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Quantum Computing Final

1. Grover's Algorithm for $n=3$

Recall that we have access to an oracle function

$$f(x) = \begin{cases} 1, & x = a \\ 0, & x \neq a \end{cases} \quad \text{-----} (1.1)$$

where x is an n-bit string. Grover's algorithm searches the $N = 2^n$ possible input strings x to find one special string $x=a$ for which $f(x)=1$. Recall that for $|x\rangle_n$ input register and $|y\rangle$ in the output register, U_f evaluates $f(x)$

$$U_f |x\rangle_n |y\rangle = |x\rangle_n |y \oplus f(x)\rangle \quad \text{---} (1.2)$$

In the given circuit,

$$W = 2|\phi\rangle\langle\phi| - \mathbb{1} \quad \text{---} (1.3)$$

where

$$|\phi\rangle = \frac{1}{2^{n/2}} \sum_{x=0}^{2^n-1} |x\rangle \quad \text{--} (1.4)$$

we can only measure the input register

$$U_f |x\rangle|-\rangle = V|x\rangle|-\rangle \quad \text{---} (1.5)$$

where

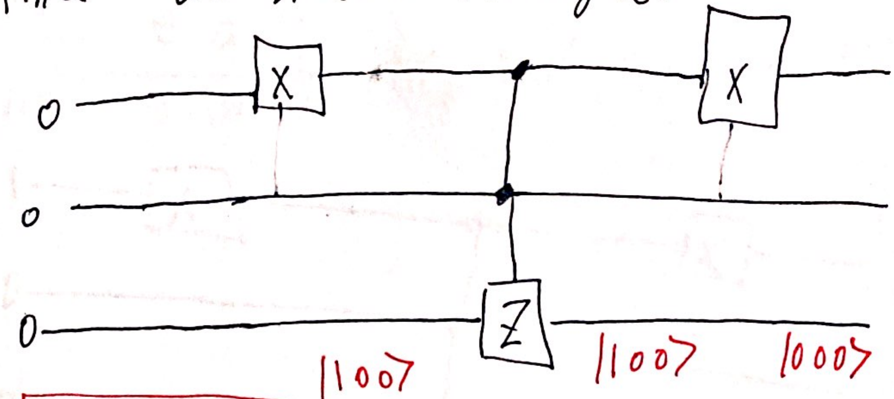
$$V = \mathbb{1} - 2|a\rangle\langle a| \quad \text{----} (1.6)$$

and

$$V|x\rangle_n = (-1)^{f(x)} |x\rangle_n = \begin{cases} |x\rangle_n ; & x \neq a \\ -|a\rangle_n ; & x = a \end{cases} \quad \text{...} (1.7)$$

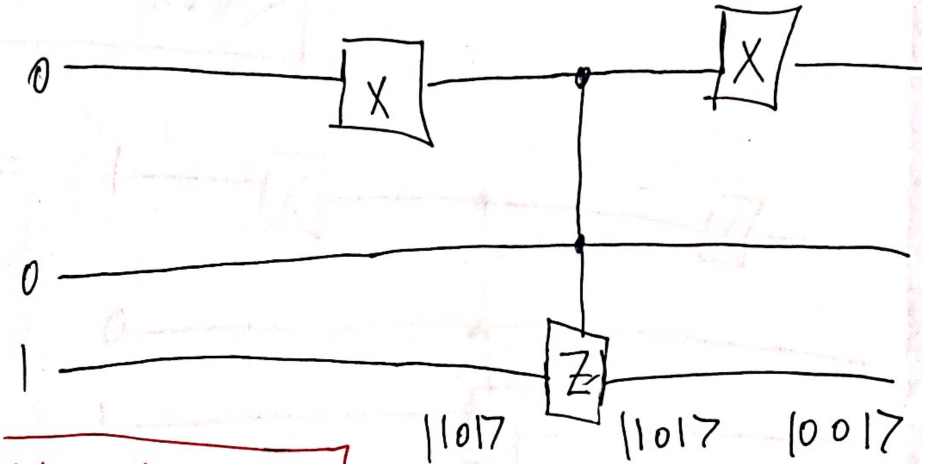
a) Given the circuit for the case $n=3$ (only in the input register!), Let's verify equation (1.6) and find the special string a.

1) $V|000\rangle =$



$\Rightarrow \boxed{V|000\rangle = |000\rangle}$

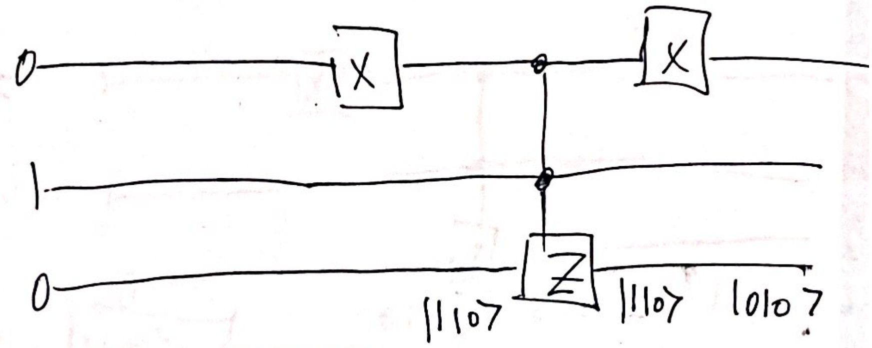
2) $V|001\rangle =$



$\Rightarrow \boxed{V|001\rangle = |001\rangle}$

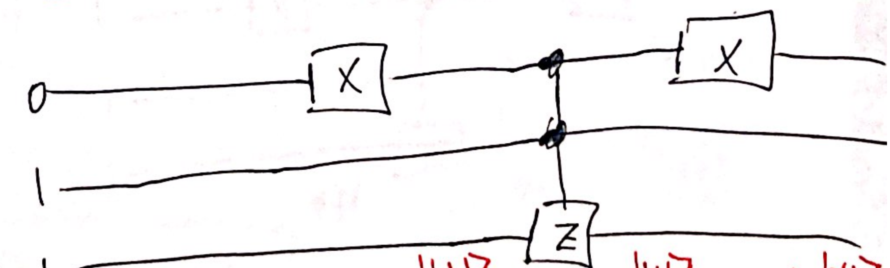
3) $V|010\rangle =$

(Note: Z didn't get applied here b/c both the target and control qubits must be 1.)



$\Rightarrow \boxed{V|010\rangle = |010\rangle}$

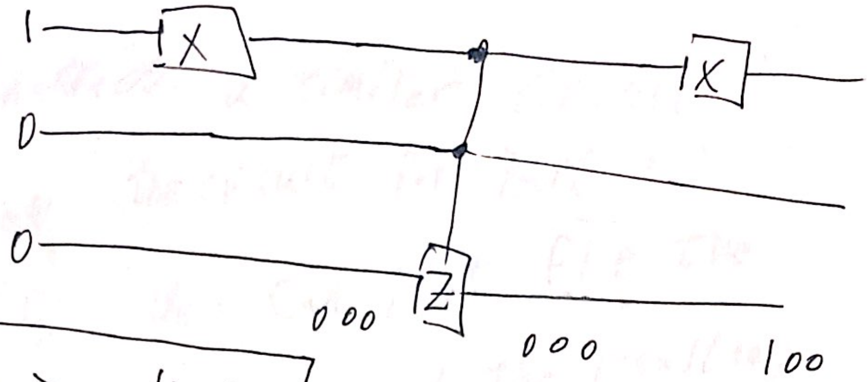
4) $V|011\rangle =$



$\Rightarrow \boxed{V|011\rangle = -|011\rangle}$

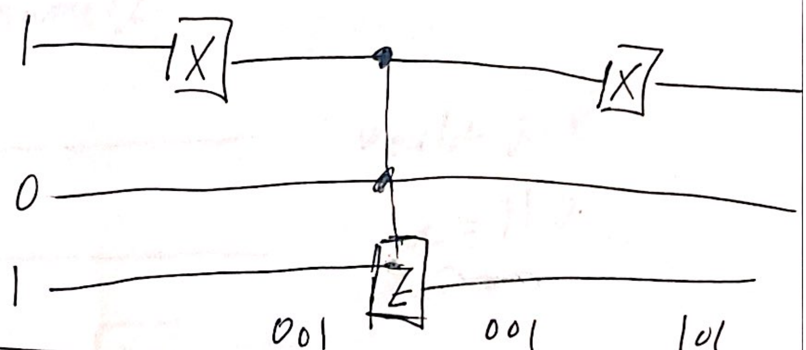
$\Rightarrow \boxed{a = 011}$ by equation (1.7) !

5) $V|100\rangle =$



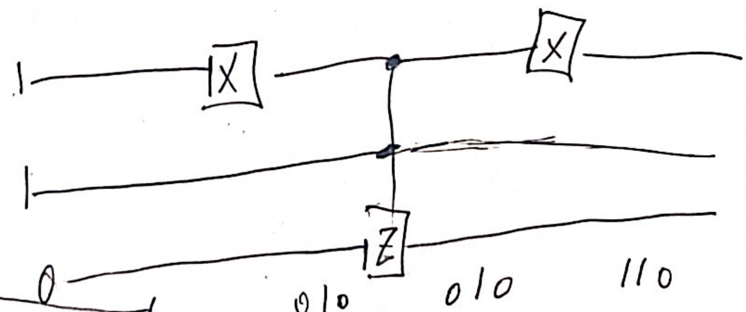
$\Rightarrow \boxed{V|100\rangle = |100\rangle}$

6) $V|101\rangle =$



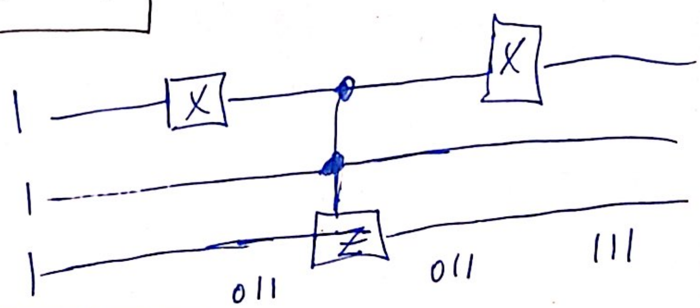
$\Rightarrow \boxed{V|101\rangle = |101\rangle}$

7) $V|110\rangle =$



$\Rightarrow \boxed{V|110\rangle = |110\rangle}$

8) $V|111\rangle =$



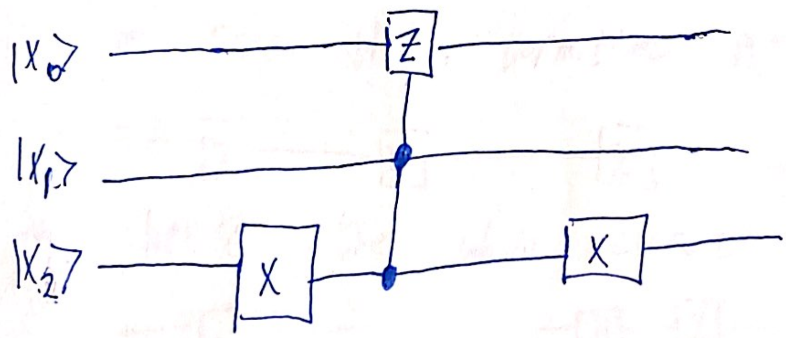
$\Rightarrow \boxed{V|111\rangle = |111\rangle}$

As we see, $V = \mathbb{1} - 2|a\rangle\langle a|$, as well.

For example, $V|a\rangle = |a\rangle - 2|a\rangle\langle a|a\rangle = -|a\rangle$
which is what we expect to get by equation (1.7).

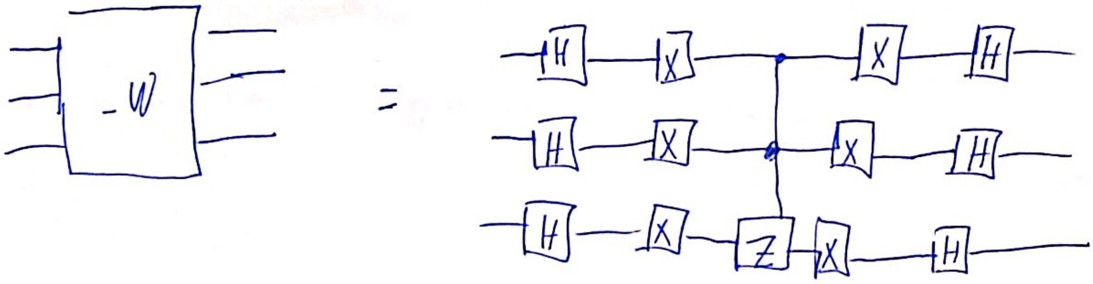
b) We can construct a similar circuit.

Since we know the circuit for part (a) gives $a = 011$, we can just flip the order of the qubits, and the resulting special string a would just have its order of qubits reversed, namely,

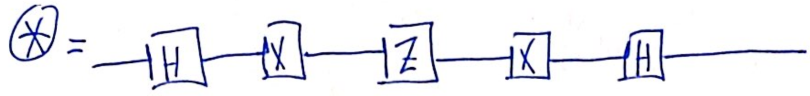


would give
$a = 110$

c)



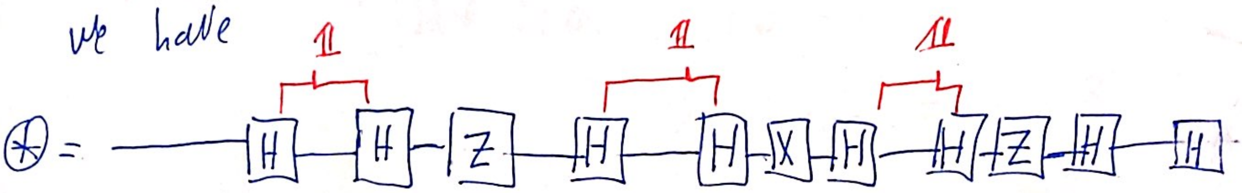
looking only at the bottom qubit,



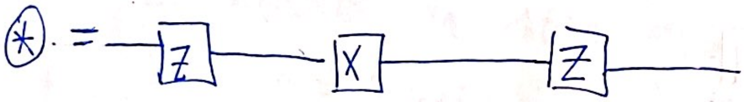
and using the relations

$$Z = HXH \quad \text{and} \quad X = HZH \quad \ldots (1.8)$$

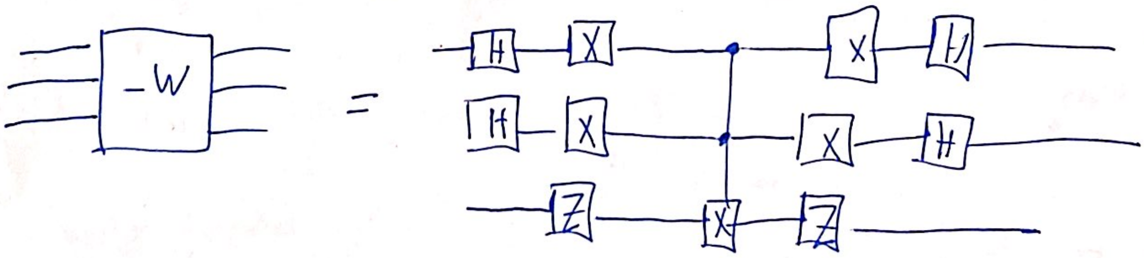
we have



and since H is Hermitian $HH = \mathbb{1}$



and hence the whole circuit is



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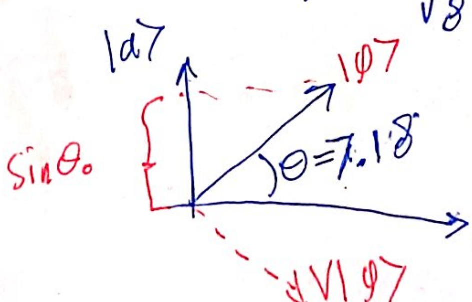
d) $\langle a|\phi\rangle = \frac{1}{2^{n/2}} = \frac{1}{\sqrt{N}}$

and since we have $n = 3 \Rightarrow N \equiv 2^n = 2^3 = 8$

$\langle a|\phi\rangle = \frac{1}{\sqrt{8}}$ $\Rightarrow \sin\theta_0 = \frac{1}{8}$

$\Rightarrow \theta_0 = \sin^{-1}\left(\frac{1}{8}\right) = \underline{7.18}$



$\Rightarrow Prob(a) = |\langle a|\phi\rangle|^2 = \frac{1}{8} = \boxed{0.125}$

e) Each iteration increases $\theta \to \theta + 2\theta$

$\Rightarrow$ First iteration : $WV|\phi\rangle \to \theta + 2\theta_0$

$$\Rightarrow \theta_1 = 7.18 + 2(7.18) = 21.54^\circ$$

$\Rightarrow$ Probability we obtain a after one iteration $= |\sin\theta_1|^2 = |\sin(21.54^\circ)|^2 = \boxed{0.188}$

f) After two Grover iterations, the angle will be added by yet (2θ), so

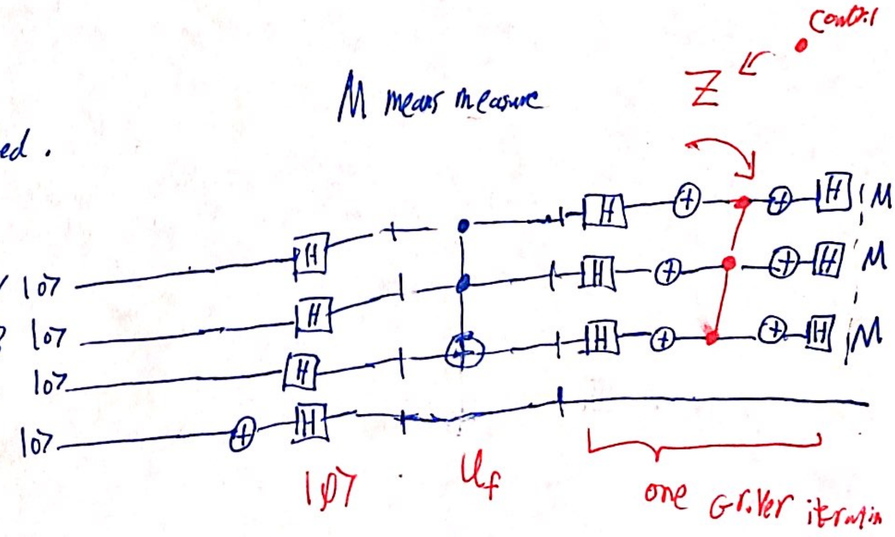
$$\theta_2 = \theta_1 + 2\theta_0 = 5 \times 7.18^\circ = 35.9$$

$\Rightarrow$ Probability we obtain a after 2 iterations

$$= |\sin\theta_2|^2 = |\sin(35.9)|^2 = \boxed{0.9487}$$

h) screenshots attached.

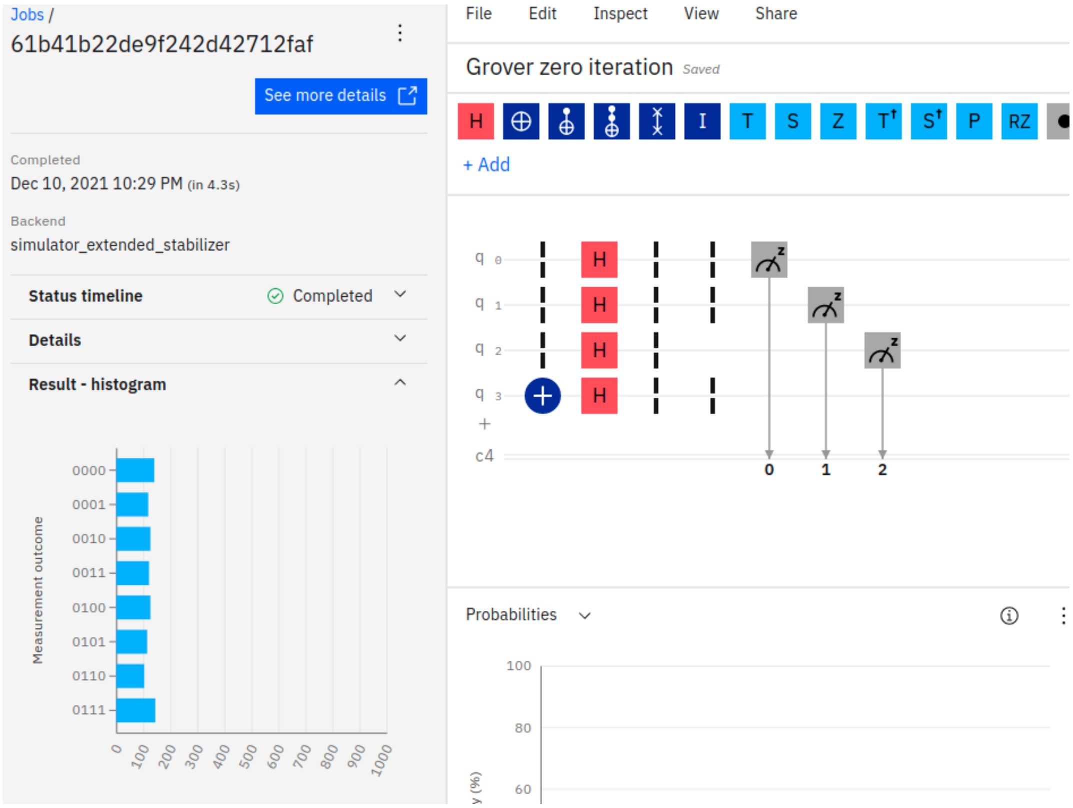
one iterations :



Result: (a) with probability.

if we overflow.t, equal Prob.

Problem 1, h)
For ZERO iterations, we have the circuit



And the frequency of 011 states is 0.122, which is very close to the expected value of 1/8=0.125.

For 1 iteration, we have the circuits

We get Frequency for 011=115/1000, which is very close to the calculated expected probability to find a, which is 0.188 !

For TWO iterations, we have the circuit

And hence we have an equal superposition of all states, since we overshot, which is what is expected.

Ali Al Kadhim
Quantum Computing
Final

1 Problem 2

1.1 Part a

Recall that for period-finding we have a function $f(x) = f(y)$ iff

$$y = x + kr \tag{1}$$

where k is an integer and r is the period. The particular function for Shors algorithm is

$$f(x) = b^x (modN) \tag{2}$$

Where $N = pq$.
Here, we start with initial state

$$|0\rangle|0\rangle|0\rangle \tag{3}$$

We then apply H on the first qubit, identity on the second and X on the third,

$$\begin{aligned} H^1 \otimes \mathbb{1}^2 \otimes X^3 |0\rangle|0\rangle|0\rangle \\ &= \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) \otimes |0\rangle \otimes |1\rangle \\ &= \boxed{\frac{1}{\sqrt{2}}(|001\rangle + |101\rangle)} \end{aligned} \tag{4}$$

Now we have the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|001\rangle + |101\rangle) \tag{5}$$

Which is supposed to mimic the output state of Shor's algorithm, ie it has the form

$$|\psi\rangle = \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} |x_0 + kr\rangle \tag{6}$$

By comparing the two states $|\psi\rangle$ above, we immediately see that $m = 2$. Therefore, equation 6 reads

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|x_0\rangle + |x_0 + r\rangle) \tag{7}$$

and we immediately see that $|x_0\rangle_n = |001\rangle$ where $n = 3$ is the number of qubits. Using standard basis for $n = 3$

$$\{000, 001, 010, 011, 100, 101, 110, 111\} \leftrightarrow \{0, 1, 2, 3, 4, 5, 6, 7\} \tag{8}$$

So that 001 corresponds to 1 so $x_0 = 1$. And 101 corresponds to 5, therefore $1 + r = 5 \rightarrow r = 4$.

1.2 Part b

Recall that the quantum Fourier transform

$$U_{FT}|x\rangle_n = \frac{1}{2^{n/2}} \sum_{y=0}^{2^n-1} e^{2\pi i xy/2^n} |y\rangle_n \tag{9}$$

So that for $n = 3$,

$$U_{FT}|x\rangle_3 = \frac{1}{2^{3/2}} \sum_{y=0}^{7} e^{2\pi i xy/2^3} |y\rangle_3 \tag{10}$$

We can write

$$|x \tag{11}$$

Now using

$$x = x_{\text{number}} = x_0^{\text{bin}} + 2x_1^{\text{bin}} + 4x_2^{\text{bin}} \tag{12}$$

Where x_{bin} is a binary number (0 or 1). And similarly,

$$y = y_0 + 2y_1 + 4y_2 \tag{13}$$

Now let's evaluate $\frac{xy}{8}$

$$\begin{aligned} \frac{xy}{8} &= \frac{(x_0 + 2x_1 + 4x_2)(y_0 + 2y_1 + 4y_2)}{8} \\ &= y_0 \left(\frac{x_0}{8} + \frac{x_1}{4} + \frac{x_2}{2} \right) + y_1 \left(\frac{x_0}{4} + \frac{x_1}{2} + x_2 \right) + y_2 \left(\frac{x_0}{2} + x_1 + 2x_2 \right) \end{aligned} \tag{14}$$

Since $e^{2\pi i xy/8}$, any integer term in $xy/8$ can be set to 1 since they drop out (since $e^{2\pi i n} = 1$). Hence

$$\begin{aligned} UFT|x_2\rangle|x_1\rangle|x_0\rangle &= \frac{1}{\sqrt{8}} \sum_{y_0=0}^{1} \sum_{y_1=0}^{1} \sum_{y_2=0}^{1} e^{2\pi i (\frac{xy}{8})} |y_2\rangle|y_1\rangle|y_0\rangle \\ &= \left(\frac{1}{\sqrt{2}} (|0\rangle + e^{2\pi i x_0/2}|1\rangle \right) \otimes \frac{1}{\sqrt{2}} \left(|0\rangle + e^{2\pi i (\frac{x_0}{4} + \frac{x_1}{2})}|1\rangle \right) \otimes \frac{1}{\sqrt{2}} \left(|0\rangle + e^{2\pi i (\frac{x_0}{8} + \frac{x_1}{4} + \frac{x_2}{2})}|1\rangle \right) \end{aligned} \tag{15}$$

Therefore, for $|x\rangle_3 = |001\rangle$, $x_2 = 0, x_1 = 0, x_0 = 1$

$$\begin{aligned} U_{FT}|001\rangle &= \left(\frac{1}{\sqrt{2}} |0\rangle + e^{2\pi i 1/2}|1\rangle \otimes \frac{1}{\sqrt{2}} \left(|0\rangle + e^{2\pi i (\frac{1}{4} + \frac{0}{2})}|1\rangle \right) \otimes \frac{1}{\sqrt{2}} \left(|0\rangle + e^{2\pi i (\frac{1}{8} + \frac{0}{4} + \frac{0}{2})}|1\rangle \right) \right. \\ &= \left(\frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \right) \otimes \frac{1}{\sqrt{2}} \left(|0\rangle + e^{\pi i/2}|1\rangle \right) \otimes \frac{1}{\sqrt{2}} \left(|0\rangle + e^{\pi i/4}|1\rangle \right) \\ &= \frac{1}{2^{3/2}} \left(|000\rangle + e^{i\pi/4}|001\rangle + e^{i\pi/2}|010\rangle + e^{3i\pi/4}|011\rangle - |100\rangle - e^{i\pi/4}|101\rangle - e^{i\pi/2}|110\rangle - e^{3\pi i/4}|111\rangle \right) \end{aligned} \tag{16}$$

Similarly, for $|x\rangle = |101\rangle$, $x_2 = 1, x_1 = 0 x_0 = 1$ so

$$\begin{aligned}
U_{FT}|101\rangle &= \left(\frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i x_0/2}|1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i(\frac{x_0}{4}+\frac{x_1}{2})}|1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i(\frac{x_0}{8}+\frac{x_1}{4}+\frac{x_2}{2})}|1\rangle\right) \\
&= \left(\frac{1}{\sqrt{2}}(|0\rangle + e^{2\pi i 1/2}|1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i(\frac{1}{4}+\frac{0}{2})}|1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i(\frac{1}{8}+\frac{0}{4}+\frac{1}{2})}|1\rangle\right) \\
&= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\pi/2}|1\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + e^{5i\pi/4}|1\rangle\right) \\
&= \frac{1}{2^{3/2}}\left(|000\rangle + e^{5i\pi/4}|001\rangle + e^{i\pi/2}|010\rangle + e^{7i\pi/4}|011\rangle - |100\rangle - e^{5i\pi/4}|101\rangle - e^{i\pi/2}|110\rangle - e^{7i\pi/4}|111\rangle\right)
\end{aligned} \tag{17}$$

Hence using the above two relations,

$$\begin{aligned}
U_{FT}|\psi\rangle &= \frac{1}{\sqrt{2}}(U_{FT}|001\rangle + U_{FT}|101\rangle) \\
&= \frac{1}{4}\Bigg[2|000\rangle + \underbrace{(e^{i\pi/4} + e^{5i\pi/4})}_{e^{i\pi/4}(1+e^{i\pi}) = e^{i\pi/4}(1-1)=0}|001\rangle + 2\underbrace{e^{i\pi/2}}_{i}|010\rangle + \underbrace{(e^{3i\pi/4} + e^{7i\pi/4})}_{e^{i\pi/4}(e^{i\pi/2}+e^{3i\pi/2}) = e^{i\pi/4}(i-i)=0}|011\rangle \\
&\quad - 2|100\rangle - \underbrace{(e^{i\pi/4} + e^{5i\pi/4})}_{0}|101\rangle - 2\underbrace{e^{i\pi/2}}_{i}|110\rangle - \underbrace{(e^{3i\pi/4} + e^{7i\pi/4})}_{0}|111\rangle\Bigg]
\end{aligned} \tag{18}$$

Therefore

$$\begin{aligned}
U_{FT}|\psi\rangle &= \frac{1}{4}\Big[2|000\rangle + 2i|010\rangle - 2|100\rangle - 2i|110\rangle\Big] \\
&= \frac{1}{2}\Big[|000\rangle + i|010\rangle - |100\rangle - i|110\rangle\Big]
\end{aligned} \tag{19}$$

We can find the probabilities for each possible result of measuring this state in the standard basis, giving equal probabilities of 1/4 for the states above

$$P(|000\rangle) = P(|010\rangle) = P(|100\rangle) = P(|110\rangle) = \frac{1}{4} \tag{20}$$

$$P(|001\rangle) = P(|011\rangle) = P(|101\rangle) = P(|111\rangle) = 0 \tag{21}$$

for our case

$$\frac{2^n}{r} = \frac{8}{4} = 2 \tag{22}$$

And the states could be written as

$$P(|0\rangle) = P(|2\rangle) = P(|4\rangle) = P(|6\rangle) = \frac{1}{4} \tag{23}$$

And they are all multiples of 2, hence the probability that the measured state y is an integer multiple of $\frac{2^n}{r} = 2$ is the sum of all the above states' individual probabilities

$$P(y \text{ is an integer multiple of } \frac{2^n}{r} = 2) = \frac{1}{4} \times 3 = 0.75 \tag{24}$$

Where above, it's summed over 3 states, since the $|0\rangle$ state corresponding to $y = 0$ is not considered an integer multiple of $\frac{2^n}{r} = 2$ (this might be incorrect, but given all our class notes, y is reacted to j which is an integer starting at 1 not 0).

1.3 Part c

We use IBM Q website to construct the Quantum Fourier Transform algorith acting on state $|\psi\rangle$ as was shown in part a. Below is the screenshot from IBM Q

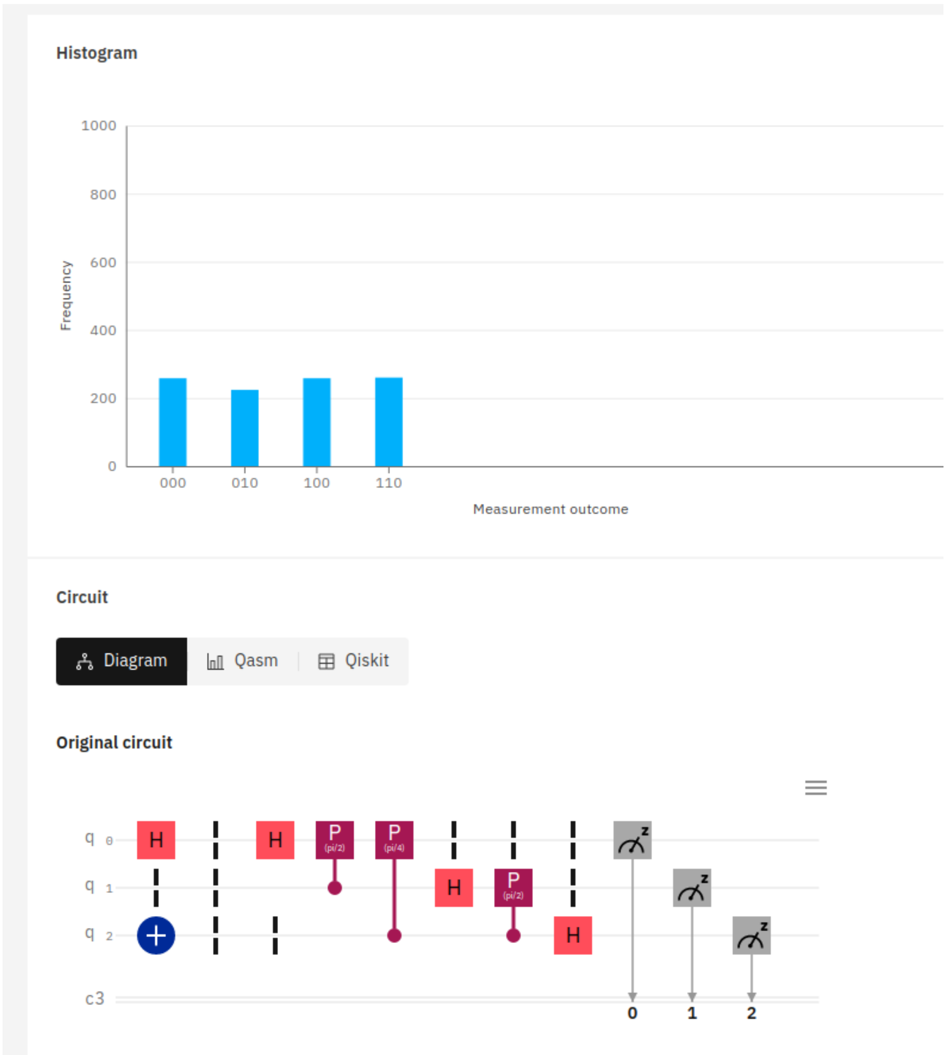


Figure 1: Screenshot of IBM Q circuit for $U_{FT}|\psi\rangle$

We see that the only states with nonzero probabilities are $|000\rangle, |010\rangle, |100\rangle, |110\rangle$, as expected. We also see that all their probabilities are roughly equal to 1/4, as we expect given our calculations in part b. The numbers given from IBM Q are: $P(|000\rangle) = 258/1000 = 0.258, P(|010\rangle) = 224/1000 = 0.224, P(|100\rangle) = 258/1000, P(|110\rangle) = 260/1000 = 0.26$, all extremely close to our expected probability of 0.25!

Problem 3

$$|\psi\rangle = \frac{1}{\sqrt{5}}(|1\rangle + |4\rangle + |7\rangle + |10\rangle + |13\rangle) \quad \text{--- (3.1)}$$

This is a 4-qubit state in standard basis.

$|0\rangle \leftrightarrow |0000\rangle$ $\qquad$ $|6\rangle \longleftrightarrow |0110\rangle$

$|1\rangle \leftrightarrow |0001\rangle$ $\qquad$ $|7\rangle \longleftrightarrow |0111\rangle$

$|2\rangle \longleftrightarrow |0010\rangle$ $\qquad$ $|8\rangle \longleftrightarrow |1000\rangle$

$|3\rangle \longleftrightarrow |0011\rangle$ $\qquad$ $|9\rangle \longleftrightarrow |1001\rangle$

$|4\rangle \longleftrightarrow |0100\rangle$ $\qquad$ $|10\rangle \longleftrightarrow |1010\rangle$

$|5\rangle \longleftrightarrow |0101\rangle$ $\qquad$ $|11\rangle \longleftrightarrow |1011\rangle$

$|15\rangle \longleftrightarrow |1111\rangle$ $\qquad$ $|12\rangle \longleftrightarrow |1100\rangle$

$|13\rangle \longleftrightarrow |1101\rangle$

$|14\rangle \longleftrightarrow |1110\rangle$

a) Here we have $r=3$, $n=4$

$$\Rightarrow m = \left\lceil \frac{2^n}{r} \right\rceil - 1 = \left\lceil \frac{16}{3} \right\rceil - 1 = 5$$

We want to find the probability that if we measure $U_{FT}|\psi\rangle$ we obtain a value of y which is within $\frac{1}{2}$ of an integer multiple of

$$\frac{2^n}{r} = \frac{16}{3} \qquad (\simeq 5.3)$$

$\Rightarrow$ Integer multiples of $\frac{16}{3}$:

- $1 \times \frac{16}{3} \Rightarrow$ our bounds are: $\left[\frac{16}{3} - \frac{1}{2}, \frac{16}{3} + \frac{1}{2}\right] = [4.83, 5.83]$

  $\Rightarrow y = 5$ is a state that we have that spans this range.

- $2 \times \frac{16}{3} \Rightarrow$ bounds are: $\left[\frac{32}{3} - \frac{1}{2}, \frac{32}{3} + \frac{1}{2}\right] = [10.16, 11.16]$

  $\Rightarrow y = 11$ is a state that spans this range.

$$\cdot\, 3 \times \frac{16}{3} \Rightarrow \text{Bounds are: } \left[\frac{48}{3} - \frac{1}{2}, \frac{48}{3} + \frac{1}{2}\right]$$

$$= [15.5, \ 16.5]$$

None of our y states span this range.

$\Rightarrow$ our y states are $y = \{5, 11\}$

Now recall that shor's algorithm applies U_{FT} to the input register

$$\frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} |x_0 + kr\rangle_n \xrightarrow{\ /n\ } \boxed{U_{FT}}$$

$$U_{FT} \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} |x_0 + kr\rangle_n = \frac{1}{2^{n/2}} \frac{1}{\sqrt{m}} \sum_{k=0}^{m-1} \sum_{y=0}^{2^n-1} e^{\frac{2i\pi(x_0+kr)y}{2^n}} |y\rangle_n$$

$$= \sum_{y=0}^{2^n-1} \underbrace{\left[\underbrace{\frac{1}{\sqrt{m}\, 2^{n/2}} e^{2i\pi x_0 y/2^n}}_{\text{squares to 1}} \sum_{k=0}^{m-1} e^{2i\pi k r y/2^n} \right]}_{h(y)\text{ : Amplitude to find } y} |y\rangle_n$$

$$\Rightarrow P(y) = |h(y)|^2 = \frac{1}{m} \frac{1}{2^n} \left| \sum_{k=0}^{m-1} e^{2i\pi k r y/2^n} \right|^2$$

So for our case,

$$P(y) = \frac{1}{5} \frac{1}{2^4} \left| \sum_{k=0}^{4} e^{6i\pi k y/16} \right|^2 = \frac{1}{80} \left| \sum_{k=0}^{4} e^{3i\pi k y/8} \right|^2$$

$$= \frac{1}{80} \left| 1 + e^{3i\pi y/8} + e^{3i\pi y/4} + e^{9i\pi y/8} + e^{\frac{3i\pi y}{2}} \right|^2$$

Using this relation,

3.3

$$P(y=5) = \frac{1}{80}\left| 1 + e^{15 i\pi/8} + e^{15 i\pi/4} + e^{45 i\pi/8} + e^{\frac{15 i\pi}{2}} \right|^2$$

$$= \frac{1}{80}\left[1 + e^{15 i\pi/8} + e^{15 i\pi/4} + e^{45 i\pi/8} + e^{15 i\pi/2}\right]$$

$$\times \left[1 + e^{-15 i\pi/8} + e^{-15 i\pi/4} + e^{-45 i\pi/8} + e^{-15 i\pi/2}\right]$$

$$P(y=5) = 0.22705$$

$$P(y=11) = \frac{1}{80}\left| 1 + e^{33 i\pi/8} + e^{33 i\pi/4} + e^{99 i\pi/8} + e^{33 i\pi/2} \right|^2$$

$$= \frac{1}{80}\left[1 + e^{33 i\pi/8} + e^{33 i\pi/4} + e^{99 i\pi/8} + e^{33 i\pi/2}\right]$$

$$\times \left[1 + e^{-33 i\pi/8} + e^{-33 i\pi/4} + e^{99 i\pi/8} + e^{-33 i\pi/2}\right]$$

$$P(y=11) = 0.22705$$

$\Rightarrow$ Total Probability $= P(y=5) + P(y=11)$

$= 0.22705 + 0.22705$

$= \boxed{0.45411}$

This satisfies our bound $P(y) \geq 0.4$ that we derived in class. #

b) The question asks us to Consider the biggest Value of y which is within $\frac{1}{2}$ of $\frac{2^n}{r} = \frac{16}{3}$, in our case this is $y = 11$.

Then $\frac{y}{16} = \frac{11}{16}$ Can be expressed as a Continued fraction.

$$\frac{11}{16} = \frac{1}{\left(\frac{16}{11}\right)}, \quad 1 \times 11 = 11, \; 16 - 11 = 5,$$

$$= \frac{1}{1 + \frac{5}{11}}$$

$$= \frac{1}{1 + \frac{1}{\left(\frac{11}{5}\right)}}, \quad 2 \times 5 = 10, \; 11 - 10 = 1$$

$$= \boxed{\frac{1}{1 + \frac{1}{2 + \frac{1}{5}}}}$$

$\Rightarrow a_0 = 1, \quad a_1 = 2, \quad a_2 = 5$

c) . $\frac{1}{a_0} = \frac{1}{1} < 4$

. $\frac{1}{a_0 + \frac{1}{a_1}} = \frac{1}{1 + \frac{1}{2}} = \frac{1}{\frac{3}{2}} = \frac{2}{3} \approx 0.66 < 4$

$\frac{y}{2^n} = \frac{11}{16} \rightarrow \left[\frac{11}{16} - \frac{1}{2^5}, \frac{11}{16} + \frac{1}{2^5}\right] = [0.656, 0.718]$

only the <u>second term</u>, $\boxed{\frac{1}{a_0 + \frac{1}{a_1}} = 0.66}$ is within $\frac{1}{2^5}$!

Question 4: Bell's Inequality

Here we explore Bell's inequality,

$$P_{AB} + P_{AC} + P_{BC} \geq 1$$

By creating entangled Bell states

$$|\Phi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

By applying the rotation

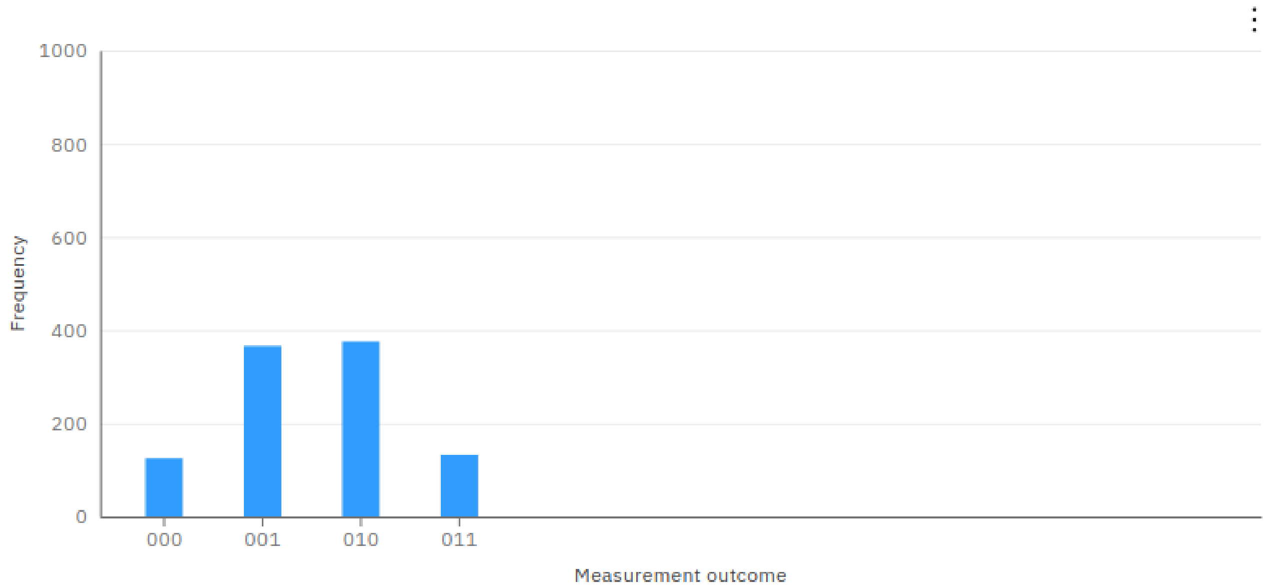
$$R_y(2\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

On our 2-qubit standard basis states.

The angle is $\theta = \pi/3$ for our first set of measurements, and hence the rotation matrix is $R_y(2\theta) = R(2\pi/3)$. For all the following, we use 1000 total experiments (measurements) for each of the circuits.

Part a)

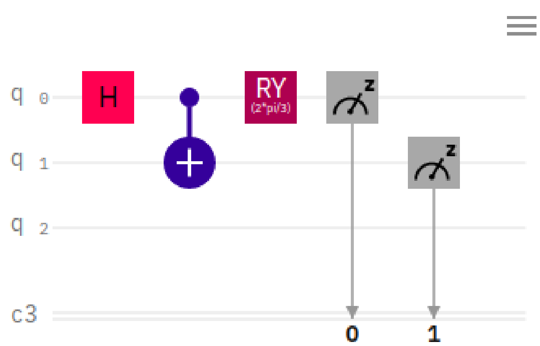
Bell Circuit BA



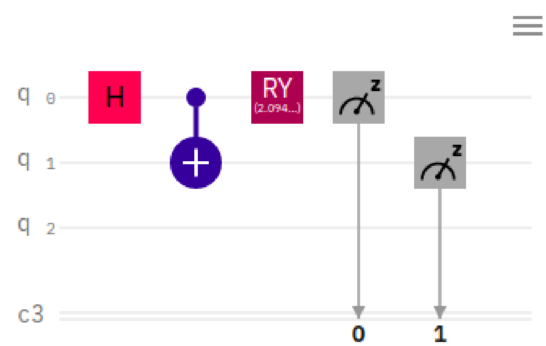
Circuit

Diagram | Qasm | Qiskit

Original circuit

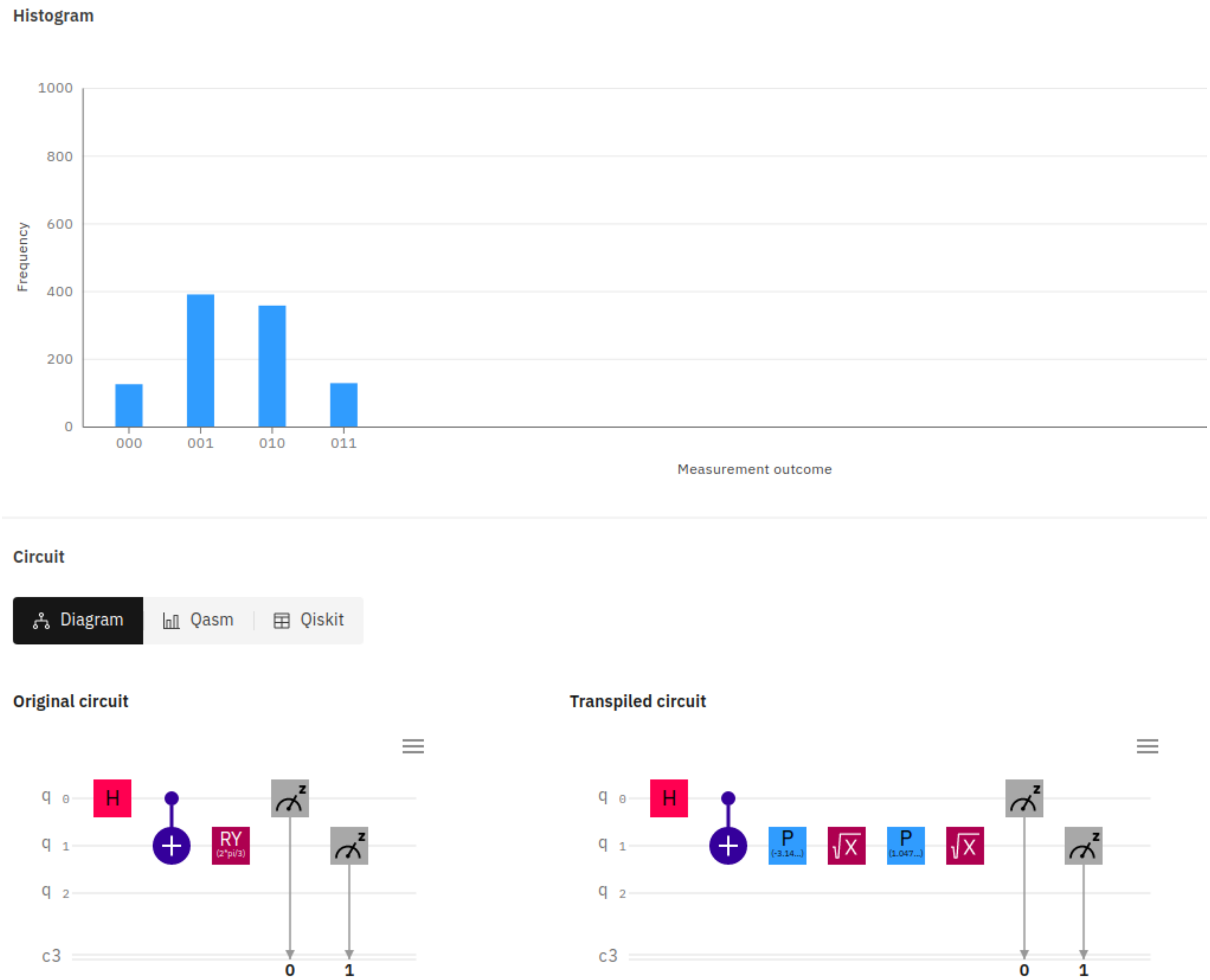


Transpiled circuit



125 had 00, 133 had 11, so (125+133)/1000 = Contribution of BA = 0.258

Bell Circuit AB



125 had 00, 128 had 11, so (125+128)/1000=0.253

so $P_{AB} = (0.253 + 0.258)/2 = 0.2555$

Bell Circuit CA

116 had 00, 114 had 11, so

(116+114)/1000=0.23

Bell Circuit AC

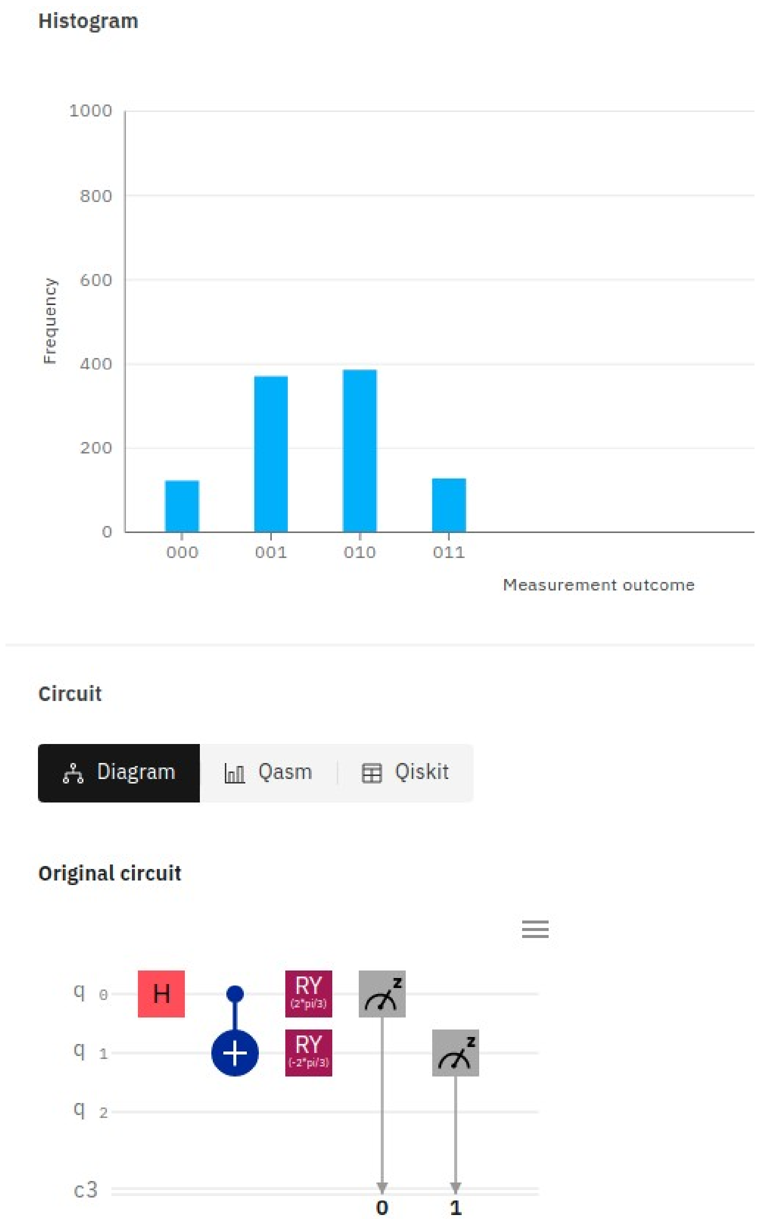
148 had 00, 120 had 11, so (148+120)/1000=0.268

So

$$P_{AC} = (0.23 + 0.268)/2 = 0.249$$

Bell Circuit BC

121 had 00, 126 had 11, so

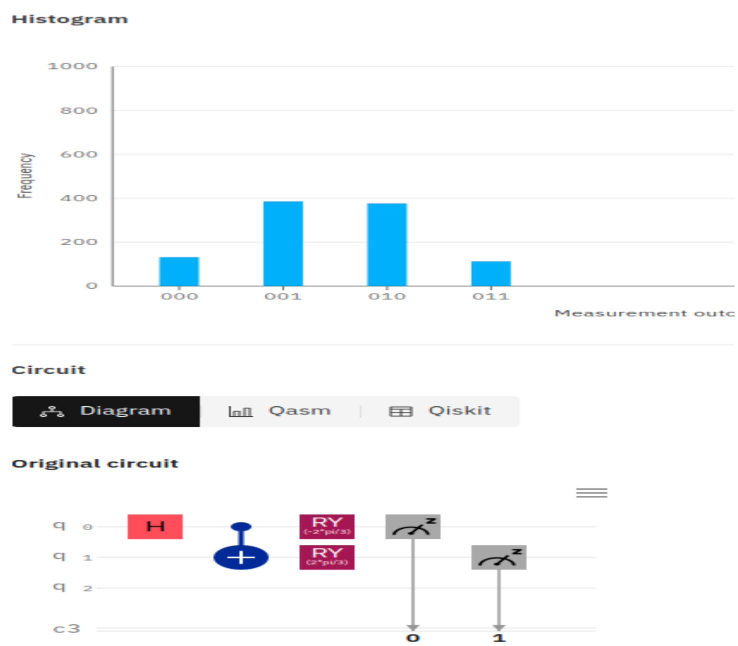


(121+126)/1000=0.247

Bell Circuit CB

130 had 00, 111 had 11, so contribution=(130+111)/1000=0.241

So $P_{AC} = (0.247 + 0.241)/2 = 0.244$

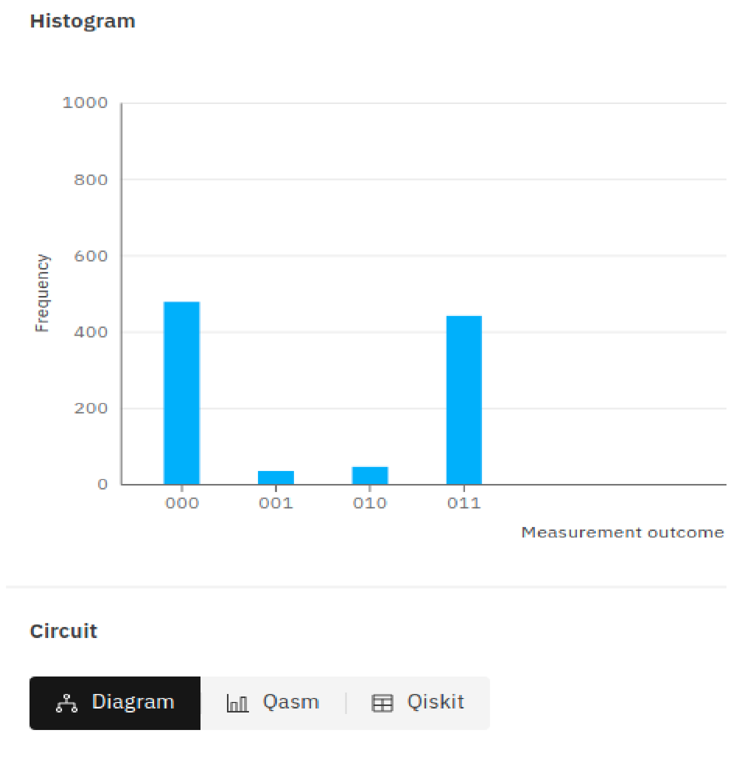


$$P_{AC} + P_{AB} + P_{BC} = 0.2555 + 0.249 + 0.244 = 0.7444999 \quad \text{for } \theta = \pi/3$$

This violates Bell's inequality, as we expect!

Part b)
Now choosing a different value for the angle, $\theta = \pi/12 \implies R_y(2\theta) = R_y(\pi/6)$, and repeating the same prodedure as in part a.

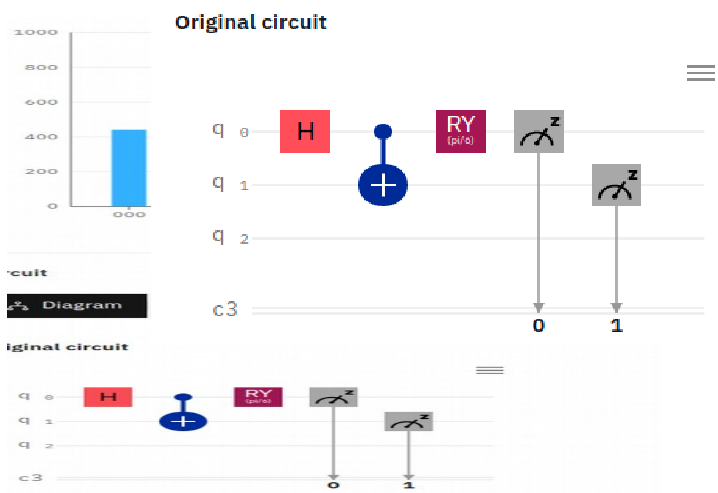
Circuit AB:



00:478 , 11:441, so contribution=(478 +441)/1000=0.933

BA Circuit:

00:440 , 11: 492, so

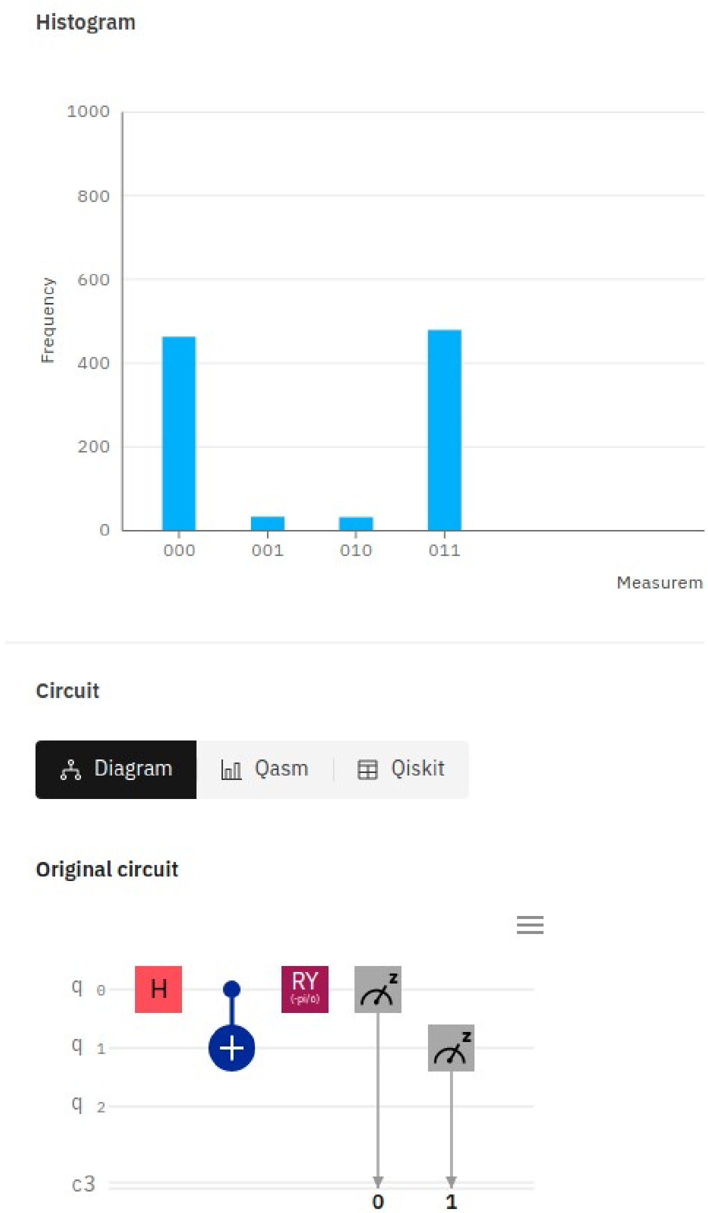


contribution=(440+492)/1000=0.932

$$P_{AB} = (0.932 + 0.933)/2 = 0.9325$$

CA

00:461, 11:478, so (461+ 478)/1000=0.939



AC

00: 495, 11:445, so (495+ 445)/1000=0.94

so

$$P_{AC} = (0.93 + 0.94)/2 = 0.935$$

BC

00:525, 11:473, so
(525+ 473)/1000=0.998

CB

00: 495, 11:505, so (495+505)/1000=1

so $P_{AC} = (0.998+1)/2 = 0.999$

Therefore $P_{AB} + P_{AC} + P_{BC} = 0.9325 + 0.935 + 0.999 = 2.8665 \text{ for} \theta = \pi/12$

Theoretical values and Comparison

Recall that a direct quantum mechanical calculation gives
$P_{AB}+P_{AC}+P_{BC}=2\cos^2\theta+\cos^2(2\theta)$

Which is the theoretical value.

For $\theta=\pi/3$, $P_{AB}+P_{AC}+P_{BC}=0.75$

For $\theta=\pi/12$,
$P_{AB}+P_{AC}+P_{BC}=2.616025404$

Below I plot the theoretical value of the the sum
$P_{AB}+P_{AC}+P_{BC}=2\cos^2\theta+\cos^2(2\theta)$ as well as my experimental values from IBM Q.