Homework 5

Ali Al Kadhim

Cosmology and Structure Formation

1)

a) Note that the age of the universe can be solved in one of two ways: I present one way to solve for the age of the universe. To see the alternate way to solve for the age, see Appendix A. I used the other way to confirm that I have solved for the age correctly!

The current age of the universe can be found via the fundamental Friedmann eq.

$$\dot{a}^2 + K = \frac{8\pi G\rho\, a^2}{3} \tag{1}$$

Where a is the scale parameter, K is the curvature constant and ρ is the total mass density in the universe. We also have the fluid equation

$$\dot{\epsilon} + 3\frac{\dot{a}}{a}(\epsilon + P) = 0 \tag{2}$$

And the equation of state

$$P = w\epsilon \tag{3}$$

Where P is the pressure and w is a constant. For a spatially flat universe, which is our assumption (and close to the real curvature of the universe), $K = 0$. We can write the Friedmann eq. (1) as

$$\frac{da}{dt} = a\, H_0 \sqrt{\Omega_{\Lambda 0} + \Omega_{K0}\left(\frac{a_0}{a}\right)^2 + \Omega_{m0}\left(\frac{a_0}{a}\right)^3 + \Omega_{R0}\left(\frac{a_0}{a}\right)^4} \tag{4}$$

Where $H_0 \equiv \sqrt{\frac{8\pi G\rho_0}{3}}$ is the current Hubble parameter (the Hubble parameter or "constant" measured today), where the Ωs are the density parameters for the different components in the universe, such that

$$\rho_{\Lambda 0} \equiv \frac{3H_0^2\Omega_\Lambda}{8\pi G}, \rho_{m0} \equiv \frac{3H_0^2}{8\pi G}, \rho_{r0} = \frac{3H_0^2\Omega_r}{8\pi G}$$

And $\Omega_\Lambda + \Omega_m + \Omega_\mathrm{r} + \Omega_K = 1$, $\Omega_K = -\frac{K}{a_0^2 H_0^2}$.

Fortunately, the energy density and pressure for the different components of the universe are additive. We may write the total energy density ϵ of the universe as

$$\epsilon = \sum_w \epsilon_w$$

Where ϵ_w represents the energy density of each component of with equation of state parameter w. The total pressure P is also the sum of the pressures of the different components=[;.

$$P = \sum_w P_w = \sum_w w\epsilon_w$$

The first way to solve for the age of the universe involves solving for the energy density ϵ for a particular component from the fluid equation (2). Since the energy densities are additive the equation of state must hold for each component

$$\dot{\epsilon}_w + \frac{3\dot{a}}{a}(\epsilon_w + P_w) = 0$$

$$\dot{\epsilon}_w + 3\frac{\dot{a}}{a}(\epsilon_w + w\epsilon_w) = 0$$

$$\frac{d\epsilon_w}{dt} + \frac{3}{a}\frac{da}{dt}(1 + w)\epsilon_w = 0$$

$$\frac{d\epsilon_w}{\epsilon_w} = -(1 + w)\frac{da}{a}$$

If we assume that w is a constant (which it is) then the solution for this simple ODE is

$$\epsilon_w(a) = \epsilon_{w,0}a^{-3(1+w)} \tag{5}$$

Where I have assumed that normalization that $a_0 = 1$ at the present day, when the energy density of the w component is $\epsilon_{w,0}$.

Hence if we have a **spatially flat universe** i.e. $K = 0$ we have the friedmann eq (1).:

$$\dot{a}^2 = \frac{8\pi G}{3c^2}\epsilon a^2$$

And plugging in the solution above $\epsilon_w(a)$ for a given component from eq. (5), we have

$$\dot{a}^2 = \frac{8\pi G\epsilon_0}{2c^2}a^{-(1+3w)} \tag{6}$$

We can solve this equation by making the ansatz that a has the power law form $a(t) \propto t^q$ and so $\dot{a} \propto t^{q-1}$ and hence the LHS $\dot{a}^2 \propto t^{2q-2}$ and the RHS is $\propto t^{-(1+3w)q}$ and equating the powers we have

$$2q - 2 = -(1 + 3w)q$$

And hence $q = \frac{2}{3+3w}$ (with the restriction that $w \neq -1$). Hence, with the proper normalization, the scale factor in a single component universe is

$$a(t) = \left(\frac{t}{t_0}\right)^{2/(3+3w)} \tag{7}$$

And $\dot{a}(t) = \frac{2}{3+3w}\left(\frac{1}{t_0}\right)^{2/(3+3w)} t^{(3w-1)/(3+3\,w)}$ and plugging this into equation (6) we have

$$\left[\frac{2}{3+3w}\left(\frac{1}{t_0}\right)^{\frac{2}{3+3w}} t^{(-3w-1)/(3+3w)}\right]^2 = \frac{8\pi G\epsilon_0}{2c^2}\left(\frac{t}{t_0}\right)^{-\frac{2(1+3w)}{3+3w}}$$

Notice that the t factor cancels, and we solve for t_0 to get

$$t_0 = \frac{1}{1+w}\left(\frac{c^2}{6\pi G\epsilon_0}\right)^{1/2}$$

The Hubble constant for this universe is

$$H_0 \equiv \frac{\dot{a}(t_0)}{a(t_0)} = \frac{\frac{2}{3+3w}\left(\frac{1}{t_0}\right)^{2/(3+3w)} t_0^{(3w-1)/(3+3\,w)}}{1}$$

Where $a(t_0)$ and $\dot{a}(t_0)$ were derived above! Since $a(t_0) = \left(\frac{t_0}{t_0}\right)^{2/(3+3w)} = 1$. Simplifying we get

$$H_0 = \frac{2}{3(1+w)}\frac{1}{t_0}$$

Hence the current age of the universe in terms of the current Hubble parameter (Hubble time) is

$$t_0 = \frac{2}{3(1+w)}\ H_0^{-1} \tag{8}$$

Depending on what single component there are other than being empty, we can use the above equation.

1) Matter or a Dark matter-dominated universe using Alternate Solution

For a matter or dark matter-dominated universe, $w = 0$. Hence, plugging $w = 0$ into equation (8) we can readily find the age of the universe as

$$\boxed{\boldsymbol{t_0 = \frac{2}{3}\ H_0^{-1}.}}$$

2) Empty Universe

An empty universe means that there exista only the curvature term in the Friedmann eq. and no energy densities for any component. This means the Fridmann eq .(1) reduces to

$$\dot{a}^2 = -K$$

Notice that ***K* has to be zero or negative**, since a positively curved space means an imaginary value for $\dot{a}$ which is forbidden. Hence we have two cases:

i) $K = 0$: One acceptable solution to this is equation has $\dot{a} = 0$ and $K = 0$.

ii) $K = -1$: $\dot{a}^2 = 1 \rightarrow\ \dot{a} = \pm 1 \rightarrow a(t) = t \rightarrow \frac{\dot{a_0}}{a_0} = \frac{1}{H_0}$

Hence $H_0 = \frac{1}{t_0}$ i.e. **the age of the universe is the Hubble time**

$$\boxed{\boldsymbol{t_0 = H_0^{-1}}}$$

3) Cosmological constant universe

We saw above that a single component universe with $w \neq -1$ has a power law dependence of a on t (see equation (7))

$$a(t) \propto t^{2/(3+3w)}$$

Now supposing we have a cosmological constant universe (a universe in which the energy density is dominated by a cosmological constant Λ only), then for such a universe the energy density has no dependence on the scale factor (and hence independent of time), therefore, Friedmann equation (1) becomes

$$\dot{a}^2 = \frac{8\pi G\epsilon_\Lambda}{3c^2}\ a^2$$

This can be rewritten in the form

$$\dot{a} = H_0 a$$

Where $H_0 = \left(\frac{8\pi G \epsilon_\Lambda}{3c^2}\right)^{1/2}$. Hence

$$\frac{da}{a} = H_0 dt \tag{8}$$

This is easily integrated

$$\ln(a) = H_0(t - t_0)$$

and hence the solution for the scale factor

$$a(t) = e^{H_0(t-t_0)}$$

So we see that this universe constantly expands, and the energy density ϵ stays constant forever. We can also plug this in to the Friedmann equation using

$\dot{a}(t) = H_0^2(t - t_0)e^{H_0(t-t_0)}$ so

$$H_0^4(t - t_0)^2 e^{2H_0(t-t_0)} = H_0 e^{H_0(t-t_0)}$$

We see that such a universe is

$$\boxed{\textbf{infinitely old}, t_0 = \infty,}$$

and expands forever.

b) Luminosity distance

The luminosity distance D_L is given by

$$D_L \equiv a_0\, l(r)\, (1 + z) \tag{9}$$

Where $l(r)$ depends on the curvature K

$$l(r) = \begin{cases} \sin(r)\,; & K = +1 \\ r; & K = 0 \\ \sinh(r)\,; & K = -1 \end{cases} \tag{10}$$

And since we are considering a flat universe case, $l(r) = r$, therefore

$$D_L = a_0\, r\, (1 + z) \tag{11}$$

Where r is the comoving distance, which may differ depending on the type of components of energy density that are present in the universe. In order to calculate this comoving distance, we go back to the R-W metric,

$$ds^2 = -c^2 dt^2 + a^2[dr^2 + l^2(r)[d\theta^2 + \sin^2\theta \ d\phi^2]\,] \tag{12}$$

And we consider a distance to a source where the separation is only in the radial direction, therefore $d\theta = d\phi = 0$ and we consider a lightlike interval, therefore $ds^2 = 0$. Hence equation (12) turns into

$$c\, dt = -a\, dr$$

And we use this to calculate r, where we replace dt by the appropriate expression from the Friedmann equation (which depends on the components). By the equation above, we integrate

$$\frac{c}{a}\int_t^{t_0} dt = -\int_r^0 dr = r$$

Where the limits are from a time t in the past to the time now t_0, and the distance from r which is away to a distance 0 i.e. “here”. Therefore we have

$$r = \frac{c}{a}\int_t^{t_0} dt \tag{13}$$

And fromFriedmann equation (4) we make the subsititution $x \equiv \frac{a}{a_0} \rightarrow dx = \frac{da}{a_0}$

Therefore Friedmann equation (4) becomes

$$dt = \frac{dx}{H_0\, x\sqrt{\Omega_{\Lambda 0} + \Omega_{m0}\, x^{-3} + \Omega_{K0}\, x^{-2} + \Omega_{R0}\, x^{-4}}} \tag{14}$$

We can use the equation above (14) to solve for the luminosity distance for a given single-component universe.

1) Matter-Dominated Universe

If we have a matter dominated universe, then $\Omega_{m0} = 1$ and all the other Ωs are =0. Hence equation (14) turns into

$$dt = \frac{dx}{H_0\, x\sqrt{x^{-3}}} = \frac{dx}{H_0\, x^{-1/2}} = \frac{dx\, x^{1/2}}{H_0}$$

And we plug the expression above into equation (13). Therefore we have

$$r = c\int_t^{t_0}\frac{dt}{a} = \frac{c}{H_0}\int_x^1 \frac{dx\, x^{1/2}}{a} \tag{15}$$

Where we are integrating from x in the past to $x = 1$ today. Also, since $x \equiv a/a_0$, we plug in $a = a_0 x$ in the integral above to get.

$$r = \frac{c}{H_0}\int_x^1 \frac{dx\, x^{1/2}}{a_0 x} = \frac{c}{H_0 a_0}\int_x^1 dx\, x^{-1/2} = \frac{2c}{H_0 a_0}\left[1 - x^{\frac{1}{2}}\right]$$

And since $x \equiv \frac{a}{a_0} = \frac{1}{1+z}$ we can write the above equation in terms of z which is measurable

$$r = \frac{2c}{H_0 a_0}\left[1 - (1+z)^{-\frac{1}{2}}\right] \tag{16}$$

Now, in order to find the luminosity distance D_L we simply plug in the comoving distance (16) into the definition of the luminosity distance D_L from equation (11), Hence we have the luminosity distance for matter-dominated universe D_L^M as

$$\boxed{\boldsymbol{D_L^M = \frac{2c}{H_0}\left[1 - (1+z)^{-1/2}\right](1+z)}} \tag{17}$$

2) Cosmological Constant Universe

And since we have only $\mathbf{\Omega_{\Lambda 0} = 1}$ and the rest zero we have

$$dt = \frac{dx}{H_0\, x}$$

Therefore

$$r = c\int_t^{t_0}\frac{dt}{a} = \frac{c}{H_0}\int_x^1 \frac{dx}{a\, x}$$

Where we are integrating from x in the past to $x = 1$ today. Also, since $x \equiv a/a_0$, we plug in $a = a_0 x$ in the integral above to get

$$r = \frac{c}{a_0 H_0}\int_x^1 \frac{dx}{x^2}$$

$$r = \frac{-c}{a_0 H_0}\left[\frac{1}{x}\right]_x^1 = \frac{c}{a_0 H_0}\left(\frac{1}{x} - 1\right) \tag{1812}$$

Hence for a cosmological constant universe, the angular diameter distance and the luminosity distance are found using the equations above

$$\boldsymbol{D_L^\Lambda = a_0 \frac{c}{a_0 H_0}\left(\frac{1}{x} - 1\right)(1+z)}$$

$$\boldsymbol{x = \frac{a}{a_0} = \frac{1}{1+z}}$$

$$\boldsymbol{\rightarrow r = \frac{c}{a_0 H_0} z}$$

$$\boxed{\boldsymbol{D_L^\Lambda = \frac{z(1+z)c}{H_0}}}$$

3) Empty Universe

For an empty universe (curvature only) we have $\Omega_{K0} = 1$ and the other Ωs zero, so

$$dt = \frac{dx}{H_0 x\sqrt{x^{-2}}}$$

Hence $dt = \frac{dx}{H_0}$ and therefore

$$r = \frac{c}{a}\int_t^{t_0} dt = \frac{c}{H_0}\int_x^1 \frac{dx}{a} = \frac{c}{H_0 a_0}\int_x^1 \frac{dx}{x} = \frac{c}{H_0\, a_0}\ln\frac{1}{x}$$

Or

$$r = \frac{c}{H_0 a_0}\ln(1+z) \tag{1913}$$

For an empty universe, there has to be a curvature term, and the curvature has to be negative, $K = -1$. For a non-flat universe, the general form of D_L is

$$D_L^{non-flat} = a_0 \sinh(r)(1+z)$$

Since $l = \sinh(r)$ for $K = -1$. Hence the above equation becomes

$$D_L^K = a_0 \sinh\left(\frac{c}{H_0 a_0} ln(1+z)\right)(1+z)$$

$$D_L^K = \frac{c}{H_0} \sinh(ln(1+z))\ (1+z)$$

Now we can simplify (if we wish) the above expression using $\sinh(x) = \frac{e^x - e^{-x}}{2}$, therefore this becomes

$$D_L^K = \frac{c}{2H_0}\left[e^{\ln(1+z)} - e^{-\ln(1+z)}\right](1+z)$$

$$D_L^K = \frac{c}{2H_0}[2(1+z)](1+z)]$$

$$\boxed{\boldsymbol{D_L^K = \frac{c}{H_0}(1+z)^2}}$$

C) Angular diameter distance

Now we wish to calculate the angular diameter distance for the different single-component universes. The angular diameter distance is defined as

$$D_A = \frac{a_0 l(r)}{(1+z)} \tag{20}$$

For flat spaces, this is

$$D_A = \frac{a_0\, r}{(1+z)} \tag{21}$$

1)Matter-Dominated Universe

Since we have already calculated the comoving distance r for matter-dominated universe, we can simply use the result and plug it in here. Using equation 16 (repeated here)

$$r = \frac{2c}{H_0 a_0}\left[1-(1+z)^{-\frac{1}{2}}\right] \tag{16 repeated}$$

Into equation (21), we have the matter-dominated angular diameter distance D_A^M.

$$D_A^M = \frac{2c}{H_0}\left[1-(1+z)^{-\frac{1}{2}}\right](1+z)^{-1}$$

Or we can simplify it if we wish as

$$\boxed{\boldsymbol{D_A^M = \frac{2c}{H_0}\left[(1+z)^{-1}-(1+z)^{-\frac{3}{2}}\right]}}$$

2) Cosmological Constant Universe

For a cosmological constant universe, r was calculated again using the previous question, and it can be used. Rewriting r:

$$\boldsymbol{r = \frac{c}{a_0 H_0} z}$$

We can plug this in to equation (21) to find the angular diameter distance for a cosmological constant universe D_A^Λ

$$\boxed{\boldsymbol{D_A^\Lambda = \frac{c}{H_0}\frac{z}{(1+z)}}}$$

3) Empty Universe

Similarly, for an empty universe with negative curvature, the angular diameter distance will be (from equation (17))

$$D_A = \frac{a_0}{(1+z)}\sinh(r)$$

Hence for an empty universe (with negative curvature), we use use the comoving)distance for empty universe (from equation 19)

$$r = \frac{c}{H_0 a_0}\ln\,(1+z)$$

Hence

$$D_A^K = \frac{a_0}{1+z}\sinh\left(\frac{c}{H_0 a_0}\ln(1+z)\right) = \frac{c}{H_0(1+z)}\sinh(\ln(1+z))$$

And using the identity $\sinh(x) = \frac{e^x - e^{-x}}{2}$ again

$$\boxed{\boldsymbol{D_A^K = \frac{c}{H_0}}}$$

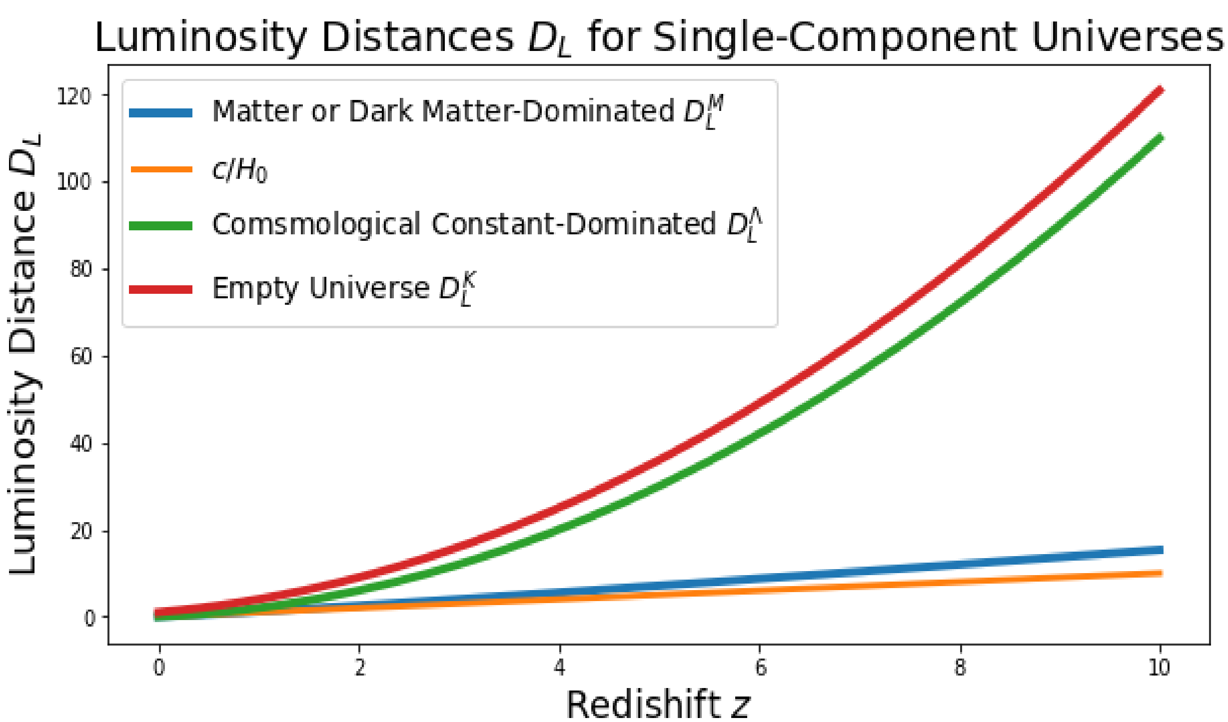
C) Plots for luminosity distances

With the following simple code, we can plot these luminosity distances in python

```
import numpy as np
import matplotlib.pyplot as plt
z=np.linspace(0.,10)
#c/H_0=1
DL_M= 2*(1- (1+z)**(-1/2))*(1+z)
DL_lambda=z*(1+z)
DL_K= (1+z)**2

plt.figure(figsize=(10, 5))
plt.plot(z,DL_M, linewidth=4)
plt.plot(z,z, linewidth=3)
plt.plot(z, DL_lambda, linewidth=4)
plt.plot(z,DL_K, linewidth=4 )

plt.legend([r"Matter or Dark Matter-Dominated $D_L^M$", "$c/H_0$",
            "Comsmological Constant-Dominated $D_L^\Lambda$",
           "Empty Universe $D_L^K$"], fontsize=14)
plt.title(r"Luminosity Distances $D_L$ for Single-Component Universes", fontsize=20)
plt.xlabel('Redishift $z$', fontsize=18)
plt.ylabel(r'Luminosity Distance $D_L$', fontsize=18)
plt.show()
```



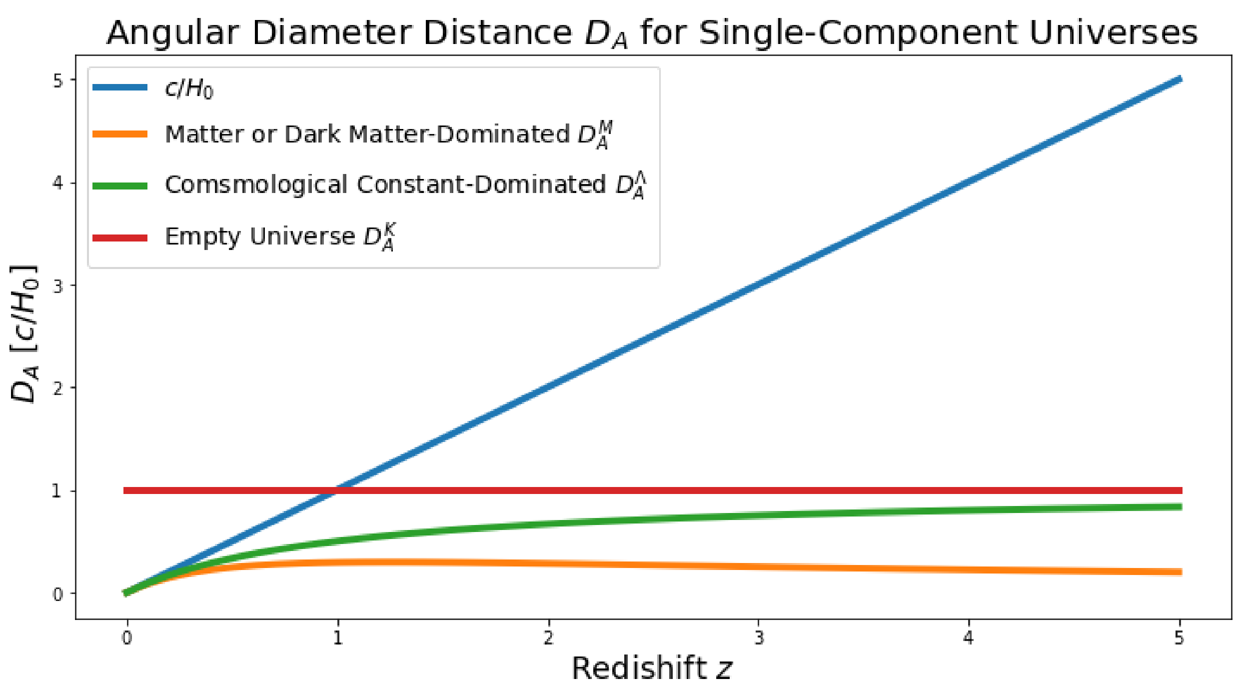
D) Plots for Angular-Diameter Distances

```
z=np.linspace(0.,5)
DA_M=2* ((1+z)**(-1) - (1+z)**(-3/2) )
DA_lambda=z/(1+z)
DA_K=np.ones(len(z))

plt.figure(figsize=(12, 6))
plt.plot(z,z, linewidth=4)
plt.plot(z, DA_M, linewidth=4)
plt.plot(z, DA_lambda, linewidth=4)
plt.plot(z, DA_K, linewidth=4)


plt.legend([r"$c/H_0$", "Matter or Dark Matter-Dominated $D_A^M$",
            "Comsmological Constant-Dominated $D_A^\Lambda$",
           "Empty Universe $D_A^K$"], fontsize=14)
plt.xlabel('Redishift $z$', fontsize=18)
plt.ylabel(" $D_A$ $[c/H_0]$", fontsize=18)

plt.title("Angular Diameter Distance $D_A$ for Single-Component Universes", fontsize=20)
```



2) Beginning of Acceleration

The observational evidence that the expansion of the universe is is speeding up and not slowing down (slowing down due to gravity and if we put positive Ωs in the Friedmann equation, as was previously thought). This means that presumable at some time in the past, which corresponds to some intermediate redshift, the universe underwent a transition to shift the universe from slowing down to speeding up (this corresponds to changing the sign in what Weiberg denotes as $q_0(z)$.) This is a very important problem in cosmology to understand, since knowing this redshift can allow us to constrain the different candidates that describe the negative pressure causing the expansion, i.e. dark energy. I thereby took extra time to understand it well before attempting to solve it, apologies to the grader.

The Friedmann equation also allows us to find the redshift at which the expansion of the universe stopped decelerating and started to accelerate. Assuming $\Omega_M = 0.25$ and $\Omega_\Lambda = 0.75$, and that $\Omega_R = 0$ we can do this. Friedmann equation can be rewritten as

$$\frac{da}{dt} = \dot{a} = a\,H_0\sqrt{\Omega_{\Lambda 0} + \Omega_{k0}\left(\frac{a_0}{a}\right)^2 + \Omega_{M0}\left(\frac{a_0}{a}\right)^2}$$

What we need to do is find the turning point, or inflection point in the Friedmann equation (since this is exactly what it means to turn from accelerating to decelerating). Hence this conditions occurs when second derivative of a is set to zero. Hence

$$\ddot{a} = \frac{d}{dt}\left[a\,H_0\sqrt{\Omega_{\Lambda 0} + \Omega_{k0}\left(\frac{a_0}{a}\right)^2 + \Omega_{M0}\left(\frac{a_0}{a}\right)^2}\right]$$

And the condition is found when $\ddot{a} = 0$, i.e. when

$$\frac{d}{dt}\left[a\,H_0\sqrt{\Omega_{\Lambda 0} + \Omega_{k0}\left(\frac{a_0}{a}\right)^2 + \Omega_{M0}\left(\frac{a_0}{a}\right)^2}\right] = 0$$

$$\frac{d}{dt}\left[a\,H_0\sqrt{\Omega_{\Lambda 0} + \Omega_{k0}\left(\frac{a_0}{a}\right)^2 + \Omega_{M0}\left(\frac{a_0}{a}\right)^2}\right] =$$

$$H_0\left[a \times \frac{d}{dt}\left(\sqrt{\Omega_{\Lambda 0} + \Omega_{k0}\left(\frac{a_0}{a}\right)^2 + \Omega_{M0}\left(\frac{a_0}{a}\right)^2}\right) + \sqrt{\Omega_{\Lambda 0} + \Omega_{k0}\left(\frac{a_0}{a}\right)^2 + \Omega_{M0}\left(\frac{a_0}{a}\right)^2} \times \left(\frac{da}{dt}\right)\right]$$

$$\frac{d}{dt}\left(\sqrt{\Omega_{\Lambda 0} + \Omega_{k0}\left(\frac{a_0}{a}\right)^2 + \Omega_{M0}\left(\frac{a_0}{a}\right)^2}\right)$$

$$= \frac{1}{2\sqrt{\Omega_{\Lambda 0} + \Omega_{k0}\left(\frac{a_0}{a}\right)^2 + \Omega_{M0}\left(\frac{a_0}{a}\right)^2}}\frac{\partial}{\partial a}\left(\Omega_{\Lambda 0} + \Omega_{k0}\left(\frac{a_0}{a}\right)^2 + \Omega_{M0}\left(\frac{a_0}{a}\right)^2\right)\frac{\partial a}{\partial t}$$

$$= \frac{1}{2\sqrt{\Omega_{\Lambda 0} + \Omega_{k0}\left(\frac{a_0}{a}\right)^2 + \Omega_{M0}\left(\frac{a_0}{a}\right)^2}}\left(-2\,\Omega_{K0}\left(\frac{a_0}{a}\right)\,a^{-2} - 3\,\Omega_{M0}\left(\frac{a_0}{a}\right)^2\,a^{-2}\right)$$

Hence

$$\frac{d}{dt}\left(\sqrt{\Omega_{\Lambda 0} + \Omega_{k0}\left(\frac{a_0}{a}\right)^2 + \Omega_{M0}\left(\frac{a_0}{a}\right)^2}\right)$$

$$= \frac{1}{2\sqrt{\Omega_{\Lambda 0} + \Omega_{k0}\left(\frac{a_0}{a}\right)^2 + \Omega_{M0}\left(\frac{a_0}{a}\right)^2}}\left(-2\Omega_{K0}\frac{a_0}{a^3} - 3\Omega_{M0}\frac{a_0^2}{a^4}\right)$$

And plugging this in to our condition, and plugging in $\frac{da}{dt} = a\,H_0\sqrt{\Omega_{\Lambda 0} + \Omega_{k0}\left(\frac{a_0}{a}\right)^2 + \Omega_{M0}\left(\frac{a_0}{a}\right)^2}$ we have

$$\frac{d}{dt}\left[a\,H_0\sqrt{\Omega_{\Lambda 0} + \Omega_{k0}\left(\frac{a_0}{a}\right)^2 + \Omega_{M0}\left(\frac{a_0}{a}\right)^2}\right]$$

$$= a\left[\frac{1}{2\sqrt{\Omega_{\Lambda 0} + \Omega_{k0}\left(\frac{a_0}{a}\right)^2 + \Omega_{M0}\left(\frac{a_0}{a}\right)^2}}\left(-2\Omega_{K0}\frac{a_0}{a^3} - 3\Omega_{M0}\frac{a_0^2}{a^4}\right)\right]$$
$$+ \sqrt{\Omega_{\Lambda 0} + \Omega_{k0}\left(\frac{a_0}{a}\right)^2 + \Omega_{M0}\left(\frac{a_0}{a}\right)^2} \times a\,H_0\sqrt{\Omega_{\Lambda 0} + \Omega_{k0}\left(\frac{a_0}{a}\right)^2 + \Omega_{M0}\left(\frac{a_0}{a}\right)^2}$$

$$= \frac{1}{2\sqrt{\Omega_{\Lambda 0} + \Omega_{k0}\left(\frac{a_0}{a}\right)^2 + \Omega_{M0}\left(\frac{a_0}{a}\right)^3}}\left(-2\Omega_{K0}\frac{a_0}{a^2} - 3\Omega_{M0}\frac{a_0^2}{a^3}\right)$$
$$+ a\,H_0\left[\Omega_{\Lambda 0} + \Omega_{k0}\left(\frac{a_0}{a}\right)^2 + \Omega_{M0}\left(\frac{a_0}{a}\right)^3\right]$$

And our condition is to set this to zero

$$\frac{1}{2\sqrt{\Omega_{\Lambda 0}+\Omega_{k0}\left(\frac{a_0}{a}\right)^2+\Omega_{M0}\left(\frac{a_0}{a}\right)^3}}\left(-2\Omega_{K0}\frac{a_0}{a^2}-3\Omega_{M0}\frac{a_0^2}{a^3}\right)$$
$$+\,a\,H_0\left[\Omega_{\Lambda 0}+\Omega_{k0}\left(\frac{a_0}{a}\right)^2+\Omega_{M0}\left(\frac{a_0}{a}\right)^3\right]=0$$

Now, we multiply by the denominator $\sqrt{\Omega_{\Lambda 0}+\Omega_{k0}\left(\frac{a_0}{a}\right)^2+\Omega_{M0}\left(\frac{a_0}{a}\right)^2}$ get rid of it, so we get

$$\frac{1}{2}\left(2\Omega_{K0}\frac{a_0}{a}+3\Omega_{M0}\frac{a_0^2}{a^3}\right)=aH_0\left[\Omega_{\Lambda}+\Omega_{K0}\left(\frac{a_0}{a}\right)^2+\Omega_{M0}\left(\frac{a_0}{a}\right)^3\right]^{\frac{3}{2}}$$

Now we substitude $\Omega_{K0}=1-\Omega_{M0}-\Omega_{\Lambda 0}$

$$(1-\Omega_{M0}-\Omega_{\Lambda})\left(\frac{a_0}{a}\right)+\frac{3}{2}\Omega_M\frac{a_0^2}{a^3}=a\,H_0\left[\Omega_{\Lambda 0}+(1-\Omega_{M0}-\Omega_{\Lambda 0})\left(\frac{a_0}{a}\right)^2+\Omega_{M0}\left(\frac{a_0}{a}\right)^3\right]^{3/2}$$

And we substitute $x=\frac{a_0}{a}$ to make it an equation of one variable and simplify the algebra

$$(1-\Omega_{M0}-\Omega_{\Lambda 0})\frac{x}{a}+\frac{3}{2}\Omega_{M0}\frac{x^2}{a}-a\,H_0[\Omega_{M0}+(1-\Omega_{M0}-\Omega_{\Lambda 0})x^2+\Omega_{M0}\,x^2]^{\frac{3}{2}}=0$$

A few more simple algebra steps and we are finished in solving for x so that

$$x=\frac{1}{1+z}=\left[\frac{2\Omega_{\Lambda 0}}{\Omega_{M0}}\right]^{-1/3}$$

So that

$$\boxed{\boldsymbol{z=\left[\frac{2\Omega_{\Lambda 0}}{\Omega_{M0}}\right]^{1/3}-1}}$$

If we neglect curvature and set $\Omega_{K0}=0$ (and $\Omega_{M0}+\Omega_{\Lambda 0}=1$ so that we can substitute$\Omega_{\Lambda 0}=\Omega_{M0}-1$), our algebra is greatly simplified, so that this transition redsdhift for a flat universe is

$$z_{flat}=\left[\frac{2(1-\Omega_{M0})}{\Omega_{M0}}\right]-1$$

If we have by assumption that $\Omega_M = 0.25$ and $\Omega_\Lambda = 0.75$ we can plug those in to find the redshift

$$\boxed{\boldsymbol{z = \left[\frac{2\Omega_\Lambda}{\Omega_M}\right]^{1/3} - 1 = \left[2 \times \frac{0.75}{0.25}\right]^{\frac{1}{3}} - 1 = 0.8171}}$$

3) Epoch of Matter-Radiation Equality

a) want to solve for when the matter density and radiation density were equal. In other words

$$\rho_M = \rho_R$$

$$\rho_{M0}\left(\frac{a_0}{a}\right)^3 = \rho_{R0}\left(\frac{a_0}{a}\right)^4$$

And since $\Omega_{M0} = \frac{\rho_{M0}}{\rho_c}$ and $\Omega_{R0} = \frac{\rho_{R0}}{\rho_c}$ where ρ_c is the critical density, we can substitute for the densities as

$$\Omega_{M0}\,\rho_c\,(1+z)^3 = \Omega_{R0}\,\rho_c\,(1+z)^4$$

$$\Omega_{M0}\,(1+z)^3 = \Omega_{R0}\,(1+z)^4$$

And substituting $\frac{a}{a_0} = \frac{1}{1+z} \rightarrow \frac{a_0}{a} = 1+z$

$$\Omega_{M0}\,(1+z)^3 = \Omega_{R0}\,(1+z)^4$$

And we can then simply solve for z from above, so that $\Omega_{M0} = \Omega_{R0}(1+z)$ or

$$z = \frac{\Omega_{M0}}{\Omega_{R0}} - 1$$

And substituting the given values, we get the (matter-radiation) equality redshift $z_{equality}$

$$\boldsymbol{z_{equality} = \frac{0.15}{4.15\times10^{-5}} - 1 = 3613.46}$$

b)

By definition, the matter density (not energy density as in the pdf!) for some redshift z is

$$\rho_R(z) = \rho_{R0}\left(\frac{a_0}{a}\right)^4 = \rho_{R0}(1+z)^4$$

Therefore at matter-radiation equality the mass density for radiation is

$$\rho_R(z_{equality}) = \rho_{R0}(1+z_{equality})^4 = 7.8\times10^{-34}\times(1+3613.46)^4 = 1.331\times10^{-19}$$

Hence

$$\boldsymbol{\rho_R(z_{equality}) = 1.331\times10^{-19}\ g\ cm^{-3}}$$

c) $K = 0$ so $\Omega_K = 0$. Assuming the universe was radiation-dominated prior to this point, then we can calculate the age of the universe at that time, i.e. the time elapsed since the big bang, until reaching that redshift. This can be done by Friedmann equation, as in the previous questions. In this case, we have a radiation-dominated universe, so

$$t(z) = \frac{1}{H_0} \int_{x=0}^{x=1/(1+z)} \frac{dx}{x\sqrt{\Omega_R x^{-4}}}$$

Where $\Omega_R = \frac{\rho_R}{\rho_c}$ so that $\Omega_{R0} = \frac{\rho_{R0}}{\rho_c}$ and if space is flat, $\rho_c = \rho_{c0}$.Also, we integrate from since $x = 0$ corresponds to infinite redshift which corresponds to the big bang.

$$t(z_{equality}) = \frac{x^2}{2\sqrt{\Omega_R}} |_0^{\frac{1}{1+z}}$$

Therefore, taking the limit and plugging in the values for Ω_{R0} and $z_{equality}$

$$t(z_{equality}) = \frac{1}{2\sqrt{\Omega_{R0}}} \left(\frac{1}{1 + z_{eq}} \right)^2 = \frac{1}{2\sqrt{4.15 \times 10^{-5} h^{-2}}} \left(\frac{1}{1 + 3613.46} \right)^2$$

Where h is the Hubble constant. Using the agreed upon value (see Efstathiou, 2014, arXiv:1311.3461v2 , for example), $h = 67.3 \frac{km}{s} \frac{1}{Mpc}$, we see that by dimensional analysis, we do get a time dimension for the answer.

$$\boxed{\boldsymbol{t(z_{equality}) = 1581.95\ h^{-1}}}$$

We can use the conversion factor 1.02×10^{-12} to convert the h value into units of inverse years. Hence

$$t(z_{equality}) = \frac{1.581.95}{67.3 \times 1.02 \times 10^{-12}}$$

With this conversion, I attain $t(z_{equality}) = 3.42 \times 10^{11}\ years$. This obviously does not correspond to our universe.