AST 4414/5416 Cosmology & Structure Formation, Spring 2021

HW Set 3

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1) Circumference, Surface Area and Volume in R-W Space

Before we begin, let's first revise and recall:

The R-W (Robertson-Walker) metric is a metric that describes the geometry of spacetime, which is homogeneous and isotropic.Iin quasi-Euclidean coordinates, this is

$$ds^2 = -d\tau^2 = -g_{\mu\nu}(x)dx^{\mu dx^{\nu}} = -c^2dt^2 - a^2(t)\left[d\,\vec{x}^2 + \frac{K(\,\vec{x}\cdot d\vec{x})^2}{1-K\vec{x}^2}\right]$$

Using spherical polar coordinates, for a (n-) sphere of radius r:

$d\vec{x}^2 = d\vec{r}^2 + r^2 d\Omega^2,\qquad d\Omega^2 = d\theta^2 + \sin^2(\theta)d\phi^2$, so

$$d\tau^2 = c^2dt^2 - a^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2 d\Omega\right]$$

So that the metric interval can be written as:

$$\boldsymbol{ds^2 = -c^2dt^2 + a^2[dr^2 + l^2 d\Omega^2]}$$

Where

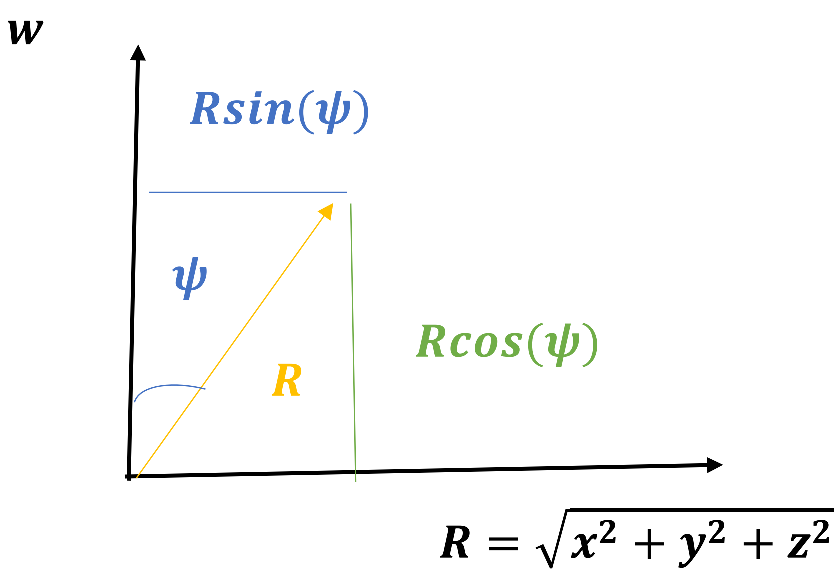
$$\begin{cases} \boldsymbol{l(r) = r \text{ for } K = 0, \textbf{flat space}} \\ \boldsymbol{l(r) = \sin(r) \textbf{ for } \mathrm{K} = +1, \textbf{spherical (positively curved) space}} \\ \boldsymbol{l(r) = \sinh(r), \textbf{for } \mathrm{K} = -1, \textbf{hyperbolic (negatively curved) space}} \end{cases}$$

Where $a(t)$is the characteristic scale factor of the uuniverse, and the above rewriting was done by the substitution $r = R\theta$ where r is the radius of the small sphere draw on the surface of the 3-sphere which was done in detail in my last homework (remember for that the metric for that spherical surface was $ds^2 = dr^2 + R^2 \sin^2\left(\frac{r}{R}\right) d\theta^2$.

Also note that if the universe had positive curvature $k = +1$, then the universe would be closed (like the surface on a sphere or hypersphere), and finite in volume (and R is the radius of the universe. This doesn't mean, however, that the universe has a boundary! (A particle going to the edge of the universe would not encounter a boundary that it can't extend, but it would eventually return to the starting point just like the surface of a sphere embedded in 3d doesn't have a boundary). On the other hand, if the curvature of space was zero $k = 0$ (i.e. flat) or negative $k = -1$ (i.e. hyperbolic) the universe would be open, it would be infinite.

The generalization of the R-W metric to 3 dimensions can be done in several ways, either geometrically, or by removing the extra dimension (since the 3 spacial dimensional metric corresponds to a 3-sphere embedded in a 4-sphere).

Let's do it geometrically since it's the easiest. Since we want the surface of a 4-sphere, we embed a 3sphere in a 4 sphere and introduce a new (polar) axis, say w. Now the 4-sphere is described by $R^2 = x^2 + y^2 + z^2 + w^2$ .We then extend a line into the surface whose coordinates we want to describe, and denote the polar angle from the polar axis w as ψ (see figure below).



If we are on the surface of a sphere of radius R we keep R constant and keep θ and ϕ as coordinates. So

$$w = Rcos\psi$$

Then the other coordinates follow by induction, except now the radius is not R but is $Rsin\psi$ so

$$x = Rsin\psi sin\theta cos\phi$$

$$y = Rsin\psi ssin\theta sin\phi$$

$$z = Rsin\psi cos\phi$$

Where

$$0 < \phi < 2\pi, 0 < \theta < \pi, 0 < \psi < \pi$$

We can keep doing this if we want to describe a very high dimensional space, invent a new axis for every new dimension and a new angle from that axis. See appendix where I also verify this result using another method.

Proper Distance

Proper distance is the distance between two events A and B in a reference frame in which they occur simultaneously ($t_A = t_B$). In other words, the proper distance between two points in spacetime is equal to the length o the spatial geodesic between them when the scale factor is fixed by the value $a(t)$. The proper distance in a fixed time t can be found by the R-W metric:

$$ds^2 = a^2(t)[dr^2 + l^2(r)d\Omega^2]$$

With the appropriate form of $l(r)$ for 3d space (3-sphere).

Along the special geodesic between the observer and another point, the angle (θ, ϕ) is fixed, so

$$ds = a(t)dr$$

So the proper distance d_{proper} is found by integrating over the comoving radial coordinate r

$$d_{proper} = a(t)\int_0^r dr = a(t)r$$

a) For each of the 3-D spaces of constant curvature derive expressions for the circumference of a circle (C), the surface area of a sphere (A), and the volume of a sphere (V), all of proper radius d.

Now we have 3d space in which there exists a 3 sphere, so we can use the results I derived above for the R-W metric for this 3-sphere,. Since the changes in the coordinates are still orthogonal to each other, we can use the pythagorian theorem, i.e. we can use $ds^2 = dz^2 + dy^2 + dz^2 + dw^2$ and plug in the coordinates derived above. We can either use that or realize geometrically that the metric for the 3-sphere is just like the metric for the 2-sphere but with the radius changed from R to $Rsin(\psi)$. Hence

$$ds^2 = R^2[d\psi^2 + \sin^2\psi(d\psi^2 + \sin^2\theta d\phi^2)]$$

Now, since we are looking at a sphere with radius r we have $r = R\psi$ (just like I did in the previous homework). So the R-W for the 3-sphere becomes:

$$\boldsymbol{ds^2_{3-sphere} = -c^2dt^2 + a^2(t)\left[dr^2 + R^2\sin^2\left(\frac{r}{R}\right)d\Omega^2\right]} \qquad \boldsymbol{; for\ k = +1}$$

$$\boldsymbol{ds^2_{3-sphere} = -c^2dt^2 + a^2(t)[dr^2 + r^2d\Omega^2]} \qquad \boldsymbol{; for\ k = 0}$$

$$\boldsymbol{ds^2_{3-sphere} = -c^2dt^2 + a^2(t)[dr^2 + R^2\sinh^2\left(\frac{r}{H}\right)]} \qquad \boldsymbol{; for\ k = -1}$$

So for **flat space**:

the space component of the metric is $ds^2 = a^2(t)[dr^2 + r^2d\Omega^2]$

$$C = a^2(t)\int_0^{2\pi} d\phi$$

$$S.A. = a^2(t)\int_0^{2\pi} d\phi\,(d^2)\int_0^{\pi}\sin^2(\theta) = 4\pi a^2(t)d^2$$

$$Vol = a^3(t)\int_0^{2\pi} d\phi\int_0^{\pi}\int_0^{d} dr\,r^2\sin(\theta)\,d\theta = \frac{4\pi a^3(t)r^3}{3}$$

And since the proper distance (proper radius) of the sphere is d, the comoving distance $r = \frac{d}{a(t)}$ so $d = a(t)r$, hence

$$vol = \frac{4\pi d^3}{3}$$

Which is what we expect for a Euclidean sphere volume!

For **positively curved (K=+1) space**

We have the metric

$$\boldsymbol{ds^2 = -c^2dt^2 + a^2(t)\left[dr^2 + R^2\sin^2\left(\frac{r}{R}\right)d\Omega^2\right]}$$

Since this is now more complicated to evaluate, let us use the general result that the volume for any surface is given by $V = \int \sqrt{|g_{ij}|}\, dx_1 dx_2 dx_3$ where $|g|$ is the determinant of the components of the metric.

So for this positively curved space we have

$$g_{ij} = a^2(t)\begin{pmatrix} 1 & 0 & 0 \\ 0 & R^2\sin^2\left(\frac{r}{R}\right) & 0 \\ 0 & 0 & R^2\sin^2\left(\frac{r}{R}\right)\sin^2(\theta) \end{pmatrix}$$

Hence $|g| = a^2R^4\sin^4\left(\frac{r}{R}\right)sin^2(\theta) \to \sqrt{g} = a(t)R^2\sin^2\left(\frac{r}{R}\right)\sin\,(\theta)$.

And we have $dx_1 = a(t)dr, dx_2 = a(t)R\sin\left(\frac{r}{R}\right)\, d\theta,\ dx_3 = a(t)R\sin\left(\frac{r}{R}\right)\sin(\theta)\ d\phi$

Therefore, if we want the circumference, we just integrate over the ϕ component, i.e. dx_3 from 0 to 2π at $\theta = \frac{\pi}{2}$ so

$$C = a(t)Rsin\left(\frac{r}{R}\right)\int_0^{2\pi} d\theta = 2\pi a(t)R\sin\left(\frac{r}{R}\right)$$

For the surface area, we integrate over dx_1 and dx_2

$$SA = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \left(a\ R\sin\left(\frac{r}{R}\right)\right)\left(aR\sin\left(\frac{r}{R}\right)\sin(\theta)\right) = 4\pi a^2(t)R^2\sin^2\left(\frac{r}{R}\right)$$

For the volume, we just integrate over dx_1, dx_2, dx_3

$$vol = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{r} dr\ (a\ dr)\ \left(a\ R\sin\left(\frac{r}{R}\right)\right)\left(aR\sin\left(\frac{r}{R}\right)\sin(\theta)\right)$$

$$vol = 4\ \pi a^3(t)R^2\int_0^{r} dr\sin^2\left(\frac{r}{R}\right)$$

Let $u = \frac{r}{R}$ so $dr = R\ du \quad \to vol = 4\pi a^3(t)R^3\int du\sin^2(u) = 4\pi a^3(t)R^3\left(\frac{u}{2} - \frac{\sin(2u)}{4}\right)$

$$vol = 4\pi a^3(t)R^3\left(\frac{r}{2R} - \frac{\sin\left(\frac{2r}{R}\right)}{4}\right)$$

NOTE: for all the above, the proper radius d can be written in terms of the comoving distance r as

$r = \frac{d}{a(t)}$ **so all the above results can be written in terms of them, so**

$$C = 2\pi a^2(t)R^2 sin^2\left(\frac{d}{a(t)R}\right)$$

$$SA = 4\pi a^2(t)R^2\ sin^2\left(\frac{d}{a(t)R}\right)$$

$$Vol = 4\pi\ a^3(t)R^3\left(\frac{d}{2a(t)R} - \frac{sin\left(\frac{2d}{a(t)R}\right)}{4}\right)$$

For **negatively curved (K=-1) space**

We have the metric

$$ds^2 = -c^2dt^2 + a^2(t)[dr^2 + R^2 \sinh^2\left(\frac{r}{H}\right)]$$

By analogy to the positively curved space, we follow the same method, so

$$C = a(t)Rsinh\left(\frac{r}{R}\right)\int_0^{2\pi} d\theta = 2\pi a(t)R\sinh\left(\frac{r}{R}\right)$$

$$SA = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \left(a\ R\sinh\left(\frac{r}{R}\right)\right)\left(aR\sinh\left(\frac{r}{R}\right)\sin(\theta)\right)$$
$$= 4\pi a^2(t)R^2\sinh^2\left(\frac{r}{R}\right)$$

$$vol = 4\ \pi a^3(t)R^2\int_0^r dr\sinh^2\left(\frac{r}{R}\right)$$

$$vol = \ 4\ \pi a^3(t)R^2\left(\frac{\sin\left(\frac{2r}{R}\right)}{4} - \frac{r}{2R}\right)$$

Similarly, these can be written in terms of the proper radius

$$C = 2\pi a^2(t)R^2 sinh^2\left(\frac{d}{a(t)R}\right)$$

$$SA = 4\pi a^2(t)R^2\ sinh^2\left(\frac{d}{a(t)R}\right)$$

$$vol = \ 4\ \pi a^3(t)R^2\left(\frac{\sin\left(\frac{2d}{a(t)R}\right)}{4} - \frac{d}{2a(t)R}\right)$$

Property	Closed	Euclidean	Open
Spacial Curvature	Positive	Zero	Negative
Circle Circumference	$< 2\ \pi R$	$2\pi R$	$> 2\pi R$
Surface Area	$< 4\pi R^2$	$4\pi R^2$	$> 4\pi R^2$
Sphere Volume	$< \frac{4}{3}\pi R^3$	$\frac{4}{3}\pi R^3$	$> \frac{4}{3}\pi R^3$
Total Volume	Finite $(2\pi^2 R^3)$	Infinite	Infinite
Surface along	Sphere	Plane	Saddle

B) What is the largest possible proper distance in the K = +1 space?

For the $K = +1$ space we have the metric

$$\boldsymbol{ds^2} = -\boldsymbol{c^2 dt^2} + \boldsymbol{a^2(t)}\left[\boldsymbol{dr^2} + \boldsymbol{R^2 \sin^2\left(\frac{r}{R}\right) d\Omega^2}\right]$$

Proper distance defines how far away something is at this time, and it is purely radial, so $\int d\Omega = 0$ and "at this time" means $\int dt = 0$, so the R-W metric gives

$$ds = a(t) dr_0$$

So the total proper distance is

$$r = \int ds = a(t) \int dr_0 = a(t)\, r_0$$

Where r_0 is the current comoving radius of the universe (hence the proper distance is the comoving radius of the universe scaled by $a(t)$.

C) What are the maximum values of C, A, and V for the K = +1 (spherical) universe, and at what values of proper radius d are they reached?

Note that I technically already derived this in part (a) ! The only difference here is that the radius of the universe R scaled by $a(t)$ can be defined as the maximum proper distance in the universe, so we can define $a(t)R = R_0$.

Circumference:

At comoving coordinate r_0 and time t, we have a radial vector of length $r = a(t)\, r_0$ (as shown above). For Tthe circumference, $dr = 0$, and we integrate $d\phi$ from 0 to 2π so

$$C_{max} = \int_0^{2\pi} d\phi a(t) R \sin\left(\frac{r_0}{R}\right) = 2\pi a(t)\, R \sin\left(\frac{r}{R}\right) = 2\pi R_0 \sin\left(\frac{r}{R}\right)$$

The surface area similarly is

$$\begin{aligned} SA_{max} &= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \left(a\, R \sin\left(\frac{r}{R}\right)\right)\left(aR \sin\left(\frac{r}{R}\right) \sin(\theta)\right) \\ &= 4\pi a^2(t) R^2 \sin^2\left(\frac{r}{R}\right) \\ &= 4\pi R_0^2 \sin^2\left(\frac{r}{R}\right) \end{aligned}$$

But the volume is different, since now we must integrate this area over dr from 0 to the antipole at $r = \pi\, a(t) R$

$$V_{max} = 4\pi R_0^2 \int_0^{\pi a(t) R} dr \sin^2\left(\frac{r}{R}\right) = 2\,\pi\, a^3(t) R^3 = 2\pi\, R_0^3$$

2) Derive the Equation for the Evolution of Energy Density

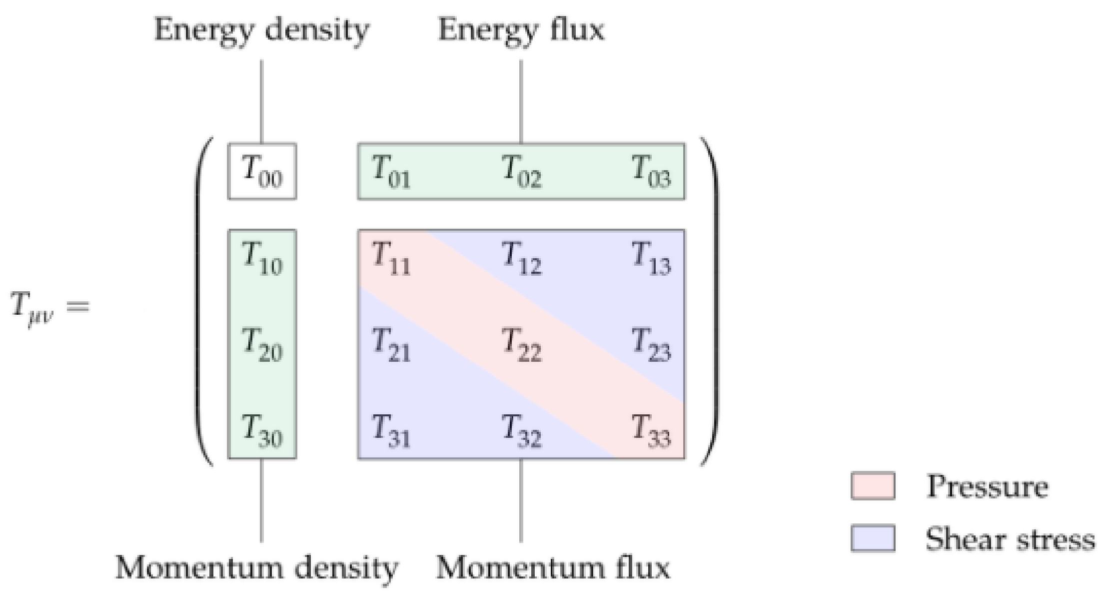
a) We start with the covariant derivative of the stress-energy tensor, $T^{\mu\nu}$

$$\nabla_\mu T^{\mu\nu} = 0$$

You have a typo! There should be a nabla ∇_μ which didn't show up in the assignment pdf.

Before we answer the question let's remember the components of the stress-energy tensor:

T^{00} is the energy density, T^{0i} is the energy flux in the i direction. T^{i0} is the momentum density in the i direction, and T^{ij} is the i −component of momentum flux in the j direction. Note also that the stress-energy tensor is symmytric" $T^{\mu\nu} = T^{\nu\mu}$.



For an ideal fluid the stress-energy tensor is:

$$T^{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) U^\mu U^\nu - P g^{\mu\nu}$$

Where ρ is the mass density of the fluid, P is the pressure of the fluid, and U^μ is its 4-velocity.

How many separate equations does this expression represent?

The number of equations is 4, since we have one equation for each component.

Note that since $T^{\mu\nu}$ is a 4×4 matrix, it has 16 components. However, $T^{\mu\nu} = T^{\nu\mu}$ ie. $T^{\mu\nu}$ is a a symmetric tensor, hence this reduces the components from 16 to 10 components:

$T^{00} = \left(\rho + \frac{P}{c^2}\right) U^0 U^0 - P g^{00}, T^{11} = \left(\rho + \frac{P}{c^2}\right) U^1 U^1 - P g^{11}, T^{22} = \left(\rho + \frac{P}{c^2}\right) U^2 U^2 - P g^{22}, T^{33} = \left(\rho + \frac{P}{c^2}\right) U^3 U^3 - P g^{33}, T^{10} = \left(\rho + \frac{P}{c^2}\right) U^1 U^0 - P g^{10}, T^{20} = \left(\rho + \frac{P}{c^2}\right) U^2 U^0 - P g^{20}, T^{21} = \left(\rho + \frac{P}{c^2}\right) U^2 U^1 - P g^{21}, T^{30} = \left(\rho + \frac{P}{c^2}\right) U^3 U^0 - P g^{30}, T^{31} = \left(\rho + \frac{P}{c^2}\right) U^3 U^1 - P g^{31}, T^{32} = \left(\rho + \frac{P}{c^2}\right) U^3 U^2 - P g^{32}$

If you consider $\mu = 0$, this represents the equation for what? What do the other equations express?

The $\mu = 0$ components represent the energy components, where T^{00} is the energy density, and T^{0i} is the energy flux, hence the $\mu = 0$ components represent *the conservation of energy equations* (or conservation of energy flux in the i direction for the T^{0i} components).

The cother components represent the momentum components, hence the other compoents repressent *the conservation of momentum equtions*, where T^{i0} represent the conservation of momentum density, and T^{ij} represent the conservation of i-th component of momentum flux in the j- direction.

Given your answer, write down the conservation equation again, but focusing only on the conservation equation for the energy density. Explicitly expand the covariant derivative in terms of the affine connection terms.

In flat space (in absence of gravity), the conservation of energy, matter and it momentum is defined by

$$\frac{\partial T_{\mu\nu}}{\partial x_\nu} = 0$$

For the energy density, we only need to consider $\mu = 0$ components. Suppose we have Cartesian coordinates, then the above equation is expanded into

$$\frac{\partial T^{00}}{\partial t} + \frac{\partial T^{0x}}{\partial x} + \frac{\partial T^{0y}}{\partial y} + \frac{\partial T^{0z}}{\partial z} = 0$$

In curved space (in the presence of a gravitational field), we shall rather have

$$\nabla_\nu T^{\mu\nu} = T^{\mu\nu}_{;\nu} = 0$$

i.e. we covariant differentiation. Where there is implicit sum over ν. Hence we need to expand in terms of the affine connections dictated by the rules of covariant differentiation:

$$\nabla_\nu T^{\mu\nu} = \frac{\partial T^{\mu\nu}}{\partial x^\nu} + \Gamma^{\mu}_{\kappa\nu} T^{\kappa\nu} + \Gamma^{\nu}_{\kappa\nu} T^{\mu\kappa} = 0$$

b) **If the Universe is indeed isotropic and homogeneous, what does this imply about the stress-energy tensor?**

Homogeneity means that at any given time, the universe looks the same at every point in space. Isotropy means that the universe looks the same whatever direction one looks. Note that a universe which is isotropic will be homogeneous, but a universe that is homogeneous *may not be isotropic.* It is believed that the universe satisfies homogeneity and isotropy.

Homogeneity greatly restricts the metrics that we can use. They must be independent of space, and solely functions of time. Also, we must restrict ourselves only to spaces of constant curvature (for which there are three: $K = -1,0,+1$.

Let's suppose we have a flat space, then the metric for this flat space is

$$ds^2 = -c^2dt^2 + a^2(t)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2]$$

And suppose we have a perfect fluid:

$$T^{\mu\nu} = \left(\rho + \frac{P}{c^2}\right)U^\mu U^\nu - Pg^{\mu\nu}$$

And suppose $U^\mu = (c, 0, 0, 0)$, **Then both the metric and the stress-energy tensor must be diagonal.**

So for this example,

$$g_{00} = -1, \qquad\qquad g_{ij} = a^2(t)\delta_{ij}$$

$$T_{00} = \rho c^2, \qquad\qquad T_{ij} = a^2(t)P\delta_{ij}$$

C) Which affine connection coefficients do you need, and what are they?

The affine connection coeffiecients are given by

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}\sum_\delta g^{\alpha\delta}\left(\frac{\partial g_{\delta\gamma}}{\partial x^\beta} + \frac{\partial g_{\beta\delta}}{\partial x^\gamma} - \frac{\partial g_{\beta\gamma}}{\partial x^\delta}\right)$$

Hence you only need the affine connection coefficients associated with T_{00} and the diagonal components T_{ij}, **hence you only need $\boldsymbol{\Gamma^0_{ij}}$ and $\boldsymbol{\Gamma^i_{0j}}$**

Suppose we have $c = 1$,and $g^{ij} = diag(1,1,1)$, then our energy momentum tensor is

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g^{ij}P & \\ 0 & & & \end{pmatrix}$$

Now expanding the covariant derivative for the zero components of the stress-energy tensor:

$$\nabla_\nu T^{\mu\nu} = \frac{\partial T^{\mu\nu}}{\partial x^\nu} + \Gamma^\mu_{\kappa\nu} T^{\kappa\nu} + \Gamma^\nu_{\kappa\nu} T^{\mu\kappa} = 0$$

Hence we only need to consider non-vanishing elements of $\frac{d}{dt}$

$$\Gamma^0_{ij} = \frac{1}{2}\sum_\delta g^{0\delta}\left(\frac{\partial g_{\delta j}}{\partial x^i} + \frac{\partial g_{i\delta}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^\delta}\right) = a(t)\dot{a}(t)\delta_{ij}$$

$$\Gamma^i_{0j} = \frac{1}{2}\sum_\delta g^{i\delta}\left(\frac{\partial g_{\delta j}}{\partial x^0} + \frac{\partial g_{0\delta}}{\partial x^j} - \frac{\partial g_{0j}}{\partial x^\delta}\right) = \frac{\dot{a}(t)}{a(t)}\delta^i_j$$

D) Bring this all together to derive the energy density equation.

For flat space the equation was derived in part (a) which is trivial to plug in for the tensor.

For curved space, $\nabla_\nu T^{\mu\nu} = \frac{\partial T^{\mu\nu}}{\partial x^\nu} + \Gamma^\mu_{\kappa\nu} T^{\kappa\nu} + \Gamma^\nu_{\kappa\nu} T^{\mu\kappa} = 0$ hence if we only consider to $\nu = 0$ components we plug in the derived affine connections above, so we have

$$\nabla_0 T^{i0} = \frac{\partial T^{i0}}{\partial x^0} + \Gamma^i_{0j} T^{0j} + \Gamma^0_{j0} T^{ij} = 0$$

$$= \frac{\partial T^{00}}{\partial t} + \frac{\partial T^{10}}{\partial t} + \frac{\partial T^{20}}{\partial t} + \frac{\partial T^{30}}{\partial t} + \frac{\dot{a}(t)}{a(t)}\delta^i_j\, T^{0j} + a(t)\dot{a}(t)\delta_{ij}\, T^{ij}$$

$$= \dot{\rho} + 3\frac{\dot{a}(t)}{a(t)}\rho + 3\frac{\dot{a}(t)}{a(t)}P = 0$$

$$\rightarrow \boldsymbol{\dot{\rho} + 3\frac{\dot{a}(t)}{a(t)}(\rho + P) = 0}$$