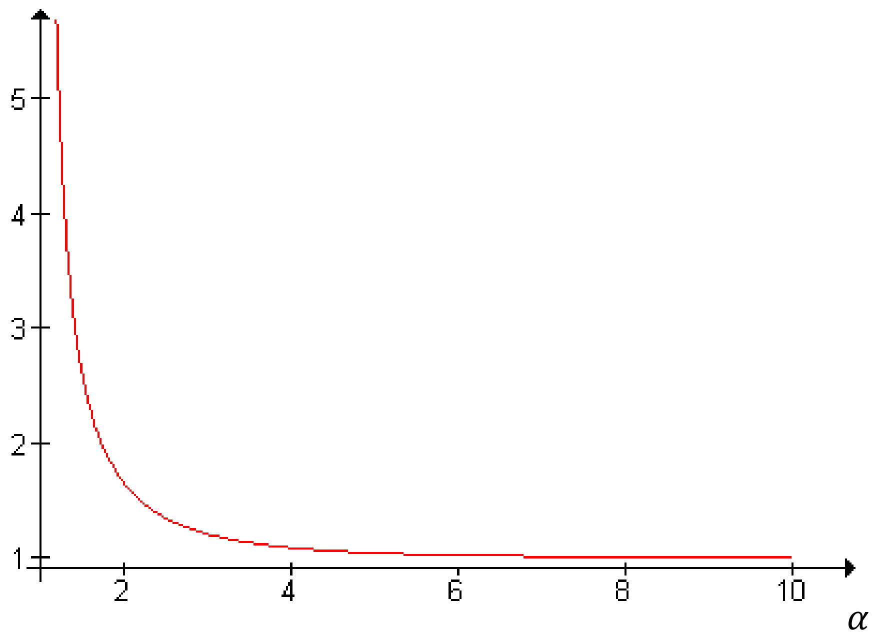
The *Riemann zeta function* ζ, named after *Bernhard Riemann*, is defined as follows:

$$\zeta(\alpha) = \sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}, \qquad \alpha \in (1, \infty)$$

Zeta function converges for $\alpha > 1$ and diverges for $\alpha \leq 1$.



The zeta function satisfies the following properties:

a. ζ is decreasing.
b. ζ is concave upward.
c. $\zeta(\alpha) \downarrow 1$ as $\alpha \uparrow \infty$.
d. $\zeta(\alpha) \uparrow \infty$ as $\alpha \downarrow 1$.

The zeta distribution with shape parameter $\alpha \in (1, \infty)$ is a discrete distribution on $\mathbb{N}_+$ with probability density function f given by:

$$f(n) = \frac{1}{\zeta(\alpha) n^{\alpha}}, \quad n \in \mathbb{N}_+$$

Suppose that N has the zeta distribution with shape parameter $\alpha \in (1, \infty)$:

- If $k \geq \alpha - 1$, $\mathbb{E}(X) = \infty$. If $k < \alpha - 1$, $\mathbb{E}(N^k) = \frac{\zeta(\alpha - k)}{\zeta(\alpha)}$.
- If $\alpha > 2$, mean of N equals $\mathbb{E}(N) = \frac{\zeta(\alpha-1)}{\zeta(\alpha)}$.
- If $\alpha > 3$, $var(N) = \frac{\zeta(\alpha-2)}{\zeta(\alpha)} - \left(\frac{\zeta(\alpha-1)}{\zeta(\alpha)}\right)^2$.

R code snippet:

```r
draw.zeta <- function(nrep, alpha)
{
    if (alpha <= 1) { stop("Alpha must be greater than 1!\n") }
    zeta <- numeric(nrep)
    for (i in 1:nrep)
    {
        index <- 0;
        while (index < 1)
        {
            u1 <- runif(1)
            u2 <- runif(1)
            x <- floor(u1 ^ (-1 / (alpha - 1)))
            t <- (1 + 1 / x) ^ (alpha - 1)
            w <- x < (t / (t - 1)) * (2 ^ (alpha - 1) - 1) / (2 ^ (alpha - 1) * u2)
            zeta[i] <- x
            index <- sum(w)
        }
    }
    zeta
}
```

Zeta (Zipf) distribution

PDF: $f(x \mid \alpha) = \dfrac{1}{\zeta(\alpha) x^{\alpha}}$ for x=1,2,3,... and α>1, where $\zeta(\alpha) = \sum_{x=1}^{\infty} x^{-\alpha}$ (Riemann zeta function). *EAA*: Modeling the frequency of random processes. *GA*: Acceptance/rejection algorithm of Devroye (1986).

$$E(X) = \frac{\zeta(\alpha - 1)}{\zeta(\alpha)},$$

$$V(X) = \frac{\zeta(a)\zeta(a-2) - (\zeta(a-1))^2}{(\zeta(a))^2}$$