Homework 6

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1 Question 1

Supposing that we have a flat universe ($K = 0 = \Omega_K$), and that our assumption (from the question) that we have a matter-dominated univese, hence $\Omega_M = 1$ and $\Omega_R = \Omega_\Lambda = 0$. Therefore, we start from the full Friedmann equation

$$\frac{da}{dt} = aH_0\sqrt{\Omega_{\Lambda 0} + \Omega_{K0}\left(\frac{a_0}{a}\right)^2 + \Omega_{m0}\left(\frac{a_0}{a}\right)^3 + \Omega_{R0}\left(\frac{a_0}{a}\right)^4}$$

Which reduces to only the matter term, i.e. it can be written as

$$\frac{da}{dt} = aH_0\left(\frac{a_0}{a}\right)^{3/2} \tag{1}$$

or

$$\frac{da}{dt} = H_0 a_0^{3/2} a^{-1/2}$$

Using the normalization $a(t_0) = a_0 = 1$, or equivalently defining $x = \frac{a}{a_0} \to dx = \frac{da}{a_0}$, which yields the same result, we have

$$\frac{da}{dt} = H_0 a^{-1/2}$$

Hence we can find the age of the universe at time t which coresponds to scale factor $a(t)$ by integrating the equation above

$$\int_0^t H_0 dt = \int_0^a da a^{1/2}$$

Hence

$$H_0 t = \frac{2a^{\frac{3}{2}}}{3}$$

Or

$$t = \frac{2a^{3/2}}{3H_0}$$

And using $\frac{a}{a_0} = \frac{1}{1+z}$ or $a = \frac{1}{1+z}$, we get

$$t = \frac{2}{3H_0(1+z)^{3/2}}$$

Now, using $H_0 = 70\frac{km}{s}\frac{1}{Mpc}$ and using that a megaparsec $Mpc = 3.0857 \times 10^{-22}m$ so that $H_0 = 70 \times \frac{1000\ m}{s}\frac{1}{3.0857\times 10^{-22}m}$ s, and $z = 2.5$, we get the age in seconds. Finally, we get the age in years by multipling by $t_{years} = t_{seconds} \times \frac{1}{60\ s\ \times 60\ min\ \times 24\ hour \times 365\ days}$ to get

$$\boxed{\boldsymbol{t = 1423166551 = 1.423 \times 10^9}\ \text{years}}$$

2 Question 2

2.1 part (a)

Heavy element abundances (or numbers) could be used to estimate ages of stars and galaxies. If a nucleus decays with rate λ, then the rate of change of the abundance at some time $A(t)$ due to the decay can be solved through the differential equation

$$\frac{dA}{dt} = -\lambda A \tag{2}$$

And the solution is easily solved by $\frac{dA}{A} = -\lambda dt$ and integrating gives $\ln(\frac{A}{A_{init}}) = -\lambda t$ and hence the solution for the abundance A at a later time is

$$A = A_{init}e^{-\lambda t}$$

Hence if we know A_{init} and could measure A then we could determine the time T to reach abundance A

$$T = \lambda^-1ln(\frac{A_{init}}{A})$$

This is difficult to measure, but it is easier to calculate the ratio of the initial abundances of two nuclei $A_{1,init}$ and $A_{2,init}$ pf two nuclei and to measure their relative present abundance A_1/A_2

$$\frac{A_1}{A_2} = \left(\frac{A_{1,init}}{A_{2,init}}\right) exp((\lambda_1 - \lambda_2)T)$$

And solving for time, we have

$$T = \frac{1}{\lambda_2 - \lambda_1}[\ln \frac{A_1}{A_2} - \ln \frac{A_{1,init}}{A_{2,init}}] \tag{3}$$

We are given the half life of Th-232 τ_{Th} and the half-life of U-238 τ_U. These can easily be converted into decay rate constants λ since the half life $\tau_{1/2} = \frac{\ln 2}{\lambda}$, hence

$$\lambda_{Th} = \frac{\ln 2}{\tau_{Th}},$$

$$\lambda_U = \frac{\ln 2}{\tau_U}$$

And we are given the present terrestrial abundance ratio of Th-232 to U-238

$$\frac{A_{Th}}{A_U} = 4.0$$

Also, we are also given the initial abundance ratio, since we assume that the production happened in an instant, therefore the abundance ration is the given production ratio. Hence

$$\frac{A_{Th,init}}{A_{U,init}} = 1.6$$

Hence, equation (3) becomes

$$T = \frac{1}{\lambda_U - \lambda_{Th}}[\ln \frac{A_{Th}}{A_U} - \ln \frac{A_{Th,init}}{A_{U,init}}] \tag{4}$$

And plugging in the given values, we get a lower bound for the age of the galaxy

$$\boxed{\boldsymbol{T = 8.796 \times 10^9} \text{ years}}$$

2.2 Part (b)

Here, we make things more complicated by assuming that in the time between the formation of the galaxy and the formation of the solar system, we have that Thorium and Uranium are decaying and being produced by supernovae all across the galaxy. then, after the solar system forms, we no longer have production, hence we only have decaying of Th and U after the formation of the solar system. Hence we have two distinct epochs:

1. **From the formation of the Galaxy to the formation of the solar system**

   Here, we are dealing with the time associated with the formation of the galaxy at $t = 0$ until the formation of the solar system, which occurs at $t = t_{\odot}$. In this epoch, we have production *and* decay of Th and U, hence we can set up the differential equation for the rate of change of the abundance of an element A_i as:

$$\frac{dA_i}{dt} = -\lambda_i A_i + p_i \tag{5}$$

   Where λ_i is the decay rate of species i and p_i is the production rate of species i (technically, p should be multiplied to a different quantity, the rate of conversion of mass into stars, which varies with time, but here we assume the rate is constant and independent of the abundance ratio.) This could be solved by replacement of substitution of variables/ Let $y_i = -\lambda_i A_i + p_i$ hence $A_i = \frac{p_i - y_i}{\lambda_i}$. Hence plugging in to equation (5), we have

$$\frac{d}{dt}\left(\frac{p_i - y_i}{\lambda_i}\right) = y_i \tag{6}$$

   Hence equation (5) becomes

$$-\frac{1}{\lambda_i}\frac{dy_i}{dt} = y_i \tag{7}$$

$$\frac{dy}{y} = -\lambda_i dt$$

   We can solve the above equation to get the associated time from the formation of the galaxy to the formation of the solar system by integrating from $t = 0$ to $t = t_{\odot}$, i.e. the time from the formation of the galaxy to the formation of the solar system is $t_1 = \int_{t=0}^{t=t_{\odot}} -\frac{1}{\lambda_i}\frac{dy_i}{y_i}dt$. But since we are given the production and decay values for the ratio of species, not individual species, it is more convenient so solve for A_i and then divide to get the ratios. Continuing with this, we get

$$\ln y = -\lambda_i t$$

   Hence the solution for y is

$$y_i = -\lambda_i A_i + p_i = e^{-\lambda_i t}$$

   And hence the abundance for species i as a function of time is

$$A_i(t) = \frac{P_i}{\lambda_i}\left(1 - e^{-\lambda_i t}\right) \tag{8}$$

   Hence we can similarly take the ratio of two species to get

$$\frac{A_1}{A_2} = \frac{\lambda_2}{\lambda_1}\frac{P_1(1 - e^{-\lambda_1 t})}{P_2(1 - e^{-\lambda_2 t})} \tag{9}$$

   Hence

$$\frac{A_{Th}}{A_U} = \left(\frac{P_{Th}}{P_U}\right) e^{-(\lambda_{Th} - \lambda_U)T} \tag{10}$$

   And hence, the time T_1, from the formation of the galaxy to the formation of the solar system is

$$T_1 = \frac{1}{\lambda_2 - \lambda_1}\ln\left(\frac{A_1}{A_2}\frac{P_2}{P_1}\right) \tag{11}$$

2. **From the formation of the solar system to today**

   The next epoch that we need to consider is the epoch from the formation of the solar system to today. In this epoch, all we have is decay, therefore we have an ODE similar to that in part (a), as in equation (3). So we have a solution for time T_2

$$T_2 = \frac{1}{\lambda_U - \lambda_{Th}} \left[\ln \frac{A_{Th}}{A_U} \right]. \tag{12}$$

   Where here we don't have the initial abundance ratio, unlike in part (a) where we assumed that the production happened in an instant.

Hence an estimate can be found for the age of the galaxy can be found using the given age of the solar system:

$$t_{solar} = t_g - t_{\odot} = 4.6 \times 10^9 \ years$$

hence

$$t_g = 4.6 \times 10^9 + T_1$$

where T_1 is from **??**, hence we can get a modest lower bound for the age of the galaxy

$$\boxed{\boldsymbol{t_g = 13.396 \times 10^9} \text{ years}}$$

3 Question 3

3.1 part (a)

Let's start by a very rough estimate, based on Ryden's Cosmology textbook, which gives a crude estimate of the energy density of starlight. In section 2.3, she argues that knowing the energy and luminosity densities of starlight could serve as a way to refute Olber's paradox. Ryden mentions that galaxy surveys tell us that luminosity density of galaxies us $nL \approx 2 \times 10^8 L_{\odot} Mpc^-3 \approx 2.6 \times 10^-33 Watts/m^-3$. And based on that she gives a rough estimate of the energy density (in section 5.1) of starlight as $\epsilon_{\gamma,0} \sim nLt_0 \sim 1 \times 10^{-15}$ J m^{-3} $\sim$ 0.007MeVm^{-3}. However, this calculation has not been done rigorously by anyone! Hence, ryden's estimate should serve as a rough guide that my answer is on the same order of magnitude.

Consider a small patch in the sky of volume dV filled with stars, there is a uniform start density

$$n_{stars} = \frac{dN_{stars}}{dV}$$

If each star has a luminosity L then the total luminosity for all the stars is

$$dL_{tot} = LdN_{stars}$$

Where N_stars is the number of stars in the patch. In Euclidian space, the(isotropic) flux from a source at r and luminosity dL is

$$dF_{\text{Euclidean}} = \frac{\mathrm{d}L}{4\pi r^2}$$

The energy density of photons ϵ_γ (equally referred to as U_* in the question) is related to the flux by

$$d\epsilon_\gamma = \frac{dF}{c}$$

Hence the total energy density of the passing photons is $\epsilon_\gamma = \int d\epsilon_\gamma$, hence

$$\begin{aligned} \epsilon_\gamma &= \int \tfrac{dF}{c} \\ &= \tfrac{1}{c} \int \tfrac{dL}{4\pi r^2} \end{aligned} \tag{13}$$

However, since we have cosmology, we have to replace r with angular diameter distance d_L , hence

$$\epsilon_y = \frac{1}{4\pi c} \int \frac{dL}{[d_L]^2} \tag{14}$$

Where r_1 is the radial coordinate of the source. Hence

$$\epsilon_\gamma = \frac{1}{4\pi c} \int \frac{L_\odot dN_{stars}}{[d_L]^2} \tag{15}$$

Where $L_\odot$ is the luminosity of the sun since we are considering solar-type stars. Now, for an infinitesimal volume element (patch) of the sky having infinitesimal volume dV and a uniform star density $n_s tars$, we have

$$dN_{\text{stars}} = n_{stars} dV = n_{stars} 4\pi dr \tag{16}$$

And $\Omega_M = \epsilon_M/\epsilon_c$, and $n_{\text{stars}} = f\rho_M = f\Omega_M \rho_{\text{crit}}$. This gives us the dimensions of Mass/Volume, but we want a number density (i.e. we want number of starts/volume), hence, since they are solar-type stars, we have to divide by the mass of the sun. Hence

$$n_{\text{stars}} = f\frac{\Omega_M \rho_{crit}}{M_\odot} \tag{17}$$

This gives us the correct units of number/volume. Hence dN_{stars} becomes

$$dN_{stars} = f\frac{\Omega_M \rho_{crit}}{M_\odot} 4\pi dr \tag{18}$$

Where dr is the trajectory of the photons on the lightlike interval ($ds^2 = 0$) on their way to the observer (here). Hence

$$dr = -cdt = -c\frac{dt}{dz}dz = -c\frac{dz}{\dot{z}} = \frac{cdz}{H(z)(1+z)} \tag{19}$$

Plugging in equation (18) into equation (15) we get

$$\epsilon_\gamma = \frac{1}{4\pi c} \int \frac{L_\odot}{[d_L]^2} f\frac{\Omega_M \ \rho_{crit}}{M_0} 4\pi dr \tag{20}$$

And now plugging in the expression for dr from equation (19) into the expression of ϵ_γ in equation (**??**) we get

$$\epsilon_\gamma = \frac{1}{4\pi c} \int \frac{L_\odot}{[d_L]^2} f\frac{\Omega_M \ \rho_{crit}}{M_\odot} 4\pi \frac{cdz}{H(z)(1+z)} \tag{21}$$

Now, we have the luminosity density d_L which is squared in the denominator of the expression above. This is troubling since it is given by $d_L = a(t_0)r_1\ (1+z)$ where r_1 is the radial coordinate of the source, hence it is not clear what to do for this radial coordinate r_1 (i.e. should we integrate it? This is poorly defined since we build our integral in terms of the redshift z.) Luckily, however, we are dealing in the regime where $z << 1$, hence we can Taylor expand d_L in the form

$$d_L = H_0^{-1} \left[z + \frac{1}{2}\left(1 - q_0\right) z^2 + \cdots \right] \tag{22}$$

Where H_0 is the current Hubble parameter. Taking the first term from equation (22), i.e. $d_L = H_0^{-1}\ z$ and plugging it into The expression in equation (21), we get a final form of our integral

$$\epsilon_\gamma = \frac{1}{4\pi c} \int_0^\infty \frac{L_\odot}{[H_0^{-1}\ z]^2} f\frac{\Omega_M \ \rho_{crit}}{M_\odot} 4\pi \frac{cdz}{H(z)(1+z)} \tag{23}$$

The above integral can be evaluated for any given values, which simplifies to

$$\epsilon_\gamma = \frac{L_\odot f \Omega_M \rho_{crit}}{M_\odot H_0^{-2}} \int_0^\infty \frac{dz}{H(z)\ z^2\ (1+z)} \tag{24}$$

Since we have a matter-dominated universe, we can substitute for $H(z)$ as

$$H(z) = H_0 \sqrt{\Omega_M (1+z)^3} \tag{25}$$

Which is equation (1.10.14) in Weinberg. Now plugging in equation (25) into equation (24) we get

$$\epsilon_\gamma = \frac{L_\odot f \Omega_M \rho_{crit}}{M_\odot H_0^{-2}} \int_0^\infty \frac{dz}{H_0 \sqrt{\Omega_M (1+z)^3}\ z^2\ (1+z)} \tag{26}$$

Which finally simplifies to a commutable integral:

$$\boxed{\epsilon_\gamma = \frac{L_\odot\ f\ \Omega_M^{1/2}\ \rho_{crit}}{M_\odot\ {}_{H_0^{-1}}} \int_0^\infty \frac{dz}{(1+z)^{5/2}\ z^2}} \tag{27}$$

!!!

The integral $\int_0^\infty \frac{dz}{(1+z)^{5/2}\ z^2}$ was evaluated numerically to equal 10.39. Plugging in the rest of the known values such as $Omega_M$, $L_\odot$ and $L_\odot$ gives us the final numerical answer.

$$\boxed{\epsilon_\gamma = 10.39 \times \frac{L_\odot\ f\ \Omega_M^{1/2}\ \rho_{crit}}{M_\odot\ {}_{H_0^{-1}}}}$$

I am not entirely sure on the correct units to represent these, so I leave it for the reader to look up their values and plug them in using convenient units.

3.2 Part (b)

If one looks to a cluster, or "clump" from outside, we are in "intergalactic space". Here, the fact that theses stars are distributed into clumps does not change the answer since they are still distributed uniformly within the clumps. Hence the answer for the energy density will be the same as that of part (a), i.e.

$$\epsilon_{\gamma,intergalactic} = 10.39 \times \frac{L_\odot\ f\ \Omega_M^{1/2}\ \rho_{crit}}{M_\odot\ {}_{H_0^{-1}}} \tag{28}$$

Now if we imagine being inside the cluster and looking outwards, then there are much more stars near the observer than far away from the observer. Now we are Within one galaxy, the stars are so close that the cosmological expansion has no effect. Hence the problem is much simpler and we can just use the Euclidean flux. So inside the galaxy we have

$$dF_{\text{Inside}} = \frac{\mathrm{d}L}{4\pi r^2}$$

Hence

$$d\epsilon_{\gamma,Inside} = \frac{1}{c} \int \frac{dL}{4\pi r^2}$$

And following exactly the same derivation as above, we note that now,

$$dL = L_\odot dN_{stars} = L_\odot dN_{stars} \tag{29}$$

But now,

$$dN_{stars} = n_{stars} dV = n_{stars} 4\pi dr = \frac{10^{11}}{\frac{4}{3}\pi r^3}\ 4\pi dr \tag{30}$$

Since the stars are uniformly distributed in this with $N = 10^{11}$. we have the energy density, where here we don't have to use the luminosity distance, hence using the above two relations we get

$$\epsilon_{\gamma,Inside} = \frac{1}{c} \int_R^0 \frac{L_\odot}{4\pi r^2}\ \frac{10^{11}}{\frac{4}{3}\pi r^3}\ 4\pi dr \tag{31}$$

Where R is the radius of the galactic sphere. Hence equation 31 becomes

$$\epsilon_{\gamma,Inside} = \frac{3\times 10^{11}\ L_\odot}{4\ \pi c}\int_R^0 \frac{dr}{r^5} \tag{32}$$

Hence

$$\boxed{\epsilon_{\gamma,Inside} = \frac{3\times 10^{11}\ L_\odot}{16\ \pi c\ R^4}}$$

3.3 Part (c)

On page 106 of his book, Weinberg writes down that the total energy density of radiation is

$$\rho_{R0} = \left[1 + 3\left(\frac{7}{8}\right)\left(\frac{4}{11}\right)^{4/3}\right]\rho_{\gamma 0} = 7.80\times 10^{-34}\ \text{g cm}^{-3}$$

Although this is mass density (which is equivalent to energy density if we set $c = 1$).These are in completely different units, so it is difficult to compare them. However, plugging in the numerical values for the constants such as $M_\odot$ we see that this is larger than the values we derived for any fraction f, probably because it was derived with a completely different method (using the Steffan-Botlzmann law) and because it also accounts for neutrinos.

4 Question 4: Faber-Jackson and Fundamental Plane Relations

The Faber-Jackson and Fundamental plane relations are methods to measure the mass-to-light ratio M/L of of galaxies in galaxy clusters (they can also be used to measure M/L for stars in individual galaxies). These measurements are done by measuring the velocity dispersion of the galaxies about the mean velocity of the cluster. For spherically-shaped galaxies, measuring the velocity dispersion works well in calculating M/L since one can use the Virial Theorem

$$2T + V = 0 \tag{33}$$

Where T is the kinetic energy of the galaxies about the center of mass, and V is the potential energy of (all pairs of) galaxies. If a system has become virialized, i.e. having no acceleration of galaxies, then the kinetic and potential terms in (33) become

$$T = \frac{1}{2}M\left\langle v^2\right\rangle, \quad V = -\frac{1}{2}GM^2\left\langle\frac{1}{r}\right\rangle \tag{34}$$

Where $< v^2 >$ is the mean square velocity relative to the center of mass, and $< 1/r >$ is the mean inverse distance separation. Hence we can solve for the total mass from the virial theorem (33)

$$M = \frac{2 < v^2 >}{G < 1/r >} \tag{35}$$

The mean square velocity $< v^2 >$ can be measured from the he Doppler shifts of the galaxies (the absorption of the 21 cm absorption line in the galaxies). Then the mean distance separation $< 1/r >$ can be measured from angular separations, since the proper distance separation $d = \theta d_A$ where θ is some angular separation and d_A is the angular diameter distance.

After attaining the mass from $< v^2 >$ and $< 1/r >$ and plugging it into the mass in equation (35), one can measure the absolute luminosity of a cluster of galaxies using

$$L = \frac{4\pi z^2 F}{H_0^2} \tag{36}$$

where F is the flux. Therefore, we can solve for the mass-to-luminosity ratio M/L for different systems, by dividing equation (35) by equation (36). An interesting outcome of this is that M/L for stars in a galaxies is much lower than M/L for clusters of galaxies, which is a clue that there must be more non-luminous mass in clusters of galaxies than is accounted for, in the form of dark matter.

The Faber-Jackson method and the Fundamental Plane method both use this idea to calculate M/L, and they both are extensions of the Tully-Fisher relation, which uses the *rotational* speed of individual galaxies as a way to measure their mass, whereas the Faber-Jackson method and the Fundamental Plane method use the *random* velocities of galaxies around their center of mass. The Fundamental plane method is an improvement to the Faber-Jackson method, by the recognition that the correlation between orbital velocity and absolute luminosity depends also on the surface brightness of the cluster, and hence on its area.

Both of these methods are ways of computing M/L, and hence possibly computing Ω_M, under the *assumption* that the mass-to-light ratio of clusters of galaxies is typical to that of the universe as a whole. Also, as explained earlier, both of these methods compute M/L by using the virial theorem, which makes the *assumption* that the particular system has reached a state of equilibrium, or become virialized, so that although individual masses are moving, there is no further statistical evolution, and in particular that the accelerations of the galaxies vanish. This assumption makes the method ineffected for (irregular) systems in which the motions of the galaxies have not reached an equilibrium. Furthermore, in measuring the mean square velocities $> v^2 >$ both of these methods make the *assumption* that the galaxies are spiral. For example, this method does not work for the cluster in Coma, which appears to be spherical.

With mass-to-light ratios of clusters $(M/L)_{cluster}$ being on the order of $200 - 350\ hM_\odot/L_\odot$, this can be compared with that of elliptical galaxies. The mass-to-light ratios of individual elliptical galaxies can be measured using this method, with the velocity dispersion $\sqrt{< v^2 >}$ taken as the velocity dispersion of stars contained in the elliptical galaxy; this gives mass-to-light-ratio $(M/L)_{ellip.Galaxy}$ on the range of 10 to 20 $hM_\odot/L_\odot$. As mentioned earlier, this gives us one clue that there must be more non-luminous mass in clusters of galaxies than is accounted for, in the form of (non-baryonic) dark matter.

In [7]:
```
#1)a Mpc = 3.0857×10^16 m
Mpc = 3.0857*10**22
H_0 = 70*1000 / Mpc #m/s
z=2.5
t = 2/(3*H_0 * (1+z)**(3/2))
t/(60*60*24*365)
```
Out[7]:
```
1423166551.3365016
```

In [4]:
```
t_half_Th=13.9*10**9
t_half_U=4.5*10**9

abundance_ratio=4
production_ratio=1.6

lambda_Th=np.log(2)/t_half_Th
lambda_U=np.log(2)/t_half_U
#Th is 1 and U is 2
T=1/(lambda_U - lambda_Th) *(np.log(abundance_ratio)-np.log(production_ratio))
T/10**9
```
Out[4]:
```
8.79644705693665
```

In [7]:
```
T_1 = 1/(lambda_U - lambda_Th) * np.log(abundance_ratio / production_ratio); T_1
T_2 = 1/(lambda_U - lambda_Th) * np.log(abundance_ratio);
print(T_1, T_2)
```
```
8796447056.936653 13308510638.297874
```

In [27]:
```
t_solar=4.6*10**9
(t_solar+T_1)/(10**9)
```
Out[27]:
```
13.396447056936653
```

Figure 1: My python code for the calculations in this assignment