Introduction Into Probability Theory

MTH 231

Lecture1

Chapter II

Fundamentals of Probability

Our Course MT231

Module leader : Dr. Nashwa Mohamed yhiea
E mail: nashwa_Mohamed@science.suez.edu.eg
Office hours : from 11 – 1 Monday
: from 9 -10, 2 - 3 Thursday in my office.

- **Assessment**

60% → **Final Unseen exam**

40% → **In class assessment , two Midterm exam and oral exam**

- Main Text book : Probability & Statistics for Engineers & Scientists, by: Walpole, Myers.

Contents of the course

Chapter I : Fundamentals of Probability

Chapter II : Random Variables and Probability

Chapter III

- Mathematical Expectation

Chapter IV

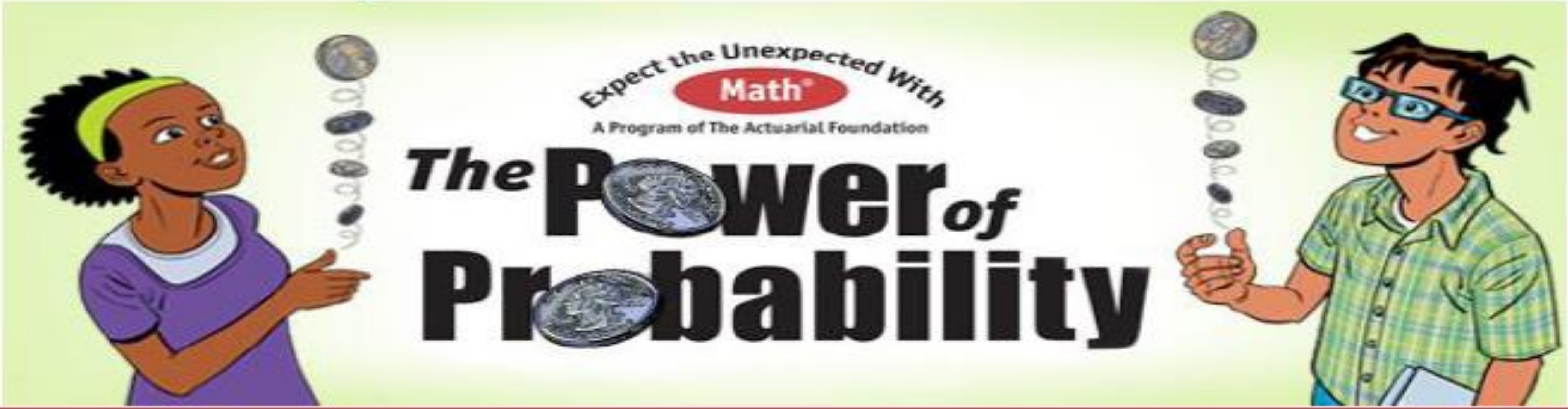
- Some Discrete Probability Distribution

Chapter V

- Some Continuous probability Distribution

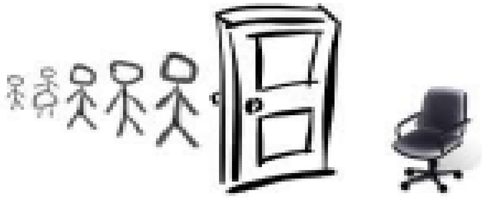
Today's lecture

- Random Experiments
- Sample Space
- Set Operations
- Probability Rules



Random Experiments

- Examples:
  - tossing a die or coin
  - counting the number of calls arriving at a telephone exchange during a fixed time period
  - choosing at random 10 people and measuring their height
  - selecting a random sample of 50 people and observing the number of left-handers
  - Answering a true or false question
  - . … . ???



Sample Space

Consider an experiment whose outcome is not predictable with certainly in advance. However, although the outcome of the experiment will not be known in advance, let us suppose that the set of all possible outcomes of an experiment is known.

➢ The set of all possible outcomes of a statistical experiment is called the **sample space and is represented by the symbol S.**

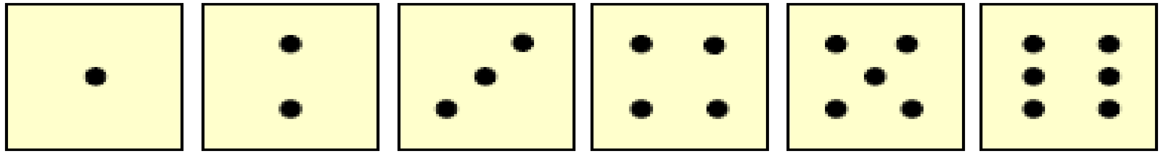
Examples

➢**Example(1): Tossing a coin once.**

S = {H, T}

➢**Example(2): Rolling a die once.**

S = {1, 2, 3, 4, 5, 6}



➢**Example(3): Tossing two coins, then the sample space consists of the following four points**

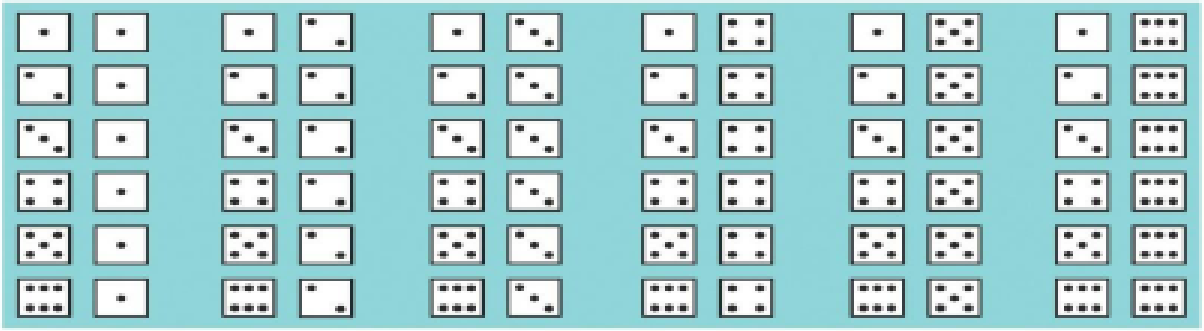
S = {(H, H), (H, T), (T, H), (T, T)}

The outcome will be (H,H) if both coins are heads , (H,T) if the first coin is heads and the second tails , (T,H) if the first is tails and the second heads , and (T,T) if both coins are tails .

Sample Space *(Continued)*

➢**Example(4) : Rolling two dice.**

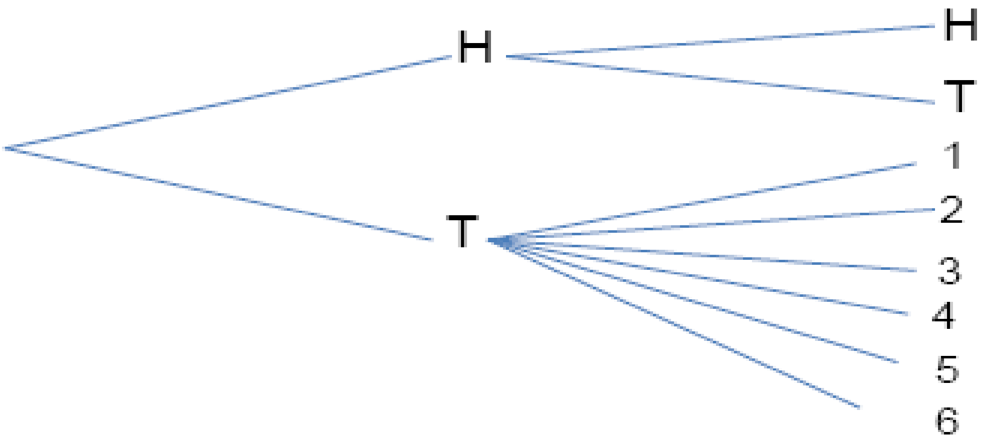
S = {(1,1), (1,2), ... , (1,6),
(2,1), (2,2), ... , (2,6),
. . . .
(6,1), (6,2), ... , (6,6)}



Or , we can write

S = { (i , j) : i , j = 1,2,3,4,5,6 } where the outcome (i , j) is said to occur if i appears on the leftmost die and j on the other die

➢**Example(5)**: An experiment consists of flipping a coin and then flipping it a second time if a head occurs. If a tail occurs on the first flip, then a die is tossed once. To list the elements of the sample space providing the most information, we construct the tree diagram as follows:



S = {HH, HT, T1, T2, T3, T4, T5, T6}.

• Sample spaces with a large or infinite number of sample points are best described by a **statement or rule method.**

➢**Example (6)** :

▪ If the possible outcomes of an experiment are the set of cities in the world with a population over 1 million, our sample space is written as

$$S = \{x \mid x \text{ is a city with a population over 1 million}\},$$

which reads "*S is the set of all x such that x is a city with a population over 1 million.*" The vertical bar is read "such that."

▪ Similarly, if *S is the set of all points (x, y) on the boundary or the interior of a circle of radius 2 with center at the* origin, we write the **rule**

$$S = \{(x, y) \mid x^2 + y^2 \le 4\}.$$

➢**Example (7)**: If the experiment consists selecting three items at random from a manufacturing process, such that each item is inspected and classified defective, D, or non defective, N. Then the sample space will be

$$S = \{(D,D,D), (D,D,N), (D,N,D), (N,D,D), (D,N,N), (N,D,N), (N,N,D), (N,N,N)\}.$$

Events

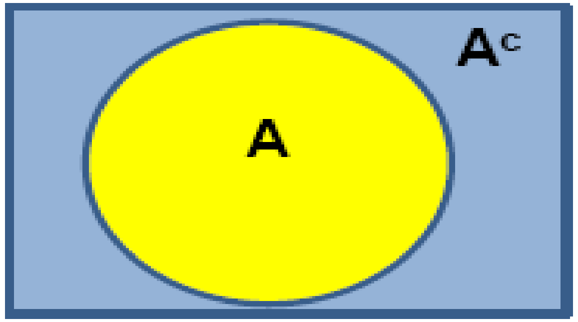
- An event, A, is a subset of a sample space. If A is an event, we say that A has occurred if it contains the outcomes that occurred $A \subseteq S$.
- If an event A contains no outcomes, then A is an impossible event.

➢ **Example (8)**: In example 3, if $A = \{(H, H), (H, T)\}$, then A is the event that a head appears on the first coin.

➢ **Example (9)**: In example 4, if $A = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$, then A is the event that the sum of the dice equals 7 .

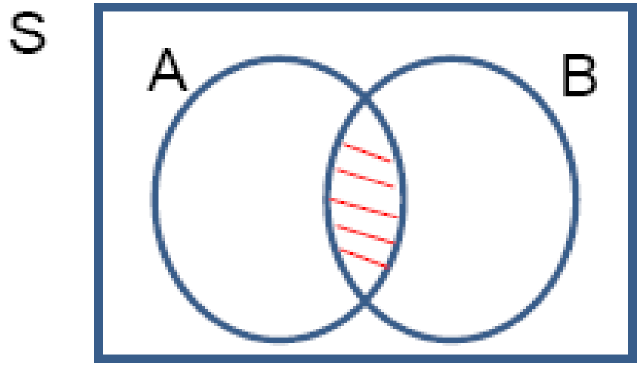
Complementary Events

The complement of an event A with respect to S is the subset of all elements of S that are not in A. We denote the complement of A by $\mathbf{A^c}$.



Blue region : $\mathbf{A^c}$

Definition: The intersection of two events A and B, denoted by $A \cap B$, or AB, is the event containing all elements that are common to A and B.



Shaded region : **AB**

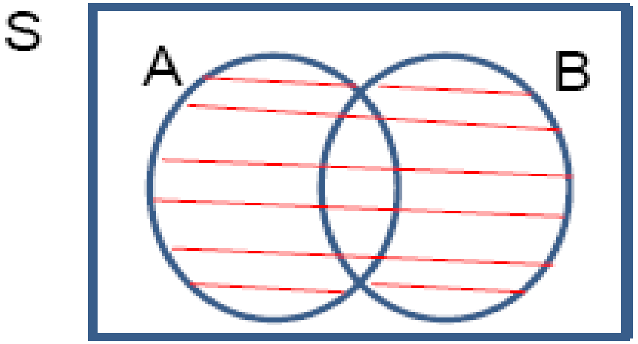
➢ **Example (10)**: In example 3, if A = {(H, H), (H, T), (T, H)} is the event that at least 1 head occurs, and B = {(H, T), (T, H), (T, T)} is the event that at least 1 tail occurs, then $A \cap B$ = {(H, T), (T, H)} is the event that exactly 1 head and 1 tail appear.

➢ **Example (11)**: In example 4, if A = { (1,6),(2,5), (3,4), (4,3), (5,2), (6,1) } , is the event that the sum of the dice equals 7 and B = { (1,5), (2,4), (3,3), (4,2), (5,1) } is the event that the sum is 6 , then the event $A \cap B = \emptyset$. That is A and B have no elements in common and therefore, cannot occur simultaneously.

Definition: Two events A and B are mutually exclusive, or disjoint if $A \cap B = \emptyset$, that is, if A and B have no elements in common.

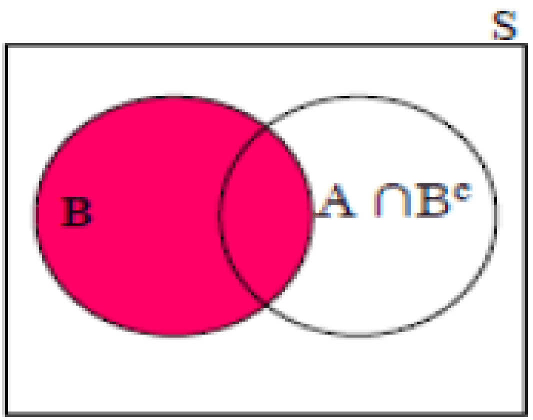
*It is clear that A and A^c are mutually exclusive.

Definition: The union of the two events A and B, denoted by $A \cup B$, is the event containing all the elements that belong to A or B or both .



Shaded region : $\mathbf{A} \cup \mathbf{B}$

Definition: The difference of the two events A and B, denoted by $A - B$, is the event containing all the elements that belong to A and not belong to B, that is A only ($A \cap B^c$).



➢ **Example (12)**: We assume the sample space **S** = $\{t \mid t \in \mathbb{R}\}$.
Let **A** = $\{t \mid t < 100\}$, **B** = $\{t \mid 50 \le t \le 200\}$, **C** = $\{t \mid t > 150\}$. Then

a) $\mathbf{A} \bigcup \mathbf{B} = \{t \mid t \le 200\}$

b) $\mathbf{A} \bigcap \mathbf{B} = \{t \mid 50 \le t < 100\}$

c) $\mathbf{B} \bigcup \mathbf{C} = \{t \mid t \ge 50\}$

d) $\mathbf{B} \bigcap \mathbf{C} = \{t \mid 150 < t \le 200\}$

e) $\mathbf{A} \bigcap \mathbf{C} = \emptyset$

f) $\mathbf{A} \bigcup \mathbf{C} = \{t \mid t < 100 \text{ or } t > 150\}$

g) $\mathbf{A}^c = \{t \mid t \ge 100\}$

h) $\mathbf{C}^c = \{t \mid t \le 150\}$

Set operations

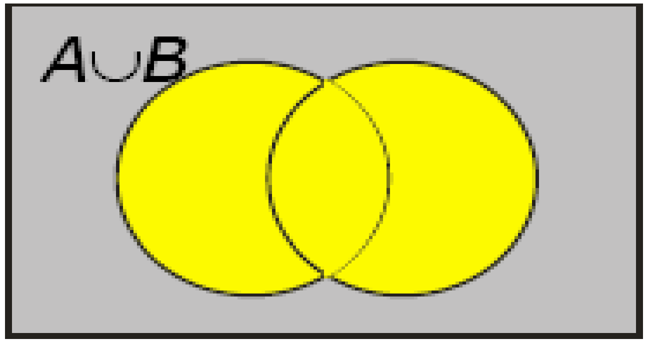
1. the set $A \cup B$ (A **union** B) is the event that A *or* B *or* both occur
2. the set $A \cap B$ (A **intersection** B) is the event that A *and* B both occur,
3. the event A^c (A **compleent**) is the event that A does *not* occur
4. If $A \subset B$ (A is a **subset** of B) then event A is said to *imply* event B
5. If $A \cap B = \emptyset$, then A and B are called **disjoint** events.

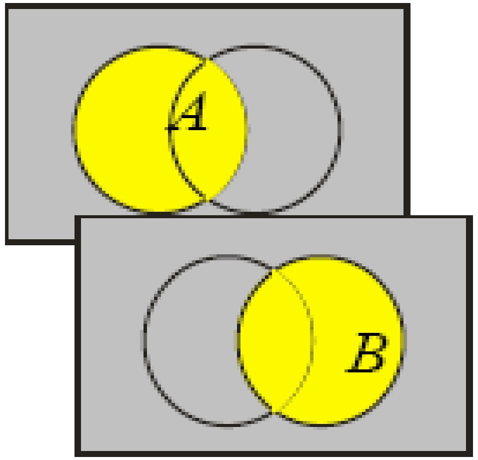
Sets Operations:

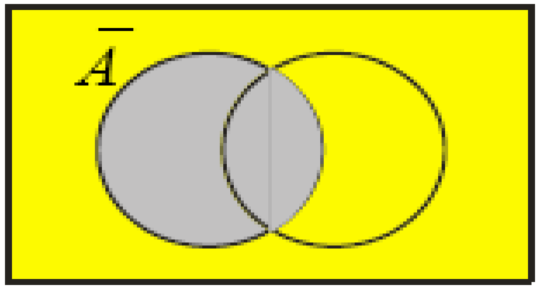
- De Morgan's laws:

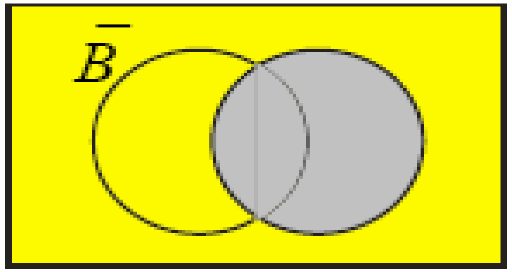
$$(1)\ \overline{A \cup B} = \overline{A} \cap \overline{B}$$

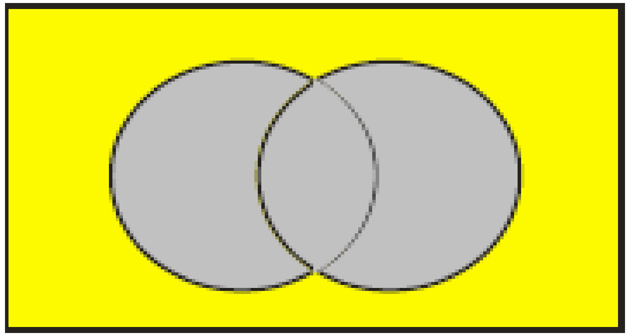
$$(2)\ \overline{A \cap B} = \overline{A} \cup \overline{B}$$





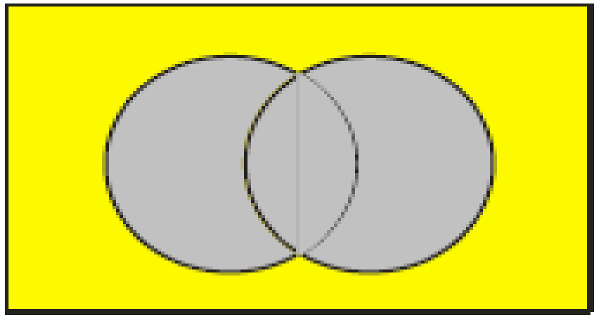






$\overline{A \cup B}$

$\overline{A \cup B}$



Some Cases About Events

- Favorable Cases
- Equally likely outcome
- Mutually Exclusive Cases

Some Expressions

- At most one event $=(\boldsymbol{A} \cap \boldsymbol{B})^{c}$,
- At least one event = $\boldsymbol{A} \cup \boldsymbol{B}$,
- One event only = $A - B = A\,B^{c}$,
- Exactly one event = $(A - B) \cup (B - A)$.

Probability Axioms

- Probability is the numerical measure of the likelihood that an event resulting from a statistical experiment will occur.
- The probability of any event must be between 0 and 1, inclusively

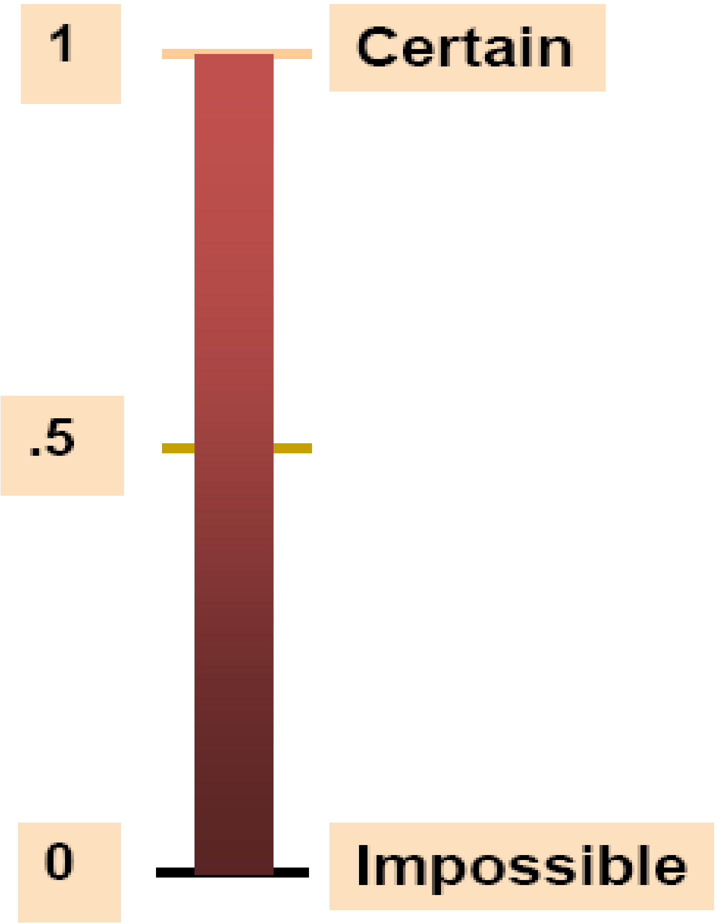
$0 \le P(A) \le 1$ For any event A

- The sum of the probabilities of all mutually exclusive events is 1

$$P(S)=1$$

- If A_1, A_2, A_3 … is a sequence of mutually exclusive events (i.e., $A_i \cap A_j = \varnothing$, $i \neq j$) then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right)=\sum_{i=1}^{\infty} P(A_i)$$



Equally Likely Outcomes

For finite sample spaces having equally likely outcomes, the probability of an event A

$$P(A) = \frac{\text{number of outcomes in the event A}}{\text{number of outcomes in the sample space S}} = \frac{n(A)}{n(S)}$$

- **Theorem** :The probability function P satisfies:

i. $P(\mathbf{A})+P(\mathbf{A}^c) = 1$ ii. $P(\mathbf{A} \cup \mathbf{B}) = P(\mathbf{A})+P(\mathbf{B})-P(A \cap B)$

Proof: i. $1 = P(\mathbf{S}) = P(A \cup A^c) = P(\mathbf{A})+P(\mathbf{A}^c)$

ii. $\mathbf{A} \cup \mathbf{B} = \mathbf{A} \cup (B \cap A^c) \Longrightarrow P(\mathbf{A} \cup \mathbf{B}) = P(\mathbf{A})+P(B \cap A^c)$ (1)

Also, $\mathbf{B} = B \cap S = B \cap (A \cup A^c) = (B \cap A) \cup (B \cap A^c)$. Hence,

$$P(\mathbf{B}) = P(B \cap A) + P(B \cap A^c) \quad (2)$$

The two eqs. (1) and (2) are combined.

➢**Example (13):** A coin is tossed twice. What is the probability that at least one head occurs?

Solution: The sample space for this experiment is

$$S = \{(H, H), (H, T), (T, H), (T, T)\} \ , \ n(S)=4$$

If the coin is balanced, each of these outcomes would be equally to occur.

If A represents the event of at least one head occurring, then

$A = \{(H, H), (H, T), (T, H)\}$, $n(A)=3$, and

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{4}$$

➢**Example (14)**: The probability that a student passes mathematics is 2/3, and the probability that he passes English is 4/9. If the probability of passing both courses is 1/4, what is the probability that the student will pass at least one of these courses?

Solution: Let **M** be the event "**passing mathematics**," and **E** the event that "**passing English**", then

$$P(M \cup E) = P(M) + P(E) - P(M \cap E) = \frac{2}{3} + \frac{4}{9} - \frac{1}{4} = \frac{31}{36}$$

➢**Example (15)**: What is the probability of getting a total of 7 or 11 when a pair of dice is tossed?

Solution: $S = \{ (i , j) : i , j = 1, 2, 3, 4, 5, 6 \}$, and $n(S) = 36$.

Let A be the event that 7 occurs and B the event that 11 comes up.

Now,

$A = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$, $B = \{(5,6), (6,5)\}$,

$n(A) = 6, \qquad n(B) = 2,$

$$P(A) = \frac{n(A)}{n(S)} = \frac{6}{36} = \frac{1}{6} \qquad\qquad P(B) = \frac{n(B)}{n(S)} = \frac{2}{36} = \frac{1}{18}$$

The events A and B are **mutually exclusive,** since a total of 7 and 11 cannot both occur on the same toss.

Therefore, $P(A \cup B) = P(A) + P(B) = \frac{1}{6} + \frac{1}{18} = \frac{2}{9}$

➢**Example (16)**: Let A and B be events with P(A)=1/2, P(B)=3/8, and P(A∩B)=1/4 . Find P(A U B), P(B^c), P(A^c ∩ B^c), P(A^c U B^c), P(A ∩ B^c)

Solution:

(i) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{2} + \frac{3}{8} - \frac{1}{4} = \frac{5}{8}$

(ii) $P(B^c) = 1 - P(B) = 1 - \frac{3}{8} = \frac{5}{8}$

(iii) $P(A^c \cap B^c) = P(A \cup B)^c = 1 - P(A \cup B) = 1 - \frac{5}{8} = \frac{3}{8}$

(iv) $P(A^c \cup B^c) = P(A \cap B)^c = 1 - P(A \cap B) = 1 - \frac{1}{4} = \frac{3}{4}$

(v) $P(A \cap B^c) = P(A) - P(A \cap B) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

Probability:

PROBABILITY RULES

Rule 1. The probability $P(A)$ of any event A satisfies $0 \leq P(A) \leq 1$.

Rule 2. If S is the sample space in a probability model, then $P(S) = 1$.

Rule 3. For any event A,

$$P(\text{A does not occur}) = 1 - P(A)$$

Rule 4. Two events A and B are **disjoint** if they have no outcomes in common and so can never occur simultaneously. If A and B are disjoint,

$$P(A \text{ or } B) = P(A) + P(B)$$

This is the **addition rule for disjoint events.**

Theorem

- Let A and B be events. Then,
  - 1. $P(\emptyset) = 0$.
  - 2. if $A \subset B$,then $P(A) \leq P(B)$.
  - 3. $P(A) \leq 1$.
  - 4. $P(A^c) = 1 - P(A)$.
  - 5. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Some properties

From general properties of sets and the properties of definition of probability, we can derive some useful properties of probability:

- ✓ If A and B are subsets of S then **$P(A \cap B^c) = P(A) - P(A \cap B)$**
- ✓ If A and B are two events of S such that $A \subset B$, then **$P(A) \leq P(B)$**
- ✓ **$P(\emptyset) = 0$, $P(S) = 1$**

De Morgan's laws:

- ✓ $P(A^c \cup B^c) = P(A \cap B)^c = 1 - P(A \cap B)$
- ✓ $P(A^c \cap B^c) = P(A \cup B)^c = 1 - P(A \cup B)$

- **Example (18)**: A class consists of 30 students, 18 of them study statistics, 10 study computer and 6 study both statistic and computer. If a student is selected at random. Calculate the probability that this student:

(a) is one of those studying at least one of the subjects.

(b) is one of those studying at most one of the subjects?

(c) does not study any of the subjects.

(d) is one of those studying only one subject.

Solution:

Let S = the event that , the student study Statistics , and

C= the event that , the student study Computer

$P(S) = 18/30 = 3/5$, $P(C) = 10/30 = 1/3$ and $P(S \cap C) = 6/30 = 1/5$

(a) $P(S \cup C) = P(S) + P(C) - P(S \cap C) = \frac{3}{5} + \frac{1}{3} - \frac{1}{5} = \frac{11}{15}$

(b) $P(S^c \cup C^c) = P(S \cap C)^c = 1 - P(S \cap C) = 1 - \frac{1}{5} = \frac{4}{5}$

(C) $P(S^c \cap C^c) = P(S \cup C)^c = 1 - P(A \cup B) = 1 - \frac{11}{15} = \frac{4}{15}$

(d) $P(S\text{ only}) + P(C\text{ only}) = P(S \cap C^c) + P(S^c \cap C)$

$= [P(S) - P(S \cap C)] + [P(C) - P(S \cap C)]$

$= P(S) + P(C) - 2P(S \cap C)$

$= \frac{3}{5} + \frac{1}{3} - \frac{2}{5} = \frac{8}{15}$

➢**Example (19):** A die is loaded in such a way that an even number is twice as likely to occur as an odd number. If A is the event that a number less than 4 occurs on a single toss of the die, find P(A).

Solution :

The sample space is $S = \{ 1, 2, 3, 4, 5, 6\}$

We assign a probability ω to each odd number and a probability 2ω to each even number ,
i.e. $P(1) = P(3) = P(5) = \omega$ and $P(2) = P(4) = P(6) =2\omega$

Since the sum of the probabilities must be 1 , we have

$P(1) + P(2)+ P(3) + P(4) + P(5) + P(6) = 1$ then we have

$\omega+ 2\omega+ \omega+ 2\omega+ \omega+ 2\omega =9\omega= 1$, so $\omega = 1/9$.

Hence $P(1) = P(3) = P(5) = /9$ and $P(2) = P(4) = P(6) = 2 / 9$.
Therefore
$A = \{ 1, 2, 3\}$ and
$P(A) = P(1) + P(2)+ P(3)= 1 / 9 + 2 / 9 + 1 / 9 = 4 / 9$

Questions!

YOU AREN'T YOUR PAST,
YOU ARE PROBABILITY
OF YOUR FUTURE

OPRAH WINFREY