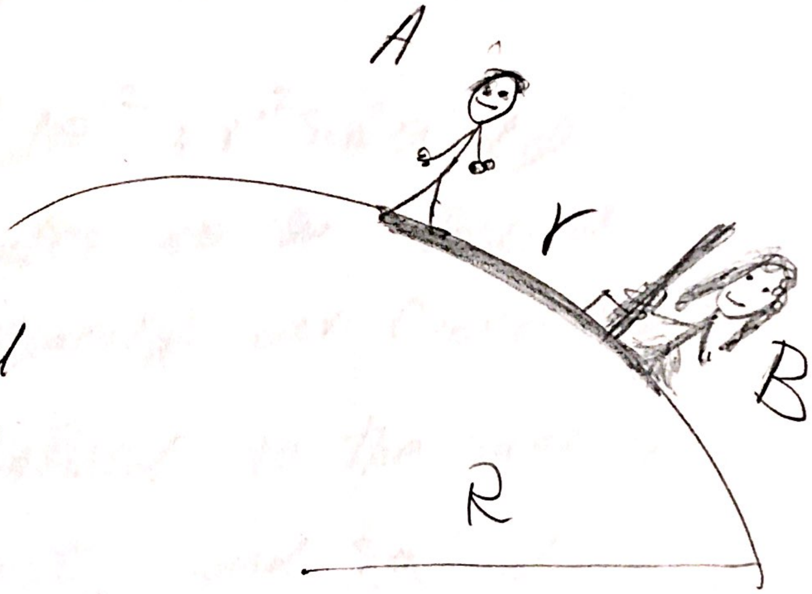
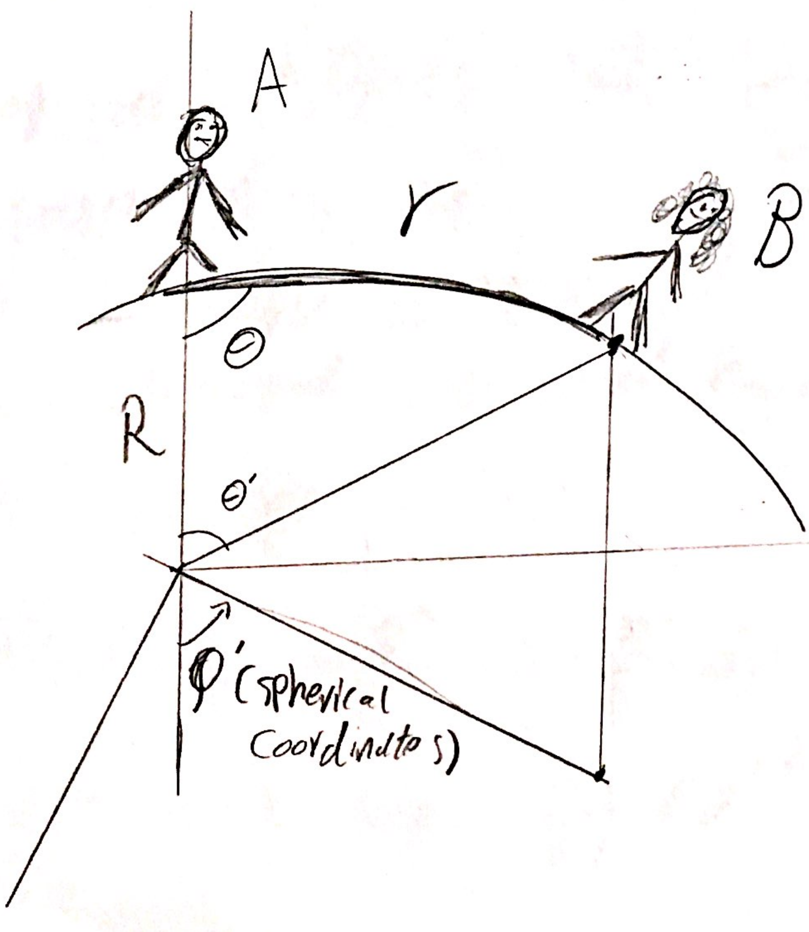
1)

a) In these coordinates, we are free to choose where our origin is.
To make use of spherical coordinates, let's set up our origin at the north Pole, then our figure becomes





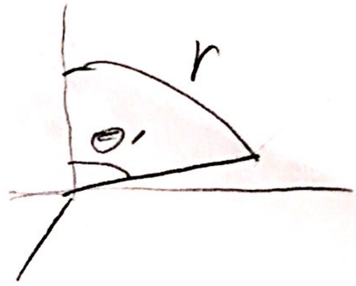
Now we can make use of spherical coordinates in 3D Euclidean space.
Recall the metric for spherical coordinates is:

$$ds^2 = dr'^2 + r'^2 d\theta'^2 + r'^2 \sin^2\theta' \, d\phi'^2$$

where the primed coordinates are the spherical coordinates, not necessarily our coordinates!

Since A and B are confined to the surface of the sphere, $r' = R = \text{Const}$, and so $dr' = 0$

$$\Rightarrow ds^2 = R^2 d\theta'^2 + R^2 \sin^2\theta' \, d\phi'^2$$



Now, by looking at the figures, note that the length r is the arc length of the angle swept by θ' in the traditional spherical coordinates. Hence

$$r = R\theta' \Rightarrow \theta' = r/R$$

$$\rightarrow ds^2 = R^2 d\left(\frac{r}{R}\right)^2 + R^2 \sin^2\left(\frac{r}{R}\right) d\phi'^2$$

$$ds^2 = dr^2 + R^2 \sin^2\left(\frac{r}{R}\right) d\phi'^2$$

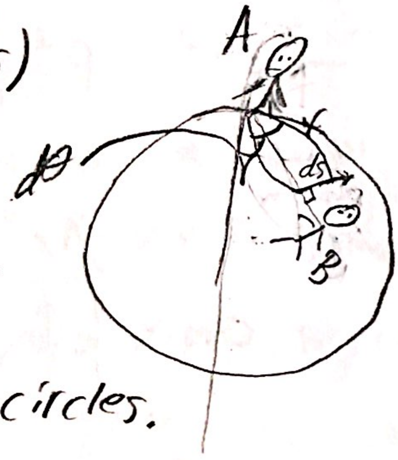
Also note that θ (in our coordinates) is the same as ϕ' (in the standard spherical coordinates) by the standard definition of spherical coordinates.

$$\Rightarrow \boxed{ds^2 = dr^2 + R^2 \sin^2\left(\frac{r}{R}\right) d\theta^2}$$

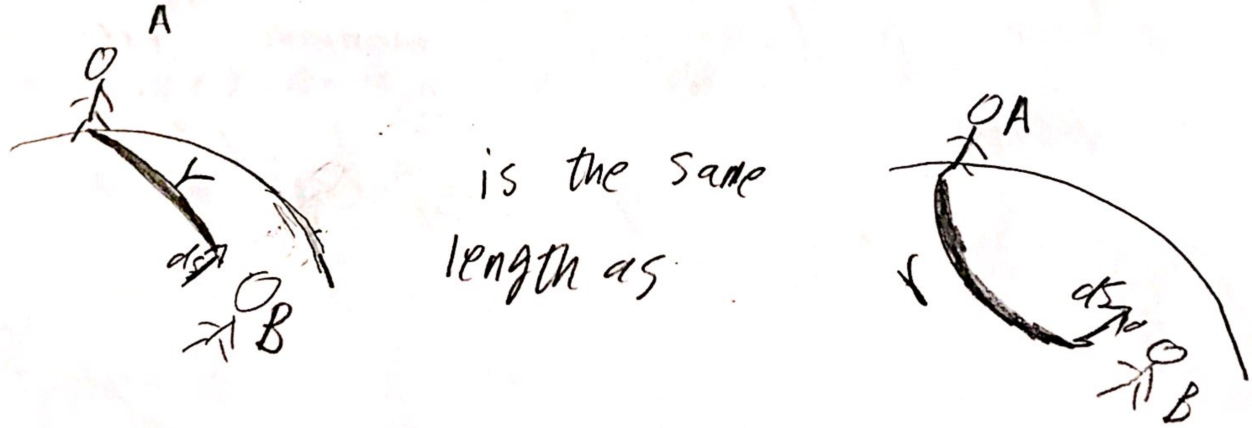
b) length of B's stick : ds

$$ds \ll R$$

Let's look at the figure from another angle (see attached 3D figures)



Since ds is very small, r will be the same distance (roughly) along the two great circles.
i.e.



Hence there is no change in r, i.e. $dr = 0$. So

$$ds^2 = R^2 \sin^2\left(\frac{r}{R}\right) d\theta^2$$

$$d\theta^2 = \frac{ds^2}{R^2 \sin^2\left(\frac{r}{R}\right)}$$ is the angular width

$$\Rightarrow \boxed{d\theta = \frac{ds}{R \sin\left(\frac{r}{R}\right)}}$$

c) Behavior of $d\theta$ as $r \to \pi R$.

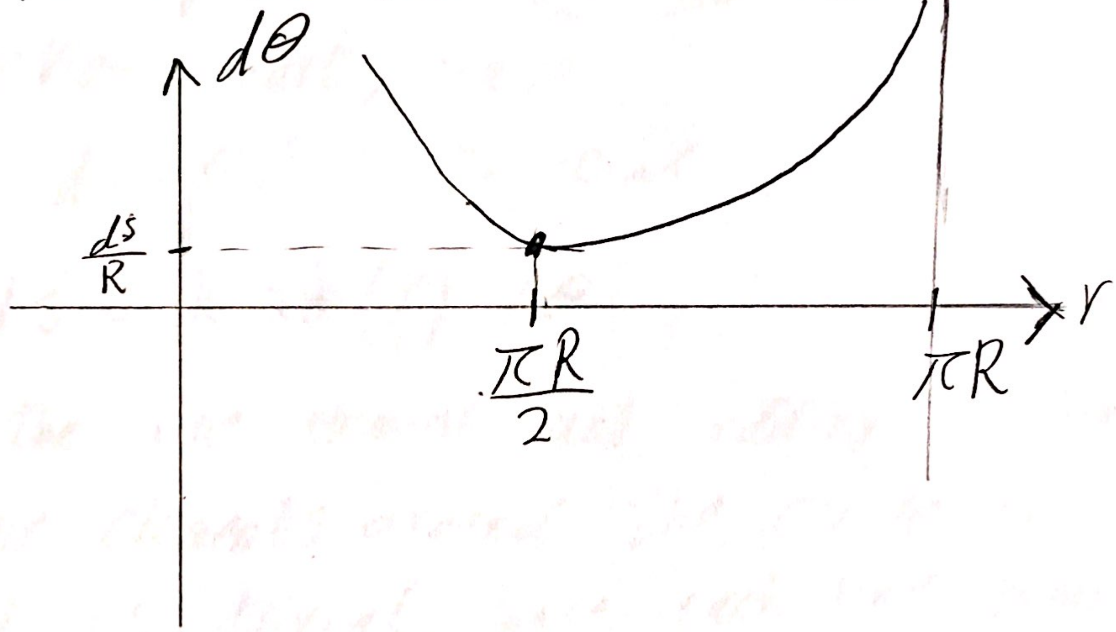
Let's look at the above equation again. Note that the minimum of $d\theta$ is when $\sin\left(\frac{r}{R}\right)$ is maximum, i.e. when $r = \frac{\pi R}{2}$, so that $\sin\left(\frac{r}{R}\right) = \sin\left(\frac{\pi}{2}\right) = 1$. So

$$\boxed{d\theta_{min} = \frac{ds}{R}, \quad \text{when} \quad r = \frac{\pi R}{2}}$$

Now when r keeps approaching πR, $d\theta$ starts increasing (since the sine argument starts decreasing).

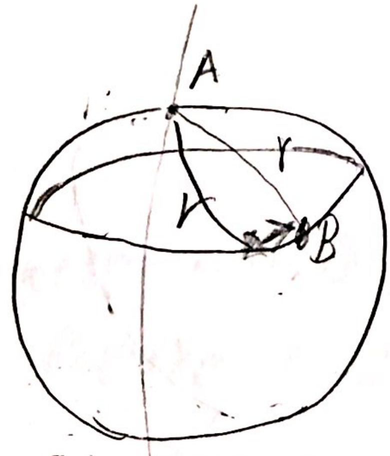
$$\boxed{d\theta_{max} \to \infty \quad \text{as} \quad r > \frac{\pi R}{2},\ r \to \pi R}$$

We can graph the dependence on r



2) a) They want to measure the circumference of a circle of radius r.

our metric:



$$ds^2 = dr^2 + R^2 \sin^2\left(\frac{r}{R}\right) d\theta^2$$

By the same argument as the previous part, since $ds \ll R$, $dr = 0$, so that

$$ds = R \sin\left(\frac{r}{R}\right) d\theta$$

This is the line element and adding together all the line elements around the circle (i.e. integrating) is trivial, since each line points to the direction of increasing θ, so the circumference is

$$C = \int_0^{2\pi} ds = \int_0^{2\pi} R \sin\left(\frac{r}{R}\right) d\theta$$

$$\Rightarrow \boxed{C = 2\pi R \sin\left(\frac{r}{R}\right)}$$

b) $R = 6371$ km, how large a circle would we need to draw to determine the surface is a sphere & not flat?

$$C = 2\pi R \sin\left(\frac{r}{R}\right)$$

if $r \ll R$, the space is approximately flat locally, so the flat-space circumference $C = 2\pi r$ holds.

The difference of the circumference for our coordinates and the circumference of the flat space will give a good measure of the size of the circle we have to draw. This difference

$$\Delta C = 2\pi R \sin\left(\frac{r}{R}\right) - 2\pi r$$

since $r \ll R$, let's Taylor expand it to first order

$$\Delta C = 2\pi R\left[\frac{r}{R} - \frac{1}{3!}\left(\frac{r}{R}\right)^3 + \dots\right] - 2\pi r$$

since $\sin(x) \simeq x - \frac{1}{3!}x^3 + \dots$

$$\Delta C = 2\pi r - \frac{1}{3!}\left(\frac{r}{R}\right)^3 - 2\pi r$$

$$\Delta C = 2\pi(r - r) - \frac{1}{3!}\left(\frac{r}{R}\right)^3$$

$$\rightarrow \Delta C = -\frac{1}{3!}\left(\frac{r}{R}\right)^3 2\pi R = -\frac{1}{3}\left(\frac{r}{R}\right)^3 \pi R$$

Note that here $\Delta C = C_{sphere} - C_{flat}$.

We could redefine it as $\Delta C = C_{flat} - C_{sphere}$ to get rid of the above negative sign, since the negative sign has no significance.

hence

$$C_{flat} - C_{sphere} = \frac{1}{3}\left(\frac{r}{R}\right)^3 \pi R \geq 1m$$

Since 1m is the accuracy of our measurement.

Hence

$$\frac{1}{3}\left(\frac{r}{R}\right)^3 \pi R \geq 1 m$$

$$r \geq \left(\frac{3}{\pi R} R^3\right)^{1/3}$$

$$\boxed{r \geq 33.842 \text{ Km}}$$

Hence the radius of the drawn circle has to be approximately 34 Km long to tell that the surface is a sphere.