\ since\ complement\ operation\ holds\ regularity,\ it\ means\ our\ language\ is\ not\ regular.$

$Thus, language\ :\ \{w \in \Sigma^*\ :\ w \neq w^r\}\ is\ not\ regular.$

$\square$

3) *Consider the following "addition checker" language* $\{a^i\, b^j\, c^{i+j} \;:\; i, j \geq 0\}$ *where the number of c's is equal to the sum of the number of a's and b's. Provide a grammar for that language and formally prove that the grammar accepts the language. You may use without proof the fact that the grammar* $S \; -> \; 0S1|\varepsilon$ *generates the language* $0^i1^i \;:\; i \geq 0.$

The grammer for this language is :
$G = (\{S,\, B\},\, \{a, b, c\},\, P,\, S)$
$S \; -> aSc \mid B$
$B \; -> bBc \mid \varepsilon$
To prove that this grammer works we have to show that $L = (G)$.
Therefore we have to show this in two steps $L \subseteq L\,(G)$ *and* $L\,(G) \subseteq L.$

To prove that $L \subseteq L\,(G)$, *we choose an arbitrary* $x \in L$ *and prove that there is a derivation of x in G, i,e.* $S \Rightarrow^* x$. *This is done by induction on the length of the word x.*
$a^i\, b^j\, c^{i+j} = a^i\, b^j\, c^j\, c^i$
The language generated by B from our production is $\{b^j c^j \;:\; j \geq 0\}$.
Thus without loss of generality we now have the language $L = \{a^iBc^i \;:\; i \geq 0\}$.
Now by using induction on i we can prove for all i, $S \Rightarrow a^iBc^i$.
Base case $(i = 0)$:
Suppose $z \in L$, *then* $z = a^iBc^i = a^0Bc^0 = B$. *So* $S \Rightarrow B$ *is a derivation of z, this holds for* $i = 0$.
Inductive Hypothesis :
Assume that there is some value of n such that, a^nBc^n *has a derivation in L.*
Induction Step :
Show that there is some value n such that $a^{n+1}Bc^{n+1}$, *by contructing the derivation.*
We start with $S \Rightarrow aSc$. *By our induction hypothesis we have that* $S \Rightarrow^* a^nBc^n$.
Using that as the ultimate derivation of B in the sentinel form aBc.
Thus, $aBc \Rightarrow^* aa^n\, b^nb\, c^nc = a^{i+1}b^{j+1}c^{i+j+1}$, *holds for the induction step and* $L \subseteq L\,(G)$.

To prove that $L\,(G) \subseteq L$, *we start with a word x that can be derived by the grammar, prove that* $x \varepsilon L$, *this is done by using induction on the number of steps in the derivation.*
If $S \Rightarrow^n x$ *then* $x = a^ib^ic^{i+j}$ *where* $i, j \geq 0$.

Base Case $(n = 2)$:
For our production to end we need to atleast two steps.
If we derive a word the only possibility according to our grammar :
$S \Rightarrow B \Rightarrow \varepsilon$, *which is* $S \Rightarrow \varepsilon$.
The word of epsilon, $x = \varepsilon$ *which* $\in L$ *if* $i, j = 0\; a^0\, b^0\, c^{0+0}$
Thus, this holds for the base case.
Inductive Hypothesis :

$Assume\, for\, all\, words\, x,\; if\, x\, is\, derived\, in\, k\, steps,\; S \Rightarrow^k x\, then\, x \in L\, where\, x\; =\; \{a^i\; b^j\; c^{i+j}\;:\; i,\, j\; \geq\; 0\;\}.$

$Induction\; Step:$

$Prove\; this\, for\, k + 1\; derivation\; steps.$

$Suppose\; that\, x \in \Sigma^*\; such\; that\, S \Rightarrow^{k+1} x.\, If\, we\, have\, to\, start\, at\, symbol\, S\, and\, is\, has\, at\, least\, one\, derivation$
$then\, we\, can\, use\, S\; \Rightarrow aSc \Rightarrow^k x.$

$Then\, we\, can\, rewrite\, x\; =\; anb\; such\; that\, S \Rightarrow^k n.\, From\, our\, inductive\, hypothesis,\; the\, word\, n\; = a^i b^j c^{i+j}.$

$This\, means\, that\, x\; =\; aa^i bb^j cc^{i+j} = a^{i+1} b^{j+1} c^{i+j+1}\; thus\, this\, holds\, for\, inductive\, step\, so,\; L\,(G)\; \subseteq\; L.$

$By\, set\, equalities\, we\, see\, that\, L\; \subseteq\; L\,(G)\; and\;\; L\,(G)\; \subseteq\; L,\; thus\, L = (G).$

□

4)(*Note*: *this question has two parts but it will be marked as though it were two separate questions.*)
In class we showed that the language $prefix(L)$ *is closed for context − free languages, meaning that if L is context − free then* $prefix(L)$ *is also context free. This does not imply, however, that if L is not regular then* $prefix(L)$ *is not regular.*

(a) *Give an example of a non − regular context − free language* L_1 *and prove that* $prefix(L_1)$ *is regular.*
From the context − free − grammer lecture note we know that palindromes are context − free − languages, and from the Pumping Lemma notes we also know that palindromes are non − regular.

Lemma to prove $L' = \{\{a,b\}^* \ : \ w \in ww^r\} \ = \ \Sigma^*$, *is a prefix for the palindrome* $L_1 = \{x \in \{a,b\}^* : \ x = x^r\}$
1) *Show for every word w in* L', *there exists y such that wy is in* L'
Direct proof:
Suppose $w \in L'$, *then there exists some* w^r *in* Σ^* *such that when w is concatenated with* w^r, ww^r *is an element of L, this holds for all w in L.*

2) *For every w word in* L', *break w into xy and prove x is in* L'
Choose w to be an arbitrary element of L_1, *where* $w \ = xy$, *x is the arbitrary prefix, since* $L' \ = \Sigma^*$.
It will be the case that x is an element of Σ^*.

Let L_1 *be a palindrome language* $= \{x \in \{a,b\}^* : \ x = x^r\}$
Description of palindromes:
(i) *it is the same forwards and backwards*
(ii) ww^r *or* waw^r *for* $w \in \Sigma^*$ *and* $a \in \Sigma$

The prefix language of (L_1), *is the set* L' *of all valid prefices of all words w, a prefix of word w is a string x such that* $W = xy$ *for some* $y \in \Sigma^*$.
Then from the description the $L' = prefix\ (L_1) \ = \{a,b\}^*$ *which is* Σ^* *so the prefix is the set some of a well known language, thus* $L' = prefix\ (L_1)$ *is regular.*
So, a non − regular context − free language L_1 *has* $prefix(L_1)$ L' *which is regular.*
□

(b) *Give an example of a non − regular context − free language* L_2 *and prove that prefix*(L_2) *is not regular.*

Lemma to prove $L' = \{0^n 1^m \ : \ m \ \leq \ n\}$ *is a prefix* $L_2 \ = \ \{0^n 1^n \ : \ n \ \geq \ 0\}$:
1) *show for every word w in* L', *there exists y such that wy is in* L'
2) *for every w word in* L', *break w into xy and prove x is in* L'
Proof by induction :
Suppose $z \in L'$, *then* $z \ = 0^n 1^m$ *where* $m \leq n$.
Base case ($n \ = 0$) :
There is only one word in L' *with* $n \ = 0$.
Choose $n = \ 0$, *and since* $m \leq n$, *it must mean that* $m \ = \ n$ *as you can not have negative words so,*
$z \ = \varepsilon$ *which is epsilon and we know that epsilon is the prefix of all languages, thus this holds for the base case.*
Inductive Hypothesis : *Assume that there is some value of n such that,* $\{0^n 1^m \ : \ m \ \leq \ n\}$ *is in* L'.
Inductive step :
$m \ \leq \ n$ *if we increment n, then we must increment m as well,* $m + 1 \ \leq \ n + 1$.
Show that this holds true for $n + 1$ *such that* $\{0^{n+1} 1^{m+1} \ : \ m \ \leq \ n\}$ *is in* L'.
$0^{n+1} 1^{m+1} = 00^n 11^m$
Any number of 0*s followed by any number of* 1*s less than or equal to the number of* 0*s is intuitively a prefix.*
To show number 1) $z = wy$ *in* L' :
Choose $w \ = 00^n$ *and* $y \ = \ 11^m$, *thus we know any number of* 0*s is a prefix of L and when* 0*s is concatenated with* 1*s where the number of* 1*s is less than equal to the number* 0*s is also a prefix of L upto and including L.*
Thus, this proves that the induction step holds and that $L' = \{0^n 1^m \ : \ m \ \leq \ n\}$ *is a prefix* $L_2 = \{0^n 1^n \ : \ n \ \geq \ 0\}$.

Let $L_2 \ = \ \{0^n 1^n \ : \ n \ \geq \ 0\}$ *be a non − regular context − free language, without proving it here as we can use the given fact in question* #3 *that the language is a context − free language as it has a grammer that creates it. We can also use the pumping lemma example* #1 *in the lecture slides that the language is non − regular.*
The language of prefix (L_2), *is the set of all valid prefices of all words w, a prefix of word w is a string x such that* $W = xy$ *for some* $y \in \Sigma^*$.
Then from the above lemma $L' = prefix\ (L_2) \ = \{0^n 1^m \ : \ m \ \leq \ n\}$, *intuitively this language is not accepted by a finite state machine, due to limitation in memory of a FSM.*
However we can prove this formally using Pumping Lemma.

Proof by contradiction.
Assume that L' *is regular. Therefore there exists some pumping length p.*
While we don't know what p is exactly, we are free to choose any word in the process of proving the language is not regular. So choose $z \ = \ 0^p 1^p$. *This word is in* L' *and it is also longer than* p : $|z| \ = \ p \ + \ p \ = \ 2p$.

Now consider all possible decompositions of $z = uvw$ where $|uv| \le p$ and $|v| \ge 1$. Because our word is $0^p 1^p$ we must have that $uv = 0^{i+j}$. That is, it can only consist of zeros. Let $v = 0^J$ and $u = 0^i$. Then $w = 0^{p-i-j}1^p$. Now consider the word $z^0 = uv^0w$. This is equal to $0^i 0^{p-i-j} 1^p$ where $j \ge 1$. But $i + p - i - j = p - j \neq p$, thus the word is of the form $0^{p-j}1^p$ and is not an element of L' as $j \ge 1$ and due to our condition being that $m \le n$, there are more or equal number of 0s than 1s. We conclude that our assumption that the language is regular is wrong. Therefore the language is not regular. Thus, the non-regular context-free language language L_2 has a prefix(L_2) L' that is not regular.

□

Note that you may specify $L1$ or $L2$ using math notation and give a grammar without doing a proof that the grammar generates the language. If you use a language that we have not shown was non-regular for $L1$ or $L2$, however, you must prove that the language you use is non-regular (i.e., with pumping lemma, etc.).

Proof formats taken directly from the notes and applied to the assignment question answers. *