Homework 8

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1 Problem 1

the condition of thermal equilibrium tells us that the entropy in a comoving volume is fixed

$$s(T)a^3 = \text{ constant} \tag{1}$$

The photon number can change as a result of bremstraulung, and since we have this epoch in thermal equilibrium, the bremstraulung is also in equilibrium, which ensures that the chemical potential of the photons (and hence by the ϕ_0 particle which is interacting with it) is zero. Note however that we do have reactions like $e^+e^- \to \gamma$ at this time, but since the number densities of electrons and positrons, which are determined by the chemical potential, are very close to each other, the chemical potentials of electrons and positrons must be (to a very good approximation) equal to each other. Hence in thermal equilibrium With equal numbers of particles and antiparticles, the number density n(p)dp of a species of fermions (such as electrons) or bosons (like photons) of mass m and momentum between p and p + dp is given by the Fermi–Dirac or Bose–Einstein distributions (with zero chemical potential)

$$n(p,T) = \frac{4\pi g p^2}{(2\pi\hbar)^3}\left(\frac{1}{\exp\left(\sqrt{p^2+m^2}/k_BT\right)\pm 1}\right) \tag{2}$$

where g is the number of spin states (spin degeneracy) of the particle and antiparticle, and the sign is + for fermions and for bosons. For relativistic particles, $p^2 >> m^2$ and so the energy density is

$$\varepsilon_{massless}(T) = g\frac{a_BT^4}{2}\begin{cases}\times 1, & \text{for bosons}\\ \times 7/8, & \text{for fermions}\end{cases} \tag{3}$$

The important thing to note is that we are familiar that the energy density for photons goes as t^4, but this energy density goes like this for *any* relativistic particle, which could be any particle in a certain energy regime.

1.1 Part a

It is amazing how since we have thermal equilibrium in the early universe, we can get powerful and exact results about this period using elementary classical thermodynamics. The entropy of a particle can be derived by the first lat of Thermodynamics. Using entropy and energy densities, the entropy in a volume V is $s(T)V$ and the energy is $\varepsilon(T)V$. Hence using the first law of thermodynamics $dE = dQ - PdV \to d(\varepsilon V) = Td(s(T)V) - PdV$ we have

$$d(s(T)V) = s(T)dV + Vs(T)dT = \frac{d(\varepsilon(T)V) + pdV}{T} \tag{4}$$

And setting the dV and dT components equal, we get

$$d(s(T)V) = s(T)dV + V\frac{ds}{dT}dT \tag{5}$$

and

$$d(\varepsilon(T)V) = \varepsilon(T)dV + V\frac{d\varepsilon}{dT}dT \tag{6}$$

And plugging Eq (6) into Eq (4) we get

$$s(T)dV + V\frac{dS}{dT}dT = \frac{\varepsilon(T)dV + V\frac{Vd\varepsilon}{dT}dT + pdV}{T} \tag{7}$$

And setting the dV terms of the LHS equal to those on the RHS, we get

$$s(T) = \frac{\varepsilon(T) + p(T)}{T} \tag{8}$$

and

$$\frac{dS}{dT} = \frac{1}{T}\frac{d\varepsilon}{dT} \tag{9}$$

If we use $c = 1$ then we can also write the entropy density in terms of mass density and pressure

$$s(T) = \frac{\rho(T) + p(T)}{T} \tag{10}$$

Where $\rho(T)$ and $P(T)$ are the energy density and pressure of a particle of mass m are given by the integrals

$$\begin{aligned} \rho(T) &= \int_0^\infty n(p,T)dp\sqrt{p^2+m^2} \\ p(T) &= \int_0^\infty n(p,T)dp\frac{p^2}{3\sqrt{p^2+m^2}} \end{aligned} \tag{11}$$

We see that for relativistic particles, $p = \frac{\varepsilon}{3} \to \varepsilon \propto T^4$, and $s \to T^3$. During any epoch in which the dominant constituent of the universe is a highly relativistic ideal gas, the entropy and energy densities are

$$s(T) = \frac{2\mathcal{N}a_{\mathcal{B}}T^3}{3} \tag{12}$$

and

$$\rho(T) = \frac{\mathcal{N}a_{\mathcal{B}}T^4}{2} \tag{13}$$

Where

$$a_B = \frac{2\pi^5 k_{\mathrm{B}}^4}{15h^3c^2} = \frac{\pi^2 k_{\mathrm{B}}^4}{60\hbar^3 c^2}, \tag{14}$$

is the Stefan boltzmann constant. And $\mathcal{N}$ is the number of particle types, counting particles and antiparticles and each spin state separately, and with an extra factor of 7/8 for fermions.

By Friedmann equation here we have

$$H^2 = \frac{8\pi G\varepsilon}{3c^2} \tag{15}$$

If the energy of this particle is much higher than the rest mass energy of all the standard model particles then the energy of all the particles after this decay will be relativistic. In this case

$$\varepsilon = \frac{\mathcal{N}}{2} a_B T^4 \tag{16}$$

so that

$$H \approx \sqrt{\frac{8\pi G a_B T^4 \mathcal{N}}{6c^2}} \tag{17}$$

With all the particles in the SM, $N \approx 100$ The important thing to note here is that in the regime where the temperature per particle is much greater than the particle's rest mass energy, ($m_{particle}c^2 << k_B T$) then the particle's motion is relativistic, and hence it behaves like radiation. In this regime we have that *any* particle will have the integral in Eq (2) giving $n(T) \propto T^3 \to S(T) \propto T^3$

$$n(T) \propto T^3,\ T >> m$$

$$n(T) \propto (Tm)^{3/2} e^{-m/kT},\ m >> T$$

The entropy density for photons is given from (12), where for photons $\mathcal{N} = 2$ since photons have just 2 spin (polarization) states [1]. Hence

$$s_\gamma(T) = \frac{4a_B T^3}{3} \tag{18}$$

where a_B is given from (14). Hence we can say as a starting point that the total entropy density is as the entropy density of just photons times some unknown function of T, $f(T)$

$$\boxed{s_{tot}(T) = \frac{4a_B T^3}{3} \times f(T)} \tag{19}$$

1.2 Part b: Low Temperature Limit

In the low temperature limit, i.e. when $k_B T << m_p c^2$ for any particle in the standard model, **then the energy was sufficient for all particles and antiparticles annihilate, i.e. there would be no photons around with enough energy to pair-produce a particle and its corresponding antiparticle** (for example, the energy needed to produce a proton is $\propto$ 1 GeV. We are assuming that we are in an energy regime where the temperature is much less than that needed to pair-produce a proton-antiproton, or any other particle-antiparticle). Hence the photons will dominate the entropy density. Hence, this reduces to the simple case for the entropy density

$$s_{low\ limit}(T) = \frac{2\mathcal{N}_\gamma a_B T^3}{3} \tag{20}$$

And $N_\gamma = 2$ since the photon has only two spin states, and it's its own antiparticle. Hence

$$\boxed{s_{low\ limit}(T) \approx \frac{4a_B T^3}{3}} \tag{21}$$

[1] More on helicity, spin, polarization in part d

1.3 Part c: High Temperature Limit

In the temperature limit, $T = 10^{16}\ K$, then all particles in the standard model are relativistic, since $k_B T >> m_p c^2$ for any particle inthe standard model p. since the heaviest particle in the standard model is the top quark, with a mass $m_t = 167$ GeV, and a temperature of $10^{10} K$ corresponds to $\ 1MeV$ hence $10^{16} K \propto 10^6 Mev = 10^3 GeV \ \ >> m_t$. Hence the entropy density is given by equation (8) where the density and pressure integrals are given in 11. In this relativistic limit we have all the relativistic particles in thermal equilibrium with the photons. This means that a photon can pair-produce any given particle along with its antiparticle (for example, $\gamma \to \mu^+\ \mu^-$, etc.) In such a case, since all particles are relativistic, their masses can be neglected. The number density of particle species i is

$$n_i = \frac{g_i}{(2\pi\hbar)^3} \int \frac{d^3 p}{e^{(p-\mu_i)/k_B T} \mp 1} \tag{22}$$

And if we have chemical equilibrium we can neglect the chemical potentials μ_i. Hence

$$n_i = 4\pi g_i \left(\frac{k_B T}{2\pi\hbar}\right)^3 \int_0^\infty \frac{x^2 dx}{e^{x-/k_B T} \mp 1} \tag{23}$$

where g_i is the number of helicity (and other sources of multiplicity) states for each species, and the sign is for bosons and + for fermions. The antiparticle density $\bar{n}_i$ will be given by the same formula, hence their difference will be

$$n_i - \bar{n}_i = 8\pi g_i \left(\frac{k_{\mathcal{B}} T}{2\pi\hbar}\right)^3 \frac{\mu_i}{k_{\mathcal{B}} T} \int_0^\infty \frac{x^2 e^x dx}{(e^x \mp 1)^2} \tag{24}$$

[2] The integral over x has the value $2/3\pi$ for bosons and $1/3\pi$ for fermions, so we can write this as

$$n_i - \bar{n}_i = f(T)\tilde{g}_i \mu_i \tag{25}$$

where

$$f(T) \equiv \frac{4\pi^3}{3} \frac{(k_B T)^2}{(2\pi\hbar)^3} \tag{26}$$

and $\tilde{g}_i$ is the number of spin states, but with an extra factor of 2 for bosons. These degeneracy factors are given in Weinberg table 3.4 in page 183.

Hence, if we follow the derivation from earlier, we would have mass density for each particle species i as

$$\rho_i(T) = \int_0^\infty n_i(p,T) dp \sqrt{p^2+m^2} = \int_0^\infty n_i(p,T) dp \sqrt{p^2+m^2} = \int_0^\infty 4\pi g_i \left(\frac{k_B T}{2\pi\hbar}\right)^3 \int_0^\infty \frac{x^2 dx}{e^{x-/k_{\mathcal{B}} T} \mp 1} dp \sqrt{p^2+m^2} \tag{27}$$

In the high temperature limit, this just reduces to densities of massless (relativistic) particles

$$\begin{aligned} \rho(T) &= g \int_0^\infty \frac{4\pi p^3 dp}{(2\pi\hbar)^3} \left(\frac{1}{\exp{(p/k_{\mathcal{B}} T)} \pm 1}\right) \\ &= \begin{cases} g a_{\mathcal{B}} T^4/2 & \text{bosons} \\ 7 g a_{\mathcal{B}} T^4/16 & \text{fermions} \end{cases} \end{aligned} \tag{28}$$

[2] Note we are pretty sure that the current values $n_{\bar{i}}, 0 \approx 0$, otherwise if $n_{\bar{i}}, 0 \approx n_i, 0$ (or close) we would all annihilate. We can also extrapolate that $n_{\bar{i}}, 0 \approx 0$ for our local patch in the universe, since significant particle-antiparticle annihilation would produce gamma rays or x-rays that we could detect in the sky

We also saw that in this relativistic limit the entropy density for all the particles is the entropy density of photons times some factor which accounts for all the particles. Hence we conclude, by Eq (12), that during this high temperature regime we have the entropy density of all particles as

$$s(T) = \underbrace{\frac{4a_{\mathcal{B}}T^3}{3}}_{s_\gamma(T)} \underbrace{\frac{\mathcal{N}_{\mathcal{SM}}}{2}}_{\text{for all SM particles}} \tag{29}$$

Where $\mathcal{N}_{\mathcal{SM}}$ now has to account for all the particles in the SM at that time. We can express the number of distinct fermion states as $\mathcal{N}_F$ and the number of distinct boson states as $\mathcal{N}_B$. Hence we can say that the high temperature limit $f(t)$ reduces to

$$s(T) = \underbrace{\frac{4a_{\mathcal{B}}T^3}{3}}_{s_\gamma(T)} \underbrace{\frac{\mathcal{N}_{\mathcal{F}} + \mathcal{N}_B}{2}}_{\text{for all SM particles}} \tag{30}$$

1.4 Part d

Let us remind ourselves that $\mathcal{N}_{\mathcal{SM}}$ is the number of particle types, counting particles and antiparticles and each spin state separately, and with an extra factor of 7/8 for fermions. The universe at the time is composed of many different particles, and we should find the N for each of them.

The intrinsic angular momentum of a particle is called spin. Fermions all have spin 1/2 while bosons all have spin 1. The spin in the direction of motion of a particle is called helicity[3]. Particles can have spin that points in the direction of its travel or opposite to it, hence fermions have helicity $\pm 1/2$.

Massless particles may exist in just one helicity state, since they are always travelling at the speed of light, hence there is no reference frame in which you could Loretnz transform to "catch up" to the particle and see it spinning in the opposite direction. Neutrinos may exist only in negative helocity states, known as left-handed states, and anti-neutrinos in positive, right-handed helicity states.

The $W^\pm$ and Z^0 both have spin 1 and are massive, hence they each have three spin states $\{\pm 1, 0\}$ and hence they both have 3 helicity states$\{\pm 1, 0\}$. Photons and gluons are massless and have spin 1, hence they have spin states $\{\pm 1, 0\}$, but they only have helicity states ± 1. This can be seen classically as only left-circularly polarized and right-circularly polarized light[4].

Clearly, as we saw in earlier discussions, photons have

$$\mathcal{N}_\gamma = 2 \tag{31}$$

Neutrinos ν (which are fermions, hence we need a factor of 7/8) have only one spin state (left-handed), and antinutrinos $\bar{\nu}$ similarly have only one spin state (right-handed). They also have no color charge. For example, for the e^+ e^- pairs we have a similar thing, but now we have only one flavor and 2 spin states, hence $\mathcal{N}_{e^+e^-} = \underbrace{\frac{7}{8}}_{\text{fermions}} \times \underbrace{1}_{\text{flavors}} \times \underbrace{2}_{\text{particle-antiparticle}} \times \underbrace{2}_{\text{spin}} = \frac{7}{2}$

We consider all 3 generations of quarks and leptons, and their antiparticles.

$$\mathcal{N}_{\nu' s} = \underbrace{\frac{7}{8}}_{\text{fermions}} \times \underbrace{3}_{\text{flavors}} \times \underbrace{2}_{\text{particle-antiparticle}} \times \underbrace{1}_{\text{spin}} = \frac{21}{4} \tag{32}$$

[3]In QM terms, it is the spin operator projected onto the momentum operator

[4]Slight caveat here that virtual photons do have mass, so they may exist in the 0 helicity state as well

Now denoting leptons as l and antileptons as $\bar{l}$, we have

$$\mathcal{N}_{l\bar{l}} = \underbrace{\frac{7}{8}}_{\text{fermions}} \times \underbrace{3}_{\text{flavors}} \times \underbrace{2}_{\text{particle-antiparticle}} \times \underbrace{2}_{\text{spin}} = \frac{21}{2} \tag{33}$$

We have weak vector bosons Z^0, W^+, W^{-1}. These are bosons (factor 1), each of them has 3 spin states, and

$$\mathcal{N}_{W^\pm} = \underbrace{1}_{\text{bosons}} \times \underbrace{1}_{\text{flavors}} \times \underbrace{2}_{\text{particle-antiparticle}} \times \underbrace{3}_{\text{spin}} = 6 \tag{34}$$

$$\mathcal{N}_{Z^0} = \underbrace{1}_{\text{bosons}} \times \underbrace{1}_{\text{flavors}} \times \underbrace{1}_{\text{particle-antiparticle}} \times \underbrace{3}_{\text{spin}} = 3 \tag{35}$$

The quarks q and antiquarks $\bar{q}$

$$\mathcal{N}_{q\bar{q}} = \underbrace{\frac{7}{8}}_{\text{fermions}} \times \underbrace{3}_{\text{flavors}} \times \underbrace{2}_{\text{particle-antiparticle}} \times \underbrace{2}_{\text{spin}} = \frac{21}{2} \tag{36}$$

For the gluons g we have 8 different color combinations, but since we are not considering color charge,

$$\mathcal{N}_{g} = \underbrace{1}_{\text{bosons}} \times \underbrace{1}_{\text{flavors}} \times \underbrace{1}_{\text{particle-antiparticle}} \times \underbrace{2}_{\text{spin}} = 2 \tag{37}$$

And the neutral inflaton scalar particle, let's call it ϕ^0 has helicity 0, hence

$$\mathcal{N}_{\phi^0} = 0 \tag{38}$$

The multiplicity factor for all the standard model particles, discounting color charge [5] is then

$$\boxed{\mathcal{N}_{SM} = \mathcal{N}_\gamma + \mathcal{N}_{\nu' s} + \mathcal{N}_{l\bar{l}} + \mathcal{N}_{W^\pm} + \mathcal{N}_{Z^0} + \mathcal{N}_{q\bar{q}} + \mathcal{N}_g = \frac{157}{4}} \tag{39}$$

And we can say that the multiplicities for the bosons and fermions are

$$\mathcal{N}_B = \mathcal{N}_\gamma + \mathcal{N}_{W^\pm} + \mathcal{N}_{Z^0} + \mathcal{N}_{\phi^0} + \mathcal{N}_g = \frac{52}{4} \tag{40}$$

$$\mathcal{N}_F = \mathcal{N}_{\nu' s} + \mathcal{N}_{l\bar{l}} + \mathcal{N}_{q\bar{q}} = \frac{105}{4} \tag{41}$$

[5] More complicated analysis would potentially require us to account for color charge, but the question does not say to consider color charge

1.5 Part e

From the equation for the mass densities in 28

$$\rho(T) = g\int_0^\infty \frac{4\pi p^3 dp}{(2\pi h)^3}\left(\frac{1}{\exp\left(p/k_k T\right)\pm 1}\right) \tag{42}$$

we have that the integral reduces to an integral for fermions and an integral for bosons. Hence, plugging in the value for a_B we can sum over the different fermionic and bosonic species, hence

$$\rho_{\text{total}} \; = \sum_{i=\text{ boson}} \mathcal{N}_i\left(\frac{\pi^2}{30}\right)T_i^4 + \sum_{i=\text{ fermion}} \frac{7}{8}\mathcal{N}_i\left(\frac{\pi^2}{30}\right)T_i^4 = \mathcal{N}_{\text{total}}\left(\frac{\pi^2}{30}\right)T^4 \tag{43}$$

Where

$$\mathcal{N}_{\text{total}} \equiv \sum_{\text{boson}} \mathcal{N}_B + \sum_{\text{fermion}} \frac{7}{8}\mathcal{N}_F. \tag{44}$$

The corresponding entropy density is given by very complicated integrals in Weinberg. For example, the entropy density of only the photons, electrons, and positrons is

$$\begin{aligned} s(T) = & \frac{4a_{\mathcal{B}}T^3}{3} + \frac{4}{T}\int_0^\infty \frac{4\pi p^2 dp}{(2\pi\hbar)^3}\left(\sqrt{p^2+m_e^2} + \frac{p^2}{3\sqrt{p^2+m_e^2}}\right) \\ & \times \frac{1}{\exp\left(\sqrt{p^2+m_e^2}/k_{\mathcal{B}}T\right)+1} \end{aligned} \tag{45}$$

(Essentially, the first term on the RHS of the equation is $S_\gamma(T)$ and the second term the equation above is the precise form of $f(T)$ for electrons, positrons and photons only)

We see that we can take a limit to have an approximate answer s$\cong \frac{1}{T}(\rho+p) = \frac{2\pi^2}{45}\mathcal{N}_{\text{total}}T^3$. We can also do it the way Weinberg calculates it in his book [1] and define the integral

$$\mathcal{S}(x) \equiv 1 + \frac{45}{2\pi^4}\int_0^\infty y^2 dy\left(\sqrt{y^2+x^2} + \frac{y^2}{3\sqrt{y^2+x^2}}\right)\frac{1}{\exp\sqrt{y^2+x^2}+1} \tag{46}$$

The entropy conservation law gives $a^3T^3\mathcal{S}\left(m_e/k_{\mathcal{B}}T\right)$ constant, and since $T_v \propto 1/a$, this means that T_ν is proportional to $T\mathcal{S}^{1/3}\left(m_e/k_{\mathcal{B}}T\right)$. The temperature in this limit was higher than all the standard model particles m_{SM}: $k_BT >> m_{SM}$, and

$$S(0) = 1 + [\mathcal{N}_{SM} - \mathcal{N}_\gamma] = 1 + \frac{149}{4} = \frac{153}{4} \tag{47}$$

So

$$T_X = (4/153)^{1/3}T\mathcal{S}^{1/3}\left(m_e/k_{\mathcal{B}}T\right) \tag{48}$$

To find the asymptotic value of T/T_X without a doing long computer calculations, we note that $S(\infty) = 1$ so for $k_BT << m_e$, Eq. (48) gives

$$\frac{T}{T_X} \;\to\; \left(\frac{153}{4}\right)^{1/3} = 3.369 \text{ K} \tag{49}$$

Now using the conservation of the entropy density, we have that the comoving entropy density must be conserved, i.e.

$$\frac{dS}{dt} \equiv \frac{d}{dt}\left(a^3\frac{\rho+P}{T}\right) = 0 \tag{50}$$

Now suppose we have a plasma with initial temperature T_{in} at scale factor a_{in}, and final temperature and scale factor, then since the only variables that depend on time are $a = a(t)$ and $T = T(t)$, Eq (50) gives us

$$\frac{T_{fin}}{T_{in}} = \left(\frac{153}{4}\right)^{1/3} \frac{a_{in}}{a_{fin}} \tag{51}$$

We can proceed as Weinberg does and neglect the scale factors, by saying $s(T)a^3 = \text{ constant }$. In such a case, the scale factor ratio is just 1 in (51). Hence using $T_{fin} = T_\gamma$ and $T_{in} = T_X$, we get

$$\boxed{\frac{T_\gamma}{T_X} = \left(\frac{153}{4}\right)^{1/3} = 3.369 \text{ K}} \tag{52}$$

References

[1] S. Weinberg, Cosmology,Oxford, UK: Oxford Univ. Pr. , 2008