Quantum Computing HW 1

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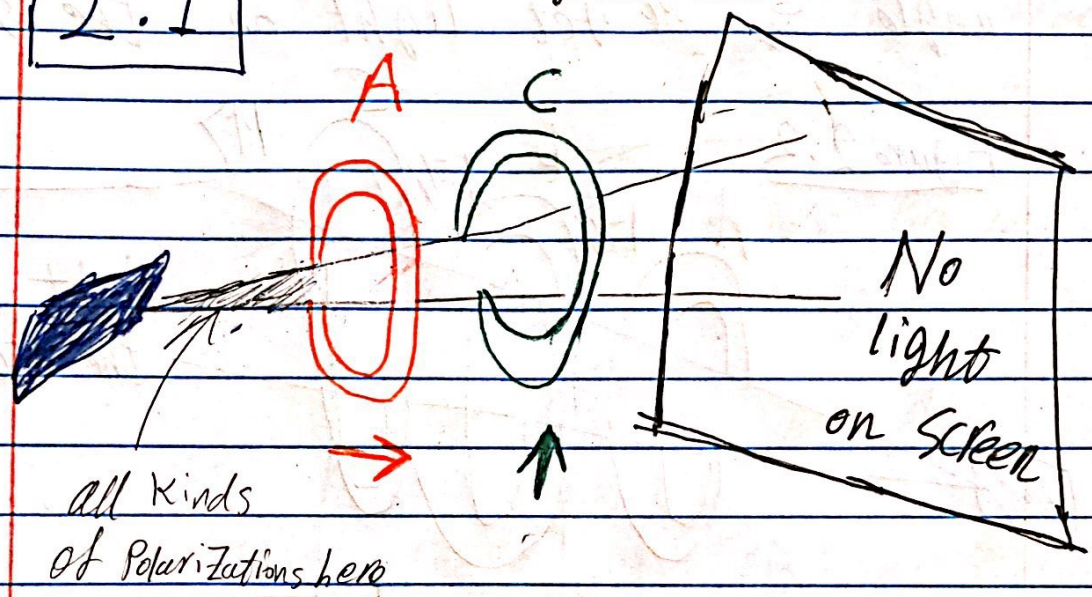
2.1

Figure 2.1



Classical Physics can explain the Phenomenon in Figure 2.1, using Malus's law, where the intensity reduces by $I = I_0 \cos^2\theta$ when passing through a Polarizer.

Quantum mechanics can also explain this Phenomenon, where we have light passing through a Polarizer A whose preferred axis is $\rightarrow$ and then passes through another Polarizer C whose preferred axis is $\uparrow$, resulting in no light being transmitted to the screen. This is because $\langle \rightarrow | \uparrow \rangle = 0$.

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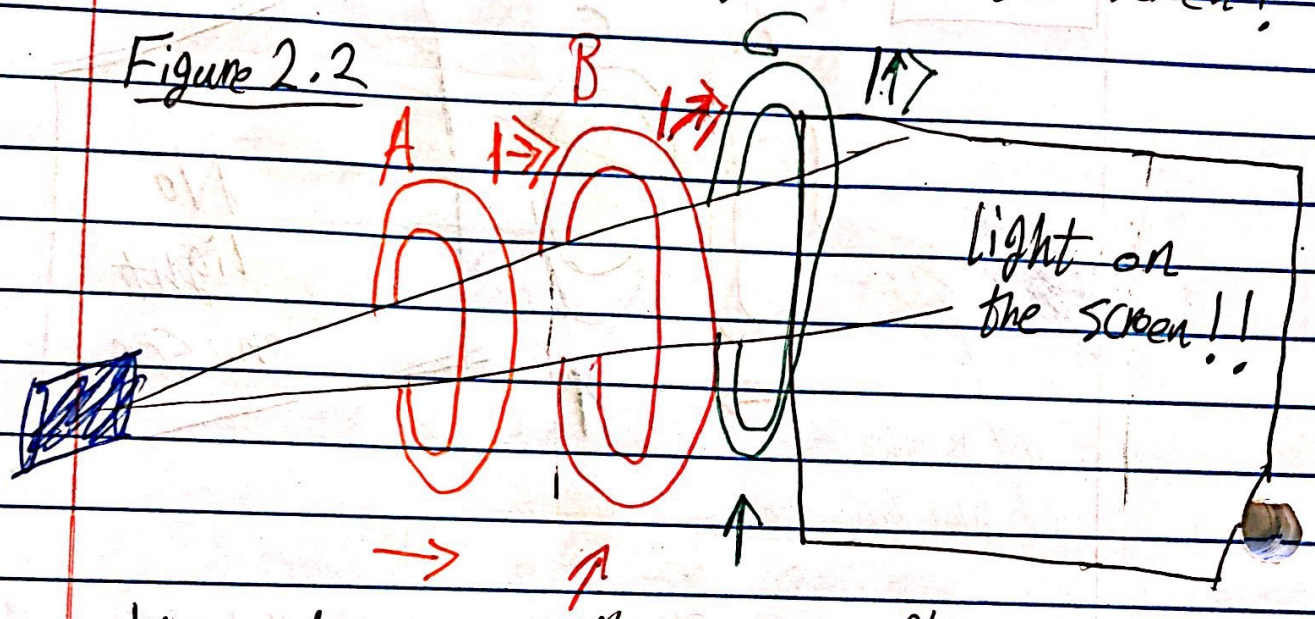
Now, something very interesting happens if we put a third polarizer B in between A and C, oriented at some angle θ, we get light on the screen!

Figure 2.2



In order to explain this phenomenon (Fig. 2.2), we need quantum mechanics.

Let the direction $|v\rangle$ of polarizer B's preferred axis be

$$|v\rangle = \cos\theta\,|\rightarrow\rangle + \sin\theta\,|\uparrow\rangle$$

The fraction of photons passing through each polarizer can be found by calculating the inner product of the incoming photon polarization state and the polarizer's polarization state, and amplitude-squaring this quantity.

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Let P_A = fraction of Photons Passing through Polarizer A

P_B = fraction of Photons Passing through Polarizer B.

P_C = fraction of Photons Passing through Polarizer C.

P_{tot} = total fraction of Photons.

And since these are disjoint Probabilities,

$$P_{tot} = (P_A \cap P_B \cap P_C) = P_A P_B P_C$$

Prior to entering A, the Photons have random Polarizations. Since we are using $\{|\uparrow\rangle, |\rightarrow\rangle\}$ as our standard basis, half of all Photons will have $|\rightarrow\rangle$ Polarization, so

$$\boxed{P_A = 0.5}$$

Photons exiting A now have $|\rightarrow\rangle$ Polariz. state, and they interact with B which has $|\uparrow\rangle$ Polarization. Hence

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$$P_B = |\langle \rightarrow | \nu \rangle|^2$$

$$= |\langle \rightarrow | [\cos\theta | \rightarrow \rangle + \sin\theta | \uparrow \rangle]|^2$$

$$\Rightarrow \boxed{P_B = \cos^2\theta}$$

Photons exiting B are now in Polarization state $|\nu\rangle$, and they interact with C, which has Polarization state $|\uparrow\rangle$. So

$$P_C = |\langle \nu | \uparrow \rangle|^2$$

$$= |[\cos\theta \langle \rightarrow | + \sin\theta \langle \uparrow |] | \uparrow \rangle|^2$$

$$= \sin^2\theta$$

Hence the total fraction of Photons that will make it to the screen is

$$\boxed{P_{tot} = 0.5 \cos^2\theta \sin^2\theta}$$

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2.3

Let's Review the Conventions for choice of bases:

- Standard bases : $\{|0\rangle, |1\rangle\}$
- Haddamard bases : $\{|+\rangle, |-\rangle\}$

and, Pg. 24 eq'n's (2.1) – (2.4):

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

$$|-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

a) $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$: Yes, it is a superposition with respect to the standard basis. Clearly it is not a superposition with respect to the Haddamard basis $\{|+\rangle, |-\rangle\}$ since $\langle +|-\rangle = 0$, where $|+\rangle$ and $|-\rangle$ are defined above.

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b) $\frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$

using 2.1-2.4

$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) + \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right)$

$= \frac{1}{2}(2|0\rangle) = |0\rangle \Rightarrow$ No, it is not a superposition of the standard basis. A superposition of the standard basis has to be a linear superposition of $|0\rangle$ and $|1\rangle$.

c) $\frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$

$= \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) - \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right]$

$= \frac{1}{2} \cdot 2|1\rangle = |1\rangle \Rightarrow$ No, not a superposition of the standard basis, for the same reason as above.

d) $\frac{\sqrt{3}}{2}|+\rangle - \frac{1}{2}|-\rangle$

$= \frac{\sqrt{3}}{2}\left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right] - \frac{1}{2}\left[\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right]$

$= \frac{\sqrt{3}}{2\sqrt{2}}|0\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|1\rangle - \frac{1}{2\sqrt{2}}|0\rangle + \frac{1}{2\sqrt{2}}|1\rangle$

$= \frac{\sqrt{3}-1}{2\sqrt{2}}|0\rangle + \frac{\sqrt{3}+1}{2\sqrt{2}}|1\rangle \Rightarrow$ Yes

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it is a superposition of the standard basis.
A basis in which it's not a superposition can be found by $\langle \psi | \phi \rangle = 0$

where $\psi = \frac{\sqrt{3}}{2} |+\rangle - \frac{1}{2} |-\rangle$,

let $\phi = a |+\rangle + b |-\rangle$

$$\langle \psi | \phi \rangle = \left(-\frac{1}{2} \langle -| + \frac{\sqrt{3}}{2} \langle +| \right) \left(a|+\rangle + b|-\rangle \right)$$

$$= \frac{\sqrt{3}}{2} a \langle +|+\rangle - \frac{1}{2} b \langle -|-\rangle$$

$$= \frac{\sqrt{3}}{2} a - \frac{1}{2} b = 0$$

$$\Rightarrow a = \frac{1}{2}, \quad b = \frac{\sqrt{3}}{2}$$

$$\Rightarrow |\phi\rangle = \frac{1}{2} |+\rangle + \frac{\sqrt{3}}{2} |-\rangle$$

and ψ is not a superposition of the basis $\{\psi, \phi\}$.

e) $\frac{1}{\sqrt{2}} \left(|i\rangle - |-i\rangle \right)$

$$= \frac{1}{\sqrt{2}} \left[\frac{1}{\sqrt{2}} \left(|0\rangle + i|1\rangle \right) - \frac{1}{\sqrt{2}} \left(|0\rangle - i|1\rangle \right) \right]$$

$$= \frac{1}{2} \left(0 + 2i|1\rangle \right) = i|1\rangle \Rightarrow \text{NO,}$$

(Same reason as b)

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f) $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ Yes, clearly it's a superpos. of $\{|0\rangle, |1\rangle\}$

We already know the above state is $|-\rangle$ in the Hadamard bases $\{|+\rangle, |-\rangle\}$, which is orthonormal so $\langle +|-\rangle = 0$, so This state is not a superposition with respect to the Hadamard basis.

2.4

from question 2.3, I showed that Parts (a) and (f) are not superpositions of the Hadamard basis $\{|+\rangle, |-\rangle\}$. It's also trivial to see that Parts (b), (c), (d) are superpositions of the Hadamard basis.
Furthermore, in Part (e),

$$\frac{1}{\sqrt{2}}(|i\rangle - |-i\rangle) = i|1\rangle = i\left[\frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)\right]$$

$$= \frac{i}{\sqrt{2}}(|+\rangle - |-\rangle)$$

so (f) is also a super pos. of H basis.

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→ Parts (b), (c), (d), (f) of 2.3 are superpositions of Hadamard basis.

$\boxed{2.6}$ In all cases, the possible measurements are the possible states in the basis (the eigenstates), and each measurement occurs with a probability of the square of the amplitude of the component along the state of interest. Colloquially one could write that the probability of observing "a state of interest" is

$$\left| \langle \text{lin. superposition of all possible states} | \text{state of interest} \rangle \right|^2$$

a) $\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle$, $\underbrace{\{|0\rangle, |1\rangle\}}$

Possible outcomes: $|0\rangle$ or $|1\rangle$

$$P(|0\rangle) = \left| \langle 0 | \left[\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle \right] \right|^2$$

$$= \left| \frac{\sqrt{3}}{2} \langle 0|0\rangle \right|^2 = \boxed{\frac{3}{4}}$$

And $P(|1\rangle) = \left| \langle 1 | \left[\frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle \right] \right|^2$

$$= \boxed{\frac{1}{4}}$$

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b) $\frac{\sqrt{3}}{2}|1\rangle - \frac{1}{2}|0\rangle$, $\underbrace{\{|0\rangle, |1\rangle\}}_{\text{Possible outcomes}}$

$$P(|1\rangle) = \left|\langle 1| \left[\frac{\sqrt{3}}{2}|1\rangle - \frac{1}{2}|0\rangle\right]\right|^2$$

$$= \boxed{\frac{3}{4}}$$

$$P(|0\rangle) = \left|\langle 0| \left[\frac{\sqrt{3}}{2}|1\rangle - \frac{1}{2}|0\rangle\right]\right|^2 = \boxed{\frac{1}{4}}$$

c) $|-i\rangle$, $\underbrace{\{|0\rangle, |1\rangle\}}_{\text{Possible outcomes}}$

$$|-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

$$P(|0\rangle) = \left|\langle 0| \left[\frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)\right]\right|^2$$

$$= \boxed{\frac{1}{2}}$$

$$P(|1\rangle) = \left|\langle 1| \left[\frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)\right]\right|^2$$

$$= \frac{(-i)}{\sqrt{2}} \times \frac{(+i)}{\sqrt{2}} = \boxed{\frac{1}{2}}$$

d) $|0\rangle$, $\underbrace{\{|+\rangle, |-\rangle\}}_{\text{Possible outcomes}}$

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$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$P(|0\rangle) = \left|\langle 0|\left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right]\right|^2$$

$$= \boxed{\frac{1}{2}}$$

$$P(|1\rangle) = \left|\langle 1|\left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right]\right|^2$$

$$= \boxed{\frac{1}{2}}$$

e) $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, $\underbrace{\{|i\rangle, |-i\rangle\}}_{\text{possible outcomes}}$

$$|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

$$P(|i\rangle) = \left|\langle i|\left[\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right]\right|^2$$

$$= \left|\left[\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)\right]\left[\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right]\right|^2$$

$$= \left|\frac{1}{2} - \frac{i}{2}\right|^2 = \left(\frac{1}{2} + \frac{i}{2}\right)\left(\frac{1}{2} - \frac{i}{2}\right)$$

$$= \frac{1}{4} - \frac{i}{4} + \frac{i}{4} + \frac{1}{4} = \boxed{\frac{1}{2}}$$

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$$P(|-i\rangle) = \langle -i| \left[\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right]$$

$$|-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

$$\Rightarrow \langle -i| = \frac{1}{\sqrt{2}}(\langle 0| + i\langle 1|)$$

$$\Rightarrow P(|-i\rangle) = \left|\left[\frac{1}{\sqrt{2}}(\langle 0| + i\langle 1|)\right]\left[\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right]\right|^2$$

$$= \left|\frac{1}{2}(1 - i)\right|^2 = \frac{1}{2}(1+i)\cdot\frac{1}{2}(1-i)$$

$$= \frac{1}{4}(1 + i - i + 1) = \frac{2}{4} = \boxed{\frac{1}{2}}$$

f) $|1\rangle$, $\underbrace{\{|i\rangle, |-i\rangle\}}_{\text{Possible outcomes}}$

$$|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

$$|-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

$$\Rightarrow |i\rangle - |-i\rangle = \frac{1}{\sqrt{2}}(2i|1\rangle) = \frac{2}{\sqrt{2}}i|1\rangle$$

$$\Rightarrow |1\rangle = \frac{1}{\sqrt{2}}(|i\rangle - |-i\rangle)$$

$$P(|i\rangle) = |\langle i|1\rangle|^2 = \left|\langle i|\frac{1}{\sqrt{2}}(|i\rangle - |-i\rangle)\right|^2$$

$$= \boxed{\frac{1}{2}}$$

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$$P(|-i\rangle) = |\langle -i|1\rangle|^2$$

$$= \left|\langle -i| \frac{1}{\sqrt{2}}(|i\rangle - |-i\rangle)\right|^2$$

$$= \boxed{\frac{1}{2}}$$

d) $|+\rangle$, $\left\{\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle, \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle\right\}$

let $\alpha = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$, $\beta = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle$ } α & β are the two possible results

$$P(\alpha) = |\langle \alpha|+\rangle|^2$$

$$= \left|\left[\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\right]\overbrace{\left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right]}^{|+\rangle}\right|^2$$

$$= \left|\frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}\right|^2 = \left(\frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}\right)\left(\frac{1}{2\sqrt{2}} + \frac{\sqrt{3}}{2\sqrt{2}}\right)$$

$$= \frac{1}{8} + \frac{\sqrt{3}}{4} + \frac{3}{8} = \frac{4 + 2\sqrt{3}}{8}$$

$$= \boxed{\frac{2 + \sqrt{3}}{4}}$$

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$$P(\beta) = |\langle \beta | + \rangle|^2$$

$$= \left| \left[\frac{\sqrt{3}}{2} \langle 0| - \frac{1}{2} \langle 1| \right] \left[\frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \right] \right|^2$$

$$= \left| \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} \right|^2$$

$$= \frac{3}{8} + \frac{1}{8} - 2\frac{\sqrt{3}}{8} = \frac{4 - 2\sqrt{3}}{8}$$

$$= \boxed{\frac{2 - \sqrt{3}}{4}}$$

2.8) Although $|1\rangle = -|1\rangle$, i.e. they are the same physical state, one cannot simply plug in $|1\rangle = -|1\rangle$ in a state which is a linear superposition of $|1\rangle$ and $|0\rangle$. This is because the negative being multiplied in a state $|\psi\rangle = -|1\rangle$ is a global phase, and hence it does not alter the outcome of any measurement that one conducts, because it will be multiplied out when computing the inner product for calculating a probability. This is different than a relative phase, which is what we have in

$\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. So

$$\underbrace{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)}_{\text{Call this } \alpha} \neq \underbrace{\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)}_{\text{Call this } \beta}$$

because you are introducing a relative phase when going from $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \rightarrow \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ between your basis states, so measuring a probability of an outcome, for example of a state γ yields different results if measured with α (i.e. $P(\gamma) = |\langle\gamma|\alpha\rangle|^2$) vs. when measured with β (i.e. $P(\beta) = |\langle\gamma|\beta\rangle|^2$), and hence α and β are not equivalent.

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