Introduction Into Probability Theory

MTH 231

Lecture 5

Chapter IV

Mathematical Expectations of Random Variable

Today's lecture

- Mathematical Expectation

- Variance

Mathematical Expectation

Definition: Let X be a discrete variable with the probability distribution
$P(x_i) = P(X = x_i)$, $i = 1, 2, \ldots, n$.
The **mean** or **expected value** of X is

$$\mu = E(X) = \sum_{i=1}^{n} x_i\, P(x_i)$$

➢**Example ( )[1]** A shipment of 8 similar microcomputers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers,
(a) Find the probability distribution for the number of defectives.
(b) Find the expected value of the random variable X

Solution: Let X be a random variable whose values x are the possible numbers of defective computers purchased by the school.
Then X = 0, 1, 2. Now,

(a) $P(0) = P(X = 0) = \dfrac{\binom{3}{0}\binom{5}{2}}{\binom{8}{2}} = \dfrac{10}{28}$ $\qquad P(1) = P(X = 1) = \dfrac{\binom{3}{1}\binom{5}{1}}{\binom{8}{2}} = \dfrac{15}{28}$

$$P(2) = P(X = 2) = \frac{\binom{3}{2}\binom{5}{0}}{\binom{8}{2}} = \frac{3}{28}$$

Thus , the probability distribution of X is

X	0	1	2
P(X = x)	10/28	15/28	3/28

(b) $\mu = E(X) = (0)(\frac{10}{28}) + (1)(\frac{15}{28}) + (2)(\frac{3}{28}) = \frac{21}{28} = \frac{3}{4}$

Theorem (1): Let X be a discrete random variable with probability function P(x). The mean or expected value of the random variable g(X) is defined by

$$E[g(X)] = \sum_{i=1}^{n} g(x)\, P(x_i)$$

Note that : If $g(X) = X^2$, then

$$E[X^2] = \sum_{i=1}^{n} x_i^2\, P(x_i)$$

Properties of Expectation

1- $E(a) = a$; (a is a constant)

2- $E(a X) = a E(X)$; (a is a constant)

3- $E(a X + b) = a E(X) + b$; (a , b are constant)

➢**Example** Find E [X − 1] ^{2} for example

Solution

$E[X-1]^2 = E[X^2 - 2X + 1] = E[X^2] - 2\,E[X] + 1$

Since E[X] = 3/4

$$E[X^2] = \sum_{i=1}^{n} x_i^2\, P(x_i) = (0)^2(\frac{10}{28}) + (1)^2(\frac{15}{28}) + (2)^2(\frac{3}{28}) = \frac{27}{28} = 0.964$$

Then ,

$$E[X-1]^2 = E[X^2] - 2E[X] + 1$$

$$= \frac{27}{28} - 2(\frac{3}{4}) + 1 = \frac{13}{28} = 0.464$$

Variance of Discrete Random Variables

Definition: Let X be a random variable with mean **E[X] = μ** .
The variance of X, denoted by Var(X) = σ^2, is given by:

$$Var(X) = \sigma^2 = E[(X-\mu)^2]$$

The positive square variance, σ, is called the **standard deviation of X.**
The variance can be simplified to give a more simple formula
as follows:

$$Var(X) = \sigma^2 = E[X^2] - (E[X])^2 = E[X^2] - \mu^2$$

Properties of Variance

1- Var (a) = 0 ; (a is a constant)

2- Var(a X) = a^2 Var(X) ; (a is a constant

3- Var(a X +b) = a^2 Var(X) ; (a , b are constants)

➢**Example** Let the random variable X represent the number of defective parts for a machine when 3 parts are sampled from a production line and tested. The following is the probability distribution of X.

x	0	1	2	3
P(x)	0.51	0.38	0.10	0.01

Find **(a)** $E(X)$ **(b)** $E(X^2)$ **(c)** $Var(X)$ **(d)** $E(2X-1)$ **(e)** $Var(3X+1)$

Solution:

(a) $\mu = E(X) = (0)(0.51) + (1)(0.38) + (2)(0.10) + (3)(0.01) = 0.61$

(b) $E(X^2) = (0)^2(0.51) + (1)^2(0.38) + (2)^2(0.10) + (3)^2(0.01) = 0.87$

(c) $Var(X) = E(X^2) - [E(X)]^2 = (0.87) - (0.61)^2 = 0.4979$.

(d) $E(2X-1) = 2\,E(X) - 1 = 2\,(0.61) - 1 = 0.22$.

(e) $Var(3X+1) = 9\,Var(X) = 9\,(0.4979) = 4.4811$.

Mathematical Expectation and Variance

Definition If X is a continuous random variable with p.d.f., f(x), then we have:

$$E[X] = \mu = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

Definition Let X be a random variable with mean E[X] = μ. The variance of X, denoted by Var(X) = σ^2, is given by:

$$\mathrm{Var}(X) = \sigma^2 = E[(X-\mu)^2]$$

The variance can be simplified to give a more simple formula as:

$$\mathrm{Var}(X) = \sigma^2 = E[X^2] - (E[X])^2 = E[X^2] - \mu^2$$

➢**Example** Consider the probability density function given in example compute the following:

(a) F(x), and then find F(1.5) , F(3), F(-1)

(b) E(X), and Var(X)

(c) E(3X+1) and Var(-3x+4)

Solution

$$f(x)=\begin{cases}\frac{3}{4}(2x-x^2) & 0<x<2\\ 0 & \text{otherwise}\end{cases}$$

(a) $F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt = \frac{3}{4}\int_{0}^{x}(2t-t^2)\,dt = \frac{3}{4}\left(t^2-\frac{1}{3}t^3\right)\Big|_0^x$

$$= \frac{3}{4}\left(x^2-\frac{1}{3}x^3\right) = \frac{1}{4}(3x^2-x^3) = \frac{x^2}{4}(3-x)$$

$$\boxed{0 < x < 2}$$

Then , we can write F(x) as

$$F(x) = \begin{cases} 0 & 0 < x \\ \dfrac{x^2}{4}(3 - x) & 0 \le x < 2 \\ 1 & x \ge 2 \end{cases}$$

$$F(1.5) = \frac{(1.5)^2}{4}(3-(1.5)) = 0.84 , \qquad F(3) = 1, \qquad F(-1) = 0$$

$$\boxed{\textbf{(b)}\ E(x) = \int_{-\infty}^{\infty} x.f(x)dx} = \int_0^2 xf(x)dx = \frac{3}{4}\int_0^2 x.(2x - x^2)dt = \frac{3}{4}(\frac{16}{3} - 4) = 1$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2.f(x)dx = \int_0^2 x^2 f(x)dx = \frac{3}{4}\int_0^2 x^2.(2x - x^2)dt = \frac{3}{4}(8 - \frac{32}{5}) = \frac{6}{5}$$

$$\Longrightarrow \quad Var(X) = \sigma^2 = E[X^2] - (E[X])^2 = \frac{6}{5} - (1)^2 = \frac{1}{5} = 0.2$$

(c) $E(3X+1) = 3\,E(X) + 1 = (3)(1) + 1 = 4$ and $Var(-3X+4) = 9\,Var(X) = 9\,(0.2) = 1.8$

➢**Example (5)**: The time, in hours, it takes to locate and repair an electrical breakdown in a certain factory is a random variable call it X, whose density function is given by

Find $E[X^3]$

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution

$$\boxed{E[X^3] = \int_{-\infty}^{\infty} x^3 f(x)\,dx} = \int_0^1 x^3 (1)\,dx = \int_0^1 x^3\,dx$$

$$= \left.\frac{x^4}{4}\right|_0^1 = \frac{1}{4} = 0.25$$

Expected Value and Variance of Sums of Random Variables

1- If $X_1, X_2, \ldots, X_n$ are random variables, then

$$E[X_1 + X_2 + \ldots + X_n] = E[X_1] + E[X_2] + \ldots + E[X_n]$$

2- If $X_1, X_2, \ldots, X_n$ are independent random variables, then

$$Var[X_1 + X_2 + \ldots + X_n] = Var[X_1] + Var[X_2] + \ldots + Var[X_n]$$

➢**Example (3)**: A construction firm has recently sent in bids for 3 jobs worth (in profits) 10, 20, and 40 (thousand) dollars. If its probabilities of winning the jobs are respectively 0.2, 0.8, and 0.3. What is the firm's expected total profit ?

Solution: Letting X_i , i = 1, 2, 3 denote the firm's profit from job i , then
Total profit = $X_1 + X_2 + X_3$, and so, **E[Total profit] = $E[X_1] + E[X_2] + E[X_3]$**

$E[X_1] = 10\,(0.2) = 2$
$E[X_2] = 20\,(0.8) = 16$
$E[X_3] = 40\,(0.3) = 12$

$\Longrightarrow$ And thus the firm's expected total profit is 2 + 16 + 12 = 30 thousand dollars.

Expectation μ

- Discrete case:

$$\mu = \sum_{\text{all } x} x_i p(x_i)$$

- Continuous case:

$$\mu = \int_{\text{all } x} x_i p(x_i) dx$$

The Variance σ^2

- Discrete case: $\sigma^2 = E\left[(Y-\mu)^2\right]$

$$= \sum_{\text{all } y} (y-\mu)^2 p(y)$$

- Continuous case: $\sigma^2 = E\left[(Y-\mu)^2\right]$

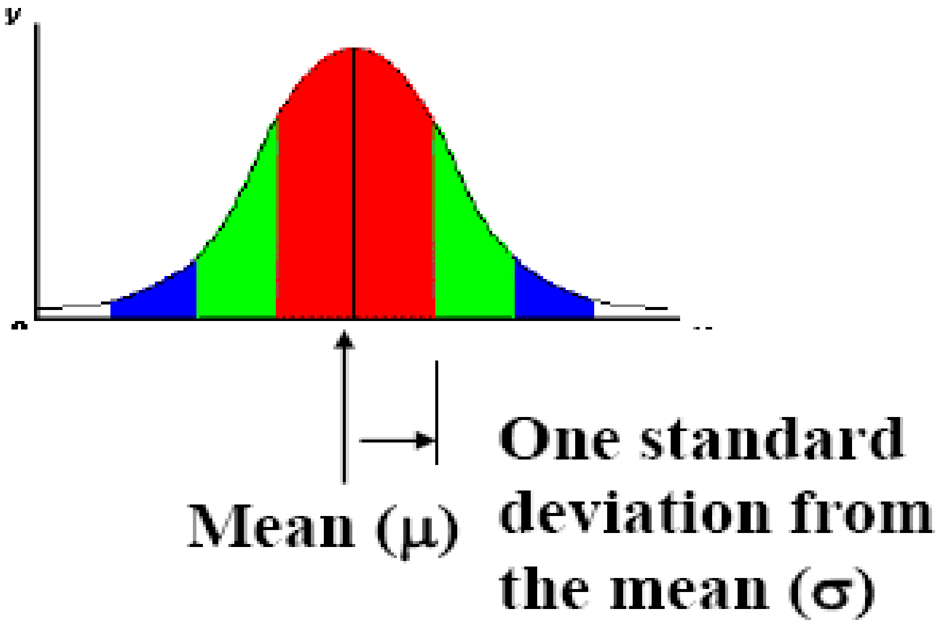
$$= \int_{all\, y} (Y-\mu)^2 f(y)dy$$

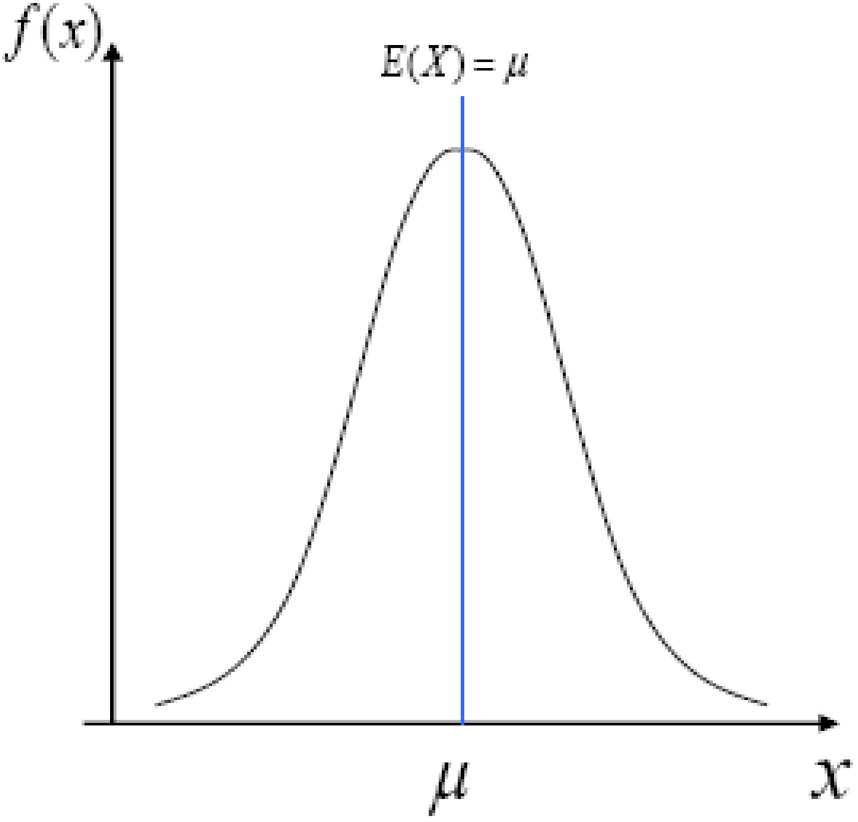
➢ Standard Deviation : $\sigma = +\sqrt{\sigma^2}$

V A R I A N C E

Interpretation

- μ indicate to the point of symmetry
- σ^2 indicate that the distribution is more spread out





In general:

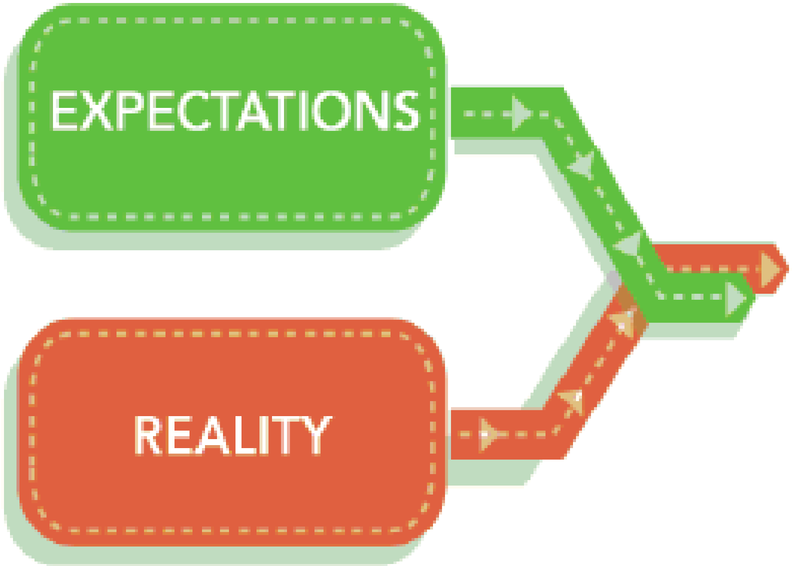
- Definition:
  - We have referred to E(X) and E(X^2) as the first and second moments of X, respectively. In general, E(X^k)) is the k-th moment of X.

  - The mean of any function g(x) is

$$E[g(X)] = \sum_{\text{all } x} g(x) \cdot p(x)$$

or

$$E[g(X)] = \int_{\text{all } x} g(x) \cdot p(x_i) dx$$



The k^{th} moment of X.

- **Definition:**

$$\mu_k = E\left(X^k\right) = \begin{cases} \sum_x x^k p(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} x^k f(x)\,dx & \text{if } X \text{ is continuous} \end{cases}$$

- **Note:**
  - The 1st moment of X is E(X)
  - The 2nd moment of X is E(X^2)
  - The 5th moment of X is E(X^5)

  ....
  ....

  - The kth moment of X is E(X^k)

Problems:

- Let X be a random variable for which $E(X) = \mu$ and $Var(X) = \sigma^2$. If c is an arbitrary constant, then Find

  $E\left((X-c)^2\right)$?

- Let X and Y be two independent random variables such that $E(X) = E(Y) = 4$ and $Var(X) = Var(Y) = 2$. If $U = 3X + 2Y$, then find $E(U)$ and $Var(U)$?

Questions!

We must become more comfortable with probability and uncertainty