HEP HW 1

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Abstract

This survey presents an overview of the advances around Tverberg's theorem, focusing on the last two decades. We discuss the topological, linear-algebraic, and combinatorial aspects of Tverberg's theorem and its applications. The survey contains several open problems and conjectures.

1 Introduction and Motivation

$W_L^{\pm}$ scattering is very sensetive to the precise mechanism of EWSB. This is because (as we will see later) *without the SM Higgs the scattering of these bosons grows steadily with the center of mass energy of the initial particles.* For large enough energies this would *violate unitarity* (limit), which is a cornerstone of the SM (this also means that the scattering amplitude has a probability larger than 1). This unitarity violation is solely due to the scattering of the longitudinal modes, specifically $W_L^{\pm}W_L^{\pm} \to W_L^{\pm}W_L^{\pm}$. For a more experimentally-driven discussion of how this kind of scattering can occur, the final states/relevant observables, etc., see Appendix. For this study, we will conclude that the theoretical study this particular process at high energies is so incredibly important, that we can derive sensetive limits on the Higgs boson mass, as was done by Quigg, Lee and Thacker in 1977. The discovery of the Higgs has radically changed our view of nature, and has made us ever more confident of the efficacy of the Standard Model.

2 Part I - Unitarity bounds and the SM Higgs

2.1 Introduction to Relevant SM Physics

We start by reviewing the gauge and fermion parts of the SM lagrangian. The SM gauge structure (without QCD, which introduces a $\times SU(3)_c\times$ term) is $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$, which comprises of weak isospin (which couples to left-handed fermions with subscript L) and hypercharge Y. Using $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$, the dyanmmics of gauge bosons are goverend by the Lagrangian

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}W^a_{\mu\nu}W^{a\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \tag{1}$$

where

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{2}$$

and

$$W^{\alpha}_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g f^{abc} W^b_\mu W^c_\nu w1 \tag{3}$$

where f^{abc} are the structure constants of the Lie Algebra associated to the non Abelian gauge symmetry Lie group, defined by the generators of the Lie algebra, which have commutation relations $\left[t^a, t^b\right] = i f^{abc} t^c$. Furthermore, for $SU(2)_L$, a, b, c run from 1 to 3 and $f^{abc} = \epsilon^{abc}$, the totally antisymmetric three-index tensor defined such that $\epsilon^{123} = 1$, so equation **??** becomes

$$W^{a}_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g \epsilon^{abc} W^b_\mu W^c_\nu \tag{4}$$

The gauge interactions are encoded in the covariant derivative

$$\mathcal{D}_\mu = \partial_\mu - i g' B_\mu Y - i g W^a_\mu T^2 \tag{5}$$

where g' is the coupling strength of the hypercharge interaction, Y is the hypercharge operator, T^a is the $SU(2)$ generator, and $T^a = \frac{\sigma^a}{2}$ where σ^a are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6}$$

	$Q_L \equiv \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	u_R	d_R	$L_L \equiv \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	e_R
Hypercharge:	1/6	2/3	−1/3	−1/2	−1
Transforms under$SU(2)_L$ as:	doublet	singlet	singlet	soublet	singlet

The Lagrangian is invariant under $U(1)_L$ and $SU(@)_L$ gauge transformations, where the $U(1)_L$ gauge transformations are

$$\mathrm{U}(1)_Y: \quad \psi \to \exp\left[i\lambda_Y(x) Y\right]\psi, \quad B_\mu \to B_\mu + \frac{1}{g'}\partial_\mu \lambda_Y(x) \tag{7}$$

and the $SU(2)_L$ gauge transformations are

$$\mathrm{SU}(2)_L: \quad \psi \to \exp\left[i\lambda^a_L(x) T^a\right]\psi, \quad W^a_\mu \to W^a_\mu + \frac{1}{g}\partial_\mu \lambda^a_L(x) + \epsilon^{abc} W^b_\mu \lambda^c_L(x) \tag{8}$$

Following the same formulation of spinor Helicity states taken by Schwarz, and using the γ matrices in the Weyl basis $\gamma^\mu_{\alpha\dot{\alpha}'} = \begin{pmatrix} 0 & \sigma^{\mu\alpha\alpha} \\ \bar{\sigma}^\mu_{\alpha\alpha} & 0 \end{pmatrix}$ [1] Note that a mass term for a gauge boson takes a form $\mathcal{L}^{\text{unbroken}}_{m_{\text{gauge boson}}} = \frac{1}{2} m_B^2 B_\mu B^\mu$ which is not gauge invariant - and hence spontaneous symmetry breaking will come to rescue such vector boson mass term later.

[1] Such that, as an example, a scattering amplitude for $e^+e^- \to \mu^+\mu^-$ is $i\mathcal{M}\left(1^-2^+3^-4^+\right) = (-ie)^2 \langle 2\gamma^\mu 1] \frac{-ig^{\mu\nu}}{s} \langle 3\gamma_\nu 4] = 2\frac{ie^2}{s}[41]\langle 23\rangle$

The SM contains three generations of chiral fermions, whose properties are given in table 2.1. Recall that the left and right handed (polarized) chiral fermion states are obtained by acting a projection operator on the unpolorized Dirac spinors:

$$\psi_R \equiv P_R \psi \tag{9}$$

$$\psi_L \equiv P_L \psi \tag{10}$$

Where the projection operators are:

$$P_R = \frac{1}{2}\left(1+\gamma^5\right), \quad P_L = \frac{1}{2}\left(1-\gamma^5\right) \tag{11}$$

We mentioned earlier that the nonzero masses of the W and Z vector bosons cannot be explained by the previous considerations and that a new ingredient has to come to explain their masses. The new ingredient is the addition of an $SU(2)_L$ doublet scalar field, which causes the spontaneous symmetry breaking of $SU(2)_L \times U(1)_Y$, commonly reffered to as the Higgs mechanism. The new scalar complex field will take the form

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} \tag{12}$$

where $\phi_1, \phi_2, \phi_3, \phi_4$ are real scalar fields. Φ will also be assigned a hypercharge $Y = \frac{1}{2}$. The Largrangian will have another term for this new ingredient

$$\mathcal{L}_\Phi = \left(\mathcal{D}_\mu \Phi\right)^\dagger \left(\mathcal{D}^\mu \Phi\right) - V(\Phi) + \mathcal{L}_{\text{Yukawa}} \tag{13}$$

wjere $\mathcal{L}_{\text{Yukawa}}$ denotes the Yukawa couplings of Φ to pairs of fermions.

The most general gauge invariant potential energy function, or scalar potential, involving Φ is

$$V(\Phi) = -\mu^2 \Phi^\dagger \Phi + \lambda \left(\Phi^\dagger \Phi\right)^2 \tag{14}$$

The case for spontaneous symmetry breaking corresponds to when $-\mu^2$ is negative and λ is positive, such that there is a degeneracy of minima (the well-known Mexican hat potential). In this case the vacuum, or minimum energy is not unvariant under $SU(2)_L \times U(1)_Y$ transformations: the gauge symmetry is spontaneously broken. We shall follow with this choice of signs, which insures the symmetry breaking.

It is trivial to see that

$$\Phi^\dagger \Phi = \frac{1}{2}\left(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2\right) \tag{15}$$

We can now minimize the potential in equation 14 to find the minimum of the field

$$\frac{\partial V}{\partial \Phi} = \frac{\partial}{\partial \Phi}(-\mu^2 \Phi^\dagger \Phi + \lambda \left(\Phi^\dagger \Phi\right)^2) = 0 \tag{16}$$

To find the minimum

$$\Phi^\dagger \Phi = \frac{\mu^2}{2\lambda} \tag{17}$$

We can now look at the scalar fields and calculate their expectation values. Consider the potential in equation 14 again

$$V = -\frac{\mu^2}{2}\left(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2\right) + \frac{\lambda}{4}\left(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2\right)^2 \tag{18}$$

We can choose the vacuum expectation values of the four fields as

$$\langle\phi_3\rangle \equiv v = \sqrt{\frac{\mu^2}{\lambda}}, \quad \langle\phi_1\rangle = \langle\phi_2\rangle = \langle\phi_4\rangle = 0 \tag{19}$$

and we can define a new scalar fields h with zero vacuum expectation value, $\langle h\rangle = 0$, according to $\phi_3 = h + v$. Then our field Φ becomes (from equation 12)

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi_1 + i\phi_2 \\ v + h + i\phi_4 \end{pmatrix} \tag{20}$$

and the potential becomes

$$V = -\frac{\mu^2}{2}\left(\phi_1^2 + \phi_2^2 + (h+v)^2 + \phi_4^2\right) + \frac{\lambda}{4}\left(\phi_1^2 + \phi_2^2 + (h+v)^2 + \phi_4^2\right)^2 \tag{21}$$

We see that ϕ_1, ϕ_2, ϕ_4 are massless, while h is massive. To learn more about the massless modes, we can write Φ in a more convenient form [2]

$$\Phi = \frac{1}{\sqrt{2}}\exp\left(\frac{i\xi^\alpha\sigma^a}{v}\right)\begin{pmatrix} 0 \\ v + h \end{pmatrix} \tag{22}$$

where h and ξ are fields σ^a are the pauli matrices where $a = 1, 2, 3$. Now consider the gauge transformations of Φ under $U(1)_Y$:

$$\mathrm{U}(1)_Y: \quad \Phi \to \exp\left(i\lambda_Y(x)\cdot\frac{1}{2}\right)\Phi \tag{23}$$

and under $SU(2)_L$:

$$\mathrm{SU}(2)_L: \quad \Phi \to \exp\left(i\lambda_L^a(x)\frac{\sigma^a}{2}\right)\Phi \tag{24}$$

If we choose $\lambda_L^a(x) = -2\xi^a/v$ then we have arrived at a gauge in which

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + h \end{pmatrix} \tag{25}$$

i.e. the degrees of freedom ξ^a have been gauged away, which is equivalent to saying the degrees of freedom of ϕ_1, ϕ_2, ϕ_3 being gauged away. Note that this gauge choice is called the unitary gauge. [3]

[2] Note that any complex number can be written as a phase and amplitude

[3] This unitary gauge is equivalent to setting $\xi = 0$ in equation 22: $\Phi = \frac{1}{\sqrt{2}}\exp\left(\frac{i\xi^\alpha\sigma^a}{v}\right)\begin{pmatrix} 0 \\ v + h \end{pmatrix} = \frac{1}{\sqrt{2}}\exp\left(\frac{i\sigma^a}{v}\right)\begin{pmatrix} 0 \\ v + h \end{pmatrix}$

Now, let's examine the kinetic term of the the gauge lagrangian $\mathcal{L} = (\mathcal{D}_\mu \Phi)^\dagger (\mathcal{D}^\mu \Phi)$. The covariant derivative here is

$$\mathcal{D}_\mu = \partial_\mu - i\frac{g'}{2}B_\mu - i\frac{g}{2}W_\mu^a \sigma^a \tag{26}$$

When acting on Φ in the unitary gauge,

$$\mathcal{D}_\mu \Phi = (\partial_\mu - i\frac{g'}{2}B_\mu - i\frac{g}{2}W_\mu^a \sigma^a \ \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v+h \end{pmatrix} \tag{27}$$

)

$$\mathcal{D}_\mu \Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} -\frac{i}{2} g \left(W_\mu^1 - iW_\mu^2\right)(v+h) \\ \partial_\mu h + \frac{i}{2}\left(gW_\mu^3 - g' B_\mu\right)(v+h) \end{pmatrix} \tag{28}$$

$$(\mathcal{D}_\mu \Phi)^\dagger = \frac{1}{\sqrt{2}} \left(\frac{i}{2} g \left(W_\mu^1 + iW_\mu^2\right)(v+h) \ , \ \partial_\mu h - \frac{i}{2}\left(gW_\mu^3 - g' B_\mu\right)(v+h) \right) \tag{29}$$

So that

$$(\mathcal{D}_\mu \Phi)^\dagger (\mathcal{D}^\mu \Phi) = \frac{1}{2}(\partial_\mu h)(\partial^\mu h) + \frac{1}{8} g^2 (v+h)^2 \left(W_\mu^1 - iW_\mu^2\right)\left(W^{1\mu} + iW^{2\mu}\right) + \frac{1}{8}(v+h)^2\left(-g' B_\mu + gW_\mu^3\right)^2 \tag{30}$$

The first term is clearly the kinetic term of a real scalar field h. Now defining the charged W bosons

$$W_\mu^+ = \frac{W_\mu^1 - iW_\mu^2}{\sqrt{2}}, \quad W_\mu^- = \frac{W_\mu^1 + iW_\mu^2}{\sqrt{2}} \tag{31}$$

And plugging these into equation 30, the second term in equation 30 becomes

$$\begin{aligned} \mathcal{L}_{\text{second term}} &= \frac{1}{8} g^2 (v+h)^2 \left(W_\mu^1 - iW_\mu^2\right)\left(W^{1\mu} + iW^{2\mu}\right) \\ &= \frac{1}{4} g^2 (v+h)^2 W_\mu^+ W^{-\mu} \\ &= \frac{g^2 v^2}{4} W_\mu^+ W^{-\mu} + \frac{g^2 v}{2} h W_\mu^+ W^{-\mu} + \frac{g^2}{4} hh W_\mu^+ W^{-\mu} \end{aligned} \tag{32}$$

Writing in this form we can clearly read off the Feynamn rules (the first term is clearly a W mass term, and the other two are interactions with the Higgs). Consider the third term in 30. Let's write the $\left(gW_\mu^3 - g' B_\mu\right)$ term as a properly normalized real field

$$\begin{aligned} \left(gW_\mu^3 - g' B_\mu\right) &= \sqrt{g^2 + g'^2} \left(\frac{g}{\sqrt{g^2 + g'^2}} W_\mu^3 - \frac{g'}{\sqrt{g^2 + g'^2}} B_\mu \right) \\ &\equiv \sqrt{g^2 + g'^2} \left(cos_w W_\mu^3 - sin_w B_\mu\right) \\ &\equiv \sqrt{g^2 + g'^2} Z_\mu \end{aligned} \tag{33}$$

where θ_W is the weak mixing angle, $\tan\theta_w = \frac{g'}{g}$. Now note that the orthogonal state to the term above

$$(\sqrt{g^2+g'^2}Z_\mu)^{\text{orthog.}} = \left(\sin_W W_\mu^3 + \cos_W B_\mu\right) \equiv A_\mu \tag{34}$$

is the photon, which aquire mass through the Higgs mechanism. The third term in equation 30 becomes

$$\begin{aligned}\mathcal{L}_{\text{third term}} &= \frac{1}{8}(v+h)^2\left(-g'B_\mu + gW_\mu^3\right)^2 \\ &= \frac{1}{8}\left(g^2+g'^2\right)(v+h)^2 Z_\mu Z^\mu \\ &= \frac{\left(g^2+g'^2\right)v^2}{8}Z_\mu Z^\mu + \frac{\left(g^2+g'^2\right)v}{4}hZ_\mu Z^\mu + \frac{\left(g^2+g'^2\right)}{8}hhZ_\mu Z^\mu\end{aligned} \tag{35}$$

The first term in 35 above is clearly the Z boson mass term, and the rest are interactions with the Higgs.

2.2 The Electroweak Lagrandian and Feynaman Rules

In order to calculate this scattering amplitude we must find the weak vector boson charges. By the definitions of Z_μ and A_μ above, i.e.

$$Z_\mu \equiv \cos\theta_w W_\mu^3 - \sin\theta_w B_\mu,\ A_\mu \equiv \sin\theta_w W_\mu^3 + \cos\theta_w B_{ji} \tag{36}$$

gives

$$B_\mu = \cos\theta_{w_\mu} A_\mu - \sin\theta_w Z_\mu,\ W_\mu^3 = \sin\theta_w A_\mu + \cos\theta_w Z_\mu \tag{37}$$

Now using $T^\pm$ as the raising and lowering operators of $SU(2)_L$, with $T^\pm = \sigma^\pm$ in the doublet representation, we have

$$W^1T^1 + W^2T^2 = \frac{1}{\sqrt{2}}\left(W^+T^+ + W^-T^-\right), \tag{38}$$

we can now rewrite the covariant derivative in equation 5 in our new basis of electroweak gauge bosons

$$\begin{aligned}\mathcal{D}_\mu &= \partial_\mu - ig'B_\mu Y - igW_\mu^a T^a \\ &= \partial_\mu - ig'(\cos\theta_{w_\mu}A_\mu - \sin\theta_w Z_\mu)Y - ig(W_\mu^1T^1 + W_\mu^2T^2) - ig(\sin\theta_w A_\mu + \cos\theta_w Z_\mu)T^3 \\ &= \partial_\mu - ig'(\cos\theta_{w_\mu}A_\mu - \sin\theta_w Z_\mu)Y - ig(\frac{1}{\sqrt{2}}\left(W^+T^+ + W^-T^-\right)) - ig(\sin\theta_w A_\mu + \cos\theta_w Z_\mu)T^3 \\ &= \partial_\mu - i\frac{g}{\sqrt{2}}\left(W_\mu^+T^+ + W_\mu^-T^-\right) - iZ_\mu\left(g\cos_W T^3 - g'\sin_W Y\right) - iA_\mu\left(g\sin_W T^3 + g'\cos_W Y\right)\end{aligned} \tag{39}$$

Here we aim to study the $2 \to 2$ scattering amplitude of $W_L^+ W_L^- \to W_L^+ W_L^-$ in the SM without the Higgs. The covariant derivative is

$$D_\mu H = \partial_\mu H - igW_\mu^a \tau^a H - \frac{1}{2} ig' B_\mu H \tag{40}$$

Where g is the $SU(2)$ coupling and g' is the $U(1)_Y$ coupling.

The final form of the Lagrangian in our doublet basis is

$$\begin{aligned} \mathcal{L}_{\text{gauge}} &= -\tfrac{1}{4} F_{j\nu}^2 - \tfrac{1}{4} Z_{\mu\nu}^2 + \tfrac{1}{2} m_Z^2 Z^\mu Z_\mu - \tfrac{1}{2} \left(\partial_\mu W_\nu^+ - \partial_\nu W_\mu^+ \right) \left(\partial_\mu W_\nu^- - \partial_\nu W_\mu^- \right) \\ &+m_W^2 W_\mu^+ W_\mu^- - ie \cot\theta_w \left[\partial_\mu Z_\nu \left(W_\mu^+ W_\nu^- - W_\nu^+ M_\mu^{r-} \right) \right. \\ &+Z_\nu \left(-W_\mu^+ \partial_\nu W_\mu^- + WV_\mu^- \partial_\nu W_\mu^+ + W_\mu^+ \partial_\mu W_\nu^- - W_\mu^- \partial_\mu W_\nu^+ \right) \big] \\ &-ie \left[\partial_\mu A_\nu \left(W_\mu^+ W_\nu^- - W_\nu^+ W_\mu^- \right) \right. \\ &+A_\nu \left(-W_\mu^+ \partial_\nu W_\mu^- + W_\mu^- \partial_\nu W_\mu^+ + W_\mu^+ \partial_\mu W^+ W_\nu^- - W_\mu^- \partial_\mu W_\nu^+ \right) \big] \\ &-\tfrac{1}{2} \tfrac{e^2}{\sin^2\theta_w} W_\mu^+ W_\mu^- W_v^+ W_\nu^- + \tfrac{1}{2} \tfrac{e^2}{\sin^2\theta_w} W_\mu^+ W_\nu^- W_\mu^+ W_\nu^- \\ &-e^2 \cot^2\theta_w \left(Z_\mu W_\mu^+ Z_\nu W_\nu^- - Z_\mu Z_\mu W_\nu^+ W_\nu^- \right) + e^2 \left(A_\mu W_\mu^+ A_\nu W_\nu^- - A_\mu A_\mu W_\nu^+ W_\nu^- \right\} \\ &+e^2 \cot\theta_w \left[A_\mu W_\mu^+ W_\nu^- Z_\nu + A_\mu W_\mu^- Z_\nu W_\nu^+ - W_\mu^+ W_\mu^- A_\nu Z_\nu \right]_\nu \end{aligned} \tag{41}$$

Now one can read straight off for the Feynman rules.

The Feynman diagrams contributing to our process are the s, t, and 4-vertex diaggrams as in figure. We can read off the vertex factors from 41 for the different couplings. The Relevant Feynman Rules for EW are:

- For propagators, see Figure 1
- For EW triple gauge interactions, see Figure 2
- For EW quartic gauge interactions, see Figure 3

Since the vector bosons are longitudinally polarized, we must take into account the polarization vectors $\epsilon_\mu^i(p)$, where $i = 1, 2, 3$. Recall that the polarization vectors must satisfy $\nu_j \epsilon_{jt}^i(p) = 0$ for any p^μ, and they are normalized as

$$\epsilon_\mu^* \epsilon_\mu = -1 \tag{42}$$

For massless vector bosons, it is convensional to take p^μ in the z-direction, so

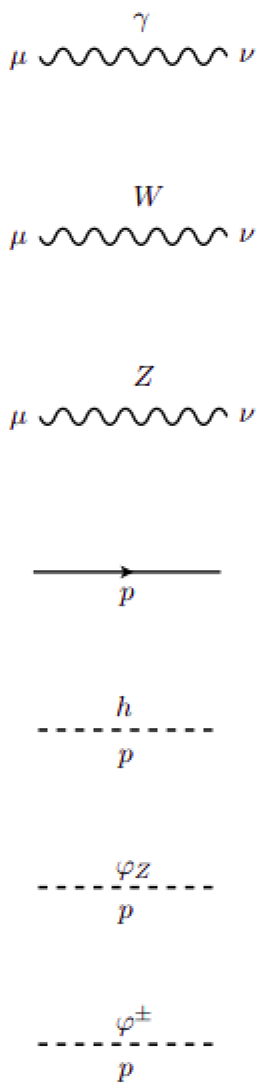
$$p^\mu = (E, 0, 0, p_z), \quad E^2 - p_z^2 = m^2 \tag{43}$$

Such that the two vectors satisfying $p_\mu \epsilon_\mu = 0$ and $\epsilon_\mu^2 = -1$ are the transverse polarizations:

$$\epsilon_\mu^1 = (0, 1, 0, 0), \quad \epsilon_\mu^2 = (0, 0, 1, 0) \tag{44}$$

and the longitudinal polarization

$$\epsilon_\mu^L = \left(\frac{p_z}{m}, 0, 0, \frac{E}{m} \right) \tag{45}$$



$$-i\left[\frac{g_{\mu\nu}}{k^2+i\epsilon}-(1-\xi)\frac{k_\mu k_\nu}{(k^2)^2}\right]$$

$$\frac{-ig_{\mu\nu}}{k^2-M_W^2+i\epsilon}$$

$$\frac{-ig_{\mu\nu}}{k^2-M_Z^2+i\epsilon}$$

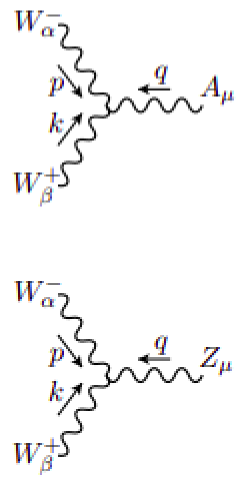
$$\frac{i(\not{p}+m_f)}{p^2-m_f^2+i\epsilon}$$

$$\frac{i}{p^2-M_h^2+i\epsilon}$$

$$\frac{i}{p^2-\xi m_Z^2+i\epsilon}$$

$$\frac{i}{p^2-\xi m_W^2+i\epsilon}$$

Figure 1: Feynman rules for electroweak propagators



$$-ie\left[g_{\alpha\beta}(p-k)_\mu + g_{\beta\mu}(k-q)_\alpha + g_{\mu\alpha}(q-p)_\beta\right]$$

$$ig\cos\theta_W\left[g_{\alpha\beta}(p-k)_\mu + g_{\beta\mu}(k-q)_\alpha + g_{\mu\alpha}(q-p)_\beta\right]$$

Figure 2: Feynman rules for electroweak triple gauge interactions

However, at high energy $E = \sqrt{p_z^2 + m^2} \to p_z$ so $\epsilon_L^\mu \to \frac{1}{m}p^\mu$, but this choice of ϵ_L^μ violates $\epsilon_L \cdot p = 0$, therefore at high energy we need to work in the subleading order, where

$$\varepsilon_L^\mu = \frac{p^\mu}{m_w} + \mathcal{O}\left(\frac{m_w}{E}\right) \tag{46}$$

The precise polarization vectors can then be written as

$$\begin{aligned}
\epsilon_1^\mu &= \frac{1}{m_W}p_1^\mu + \frac{2m_W}{u-2m_W^2}p_4^\mu \\
\epsilon_2^\mu &= \frac{1}{m_Z}p_2^\mu + \frac{2m_Z}{u-2m_Z^2}p_3^\mu \\
\epsilon_3^\mu &= \frac{1}{m_W}p_3^\mu + \frac{2m_W}{u-2m_W^2}p_2^\mu \\
\epsilon_4^\mu &= \frac{1}{m_Z}p_4^\mu + \frac{2m_Z}{u-2m_Z^2}p_1^\mu
\end{aligned} \tag{47}$$

Let us take our 4-momentum to point in the z direction, $p^\mu = (E, 0, 0, p_z)$, so that $E = \sqrt{p_z^2 + m^2}$, then we have two transverse polarization states

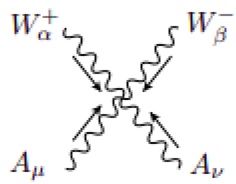
$$\epsilon_{T1}^\mu = (0,1,0,0),\ \ \epsilon_{T2}^\mu = (0,0,1,0), \quad \text{transverse polarization states} \tag{48}$$

and one longitudinal polarization state

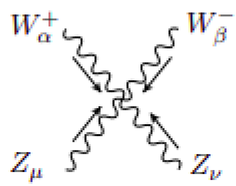
$$\epsilon_L^\mu = \frac{1}{m}\left(p_z, 0, 0, E\right), \quad \text{longitudinal polarization state} \tag{49}$$

These satisfy

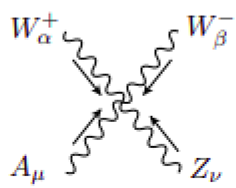
$$\epsilon_i \cdot p = 0 \tag{50}$$



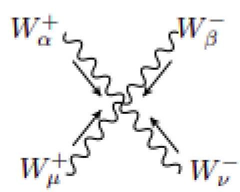
$$-ie^2\left[2g_{\alpha\beta}g_{\mu\mu} - g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right]$$



$$-ig^2\ \cos^2\theta_W\left[2g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right]$$



$$ieg\cos\theta_W\left[2g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right]$$



$$ig^2\left[2g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\nu}g_{\beta\mu}\right]$$

Figure 3: Feynman rules for electroweak quartic interactions

and $\epsilon_\mu^*\epsilon_\mu = -1$ so that

$$\epsilon_i \cdot \epsilon_j^\star = -\delta_{ij} \tag{51}$$

At high energies, $E >> m$ so $E = \sqrt{p_z^2 + m^2} \to p_z$, therefore equation 49 becomes

$$\epsilon_L^\mu = \frac{1}{m}\left(p_z, 0, 0, E\right) \ \to \frac{1}{m} p^\mu \tag{52}$$

However the diagrams for $W_L^- W_L^+ \to W_L^- W_L^+$ scattering are n figure: through the s- channel, where there is an exchange of a phton γ or a Z boson, the t-channel, where there is an exchange of a phton γ or a Z boson, the 4- point vertex (4 external W_L legs). [4] We will later see that if we add diagrams with Higgs exchanges in the s- and t-channels, we will be rescued from the high energy divergences. Suppose we have the s-channel, where upper left, lower left, upper right, lower right momenta are p_1, p_2, p_3, p_4 respectively, and the propagator momentum is k. so the first vertext will be $V(p_1, p_2, k, \mu_1, \mu_2, \lambda)$ and the second vertex is $V(p_3, p_4, k, \mu_3, \mu_4, \lambda)$, and these will be multiplied by their polarization vectors $\epsilon^L_{\mu_1}(p_1)$ ...$\epsilon^L_{\mu_4}(p_4)$. The calculation can be simplified using the Mandelstan variables. These are defined as st, u and will be used in our calculations a lot. These are defined for incoming momenta p_1, p_2 and outgoing momenta p_3, p_4 as

$$\begin{aligned} s &\equiv (p_1 + p_2)^2 = (p_3 + p_4)^2 \\ t &\equiv (p_1 - p_3)^2 = (p_2 - p_4)^2 \\ u &\equiv (p_1 - p_4)^2 = (p_2 - p_3)^2 \end{aligned} \tag{53}$$

And they satisfy

$$s + t + u = \sum_i m_i^2 \tag{54}$$

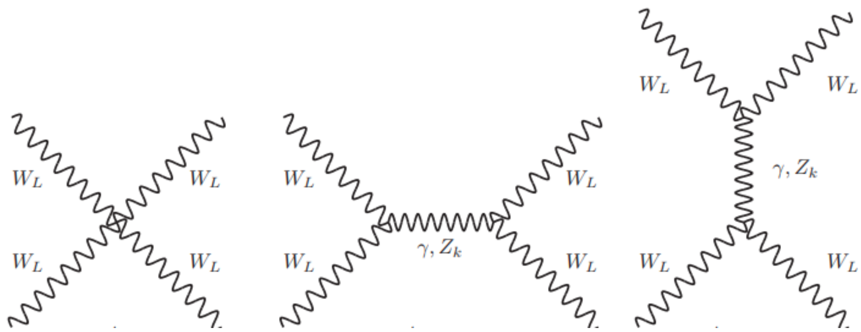
Where m_i are the invariant masses of the particles. Therefore in our case,

$$s + t + u = 4M_W^2 \tag{55}$$

and conservation of energy, for example $p_4 = -p_1 - p_2 - p_3$.

[4] This is similar to $W_L^+ Z_L \to W_L^+ Z_L$ scattering as shown in the Schwartz book. The difference is that have s, t channels and the 4-point vertex, whereas the $W_L^+ Z_L \to W_L^+ Z_L$ has to s and u diagrams, and it only involves the exchange of W in the propagators, whereas we have the exchanges of γ or Z.

Figure 4: $W_L^- W_L^+ \to W_L^- W_L^+$ diagrams without the Higgs (by exchange of a z or γ) and the 4-point vertex



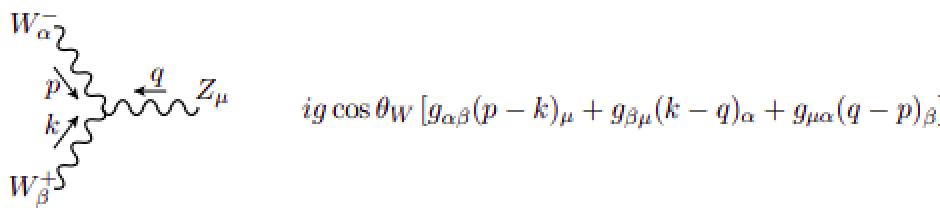


Figure 5: Feynman rule for WWZ vertex

3 $W_L^- W_L^+ \to W_L^- W_L^+$ without the Higgs

3.1 $\mathbf{M}_s^{\gamma/Z}$

For the s-channel scattering amplitude we have the exchange of either a γ or a Z boson. We have two vertices and a propagator in our amplitude: a WWZ, ZWW vertices and either a γ or a Z propagator.. Let's rewrite the Feynman rule for WWZ vertex, as shown in Figure 5

Now using

$$e = g\sin\theta_W = g'\cos\theta_W \tag{56}$$

We have that $g = e\frac{m_W}{\sin\theta_W}$. Plugging this definition of g in the Feynman rule in Figure 5, we get the Feynamn rule

$$V_{WWZ} = ie\cot\theta_W\,[g_{\alpha\beta}(p-k)_\mu + g_{\beta\mu}(k-q)_\alpha + g_{\mu\alpha}(q-p)_\beta] \tag{57}$$

On the other end (the ZWW vertex for the outgoing particles) we have a the same Feynamn rule, just with different exchanged momenta.

Note also that the relations

$$g_{WW\gamma} = e,\ g_{WWZ} = e\cot(\theta_W),\ m_Z = \frac{m_W}{\cos(\theta)},\ g = e\frac{m_W}{\sin(\theta_W)} \tag{58}$$

Where g_{VVV} is the coupling constant for vectors VVV and g is the electroweak coupling constant, will be useful in simplifying our terms, and have been used in my Form code.

We use equation 57 with renaming the incoming and outgoing momenta just to have the same expression as that in Schwartz. Hence for the first vertex (WWZ for the incoming particles, by the exchange of a Z boson) we have

$$V^{(Z)}_{WWZ} = -ie\cot\theta_w \left[g^{\mu\nu}\left(p_1 - p_2\right)^{\lambda} + g^{\nu\lambda}\left(p_2 - k\right)^{\mu} + g^{\lambda\mu}\left(k - p_1\right)^{\nu} \right] \tag{59}$$

where k is the momentum of the exchanged Z/γ boson (p_3 in Schwartz),

such that **??** is the same as equation (29.12) in Schwartz. In a similar fashion, we have a similar expression for the WZW vertex for the final states, with exchanging the momenta $p_1 \to p_3, p_2 \to p_4$.

$$V^{(Z)}_{ZWW} = -ie\cot\theta_w \left[g^{\mu\nu}\left(p_3 - p_4\right)^{\lambda} + g^{\nu\lambda}\left(p_2 - p_3\right)^{\mu} + g^{\lambda\mu}\left(p_3 - p_1\right)^{\nu} \right] \tag{60}$$

If we have the same vertex as above, but we have a photon propagatos, we have the same vertex except the differing photon coupling

$$V^{(\gamma)}_{ZWW} = -ie\cot\theta_w \left[g^{\mu\nu}\left(p_3 - p_4\right)^{\lambda} + g^{\nu\lambda}\left(p_2 - p_3\right)^{\mu} + g^{\lambda\mu}\left(p_3 - p_1\right)^{\nu} \right] \tag{61}$$

We also have the Z boson propagator, whose Feynman rule is listed in Figure 1,

$$\begin{aligned} \Delta^{\mu\nu}_{Z} = \frac{-i}{k^2 - m_Z^2 + i\epsilon}\left[g_{\mu\nu} + (\alpha - 1)\frac{k_\mu k_\nu}{k^2 - \alpha m_Z^2} \right] &\overset{\text{Unitary}}{=} \frac{-i}{k^2 - m_Z^2 + i\epsilon}\left[g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2 - m_Z^2} \right] \\ &= \frac{-i}{k^2 - m_Z^2}\left(g_{\mu\nu} + \frac{k_\mu k_\nu}{m_Z^2} \right) \end{aligned} \tag{62}$$

For a Z boson going from indeces $\mu \to \nu$ with exchanged momentum k. Here we have used the unitarity gauge ($\alpha \to \infty$). Similarly, we have the photon propagator

$$\Delta^{A}_{\mu\nu} = \frac{-i}{k^2 + i\epsilon}\left[g_{\mu\nu} + (\alpha - 1)\frac{k_\mu k_\nu}{k^2} \right] \overset{\text{Unitary}}{=} \frac{-ig_{\mu\nu}}{k^2} \tag{63}$$

There is also going to be a 4-point diagram for $W_L^+ W_L^- \to W_L^+ W_L^-$ with the Feynamn rule in Fugure (). Rewriting this in the Schwartz convention, where W^- is taken to be the particle (with momentum going towards the vertex) and W^+ is taken to be its antiparticle (with momentum going away from the vertex). Then for $W_\alpha^+ W_\beta^- \to W_\mu^+ W_\nu^-$ we have the 4-point vertex

$$V_4 = i\cot^2\theta_W \left(g_{\alpha\mu} g_{\beta\nu} + g_{\alpha\nu} g_{\nu\beta} - 2 g_{\alpha\beta} g_{\mu\nu} \right) \tag{64}$$

Hence the amplitude will have two vertex factors sch as **??** and the Z boson propagator. Now inserting the polarizarion vectors in our process of $\epsilon_1(\mu_1)\epsilon_2(\mu_2) \to \epsilon_3(\mu_3)\epsilon_4(\mu_4)$, the final expression for the matrix element of $W_L W_L \to W_L W_L$ by exchange of a Z boson will be

$$-i\mathcal{M}_s^{(Z)} = \underbrace{\left[-ie\cot\theta_w\right]^2}_{\text{coupling}} \times \underbrace{\epsilon_1(\mu_1)\epsilon_2(\mu_2)\epsilon_3^*(\mu_3)\epsilon_4^*(\mu_4)}_{\text{Polarization vectors}} \times \left[\underbrace{g^{\mu\nu}\left(p_1-p_2\right)^\lambda + g^{\nu\lambda}\left(p_2-p_3\right)^\mu + g^{\lambda\mu}\left(p_3-p_1\right)^\nu}_{V_{WWZ}}\right] \times \underbrace{\frac{-i}{k^2-m_Z^2}\left(g_{\mu\nu}+\frac{k_\mu k_\nu}{m_Z^2}\right)}_{\text{Z propagator}} \times \underbrace{\left[g^{\mu\nu}\left(p_3-p_4\right)^\lambda + g^{\nu\lambda}\left(p_2-p_3\right)^\mu + g^{\lambda\mu}\left(p_3-p_1\right)^\nu\right]}_{V_{ZWW}} \tag{65}$$

All the other amplitudes follow the same structure. [5] The Form code to calculate this amplitude is attached (as ww-ww$_n oHiggs.frm).Pleasenotethatthefollowingsimplificationsweremade(fortheZpropagatordic$

$$p_1p_1 = m_w^2,\ p_2p_2 = m_w^2,\ p_3p_3 = m_z^2 \tag{67}$$

$$s = (p_1+p_2)^2 = p_1^2+p_2^2+2p_1p_2 \implies p_1p_2 = \frac{s-p_1^2-p_2^2}{2} = \frac{s-2m_w^2}{2} \tag{68}$$

$$t = (p_1-p_3)^2 = p_1^2+p_3^2-2p_1p_3 \implies p_1p_3 = -\left(\frac{t-p_1^2-p_3^2}{2}\right) = -\left(\frac{t-2m_w^2}{2}\right) \tag{69}$$

$$u = (p_2-p_3)^2 = p_2^2+p_3^2-2p_2p_3 \implies p_2p_3 = -\left(\frac{u-p_2^2-p_3^2}{2}\right) = -\left(\frac{u-2m_w^2}{2}\right) \tag{70}$$

$$s+t+u = \sum_i m_i^2 = 4m_w^2 \implies u = -s-t+4m_w^2 \tag{71}$$

We further make simplifications by keeping only leading terms in the high-energy limit ($s >> m_W^2$). To achieve that, we rescale all invariants by a suitable energy factor E^n and then drop all terms proportional to negative powers of E. This requires to introduce different symbols for the rescaled invariants. Since the invariants s,t,u scale as E^2, then for any power of n we can rescale the invariants accordingly

$$\{s^n,t^n,u^n\} \to \{s^n,t^n,u^n\} \times E^{2n} \tag{72}$$

Lastly, we make the high energy approxamations in our final expressions as

$$t = -s-u+4m_W^2 \tag{73}$$

[5] If we were to expand the expression for the amplitude above, the s-channel will have scattering amplitude

$$\begin{aligned} i\mathcal{M}_s = &-ig^2\left(\frac{\sin^2(\theta_w)}{(p_1+p_2)^2}+\frac{\cos^2(\theta_w)}{(p_1+p_2)^2-m_Z^2}\right)\left[(p_1-p_2)^\mu\,\epsilon(p_1)\cdot\epsilon(p_2)+2p_2\cdot\epsilon(p_1)\,\epsilon^\mu(p_2)-2p_1\cdot\epsilon(p_2)\,\epsilon^\mu(p_1)\right] \\ &\times\left[(p_4-p_3)_\mu\,\epsilon(p_3)\cdot\epsilon(p_4)-2p_4\cdot\epsilon(p_3)\,\epsilon_\mu(p_4)+2p_3\cdot\epsilon(p_4)\,\epsilon_\mu(p_3)\right] \end{aligned} \tag{66}$$

$$\frac{1}{t}=-\frac{1}{s+u}\left[1+\frac{4m_W^2}{E^2}\right](s+u) \tag{74}$$

For the s channel, we have the exchanged momentum

$$k_s=p_1+p_2=p_3+p_4 \tag{75}$$

The final expression (from Form) for $\mathcal{M}_s^{(Z/\gamma)}$ is $\mathcal{M}_s=\mathcal{M}_s^Z+\mathcal{M}_s^\gamma$

$$\begin{aligned}
\mathcal{M}_s = & \\
& + gwwg^2 * mw^-4 * (-1/2 * s * t - 1/4 * s^2) \\
& + \mathrm{gwwg}^2 * mw^-2 * (2 * t^2 * [s+t]^-1 + s + 3 * s * t * [s+t]^-1 - s^2 * [s+t]^-1) \\
& + \mathrm{gwwz}^2 * mw^-4 * (-1/2 * s * t - 1/4 * s^2) \\
& + \mathrm{gwwz}^2 * mw^-4 * mz^2 * (-1/2 * t - 1/4 * s) \\
& + \mathrm{gwwz}^2 * mw^-2 * (2 * t^2 * [s+t]^-1 + s + 3 * s * t * [s+t]^-1 - s^2 * [s+t]^-1)
\end{aligned} \tag{76}$$

3.2 $\mathbf{M}_t^{\gamma/Z}$

For the t channel we have

$$k_t=p_1-p_3=p_4-p_3 \tag{77}$$

Along with the s-channel diagram, there is also a t-channel amplitude with an exchange of either a photon or a Z boson. Hence we use the same expressions ans simplifications as above, except the propagators and vertex factors will be those associated with k_t as opposed to k_s (as well as the differing signs of momenta). [6] The final expression from form for $\mathcal{M}_t$ is $\mathcal{M}_t=\mathcal{M}_s^Z+\mathcal{M}_t^\gamma$

[6] If we were to expand the above amplitude We find the scattering amplitude for the t-channel:

$$\begin{aligned}
i\mathcal{M}_t = & -ig^2\left(\frac{\sin^2(\theta_w)}{(p_1-p_3)^2}+\frac{\cos^2(\theta_w)}{(p_1-p_3)^2-m_Z^2}\right)\left[(p_1+p_3)^\mu\,\epsilon(p_1)\cdot\epsilon(p_3)-2p_3\cdot\epsilon(p_1)\,\epsilon^\mu(p_3)-2p_1\cdot\epsilon(p_3)\,\epsilon^\mu(p_1)\right] \\
& \times\left[(p_2+p_4)_\mu\,\epsilon(p_2)\cdot\epsilon(p_4)-2p_4\cdot\epsilon(p_2)\,\epsilon_\mu(p_4)-2p_2\cdot\epsilon(p_4)\,\epsilon_\mu(p_2)\right]
\end{aligned} \tag{78}$$

$$
\begin{aligned}
Mt = & \quad +gwwg^2*mw^-4*(-1/4*t^2 \\
&+gwwg^2*mw^-2*(t-t^2*[s+t]^-1+3*s*t*[s+t]^-1+2*s^2*[s+t]^-1) \\
&+gwwz^2*mw^-4*(-1/4*t^2-1/2*s*t) \\
&+gwwz^2*mw^-4*mz^2*(-1/4*t-1/2*s) \\
&+gwwz^2*mw^-2*(t-t^2*[s+t]^-1+3*s*t*[s+t]^-1+2*s^2*[s+t]^-1)
\end{aligned}
\tag{79}
$$

3.3 $\mathbf{M}_4$

[7]

$$
\begin{aligned}
M4 = & \\
&+gwwww*mw^-4*(1/4*t^2+s*t+1/4*s^2) \\
& \\
&+\mathrm{gwwww*mw}^-2*(-t-s-4*s*t*[s+t]^-1);
\end{aligned}
\tag{81}
$$

[7] If we were to expand the expression for the amplitude, the 4-vertex scattering amplitude:

$$
i\mathcal{M}_4 = ig^2\left[2\epsilon\left(p_2\right)\cdot\epsilon\left(p_3\right)\epsilon\left(p_1\right)\cdot\epsilon\left(p_4\right)-\epsilon\left(p_2\right)\cdot\epsilon\left(p_1\right)\epsilon\left(p_3\right)\cdot\epsilon\left(p_4\right)-\epsilon\left(p_2\right)\cdot\epsilon\left(p_4\right)\epsilon\left(p_1\right)\cdot\epsilon\left(p_3\right)\right] \tag{80}
$$

3.4 Total Amplitude

Now, the form code has substituted all of our definitions of the Mandelstam variables, defined above based on

$$s = (p_1 + p_2)^2 \,, t = (p_1 - p_3)^2 \tag{82}$$

And it has substituted all the propagator/vertex definitions and the polarization vectorg. Doing a little more algebra to simplify the final expressions of the amplitudes, so that we have $\frac{g^2}{4m_W^2}$ factored out, we have

$$i\mathcal{M}_t^{(Z/\gamma)} = -i\frac{g^2}{4m_W^4}\left[(s-u)t - 3m_W^2(s-u) + \frac{8m_W^2}{s}u^2\right] \tag{83}$$

$$i\mathcal{M}_s^{(Z/\gamma)} = -i\frac{g^2}{4m_W^4}\left[s(t-u) - 3m_W^2(t-u)\right] \tag{84}$$

$$i\mathcal{M}_4 = i\frac{g^2}{4m_W^4}\left[s^2 + 4st + t^2 - 4m_w^2(s+t) - \frac{8m_W^2}{s}ut\right] \tag{85}$$

Thus the total total scattering amplitude, assuming no Higgs boson, is the sum of these three amplitudes

$$\begin{aligned} i\mathcal{M}_{\text{Total}}^{\text{No Higgs}}(W_L^- W_L^+ \to W_L^- W_L^+) &= i\mathcal{M}_t^{(Z/\gamma)} + i\mathcal{M}_s^{(Z/\gamma)} + i\mathcal{M}_4 \\ &= -i\frac{g^2}{4m_W^4}[(s-u)t - 3m_W^2(s-u) + \frac{8m_W^2}{s}u^2 + s(t-u) - 3m_W^2(t-u) \\ &- s^2 - 4st - t^2 + 4m_w^2(s+t) + \frac{8m_W^2}{s}ut + s(t-u) - 3m_W^2(t-u)] \\ &= -i\frac{g^2}{4m_W^4}[-2st - ut + 3m_W^2(-s-t+2u) - s^2 - t^2 + 4m_W^2(s+t) \\ &+ \frac{8m_W^2}{s}ut + s(t-u) - 3m_W^2(t-u)] \end{aligned} \tag{86}$$

And using the relations we derived earlier in 68 - 70

$$p_1p_2 = p_3p_4 = \frac{s}{2} - m_W^2 \tag{87}$$

$$p_1p_3 = p_2p_4 = -\frac{t}{2} + m_W^2 \tag{88}$$

$$p_2p_3 = -\frac{u}{2} + m_W^2 \tag{89}$$

The result for the total scattering could be reduced even more. Let's keep everthing in terms of one of the Mandelstam variables, u.

$$
\begin{aligned}
i\mathcal{M}^{\text{No Higgs}}_{\text{Total}}(W^-_L W^+_L \to W^-_L W^+_L) = -i\frac{g^2}{4m_W^4}\Big[& -2[(p_1^2+p_2^2+2p_1p_2\cos\theta)(p_1^2+p_3^2-2p_1p_3\cos\theta)] \\
& - u(p_1^2+p_3^2-2p_1p_3\cos\theta) \\
& + 3m_W^2[(p)1^2 - p_2^2 - 2p_1p_2 cos \ - \ p_1^2 \ - \ p_3^2 \ + \ 2p_1p_3\cos\theta \ - \ 2u] \\
& - [(p_1^2+p_2^2+2p_1p_2\cos\theta)(p_1^2+p_2^2-2p_1p_2\cos\theta)] \\
& - [(p_1^2+p_3^3-2p_1p_3\cos\,)(p_1^2+p_3^3-2p_1p_3\cos\theta)] \\
& + 3m_W^2[p_1^2+p_2^2+2p_1p_2\cos\theta+p_1^2+p_3^2-2p_1p_3\cos\theta] \\
& + 8m_W^2\frac{p_1^2+p_3^2-2p_1p_3\cos\theta}{p_1^2+p_2^2+2p_1p_2\cos\theta}\ u\Big]
\end{aligned} \tag{90}
$$

The cancellation could be seen from above, or much more neatly from 86. Let's rewrite this equation and see the cancellations explicitly [8]

$$
\begin{aligned}
i\mathcal{M}^{\text{No Higgs}}_{\text{Total}}(W^-_L W^+_L \to W^-_L W^+_L) &= -i\frac{g^2}{4m_W^4}[-2st-ut+3m_W^2(-s-t+2u)-s^2-t^2+4m_W^2(s+t) \\
&+ \frac{8m_W^2}{s}ut + s(t-u) - 3m_W^2(t-u)] \\
&= -i\frac{g^2}{4m_W^4}[-m_W^2(s+t)+6m_W^2u+\frac{8m_W^2}{s}ut+6m_W^2u-s^2-t^2 \\
&- 2st-(s+t)^2-ut+s(t-u)-3m_W^2(t-u)] \\
&= -i\frac{g^2}{4m_W^4}[-(s+t)^2+(s+t)u+4(u-4m_W^2)+\frac{8(t+u)u}{s}-3(s+t-2u)] \\
&= -i\frac{g^2}{4m_W^4}[-(s+t)^2-(s+t)(-s-t+4m_W^2)-4u+16m_W^2 \\
&+ 8\frac{(-s+4m_W^2)u}{s}-3(-3u+4m_W^2)] \\
&= -i\frac{g^2}{4m_W^4}[(u-4m_W^2)-3u+4m_W^2+\frac{32m_W^2u}{s}]
\end{aligned} \tag{91}
$$

Notice this problematic high-energy behavior, where the matrix element still grows with energy. Notice also how the strongest high-energy growth, the E^4 terms, have been cancelled by summing the diagrams, which is a consequence of gauge-invariance. [9]. Nevertheless, we see that the matrix element still grows with energy. These terms are the terms where two mandelestan variables are multiplied together (each mandestan variable is $\propto E^2$). This cancellation can be be seen by either

[8]In these simplifications, we'll just use $s+t+u=4m_W^2$.

[9]One would naively expect there to be E^4 terms from the 4-point vertex

$$\mathcal{M}_4^{\text{naive}} \propto \epsilon_L\epsilon_L\epsilon_L\epsilon_L \propto s^2$$

equation, but much more cleanly from the 86 equation. The only terms left are quadratic in energy, writing this in terms of u, we have

$$i\mathcal{M}^{\text{No Higgs}}_{\text{Total}}(W^-_L W^+_L \to W^-_L W^+_L) \approx -i\frac{g^2}{4m_W^2}\left(u - 12m_W^2 + \frac{32m_W^2 u}{s}\right) \tag{92}$$

The terms $-12m_W^2 + 32m_W^2 u/s$ are not important since they are just constants. What is important is to look at the linear term in u, meaning the quadratic term in energy.

$$i\mathcal{M}^{\text{No Higgs}}_{\text{Total}}(W^-_L W^+_L \to W^-_L W^+_L) \approx -i\frac{g^2}{4m_W^2}u + \mathcal{O}\left((E/m_W)^0\right) \tag{93}$$

Where E denotes the scattering energy and so the $\mathcal{O}\left((E/m_W)^0\right)$ is the next-to-leading order. Notice that 93 essentialy says that although the $\mathcal{O}\left((E^4/m_W^4)\right)$ terms have been cancelled, we are still left with a term of order $\mathcal{O}\left((E^2/m_W^2)\right)$, so we are still in trouble because the energy still grows without bounds!

4 $W_L^- W_L^+ \to W_L^- W_L^+$ with Only Higgs Exchanges

If we include the Higgs boson, the high energy behavior will be tamed down. To see this cancellation, let us see how the inclusion of Higgs interactions would affect our amplitudes.

When including Higgs interactions, we only have two extra diagrams: an s-channel diagram by the exchange of a Higgs and a t-channel diagram by the exchange of a Higgs. The vertex for a WWZ interaction is very simply:

$$V_{WWh} = -i\frac{e}{\sin\theta_w} m_W g^{\mu\nu} = -igm_W g^{\mu\nu} \tag{94}$$

And can be seen in Figure 6, which shows the Feynman rules for the interactions of Higgs bosons to gauge bosons. Note also that

$$g = 2m_W(G_F\sqrt{2})^{1/2} \tag{95}$$

Therefore the vertex above can also be written as $V_{WWh} = -2im_W^2(G_F\sqrt{2})^{1/2}g^{\mu\nu}$. The propagator for the Higgs is also very simple,

$$\Delta^H_{\mu\nu} = \frac{1}{k^2 - m_H^2} \tag{96}$$

Where k^2 is the momentum exchanged ($k^2 = s$ or $k^2 = t$ for our diagrams). The coupling constant for the WWH vertex is $g^H_{WWH} = g^{\mu\nu} = gm_W g^{\mu\nu}$. Using these rules,

The s-channel amplitude with only Higgs exchange is

$$i\mathcal{M}_s^H = i\frac{g^2}{m_W^2}(-\frac{1}{4}s) \tag{97}$$

The t-channel amplitude with only Higgs exchange is

$$i\mathcal{M}_t^H = i\frac{g^2}{m_W^2}(\frac{1}{4}u + \frac{1}{4}s) \tag{98}$$

Combining them, we have the total Higgs contribution to the amplitude

$$i\mathcal{M}_{\text{Total}}^{\text{only Higgs}} = -\frac{g^2}{4m_W^2}(s+u) \quad = -i\frac{g^2}{4m_W^2}(-u + 4m_W^2) \tag{99}$$

We immediately see from the equation above that the linear $+\frac{g^2}{4m_W^2}u$ term cancels the $-\frac{g^2}{4m_W^2}u$ term from equation 92.

$$\begin{aligned} i\mathcal{M}_{\text{Total}} &= i\mathcal{M}_{\text{Total}}^{\text{only Higgs}} + i\mathcal{M}_{\text{Total}}^{\text{No Higgs}} \\ &= -i\frac{g^2}{4m_W^2}(-8m_W^2 + 32m_W^2 u/s) \end{aligned} \tag{100}$$

Hence the bad energy behavior which is growing without bound has been cancelled! The inclusion of the Higgs has remarkably implied that the amplitude has a finite high-energy limit, and does not grow without bounds as it was prior to its inclusion.

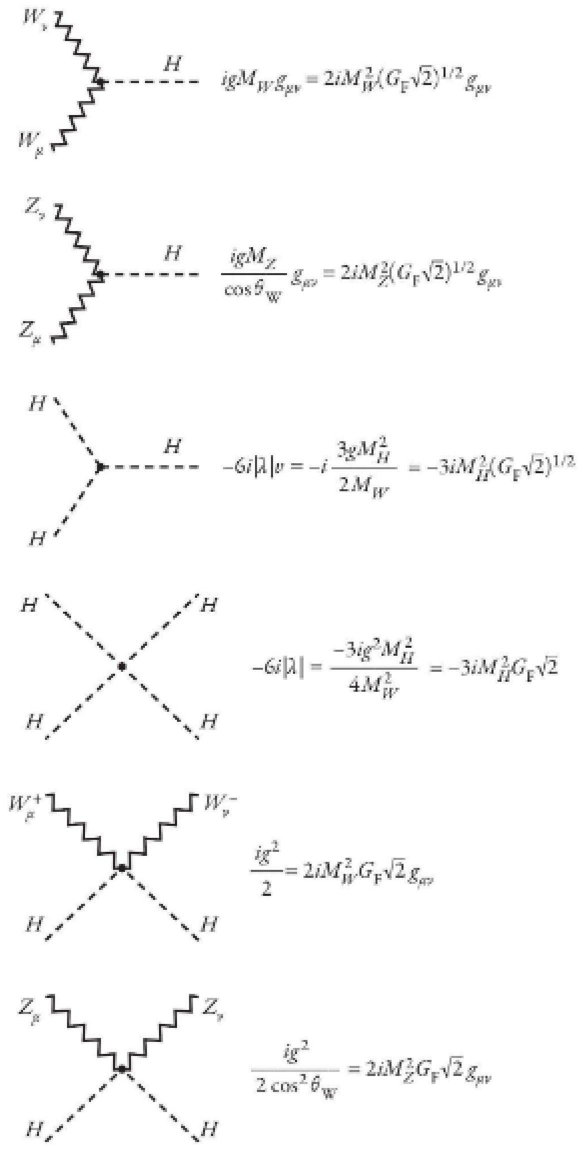


Figure 6: Feynman rules for the interactions of Higgs bosons to gauge bosons

5 Goldstone Boson Scattering

5.1 introduction

We saw that the divergences in the massive weak vector bosons in the high energy limit are associated with physical states of spin-1 particles carrying longitudinal polarization, i.e. zero helicity. We also saw that through the Higgs mechanism, divergent Feynman graphs are cancelled, so that the tree-level scattering amplitudes are bounded in the high energy limit. We will see that the Equivalence Theorem relates the physical vector boson scattering amplitude to the scattering of unphysical Goldstone scalars (unphysical because they actually disappear from the physical spectrum as a result of the Higgs mechanism). [Quigg]

Since the scattering of a massless gauge boson does not produce the problematic energy growth, the unphysical growth has to come from the longitudinal degree of freedom, i.e. the Goldstone bosons. To see this, let's review the Higgs mechanism. Recall that the Lagrangian in the Higgs sector involves the complex scalar $SU(2)$ doublet Φ

$$\Phi = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix} \tag{101}$$

Where ϕ^0 is defined to have zero charge. Upon EWSB, we have the doublet parameterized as

$$\Phi = \begin{pmatrix} -iw^+ \\ \frac{1}{\sqrt{2}}(v + H + iz) \end{pmatrix} \tag{102}$$

Where ω^+ is complex and H and z are real. Here it is clear to see that the 4 degrees of freedom in the original complex scalar doublet have given mass to the longitudinal vector bosons, and one scalar Higgs is left.

with vacuum expectation value v arising from the self-interaction

$$V(\Phi) = \lambda \left[|\Phi|^2 - v^2/2\right]^2 \tag{103}$$

In the EWSB, the W and Z bosons aquire mass, the scalar field components $\omega^\pm$ and z become the longitudinal components $W_L^\pm$ and Z_L while H survivves as a real $SU(2)$ singlet field, with mass

$$m_H = \sqrt{2\lambda}v \implies \lambda = \frac{m_H^2}{2v^2} = \left[\frac{m_H}{348GeV}\right]^2 \tag{104}$$

Hence λ is large if H is heavy, implying that strong scattering of $\omega^\pm$ and z become $W_L^\pm$ and Z_L. According to the electroweak equivalence theorem, the scattering amplitudes for longitudinal gauge bosons can be expressed in terms of the scattering amplitudes for the corresponding Goldstone bosons, i.e.

$$\mathcal{M}\left(V_L^1 \ldots V_L^n \to V_L^1 \ldots V_L^m\right) = (i)^n(-i)^m \mathcal{M}\left(\omega^1 \ldots \omega^n \to \omega^1 \ldots \omega^m\right) + O\left(\frac{M_V^2}{s}\right) \tag{105}$$

where ω^i $(i = 1, 2, 3)$ is the Goldstone boson associated to the longitudinal component of the gauge boson V^i. Hence, according to this theorem, the $W_L^+ W_L^- \to W_L^+ W_L^-$ scattering at high energy must satisfy

$$\mathcal{M}\left(W_L^+ W_L^- \to W_L^+ W_L^-\right) = \mathcal{M}\left(\omega^+ \omega^- \to \omega^+ \omega^-\right) + O\left(\frac{M_W^2}{s}\right) \tag{106}$$

5.2 R-Gauges

In order to use the equivalence, we have to use a formal description of the SM, in which the unphysical Goldstone fields are preserved as auxiliary fields. This is achieved by renormalizable (or R) gauges, like the t'Hooft-Feynman gauge or the Lorentz gauge. We have to use these gauges, which differ from the unitary gaguge used before, since fixing the unitary gauge means to eliminate all the Goldstone bosons that would emerge later in the Higgs mechanism. Recall what we wrote down for the propagator of the Z boson in the unitary gauge

$$\Delta_{\mu\nu}^{Z} = \frac{-i}{k^2 - m_Z^2}\left(g_{\mu\nu} + \frac{k_\mu k_\nu}{m_Z^2}\right) \tag{107}$$

The fact that the propagator does not decrease as $k \to \infty$ leads to divergent behaviour of Feynman diagrams (it has problematic UV behaviour). R-gauges are used to tame these energy divergences. Specifically, this is done by quantizing the gauge condition G of the form

$$G = \frac{1}{\sqrt{\xi}}\left(\partial_\mu A^\mu + \xi g v \phi_2\right) \tag{108}$$

In the generating functional

$$Z[J] = C\int DAD\phi_1 D\phi_2 \exp\left[i\int\left(\mathcal{L} - \frac{1}{2}G^2\right)\right]\det\left(\frac{\delta G}{\delta\alpha}\right) \tag{109}$$

Where C is an overall constant, xi is an arbitrary parameter, and α is the gauge transformation parameter (in gauge transformations like $\phi(x) \to e^{i\alpha(x)}\phi(x), A^\mu(x) \to A^\mu(x) + \frac{1}{g}\partial^\mu\alpha(x)$). From writing the generating functional in terms of integration over the goldstone bosons, we will get a gauge fixing term in the Lagrangian.

Now, by shifting the lower component of 102 one gets mass terms of the vector bosons. The gauge-fixing term may be taken to be

$$\begin{aligned}\mathcal{L}_{gaugefixing} = &-\tfrac{1}{2\xi}\left|\partial^\mu W_\mu^- - \xi m_W w^-\right|^2 - \tfrac{1}{2\xi}\left|\partial^\mu W_\mu^+ - \xi m_W w^+\right|^2 \\ &-\tfrac{1}{2\eta}\left(\partial^\mu Z_\mu - \eta m_Z z\right)^2 - \tfrac{1}{2\alpha}\left(\partial^\mu A_\mu\right)^2\end{aligned} \tag{110}$$

Now let's see how using kinetic terms in the Lagrangian differ when we use $\mathcal{L}_{SM}$ vs when we use $\mathcal{L}_{gaugefixing}$. Substituting 102 into $\mathcal{L}_{SM}$ we have

$$\left(D_\mu\Phi\right)^\dagger\left(D^\mu\Phi\right) = m_W\left(\partial^\mu w^- W_\mu^+ + \partial^\mu w^+ W_\mu^-\right) + m_Z \partial^\mu z Z_\mu + \ldots \tag{111}$$

On the other hand, 110 also produces kinetic terms

$$\mathcal{L}_{gaugefixing} = m_W\left(w^-\partial^\mu W_\mu^+ + w^+\partial^\mu W_\mu^-\right) + m_Z z\partial^\mu Z_\mu + \ldots \tag{112}$$

Adding the two lagrangians

$$\mathcal{L}_{gaugefixing} = m_W \left(w^- \partial^\mu W_\mu^+ + w^+ \partial^\mu W_\mu^- \right) + m_Z z \partial^\mu Z_\mu + \dots \tag{113}$$

And we can get the kinetic terms for $w^\pm$ and z

$$\mathcal{L}_{Higgs} + \mathcal{L}_{g.f.} = \partial^\mu w^- \partial_\mu w^+ + \frac{1}{2} \partial^\mu z \partial_\mu z - \xi m_W^2 w^- w^+ - \frac{1}{2} m m_Z^2 z^2 + \dots \tag{114}$$

Where we can see that we get terms $m_w^2 = \xi m_W^2$, $m_z^2 = \eta m_Z^2$, i.e. the unphysical gauge bosons have masses that depend on the arbitrary parameter ξ and η. Let's write all the quadratic terms of all the relevant EW interactions after this gauge fixing

$$\begin{aligned} \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{gauge fixing}} = & -\frac{1}{2} W_{\mu\nu}^- W^{+\mu\nu} - \frac{1}{\xi} \left(\partial \cdot W^- \right) \left(\partial \cdot W^+ \right) + m_W^2 W_\mu^- W^{+\mu} \\ & - \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} - \frac{1}{2\eta} (\partial \cdot Z)^2 + \frac{1}{2} m_Z^2 Z_\mu Z^\mu \\ & - \frac{1}{4} A_{\mu\nu} A^{\mu\nu} - \frac{1}{2\alpha} (\partial \cdot A)^2 + \dots \end{aligned} \tag{115}$$

5.3 Calculating the Scattering

The propagators can now be derived by using the generating functional 109 with the lagrangian in 115. For the vector bosons we get

$$D_{\mu\nu}^{(W)}(k) = \left[-g_{\mu\nu} + (1 - \xi) \left(k^2 - \xi m_W^2 \right)^{-1} k_\mu k_\nu \right] \frac{1}{k^2 - m_W^2 + i\varepsilon} \tag{116}$$

$$D_{\mu\nu}^{(Z)}(k) = \left[-g_{\mu\nu} + (1 - \eta) \left(k^2 - \eta m_W^2 \right)^{-1} k_\mu k_\nu \right] \frac{1}{k^2 - m_Z^2 + i\varepsilon} \tag{117}$$

$$D_{\mu\nu}^{(A)}(k) = \left[-g_{\mu\nu} + (1 - \alpha) \left(k^2 \right)^{-1} k_\mu k_\nu \right] \frac{1}{k^2 + i\varepsilon} \tag{118}$$

And for the the propagators of the goldstone bosons we get

$$\Delta^{(w)}(k) = \frac{1}{k^2 - \xi m_W^2 + i\varepsilon} \tag{119}$$

$$\Delta^{(w)}(k) = \frac{1}{k^2 - \xi m_W^2 + i\varepsilon} \tag{120}$$

The most convenient choice is $\xi = \eta = \alpha = 1$, which gives:

$$\Delta^{(w)}(k) = \frac{1}{k^2 - m_W^2 + i\varepsilon}, \ \Delta^{(w)}(k) = \frac{1}{k^2 - m_W^2 + i\varepsilon} \tag{121}$$

Which is known as the Feynman thooft gauge. Note that the masses of the unphysical scalar golstone bosons $w^\pm, z$ are equivalent to the masses of their vector boson counterparts $W^\pm, Z$. Notice, also, that by taking $\xi, \eta \to \infty$, which corresponds to the physical unitary gauge, the propagators of the goldstone bosons vanish (they can't propagate anymore), which makes sense since these unphysical scalars should not exist in the physical unitary picture.

We read off the two goldstone boson - vector boson interaction from the lagrangian

$$\mathcal{L}_{w^-w^+V} = iew^- \overleftrightarrow{\partial^\mu} w^+ A_\mu + i\frac{g}{\cos\vartheta_W}\left(\frac{1}{2} - \sin^2\vartheta_W\right) w^- \overleftrightarrow{\partial^\mu} w^+ Z_\mu \tag{122}$$

We also have Yukawa interactions for a lepton l with the Goldstone bosons, which we will not write since they are irrelevant for our question. For the scalar (self-interactions) we have

$$\mathcal{L}_{\text{scalar}} = -\lambda v H\left(2w^-w^+ + z^2 + H^2\right) - \frac{1}{4}\lambda\left(2w^-w^+ + z^2 + H^2\right)^2 \tag{123}$$

From these Lagrangians one can derive the coupling and the Feynman rules for the Goldstone bosons. For example,

$$\begin{aligned}\mathcal{M}(H \to w^+w^-) &= -2\lambda v \\ &= -\cancel{2}\underbrace{[\frac{m_H^2}{\cancel{2}v^{\cancel{2}}}]}_{=\lambda}\cancel{v} \\ &= -m_H^2\underbrace{\frac{g}{2m_W}}_{=\frac{1}{v}} \\ &= -\frac{m_H^2 g}{2m_W}\end{aligned} \tag{124}$$

The vertex factor for w^+w^-h can be read off as

$$V_{w^+w^-h} = -\frac{m_h^2 g}{2m_W} \tag{125}$$

The 4-point Goldstone boson scattering vertex can be read off as

$$V_{w^+w^-w^+w^-} = -\frac{m_h^2 g^2}{m_W^2} \tag{126}$$

And the Higgs propagator is independent of the gauge

$$\Delta^h(k) = \frac{1}{k^2 - m_h^2} \tag{127}$$

Finally, we have all the ingredients that are needed to calculate $\mathcal{M}(w^+w^- \to w^+w^-)$. The diagrams for this process are [Quigg]: s-channel by exchange of Higgs, t-channel by exchange of Higgs, and the 4-point vertex. The usual $p_1^2 = p_2^2 = m_w^2 = m_W^2$. The relations 68 - 70 hold here as well. We get

$$\mathcal{M}_s = \left[-\frac{m_h^2 g}{2m_W}\right]\left[\frac{1}{(p_1+p_2)^2 - m_h^2}\right]\left[-\frac{m_h^2 g}{2m_W}\right] = \frac{m_h^4 g^2}{4m_W^2}\frac{1}{s - m_h^2} \tag{128}$$

$$\mathcal{M}_t = \left[-\frac{m_h^2 g}{2m_W}\right]\left[\frac{1}{(p_1-p_3)^2 - m_h^2}\right]\left[-\frac{m_h^2 g}{2m_W}\right] \tag{129}$$

$$\mathcal{M}_4 = -\frac{m_h^2 g^2}{m_W^2} \tag{130}$$

Hence the total amplitude [10]

$$\begin{aligned}\mathcal{M}(w^+w^- \to w^+w^-) &= \frac{m_h^2}{4m_W^2} g^2 \left(\frac{s}{s-m_h^2} + \frac{t}{t-m_h^2} \right) = \frac{m_h^2}{\cancel{4m_W^2}} \underbrace{\cancel{4m_W^2} G_F \sqrt{2}}_{g^2} \left(\frac{s}{s-m_h^2} + \frac{t}{t-m_h^2} \right) \\ &= \sqrt{2} G_F m_h^2 \left(\frac{s}{s-m_h^2} + \frac{t}{t-m_h^2} \right)\end{aligned} \tag{131}$$

Which agrees to Quigg's result, and our result for $\mathcal{M}(W_L^+ W_L^- \to W_L^+ W_L^-)$, and we see the cancellation of the high-energy divergence - this doesn't grow with energy anymore, for the same reasons as the previously discussed $W_L^+ W_L^- \to W_L^+ W_L^-$!

[10]Of course, terms like

6 Unitarity, Partial wave expansion, Higgs Mass Bound and Further Discussion

Unitarity is a property of quantum mechanics which ensures the sum of probabilities of all possible final states evolving from a particular initial state is always equal to 1, and so it must hold for any acceptable theory. We will see that unitaity will constrain the amplitude to stay small enough at any given energy. To see this, consider that the scattering amplitude can be expanded into partial wave expansion

$$\mathcal{M}(\theta) = 16\pi \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta) a_l \tag{132}$$

Where a_l are the partial wave amplitudes of the elastic scattering of two particles, and $P_l(\cos\theta)$ are the Legendre polynomials that satisfy $P_J(1) = 1$ and $\int_{-1}^{1} P_j(\cos\theta) P_k(\cos\theta) d\cos\theta = \frac{2}{2j+1}\delta_{jk}$. Now consider a $2 \to 2$ elastic scattering. The cross section for this process is

$$\sigma_{\text{tot}}\,(AB \to AB) = \frac{1}{32\pi E_{\text{CM}}^2} \int d\cos\theta |\mathcal{M}(\theta)|^2 \tag{133}$$

and using the partial wave expansion we have

$$\sigma_{\text{tot}} \; = \frac{16\pi}{E_{\text{CM}}^2} \sum_{j=0}^{\infty} (2j+1) \left|a_j\right|^2 \tag{134}$$

Where the orthogonality relation $\int \mathrm{d}\cos\theta P_\ell P_{\ell'} = \delta_{\ell\ell'}$ has been used. The optical theorem says that the cross section is proportional to $\mathcal{M}$ in the forward direction

$$\begin{aligned}
\sigma = \frac{1}{E_{CM}^2} \mathrm{Im}[\mathcal{M}(\theta)] &= \frac{16\pi}{E_{CM}^2} \sum_{j=0}^{\infty} (2j+1) \left|a_j\right|^2 \\
&\implies |a_l|^2 = \mathrm{Im}(a_l) \\
&\implies [\mathrm{Re}(a_l)]^2 + [\mathrm{Im}(a_l)]^2 \\
&\implies [\mathrm{Re}(a_l)]^2 + [\mathrm{Im}(a_l) - \frac{1}{2}]^2 = \frac{1}{4}
\end{aligned} \tag{135}$$

This is the equation of a circle of radius $\frac{1}{2}$ and center $(0, \frac{1}{2})$ in teh plane $[\mathrm{Re}(a_l), \mathrm{Im}(a_l)]$. The real part is between $-\frac{1}{2}$ and $\frac{1}{2}$. This is the unitarity condition

$$|\mathrm{Re}(a_l)| \leq \frac{1}{2}, \quad |a_l| \leq 1 \tag{136}$$

for all l. [11]

We can do this expansion for our total amplitude without the Higgs

$$i\mathcal{M}_{\text{Total}}^{\text{No Higgs}}(W_L^- W_L^+ \to W_L^- W_L^+) = 16\pi(a_0 + 3a_1\cos\theta + ...) \tag{137}$$

[11] Note that the partial wave unitarity condition also implies $0 \leq \mathrm{Im}(a_l) \leq 1$ for all l.

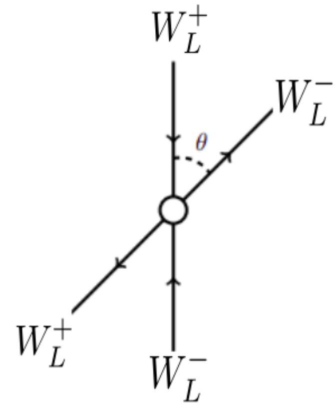


Figure 7: $W_L^- W_L^+$ scattering in the CM frame

Now let's write 92 here again

$$i\mathcal{M}_{\text{Total}}^{\text{No Higgs}}(W_L^- W_L^+ \to W_L^- W_L^+) \approx -i\frac{g^2}{4m_W^2}\left(u - 12m_W^2 + 32m_W^2 u/s\right) \approx -i\frac{g^2}{4m_W^2}u \tag{138}$$

If we equate the coefficients from the partial wave expansion to the coefficients in our total amplitude, we see that the divergent behavior (in energy) is confined to the $J = 0, 1$ and 2 partial waves. Since unitarity gives us a constraint using only the $J = 0$ partial wave, we need only consider the a_0 term.

Since our expression only involves u, we can arrive at an equation at high energy using E_{CM} instead. in the high energy limit ($s >> m_W$) we have $m_{W^\pm} = 0$, then $E = |\vec{p}|$. Looking at Figure 7 we see the four momenta of the particles in the interaction are

$$p_1 = E(1, 0, 0, 1), \tag{139}$$

$$p_2 = E(1, 0, 0, -1) \tag{140}$$

$$p_3 = E(1, 0, -\sin\theta, -\cos\theta) \tag{141}$$

$$p_4 = E(1, 0, \sin\theta, \cos\theta) \tag{142}$$

Where E is the energy and θ is the angle beween the incoming W^+W^- and the outgoing W^+W^- as shown in Figure. The total energy in the center of mass frame is

$$E_{CM} = E_{p_1} + E_{p_2} = E_{p_3} + E_{p_4} = 2E \implies E = \frac{E_{CM}}{2} \tag{143}$$

$$
\begin{aligned}
u &= (p_2 - p_3)^2 \\
&= p_{2\mu}p^{2\mu} + p_{3\mu}p^{3\mu} - 2p_{2\mu}p^{3\mu} \\
&= E^2 - E^2 + E^2 - E^2((-\sin^2\theta)^2 + (-\cos^2\theta)) - 2(E^2 - (-E)(-E\cos\theta)) \\
&= -2E^2(1-\cos\theta) \\
&= -\frac{E_{CM}^2}{2}(1-\cos\theta)
\end{aligned}
\tag{144}
$$

Plugging this in 138 we have

$$
\begin{aligned}
i\mathcal{M}_{\text{Total}}^{\text{No Higgs}}(W_L^- W_L^+ \to W_L^- W_L^+) &= -i\frac{g^2}{4m_W^2}\left[-\frac{E_{CM}^2}{2}(1-\cos\theta)\right] \\
&= i\frac{g^2E_{CM}^2}{8m_W^2}(1-\cos\theta)
\end{aligned}
\tag{145}
$$

Therefore, using 137 we equate the a_0 (again the a_0 is the one without the $\cos\theta$).

$$
16\pi a_0 = i\frac{g^2E_{CM}^2}{8m_W^2} \tag{146}
$$

Hence

$$
a_0 = i\frac{g^2E_{CM}^2}{128\pi m_W^2} \tag{147}
$$

Since we have an imaginary expression, we need to use the unitarity condition $Re(a_0) \leq \frac{1}{2}$ where $Re(a_0) = \frac{g^2E_{CM}^2}{128\pi m_W^2}$ Hence

$$
\frac{g^2E_{CM}^2}{128\pi m_W^2} \leq \frac{1}{2} \implies E_{CM}^2 = \frac{64\pi m_W^2}{g^2} \tag{148}
$$

Now plugging in $g \approx 0.64$, $m_W \approx 80.399$ GeV we can solve for the energy at which *perturbative unitarity* is violated [12]

$$
\begin{aligned}
E_{CM} &\approx \sqrt{\frac{64\pi(80.399)^2}{(0.64)^2}}GeV \\
&= \boxed{1.781 \text{ TeV}}
\end{aligned}
\tag{149}
$$

This is a very important result and it tells us that new physics must show up below 1.78 TeV in order for unitarity to hold. One can also derive a bound on the Higgs mass using this partial wave expansion. Without using the last approximation, and here assuming $s, m_H^2 >> m_W^2, m_Z^2$ (not using

[12]To answer the question explicitly, this implies that perturbative unitarity would be violated at this energy if there were not Higgs boson. That is, scattering amplitudes cannot be calculated reliably in perturbation theory. This means that scattering amplitudes in electroweak theory (non-renormalizable to begin with) must have important contributions from loops at this scale, and perturbation theory breaks down at this scale.

the previous approximation $s >> m_H^2, m_W^2, m_Z^2$), we have the total amplitude (the sum of the s and t channels) for the Higgs exchange as

$$\begin{aligned}
i\mathcal{M}_{\text{Total}}^{\text{only Higgs}} &= -i\frac{g^2}{4m_W^2}\left[\frac{\left(s-2m_W^2\right)^2}{s-m_H^2}+\frac{\left(t-2m_W^2\right)^2}{t-m_H^2}\right] \\
&\approx -i\frac{g^2}{4m_W^2}\left[\frac{s^2}{s-m_H^2}+\frac{t^2}{t-m_H^2}\right] \\
&= -i\frac{\cancel{4m_W^2}G_F\sqrt{2}}{\cancel{4m_W^2}}\left[\frac{s^2}{s-m_H^2}+\frac{t^2}{t-m_H^2}\right] \\
&\approx -i\sqrt{2}G_{\mathrm{F}}m_H^2\left[\frac{s}{s-M_H^2}+\frac{t}{t-M_H^2}\right]
\end{aligned} \tag{150}$$

Taking the approximation above, we need only consider the leading term

$$\mathcal{M}_{\text{Total}}^{\text{only Higgs}} = -\frac{4}{\sqrt{2}}G_F m_H^2 \tag{151}$$

And relating this to the first partial wave expansion, we have

$$16\pi a_0 = -\frac{4}{\sqrt{2}}G_F m_H^2 \implies a_0 = -\frac{G_F m_H^2}{4\sqrt{2}\pi} \tag{152}$$

Now using the unitarity condition $|a_0| \leq 1$ (since we don't have an imaginary first order term in the amplitude)

$$|a_0| \leq 1 \implies \frac{G_F m_H^2}{4\sqrt{2}\pi} \leq 1 \tag{153}$$

And solving for the Higgs mass we have

$$m_H \leq \sqrt{\frac{4\sqrt{2}\pi}{G_F}} \tag{154}$$

Now using $G_F = 1.166 \times 10^{-5}\text{GeV}^{-2}$ and plugging it into the equation above, we get a limit for the mass of the Higgs:

$$m_H \leq 1.034.7 \text{ TeV} \tag{155}$$

7 Physics Beyond the SM and Unitarity Bounds

The high energy behavior of $W_L^+ W_L^-$ scattering is sensitive to Higgs-gauge boson couplings, therefore couplings that differ than those of the SM should show up experimentally in vector boson scattering. These effective couplings that differ from those of the SM couplings may be a manifestation of new interactions between gague bosons. This is why the couplings between the Higgs and gauge bosons is an exceptional way to look for new physics and should be rigorously tested.

The SM effective field theory (SMEFT) approach assumes that new effective couplings can be written as

$$L_{EFT} = L_{SM} + c_5 \frac{\mathcal{O}_5}{\Lambda} + c_6 \frac{\mathcal{O}_6}{\Lambda^2} + ... \tag{156}$$

Where $\mathcal{O}_i$ are operators of dimension 5, c_i are are the associated coefficients of those operators, and Γ is the energy scale of new physics. Assuming that we have an additional interaction between the Higgs and the gauge bosons

$$\mathcal{L}_{SMEFT} = \mathcal{L}_{\text{SM}} + \delta_{W1} \frac{2m_W^2}{v} H W^{\mu +} W_\mu^- + ... \tag{157}$$

Where the dots .... represents other terms that do not impact the HWW coupling. Note that this operator may affect couplings other than HWW! Many models have been proposed recently, e.g Strongly Interacting Little Higgs models, etc. The procedure to analyze the effect of this following from SMEFT. All new interactions are described in a way that depends only on the factors in front of the operator, $\frac{2m_W^2 v^-1}{\Gamma}$. We can rewrite this additional term and combine it with the HWW term in the SM Lagrangian. Recall that the SM Lagrangian has the piece which governs the HWW interaction

$$L_{SM} = ... + \frac{2m_W}{v} H W_\mu^+ W_\mu^- + ... \tag{158}$$

Hence the coupling under SM is $c_{SM} = \frac{2m_W}{v}$. Under our new Lagrangian, L_{SMEFT} in 157, everything will stay the same, except the Higgs-gauge boson interaction term will get a new effective coupling

$$\begin{aligned} L_{SMEFT} = \ & ... \ + (\underbrace{\frac{2m_W^2}{v}}_{SM} + \underbrace{\delta_{W1} \frac{2m_W^2}{v}}_{EFT}) H W_\mu^+ W_\mu^- + \ .... \\ = & ... \ + \frac{2m_W^2}{v}(1 + \delta_{W1}) \ + \ ... \end{aligned} \tag{159}$$

Hence our new coupling for this effecting Lagrangian for the HWW interaction is

$$C_{SMEFT}^{HWW} = \frac{2m_W^2}{v}(1 + \delta_{W1}) \tag{160}$$

This new coupling that differs than the one shows up in the high energy behavior of WW scattering. Under this model, the quadratic terms in the WW scattering amplitude *are not completely*

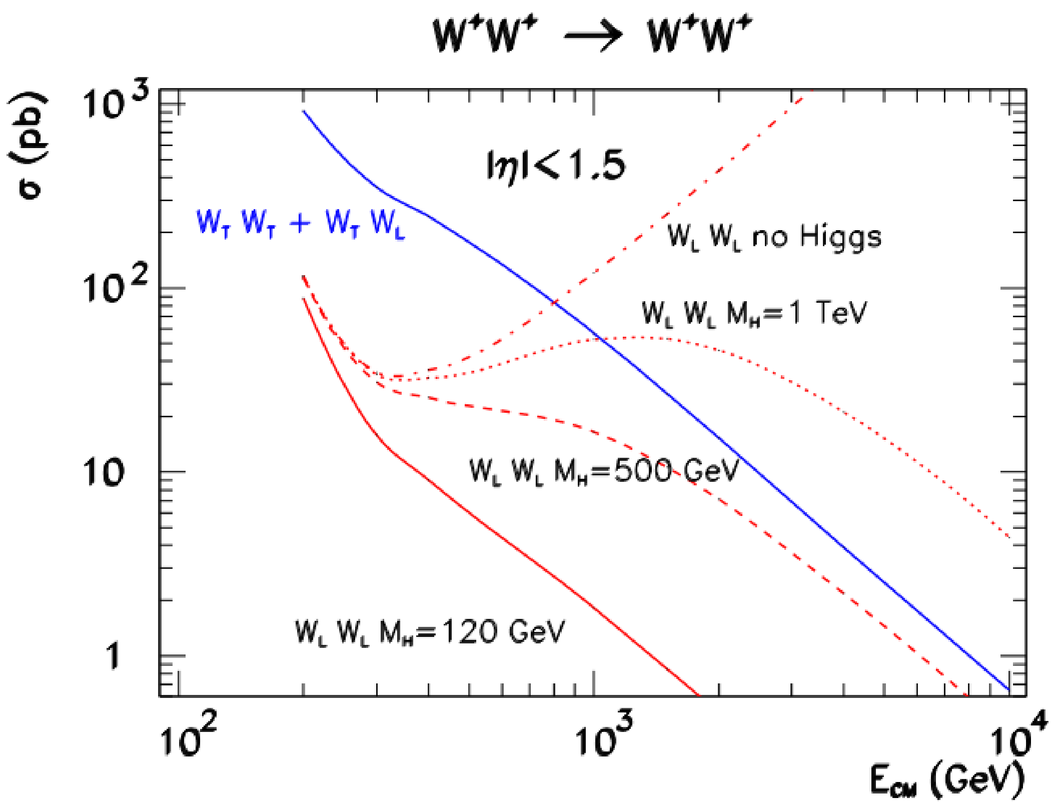


Figure 8: WWscattering cross sections as a function of the center of mass energy for different final (and initial) state polarizations and for different Higgs masses. The figures are from the very nice paper by **??**

canceled when the Higgs exchange diagrams are added and this incomplete cancelation must show up at a high enough energy. This model therefore unitarizes the amplitudes only partially. In other words, it only defers the unitarity crisis to higher energies (it tames the energy-growing behaviour). The WW scattering amplitudes still grows with energy above the Higgs mass, albiet slower. In general, the total cross section will rise up to the Higgs mass (not if MH ¡ 2MW), fall past the Higgs mass and rise up again at some energy, see figure

Since each of Feynman graphs that we consider has two vertices, each with a coupling constant, the scattering amplitude scales with the coupling constant squared $\mathcal{M}_{SMEFT}^{Higgs} \propto C_{SMEFT}^{HWW}$. We can use our scattering amplitude result for the SM, and add the contributions of Higgs exchange, where the couplings of HWW are now governed by our SMEFT Lagrangian. Hence the total amplitude is

$$\mathcal{M}_{Total}(W_L^- W_L^+ \to W_L^- W_L^+) = \mathcal{M}_{SM,Total}^{NoHiggs}(W_L^- W_L^+ \to W_L^- W_L^+) + \mathcal{M}_{SMEFT,Total}^{OnlyHiggsHWW}(W_L^- W_L^+ \to W_L^- W_L^+) \tag{161}$$

Where $\mathcal{M}_{SM,Total}^{NoHiggs}(W_L^- W_L^+)$ is the same result that we got earlier in 92.

$$i\mathcal{M}_{\text{Total}}^{\text{No Higgs}}(W_L^- W_L^+ \to W_L^- W_L^+) \approx -i\frac{g^2}{4m_W^2}\left(u - 12m_W^2 + \frac{32m_W^2 u}{s}\right) \tag{162}$$

Furthermore,

$$\begin{aligned}\mathcal{M}_{SMEFT,Total}^{OnlyHiggsHWW}(W_L^- W_L^+ \to W_L^- W_L^+) &\propto [\frac{2m_W^2}{v}(1+\delta_{W1})]^2 \\ &= \mathcal{M}_{SM,Total}^{OnlyHiggs}\ (1+\delta_{W1}) \\ &= -\frac{g^2}{4m_W^2}(-u+4m_W^2)(1+\delta_{W1})\end{aligned} \tag{163}$$

Where we used our result in 99 in the last line. Hence clearly we see that the high-energy growth is not cancelled, namely

$$\mathcal{M}_{SMEFT,Total}^{OnlyHiggsHWW}(W_L^- W_L^+ \to W_L^- W_L^+) = \frac{g^2}{4m_W^2}u\ (1+\delta_{W1}) + \mathcal{O}\left((E/m_W)^0\right) \tag{164}$$

Where $u \propto E^2$, so clearly the amplitude grows with energy, and the high-energy growth does not cancel as in the SM case. If we use the same unitarity arguments as before, and substitute $u = -\frac{E_{CM}^2}{2}(1-\cos(\theta))$ we have

$$\begin{aligned}\mathcal{M}_{SMEFT,Total}^{OnlyHiggsHWW}(W_L^- W_L^+ \to W_L^- W_L^+) &= \frac{g^2}{4m_W^2}\frac{E_{CM}^2}{2}(1-\cos(\theta))\ (1+\delta_{W1}) + \mathcal{O}\left((E/m_W)^0\right) \\ &= \frac{g^2}{4m_W^2}\frac{E_{CM}^2}{2}\ (1+\delta_{W1} - \cos(\theta) - \cos(\theta)\delta_{W1})\end{aligned} \tag{165}$$

by the previous discussion, and where we dropped the unimortant (w.r.t energy) $\left((E/m_W)^0\right)$. Therefore, by the same unitarity argument, we can equation the a_0 term

$$16\pi a_0 = \frac{g^2 E_{CM}^2}{8m_W^2}(1+\delta_{W1}) \tag{166}$$

Now using the unitarity condition $a_0 \leq 1$ we have

$$16\pi = \frac{g^2 E_{max}^2}{8m_W^2}(1+\delta_{W1}) \implies \boxed{E_{max} = \frac{128\pi m_W^2}{g^2(1+\delta)}} \tag{167}$$

Where E_{max} is the maximum energy at which unitarity still holds. Plugging in the values $g \approx 0.64,\ m_W \approx 80.385$ we can get a numerical value

$$E_{max} \approx \frac{6.343}{1+\delta_{W1}}\ PeV. \tag{168}$$

Hence we see that since E_{max} and δ_{W1} are inversely proportional, being able to measure smaller δ_{W1} through improved precision pushes the new-physics for E_{max} higher. Any calculation of δ_{W1} is significant, since measuring $\delta_{W1} = 0$ will excule new physics until higher precision (smaller δ_{W1}) which corresponds to higher E_{max}.

References

[1] Schwartz, M. (2013). Quantum Field Theory and the Standard Model. Cambridge: Cambridge University Press. doi:10.1017/9781139540940

[2] Lee, B W; Quigg, C; Thacker, H B. (1977) "Weak interactions at very high energies: The role of the Higgs-boson mass", Phys. Rev., D, 16:5

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8 Appendix 0: Understanding and Program verification for $WZ \to WZ$

The Form code, wz-wz.frm, provided by Dr. Reina, gives the final answer for the $WZ \to WZ$ t, s channels and the 4-point amplitude. There are a few algebra steps in reducing the output from Form to the equations listen in Schwartz: equations (29.22) - (29.24), these steps are provided here. As an example, consider the final expression for the s-channel amplitude:

$Ms = +\, gwwz^2 * mw^{-4} * mz^2 * E^2 * (-1/4 * s)$

$+$ gwwz$^2 * mw^{-2} * mz^-2 * E^4 * (1/2 * s * u + 1/4 * s^2)$
$+$ gwwz$^2 * mw^{-2} * [s+u]^-1 * E^2 * (s^2)$

$+$ gwwz$^2 * mw^{-2} * E^2 * (-u - 1/2 * s)$

$+$ gwwz$^2 * mz^{-2} * [s+u]^-1 * E^2 * (s^2)$

$+$ gwwz$^2 * mz^{-2} * E^2 * (-1/2 * u - s)$; (169)

Factoring out $\frac{g^2_{wwz}}{4m^2_W m^2_z}$ we have

$$\begin{aligned} M_S = \frac{g^2_{wwz}}{4m^2_W m^2_z} \Bigg[& -\frac{m^2_z s}{m^2_W} + 2su + s^2 \\ & + \frac{4m^2_z s^2}{s+u} - 2m^2_z u - 4m^2_z s \\ & + \frac{4m^2_w s^2}{s+u} - 2m^2_w u - 4m^2_w s \end{aligned} \tag{170}$$

Factoring the m^2_z and the m^2_w terms we have

$$\begin{aligned} M_S = \frac{g^2_{wwz}}{4m_W w 2m^2_z} \Bigg[& -\frac{m^2_z s}{m^2_w} + 2su + s^2 \\ & + 2m^2_z (\frac{2s^2 - us - u^2 - 2us - 2u^2}{s+u}) \\ & + 2m^2_w (\frac{2s^2 - us - u^2 - 2us - 2s^2}{s+u}) \end{aligned} \tag{171}$$

And the final bit of simplification,

$$M_s = \frac{g^2_{wwz}}{4m_W w 2m^2_z} \left(-\frac{m^2_z s}{m^2_w} + 2su + s^2 + 2m^2_z \frac{2s^2 - 3us - 3u^2}{s+u} + 2m^2_w \frac{-3us - u^2}{s+u} \right) \tag{172}$$

Which is exactly equation (29.22) in Schwartz! This verification was the first step in the workflow of this project. Similar algebraic steps were followed in our $W_L^+W_L^-$ scattering as outlined above.

9 Appendix 1

9.1 Polarization

Polarization describes the alignment of particles with respect to their direction of travel. The direction of travel is chosen to be the momentum vector of the particle. Helicity h then is quantified as the projection of the spin S onto the direction of the the particle's direction of motion $\vec{p}$

$$h = \vec{S} \cdot \frac{\vec{p}}{|p|} \tag{173}$$

This quantum number is only Lorentz-invariant for massless particles; this is because for a massless particle one cannot boost into a frame of reference for which its helicity switches, since it's always travelling at the speed of light (one can never "catch up" to it). Whereas for massive particles one can boost to a frame of reference in which one is travelling faster than the particle and hence its helicity switches.

For a massive particle with spin S, there are $2S + 1$ possible eignevalues for its helicity (massless particles only have $\pm S$). Since we are interested in $W^{\pm}$ and Z bosons, they have spin 1 and hence three possible values for h being 0 and ± 1. Helicity 0 means that the spin is orthogonal to the direction of momentum and is called longitudinally polarized, Hence a vector boson V with this polarization is denoted V_L. The ± 1 helicity states corresponds to a spin thatis aligned/antialigned with the direction of momentum, and are called right and left-handed polarizations, or transversal polarizations. A vector boson V with these helicities is denoted as V_T.

10 Appendix 2: Form Code

10.1 Form code for $W_L^+W_L^- \to W_L^+W_L^-$ Without the Higgs

10.2 Form code for $W_L^+W_L^- \to W_L^+W_L^-$ With the Higgs Channels Only

11 Appendix 3: Alternative derivation of unitarity limits

Using $u = -t - s + 4m_W^2$, and the relations in 68-70 we have

$$\begin{aligned} s &= 2m_W^2 + 2p_1p_4\cos\theta \\ &= 2m_W^2 + 2(\frac{s}{2} - m_W^2)\cos\theta \\ &= 4m_W^2 - 4m_W^2\cos\theta \end{aligned} \tag{174}$$

```
nwrite statistics;
V p1,p2,p3,p4,ks,kt,p,k;
I mu1,mu2,mu3,mu4,nu1,nu2,nu3,nu4,mu,nu,la,al,be;
S gwwz,gwwg,gwwww,g,s,t,u,mw,mz,[s-mz^2],[t-mz^2],[u-2*mw^2],[s+t],E,[t],[s],[u],n;
V eps1,eps2,eps3,eps4;
CF Vwwz,Vwwg,Vwwww,propz,propg,Asg,Asz,Atg,Atz,A4,A4test;
** Define amplitude structure
** [overall factor from couplings]*[polarization vectors]*[all the rest]
L Msz = -gwwz^2*eps1(mu1)*eps2(mu2)*eps3(mu3)*eps4(mu4)*Asz(p1,p2,p3,p4,mu1,mu2,mu3,mu4);
L Msg = -gwwg^2*eps1(mu1)*eps2(mu2)*eps3(mu3)*eps4(mu4)*Asg(p1,p2,p3,p4,mu1,mu2,mu3,mu4);
L Ms = Msz+Msg;
L Mtz = -gwwz^2*eps1(mu1)*eps2(mu2)*eps3(mu3)*eps4(mu4)*Atz(p1,p2,p3,p4,mu1,mu2,mu3,mu4);
L Mtg = -gwwg^2*eps1(mu1)*eps2(mu2)*eps3(mu3)*eps4(mu4)*Atg(p1,p2,p3,p4,mu1,mu2,mu3,mu4);
L Mt = Mtz+Mtg;
L M4 = gwwww*eps1(mu1)*eps2(mu2)*eps3(mu3)*eps4(mu4)*A4(mu1,mu2,mu3,mu4);
L M = Ms+Mt+M4;
** Express [all the rest] in terms of vertices and propagators:
** Vwwz: WWZ vertex
** Vwwg: WWg vertex
** Vwwww: WWWW vertex
** propz: Z propagator
** propg: gamma propagator
id Asz(p1?,p2?,p3?,p4?,mu1?,mu2?,mu3?,mu4?)=Vwwz(p1,p2,-ks,mu1,mu2,nu)*Vwwz(-p4,-p3,ks,mu4,mu3,la)*propz(s,ks,nu,la);
id Asg(p1?,p2?,p3?,p4?,mu1?,mu2?,mu3?,mu4?)=Vwwg(p1,p2,-ks,mu1,mu2,nu)*Vwwg(-p4,-p3,ks,mu4,mu3,la)*propg(s,ks,nu,la);
id Atz(p1?,p2?,p3?,p4?,mu1?,mu2?,mu3?,mu4?)=Vwwz(p1,-p3,-kt,mu1,mu3,nu)*Vwwz(-p4,p2,kt,mu4,mu2,la)*propz(t,kt,nu,la);
id Atg(p1?,p2?,p3?,p4?,mu1?,mu2?,mu3?,mu4?)=Vwwg(p1,-p3,-kt,mu1,mu3,nu)*Vwwg(-p4,p2,kt,mu4,mu2,la)*propg(t,kt,nu,la);
id A4(mu1?,mu2?,mu3?,mu4?)=Vwwww(mu1,mu3,mu4,mu2);
** Feynman rules for vertices and propagators
id Vwwz(p1?,p2?,p3?,mu1?,mu2?,mu3?)=d_(mu1,mu2)*(p1(mu3)-p2(mu3))+d_(mu2,mu3)*(p2(mu1)-p3(mu1))+d_(mu3,mu1)*(p3(mu2)-p1(mu2));
id Vwwg(p1?,p2?,p3?,mu1?,mu2?,mu3?)=d_(mu1,mu2)*(p1(mu3)-p2(mu3))+d_(mu2,mu3)*(p2(mu1)-p3(mu1))+d_(mu3,mu1)*(p3(mu2)-p1(mu2));
id Vwwww(mu?,nu?,al?,be?)=-d_(be,al)*d_(mu,nu)-d_(nu,al)*d_(be,mu)+2*d_(mu,al)*d_(be,nu);
id propz(s,k?,nu?,la?)=[s-mz^2]^-1*(-d_(nu,la)+k(nu)*k(la)/mz^2);
id propz(t,k?,nu?,la?)=[t-mz^2]^-1*(-d_(nu,la)+k(nu)*k(la)/mz^2);
id propg(s,k?,nu?,la?)=s^-1*(-d_(nu,la));
id propg(t,k?,nu?,la?)=t^-1*(-d_(nu,la));
** Use that p*eps(p)=0, and substitute expression for polarization vectors
id p1.eps1=0;
id p2.eps2=0;
id p3.eps3=0;
id p4.eps4=0;
id eps1.p?=p1.p/mw+2*mw/[u-2*mw^2]*p4.p;
id eps2.p?=p2.p/mw+2*mw/[u-2*mw^2]*p3.p;
id eps3.p?=p3.p/mw+2*mw/[u-2*mw^2]*p2.p;
id eps4.p?=p4.p/mw+2*mw/[u-2*mw^2]*p1.p;
** Simplify result using simple kinematic relations and express it in terms of {s,t,u}
repeat;
id p?.ks=p.p1+p.p2;
id p?.kt=p.p1-p.p3;
id p4.k?=-p3.k+p1.k+p2.k;
id p1.p1=mw^2;
id p2.p2=mw^2;
id p3.p3=mw^2;
id p1.p2=1/2*(s-2*mw^2);
id p1.p3=-1/2*(t-2*mw^2);
id p2.p3=-1/2*(u-2*mw^2);
id u=-s-t+4*mw^2;
endrepeat;
** Try further simplifications and expand denominators in the large-energy limit {s,t,u}>>{MW,MZ}
repeat;
id s*[s-mz^2]^-1=1+mz^2*[s-mz^2]^-1;
id t*[t-mz^2]^-1=1+mz^2*[t-mz^2]^-1;
id u*[u-2*mw^2]^-1=1+2*mw^2*[u-2*mw^2]^-1;
endrepeat;
.sort
repeat;
id [s-mz^2]^-1=1/s*(1+mz^2/s+mz^4*s^-2);
id [t-mz^2]^-1=1/t*(1+mz^2/t+mz^4*t^-2);
id [u-2*mw^2]^-1=1/u*(1+2*mw^2/u+4*mw^4/u^2);
** Keep only leading terms in the large-energy limit. To achieve that, rescale all invariants by a suitable energy factor (E^n) and then drop all terms proportional to negative powers of E. This requires to introduce different symbols for the rescaled invariants.
id t=E^2*[t];
id s=E^2*[s];
id u=E^2*[u];
id t^-1=E^-2*[t]^-1;
id s^-1=E^-2*[s]^-1;
id u^-1=E^-2*[u]^-1;
endrepeat;
id E^n?neg0_ = 0;
** Revert back to {s,t,u}. To compare with Schwartz's results, eliminate u in favor of {s,t}.
repeat;
id [s]=s;
id [s]^-1=s^-1;
id [t]=t;
id [t]^-1=t^-1;
id [u]=-s-t+(4*mw^2)/E^2;
id [u]^-1=-[s+t]^-1*(1+4*mw^2/E^2/[s+t]);
endrepeat;
id E^n?neg0_ = 0;
id E = 1;
** id gwwg^2 = -gwwz^2 + g^2;
** id gwwz^2 = g^2*mw^2*mz^-2;

** id gwwww = g^2;
bracket gwwz,gwwg,gwwww,[s-mz^2],[t-mz^2],[u-2*mw^2],mw,mz,E;
print;
.end
```

Figure 9: Form code for $W_L^+ W_L^- \to W_L^+ W_L^-$ Without the Higgs

```
#write statistics;
V p1,p2,p3,p4,ks,kt,p,k;
I mu1,mu2,mu3,mu4,nu1,nu2,nu3,nu4,mu,nu,la,al,be;
S gwwh,s,t,u,mw,mh,[s-mh^2],[t-mh^2],[t-2*mw^2],[s+u],E,[t],[s],[u],n;
V eps1,eps2,eps3,eps4;
CF Vwwh,proph,Ash,Ath,g;
** Define amplitude structure:
** [overall factor from couplings]*[polarization vectors]*[all the rest]
L Msh = -gwwh^2*eps1(mu1)*eps2(mu2)*eps3(mu3)*eps4(mu4)*Ash(p1,p2,p3,p4,mu1,mu2,mu3,mu4);
L Mth = -gwwh^2*eps1(mu1)*eps2(mu2)*eps3(mu3)*eps4(mu4)*Ath(p1,p2,p3,p4,mu1,mu2,mu3,mu4);
L M = Msh+Mth;
** Express [all the rest] in terms of vertices and propagators:
** Vwwh: WWH vertex
** proph: H propagator
id Ash(p1?,p2?,p3?,p4?,mu1?,mu2?,mu3?,mu4?)=Vwwh(-ks,p1,p2,mu1,mu2)*Vwwh(-p3,ks,-p4,mu3,mu4)*proph(s,ks);
id Ath(p1?,p2?,p3?,p4?,mu1?,mu2?,mu3?,mu4?)=Vwwh(p1,-kt,-p3,mu1,mu3)*Vwwh(-p4,kt,p2,mu4,mu2)*proph(t,kt);
** Feynman rules For vertices and propagators
id Vwwh(p1?,p2?,k?,mu1?,mu2?)=d_(mu1,mu2);
id proph(s,k?)=1/[s-mh^2];
id proph(t,k?)=1/[t-mh^2];
id p1.eps1=0;
id p2.eps2=0;
id p3.eps3=0;
id p4.eps4=0;
id eps1.p?=p1.p/mw-2*mw/s*p2.p;
id eps2.p?=p2.p/mw-2*mw/s*p1.p;
id eps3.p?=p3.p/mw-2*mw/s*p4.p;
id eps4.p?=p4.p/mw-2*mw/s*p3.p;
repeat;
id p?.ks=p.p1+p.p2;
id p?.kt=p.p1-p.p3;
id p4.k?=-p3.k+p1.k+p2.k;
id p1.p1=mw^2;
id p2.p2=mw^2;
id p3.p3=mw^2;
id p1.p2=1/2*(s-2*mw^2);
id p1.p3=-1/2*(t-2*mw^2);
id p2.p3=-1/2*(u-2*mw^2);
id u=-s-t+4*mw^2;
endrepeat;
repeat;
id s*[s-mh^2]^-1=1+mh^2*[s-mh^2]^-1;
id t*[t-mh^2]^-1=1+mh^2*[t-mh^2]^-1;
id t*[t-2*mw^2]^-1=1+2*mw^2*[t-2*mw^2]^-1;
endrepeat;
.sort
repeat;
id [s-mh^2]^-1=1/s*(1+mh^2/s);
id [t-mh^2]^-1=1/t*(1+mh^2/t);
id [t-2*mw^2]^-1=1/t*(1+2*mw^2/t);
id t=E^2*[t];
id s=E^2*[s];
id u=E^2*[u];
id t^-1=E^-2*[t]^-1;
id s^-1=E^-2*[s]^-1;
id u^-1=E^-2*[u]^-1;
endrepeat;
id E^n?neg0_ = 0;
repeat;
id [s]=s;
id [s]^-1=s^-1;
id [t]=-s-u+(4*mw^2)/E^2;
id [t]^-1=-[s+u]^-1*(1+4*mw^2/E^2/[s+u]);
id [u]=u;
id [u]^-1=u^-1;
endrepeat;
id E^n?neg0_ = 0;
id E = 1;
id gwwh^2 = g^2*mw^2;

bracket g,[s-mh^2],[t-mh^2],[t-2*mw^2],mw,mh,E;
print;
.end
```

Figure 10: Form code for $W_L^+ W_L^- \to W_L^+ W_L^-$ With the Higgs Channels Only

Therefore we see that s has a term that depends on $\cos\theta$ and another term that does not. Now solving for t we have

$$\begin{aligned} t &= -s - u + 4m_W^2 \\ &= -p_1^2 - p_2^2 - 2p_1p_2\cos\theta - p_1^2 - p_4^2 + 2p_1p_4\cos\theta + 4m_W^2 \\ &= -m_W^2 - m_W^2 - 2(\frac{s}{2} - m_W^2)\cos\theta - m_W^2 - m_W^2 - 2(\frac{s}{2} - m_W^2)\cos\theta + 4m_W^2 \\ &= \cancel{-4m_W^2} - s\cos\theta + 2m_W^2\cos\theta - s\cos\ + 2m_W^2\cos\theta + \cancel{4m_W^2} \\ &= 2\cos\theta(-s + 2m_W^2) \end{aligned} \tag{175}$$

Hence we see that t has only $\cos\theta$-dependent terms. Using these results, we have

$$\begin{aligned} u &= -t - s + 4m_W^2 \\ &= 2\cos\theta\ s - 4m_W^2\cos\theta + 4m_W^2\cos\theta \\ &= \underbrace{-4m_W^2}_{a_0} + \underbrace{(2s - 4m_W^2 + u)}_{a_1}\cos\theta \end{aligned} \tag{176}$$

Hence we see that our terms in the partial wave expansion are $a_0 = -4m_W^2$ and $a_1 = 2s - 4m_W^2 + u$. Plugging this into **??** for the first (0th) term we have

$$a_0^{\text{Without Higgs}} = \frac{g^2E^2}{m_W^2} \tag{177}$$

And plugging in $a_0 = \frac{1}{2}$ we can find the energy at which unitarity breaks

$$\frac{1}{2} = \frac{g^2E^2}{M_W^2} \implies E = \frac{m_W}{\sqrt{2}\ g} \tag{178}$$

12 Acknowledgements

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