

cpsc 313 assignment 4

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TOTAL POINTS

48 / 50

QUESTION 1

1 question 1 10 / 10

✓ - 0 pts Correct

- 2 pts Minor Errors
- 2 pts Chosen word not in language
- 2 pts Missing proper descriptions of steps
- 2 pts Tried to prove the wrong thing
- 4 pts Pumping length n specified instead of generic
- 4 pts Missing possible decompositions
- 4 pts Decomposition incorrect
- 10 pts Incorrect
- 10 pts Missing

QUESTION 2

2 question 2 10 / 10

✓ - 0 pts Correct

- 1 pts Small error in proving complement
- 3 pts Major error in proving complement
- 5 pts Not proving language given
- 7 pts Proved the complement instead of the language given
- 8 pts Not covering all possible decompositions
- 9 pts Mostly Incorrect
- 10 pts Incorrect
- 10 pts Missing
- 2 pts error in proving complement

QUESTION 3

3 question 3 8 / 10

- 0 pts Correct
- 1 pts minor error/missing minor step
- 2 pts sloppiness in one or more proofs
- 3 pts Major Sloppiness in one or more of the proofs
- 3 pts Wrong/No Grammar

$L(S) \subseteq L$

- 2 pts Wrong base case
- 1 pts Errors in hypothesis

✓ - 2 pts Errors in inductive step

$L \subseteq L(S)$

- 2 pts Wrong base case
- 1 pts Errors in the hypothesis
- 2 pts Errors in the induction step
- 5 pts Half Proof is wrong/missing
- 6 pts Incomplete proofs
- 10 pts Missing/Incorrect
- + 2 pts (only a correct grammar provided)

- 1 what is b ??
- 2 where did this b come from?
- 3 this is not in the language

QUESTION 4

4 question 4a 10 / 10

✓ - 0 pts Correct

- 1 pts Minor error/ Minor sloppiness
- 2 pts sloppiness in the proof
- 2 pts Incorrect proof of language not regular and context free
- 3 pts major sloppiness in the proof
- 3 pts Incorrect Language Choice
- 3 pts Incorrect proof that prefix is regular
- 4 pts Incorrect/No proof of correct Prefix
- 10 pts Incorrect/No Answer

QUESTION 5

5 question 4b 10 / 10

✓ - 0 pts Correct

- 1 pts Minor error/ Minor sloppiness

- **2 pts** sloppiness in the proof
- **3 pts** major sloppiness in the proof
- **2 pts** Incorrect proof of language not regular and context free

- **3 pts** Incorrect Language Choice
- **3 pts** Incorrect proof that prefix is not regular
- **2 pts** Incorrect prefix
- **4 pts** Incorrect/No proof of correct Prefix
- **10 pts** Incorrect/No Answer

1) Prove that the following language is not regular : $\{a^i b^j a^i b^i : i \leq j\}$.

From the pumping lemma examples :

$L = \{a^n b^n a^n b^n : n \geq 0\}$. We will use the pumping lemma to show that L is not regular.

Proof by contradiction. Assume that L is regular. Therefore there exists some pumping length n .

While we don't know what n is exactly, we are free to choose any word in the process of proving the language is not regular. So choose $z = a^n b^n a^n b^n$. This word is in L and it is also longer than n : $|z| = n + n + n + n = 4n$. Now consider all possible decompositions of $z = uvw$ where $|uv| \leq n$ and $|v| \geq 1$. Because our word is $a^n b^n a^n b^n$ we must have that $uv = a^{i+j}$. That is, it can only consist of a . Let $v = a^j$ and $u = a^i$. Then $w = a^{n-i-j} b^n a^n b^n$. Now consider the word $z^0 = uv^0w$.

This is equal to $a^i a^{n-i-j} b^n a^n b^n$ where $j \geq 1$. But $i + n - i - j = n - j \neq n$.

Thus the word is not of the form $a^n b^n a^n b^n$ and is not an element of L . We conclude that our assumption that the language is regular is wrong. Therefore the language is not regular.

□

1 question 1 10 / 10

✓ - 0 pts Correct

- 2 pts Minor Errors
- 2 pts Chosen word not in language
- 2 pts Missing proper descriptions of steps
- 2 pts Tried to prove the wrong thing
- 4 pts Pumping length n^4 specified instead of generic
- 4 pts Missing possible decompositions
- 4 pts Decomposition incorrect
- 10 pts Incorrect
- 10 pts Missing

2) Prove that the following language is not regular : $\{w \in \Sigma^* : w \neq w^r\}$.

From the pumping lemma example #3, complement closure of regularity, definition of palindromes and palindromes not being regular from the lecture notes :

$L = \{w \in \Sigma^* : w \neq w^r\}$ and by taking the complement of the language we preserve the regularity of the language, the complement is : $\bar{L} = \{w \in \Sigma^* : w = w^r\}$. We see that this is a language of palindromes, based on the lecture notes of pumping lemma example #3 we know that palindromes are not regular and since complement operation holds regularity, it means our language is not regular. Thus, language : $\{w \in \Sigma^* : w \neq w^r\}$ is not regular.

□

2 question 2 10 / 10

✓ - 0 pts Correct

- 1 pts Small error in proving complement
- 3 pts Major error in proving complement
- 5 pts Not proving language given
- 7 pts Proved the complement instead of the language given
- 8 pts Not covering all possible decompositions
- 9 pts Mostly Incorrect
- 10 pts Incorrect
- 10 pts Missing
- 2 pts error in proving complement

3) Consider the following "addition checker" language $\{a^i b^j c^{i+j} : i, j \geq 0\}$ where the number of c's is equal to the sum of the number of a's and b's. Provide a grammar for that language and formally prove that the grammar accepts the language. You may use without proof the fact that the grammar $S \rightarrow 0S1 \mid \epsilon$ generates the language $0^i 1^i : i \geq 0$.

The grammar for this language is :

$$G = (\{S, B\}, \{a, b, c\}, P, S)$$

$$S \rightarrow aSc \mid B$$

$$B \rightarrow bBc \mid \epsilon$$

To prove that this grammar works we have to show that $L = L(G)$.

Therefore we have to show this in two steps $L \subseteq L(G)$ and $L(G) \subseteq L$.

To prove that $L \subseteq L(G)$, we choose an arbitrary $x \in L$ and prove that there is a derivation of x in G , i.e. $S \Rightarrow^* x$. This is done by induction on the length of the word x .

$$a^i b^j c^{i+j} = a^i b^j c^i c^j$$

The language generated by B from our production is $\{b^j c^j : j \geq 0\}$.

Thus without loss of generality we now have the language $L = \{a^i B c^i : i \geq 0\}$.

Now by using induction on i we can prove for all i , $S \Rightarrow a^i B c^i$.

Base case ($i = 0$) :

Suppose $z \in L$, then $z = a^i B c^i = a^0 B c^0 = B$. So $S \Rightarrow B$ is a derivation of z , this holds for $i = 0$.

Inductive Hypothesis :

Assume that there is some value of n such that, $a^n B c^n$ has a derivation in L .

Induction Step :

Show that there is some value n such that $a^{n+1} B c^{n+1}$, by constructing the derivation.

We start with $S \Rightarrow aSc$. By our induction hypothesis we have that $S \Rightarrow^* a^n B c^n$.

Using that as the ultimate derivation of B in the sentinel form aBc .

Thus, $aBc \Rightarrow^* aa^n b^n b c^n c = a^{i+1} b^{j+1} c^{i+j+1}$, holds for the induction step and $L \subseteq L(G)$.

To prove that $L(G) \subseteq L$, we start with a word x that can be derived by the grammar, prove that $x \in L$, this is done by using induction on the number of steps in the derivation.

If $S \Rightarrow^n x$ then $x = a^i b^j c^{i+j}$ where $i, j \geq 0$.

Base Case ($n = 2$) :

For our production to end we need to atleast two steps.

If we derive a word the only possibility according to our grammar :

$$S \Rightarrow B \Rightarrow \epsilon, \text{ which is } S \Rightarrow \epsilon.$$

The word of epsilon, $x = \epsilon$ which $\in L$ if $i, j = 0$ $a^0 b^0 c^{0+0}$

Thus, this holds for the base case.

Inductive Hypothesis :

Assume for all words x , if x is derived in k steps, $S \Rightarrow^k x$ then $x \in L$ where $x = \{a^i b^j c^{i+j} : i, j \geq 0\}$.

Induction Step :

Prove this for $k + 1$ derivation steps.

Suppose that $x \in \Sigma^*$ such that $S \Rightarrow^{k+1} x$. If we have to start at symbol S and it has at least one derivation then we can use $S \Rightarrow aSc \Rightarrow^k x$.

Then we can rewrite $x = anb$ such that $S \Rightarrow^k n$. From our inductive hypothesis, the word $n = a^i b^j c^{i+j}$.

This means that $x = aa^i b b^j c c^{i+j} = a^{i+1} b^{j+1} c^{i+j+1}$ thus this holds for inductive step so, $L(G) \subseteq L$.

By set equalities we see that $L \subseteq L(G)$ and $L(G) \subseteq L$, thus $L = L(G)$.

□

3 question 3 8 / 10

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$L(S) \subseteq L$

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- 1 what is b ??
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4)(Note : this question has two parts but it will be marked as though it were two separate questions.)
 In class we showed that the language $\text{prefix}(L)$ is closed for context-free languages, meaning that if L is context-free then $\text{prefix}(L)$ is also context free. This does not imply, however, that if L is not regular then $\text{prefix}(L)$ is not regular.

(a) Give an example of a non-regular context-free language L_1 and prove that $\text{prefix}(L_1)$ is regular.
 From the context-free-grammar lecture note we know that palindromes are context-free-languages, and from the Pumping Lemma notes we also know that palindromes are non-regular.

Lemma to prove $L' = \{ \{a, b\}^* : w \in ww^r \} = \Sigma^*$, is a prefix for the palindrome $L_1 = \{x \in \{a, b\}^* : x = x^r\}$

1) Show for every word w in L' , there exists y such that wy is in L'

Direct proof :

Suppose $w \in L'$, then there exists some w^r in Σ^* such that when w is concatenated with w^r , ww^r is an element of L , this holds for all w in L .

2) For every w word in L' , break w into xy and prove x is in L'

Choose w to be an arbitrary element of L_1 , where $w = xy$, x is the arbitrary prefix, since $L' = \Sigma^*$.
 It will be the case that x is an element of Σ^* .

Let L_1 be a palindrome language $= \{x \in \{a, b\}^* : x = x^r\}$

Description of palindromes :

(i) it is the same forwards and backwards

(ii) ww^r or waw^r for $w \in \Sigma^*$ and $a \in \Sigma$

The prefix language of (L_1) , is the set L' of all valid prefixes of all words w , a prefix of word w is a string x such that $w = xy$ for some $y \in \Sigma^*$.

Then from the description the $L' = \text{prefix}(L_1) = \{a, b\}^*$ which is Σ^* so the prefix is the set some of a well known language, thus $L' = \text{prefix}(L_1)$ is regular.

So, a non-regular context-free language L_1 has $\text{prefix}(L_1)$ L' which is regular.

□

4 question 4a 10 / 10

✓ - **0 pts** Correct

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- **2 pts** Incorrect proof of language not regular and context free
- **3 pts** major sloppiness in the proof
- **3 pts** Incorrect Language Choice
- **3 pts** Incorrect proof that prefix is regular
- **4 pts** Incorrect/No proof of correct Prefix
- **10 pts** Incorrect/No Answer

(b) Give an example of a non-regular context-free language L_2 and prove that $\text{prefix}(L_2)$ is not regular.

Lemma to prove $L' = \{0^n 1^m : m \leq n\}$ is a prefix $L_2 = \{0^n 1^n : n \geq 0\}$:

1) show for every word w in L' , there exists y such that wy is in L'

2) for every word w in L' , break w into xy and prove x is in L'

Proof by induction :

Suppose $z \in L'$, then $z = 0^n 1^m$ where $m \leq n$.

Base case ($n = 0$) :

There is only one word in L' with $n = 0$.

Choose $n = 0$, and since $m \leq n$, it must mean that $m = n$ as you can not have negative words so,

$z = \epsilon$ which is epsilon and we know that epsilon is the prefix of all languages, thus this holds for the base case.

Inductive Hypothesis : Assume that there is some value of n such that, $\{0^n 1^m : m \leq n\}$ is in L' .

Inductive step :

$m \leq n$ if we increment n , then we must increment m as well, $m + 1 \leq n + 1$.

Show that this holds true for $n + 1$ such that $\{0^{n+1} 1^{m+1} : m \leq n\}$ is in L' .

$$0^{n+1} 1^{m+1} = 00^n 11^m$$

Any number of 0s followed by any number of 1s less than or equal to the number of 0s is intuitively a prefix.

To show number 1) $z = wy$ in L' :

Choose $w = 00^n$ and $y = 11^m$, thus we know any number of 0s is a prefix of L and when 0s is concatenated with 1s where the number of 1s is less than equal to the number 0s is also a prefix of L upto and including L .

Thus, this proves that the induction step holds and that $L' = \{0^n 1^m : m \leq n\}$ is a prefix $L_2 = \{0^n 1^n : n \geq 0\}$.

Let $L_2 = \{0^n 1^n : n \geq 0\}$ be a non-regular context-free language, without proving it here as we can use the given fact in question #3 that the language is a context-free language as it has a grammar that creates it. We can also use the pumping lemma example #1 in the lecture slides that the language is non-regular.

The language of prefix (L_2), is the set of all valid prefixes of all words w , a prefix of word w is a string x such that $W = xy$ for some $y \in \Sigma^*$.

Then from the above lemma $L' = \text{prefix}(L_2) = \{0^n 1^m : m \leq n\}$, intuitively this language is not accepted by a finite state machine, due to limitation in memory of a FSM.

However we can prove this formally using Pumping Lemma.

Proof by contradiction.

Assume that L' is regular. Therefore there exists some pumping length p .

While we don't know what p is exactly, we are free to choose any word in the process of proving the language is not regular. So choose $z = 0^p 1^p$. This word is in L' and it is also longer than

$$p : |z| = p + p = 2p.$$

Now consider all possible decompositions of $z = uvw$ where $|uv| \leq p$ and $|v| \geq 1$. Because our word is $0^p 1^p$ we must have that $uv = 0^{i+j}$. That is, it can only consist of zeros. Let $v = 0^j$ and $u = 0^i$. Then $w = 0^{p-i-j} 1^p$. Now consider the word $z^0 = uv^0 w$. This is equal to $0^i 0^{p-i-j} 1^p$ where $j \geq 1$. But $i + p - i - j = p - j \neq p$, thus the word is of the form $0^{p-j} 1^p$ and is not an element of L' as $j \geq 1$ and due to our condition being that $m \leq n$, there are more or equal number of 0s than 1s. We conclude that our assumption that the language is regular is wrong. Therefore the language is not regular. Thus, the non-regular context-free language language L_2 has a prefix(L_2) L' that is not regular.

□

Note that you may specify L_1 or L_2 using math notation and give a grammar without doing a proof that the grammar generates the language. If you use a language that we have not shown was non-regular for L_1 or L_2 , however, you must prove that the language you use is non-regular (i.e., with pumping lemma, etc.).

Proof formats taken directly from the notes and applied to the assignment question answers. *

5 question 4b 10 / 10

✓ - 0 pts Correct

- 1 pts Minor error/ Minor sloppiness
- 2 pts sloppiness in the proof
- 3 pts major sloppiness in the proof
- 2 pts Incorrect proof of language not regular and context free
- 3 pts Incorrect Language Choice
- 3 pts Incorrect proof that prefix is not regular
- 2 pts Incorrect prefix
- 4 pts Incorrect/No proof of correct Prefix
- 10 pts Incorrect/No Answer