Introduction Into Probability Theory

MTH 231
Lecture2
Chapter II

Fundamentals of Probability

Lecture Goals

After completing this lecture, we should be able to:

- Explain basic probability concepts
- Apply common rules of probability
- Compute conditional probabilities
- Determine whether events are statistically independent
- Use Bayes' Theorem for conditional probabilities

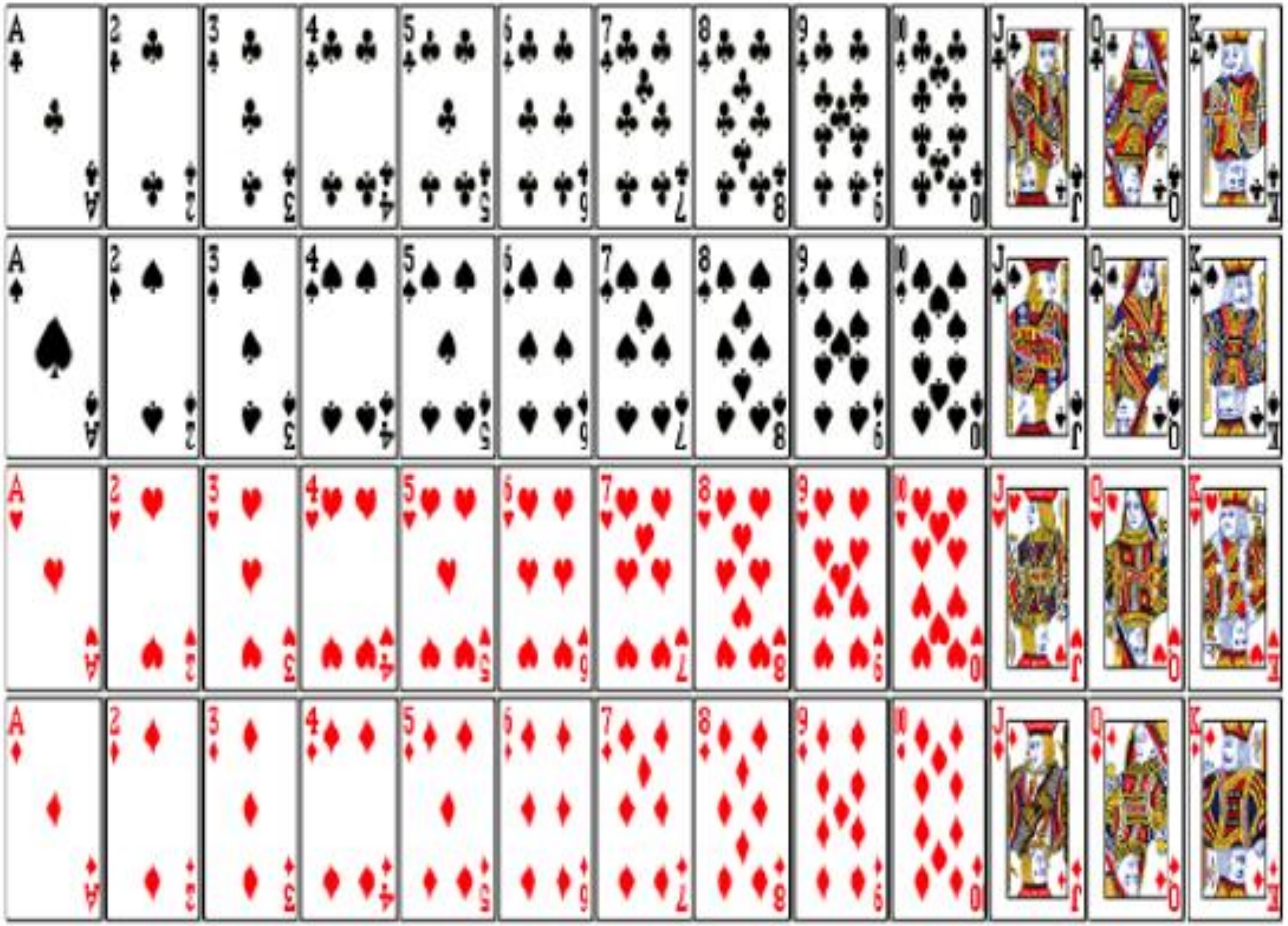
Topics are:

✓**Counting Rules**
✓ **Conditional Probability**
✓**Theorem of Total Probability**
✓**Baye's Theorem**
✓ **Independent Events**

➢Example (1):

A card is drawn from a standard deck. Find the probabilities of the following events:

1. Getting a queen.
2. Getting a club.
3. Getting a number.



Solution

We recall that the Sample Space is the collection of all possible events

i.e. All 52 cards of the bridge deck.

Note that n(S) = 52

1. A is the event of getting a queen.
   P(A) = n(A) / n(S) = 4 / 52 = 1 / 13

2. B is the event of getting a club.
   P(B) = n(B) / n(S) = 13 / 52 = 1 / 4

3. C is the event of getting a number
   P(C) = n(C) / n(S) = 40 / 52 = 10 / 13

Multiplication Rule

If an operation can be performed in n1 ways, and if for each of these a second operation can be performed in n2 ways, then the two operations can be performed together in n1 n2 ways. The rule is sometimes called the ***Basic Principle of Counting.***

➢ **Example(2)** : A salesman needs to get from city **A** to city **B** and then to city **C**. Three roads lead from **A** to **B**, and two lead from **B** to **C**. How many routes can the salesman take to accomplish his task ?

Solution

The event of getting from **A** to **C** is a sequence of two events.
The first (from **A** to **B**) can occur in 3 different ways.
The second (from **B** to **C**) can occur in 2 different ways.
Thus, getting from **A** to **C** can occur in
$3 \times 2 = 6$ different ways

➢**Example (3):** How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and
the final 4 by numbers ?

Solution

Using the Multiplication Rule , we have
26 . 26 . 26 . 10. 10 .10 .10 = 175760000 possible license plates .

➢**Example (4):** in example (3), how many license plates would be possible if no letter or digit can be repeated?

Solution

In this case there would be
26 . 25 . 24 . 10. 9 .8 .7 = 78624000 possible license plates .

➢**Example (5):** in example (3) how many license plates would be possible if repetitions are allowed, the three letters are vowels and all digits are even?

Solution

In this case there would be
5 . 5 . 5 . 5. 5 .5 .5 =78125 possible license plates

Permutations

The number of **arrangements** of **size** r from a set of n **distinct** objects is given by

$$\boxed{{}^{n}P_{r} = \frac{n!}{(n-r)!}; \qquad 0 \le r \le n}$$

Note that

If $r = n$

$${}^{n}P_{n} = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n!$$

If $r = 0$

$${}^{n}P_{0} = \frac{n!}{n!} = 1$$

➢**Example (6)**: In a class of ten students, six are to be chosen and seated in a row for a picture. How many different pictures are possible?

Solution

Thus, there are

$10 \times 9 \times 8 \times 7 \times 6 \times 5 = 151200$ different pictures.

Note that:

$$10 \times 9 \times 8 \times 7 \times 6 \times 5 = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times \frac{4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1}$$

$$= \frac{10!}{4!} = {}^{10}P_6 = 151200$$

➢**Example (7):** Mr. Jones has 10 books that he is going to put on his bookshelf. Of these, 4 are mathematics books, 3 are chemistry books, 2 are history books, and 1 is a language book. Jones wants to arrange his books so that all the books dealing with the same subject are together on the shelf. How many different arrangements are possible?

Solution

There are 4! 3! 2! 1! Arrangements such that the mathematics books are first in line, then the chemistry books, then the history books, and then the language book. Similarly, for each ordering of the subjects, there are 4! 3! 2! 1! Possible arrangements. Hence, as there are **4! Possible orderings of the subjects**, the desired answer is

4! 4! 3! 2! 1! = 6912

➢**Example (8)**: A class in probability theory consists of 6 men and 4 women. An examination is given, and the students are ranked according to their performance. Assume that no two students obtain the same score.

(a) How many different rankings are possible?

(b) If the men are ranked just among themselves and the women among themselves, how many different rankings are possible?

(c) What is the probability that women receive the top 4 scores ?

Solution

(a) As each ranking corresponding to a particular ordered arrangement of the 10 people, we see that the answer to this part is **10! = 3628800**.

(b) As there are 6! possible ranking of the men among themselves and 4! possible ranking of the women among themselves, it follows from there are **2 (6!)(4!)** = 2 (720) (24) = 34560 possible ranking in this case.

(c) Let A be the event that women receive the top 4 scores, then,

n(A) = (4!) (6!) = (24)(720) = 17280 . **n(S) = 10!** = 3628800, so

$$P(A) = \frac{n(A)}{n(S)} = \frac{4!\,6!}{10!} = \frac{1}{210}$$

Permutation with repetition

The number of ways of partitioning a set of **n** objects into **r** cells with n_1 elements in the first cell, n_2 elements in the second, and so forth, is

$$\frac{n!}{n_1!\ n_2!\ \dots\ n_r!}$$

where $n_1 + n_2 + \cdots + n_r = n$.

i.e. The number of distinct permutations of n objects of which n_1 are alike, n_2 are alike ,…, n_r are alike.

- **Example (9):** What is the number of permutations of the letters in the word "ball"?

Note that:

(i) The answer is not 4! since we do not have 4 distinct objects.

(ii) We have a set of only 3 distinct objects .But note that the answer is not 3!. The permutations here involve repetitions

Solution

- Suppose that there are no repetitions:

ball

- In this case, there are 4! permutations.
- **But for each of these permutations, there is exactly one permutation where l and l switch positions.**
- These are, really, **the same permutations**.
- Thus, the number of permutations of the letters in "ball" is

$$4!/2! = 12$$

➢**Example (10)**: How many distinct permutations can formed from all the letters of each word (i) them (ii) that (iii) radar (iv) unusual (v) sociological

Solution

(i) $4! = 24$ (ii) $\frac{4!}{2!} = \frac{24}{2} = 12$ (iii) $\frac{5!}{2!\ 2!} = \frac{120}{4} = 30$ (iv) $\frac{7!}{3!} = \frac{5040}{6} = 840$

(v) $\frac{12\,!}{3!2!2!2!}$

Combinations

The number of selections of size r from a set of n distinct objects is given by

$$^{n}C_{r} = \binom{n}{r} = \frac{n!}{(n-r)!\, r!}$$

➢**Example (11):** A committee of 3 is to be selected from a group of 20 people. How many different committees are possible?

Solution

There are $\binom{20}{3} = \frac{20!}{3!\,17!} = 1140$ possible committees.

➢ **Example (12):** 5 cards are drawn from a standard deck. Find the probability that they consists of 2 aces and 3 jacks.

Solution

Let A be the event that we have 2 aces and 3 jacks, the number of ways of being dealt 2 aces from 4 is $\binom{4}{2} = \frac{4!}{2!\,2!} = 6$

And the number of ways of being dealt 3 jacks from 4 is $\binom{4}{3} = \frac{4!}{3!\,1!} = 4$

By the multiplication rule we have $n(A) = \binom{4}{2}\binom{4}{3} = (6)(4) = 24$

The total number of 5-cards, all of which are equally likely, is

$$n(S) = \binom{52}{5} = 2598960$$

Therefore, the probability of event A of getting 2 aces 3 the and jacks is

$$P(A) = \frac{n(A)}{n(S)} = \frac{\binom{4}{2}\binom{4}{3}}{\binom{52}{5}} = \frac{24}{2598960} = 0.9x10^{-5}$$

➢**Example (13)**: A committee of 5 is to be selected from a group of 6 men and 9 women. If the selection is made randomly, what is the probability that the committee consists of 3 men and 2 women?

Solution: If A represents the event of selecting the committee, then

$$n(S) = \binom{15}{5} \qquad , \text{and } n(A) = \binom{6}{3}\binom{9}{2}$$

Therefore,
$$P(A) = \frac{n(A)}{n(S)} = \frac{\binom{6}{3}\binom{9}{2}}{\binom{15}{5}} = \frac{240}{1001}$$

Conditional Probability

The probability of an event B occurring when it is known that some event A has occurred is called a conditional probability and is denoted by P(B / A). The symbol P(B / A) is usually read

"B occurs given the probability that A occurs"

or simply

"the probability of B, given A "

Definition: The conditional probability of B, given A, denoted by is defined by

$$P(B/A) = \frac{P(A \cap B)}{P(A)} \quad \text{if } P(A) > 0$$

*A general formula for P(A / B) that is valid for all events A and B is derived by the same manner as just described. Namely, if the event A occurs, then in order for B to occur it is necessary that the actual occurrence be a point in both B and A ; that is, it must be in $A \cap B$.
Now, since we know that A has occurred , it follows that A becomes our **new (reduced) sample space.**

➢**Example (14)**: A coin is flipped twice. If we assume that all four points in the sample space S = {(H, H), (H, T), (T, H), (T, T)} are equally likely, what is the conditional probability that both flips result in heads, given that the first flip does?

Solution

If B = {(H, H)} denotes the event that both flips land heads, and A = {(H, H), (H, T)}
the event that the first flip lands heads, then the desired probability is given by

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

➢**Example (15):** Suppose that an office has 100 calculating machines. Some of these machines are electric **(E)** while others are manual **(M)**. And some of the machines are new **(N)** while others are used **(U).** The following table gives the number of machines in each category . A person enters the office, picks a machine at random, and discovers that it is new. What is the probability that it is electric ?

	E	M	Total
N	40	30	70
U	20	10	30
Total	60	40	100

Solution

We observe that n(S) = 100, n(E) = 60, n(M) =40 , n(N) = 70 , n(U) = 30. In terms of the notation introduced we wish to compute P(E / N)

$$P(E/N) = \frac{P(E \cap N)}{P(N)} = \frac{40/100}{70/100} = \frac{4}{7}$$

➢**Example (16)**: A bin contains 5 defective transistors (that immediately fail when put in use), 10 partially defective (that fail after a couple of hours of use), and 25 acceptable transistors. A transistor is chosen at random from the bin and put into use. If it does not immediately fail, what is the probability it is acceptable ?

Solution: Since the transistor did not immediately fail, we know that it is not one of the 5 defectives and so the desired probability is:

$$P(\text{acceptable}\,/\text{not defective}) = \frac{P(\text{acceptable}\,,\text{not defective})}{P(\text{not defective})}$$

$$= \frac{P(\text{acceptable})}{P(\text{not defective})}$$

$$P(\text{acceptable}/\text{not defective}) = \frac{25/40}{35/40} = \frac{25}{35} = \frac{5}{7}$$

Note that :

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

By multiplying both sides of the equation by P(A) we obtain that

$$P(A \cap B) = P(A)P(B/A)$$

In words, the last equation states that the probability that both A and B occur is equal to the probability that A occurs multiplied by the conditional probability of B given that A occurred . This equation is often quite useful in computing the probability of the intersection of events . This is illustrated by the following example .

➢**Example (17):** Ms. Perez figures that there is a 30 percent chance that her company will set up a branch office in Phoenix. If it does, she is 60 percent certain that she will be made manager of this new operation . What is the probability that Perez will be a Phoenix branch office manager ?

Solution: Let

B = the event that the company sets up a branch in Phoenix,

M = the event that Perez is made the Phoenix manager,

The desired probability is **$P(B \cap M)$,** which is obtained as follows:

$$P(B \cap M) = P(B)\, P(M / B) = (0.3)(0.6) = 0.18$$

Hence, there is an 18 percent chance that Perez will be the Phoenix manager .

Theorem of Total Probability

The events E_1 , E_2 , …, E_n are called a partition of the sample space S if $E_i \cap E_j = \varphi$ for all $i \neq j$, and $E_1 \cup E_2 \cup \ldots \cup E_n = S$.
Thus, a partition cuts the whole sample space into mutually exclusive pieces. Figure 1, gives a Venn diagram with n = 7 events in the partition.

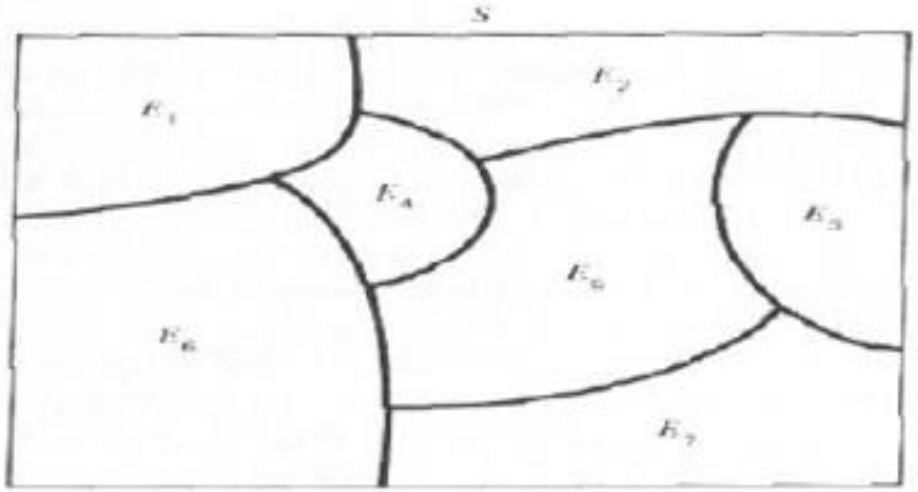


Figure 1

If $A \subset S$, is any event and $E_1, E_2, \ldots, E_n$ is a partition of S, then $E_1, E_2, \ldots, E_n$ also partition A; that is

$$A = (A \cap E_1) \cup (A \cap E_2) \cup \ldots \cup (A \cap E_n)$$

and of course,

$$(A \cap E_i) \cap (A \cap E_j) = \varphi \qquad \text{for all } i \neq j$$

Figure 2, pictures this partitioning of the event A, again with n = 7 events in the partition.

It then follows that we can write

$$P(A) = P(A \cap E_1) + P(A \cap E_2) + \ldots + P(A \cap E_n)$$

$$= \sum_{i=1}^{n} P(A \cap E_i) = \sum_{i=1}^{n} P(A/E_i)P(E_i)$$

which is known as the theorem of total probability.

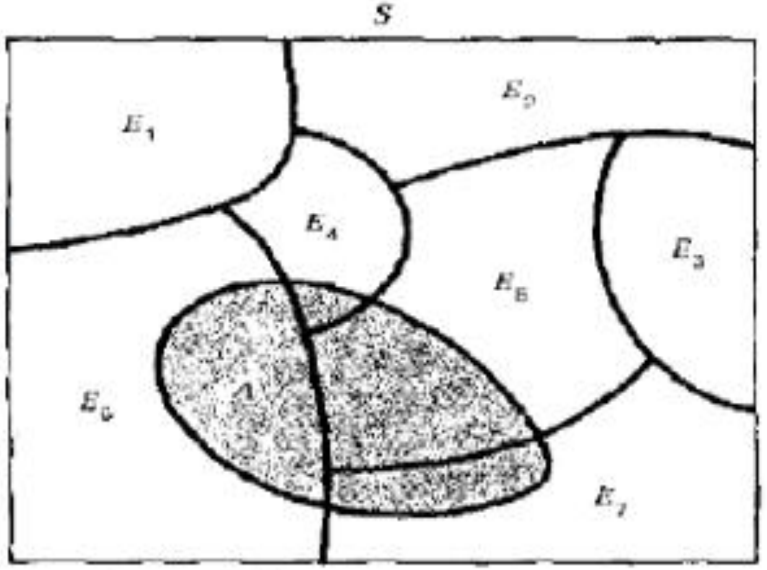


Figure 2

➢**Example (18):** A calculator manufacturer buys the same integrated circuit from three different suppliers. Call them **I**, **II**, **III**. From past experience, 1 percent of the circuits supplied by **I** have been defective, 3 percent of those supplied by **II** have been defective, and 4 percent of those supplied by **III** have been defective. Granted that this manufacturer buys 30 percent of his circuits from **I**, 50 percent from **II**, and the rest from **III**. What is the probability that an integrated circuit, checked just before final assembly into a calculator is found to be defective.

Solution Let D : The event that the chip is found defective , and let E_1 , E_2 , E_3 be the events that the chip selected was manufactured by I , II , III , respectively , and we have

$$P(E_1) = 0.3 \text{ ، } P(E_2) = 0.5 \text{ ، } P(E_3) = 0.2 \text{ ،}$$

$$P(D/E_1) = 0.01 \text{ ، } P(D/E_2) = 0.03 \text{ ، } P(D/E_3) = 0.04$$

$$\begin{aligned} P(D) &= P(D/E_1)P(E_1) + P(D/E_2)P(E_2) + P(D/E_3)P(E_3) \\ &= (0.01)(0.3) + (0.03)(0.5) + (0.04)(0.2) \\ &= 0.003 + 0.015 + 0.008 = 0.026 \end{aligned}$$

Bayes' Rule

The theorem of total probability can be used to easily establish Bayes' theorem, named after the Reverend T. Bayes; the result is commonly credited with first being published in 1764 in Bayes' posthumous memoirs.

Theorem (1): Let $E_1, E_2, \ldots, E_n$ be a partition of S . Then for any event $A \subset S$

$$P(E_i / A) = \frac{P(E_i) P(A / E_i)}{\sum_{j=1}^{N} P(E_j) P(A / E_j)}, \quad i = 1, 2, \ldots, n$$

Proof: By definition

$$P(E_i/A) = \frac{P(E_i \cap A)}{P(A)}$$

and since,

$$P(E_i \cap A) = P(A/E_i)P(E_i) \quad \text{and}$$

$$P(A) = \sum_{i=1}^{n} P(A/E_i)P(E_i)$$

Then the result follows immediately, then

$$P(E_i/A) = \frac{P(E_i \cap A)}{P(A)} = \frac{P(A/E_i)P(E_i)}{\sum_{i=1}^{n} P(A/E_i)P(E_i)}$$

➢**Example (19)**: Let us assume the same situation as example (18). If one circuit is selected from the box and tested, and found defective. Find the probability that it came from **(a)** box I **(b)** box II **(c)** box III

Solution

(a) $$P(E_1/D) = \frac{P(D/E_1)P(E_1)}{P(D/E_1)P(E_1)+P(D/E_2)P(E_2)+P(D/E_3)P(E_3)}$$

$$= \frac{(0.3)(0.01)}{(0.3)(0.01)+(0.5)(0.03)+(0.2)(0.04)} = 0.115$$

(b) $$P(E_2/D) = \frac{P(D/E_2)P(E_2)}{P(D/E_1)P(E_1)+P(D/E_2)P(E_2)+P(D/E_3)P(E_3)}$$

$$= \frac{(0.5)(0.03)}{(0.3)(0.01)+(0.5)(0.03)+(0.2)(0.04)} = 0.577$$

(c) $$P(E_3/D) = \frac{P(D/E_3)P(E_3)}{P(D/E_1)P(E_1)+P(D/E_2)P(E_2)+P(D/E_3)P(E_3)} = 0.308$$

Independent Events

Two events A and B are **independent** if and only if

$$P(B|A) = P(B) \quad \text{or} \quad P(A|B) = P(A),$$

assuming the existences of the conditional probabilities. Otherwise, A and B are **dependent**.

In other words, A and B are independent if and only if

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = P(A) \quad \text{then, } P(A \cap B) = P(A)\,P(B)$$

and

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = P(B) \quad \text{then, } P(A \cap B) = P(A)\,P(B)$$

We thus have the following definition.

Definition: Two events A and B are said to be independent if

$$P(A \cap B) = P(A)\,P(B)$$

➢**Example (20)**: An electrical system consists of four components as illustrated in Figure 3. The system works if components A and B work and either of the components C or D works. The reliability (probability of working) of each component is also shown in Figure 3. Find the probability that **(a)** the entire system works and **(b)** the component C does not work, given that the entire system works. Assume that the four components work independently.

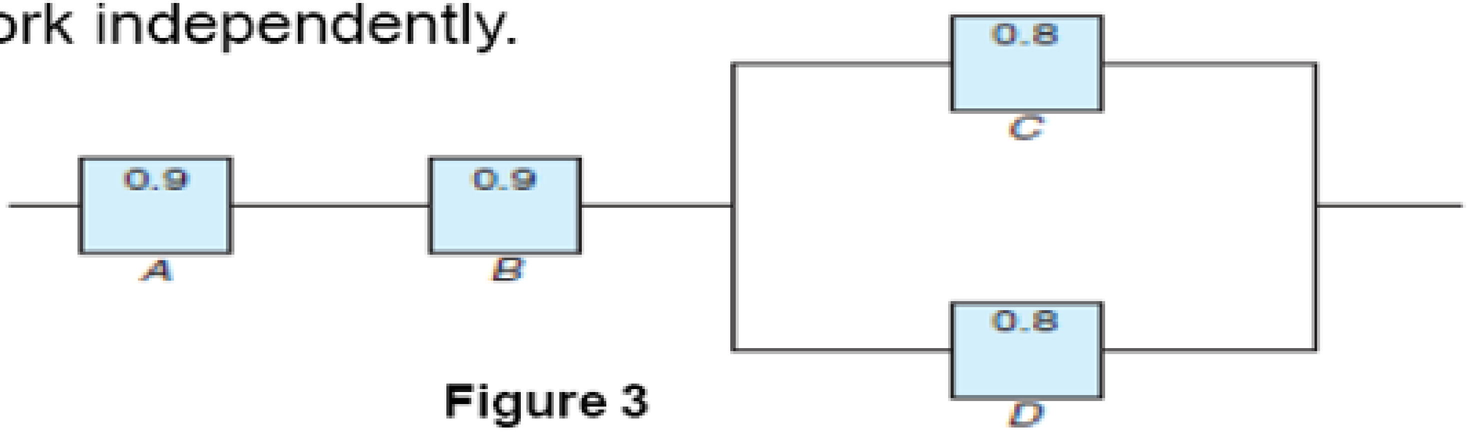


Figure 3

Solution: In this configuration of the system, A, B, and the subsystem C and D constitute a serial circuit system, whereas the subsystem C and D itself is a parallel circuit system.

(a) Clearly the probability that the entire system works can be calculated as follows:

$$
\begin{aligned}
P[A \cap B \cap (C \cup D)] &= P(A)\, P(B)\, P(C \cup D) = P(A)\, P(B)\, [1 - P(C^c \cap D^c)] \\
&= P(A)\, P(B)\, [1 - P(C^c)\, P(D^c)] \\
&= (0.9)\, (0.9)\, [1 - (1 - 0.8)\, (1 - 0.8)] = 0.7776
\end{aligned}
$$

Rel. of the whole system = 0.7776.

The equalities above hold because of the independence among the four components.

(b) To calculate the conditional probability in this case, notice that

$$
P = \frac{P(\text{the system works but } C \text{ does not work})}{P(\text{the system works})}
$$

$$
= \frac{P(A \cap B \cap C^c \cap D)}{P(\text{the system works})} = \frac{(0.9)(0.9)(1 - 0.8)(0.8)}{0.7776} = 0.1667.
$$

➢**Example (21)**: Suppose the diagram of an electrical system is as given in Figure 4. What is the probability that the system works? Assume the components fail independently.

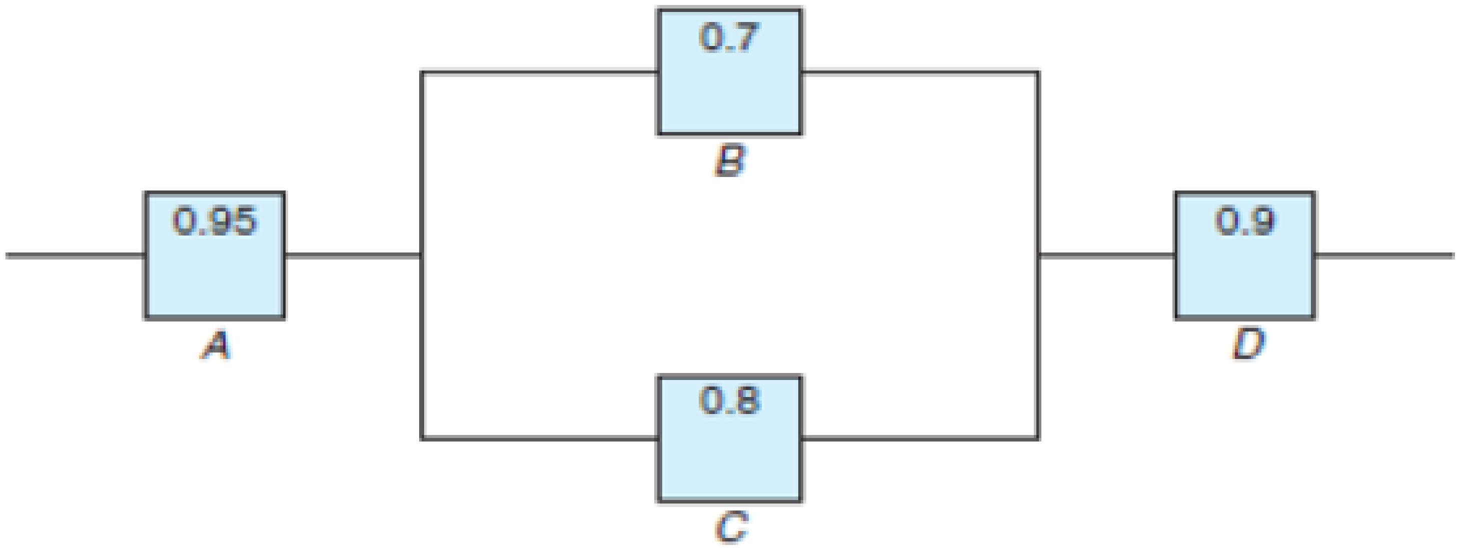


Figure 4

$$P = (0.95)\ [1 - (1 - 0.7)\,(1 - 0.8)]\,(0.9) = 0.8037. \quad \text{(How?)}$$

➢**Example (22)**: Suppose that we toss 2 fair dice. Let A denote the event that the sum of the dice is 6 and B denote the event that the first die equals 4. Is A and B independent?

Solution: **S = { (i , j) : i , j = 1,2,3,4,5,6 } , and n(S) = 36 .**

A = { (1, 5) , (5,1) , (2,4) , (4,2) , (3,3) } ,

n(A) = 5 , P(A) = 5/36 , and

B = { (4,1) , (4,2) , (4,3) , (4,4) , (4,5) , (4,6) } ,

n(B) =6 , P(B) = 6/36 = 1/6 ,

and

$A \cap B = \{ (4,2) \}$, $n(A \cap B) = 1$, $P(A \cap B) = 1/36$

Now, $P(A \cap B) = 1/36$ and $P(A)\,P(B) = (\frac{5}{36})(\frac{1}{6}) = \frac{5}{216}$

then . $P(A \cap B) \neq P(A)P(B)$. Hence A and B are not independent.

Definition: The three events A, B, and C are said to be independent if:

1- $P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$

2- $P(A \cap B) = P(A)\,P(B)$

3- $P(A \cap C) = P(A)\,P(C)$

4- $P(B \cap C) = P(B)\,P(C)$

➢**Example (23)**: Assume a fair coin is flipped two times and define the three events: **A: Head on first flip**, **B: Head on second flip**, and **C: Same face on both flips**. Are A , B , and C independent ?

Solution:

$A = \{(H, H), (H, T)\}$ $\quad B = \{(H, H), (T, H)\}$ $\quad C = \{(H, H) , (T, T)\}$

$P(A) = P(B) = P(C) = 1/2$ and that

$P(A \cap B) = P(A \cap C) = P(B \cap C) = 1/4$

so equations 2,3, and 4 of the last definition are satisfied . But since $P(A \cap B \cap C) = 1/4 \neq (1/2)^3$.

Then A, B, and C are called pairwise independent events

Questions!

YOU AREN'T YOUR PAST,
YOU ARE PROBABILITY
OF YOUR FUTURE

OPRAH WINFREY