# Introduction Into Probability Theory

MTH 231
Lecture 4
Chapter III

## Joint Probability Distributions

# Today's lecture

- Joint Density Functions
- Marginal Distributions
- Conditional Probability
- Independence
- The Covariance

# Joint Probability Distributions

- Definition: The function $P(x, y)$ is a joint probability distribution function of two *discrete r.v's*. $X$ and $Y$ if:

1. $P(x, y) \geq 0$ for all $(x, y)$
2. $\sum_x \sum_y P(x, y) = 1$
3. $P(X = x, Y = y) = P(x, y)$
4. For any region $A$ in the $XY$ plane,

$$P[(X, Y) \in A] = \sum \sum P(x, y)$$

*For the discrete case.*

# Joint *PMF*

- Example: Let the joint *pmf* of $X$ and $Y$ be give by

$$P(x,y)=\begin{cases}\alpha(x^2+y^2), & \text{if } (x,y)=(1,1),(1,2),(2,3),(3,3)\\ 0, & \text{otherwise.}\end{cases}$$

a) Find the value of $\alpha$ ?

b) Calculate $P(X>Y)$, $P(X+Y\leq 4)$, and $P(Y\geq X)$.

# Joint PMF

- Solution:

a)

$$\sum_x \sum_y P(x,y) = \sum_{(x,y)} P(x,y) = 1$$

$$= \boldsymbol{\alpha}[\boldsymbol{P}(\mathbf{1},\mathbf{1}) + \boldsymbol{P}(\mathbf{1},\mathbf{2}) + \boldsymbol{P}(\mathbf{2},\mathbf{3}) + \boldsymbol{P}(\mathbf{3},\mathbf{3})]$$

$$= 38\alpha,$$

$$\Rightarrow \alpha = 1/38.$$

b) $P(X > Y) = 0$, $P(X + Y \leq 4) = 7/38$, and $P(Y \geq X) = 1$. Why?

# Joint Density Functions

- **Definition:** The function $f(x,y)$ is a joint probability density function of the *continuous random variables* $X$ and $Y$ if:

1. $f(x, y) \geq 0$ for all $(x, y)$
2. $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dx\,dy = 1$$
3. P(X <x, Y<y) = $f(x, y)$
4. $$P[(X, Y) \in A] = \int\int f(x, y)\,dx\,dy$$

*For the continous case.*

# Joint PDF

- Example: A candy company distributed boxes of chocolates with a mixture of creams, toffees, and nuts coated in both light and dark chocolate. Select a box randomly, let $X$ and $Y$, be the proportions of the light and dark chocolates that are creams respectively. Suppose that the joint density function of $X$ and $Y$ is

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y), & 0 \le x \le 1, 0 \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

a) Verify whether $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$ ?

b) Find $P[(X,Y) \in A]$, where A is the region $\{(x,y) \mid 0<x<½, ¼<y<½\}$?

# Joint PDF

- Solution:

a)
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy = \int_0^1\int_0^1 \frac{2}{5}(2x+3y)\,dxdy =$$

$$= \int_0^1 \left.\frac{2x^2}{5} + \frac{6xy}{5}\right|_{x=0}^{x=1} dy$$

$$= \int_0^1 \left(\frac{2}{5} + \frac{6y}{5}\right) dy = \left.\frac{2y}{5} + \frac{3y^2}{5}\right|_0^1$$

$$= \frac{2}{5} + \frac{3}{5} = 1$$

# Continue

b) $P[(X,Y)\in A] = P(0 < X < \tfrac{1}{2}, \tfrac{1}{4} < Y < \tfrac{1}{2}) =$

$$= \int_{\frac{1}{4}}^{\frac{1}{2}} \int_{0}^{\frac{1}{2}} \frac{2}{5}(2x+3y)\,dx\,dy$$

$$= \int_{\frac{1}{4}}^{\frac{1}{2}} \left. \frac{2x^2}{5} + \frac{6xy}{5} \right|_{x=0}^{x=\frac{1}{2}} dy$$

$$= \int_{\frac{1}{4}}^{\frac{1}{2}} \left( \frac{1}{10} + \frac{3y}{5} \right) dy = \left. \frac{y}{10} + \frac{3y^2}{10} \right|_{\frac{1}{4}}^{\frac{1}{2}}$$

$$= \frac{1}{10}\left[ \left( \frac{1}{2} + \frac{3}{4} \right) - \left( \frac{1}{4} + \frac{3}{16} \right) \right] = \frac{13}{160}$$

# Marginal Distributions

- Definition: The individual (marginal) probability mass functions of $X$ alone and of $Y$ alone are:

  - $$P(x) = \sum_{y} p(x, y)$$ and

  - $$P(y) = \sum_{x} p(x, y)$$

*For the discrete case.*

# Marginal Distributions

**Definition:** The individual (marginal) probability density functions of $X$ alone and of $Y$ alone are

- $g(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$ and
- $h(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$

For the continuous case

# Statistical Independence

- Definition: Let $X$ and $Y$ be two discrete (or continous) random variables, with joint probability mass (or density) function $P(x, y)$ (or $f(x,y)$). The random variables $X$ and $Y$ are said to be statistically independent if and only if:

$$\mathrm{P}(x, \mathrm{y}) = \mathrm{P}_{\mathrm{X}}(x).\mathrm{P}_{\mathrm{Y}}(y) \quad \text{or} \quad f_{XY}(x,y) = f_X(x).f_Y(y)$$

For the discrete case.

For the continuous case

# Conditional Probability Distributions

- Let $X$ and $Y$ be two discrete random variables, with joint PMF $P(x,y)$ and marginal probability mass functions $P_X(x)$ and $P_Y(y)$.
  - The conditional probability mass function of the random variable $Y$, given that $X = x$, is

$$P(y \mid x) = \frac{\mathrm{P}(x, y)}{\mathrm{P}_{\mathrm{X}}(x)}$$

  - *Similarly*, the conditional PDF of the random variable $X$, given that $Y = y$, is

$$P(x \mid y) = \frac{\mathrm{P}(x, y)}{\mathrm{P}_{\mathrm{Y}}(y)}$$

# Covariance

- Definition:

  The covariance between any two jointly distributed random variables $X$ and $Y$, denoted by Cov($X$, $Y$), is defined by

$$Cov(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$$

$$= E[XY] - \mu_X \mu_Y$$

where $\mu_X = \mathrm{E}[X]$ and $\mu_Y = \mathrm{E}[Y]$

# Joint PDF

- **Example:** If $X$ and $Y$ have the joint density function

$$f(x,y)=\begin{cases}\dfrac{x(1+3y^2)}{4}, & 0<x<2, 0<y<1\\ 0, & \text{O.W.}\end{cases}$$

a) Find $g(x)$, $h(y)$, $f(y|x)$, $f(x|y)$,

b) Calculate $P(1/4<X<1/2 \mid Y=1/3)$,

c) Evaluate $Var(X)$,

d) Evaluate $Cov(X,Y)$.

# Joint PDF

a) Solution: By definition,

$$g(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^1 \frac{x(1+3y^2)}{4} dy = \frac{xy}{4} + \frac{xy^3}{4}\Bigg|_{y=0}^{y=1} = \frac{x}{2}, \quad 0 < \mathrm{x} < 2$$

and $$h(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_0^2 \frac{x(1+3y^2)}{4} dx = \frac{x^2}{8} + \frac{3x^2y^2}{8}\Bigg|_{x=0}^{x=2} = \frac{1+3y^2}{2}, \quad 0 < \mathrm{y} < 1$$

Then, $$f(y \mid x) = \frac{f(x,y)}{g(x)} = \frac{x(1+3y^2)/4}{x/2} = \frac{1}{2}(1+3y^2), \quad 0 < \mathrm{y} < 1$$

and $$f(x \mid y) = \frac{f(x,y)}{h(y)} = \frac{x(1+3y^2)/4}{(1+3y^2)/2} = \frac{x}{2}, \quad 0 < \mathrm{x} < 2$$

# Joint PDF

b) Since X and Y are independent, then

$$P\left(\frac{1}{4} < X < \frac{1}{2} \mid Y = \frac{1}{3}\right) = \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{x}{2}\,dx = \frac{3}{64}.$$

c) Since $$E(X) = \int_0^2 x.g(x)dx = \int_0^2 \frac{x^2}{2}dx = \frac{4}{3},$$

$$E(X^2) = \int_0^2 x.^2 g(x)dx = \int_0^2 \frac{x3}{2}dx = 2,$$

then $$Var(X) = E(X^2) - (E(X))^2 = \frac{2}{3}.$$

d) *Cov(X,Y)= 0.* *Why?*

# Joint PDF

- Example: If $X$ and $Y$ have the joint density function

$$f(x\,,\,y)=\begin{cases} 10\,x\,y^2\;, & 0 < x < y < 1, \\ 0, & otherwise \end{cases}$$

a) Find the marginal densities g(x) and h(y).

b) Find $f\,(y|x)$.

c) Find $P(Y > \frac{1}{2}$ | X=0.25).

# Joint PDF

## Solution

a) g(x)=$\int_x^1 f(x,y)dy = \int_x^1 10\, x\, y\, dy = \frac{10}{3}x(1-x^3).$

and

$$h(y) = \int_0^y f(x,y)dx = 5\, y^4.$$

b) $$f(y\,|x) = \frac{f(y,x)}{g(x)} = \frac{3\, y^2}{(1- x^3)}.$$

# Joint PDF

c)

$$P\left(Y > \frac{1}{2} \middle| X = 0.25\right) =$$

$$\int_{\frac{1}{2}}^{1} f(Y\,|X)dy = \int_{\frac{1}{2}}^{1} \frac{3\,y^2}{(1 - 0.25^3)}\,dy = \frac{8}{9}\,.$$

# Problems:

1) Consider the joint density function of X and Y is

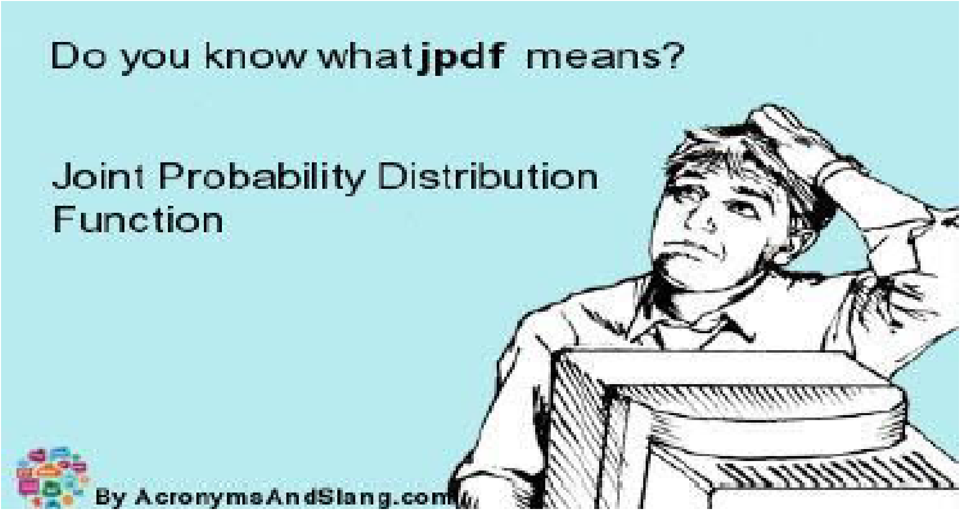
$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y), & 0 \le x \le 1, 0 \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

a) Find $g(x)$, $h(y)$,

b) Find $f(y|x)$, $f(x|y)$,

c) Evaluate $Var(X)$ and $Var(Y)$,

d) Evaluate $Cov(X, Y)$.



# Problems:

2) Let *f(x,y)* be the joint pdf of two random variables *X* and *Y*.
If *f(x,y)* is given by:

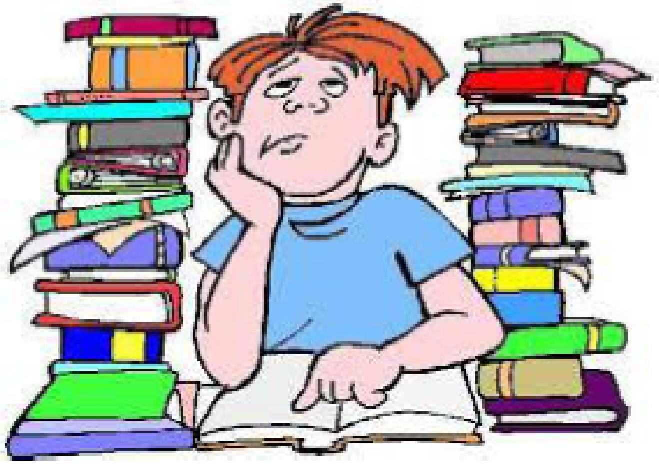
$$f(x,y) = \begin{cases} \frac{12}{5}x(2-x-y) & \text{if } 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

a) Find $P(X \mid Y = y)$, where $0 \le y \le 1$,

b) Calculate $P(X > 0.5 \mid Y = 0.5)$,

c) Evaluate $E(X \mid Y = 0.5)$,

d) Evaluate $Var(X \mid Y = 0.5)$.



# Questions!

PRACTICE!
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