Homework 7

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1 Problem 1

Find the optical depth of the Thompson scattering τ_T along the line of sight to a source at redshift z_e. Assume a flat universe with $\Omega_B \approx 0.02$

Solution very distant and early quasars give us the opportunity to study the cosmic gas frm which the fist galaxies and clusters of galaxies formed, by observing the way in which the cosmic gas absorbs the distant light coming from the quasars. The intergalactic gas, which is at some non-zero temperature $T(t)$ at time t will absorb the incoming light at frequence ν as well as emit photons to the light by the process of stimulkated emiision. If the intergalactic gas absorbs the light of frequency ν at a rate $\Lambda(\nu, t)$ then at time t the frequency would redshift by a factor $\nu_1 a(t_1)/a(t)$ where ν_1 and t_1 are the frequency and time at which the light was emitted from the quasar. Then the intensity of the light ray will change as

$$\dot{I}(t) = -\left[1 - \exp\left(-\frac{hv_1 a(t_1)}{k_{\mathcal{B}} T(t) a(t)}\right)\right] \Lambda\left(v_1 a(t_1)/a(t), t\right) I(t) \tag{1}$$

Which is Weinberg (1.10.1). The intensity observed at earth (at time t_0) will be

$$I(t_0) = \exp(-\tau) I(t_1) \tag{2}$$

Where τ is the optical depth, integrated from the time of emission to the time of observation

$$\tau = \int_{t_1}^{t_0} \left[1 - \exp\left(-\frac{hv_1 a(t_1)}{k_B T(t) a(t)}\right)\right] \Lambda\left(v_1 a(t_1)/a(t), t\right) dt \tag{3}$$

This is particularly useful to study the intergalactic gas through the analysis of the absorption of Hydrogen, also known as the Lyman-α forest.

The derivation above for the optical scattering can be derived using other useful quantities. When we examine the light of intergalactic gas, the photons we collect have been travelling straight toward us since the last time they scattered from a free electron. During a brief time interval $t \to t + dt$, the probability that a photon undergoes scattering is $dP = \Gamma(t) dt$ where $\Gamma(t)$ is the scattering rate at time t. Thus if we detect a photon at time t_0, the expected number of scatterings it has undergone since an earlier time t is the optical depth, which can be written as

$$\tau(t) = \int_t^{t_0} \Gamma(t) dt \tag{4}$$

Suppose we have Thomson scattering, i.e a process with the interaction

$$\gamma + e^- \to \gamma + e^-$$

The cross section for Thomson scattering is the Thomson scattering cross section $\sigma_T = 0.66525 \times 10^-24\ \text{cm}^2$. The mean free path of a photon (the mean distance it travels before scattering from a free electron) is

$$\lambda = \frac{1}{n_e \sigma_T} \tag{5}$$

Where n_e is the number density of electrons. Since photons travel at the speed of light c, the rate at which a photon undergoes scattering interactions is

$$\Gamma = \frac{c}{\lambda} = n_e \sigma_T c \tag{6}$$

Then, plugging in the expression of equation (6) into equation (4), we get

$$\tau(t) = \int_{t_e}^{t_0} n_e \sigma_T c dt \tag{7}$$

Where t_e is the time of emission of the light, corresponding to a source of redshift z_e. We can change the variables of integration in equation (7) by noting that

$$a = (1+z)^-1 \ \rightarrow \frac{da}{dt} = -(1+z)^{-2}\ \frac{dz}{dt}$$

And since $\frac{da}{dt} = H(t)\ a(t)$, we get

$$H(t)\ a(t) = -(1+z)^-2\ \frac{dz}{dt}$$

Hence

$$dt = -\ \frac{dz}{H(z)\ (1+z)} \tag{8}$$

Hence we can plug this expression into equation (7) to get

$$\tau(z_e) = -\int_{z_e}^{z_0} n_e \sigma_T c\ \frac{dz}{H(z)\ (1+z)} \tag{9}$$

Now, assuming that the universe is consisting of protons and electrons, hence the universe contains no elements other than hydrogen. The hydrogen can take the form of a neutral atom (H), or of naked hydrogen nucleus, otherwise known as the proton p. To maintain charge neutrality in this hydrogen-only universe, the number of free electrons must be equal to the number of free protons.

$$n_e = n_p \tag{10}$$

The degree to which the baryonic content was ionized can be expressed as X

$$X \equiv \frac{n_p}{n_p + n_{\rm H}} = \frac{n_p}{n_{\rm bary}} = \frac{n_e}{n_{\rm bary}} \tag{11}$$

Since we are concerned with the electron-photon scattering, we note that the number density dilutes with the expansion of the universe

$$n_e(z) = n_{e,0}(1+z)^3 \tag{12}$$

And by the above argument, since the universe is fully ionized, and only made of hydrogen, the baryonic content of the universe takes itself in the form of free protons, and since $n_e = n_p$, we have

$$n_e(z) = n_{B,0}(1+z)^3 \tag{13}$$

Where $n_{B,0}$ is the current number density of this universe now. Also, since $n_{B,0} = \frac{\Omega_{B,0}\ \rho_{crit,0}}{m_H}$ we have

$$n_e(z) = \Omega_{B,0}\rho_{crit,0}m_H(1+z)^3 \tag{14}$$

and $\rho_{crit,0} = \frac{3H_0^2}{8\pi G}$, therefore,

$$n_e(z) = \frac{3H_0^2\Omega_{B,0}}{8\pi G m_H}(1+z)^3 \tag{15}$$

Hence we can plug in the number density of electrons from equation (15) into the optical depth in equation (9) (and flipping the limits using the negative sign) to get

$$\tau(z_e) = \int_{z_0=0}^{z_e} \frac{3H_0^2\Omega_{B,0}}{8\pi G m_H}(1+z)^3\sigma_T c\ \frac{dz}{H(z)\ (1+z)} \tag{16}$$

Now, using the Hubble parameter as a function of z

$$H(z) = H_0\sqrt{\Omega_{M,0}(1+z)^3 + \Omega_\Lambda} \tag{17}$$

And using it in equation (16) we get

$$\tau(z_e) = \frac{3H_0\Omega_{\mathrm{B},0}}{8\pi G m_H}\sigma_T c\int_0^{z_e} \frac{(1+z)^2 dz}{\sqrt{\Omega_{\mathrm{B},0}(1+z)^3 + \Omega_\Lambda}} \tag{18}$$

Now for plugging in the values, we want to simplify the constant in front of the integral. The constant in front is $\frac{2c\rho_{c0}\Omega_B\sigma_T}{3\Omega_M H_0 m_H} = \frac{2c\Omega_B\sigma_T}{3\Omega_M H_0 m_H}\ \frac{3H_0^2}{8\pi G}\ \frac{cH_0\Omega_B\sigma_T}{4\pi G\Omega_M m_H}$. We now want to simplify the constant in terms of h. Since $\Omega_B\ h^2 = 0.02$, and the Hubble constant $h = 0.7$, so $\Omega_B h = 0.03$. And using the usual values of $\Omega_M = 0.3, \Omega_\Lambda = 0.7$, the constant in front of the integral reduces to $\frac{0.046\times\Omega_B h}{\Omega_M}$. Hence, equation (18) becomes

$$\tau(z_e) = \frac{0.046\times\Omega_B h}{\Omega_M}\ \int_0^{z_e} \frac{(1+z)^2 dz}{\sqrt{\Omega_{\mathrm{B},0}(1+z)^3 + \Omega_\Lambda}} \tag{19}$$

Equation (19) can be used to numerically calculate the integral for the optical depth $\tau(z_e)$. For example, see my code in figure 1 for numerically calculating this integral. Note also my table for the integral and the error for the values of the integral versus different values of z. We also calculate it for explicit values of $z_e = 10, 100, 1000$ to get

$$\boxed{\tau(z_e = 10) \approx 0.0971111,\ \ \tau(z_e = 100) \approx 2.83646,\ \ \tau(z_e = 1000) \approx 88.6548} \tag{20}$$

See Figure 2 where I plot the optical depth $\tau(z_e)$ versus z from 0 to 2000. The plot makes sense as the light would interact with more intergalactic gas as redshift, and hence distance, increase to the observer.

2 Question 2

2.1 Part (a)

If we look far enough backward in time we come to an era when it was too hot for electrons to be bound into atoms. The rapid collisions of photons with free electrons would have kept radiation in thermal equilibrium with the hot matter in the form of an opaque plasma. As time passed, the matter became cooler and less

```
In [28]:
import numpy as np; from scipy.integrate import quad; import matplotlib.pyplot as plt
sigma_T =6.6525*10**(-29) #m^2
Omega_B = 0.03; Omega_lambda = 0.7
m_H = 1.67 *10**(-27) #kg
#o Mpc = 3.0857x10^16 m
Mpc = 3.0857*10**22
H_0 = 70*1000 / Mpc #m/s
c = 2.998*10**(8)
G = 6.6743*10**(-11) #Nm^2 / kg^2
integral_list = np.zeros(2000)

def integrand(z, Omega_M = 0.3):
    return (0.046*Omega_B/(2*Omega_M)) * ((1+z)**2)/(np.sqrt(Omega_M * (1+z)**3 + Omega_lambda

for i in range(0, 2000):
    integral, err = quad(integrand, 0, i)
    integral_list[i] = integral

    print('z = %-10g, integral = %-10g, error = %-10g' % (i, integral, err))
#plt.savefig('tau.png')
```

```
z = 0         , integral = 0         , error = 0
z = 1         , integral = 0.00388793, error = 4.31647e-17
z = 2         , integral = 0.0100509 , error = 1.11587e-16
z = 3         , integral = 0.0176893 , error = 1.54603e-13
z = 4         , integral = 0.0264787 , error = 8.68399e-12
z = 5         , integral = 0.0362542 , error = 2.89153e-10
z = 6         , integral = 0.0469119 , error = 6.84316e-10
z = 7         , integral = 0.0583779 , error = 1.93285e-09
z = 8         , integral = 0.0705955 , error = 8.68448e-12
z = 9         , integral = 0.0835192 , error = 7.67477e-11
z = 10        , integral = 0.0971111 , error = 2.89153e-10
z = 11        , integral = 0.111339  , error = 6.12739e-10
z = 12        , integral = 0.126176  , error = 6.84317e-10
z = 13        , integral = 0.141596  , error = 8.143e-11
z = 14        , integral = 0.15758   , error = 1.93285e-09
z = 15        , integral = 0.174106  , error = 1.09783e-08
z = 16        , integral = 0.191158  , error = 8.68582e-12
z = 17        , integral = 0.20872   , error = 2.98847e-11
z = 18        , integral = 0.226778  , error = 7.67493e-11
```

Figure 1: My python code for the calculation of $\tau(z_e)$

dense, and eventually the radiation began a free expansion. The number density of photons in equilibrium with matter at temperature T at photon frequency between ν and $\nu + d\nu$ is given by the black-body spectrum

$$n_T(\nu)dv = \frac{8\pi\nu^2 d\nu}{\exp\left(h\nu/k_B T\right) - 1} \tag{21}$$

This spectrum kept evolving under this transition. Hence there was a time t_L when radiation suddenly went from being in thermal equilibrium with matter to a free expansion, where the L stands for "last scattering". Since the frequency of light is inversely related to the scale parameter a (since the wavelength will be stretched out by the expansion of the universe), a photon that has a frequency ν at some later time t would have had a frequency $va(t)/a\left(t_L\right)$ at the time the radiation went out of equilibrium with matter and so the number density at time $t$of photons with frequency between ν and $\nu + d\nu$ would be

$$n(\nu, t)d\nu = \left(a\left(t_L\right)/a(t)\right)^3 n_{T\left(t_L\right)}\left(va(t)/a\left(t_L\right)\right) d\left(va(t)/a\left(t_L\right)\right) \tag{22}$$

Where the factor $\left(a\left(t_L\right)/a(t)\right)^3$ comes because the number density of photons gets diluted as the universe expands. If we replace ν with $\nu a(t)/a\left(t_L\right)$ in the original expression for the spectrum in equation (21) and do the extra algebra in equation (22), we attain the expression

$$n(\nu, t)dv = \frac{8\pi\nu^2 dv}{\exp\left(h\nu/k_B T(t)\right) - 1} = n_{T(t)}(\nu)d\nu \tag{23}$$

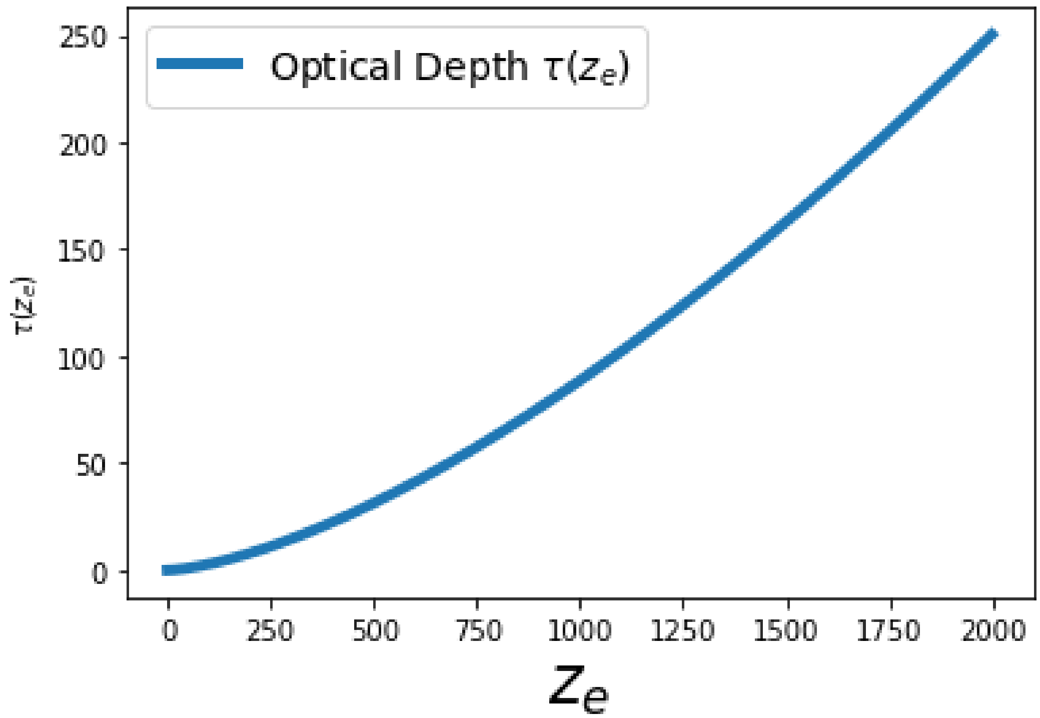

Figure 2: My plot for $\tau(z_e)$ versus z

Where

$$T(t) = T\left(t_L\right) a\left(t_L\right) / a(t) \tag{24}$$

Hence

$$a(t) = \frac{T(t_L) a(t_L)}{T(t)} \tag{25}$$

And since, by definition,

$$H = \frac{\dot{a}}{a} \tag{26}$$

And using equation (25) and the chain rule, we get the derivative of $a(t)$

$$\dot{a} = -\frac{T(t_L) a(t_L)}{T^2(t)} \; \frac{dT}{dt} \tag{27}$$

And using (27) into the definition of H in Eq. (**??**) we get

$$H = \frac{\dot{a}(t)}{a(t)} = - \; \frac{T(t_L) a(t_L)}{T^2(t)} \frac{dT}{dt} \; \frac{T(t)}{T(t_L) a(t_L)} \tag{28}$$

Hence

$$H = -\frac{1}{T} \; \frac{dT}{dt} \tag{29}$$

And hence

$$\boxed{\frac{dt}{dT} = -\frac{1}{HT}} \tag{30}$$

Which is equation (2.3.26) in Weinberg. Q.E.D.

2.2 Part (b)

In studying the CMB we need to consider the early universe, where the density of the vacuum energy was negligible, and the universe was dominated by radiation and matter, hence we neglect the vacuum energy and assume we are in a flat universe. In a universe consisting of matter and radiation densities, the Hubble parameter is written as

$$H(z) = H_0\sqrt{\Omega_M(1+z)^3 + \Omega_R(1+z)^4} \tag{31}$$

And since

$$\frac{a(t)}{a(t_0)} = (1+z)^{-1} \tag{32}$$

Eq. (31) can be written as

$$H(t) = H_0\sqrt{\Omega_M\left(\frac{a(t)}{a(t_0)}\right)^{-3} + \Omega_R\left(\frac{a(t)}{a(t_0)}\right)^{-4}} \tag{33}$$

And using the inverse relationship between a and T in Eq. (25), we have

$$\frac{a(t)}{a(t_0)} = \frac{T(t_0)}{T(t)} \tag{34}$$

And plugging in E. (34) into the expression for H in E. (33) we get the expression

$$H(t) = H_0\sqrt{\Omega_M\left(\frac{T(t_0)}{T(t)}\right)^{-3} + \Omega_R\left(\frac{T(t_0)}{T(t)}\right)^{-4}} = H_0\sqrt{\Omega_M\left(\frac{T(t)}{T(t_0)}\right)^{3} + \Omega_R\left(\frac{T(t)}{T(t_0)}\right)^{4}} \tag{35}$$

The density of radiation consisted of neutrinos and photons in the early universe, but at temperatures where neutrinos are important their mass is negligible. Hence, Eq. (35) becomes

$$\boxed{H(t) = H_0\sqrt{\Omega_M\left(\frac{T(t)}{T_\gamma(t_0)}\right)^{3} + \Omega_R\left(\frac{T(t)}{T_\gamma(t_0)}\right)^{4}}} \tag{36}$$

Where $T_\gamma(t_0)$ is the temperature of the CMB photons now. This is equation (2.3.28) in Weinbgerg. Q.E.D

3 Question 3: Angular Distance to the CMB

3.1 Part a

The angular diameter distance d_A is defined as $d_A = \frac{d_L}{(1+z)^2} = \frac{S_k(r)}{1+z}$ and since we have a flat universe, $S_k(r) = r$ hence

$$d_A = \frac{a_0\ r}{1+z} \tag{37}$$

And, as with previous homeworks, we start with the FRW metric, and we take the radial separation ($d\Omega = 0$) and we take a lightlike interval so $ds^2 = 0$, so

$$cdt = -adr \tag{38}$$

Hence

$$cdt = -adr \tag{39}$$

And we write Friedmann equation in the following form

$$\frac{da}{dt} = aH_0\sqrt{\Omega_{\Lambda 0} + \Omega_{K0}\left(\frac{a_0}{a}\right)^2 + \Omega_{m0}\left(\frac{a_0}{a}\right)^3 + \Omega_{R0}\left(\frac{a_0}{a}\right)^4} \tag{40}$$

where $H_0 \equiv \sqrt{\frac{8\pi G p_0}{3}}$ and the Ωs are given by

$$\rho_{\Lambda 0} \equiv \frac{3H_0^2\Omega_\Lambda}{8\pi G}, \rho_{m0} \equiv \frac{3H_0^2}{8\pi G}, \rho_{r0} = \frac{3H_0^2\Omega_r}{8\pi G} \tag{41}$$

We can make the substitution $x \equiv \frac{a}{a_0} \rightarrow dx = \frac{da}{a_0}$, so that Friedmann equation (40) becomes

$$dt = \frac{dx}{H_0 x\sqrt{\Omega_{\Lambda 0} + \Omega_{m0}x^{-3} + \Omega_{K0}x^{-2} + \Omega_{R0}x^{-4}}} \tag{42}$$

Hence we can calculate r as a function of x or equivalently of z

$$r(z) = \int_{t(z)}^{t_0} \frac{dt}{a(t)} \tag{43}$$

Hence if we have nonzero parameters for $\Omega_M, \Omega_R\Omega_\Lambda$ then if we take the integral from some z in the past to now ($z = 1$), we have

$$r(z) = \frac{1}{a_0 H_0}\int_{1/(1+z)}^{1} \frac{dx}{x^2\sqrt{\Omega_\Lambda + \Omega_M x^{-3} + \Omega_R x^{-4}}} \tag{44}$$

Hence

$$d_A(z) = \frac{a_0}{1+z} = \frac{1}{H_0}\int_{1/(1+z)}^{1} \frac{dx}{x^2\sqrt{\Omega_\Lambda + \Omega_M x + \Omega_R x^{-4}}} \tag{45}$$

And putting the x in the square root, we have

$$d_A(z) = \frac{1}{1+z} = \frac{1}{H_0}\int_{1/(1+z)}^{1} \frac{dx}{\sqrt{\Omega_\Lambda x^4 + \Omega_M x^{-3} + \Omega_R}} = \frac{a_0}{1+z} = \frac{1}{H_0}\int_{1/(1+z)}^{1} \frac{dx}{\sqrt{0.76x^4 + 0.24x + 8.49 \times 10^{-5}}} \tag{46}$$

Where in the last step I also used the normalization that $a_0 = a(t=0) = 1$. This integral in (46) can be calculated numerically for $z = 1100$. Figure 3 demonstrates the very simple code necessary to calculate this integral

And we have a a factor of $\frac{c}{(z+1)H_0} = \frac{c}{1101\ H_0}$, hence

$$\boxed{d_A(z) = \frac{3.4728\ c}{1101\ H_0}} \tag{47}$$

```
In [7]:  from scipy.integrate import quad; import numpy as np

         def dA(x):
             return  1/np.sqrt(0.76*(x**4) + 0.24*x + 8.49*10**(-5) )
         integral, error = quad(dA, (1/1101), 1);
         integral #* (1/1001)

Out[7]:  3.472831142007171
```

Figure 3: My python code for the calculation of $d_A(z_L)$

And $c/H_0 = 2997.9h^{-1} = Mpc$ (see conversions in my code), hence

$$d_A(z) = \frac{9.46}{h}\ Mpc \tag{48}$$

And for $H_0 = 70\ km/s$, hence $h = 0.7$ and therefore

$$\boxed{d_A(z) = 13.514\ Mpc} \tag{49}$$

3.2 Part b

It is a very intriguing and counter-intuitive idea that the angular distance, which should be a proxy for the physical distance, is smaller to the CMS (the surface of last scattering) than it is for the nearby galaxies. For example, M100 is about 20 Mpc away and the angular diameter distance to the CMB (presumably the farthest thing in the whole observable universe) is only 13.5 Mpc. How can this be? In order to get a clearer answer on what this means, we have to revisit the meaning of the angular diameter distance. It is defined in terms of ab objects physical transverse size x and the angular size of the object as viewed from earth

$$d_A = \frac{\text{Transverse physical size}}{\text{Angular size}} = \frac{x}{\theta}$$

But note that the angular diameter distance is also expressed in terms of redshift, and for a flat universe

$$d_A = \frac{r}{1+z}$$

And note that beyond a certain redshift, the angular diameter distance gets smaller with increasing distance. This was plotted by me in the previous homework. See Fig 4 for an example of how d_A starts decreasing beyond a certain redshift Now when it is related to redshift, which is a measure of time, it becomes more clear. The reason is that the farther back in time (and hence the greater value of the redshift), the closer things were in the universe, hence the distance to objects at the time of emission were smaller. This coupled with the fact that objects whose distance is small (at the time of emission) subtend larger angles, and hence there is a larger angular size associated with these objects, explains this peculiar behavior of the angular diameter distance for large redshifts.

3.3 part c

By definition, Weinberg (1.4.12) relates d_A and d_L

$$d_A/d_L = (1+z)^{-2} \tag{50}$$

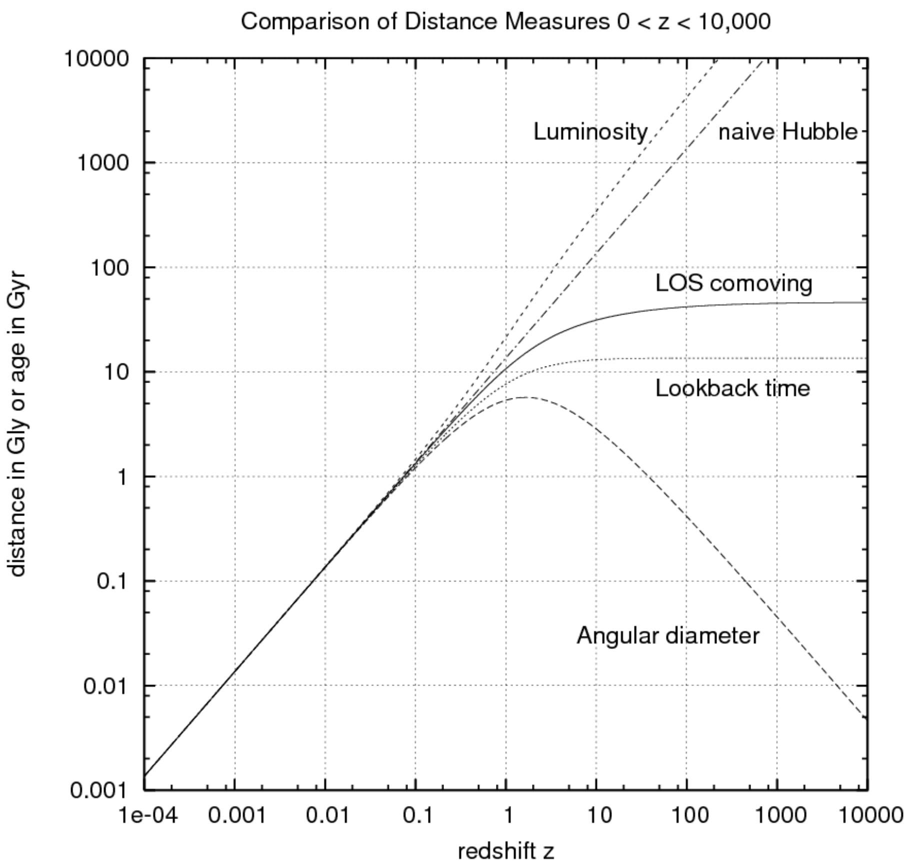

Figure 4: d_A vs z plot example

Hence

$$d_L = d_A * 1 + z_L)^2 = 1.6 \times 10^7 Mpc \tag{51}$$

4 Question 4

4.1 Part a

The Saha equation can be used to relate the number densities of 2 successive stages of ionization

$$\frac{n_{j+1} n_e}{n_j} = \frac{2U_{j+1}(T)}{U_j(T)} \left(\frac{m_e k_B T}{2\pi\hbar^2} \right)^{3/2} e^{-\chi_{j,j+1}/k_B T} \tag{52}$$

Where $\chi_{j,j+1}$ is the ionization potential for going between the lower and upper ionization stages, and U_j, U_{j+1} are the partition functions for the lower and upper ionization stages. In our case, we look at the ionization stages of a proton p going to neutral Hydrogen then to neutral Helium and then ionized helium. We have HI is neutral Hydrogen, He_I is neutral Helium, $He_I I$ is once-ionized helium, $He_I II$ is twice ionized helium.

Hydrogen 1s, 2p, etc. are all internal states of Hydrogen.

The Saha equation describes the degree of ionization of this plasma by relating the population of ions to that of neutrals. If this is a plasma that is partially ionized (some atoms will be ionized and others will be excited at the same time in this gas) and made with a mixture of Hydrogen and Helium then it must contain the following particles

1. Electrons
2. Neutral Hydrogen (H_I)
3. ionized Hydrogen (H_{II})
4. Neutral Helium (He_I)
5. Singly ionized Helium (He_{II})
6. Doubly ionized Helium (He_{III})

4.2 Part b

If the mixture of nuclei is 76% H and 24% He by weight, then we can write the following equations that represent that

$$0.76\ n_B = n_p + n_{HI} \tag{53}$$

Or equivalently,

$$0.76\rho_B = \rho_p + \rho_{HI}$$

equation (53) represents the Hydrogen content of the Baryon number density, where the total charge conservation is assumed. We can write a similar equation for the Helium content, where now the Helium is 24 % by weight, and we have a 4 factor since each Helium atom has 4 times the atomic mass of one hydrogen atom, so

$$0.24n_B = 4(n_{HeI} + n_{HeII} + n_{HeIII}) \tag{54}$$

Furthermore, we can write an equation for the electron number density in terms of the ion densities. We have 4 types of ions, each with an appropriate factor for its positive atomic charge, and assuming total conservation of electric charge implies that

$$n_e = n_{HII} + 1n_{HeII} + 2n_{HeIII} + 1n_{HI} \tag{55}$$

4.3 Part c

We start with the Saha equation

$$\frac{n_{j+1}n_e}{n_j} = \frac{2U_{j+1}(T)}{U_j(T)} \left(\frac{m_e k_B T}{2\pi\hbar^2}\right)^{3/2} e^{-\chi_{j,j+1}/k_B T} \tag{56}$$

With $j = P$ and $j + 1 = HI$, we have

$$\frac{n_{HI}\ n_e}{n_P} = \frac{2U_{HI}(T)}{U_P(T)} \left(\frac{m_e k_B T}{2\pi\hbar^2}\right)^{3/2} e^{-\chi_{P,H}/k_I BT} \tag{57}$$

And plugging in the values for the U and χ we have

$$\boxed{\frac{n_{HI} n_e}{n_P} = \frac{2 \times 4}{2} \left(\frac{m_e k_B T}{2\pi\hbar^2} \right)^{3/2} e^{-13.6\ eV/k_B T}} \tag{58}$$

Now using $j = HeI$ and $J+1 = HeII$ we have

$$\frac{n_{HeII}\ n_e}{n_{HeI}} = \frac{2U_{HeII}(T)}{U_{HeI}(T)} \left(\frac{m_e k_B T}{2\pi\hbar^2} \right)^{3/2} e^{-\chi_{HeI,HeII}/k_I BT} \tag{59}$$

And plugging in for U and χ we have

$$\boxed{\frac{n_{HeII}\ n_e}{n_{HeI}} = \frac{2 \times 2}{1} \left(\frac{m_e k_B T}{2\pi\hbar^2} \right)^{3/2} e^{-24.6\ eV/k_B T}} \tag{60}$$

And the third equation is the transition from $HeII$ to $HeIII$

$$\boxed{\frac{n_{HeIII}\ n_e}{n_{HeII}} = \frac{2 \times 1}{2} \left(\frac{m_e k_B T}{2\pi\hbar^2} \right)^{3/2} e^{-54.4\ eV/k_B T}} \tag{61}$$

4.4 Part d

If we assume that Helium was half HeII and half HeIII, then we start by looking at equation 54 without HeI, ie.

$$0.24 n_B = 4 n_{HeII} \rightarrow n_{HeII} = \frac{0.24}{4} n_B \tag{62}$$

And the other half of the universe is He_{HeIII}

$$0.24 n_B = 4 n_{HeIII} \rightarrow n_{HeIII} = \frac{0.24}{4} n_B \tag{63}$$

And we can relate these to n_e by using equation 55, where now, $n_{HeII} + n_{HeIII} = \frac{0.24}{4} n_B$ and since $n_{HeII} = n_{HeIII}$, we have $n_{HeII} = n_{HeIII} = 0.03 n_B$, so

$$n_e = 1 n_{HII} + 1 n_{HeII} + 2 n_{HeIII} = 0.76 n_B + 0.03 n_B + 2(0.03) = 0.85 n_B \tag{64}$$

Hence the Saha equation in this case is

$$\frac{n_{HeIII} n_e}{n_{HeII}} = \left(\frac{m_c k_B}{2\pi\hbar^2} \right)^{3/2} T^{3/2} e^{-\chi_{He} II, HeII/k_B T} \tag{65}$$

The left hand side reduces to just n_e since $n_{HeII} = n_{HeIII}$. And plugging in the result that $n_e = 0.85 n_B$ we have

$$0.85 n_B = \left(\frac{m_c k_B}{2\pi\hbar^2} \right)^{3/2} T^{3/2} e^{-\chi_{He} II, HeII/k_B T} \tag{66}$$

And now, using the relation that $n_B = n_{B0} (\frac{T}{T_{\gamma,0}})^3$ we have

$$0.85 \times n_{B0} (\frac{T}{T_{\gamma,0}})^3 = \left(\frac{m_c k_B}{2\pi\hbar^2} \right)^{3/2} T^{3/2} e^{-\chi_{He} II, HeII/k_B T} \tag{67}$$

And defining $\beta = \left(\frac{m_e k_B}{2\pi\hbar^2}\right)^{3/2}$, we get

$$0.85 \times n_{B0}(\frac{T}{T_{\gamma,0}})^3 = \beta T^{3/2} e^{-\chi_{HeII,HeII}/k_B T} \tag{68}$$

Or

$$\boxed{T^{3/2} e^{\chi_{H-1I,H\in IH}/k_B T} = \frac{\beta T_{\gamma 0}^3}{0.85 n_{B0}}} \tag{69}$$

Which is the desired result.

Now for the transition of $HeII$ to HeI, we have

$$n_{HeII} = n_{HeI} = 0.03 n_B$$

and

$$n_{HII} = 0.76 n_B$$

and

$$n_{HI} = n_{HeIII} = 0$$

Hence the analogy to equation (55) here becomes

$$n_e = n_{HII} + n_{HeII} = 0.76 n_B + 0.03 n_B = 0.79\ n_B \tag{70}$$

Hence the Saha equation (52) becomes

$$\frac{n_{HeII} n_e}{n_{HeI}} = \frac{2U_{HeII}}{U_{HeI}}\ \beta T^{3/2} e^{-\chi_{HeI,HeII}/k_B T} \tag{71}$$

And since $n_{HeII} = n_{HeI}$ the left hand side is simply ne which is equal to $0.79 n_B$. And since $U_{HeII} = 4$ and $U_{HeI} = 1$, this becomes

$$0.79\ n_B = 4\beta T^{3/2} e^{-\chi_{HeI,HeII}/k_B T} \tag{72}$$

And in analogy to part a, $n_B = n_{B,0}(\frac{T}{T_{\gamma,0}})^3$ so

$$0.79\ n_{B,0}(\frac{T}{T_{\gamma,0}})^3 = 4\beta T^{3/2} e^{-\chi_{HeI,HeII}/k_B T} \tag{73}$$

Or

$$\boxed{T^{3/2} e^{\chi_{HeI,HeII}/k_B T} = \frac{4\beta T_{\gamma 0}^3}{0.79\ n_{B0}}} \tag{74}$$

Which is the desired result.

4.5 Part e

For this question we use equation (2.1.13) from Weinberg,

$$n_{B0} = \frac{3\Omega_B H_0^2}{8\pi G m_N} = 1.123 \times 10^{-5} \Omega_B h^2 \text{ nucleons /cm}^3 \tag{75}$$

And we numerically solve for the temperature in which He recombines from double to singly ionized $T_{HeIII,HeII}$ and then for the temperature in which it recombines from singly ionized to neutral $T_{HeII,HeI}$.

In solving this, we must make the appropriate conversions in the units before numerically solving for the temperature. Please see the document at [1] which helped me very much in making the conversions from mks to the astronomical cgs units. See also my code which applies these conversion. Now if we assume $\Omega_B h^2 = 0.02$ then n_{B0} is simply

$$n_{B0} = \frac{3\Omega_B H_0^2}{8\pi G m_N} = 1.13 \times 10^{-5} \times 0.02 = 2.26 \times 10^{-7} cm^{-7}$$

We also know that the current temperature of the CMB is $T_{\gamma,0} = 2.725\ K$

Then I numerically solve for T using my own root-finding function, as well as using *Scipy*'s *optimize* package (particularly using the *fsolve* function). My code which numerically solves for the temperature at recombination is provided in Figure 5.

```
import numpy as np; import matplotlib.pyplot as plt;  from scipy.optimize import fsolve
#Define all the constants and convert to cgs units
kB = 1.381*10**(-16) ; T_gamma = 2.725;
h= 6.626*10**(-27)  ; hbar = h/(2*np.pi); me = 9.1096*10**(-28)
U_He3 = 1; U_He2 = 2; X_23 = 20; n_He2 = 2; n_He3=3; n_e = 3
n_B0 = 1.13*10**(-5) * 0.02
beta = ((me * kB)/(2*np.pi* hbar**2))**(3/2)
X = 54.4 #eV
X_Joule = X * 1.60218*10**(-19)
X_ergs = 54.4 * 1.602*10**(-12)
X_23_ergs = 24.6 * 1.602*10**(-12)
Mpc = 3.0857*10**22
H_0 = 70*1000 / Mpc
#Define linear space for temperature for plotting
x=np.linspace(-100,20000)
#these are transition energies chi_{HeII, HeIII}, etc. converted to  ergs.
X_21_ergs = 24.6 * 1.602*10**(-12)
X_32_ergs = 54.4 * 1.602*10**(-12)

#Define the corresponding Saha function, where it is moved to the LHS so that RHS=0
def f(T):
   return T**(3/2) * np.exp( X_32_ergs/(kB*T) ) - beta * (T_gamma**3)/(0.85*n_B0)
#start plotting
plt.figure()
plt.plot(x, f(x), label=r"$T^{3 / 2} e^{\chi_{H e I, H e I I} / k_{B} T}$", linewidth=3)
#plt.plot(x,  beta * (T_gamma**3)/(0.85*n_B0) *np.ones(len(x)), label="y=0")
plt.plot(x, x*0, linewidth=2, label = "0")
plt.legend(fontsize=14)
plt.xlabel('T (K)')
plt.ylabel('f(T)')

#Find numerical solution
sol=fsolve(f, 10000)
sol
```

```
/usr/local/lib/python3.7/dist-packages/ipykernel_launcher.py:22: RuntimeWarning: invalid value encountered in power
/usr/local/lib/python3.7/dist-packages/ipykernel_launcher.py:22: RuntimeWarning: overflow encountered in exp
array([16028.75090448])
```

Figure 5: My python code for the calculation of of the temperature of recombination $HeIII \rightarrow HeII$

This particular function requires an additional parameter, which is the initial guess of the root of the function. This initial guess is best found by plotting the function, along with the zero value, and seeing where they intercept, for a rough idea of where the root of the function is. For example, see Figure 6 where the function is plotted for a range of temperatures, and using its intercept with zero, a good guess for the root is T=10,000 K, for which the solution is given as 16,028, as can be seen in my code.

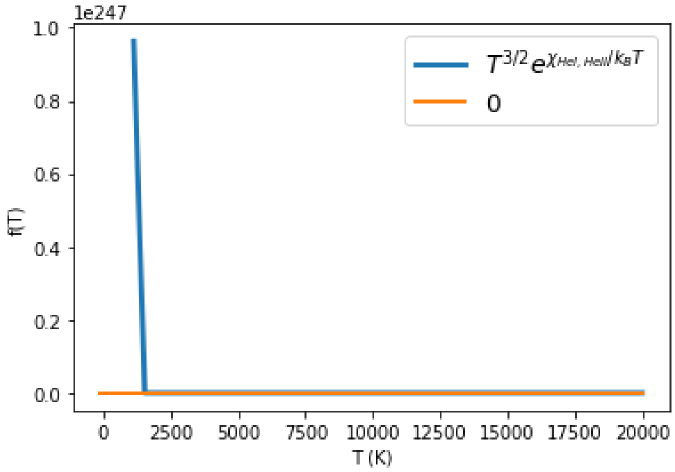

Figure 6: plot for where the temperature of recombination intercepts zero, for a better guess for the function's root

Hence the temperature at which $HeIII$ recombines with $HeII$ is

$$\boxed{T_{HeIII,HeII} = 16,028.75\ K} \tag{76}$$

The exact same method is now used for finding the temperature at which $HeII$ recombines with HeI, but now the equation is

$$T^{3/2}e^{\chi_{HeI,HeII}/k_BT} = \frac{4\beta T_{\gamma 0}^3}{0.79\ n_{B0}}$$

See my code in Figure 7 in which I attain

$$\boxed{T_{HeII,HeI} = 6774.86\ K} \tag{77}$$

4.6 Part e

We know that the redshift are related to the temperature in the following way:

$$\frac{T}{T_\gamma 0} = (1+z) \tag{78}$$

Hence

$$\frac{T_{HeIII,HeII}}{T_\gamma 0} = (1+z_{HeIII,HeII}) \tag{79}$$

Hence the redshift at which $HeIII$ recombines with $HeII$ is

$$\boxed{z_{HeIII,HeII} = \frac{T_{HeIII,HeII}}{T_\gamma 0} - 1 = \frac{16028.75}{2.725} - 1 = 5881.1} \tag{80}$$

```python
import numpy as np; import matplotlib.pyplot as plt;  from scipy.optimize import fsolve
#Define all the constants and convert to cgs units
kB = 1.381*10**(-16) ; T_gamma = 2.725;
h= 6.626*10**(-27)  ; hbar = h/(2*np.pi); me = 9.1096*10**(-28)
U_He3 = 1; U_He2 = 2; X_23 = 20; n_He2 = 2; n_He3=3; n_e = 3
n_B0 = 1.13*10**(-5) * 0.02
beta = ((me * kB)/(2*np.pi* hbar**2))**(3/2)
X = 54.4 #eV
X_Joule = X * 1.60218*10**(-19)
X_ergs = 54.4 * 1.602*10**(-12)
X_23_ergs = 24.6 * 1.602*10**(-12)
Mpc = 3.0857*10**22
H_0 = 70*1000 / Mpc
#Define linear space for temperature for plotting
x=np.linspace(-100,20000)
#these are transition energies chi_{HeII, HeIII}, etc. converted to  ergs.
X_21_ergs = 24.6 * 1.602*10**(-12)
X_32_ergs = 54.4 * 1.602*10**(-12)

#Define the corresponding Saha function, where it is moved to the LHS so that RHS=0
def f(T):
   return T**(3/2) * np.exp( X_21_ergs/(kB*T) ) - 4*beta * (T_gamma**3)/(0.79*n_B0)
#start plotting
plt.figure()
plt.plot(x, f(x), label=r"$T^{3 / 2} e^{\chi_{H e I, H e I I} / k_{B} T}$", linewidth=3)
#plt.plot(x,  beta * (T_gamma**3)/(0.85*n_B0) *np.ones(len(x)), label="y=0")
plt.plot(x, x*0, linewidth=2, label = "0")
plt.legend(fontsize=14)
plt.xlabel('T (K)')
plt.ylabel('f(T)')

#Find numerical solution
sol=fsolve(f, 3000)
sol
```

```
/usr/local/lib/python3.7/dist-packages/ipykernel_launcher.py:22: RuntimeWarning: invalid value encountered in power
/usr/local/lib/python3.7/dist-packages/ipykernel_launcher.py:22: RuntimeWarning: overflow encountered in exp
array([6774.85905938])
```

Figure 7: My python code for the calculation of of the temperature of recombination $HeII \rightarrow HeI$

Similarly, the redshift at which $HeII$ recombines with HeI is

$$\boxed{z_{HeIII,HeII} = \frac{T_{HeIII,HeII}}{T_{\gamma}0} - 1 = \frac{6774.86}{2.725} - 1 = 2485.19} \tag{81}$$

References

[1] ASTRONOMICAL CONSTANTS AND CONVERSION FACTORS, http://www.astro.uwo.ca/~basu/teach/ast020/docs/a020datasheet.pdf