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1.C)

The following is the reasoning and explanation of Deutsch's algorithm in the context of application of each of the 4 circuits that are provided in the question.

By the Deutsch's algorithm, by measuring the input register, and applying U_f only once, we can tell whether or not $f(0) = f(1)$, namely whether the function is balanced. Another way to view Deutsch's algorithm is that it can tell us whether the function f is constant.

Recall that how this works is the following: you start with a two qubit state $|+\rangle|-\rangle$ (by applying H on the top and bottom qubits), and then apply U_f, which, acted upon an arbitrary state $|x\rangle|y\rangle$ gives $|x\rangle|f(x) \oplus y\rangle$, so when applied to $|+\rangle|-\rangle$, this gives

$$U_f(|+\rangle|-\rangle) = U_f\left(\frac{1}{2}(|0\rangle + |1\rangle)(|0\rangle - |1\rangle)\right)$$

$$= \frac{1}{2}(|0\rangle(|0 \oplus f(0)\rangle - |1 \oplus f(0)\rangle) + |1\rangle(|0 \oplus f(1)\rangle - |1 \oplus f(1)\rangle)).$$

namely,

$$U_f(|+\rangle|-\rangle) = \frac{1}{2}\sum_{x=0}^{1}|x\rangle(|0 \oplus f(x)\rangle - |1 \oplus f(x)\rangle)$$

Factoring the state $|x\rangle$ like this allows us to see the two cases for $f(x)$:

- if $f(x) = 0$, $\frac{1}{\sqrt{2}}(|0 \oplus f(x)\rangle - |1 \oplus f(x)\rangle) = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle$
- if $f(x) = 1$, $\frac{1}{\sqrt{2}}(|0 \oplus f(x)\rangle - |1 \oplus f(x)\rangle) = \frac{1}{\sqrt{2}}(|1\rangle - |0\rangle) = -|-\rangle$

And hence

$$U_f\left(\frac{1}{\sqrt{2}}\sum_{x=0}^{1}|x\rangle|-\rangle\right) = \frac{1}{\sqrt{2}}\sum_{x=0}^{1}(-1)^{f(x)}|x\rangle|-\rangle$$

Now, if $f(x)$ is a constant, then $(-1)^{f(x)} = (-1)^{\text{Const.}}$, which is a physically meaningless phase that multiplies the state, and that does not affect the outcome of the measurement. Therefore the state stays in the state $|+\rangle|-\rangle$ prior to the application of U_f. If $f(x)$ is not constant, however, $(-1)^{f(x)}$ cancels exactly one state in the superposition, so the state ends up in $|-\rangle|-\rangle$. Now if we apply H to the first qubit and measure the first qubit, we will with certainty obtain $|0\rangle$ if $f(x)$ is a constant (since $H|+\rangle = |0\rangle$), and we obtain with certainty $|1\rangle$ if $f(x)$ is not constant (since $H|-\rangle = |1\rangle$).

We apply this theory to the following circuits in Problem 1.C

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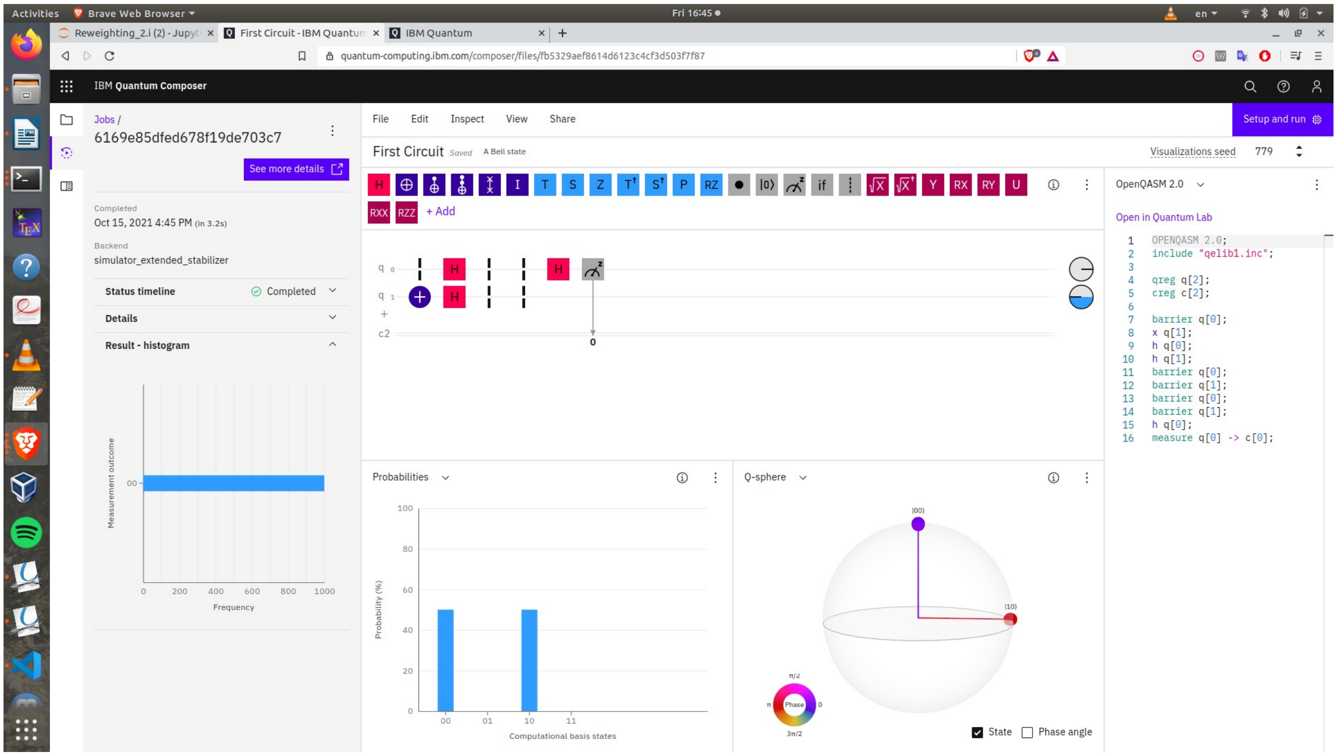
First Circuit : we expect f to be balanced since $f_0(0) = f_0(1)$, hence we expect the first qubit to be $|0\rangle$



We get result 00 with 100% certainty, which means that qubit $|q_0\rangle$ is in state $|0\rangle$ with 100% certainty, which is what we expect since it's balanced.

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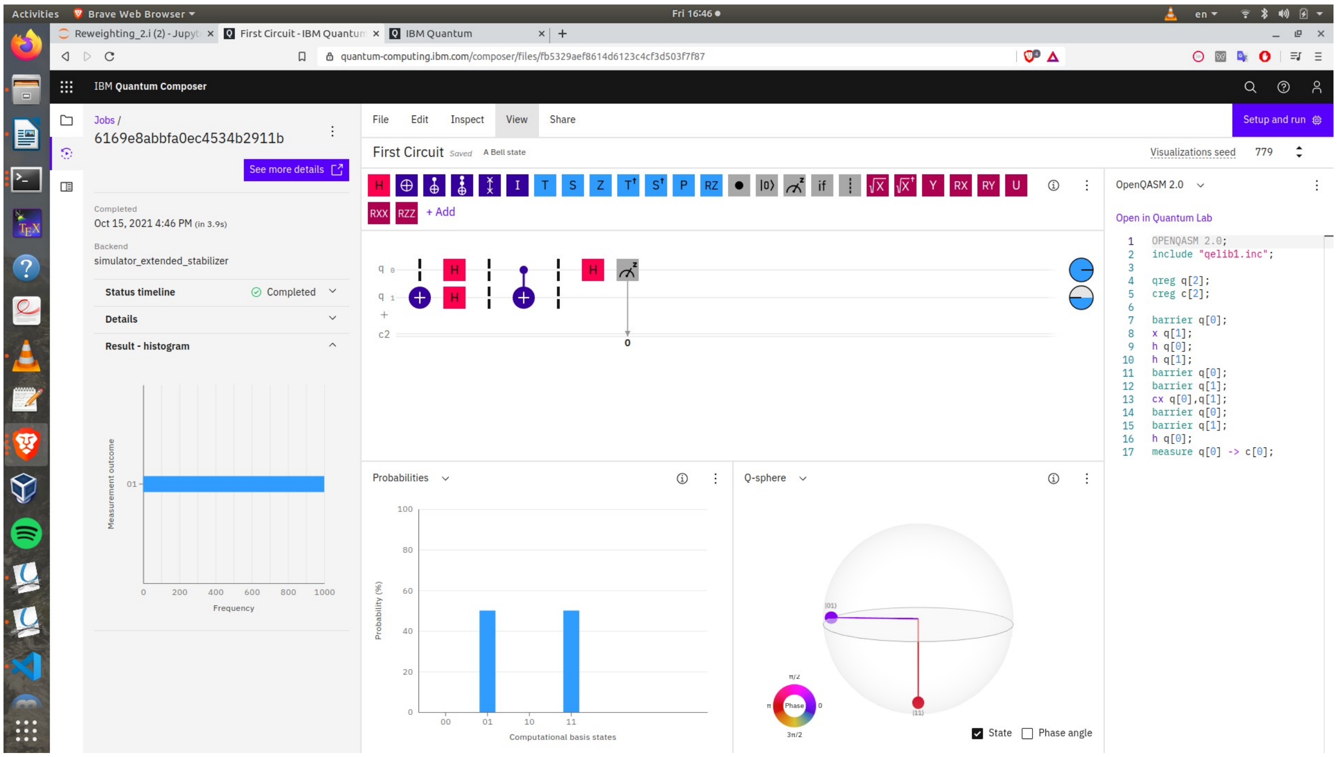
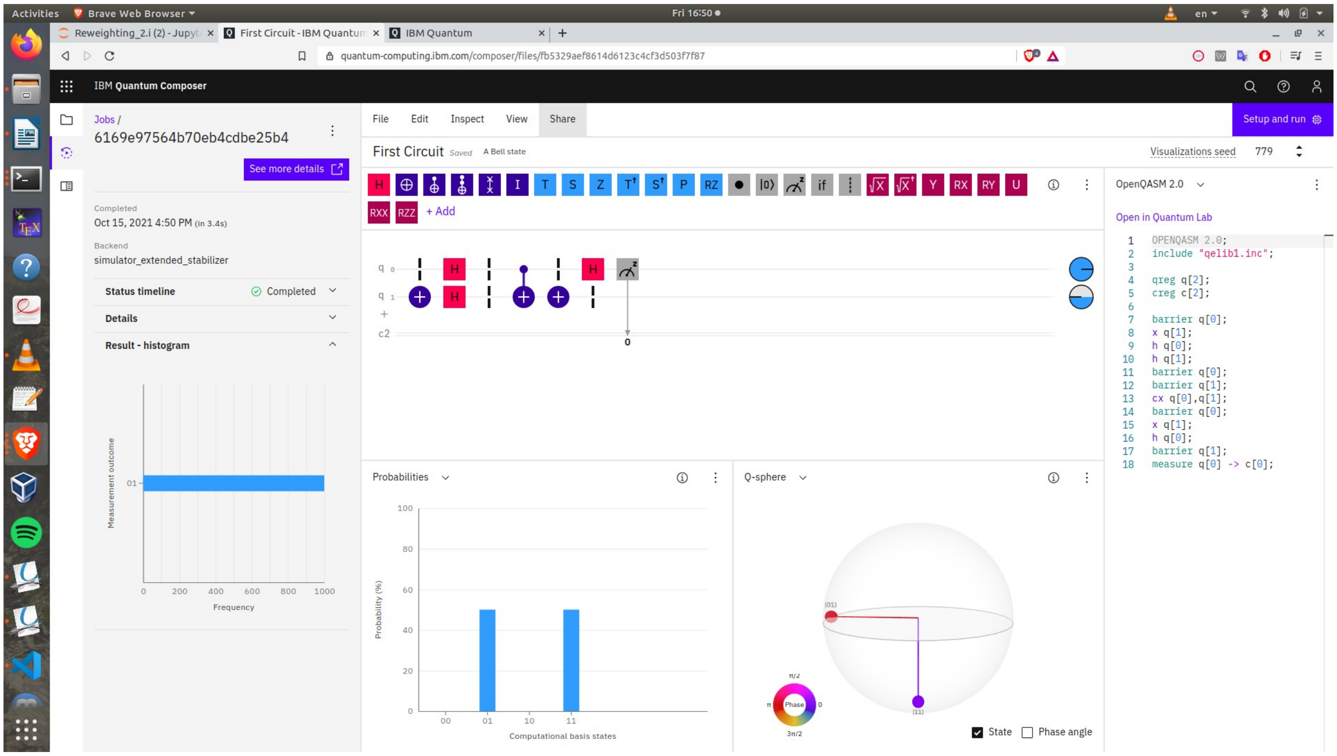
Second Circuit: **we expect f to be unbalanced since $f_1(0) \neq f_1(1)$, hence we expect the first qubit to be in state $|1\rangle$ with certainty**



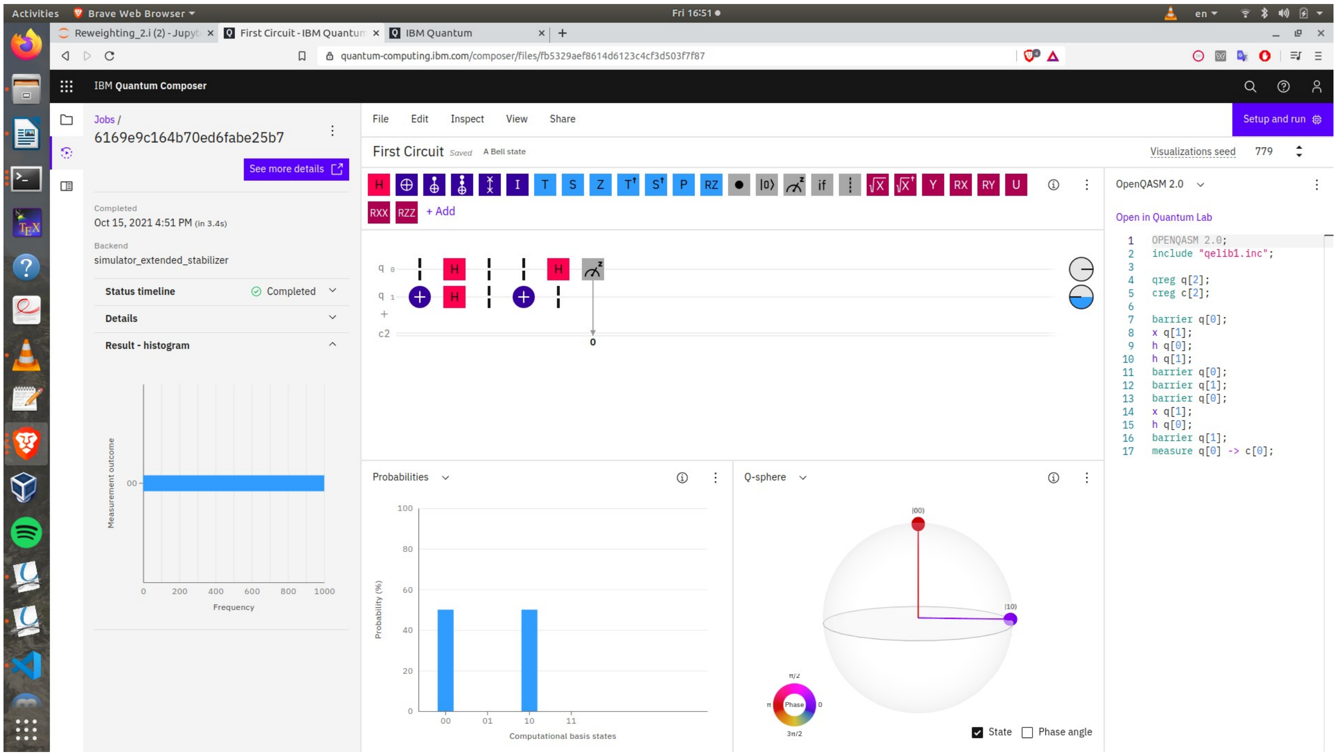
We get result 01 with 100% certainty, which means that qubit $|q_0\rangle$ is in state $|1\rangle$ with 100% certainty, which is what we expect since it's unbalanced.

Third circuit: we expect f to be unbalanced since $f_2(0) \neq f_2(1)$ hence we expect the first qubit to be in state $|1\rangle$ with 100% certainty.



The first qubit is in state $|1\rangle$ with 100% certainty, as expected.

Fourth circuit: we expect f to be balanced since $f_3(0) = f_3(1)$, hence we expect the first qubit to be $|0\rangle$.



The first qubit is in state $|0\rangle$ with 100% certainty, as expected.