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Quantum Computing
Problem Set 4

4.1 P. 4.2

a) $H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\right)$

b) $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|$

c) $Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = |0\rangle\langle 1| - |1\rangle\langle 0|$

d) $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$

e) $\begin{pmatrix} 23 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9 \end{pmatrix} = 23\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}(1\ 0\ 0\ 0) - 5\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}(0\ 1\ 0\ 0)$

$+ 9\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}(0\ 0\ 0\ 1)$

$= 23\,|00\rangle\langle 00| - 5\,|01\rangle\langle 01| + 9\,|11\rangle\langle 11|$

f. $X \otimes X = (|0\rangle\langle 1| + |1\rangle\langle 0|) \otimes (|0\rangle\langle 1| + |1\rangle\langle 0|)$

$= |0\rangle\langle 1| \otimes |0\rangle\langle 1| + |0\rangle\langle 1| \otimes |1\rangle\langle 0| + |1\rangle\langle 0| \otimes |0\rangle\langle 1| + |1\rangle\langle 0| \otimes |1\rangle\langle 0|$

$= |00\rangle\langle 11| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 00|$

g. $X \otimes Z = (|0\rangle\langle 1| + |1\rangle\langle 0|) \otimes (|0\rangle\langle 0| - |1\rangle\langle 1|)$

$= |0\rangle\langle 1| \otimes |0\rangle\langle 0| - |0\rangle\langle 1| \otimes |1\rangle\langle 1|$
$+ |1\rangle\langle 0| \otimes |0\rangle\langle 0| - |1\rangle\langle 0| \otimes |1\rangle\langle 1|$

$= |00\rangle\langle 10| - |01\rangle\langle 11| + |10\rangle\langle 00| - |11\rangle\langle 01|$

h. $H \otimes H = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)$
$\otimes \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|)$

$= \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 01| + |01\rangle\langle 00| - |01\rangle\langle 01|$
$+ |00\rangle\langle 10| + |00\rangle\langle 11| + |01\rangle\langle 10| - |01\rangle\langle 11|$
$+ |10\rangle\langle 00| + |10\rangle\langle 01| + |11\rangle\langle 00| - |11\rangle\langle 01|$
$- |10\rangle\langle 10| - |10\rangle\langle 11| - |11\rangle\langle 10| + |11\rangle\langle 11|)$

~~i. $P_1 : V \to S_1$ where S_1 is spanned by $\{|+\rangle|+\rangle, |-\rangle|-\rangle\}$~~

Let ~~V be the Vector~~ space associated with a two-qubit system, and $|\varphi\rangle = a_{++}|++\rangle + a_{+-}|+-\rangle + a_{-+}|-+\rangle + a_{--}|--\rangle$ be an arbitrary two-qubit state.

i. $\cdot P_1 : V \to S_1$, where S_1 is spanned by $\{|+\rangle|+\rangle, |-\rangle|-\rangle\}$

Let V be the vector space associated with a two-qubit system in this subspace, and

$|\varphi_1\rangle = a_{++}|++\rangle + a_{--}|--\rangle$

be an arbitrary two-qubit state in this subspace.
Then the measurement V will have decomposition

$V = S_{++} \oplus S_{--}$, where S_{ij} is the one-dimensional complex subspace spanned by $|ij\rangle$.
The related projection operators $P_{ij} : V \to S_{ij}$ are

$P_{++} = |++\rangle\langle++|$, $P_{--} = |--\rangle\langle--|$

$\Rightarrow P_1 = P_{++} + P_{--}$

$P_1 = |++\rangle\langle++| + |--\rangle\langle--|$

$\cdot P_2 : V \to S_2$ where S_2 is spanned by $\{|+\rangle|-\rangle, |-\rangle|+\rangle\}$
Following the same procedure as above, V will have decomposition $V = S_{+-} \oplus S_{-+}$, with operators

$P_{+-} = |+-\rangle\langle+-|$, $P_{-+} = |-+\rangle\langle-+|$

$\Rightarrow P_2 = P_{+-} + P_{-+}$

$P_2 = |+-\rangle\langle+-| + |-+\rangle\langle-+|$

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4.2 P. 4.3

Let P_s be a projection operator from an n-dimensional vector space V onto an s-dimensional subspace $S \in V$ with basis $\{|\alpha_0\rangle, \ldots, |\alpha_{s-1}\rangle\}$. Then

$$P_s = \sum_{i=1}^{s-1} |\alpha_i\rangle\langle\alpha_i|$$

$$= |\alpha_0\rangle\langle\alpha_0| + \cdots + |\alpha_{s-1}\rangle\langle\alpha_{s-1}|$$

We want to show that $P_s = P_s^+$.

$$P_s^+ = \left(|\alpha_0\rangle\langle\alpha_0| + \cdots + |\alpha_{s-1}\rangle\langle\alpha_{s-1}|\right)^+$$

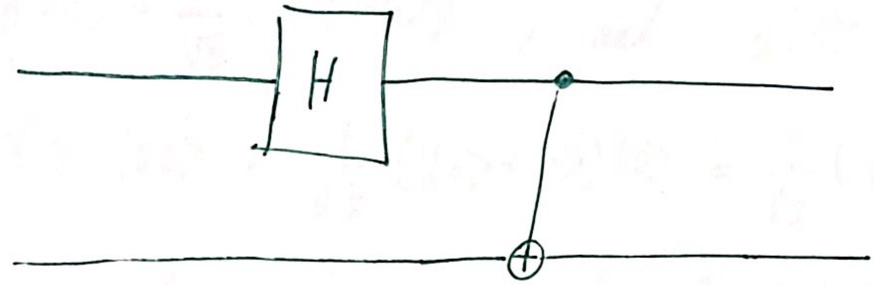
$$= \left((|\alpha_0\rangle\langle\alpha_0|)^*\right)^T + \cdots + \left((|\alpha_{s-1}\rangle\langle\alpha_{s-1}|)^*\right)^T$$

$$= |\alpha_0\rangle\langle\alpha_0| + \cdots + |\alpha_{s-1}\rangle\langle\alpha_{s-1}|$$

$$= P_s$$

$\square$

4.3 Bell Circuit



Re Call the representation of the Haddamard Transformation

$$H = \frac{1}{\sqrt{2}}\left(|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1|\right) \quad \text{----} \quad ①$$

and the C_{Not} transformation

$$C_{Not} = |00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11|$$

We want to show:

i) $|00\rangle \longrightarrow \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) = |\varphi^{+}\rangle$

. starting with input state $|00\rangle$, we act with H on the first qubit, and with $\mathbb{1}$ on the second qubit

$$H \otimes \mathbb{1}\, |00\rangle = \frac{1}{\sqrt{2}}\underbrace{\left(|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1|\right)}_{H} \otimes \underbrace{\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)}_{\mathbb{1}}$$

$$[\, |0\ 0\rangle \,]$$

where I have used

$$\mathbb{1} = |0\rangle\langle 0| + |1\rangle\langle 1| \quad \text{- - - -} \quad ③$$

Then, using equations ①, ②, ③,

$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, and $\mathbb{1}|0\rangle = |0\rangle$, Hence

$$H \otimes \mathbb{1}\,|00\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

Now we act with C_{Not} (using equation ③)

$$C_{Not}\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

$$= \left[|00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11|\right]\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

$$= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \equiv |\phi^+\rangle$$

□

ii) starting with input state $|01\rangle$, we follow the same Procedure as Part (i).

$$\underset{\text{}}{H \otimes \mathbb{1}}\,|01\rangle = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1|)$$

$$\otimes (|0\rangle\langle 0| + |1\rangle\langle 1|)\,|01\rangle$$

$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\,|1\rangle$$

$$= \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle)$$

$$C_{Not}\frac{1}{\sqrt{2}}(|01\rangle + |11\rangle) = (|00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11|)\frac{1}{\sqrt{2}}(|01\rangle + |11\rangle)$$

$$C_{Not} \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \equiv |\psi^+\rangle$$

$\square$

iii) Now starting with input state $|10\rangle$

$$H \otimes \mathbb{1} |10\rangle = \frac{1}{\sqrt{2}}\underbrace{(|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1|)}$$

$$\otimes \underbrace{(|0\rangle\langle 0| + |1\rangle\langle 1|)} \; |10\rangle$$

$$= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\,|0\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |10\rangle)$$

$$C_{Not} \frac{1}{\sqrt{2}}(|00\rangle - |10\rangle) = (|00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11|)$$

$$\left[\frac{1}{\sqrt{2}}(|00\rangle - |10\rangle)\right]$$

$$= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \equiv |\phi^-\rangle$$

$\square$

iv) starting with $|11\rangle$,

$$H \otimes \mathbb{1} |11\rangle = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1|)(|0\rangle\langle 0| + |1\rangle\langle 1|)\,|11\rangle$$

$$= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\,|1\rangle$$

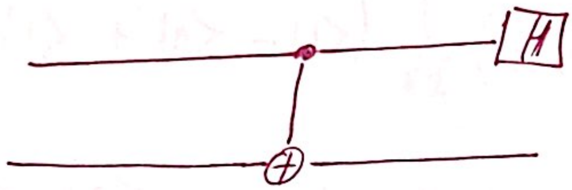
$$= \frac{1}{\sqrt{2}}(|01\rangle - |11\rangle)$$

$$C_{Not} \frac{1}{\sqrt{2}}(|01\rangle - |11\rangle) = (|00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11|)$$

$$\left[\frac{1}{\sqrt{2}}(|01\rangle - |11\rangle)\right]$$

$$\Rightarrow C_{Not}\frac{1}{\sqrt{2}}(|01\rangle - |11\rangle) = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \equiv |\Psi^-\rangle \qquad \square$$

Now, for the "reverse Bell circuit"



This is equivalent to first applying C_{Not} and then applying H on the first qubit and $\mathbb{1}$ on the second qubit.
We want to show:

v) starting with input state $|\phi^+\rangle$

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$C_{Not}|\phi^+\rangle = (|00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11|)\left[\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\right]$$

$$= \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$$

$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle$$

Now, as stated above, we act H on the first qubit and with $\mathbb{1}$ on the second qubit.

$$H \otimes \mathbb{1} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle$$

$$= \frac{1}{\sqrt{2}} \underbrace{(|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1|)}_{} \otimes \underbrace{(|0\rangle\langle 0| + |1\rangle\langle 1|)}_{}$$

$$[\ \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle$$

$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle + |0\rangle - |1\rangle) \frac{1}{\sqrt{2}}(|0\rangle)$$

$$= \frac{1}{\cancel{2}}(\cancel{2}|0\rangle)|0\rangle = |00\rangle$$

$\square$

vi) Starting with $|\Psi^+\rangle \equiv \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

$$C_{Not}|\Psi^+\rangle = (|00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11|)\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$= \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle)$$

$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |1\rangle$$

$$H \otimes \mathbb{1} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |1\rangle$$

$$= \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1|) \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|)\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |1\rangle$$

$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle + |0\rangle - |1\rangle) \frac{1}{\sqrt{2}}(|1\rangle)$$

$$= \frac{1}{2}(2|0\rangle)|1\rangle = |01\rangle \qquad \square$$

Vii) Starting with $|\phi^-\rangle \equiv \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$

$$C_{Not}|\phi^-\rangle = (|00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11|)\left[\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)\right]$$

$$= \frac{1}{\sqrt{2}}(|00\rangle - |10\rangle)$$

$$= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes |0\rangle$$

$$H \otimes \mathbb{1} \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes |0\rangle$$

$$= \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1|) \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|)$$

$$\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes |0\rangle$$

$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle - |0\rangle + |1\rangle)\frac{1}{\sqrt{2}}(|0\rangle)$$

$$= \frac{1}{\cancel{2}}(\cancel{2}|1\rangle)|0\rangle = |10\rangle \qquad \square$$

Viii) Starting with $|\psi^-\rangle \equiv \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$

$$C_{Not}|\psi^-\rangle = (|00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11|)\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

$$= \frac{1}{\sqrt{2}}(|01\rangle - |11\rangle)$$

$$= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes |1\rangle$$

3.7

$$H \otimes \mathbb{1} \ \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \otimes |1\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1|\right) \otimes \left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$$

$$\left[\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \otimes |1\rangle\right.$$

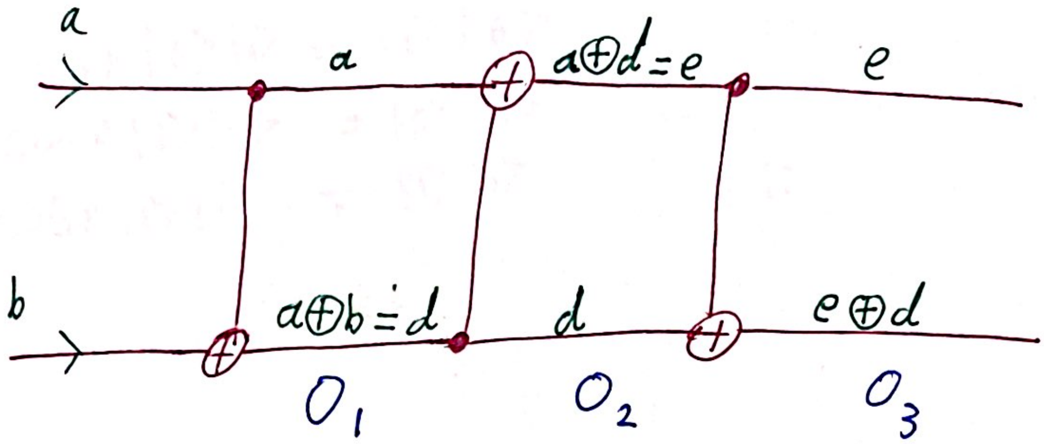
$$= \frac{1}{2}\left(|0\rangle + |1\rangle - |0\rangle + |1\rangle\right)\left(|1\rangle\right)$$

$$= \frac{1}{2}\left(2|1\rangle\right)|1\rangle = |11\rangle$$

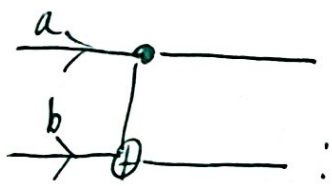
$\square$

4.4

We have the circuit



We have 2 incoming qubits (input) a and b.



: Does not change the qubit

: $\oplus$ applies $a \oplus b \pmod 2$

We can represent the 3 operations on qubits a & b by operators O_1, O_2, O_3. We want to prove this circuit swaps $|\psi\rangle$ and $|\phi\rangle$, i.e. $|\psi\rangle|\phi\rangle \to |\phi\rangle|\psi\rangle$. We can prove this by considering the 2-qubit states in the table below:

Input		O_1		O_2		O_3	
a	b	c=a	$a \oplus b = d$	$a \oplus d = e$	d	e	$e \oplus d$
0	0	0	0	0	0	0	0
0	1	0	1	1	1	1	0
1	0	1	1	0	1	0	1
1	1	1	0	1	0	1	1

Hence We Verified the required transformation,

$SWAP\ |\psi\rangle \otimes |\phi\rangle = |\phi\rangle \otimes |\psi\rangle$, i.e.

$$SWAP\ |0\rangle|0\rangle = |0\rangle|0\rangle$$
$$SWAP\ |0\rangle|1\rangle = |1\rangle|0\rangle$$
$$SWAP\ |1\rangle|0\rangle = |0\rangle|1\rangle$$
$$SWAP\ |1\rangle|1\rangle = |1\rangle|1\rangle$$

$\square$