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Global Consensus Control of Lipschitz Nonlinear Multi-Agent Systems

Zhongkui Li *, **Xiangdong Liu** *, **Mengyin Fu** *

* *School of Automation, Beijing Institute of Technology, Beijing 100081, P. R. China (e-mail: zhongkli@gmail.com).*

Abstract: This paper addresses the distributed global consensus problem of a class of nonlinear multi-agent systems with Lipschitz nonlinearity. A distributed consensus protocol based on the relative states of neighboring agents is proposed. The global consensus problem of a Lipschitz nonlinear multi-agent system is cast into the feasibility of a set of matrix inequalities having the same dimension as that of a single agent. The notion of global consensus region is then introduced and analyzed. It is pointed out through numerical examples that the global consensus region can serve as a measure for the robustness of consensus with respect to the communication topology. A necessary and sufficient condition for the existence of a protocol having an unbounded global consensus region is derived. A two-step procedure is further presented for constructing such a protocol. The effectiveness of the theoretical results is demonstrated through a network of single-link manipulators.

1. INTRODUCTION

In recent years, the consensus control problem of multi-agent systems has received compelling attention from various scientific communities, for its potential applications in such broad areas as satellite formation flying, cooperative unmanned air vehicles, and sensor networks, to name just a few Olfati-Saber et al. [2007], Ren et al. [2007a]. The main idea of consensus is to develop distributed control policies that enables a group of agents to reach an agreement on certain quantities of interest.

Consensus problems have been extensively studied by numerous researchers from various perspectives. A simple model is proposed in Vicsek et al. [1995] for phase transition of a group of self-driven particles with numerical demonstration of the complexity of the model. A theoretical explanation is provided in Jadbabaie et al. [2003] for the behavior observed in Vicsek et al. [1995] by using graph theory. In Olfati-Saber and Murray [2004], a general framework of the consensus problem for networks of dynamic agents with fixed or switching topologies is addressed. The conditions given by Olfati-Saber and Murray [2004] are further relaxed in Ren and Beard [2005]. In Hong et al. [2008], tracking control for multi-agent consensus with an active leader is considered, where a local controller is designed together with a neighbor-based state-estimation rule. The controlled agreement problem for multi-agent networks is considered from a graph-theoretic perspective in Rahmani et al. [2009]. Some predictive mechanisms are introduced in Zhang et al. [2008] to achieve ultrafast consensus. A distributed algorithm is proposed in Cortés [2008] to asymptotically achieve consensus in finite time. The consensus problem of networks of double- and high-order integrator agents is studied in Ren [2008], Lin and Jia [2009], Xie and Wang [2007], Ren et al. [2007b]. Distributed consensus control of multi-agent systems with general linear dynamics is concerned in Li et al. [2010, 2011, in press, 2011], Tuna [2009], Scardovi and Sepulchre [2009].

This paper considers the distributed global consensus problem of a class of nonlinear multi-agent systems with Lipschitz nonlinearity and undirected communication topologies. A distributed consensus protocol is proposed, based on the relative states of neighboring agents. It is shown that the global consensus problem of a Lipschitz nonlinear multi-agent system can be cast into the test of a set of matrix inequalities having the same dimension as that of a single agent. A distinct feature of this paper is that by introducing a positive scalar, i.e., the coupling strength, the notion of global consensus region is characterized and analyzed. The global consensus region can be regarded as an extension of the consensus region introduced in Li et al. [2010, 2011] for linear multi-agent systems to the case with nonlinear multi-agent systems.

It is pointed out through numerical examples that the global consensus region can serve as a measure for the robustness of consensus with respect to the communication topology. In order to ensure consensus a desired robustness margin, the global consensus region should be large enough. A necessary and sufficient condition for the existence of a protocol having an unbounded global consensus region is derived. A two-step procedure is further presented for constructing such a protocol, which maintains a favorable decoupling property. A network of single-link manipulators are utilized to illustrate the effectiveness of the theoretical results.

It should be mentioned that consensus problems of multi-agent systems with Lipschitz nonlinearity were also considered in Yu et al. [2010], Song et al. [2010]. However, the agent dynamics are restricted to be second-order there. By contrast, the results derived in this paper are applicable to any high-order Lipschitz nonlinear multi-agent system. Furthermore, a novel global-consensus-region-based consensus protocol design procedure is provided here.

The rest of this paper is organized as follows. Some basic notation and useful results of the graph theory are reviewed in Section 2. The problem is formulated in Section 3. The global consensus region is analyzed and the consensus protocol is designed in Section 4. Section 5 concludes the paper.

2. CONCEPTS AND NOTATION

Let $\mathbf{R}^{n\times n}$ be the set of $n\times n$ real matrices. $\mathbf{R}_+$ denotes the set of positive numbers. The superscript T means the transpose for real matrices. I_N represents the identity matrix of dimension N. Denote by $\mathbf{1}\in\mathbf{R}^p$ the vector with all entries equal to one. Matrices, if not explicitly stated, are assumed to have compatible dimensions. $\mathrm{diag}(A_1,\cdots,A_n)$ represents a block-diagonal matrix with matrices $A_i, i=1,\cdots,n$, on its diagonal. The matrix inequality $A>B$ means that A and B are square Hermitian matrices and $A-B$ is positive definite. $A\otimes B$ denotes the Kronecker product of matrices A and B. A matrix is Hurwitz if all of its eigenvalues have negative real parts.

An undirected graph $\mathcal{G}$ is a pair $(\mathcal{V},\mathcal{E})$, where $\mathcal{V}=\{1,\cdots,p\}$ is the set of nodes and $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ is the set of unordered pairs of nodes, called edges. Two nodes i, j are adjacent, or neighboring, if (i,j) is an edge of graph $\mathcal{G}$. A path on $\mathcal{G}$ from node i_1 to node i_l is a sequence of ordered edges of the form (i_k,i_{k+1}), $k=1,\cdots,l-1$. Graph $\mathcal{G}$ is called connected if there exists a path between every pair of distinct nodes, otherwise disconnected.

Suppose that there are m nodes in a graph. The adjacency matrix $A=(a_{ij})\in\mathbf{R}^{m\times m}$ is defined by $a_{ii}=0$, $a_{ij}=a_{ji}=1$ if $(j,i)\in\mathcal{E}$ and 0 otherwise. The Laplacian matrix $\mathcal{L}\in\mathbf{R}^{m\times m}$ is defined as $\mathcal{L}_{ii}=\sum_{j\neq i}a_{ij}$, $\mathcal{L}_{ij}=-a_{ij}$ for $i\neq j$. For an undirected graph, both its adjacency matrix and its Laplacian matrix are symmetric.

Lemma 1. Olfati-Saber and Murray [2004], Ren and Beard [2005] Zero is an eigenvalue of $\mathcal{L}$ with $\mathbf{1}$ as the corresponding right eigenvector and all the nonzero eigenvalues are positive. Furthermore, zero is a simple eigenvalue of $\mathcal{L}$ if and only if the graph is connected.

3. PROBLEM STATEMENT

Consider a group of N identical nonlinear agents, described by

$$\dot{x}_i=Ax_i+Df(x_i)+Bu_i,\qquad i=1,2,\cdots,N,\tag{1}$$

where $x_i\in\mathbf{R}^n$, $u_i\in\mathbf{R}^p$ are the state and the control input of the i-th agent, respectively, A, B, D, are constant matrices with compatible dimensions, and the nonlinear function $f(x_i)$ is assumed to satisfy the Lipschitz condition with a Lipschitz constant γ, i.e.,

$$\|f(x)-f(y)\|\leq\gamma\|x-y\|,\quad\forall\,x,y\in\mathbf{R}^n.\tag{2}$$

The communication topology among the N agents is represented by an undirected graph $\mathcal{G}$ consisting of the node set $\mathcal{V}=\{1,\cdots,N\}$ and the edge set $\mathcal{E}\subset\mathcal{V}\times\mathcal{V}$. An edge (i,j) $(i\neq j)$ means that agents i and j can obtain information from one another.

It is assumed that at each time instant each agent knows the relative states of its neighboring agent with respect to itself. In order to achieve consensus, the following distributed consensus protocol is proposed:

$$u_i=cK\sum_{j=1}^{N}a_{ij}(x_i-x_j),\tag{3}$$

where $c>0$ denotes the coupling strength, $K\in\mathbf{R}^{p\times n}$ is feedback gain matrix to be determined, and $(a_{ij})_{N\times N}$ is the adjacency matrix of graph $\mathcal{G}$.

The objective is to find a consensus protocol (3) such that the N agents in (1) can achieve global consensus, in the sense of $\lim_{t\to\infty}\|x_i(t)-x_j(t)\|=0,\quad\forall\,i,j=1,2,\cdots,N.$

Let $\bar{x}=\frac{1}{N}\sum_{j=1}^{N}x_j$, $e_i=x_i-\bar{x}$, and $e=[e_1^T,\cdots,e_N^T]^T$. Then, one gets $e=\left((I_N-\frac{1}{N}\mathbf{1}\mathbf{1}^T)\otimes I_n\right)x$. It is easy to see that 0 is a simple eigenvalue of $I_N-\frac{1}{N}\mathbf{1}\mathbf{1}^T$ with $\mathbf{1}$ as the corresponding right eigenvector, and 1 is another eigenvalue with multiplicity $N-1$. Then, it follows that $e=0$ if and only if $x_1=\cdots=x_N$. Therefore, the global consensus of agents (1) under protocol (3) can be reduced to the asymptotical stability of $e(t)$, which evolves according to the following dynamics:

$$\dot{e}_i=Ae_i+Df(x_i)-\frac{1}{N}\sum_{j=1}^{N}Df(x_j)+c\sum_{j=1}^{N}\mathcal{L}_{ij}BKe_j,\qquad i=1,2,\cdots,N,\tag{4}$$

where $\mathcal{L}=(\mathcal{L}_{ij})_{N\times N}$ is the Laplacian matrix associated with graph $\mathcal{G}$.

The following presents a sufficient condition for the global consensus problem of (4).

Theorem 2. Assume that graph $\mathcal{G}$ is connected. Then, the N agents in (1) can reach global consensus under protocol (3), if there exist a matrix $P>0$ such that

$$(A+c\lambda_iBK)P+P(A+c\lambda_iBK)^T+\gamma^2DD^T+P^2<0,\qquad i=2,3,\cdots,N,\tag{5}$$

where λ_i, $i=2,3,\cdots,N$, are the nonzero eigenvalues of $\mathcal{L}$.

Proof. Consider the Lyapunov function candidate

$$V(t)=\sum_{i=1}^{N}e_i^T(t)P^{-1}e_i(t).\tag{6}$$

The time derivative of $V(t)$ along the trajectory of (4) is given by

$$\begin{aligned}\dot{V}&=2\sum_{i=1}^{N}e_i^TP^{-1}\left[Ae_i+Df(x_i)-\frac{1}{N}\sum_{j=1}^{N}Df(x_j)\right.\\&\quad\left.+c\sum_{j=1}^{N}\mathcal{L}_{ij}BKe_j\right]\\&=2e^T(I_N\otimes P^{-1})(I_N\otimes A+c\mathcal{L}\otimes BK)e+2\sum_{i=1}^{N}e_i^T(t)\\&\quad\times P^{-1}D\left[f(x_i)-f(\bar{x})+f(\bar{x})-\frac{1}{N}\sum_{j=1}^{N}f(x_j)\right].\end{aligned}\tag{7}$$

Using the Lipschitz condition (2) gives

$$\begin{aligned} 2e_i^T P^{-1} D(f(x_i) - f(\bar{x})) &\le 2\gamma \|e_i^T P\| \|e_i\| \\ &\le \gamma^2 e_i^T P^{-1} DD^T P^{-1} e_i + e_i^T e_i. \end{aligned} \tag{8}$$

Since $\sum_{i=1} e_i^T = 0$, one has

$$\sum_{i=1} e_i^T P^{-1} D \left(f(\bar{x}) - \frac{1}{N} \sum_{j=1}^{N} f(x_j) \right) = 0. \tag{9}$$

Let $\tilde{e}_i = P^{-1} e_i$, $\tilde{e} = [\tilde{e}_1^T, \cdots, \tilde{e}_N^T]^T$. By noting (8) and (9), it follows from (7) that

$$\begin{aligned} \dot{V} = \tilde{e}^T [&I_N \otimes (AP + PA^T + \gamma^2 DD^T + P^2) \\ &+ c\mathcal{L} \otimes BKP]\tilde{e}. \end{aligned} \tag{10}$$

Let $U \in \mathbf{R}^{N \times N}$ be such a unitary matrix that $U^T \mathcal{L} U = \Lambda = \operatorname{diag}(0, \lambda_2, \cdots, \lambda_N)$. Since the right and left eigenvectors of $\mathcal{L}$ corresponding to eigenvalue 0 are $\mathbf{1}$ and $\mathbf{1}^T$, respectively, one can choose $U = [\frac{\mathbf{1}}{\sqrt{N}}\ Y_1]$ and $U^T = \begin{bmatrix} \frac{\mathbf{1}^T}{\sqrt{N}} \\ Y_2 \end{bmatrix}$, with $Y_1 \in \mathbf{R}^{N \times (N-1)}$, $Y_2 \in \mathbf{R}^{(N-1) \times N}$. Let $\xi(t) \triangleq [\xi_1^T, \cdots, \xi_N^T]^T = (U^T \otimes I_n) e(t)$. By the definition of $e(t)$, it is easy to see that $\xi_1 = (\frac{\mathbf{1}^T}{\sqrt{N}} \otimes I_n) e = 0$. Then, one has

$$\begin{aligned} \dot{V} &\le 2e^T \left[(U \otimes I_n)(I_N \otimes AP + c\mathcal{L} \otimes BKP)(U^T \otimes I_n) \right] e \\ &\quad + e^T (\gamma^2 I_N \otimes DD^T + P^2) e \\ &\le 2\xi^T \left[(I_N \otimes AP + c\Lambda \otimes BKP) \right] \xi \\ &\quad + \xi^T (\gamma^2 I_N \otimes DD^T + P^2) \xi \\ &= \sum_{i=2}^{N} \xi_i^T [(A + c\lambda_i BK)P + P(A + c\lambda_i BK)^T \\ &\quad + \gamma^2 DD^T + P^2] \xi_i \\ &= W(\xi(t)) \le 0. \end{aligned} \tag{11}$$

Note that $V(t)$ is positive definite and radically unbounded. By the LaSalle-Yoshizawa theorem (Theorem 8.4 in Khalil [2002]), one gets $\lim_{t \to \infty} W(\xi(t)) = 0$, implying that $\lim_{t \to \infty} \xi_i = 0$, $i = 2, \cdots, N$. By noticing that $\xi_1 \equiv 0$ and $e(t) = (U \otimes I_n)\xi(t)$, it follows that $\lim_{t \to \infty} e(t) = 0$, i.e., the global consensus of network (4) is achieved. ■

Remark 1. This theorem converts the global consensus of the N nonautonomous agents (1) under protocol (3) to the feasibility of a set of matrix inequalities with the same low dimensions as a single agent. The effects of the communication topology on the consensus are characterized by the nonzero eigenvalues of the corresponding Laplacian matrix $\mathcal{L}$.

4. GLOBAL CONSENSUS REGION

As seen in Theorem 2, the global consensus of the given agents (1) under protocol (3) depends on the feedback gain matrix K, the coupling strength c, and the nonzero eigenvalues λ_i of the Laplacian matrix associated with graph $\mathcal{G}$. In order to analyze their correlated effects on consensus, the notion of global consensus region is introduced.

Definition 1. The region $\mathcal{S}$ of the parameter $\sigma \in \mathbf{R}_+$, such that any of the following two statements holds:

1) there exists a $P > 0$ satisfying

$$(A + \sigma BK)P + P(A + \sigma BK)^T + \gamma^2 DD^T + P^2 < 0, \tag{12}$$

2) matrix $A + \sigma BK$ is Hurwitz and $\|D^T(sI - A - \sigma BK)^{-1}\|_\infty < \frac{1}{\gamma}$,

is called the global consensus region of network (4).

In the above definition, the equivalence between conditions 1) and 2) can be established by simply utilizing the Bounded Real lemma Zhou and Doyle [1998], as also pointed out in Rajamani [1998].

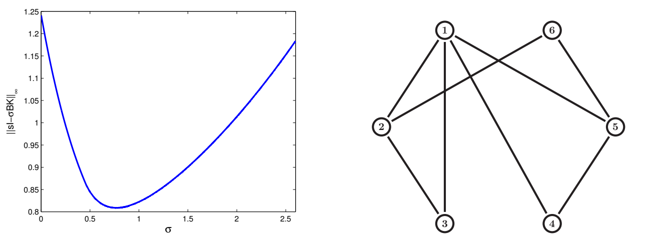
The above definition generalizes the consensus region notion previously introduced in Li et al. [2010, 2011] for linear multi-agent systems to the Lipschitz nonlinear multi-agent systems as described in (1). Also, the consensus region discussed here is similar in certain sense to the synchronized regions studied in Pecora and Carroll [1998], Duan et al. [2009, 2007] for linearized dynamical networks.

According to Theorem 2, one has,

Corollary 3. The agent network (4) with a connected graph $\mathcal{G}$ can reach global consensus, if $c\lambda_i \in \mathcal{S}$, for $i = 2, 3, \cdots, N$.

The importance of the notion of global consensus defined as above lies in that it decouples the global consensus problem into two parts: (1) design the feedback gain matrix K to yield a desirable consensus region; (2) appropriately select the coupling strength c such that $c\lambda_i$, $i = 2, 3, \cdots, N$, lie within this region. This decoupling feature will simplify the consensus protocol design, which will be detailedly discussed in the sequel.

Given the agent dynamics, the global consensus region $\mathcal{S}$ depends only on the feedback gain matrix K. The consensus region $\mathcal{S}$ for network (4), if it exists, can be bounded, unbounded, or consists of several disconnected subregions. Detailed discussions on different types of global consensus regions can be carried out by following similar steps as in Li et al. [2010] for linear multi-agent systems. For conciseness, only an example having a bounded global consensus region is given as follows.



(a) (b)

Fig. 1. (a) The bounded global consensus region; (b) the communication topology.

Example 1. The agent dynamics and the consensus protocol are given by (1) and (3), respectively, with

$$A = \begin{bmatrix} -2 & 1.35 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad K = [1\ 0.65], \quad D = I,$$

and the Lipschitz constant γ in (2) is equal to $\frac{1}{0.9}$.

The characteristic equation of $A + \sigma BK$ is $s^2 + (3 - 0.35\sigma)s + 1.65\sigma + 2 = 0$. Therefore, matrix $A + \sigma BK$ is Hurwitz for $\sigma \in \mathbf{R}_+$ if and only if $0 < \sigma < \frac{3}{0.35}$. By depicting $\|D^T(sI - A - \sigma BK)^{-1}\|_\infty$ with respect to parameter σ as in Fig. 1 (a), the global consensus region $\mathcal{S}$ can be obtained as $\mathcal{S} = (0.3885, 1.4961)$. Clearly, $\mathcal{S}$ is bounded in this example.

For illustration, let the communication topology $\mathcal{G}$ be given as in Fig. 1 (b), with Laplacian matrix

$$\mathcal{L} = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 2 & -1 & 0 \\ -1 & 0 & 0 & -1 & 3 & -1 \\ -1 & 0 & 0 & 0 & -1 & 2 \end{bmatrix},$$

whose nonzero eigenvalues are 1.382, 1.6972, 3.618, 4, 5.3028. Thus, protocol (3) given as above solves the global consensus problem for the graph in Fig. 1 (b) if the coupling strength $c \in (0.2811,\ 0.2821)$.

Let's see how modifications of the communication topology affect the global consensus problem by considering the following simple cases:

- An edge between node 3 and node 6 is added, i.e., more information exchanges exist inside the network. The minimal and maximal nonzero eigenvalues of the resulting Laplacian matrix are 1.4384 and 5.5616, respectively. Thus, no coupling strength c can be found to satisfy Corollary 3, i.e., global consensus may not be guaranteed in this case. It should be noted that consensus maybe happens since Corollary 3 involves certain conservatism when being used for consensus analysis.
- The edge between node 2 and node 3 is removed, i.e., less information links in the network. The minimal and maximal nonzero eigenvalues of the resulting Laplacian matrix become 0.8817 and 5.2688, respectively. In this case, consensus may not be guaranteed either.

This simple example implies that, for bounded global consensus regions, the distributed protocol (3), if not well designed, can be quite fragile to variations of the network's communication topology. Hence, it is desirable that the global consensus region to be large enough in order to ensure consensus a desired robustness margin with respect to the communication topology. One convenient and desirable choice is to design the consensus protocol (3) to yield an unbounded global consensus region.

The following gives a necessary and sufficient condition for the existence of a protocol (3) guaranteeing an unbounded global consensus region.

Theorem 4. There exists a consensus protocol (3) yielding an unbounded global consensus region for network (4), if and only if there exist a matrix $P > 0$ and a scalar $\tau > 0$ such that

$$\begin{bmatrix} AP + PA^T - \tau BB^T + \gamma^2 DD^T & P \\ P & -I \end{bmatrix} < 0. \tag{13}$$

Then, the global synchronized region is in the form of $\mathcal{S} = [\tau,\ \infty)$.

Proof: (Necessity) By Definition 1, if network (4) has an unbounded global consensus region, then

$$(A + \sigma BK)P + P(A + \sigma BK)^T + \gamma^2 DD^T + P^2 < 0$$

for some matrices $P > 0$, K and some scalar σ. Since K is to be designed, without loss of generality, choose $\sigma = 1$. Let $Y = KP$. Then, the above inequality becomes

$$AP + PA^T + BY + Y^T B^T + \gamma^2 DD^T + P^2 < 0. \tag{14}$$

By Finsler's Lemma Iwasaki and Skelton [1994], there exists a matrix Y satisfying (14) if and only if there exists a scalar $\tau > 0$ such that

$$AP + PA^T - \tau BB^T + \gamma^2 DD^T + P^2 < 0, \tag{15}$$

which, in virtue of the Schur complement lemma Boyd et al. [1994], is equivalent to that there exist a matrix $P > 0$ and a scalar $\tau > 0$ such that (13) holds.

(Sufficiency) Take $K = -\frac{1}{2}B^T P^{-1}$. For $c\lambda_i \geq \tau$, $i = 2, 3, \cdots, N$, it follows from (15) that

$$\begin{aligned} &(A + c\lambda_i BK)P + P(A + c\lambda_i BK)^T + \gamma^2 DD^T + P^2 \\ &\quad = AP + PA^T - c\lambda_i BB^T + \gamma^2 DD^T + P^2 < 0, \end{aligned} \tag{16}$$

implying that network (4) with K as above has an unbounded global consensus region $[\tau, \infty)$. ■

A two-step procedure is presented to determine a consensus protocol (3) satisfying Theorem 2.

Algorithm 1. Assume that graph $\mathcal{G}$ is connected. If the condition in Theorem 4 is satisfied, the consensus protocol (3) can be constructed for network (4) to achieve global consensus, as follows:

1) Solve LMI (13) to get a matrix $P > 0$ and a scalar $\tau > 0$. Then, choose $K = -\frac{1}{2}B^T P^{-1}$.
2) Select the coupling strength $c \geq \frac{\tau}{\min\limits_{i=2,\cdots,N} \lambda_i}$, where λ_i, $i = 2, 3, \cdots, N$, are the nonzero eigenvalues of $\mathcal{L}$.

Remark 2. This global-consensus-region-based protocol design procedure has a favorable decoupling feature. To be specific, step 1) deals only with the agent dynamics in (1), leaving the communication topology of the agent network to be handled in step 2) by adjusting the coupling strength. One consequence of this decoupling property is that the protocol (3) designed for one given communication graph can be used directly to any other connected graphs, with the only task of appropriately selecting the coupling strength. Therefore, the consensus protocols derived by Algorithm 1 are scalable to some extent, i.e., without having to redesign the whole protocol when an agent is added or removed.

Remark 3. Consensus problems of multi-agent systems with Lipschitz nonlinearity were considered in Yu et al. [2010], Song et al. [2010]. However, the agent dynamics are restricted to be second-order there. By contrast, Theorem 2 and Algorithm 1 derived in this section are applicable to any high-order Lipschitz nonlinear multi-agent system. Furthermore, a novel consensus protocol design procedure, namely, based on the notion of global consensus region, is provided here.

Example 2. Consider a network of 6 single-link manipulators with revolute joints actuated by a DC motor. The dynamics of the i-th manipulator is described by (1), with Rajamani and Cho [1998], Zhu and Han [2002]

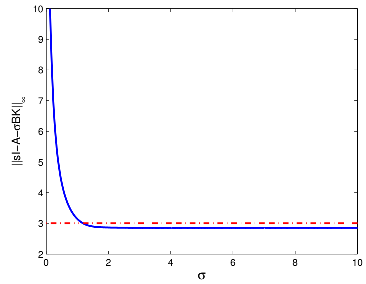


Fig. 2. The unbounded global consensus region.

$$x_i = \begin{bmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \\ x_{i4} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.26 & 48.6 & 0 \\ 0 & 0 & 0 & 10 \\ 1.95 & 0 & -1.95 & 0 \end{bmatrix}, \quad D = I,$$
$$B = [0\ 21.6\ 0\ 0]^T, \quad f(x_i) = [0\ 0\ 0\ -0.333\sin(x_{i1})]^T.$$

Clearly, $f(x_i)$ here satisfies (2) with a Lipschitz constant $\gamma = 0.333$.

Solving LMI (13) using the LMI toolbox of Matlab gives a feasible solution:

$$P = \begin{bmatrix} 0.4644 & -0.9947 & 0.4202 & -0.0916 \\ -0.9947 & 109.3616 & 0.0053 & -0.0022 \\ 0.4202 & 0.0053 & 0.6266 & -0.0563 \\ -0.0916 & -0.0022 & -0.0563 & 0.0492 \end{bmatrix}, \tau = 81.798.$$

Thus, by Algorithm 1, the feedback gain matrix of (3) is chosen as

$$K = [-0.9170\ -0.1071\ 0.5151\ -1.1228].$$

By Theorem 4, protocol (3) with such a matrix K has an unbounded global consensus region $[5.1868, \infty)$. This can be also verified in another way by depicting the $\|(sI - A - \sigma BK)^{-1}\|_\infty$ with respect to scalar σ in Fig. 2. For the graph in Fig. 1 (b), protocol (3) with K chosen here solves the global consensus problem, if the coupling strength $c \geq 59.1881$. The state trajectories of the 6 manipulators under protocol (3) with K given as above and $c = 60$ are depicted in Fig. 3, from which one can see that the global consensus is indeed achieved.

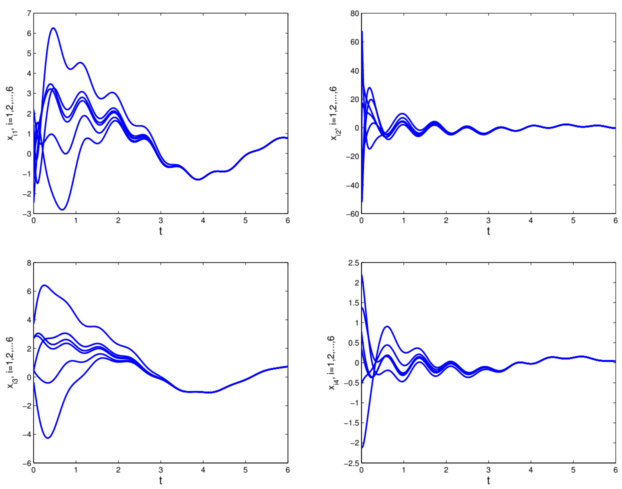


Fig. 3. The 6 manipulators reach global consensus.

5. CONCLUSION

This paper has considered the distributed global consensus problem of a class of nonlinear multi-agent systems with Lipschitz nonlinearity. A distributed consensus protocol based on the relative states of neighboring agents has been proposed. The notion of global consensus region has been introduced and analyzed. A necessary and sufficient condition for the existence of a protocol having an unbounded global consensus region has been derived, which ensure the consensus a desired margin of robustness with respect to the communication topology. A two-step procedure for constructing such a protocol has further been presented.

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