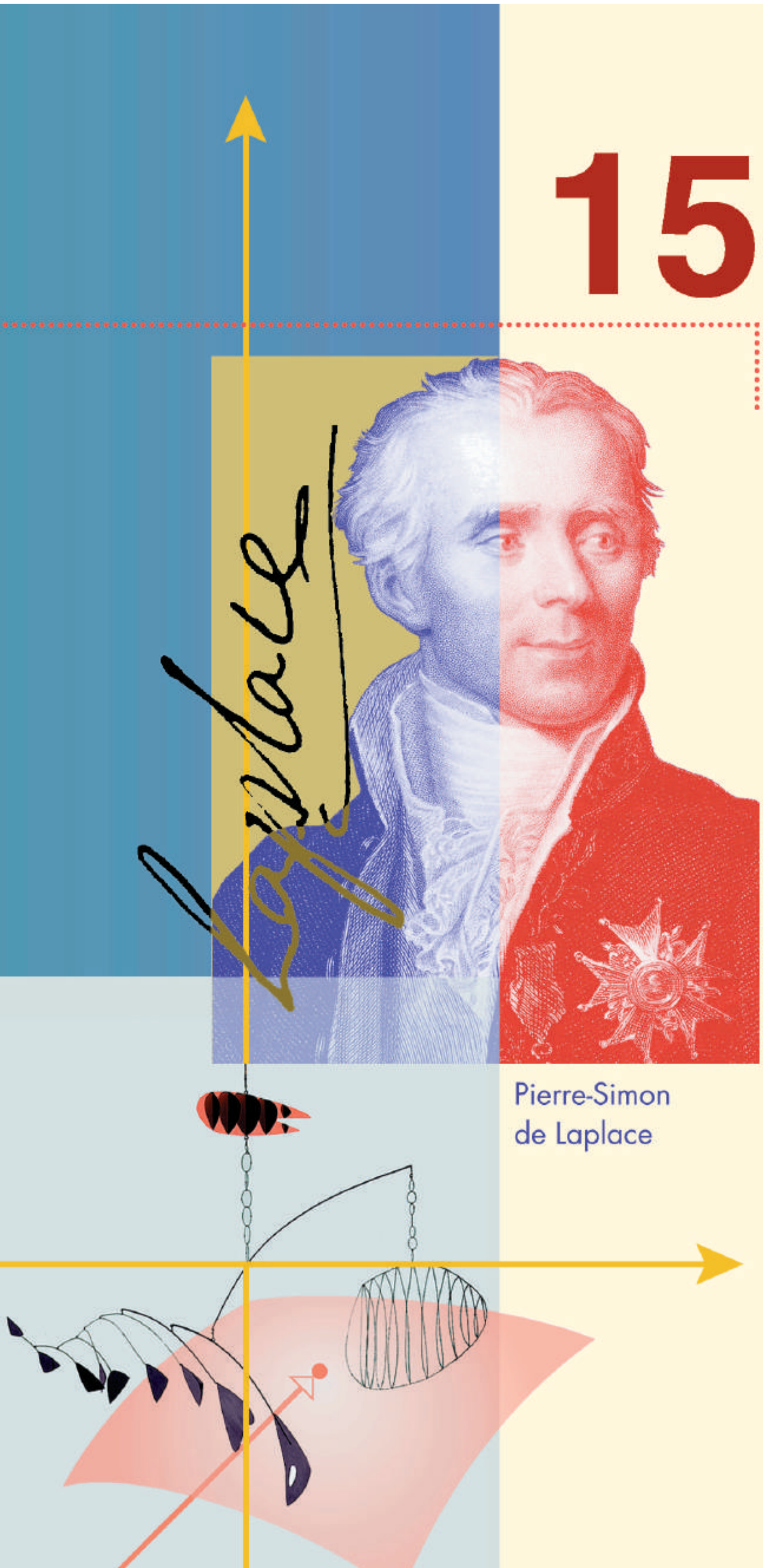


Pierre-Simon de Laplace

Alexander Calder, "Lobster Trap and Fish Tail," 1939. Hanging mobile: painted steel wire and sheet aluminum, about 6" high x 9'6" diameter. The Museum of Modern Art, New York. Commissioned by the Advisory Committee for the stairwell of the Museum. Photograph ©1998, The Museum of Modern Art, New York.

Calder's work reflects his intuitive sense for finding the perfect aesthetic and physical balance of complex objects.

15

MULTIPLE INTEGRALS

In this chapter we will extend the concept of a definite integral to functions of two and three variables. Whereas functions of one variable are usually integrated over intervals, functions of two variables are usually integrated over regions in 2-space and functions of three variables over regions in 3-space. Calculating such integrals will require some new techniques that will be a central focus in this chapter. Once we have developed the basic methods for integrating functions of two and three variables, we will show how such integrals can be used to calculate surface areas and volumes of solids; and we will also show how they can be used to find masses and centers of gravity of flat plates and three-dimensional solids. In addition to our study of integration, we will generalize the concept of a parametric curve in 2-space to a parametric surface in 3-space. This will allow us to work with a wider variety of surfaces than previously possible and will provide a powerful tool for generating surfaces using computers and other graphing utilities.

15.1 DOUBLE INTEGRALS

The notion of a definite integral can be extended to functions of two or more variables. In this section we will discuss the double integral, which is the extension to functions of two variables.

VOLUME

Recall that the definite integral of a function of one variable

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k \tag{1}$$

arose from the problem of finding areas under curves. [In the rightmost expression in (1), we use the "limit as $n \to +\infty$" to encapsulate the process by which we increase the number of subintervals of $[a, b]$ in such a way that the lengths of the subintervals approach zero.] Integrals of functions of two variables arise from the problem of finding volumes under surfaces:

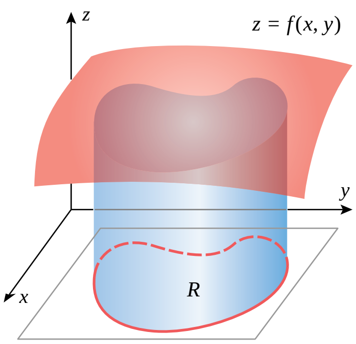


Figure 15.1.1

> **15.1.1 THE VOLUME PROBLEM.** Given a function f of two variables that is continuous and nonnegative on a region R in the xy-plane, find the volume of the solid enclosed between the surface $z = f(x, y)$ and the region R (Figure 15.1.1).

Later, we will place more restrictions on the region R, but for now we will just assume that the entire region can be enclosed within some suitably large rectangle with sides parallel to the coordinate axes. This ensures that R does not extend indefinitely in any direction.

The procedure for finding the volume V of the solid in Figure 15.1.1 will be similar to the limiting process used for finding areas, except that now the approximating elements will be rectangular parallelepipeds rather than rectangles. We proceed as follows:

- Using lines parallel to the coordinate axes, divide the rectangle enclosing the region R into subrectangles, and exclude from consideration all those subrectangles that contain any points outside of R. This leaves only rectangles that are subsets of R (Figure 15.1.2). Assume that there are n such rectangles, and denote the area of the kth such rectangle by ΔA_k.
- Choose any arbitrary point in each subrectangle, and denote the point in the kth subrectangle by (x_k^*, y_k^*). As shown in Figure 15.1.3, the product $f(x_k^*, y_k^*)\Delta A_k$ is the volume of a rectangular parallelepiped with base area ΔA_k and height $f(x_k^*, y_k^*)$, so the sum

$$\sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$$

can be viewed as an approximation to the volume V of the entire solid.

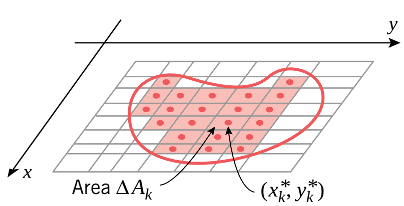


Figure 15.1.2

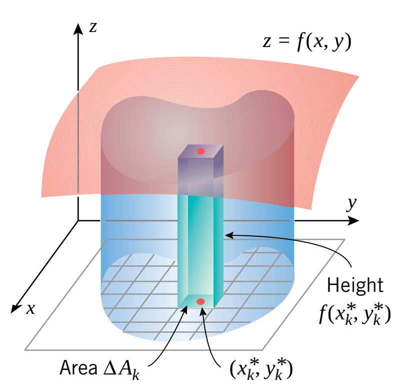


Figure 15.1.3

- There are two sources of error in the approximation: first, the parallelepipeds have flat tops, whereas the surface $z = f(x, y)$ may be curved; second, the rectangles that form the bases of the parallelepipeds may not completely cover the region R. However, if we repeat the above process with more and more subdivisions in such a way that both the lengths and the widths of the subrectangles approach zero, then it is plausible that the errors of both types approach zero, and the exact volume of the solid will be

$$V = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k$$

This suggests the following definition.

15.1.2 DEFINITION (*Volume Under a Surface*). If f is a function of two variables that is continuous and nonnegative on a region R in the xy-plane, then the volume of the solid enclosed between the surface $z = f(x, y)$ and the region R is defined by

$$V = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k \tag{2}$$

Here, $n \to +\infty$ indicates the process of increasing the number of subrectangles of the rectangle enclosing R in such a way that both the lengths and the widths of the subrectangles approach zero.

REMARK. Although this definition is satisfactory for our present purposes, there are various issues that would have to be resolved before it could be regarded as a rigorous mathematical definition. For example, we would have to prove that the limit actually exists and that its value does not depend on how the points $(x_1^*, y_1^*), (x_2^*, y_2^*), \ldots, (x_n^*, y_n^*)$ are chosen. It can be proved that this is true if f is continuous on the region R and this region is not too "complicated." The details are beyond the scope of this text.

It is assumed in Definition 15.1.2 that f is nonnegative on the region R. If f is continuous on R and has both positive and negative values, then the limit

$$\lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k \tag{3}$$

no longer represents the volume between R and the surface $z = f(x, y)$; rather, it represents a *difference* of volumes—the volume between R and the portion of the surface that is above the xy-plane minus the volume between R and the portion of the surface below the xy-plane. We call this the ***net signed volume*** between the region R and the surface $z = f(x, y)$.

DEFINITION OF A DOUBLE INTEGRAL

As in Definition 15.1.2, the notation $n \to +\infty$ in (3) encapsulates a process in which the enclosing rectangle for R is repeatedly subdivided in such a way that both the lengths and the widths of the subrectangles approach zero. Note that subdividing so that the subrectangle lengths approach zero forces the mesh of the partition of the length of the enclosing rectangle for R to approach zero. Similarly, subdividing so that the subrectangle widths approach zero forces the mesh of the partition of the width of the enclosing rectangle for R to approach zero. Thus, we have extended the notion conveyed by Formula (1) where the definite integral of a one-variable function is expressed as a limit of Riemann sums. By extension, the sums in (3) are also called ***Riemann sums***, and the limit of the Riemann sums is denoted by

$$\iint_R f(x, y)\, dA = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*)\Delta A_k \tag{4}$$

which is called the ***double integral*** of $f(x, y)$ over R.

If f is continuous and nonnegative on the region R, then the volume formula in (2) can be expressed as

$$V = \iint_R f(x, y)\, dA \tag{5}$$

If f has both positive and negative values on R, then a positive value for the double integral of f over R means that there is more volume above R than below, a negative value for the double integral means that there is more volume below than above, and a value of zero means that the volume above is the same as the volume below.

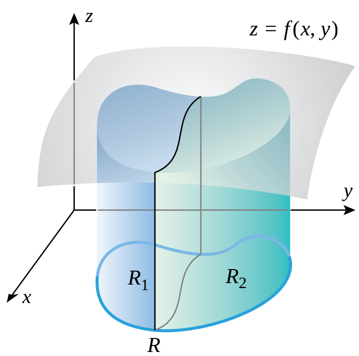
PROPERTIES OF DOUBLE INTEGRALS

To distinguish between double integrals of functions of two variables and definite integrals of functions of one variable, we will refer to the latter as ***single integrals***. Because double integrals, like single integrals, are defined as limits, they inherit many of the properties of limits. The following results, which we state without proof, are analogs of those in Theorem 5.5.4.

$$\iint_R cf(x, y)\, dA = c \iint_R f(x, y)\, dA \quad (c \text{ a constant}) \tag{6}$$

$$\iint_R [f(x, y) + g(x, y)]\, dA = \iint_R f(x, y)\, dA + \iint_R g(x, y)\, dA \tag{7}$$

$$\iint_R [f(x, y) - g(x, y)]\, dA = \iint_R f(x, y)\, dA - \iint_R g(x, y)\, dA \tag{8}$$



The volume of the entire solid is the sum of the volumes of the solids above R_1 and R_2.

Figure 15.1.4

It is evident intuitively that if $f(x, y)$ is nonnegative on a region R, then subdividing R into two regions R_1 and R_2 has the effect of subdividing the solid between R and $z = f(x, y)$ into two solids, the sum of whose volumes is the volume of the entire solid (Figure 15.1.4). This suggests the following result, which holds even if f has negative values:

$$\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA \tag{9}$$

The proof of this result will be omitted.

EVALUATING DOUBLE INTEGRALS

Except in the simplest cases, it is impractical to obtain the value of a double integral from the limit in (4). However, we will now show how to evaluate double integrals by calculating two successive single integrals. For the rest of this section, we will limit our discussion to the case where R is a rectangle; in the next section we will consider double integrals over more complicated regions.

The partial derivatives of a function $f(x, y)$ are calculated by holding one of the variables fixed and differentiating with respect to the other variable. Let us consider the reverse of this process, ***partial integration***. The symbols

$$\int_a^b f(x, y)\, dx \quad \text{and} \quad \int_c^d f(x, y)\, dy$$

denote ***partial definite integrals***; the first integral, called the ***partial definite integral with respect to x***, is evaluated by holding y fixed and integrating with respect to x, and the second integral, called the ***partial definite integral with respect to y***, is evaluated by holding x

fixed and integrating with respect to y. As the following example shows, the partial definite integral with respect to x is a function of y, and the partial definite integral with respect to y is a function of x.

Example 1

$$\int_0^1 xy^2\,dx = y^2\int_0^1 x\,dx = \frac{y^2x^2}{2}\bigg]_{x=0}^{1} = \frac{y^2}{2}$$

$$\int_0^1 xy^2\,dy = x\int_0^1 y^2\,dy = \frac{xy^3}{3}\bigg]_{y=0}^{1} = \frac{x}{3}$$ ◀

A partial definite integral with respect to x is a function of y and hence can be integrated with respect to y; similarly, a partial definite integral with respect to y can be integrated with respect to x. This two-stage integration process is called ***iterated*** (or ***repeated***) ***integration***. We introduce the following notation:

$$\int_c^d\int_a^b f(x,y)\,dx\,dy = \int_c^d\left[\int_a^b f(x,y)\,dx\right]dy \tag{10}$$

$$\int_a^b\int_c^d f(x,y)\,dy\,dx = \int_a^b\left[\int_c^d f(x,y)\,dy\right]dx \tag{11}$$

These integrals are called ***iterated integrals***.

Example 2 Evaluate

(a) $\displaystyle\int_0^3\int_1^2 (1+8xy)\,dy\,dx$ (b) $\displaystyle\int_1^2\int_0^3 (1+8xy)\,dx\,dy$

Solution (a).

$$\begin{aligned}\int_0^3\int_1^2 (1+8xy)\,dy\,dx &= \int_0^3\left[\int_1^2 (1+8xy)\,dy\right]dx\\ &= \int_0^3 \left[y+4xy^2\right]_{y=1}^{2}dx\\ &= \int_0^3 [(2+16x)-(1+4x)]\,dx\\ &= \int_0^3 (1+12x)\,dx = (x+6x^2)\Big]_0^3 = 57\end{aligned}$$

Solution (b).

$$\begin{aligned}\int_1^2\int_0^3 (1+8xy)\,dx\,dy &= \int_1^2\left[\int_0^3 (1+8xy)\,dx\right]dy\\ &= \int_1^2 \left[x+4x^2y\right]_{x=0}^{3}dy\\ &= \int_1^2 (3+36y)\,dy = (3y+18y^2)\Big]_1^2 = 57\end{aligned}$$ ◀

The following theorem shows that it is no accident that the two iterated integrals in the last example have the same value.

15.1.3 THEOREM. *Let R be the rectangle defined by the inequalities*

$$a \le x \le b, \quad c \le y \le d$$

If $f(x, y)$ is continuous on this rectangle, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx$$

This important theorem allows us to evaluate a double integral over a rectangle by converting it to an iterated integral. This can be done in two ways, both of which produce the value of the double integral. We will not formally prove this result; however, we will give a geometric motivation of the result for the case where $f(x, y)$ is nonnegative on R. In this case the double integral can be interpreted as the volume of the solid S bounded above by the surface $z = f(x, y)$ and below by the region R, so it suffices to show that the two iterated integrals also represent this volume.

For a fixed value of y, the function $f(x, y)$ is a function of x, and hence the integral

$$A(y) = \int_a^b f(x, y)\, dx$$

represents the area under the graph of this function of x. This area, shown in yellow in Figure 15.1.5, is the cross-sectional area at y of the solid S bounded above by $z = f(x, y)$ and below by the region R. Thus, by the method of slicing discussed in Section 6.2, the volume V of the solid S is

$$V = \int_c^d A(y)\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx\right] dy = \int_c^d \int_a^b f(x, y)\, dx\, dy \tag{12}$$

Similarly, the integral

$$A(x) = \int_c^d f(x, y)\, dy$$

represents the area of the cross section of S at x (Figure 15.1.6), and the method of slicing again yields

$$V = \int_a^b A(x)\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy\right] dx = \int_a^b \int_c^d f(x, y)\, dy\, dx \tag{13}$$

This establishes the result in Theorem 15.1.3 for the case where $f(x, y)$ is continuous and nonnegative on R.

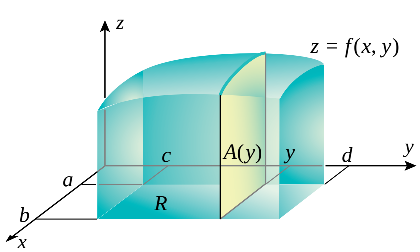


Figure 15.1.5

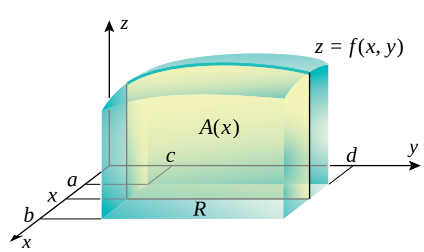


Figure 15.1.6

Example 3 Evaluate the double integral

$$\iint_R y^2 x\, dA$$

over the rectangle $R = \{(x, y) : -3 \le x \le 2, 0 \le y \le 1\}$.

Solution. In view of Theorem 15.1.3, the value of the double integral may be obtained from either of the iterated integrals

$$\int_{-3}^{2}\int_{0}^{1} y^2 x\,dy\,dx \quad \text{or} \quad \int_{0}^{1}\int_{-3}^{2} y^2 x\,dx\,dy \tag{14}$$

Using the first of these, we obtain

$$\begin{aligned}\iint_R y^2 x\,dA &= \int_{-3}^{2}\int_{0}^{1} y^2 x\,dy\,dx = \int_{-3}^{2}\left[\frac{1}{3}y^3 x\right]_{y=0}^{1} dx \\ &= \int_{-3}^{2}\frac{1}{3}x\,dx = \frac{x^2}{6}\bigg]_{-3}^{2} = -\frac{5}{6}\end{aligned}$$

You can check this result by evaluating the second integral in (14). ◀

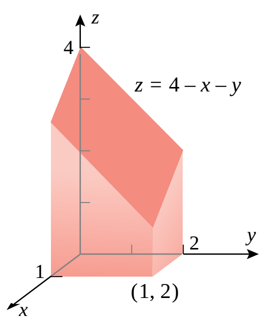


Figure 15.1.7

REMARK. We will often express the rectangle $\{(x, y) : a \le x \le b, c \le y \le d\}$ as $[a, b] \times [c, d]$ for simplicity.

Example 4 Use a double integral to find the volume of the solid that is bounded above by the plane $z = 4 - x - y$ and below by the rectangle $R = [0, 1] \times [0, 2]$ (Figure 15.1.7).

Solution.

$$\begin{aligned}V &= \iint_R (4 - x - y)\,dA = \int_{0}^{2}\int_{0}^{1}(4 - x - y)\,dx\,dy \\ &= \int_{0}^{2}\left[4x - \frac{x^2}{2} - xy\right]_{x=0}^{1} dy = \int_{0}^{2}\left(\frac{7}{2} - y\right) dy \\ &= \left[\frac{7}{2}y - \frac{y^2}{2}\right]_{0}^{2} = 5\end{aligned}$$

The volume can also be obtained by first integrating with respect to y and then with respect to x. ◀

FOR THE READER. Most computer algebra systems have a built-in capability for computing iterated double integrals. If you have a CAS, read the relevant documentation and use the CAS to check Examples 3 and 4.

EXERCISE SET 15.1 [C] CAS

In Exercises 1–12, evaluate the iterated integrals.

1. $\int_{0}^{1}\int_{0}^{2}(x + 3)\,dy\,dx$

2. $\int_{1}^{3}\int_{-1}^{1}(2x - 4y)\,dy\,dx$

3. $\int_{2}^{4}\int_{0}^{1} x^2 y\,dx\,dy$

4. $\int_{-2}^{0}\int_{-1}^{2}(x^2 + y^2)\,dx\,dy$

5. $\int_{0}^{\ln 3}\int_{0}^{\ln 2} e^{x+y}\,dy\,dx$

6. $\int_{0}^{2}\int_{0}^{1} y\sin x\,dy\,dx$

7. $\int_{-1}^{0}\int_{2}^{5} dx\,dy$

8. $\int_{4}^{6}\int_{-3}^{7} dy\,dx$

9. $\int_{0}^{1}\int_{0}^{1}\frac{x}{(xy + 1)^2}\,dy\,dx$

10. $\int_{\pi/2}^{\pi}\int_{1}^{2} x\cos xy\,dy\,dx$

11. $\int_{0}^{\ln 2}\int_{0}^{1} xye^{y^2 x}\,dy\,dx$

12. $\int_{3}^{4}\int_{1}^{2}\frac{1}{(x + y)^2}\,dy\,dx$

In Exercises 13–16, evaluate the double integral over the rectangular region R.

13. $\displaystyle\iint_R 4xy^3\,dA;\;\; R = \{(x, y) : -1 \le x \le 1, -2 \le y \le 2\}$

14. $\displaystyle\iint_R \frac{xy}{\sqrt{x^2+y^2+1}}\,dA;$
$R = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$

15. $\displaystyle\iint_R x\sqrt{1-x^2}\,dA;\;\; R = \{(x, y) : 0 \le x \le 1, 2 \le y \le 3\}$

16. $\displaystyle\iint_R (x \sin y - y \sin x)\,dA;$
$R = \{(x, y) : 0 \le x \le \pi/2, 0 \le y \le \pi/3\}$

17. (a) Let $f(x, y) = x^2 + y$, and as shown in the accompanying figure, let the rectangle $R = [0, 2] \times [0, 2]$ be subdivided into 16 subrectangles. Take (x_k^*, y_k^*) to be the center of the kth rectangle, and approximate the double integral of f over R by the resulting Riemann sum.
(b) Compare the result in part (a) to the exact value of the integral.

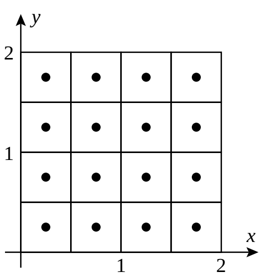


Figure Ex-17

18. (a) Let $f(x, y) = x - 2y$, and as shown in Figure Ex-17, let the rectangle $R = [0, 2] \times [0, 2]$ be subdivided into 16 subrectangles. Take (x_k^*, y_k^*) to be the center of the kth rectangle, and approximate the double integral of f over R by the resulting Riemann sum.
(b) Compare the result in part (a) to the exact value of the integral.

In Exercises 19–22, use a double integral to find the volume.

19. The volume under the plane $z = 2x + y$ and over the rectangle $R = \{(x, y) : 3 \le x \le 5, 1 \le y \le 2\}$.

20. The volume under the surface $z = 3x^3 + 3x^2y$ and over the rectangle $R = \{(x, y) : 1 \le x \le 3, 0 \le y \le 2\}$.

21. The volume of the solid enclosed by the surface $z = x^2$ and the planes $x = 0$, $x = 2$, $y = 3$, $y = 0$, and $z = 0$.

22. The volume in the first octant bounded by the coordinate planes, the plane $y = 4$, and the plane $(x/3) + (z/5) = 1$.

In Exercises 23 and 24, each iterated integral represents the volume of a solid. Make a sketch of the solid. (You do *not* have to find the volume.)

23. (a) $\displaystyle\int_0^5\int_1^2 4\,dx\,dy$ (b) $\displaystyle\int_0^3\int_0^4 \sqrt{25-x^2-y^2}\,dy\,dx$

24. (a) $\displaystyle\int_0^1\int_0^1 (2-x-y)\,dy\,dx$ (b) $\displaystyle\int_{-2}^2\int_{-2}^2 (x^2+y^2)\,dx\,dy$

25. Evaluate the integral by choosing a convenient order of integration:
$$\iint_R x\cos(xy)\cos^2\pi x\,dA;\; R = \left[0, \tfrac{1}{2}\right] \times [0, \pi]$$

26. (a) Sketch the solid in the first octant that is enclosed by the planes $x = 0$, $z = 0$, $x = 5$, $z - y = 0$, and $z = -2y + 6$.
(b) Find the volume of the solid by breaking it into two parts.

The ***average value*** or ***mean value*** of a continuous function $f(x, y)$ over a rectangle $R = [a, b] \times [c, d]$ is defined as
$$f_{\text{ave}} = \frac{1}{A(R)}\iint_R f(x, y)\,dA$$
where $A(R) = (b - a)(d - c)$ is the area of the rectangle R (compare to Definition 5.7.5). Use this definition in Exercises 27–30.

27. Find the average value of $f(x, y) = y \sin xy$ over the rectangle $[0, 1] \times [0, \pi/2]$.

28. Find the average value of $f(x, y) = x(x^2 + y)^{1/2}$ over the interval $[0, 1] \times [0, 3]$.

29. Suppose that the temperature in degrees Celsius at a point (x, y) on a flat metal plate is $T(x, y) = 10 - 8x^2 - 2y^2$, where x and y are in meters. Find the average temperature of the rectangular portion of the plate for which $0 \le x \le 1$ and $0 \le y \le 2$.

30. Show that if $f(x, y)$ is constant on the rectangle $R = [a, b] \times [c, d]$, say $f(x, y) = k$, then $f_{\text{ave}} = k$ over R.

Most computer algebra systems have commands for approximating double integrals numerically. For Exercises 31 and 32, read the relevant documentation and use a CAS to find a numerical approximation of the double integral.

C **31.** $\displaystyle\int_0^2\int_0^1 \sin\sqrt{x^3+y^3}\,dx\,dy$

C **32.** $\displaystyle\int_{-1}^1\int_{-1}^1 e^{-(x^2+y^2)}\,dx\,dy$

33. In this exercise, suppose that $f(x, y) = g(x)h(y)$ and $R = \{(x, y) :\; a \le x \le b, c \le y \le d\}$. Show that
$$\iint_R f(x, y)\,dA = \left[\int_a^b g(x)\,dx\right]\left[\int_c^d h(y)\,dy\right]$$

34. Use the result in Exercise 33 to evaluate the integral
$$\int_0^{\ln 2}\int_{-1}^1 \sqrt{e^y+1}\tan x\,dx\,dy$$
by inspection. Explain your reasoning.

C **35.** Use a CAS to evaluate the iterated integrals
$$\int_0^1 \int_0^1 \frac{y-x}{(x+y)^3}\,dx\,dy \quad \text{and} \quad \int_0^1 \int_0^1 \frac{y-x}{(x+y)^3}\,dy\,dx$$
Does this violate Theorem 15.1.3? Explain.

C **36.** Use a CAS to show that the volume V under the surface $z = xy^3 \sin xy$ over the rectangle shown in the accompanying figure is $V = 3/\pi$.

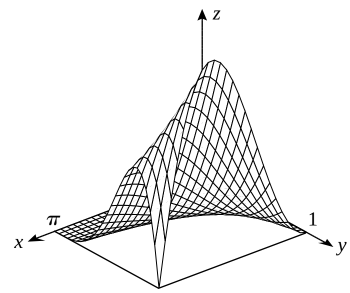


Figure Ex-36

15.2 DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS

In this section we will show how to evaluate double integrals over regions other than rectangles.

ITERATED INTEGRALS WITH NONCONSTANT LIMITS OF INTEGRATION

Later in this section we will see that double integrals over nonrectangular regions can often be evaluated as iterated integrals of the following types:

$$\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right] dx \tag{1}$$

$$\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right] dy \tag{2}$$

We begin with an example that illustrates how to evaluate such integrals.

Example 1 Evaluate

(a) $\displaystyle\int_0^2 \int_{x^2}^{x} y^2 x\,dy\,dx$ (b) $\displaystyle\int_0^\pi \int_0^{\cos y} x \sin y\,dx\,dy$

Solution (a).

$$\int_0^2 \int_{x^2}^{x} y^2 x\,dy\,dx = \int_0^2 \left[\int_{x^2}^{x} y^2 x\,dy\right] dx = \int_0^2 \left[\frac{y^3 x}{3}\right]_{y=x^2}^{x} dx$$
$$= \int_0^2 \left(\frac{x^4}{3} - \frac{x^7}{3}\right) dx = \left[\frac{x^5}{15} - \frac{x^8}{24}\right]_0^2$$
$$= \frac{32}{15} - \frac{256}{24} = -\frac{128}{15}$$

Solution (b).

$$\int_0^\pi \int_0^{\cos y} x \sin y\,dx\,dy = \int_0^\pi \left[\int_0^{\cos y} x \sin y\,dx\right] dy = \int_0^\pi \left[\frac{x^2}{2}\sin y\right]_{x=0}^{\cos y} dy$$
$$= \int_0^\pi \frac{1}{2}\cos^2 y \sin y\,dy = \left[-\frac{1}{6}\cos^3 y\right]_0^\pi = \frac{1}{3}$$
◀

DOUBLE INTEGRALS OVER NONRECTANGULAR REGIONS

Plane regions can be extremely complex, and the theory of double integrals over very general regions is a topic for advanced courses in mathematics. We will limit our study of double integrals to two basic types of regions, which we will call *type I* and *type II*; they are defined as follows:

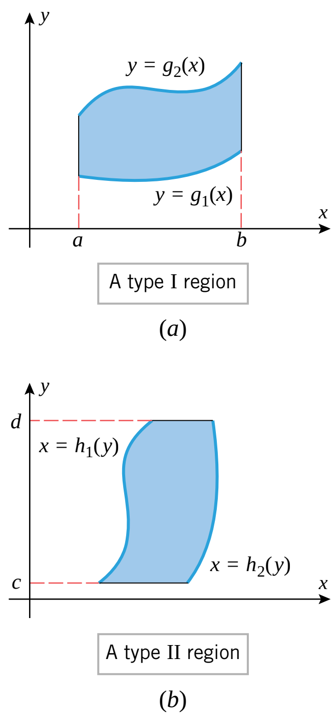


Figure 15.2.1

15.2.1 DEFINITION.

(a) A ***type I region*** is bounded on the left and right by vertical lines $x = a$ and $x = b$ and is bounded below and above by continuous curves $y = g_1(x)$ and $y = g_2(x)$, where $g_1(x) \le g_2(x)$ for $a \le x \le b$ (Figure 15.2.1*a*).

(b) A ***type II region*** is bounded below and above by horizontal lines $y = c$ and $y = d$ and is bounded on the left and right by continuous curves $x = h_1(y)$ and $x = h_2(y)$ satisfying $h_1(y) \le h_2(y)$ for $c \le y \le d$ (Figure 15.2.1*b*).

The following theorem will enable us to evaluate double integrals over type I and type II regions using iterated integrals.

15.2.2 THEOREM.

(*a*) *If R is a type I region on which $f(x, y)$ is continuous, then*

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx \tag{3}$$

(*b*) *If R is a type II region on which $f(x, y)$ is continuous, then*

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy \tag{4}$$

We will not prove this theorem, but for the case where $f(x, y)$ is nonnegative on the region R, it can be made plausible by a geometric argument that is similar to that given for Theorem 15.1.3. Since $f(x, y)$ is nonnegative, the double integral can be interpreted as the volume of the solid S that is bounded above by the surface $z = f(x, y)$ and below by the region R, so it suffices to show that the iterated integrals also represent this volume. Consider the iterated integral in (3), for example. For a fixed value of x, the function $f(x, y)$ is a function of y, and hence the integral

$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy$$

represents the area under the graph of this function of y between $y = g_1(x)$ and $y = g_2(x)$. This area, shown in yellow in Figure 15.2.2, is the cross-sectional area at x of the solid S, and hence by the method of slicing, the volume V of the solid S is

$$V = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

which shows that in (3) the iterated integral is equal to the double integral. Similarly for (4).

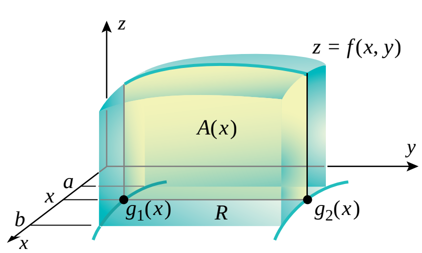


Figure 15.2.2

SETTING UP LIMITS OF INTEGRATION FOR EVALUATING DOUBLE INTEGRALS

To apply Theorem 15.2.2, it is helpful to start with a two-dimensional sketch of the region R. [It is not necessary to graph $f(x, y)$.] For a type I region, the limits of integration in Formula (3) can be obtained as follows:

Step 1. Since x is held fixed for the first integration, we draw a vertical line through the region R at an arbitrary fixed value x (Figure 15.2.3). This line crosses the boundary of R twice. The lower point of intersection is on the curve $y = g_1(x)$ and the higher point is on the curve $y = g_2(x)$. These two intersections determine the lower and upper y-limits of integration in Formula (3).

Step 2. Imagine moving the line drawn in Step 1 first to the left and then to the right (Figure 15.2.3). The leftmost position where the line intersects the region R is $x = a$ and the rightmost position where the line intersects the region R is $x = b$. This yields the limits for the x-integration in Formula (3).

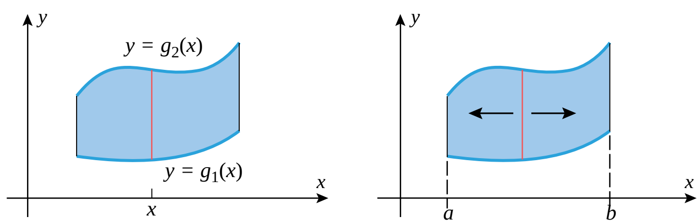


Figure 15.2.3

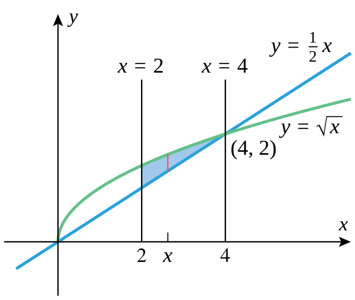


Figure 15.2.4

Example 2 Evaluate

$$\iint_R xy\, dA$$

over the region R enclosed between $y = \frac{1}{2}x$, $y = \sqrt{x}$, $x = 2$, and $x = 4$.

Solution. We view R as a type I region. The region R and a vertical line corresponding to a fixed x are shown in Figure 15.2.4. This line meets the region R at the lower boundary $y = \frac{1}{2}x$ and the upper boundary $y = \sqrt{x}$. These are the y-limits of integration. Moving this line first left and then right yields the x-limits of integration, $x = 2$ and $x = 4$. Thus,

$$\iint_R xy\, dA = \int_2^4 \int_{x/2}^{\sqrt{x}} xy\, dy\, dx = \int_2^4 \left[\frac{xy^2}{2}\right]_{y=x/2}^{\sqrt{x}} dx = \int_2^4 \left(\frac{x^2}{2} - \frac{x^3}{8}\right) dx$$

$$= \left[\frac{x^3}{6} - \frac{x^4}{32}\right]_2^4 = \left(\frac{64}{6} - \frac{256}{32}\right) - \left(\frac{8}{6} - \frac{16}{32}\right) = \frac{11}{6} \quad \blacktriangleleft$$

If R is a type II region, then the limits of integration in Formula (4) can be obtained as follows:

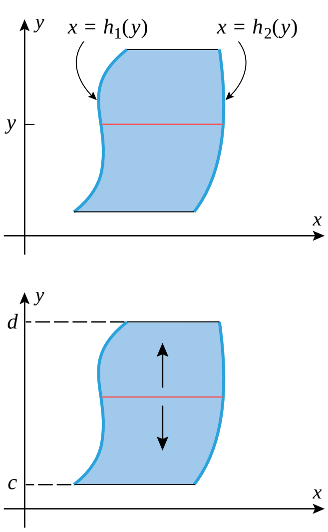


Figure 15.2.5

Step 1. Since y is held fixed for the first integration, we draw a horizontal line through the region R at a fixed value y (Figure 15.2.5). This line crosses the boundary of R twice. The leftmost point of intersection is on the curve $x = h_1(y)$ and the rightmost point is on the curve $x = h_2(y)$. These intersections determine the x-limits of integration in (4).

Step 2. Imagine moving the line drawn in Step 1 first down and then up (Figure 15.2.5). The lowest position where the line intersects the region R is $y = c$, and the highest position where the line intersects the region R is $y = d$. This yields the y-limits of integration in (4).

Example 3 Evaluate

$$\iint_R (2x - y^2)\, dA$$

over the triangular region R enclosed between the lines $y = -x + 1$, $y = x + 1$, and $y = 3$.

Solution. We view R as a type II region. The region R and a horizontal line corresponding to a fixed y are shown in Figure 15.2.6. This line meets the region R at its left-hand boundary $x = 1 - y$ and its right-hand boundary $x = y - 1$. These are the x-limits of integration. Moving this line first down and then up yields the y-limits, $y = 1$ and $y = 3$. Thus,

$$\begin{aligned}\iint_R (2x - y^2)\, dA &= \int_1^3 \int_{1-y}^{y-1} (2x - y^2)\, dx\, dy = \int_1^3 \left[x^2 - y^2 x\right]_{x=1-y}^{y-1} dy \\ &= \int_1^3 [(1 - 2y + 2y^2 - y^3) - (1 - 2y + y^3)]\, dy \\ &= \int_1^3 (2y^2 - 2y^3)\, dy = \left[\frac{2y^3}{3} - \frac{y^4}{2}\right]_1^3 = -\frac{68}{3}\end{aligned}$$ ◀

REMARK. To integrate over a type II region, the left- and right-hand boundaries must be expressed in the form $x = h_1(y)$ and $x = h_2(y)$. This is why we rewrote the boundary equations $y = -x + 1$ and $y = x + 1$ as $x = 1 - y$ and $x = y - 1$ in the last example.

In Example 3 we could have treated R as a type I region, but with an added complication: Viewed as a type I region, the upper boundary of R is the line $y = 3$ (Figure 15.2.7) and the lower boundary consists of two parts, the line $y = -x + 1$ to the left of the origin and the line $y = x + 1$ to the right of the origin. To carry out the integration it is necessary to decompose the region R into two parts, R_1 and R_2, as shown in Figure 15.2.7, and write

$$\begin{aligned}\iint_R (2x - y^2)\, dA &= \iint_{R_1} (2x - y^2)\, dA + \iint_{R_2} (2x - y^2)\, dA \\ &= \int_{-2}^{0} \int_{-x+1}^{3} (2x - y^2)\, dy\, dx + \int_0^2 \int_{x+1}^{3} (2x - y^2)\, dy\, dx\end{aligned}$$

This will yield the same result that was obtained in Example 3.

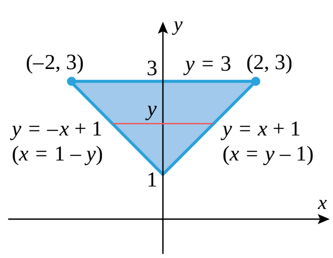


Figure 15.2.6

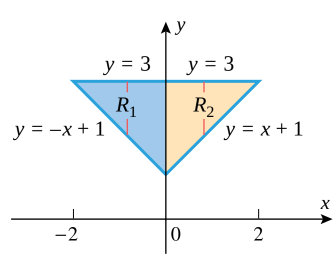


Figure 15.2.7

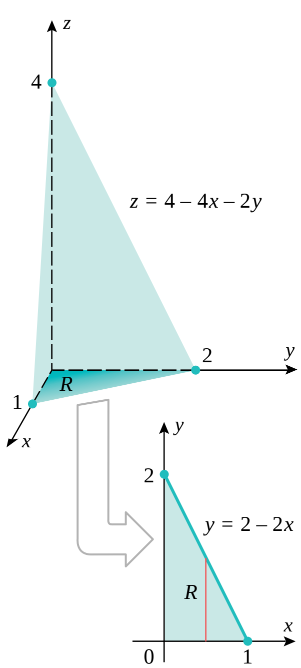


Figure 15.2.8

Example 4 Use a double integral to find the volume of the tetrahedron bounded by the coordinate planes and the plane $z = 4 - 4x - 2y$.

Solution. The tetrahedron in question is bounded above by the plane

$$z = 4 - 4x - 2y \tag{5}$$

and below by the triangular region R shown in Figure 15.2.8. Thus, the volume is given by

$$V = \iint_R (4 - 4x - 2y)\, dA$$

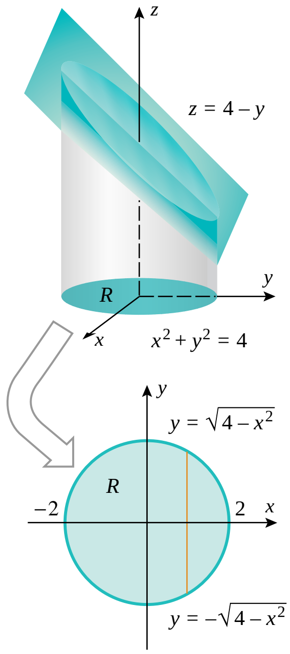


Figure 15.2.9

The region R is bounded by the x-axis, the y-axis, and the line $y = 2 - 2x$ [set $z = 0$ in (5)], so that treating R as a type I region yields

$$V = \iint_R (4 - 4x - 2y)\, dA = \int_0^1 \int_0^{2-2x} (4 - 4x - 2y)\, dy\, dx$$

$$= \int_0^1 \left[4y - 4xy - y^2\right]_{y=0}^{2-2x} dx = \int_0^1 (4 - 8x + 4x^2)\, dx = \frac{4}{3} \qquad ◀$$

Example 5 Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.

Solution. The solid shown in Figure 15.2.9 is bounded above by the plane $z = 4 - y$ and below by the region R within the circle $x^2 + y^2 = 4$. The volume is given by

$$V = \iint_R (4 - y)\, dA$$

Treating R as a type I region we obtain

$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - y)\, dy\, dx = \int_{-2}^{2} \left[4y - \frac{1}{2}y^2\right]_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} 8\sqrt{4 - x^2}\, dx = 8(2\pi) = 16\pi \qquad ◀$$

See Formula (3) of Section 8.4.

REVERSING THE ORDER OF INTEGRATION

Sometimes the evaluation of an iterated integral can be simplified by reversing the order of integration. The next example illustrates how this is done.

Example 6 Since there is no elementary antiderivative of e^{x^2}, the integral

$$\int_0^2 \int_{y/2}^{1} e^{x^2}\, dx\, dy$$

cannot be evaluated by performing the x-integration first. Evaluate this integral by expressing it as an equivalent iterated integral with the order of integration reversed.

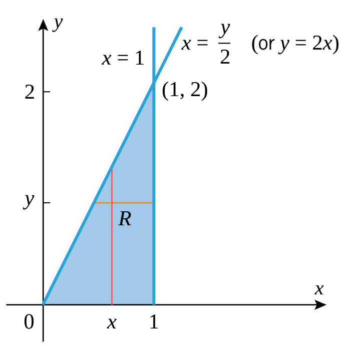


Figure 15.2.10

Solution. For the inside integration, y is fixed and x varies from the line $x = y/2$ to the line $x = 1$ (Figure 15.2.10). For the outside integration, y varies from 0 to 2, so the given iterated integral is equal to a double integral over the triangular region R in Figure 15.2.10.

To reverse the order of integration, we treat R as a type I region, which enables us to write the given integral as

$$\int_0^2 \int_{y/2}^{1} e^{x^2}\, dx\, dy = \iint_R e^{x^2}\, dA = \int_0^1 \int_0^{2x} e^{x^2}\, dy\, dx = \int_0^1 \left[e^{x^2} y\right]_{y=0}^{2x} dx$$

$$= \int_0^1 2xe^{x^2}\, dx = e^{x^2}\Big]_0^1 = e - 1 \qquad ◀$$

AREA CALCULATED AS A DOUBLE INTEGRAL

Although double integrals arose in the context of calculating volumes, they can also be used to calculate areas. To see why this is so, recall that a *right cylinder* is a solid that is generated when a plane region is translated along a line that is perpendicular to the region. In Formula (2) of Section 6.2 we stated that the volume V of a right cylinder with cross-sectional area A and height h is

$$V = A \cdot h \tag{6}$$

Now suppose that we are interested in finding the area A of a region R in the xy-plane. If we translate the region R upward 1 unit, then the resulting solid will be a right cylinder that

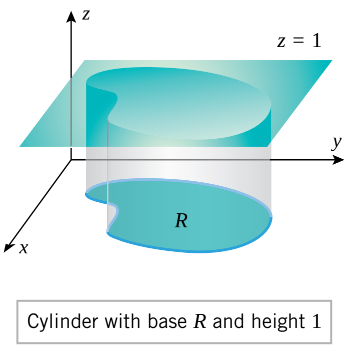


Figure 15.2.11

has cross-sectional area A, base R, and the plane $z = 1$ as its top (Figure 15.2.11). Thus, it follows from (6) that

$$\iint_R 1\,dA = (\text{area of } R)\cdot 1$$

which we can rewrite as

$$\text{area of } R = \iint_R 1\,dA = \iint_R dA \tag{7}$$

REMARK. Formula (7) is sometimes confusing because it equates an area and a volume; the formula is intended to equate only the *numerical values* of the area and volume and not the units, which must, of course, be different.

Example 7 Use a double integral to find the area of the region R enclosed between the parabola $y = \frac{1}{2}x^2$ and the line $y = 2x$.

Solution. The region R may be treated equally well as type I (Figure 15.2.12*a*) or type II (Figure 15.2.12*b*). Treating R as type I yields

$$\begin{aligned}\text{area of } R &= \iint_R dA = \int_0^4 \int_{x^2/2}^{2x} dy\,dx = \int_0^4 [y]_{y=x^2/2}^{2x}\,dx\\ &= \int_0^4 \left(2x - \frac{1}{2}x^2\right) dx = \left[x^2 - \frac{x^3}{6}\right]_0^4 = \frac{16}{3}\end{aligned}$$

Treating R as type II yields

$$\begin{aligned}\text{area of } R &= \iint_R dA = \int_0^8 \int_{y/2}^{\sqrt{2y}} dx\,dy = \int_0^8 [x]_{x=y/2}^{\sqrt{2y}}\,dy\\ &= \int_0^8 \left(\sqrt{2y} - \frac{1}{2}y\right) dy = \left[\frac{2\sqrt{2}}{3}y^{3/2} - \frac{y^2}{4}\right]_0^8 = \frac{16}{3}\end{aligned}$$ ◀

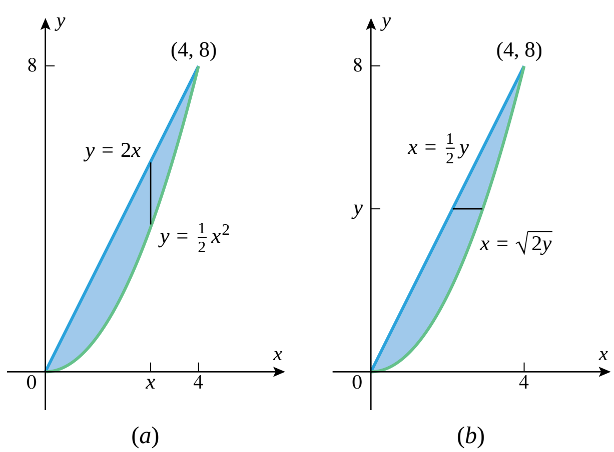


Figure 15.2.12

EXERCISE SET 15.2 Graphing Utility C CAS

In Exercises 1–10, evaluate the iterated integral.

1. $\displaystyle\int_0^1 \int_{x^2}^{x} xy^2\,dy\,dx$

2. $\displaystyle\int_1^{3/2} \int_y^{3-y} y\,dx\,dy$

3. $\displaystyle\int_0^3 \int_0^{\sqrt{9-y^2}} y\,dx\,dy$

4. $\displaystyle\int_{1/4}^1 \int_{x^2}^{x} \sqrt{\frac{x}{y}}\,dy\,dx$

5. $\displaystyle\int_{\sqrt{\pi}}^{\sqrt{2\pi}} \int_0^{x^3} \sin\frac{y}{x}\,dy\,dx$

6. $\displaystyle\int_{-1}^1 \int_{-x^2}^{x^2} (x^2 - y)\,dy\,dx$

7. $\displaystyle\int_{\pi/2}^{\pi} \int_0^{x^2} \frac{1}{x}\cos\frac{y}{x}\,dy\,dx$

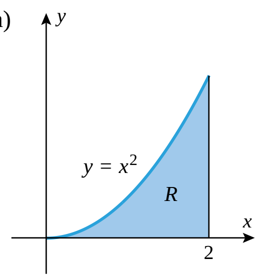
8. $\displaystyle\int_0^1 \int_0^x e^{x^2}\,dy\,dx$

9. $\displaystyle\int_0^1 \int_0^x y\sqrt{x^2 - y^2}\,dy\,dx$

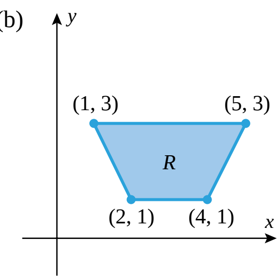
10. $\displaystyle\int_1^2 \int_0^{y^2} e^{x/y^2}\,dx\,dy$

11. In each part, find $\displaystyle\iint_R xy\,dA$ over the shaded region R.

(a)

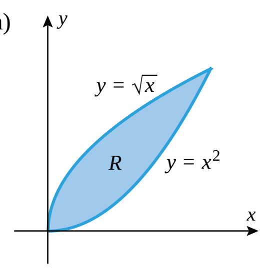


(b)

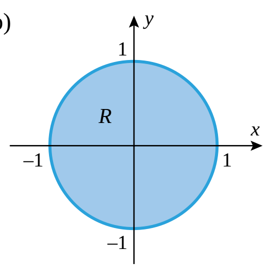


12. In each part, find $\displaystyle\iint_R (x + y)\,dA$ over the shaded region R.

(a)



(b)



In Exercises 13–16, evaluate the double integral in two ways using iterated integrals: (a) viewing R as a type I region, and (b) viewing R as a type II region.

13. $\displaystyle\iint_R x^2\,dA$; R is the region bounded by $y = 16/x$, $y = x$, and $x = 8$.

14. $\displaystyle\iint_R xy^2\,dA$; R is the region enclosed by $y = 1$, $y = 2$, $x = 0$, and $y = x$.

15. $\displaystyle\iint_R (3x - 2y)\,dA$; R is the region enclosed by the circle $x^2 + y^2 = 1$.

16. $\displaystyle\iint_R y\,dA$; R is the region in the first quadrant enclosed between the circle $x^2 + y^2 = 25$ and the line $x + y = 5$.

In Exercises 17–22, evaluate the double integral.

17. $\displaystyle\iint_R x(1 + y^2)^{-1/2}\,dA$; R is the region in the first quadrant enclosed by $y = x^2$, $y = 4$, and $x = 0$.

18. $\displaystyle\iint_R x\cos y\,dA$; R is the triangular region bounded by the lines $y = x$, $y = 0$, and $x = \pi$.

19. $\displaystyle\iint_R xy\,dA$; R is the region enclosed by $y = \sqrt{x}$, $y = 6 - x$, and $y = 0$.

20. $\displaystyle\iint_R x\,dA$; R is the region enclosed by $y = \sin^{-1} x$, $x = 1/\sqrt{2}$, and $y = 0$.

21. $\displaystyle\iint_R (x - 1)\,dA$; R is the region in the first quadrant enclosed between $y = x$ and $y = x^3$.

22. $\displaystyle\iint_R x^2\,dA$; R is the region in the first quadrant enclosed by $xy = 1$, $y = x$, and $y = 2x$.

23. (a) By hand or with the help of a graphing utility, make a sketch of the region R enclosed between the curves $y = x + 2$ and $y = e^x$.
(b) Estimate the intersections of the curves in part (a).
(c) Viewing R as a type I region, estimate $\displaystyle\iint_R x\,dA$.
(d) Viewing R as a type II region, estimate $\displaystyle\iint_R x\,dA$.

24. (a) By hand or with the help of a graphing utility, make a sketch of the region R enclosed between the curves $y = 4x^3 - x^4$ and $y = 3 - 4x + 4x^2$.
(b) Find the intersections of the curves in part (a).
(c) Find $\displaystyle\iint_R x\,dA$.

In Exercises 25–28, use double integration to find the area of the plane region enclosed by the given curves.

25. $y = \sin x$ and $y = \cos x$, for $0 \le x \le \pi/4$.

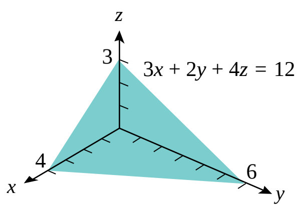
26. $y^2 = -x$ and $3y - x = 4$.

27. $y^2 = 9 - x$ and $y^2 = 9 - 9x$.

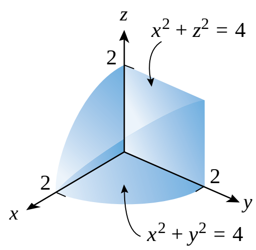
28. $y = \cosh x$, $y = \sinh x$, $x = 0$, and $x = 1$.

In Exercises 29 and 30, use double integration to find the volume of the solid.

29.



30.



In Exercises 31–38, use double integration to find the volume of each solid.

31. The solid bounded by the cylinder $x^2 + y^2 = 9$ and the planes $z = 0$ and $z = 3 - x$.

32. The solid in the first octant bounded above by the paraboloid $z = x^2 + 3y^2$, below by the plane $z = 0$, and laterally by $y = x^2$ and $y = x$.

33. The solid bounded above by the paraboloid $z = 9x^2 + y^2$, below by the plane $z = 0$, and laterally by the planes $x = 0$, $y = 0$, $x = 3$, and $y = 2$.

34. The solid enclosed by $y^2 = x$, $z = 0$, and $x + z = 1$.

35. The wedge cut from the cylinder $4x^2 + y^2 = 9$ by the planes $z = 0$ and $z = y + 3$.

36. The solid in the first octant bounded above by $z = 9 - x^2$, below by $z = 0$, and laterally by $y^2 = 3x$.

37. The solid that is common to the cylinders $x^2 + y^2 = 25$ and $x^2 + z^2 = 25$.

38. The solid bounded above by the paraboloid $z = x^2 + y^2$, bounded laterally by the circular cylinder $x^2 + (y-1)^2 = 1$, and bounded below by the xy-plane.

In Exercises 39 and 40, use a double integral and a CAS to find the volume of the solid.

39. The solid bounded above by the paraboloid $z = 1 - x^2 - y^2$ and below by the xy-plane.

40. The solid in the first octant that is bounded by the paraboloid $z = x^2 + y^2$, the cylinder $x^2 + y^2 = 4$ and the coordinate planes.

In Exercises 41–46, express the integral as an equivalent integral with the order of integration reversed.

41. $\displaystyle\int_0^2 \int_0^{\sqrt{x}} f(x, y)\, dy\, dx$

42. $\displaystyle\int_0^4 \int_{2y}^{8} f(x, y)\, dx\, dy$

43. $\displaystyle\int_0^2 \int_1^{e^y} f(x, y)\, dx\, dy$

44. $\displaystyle\int_1^e \int_0^{\ln x} f(x, y)\, dy\, dx$

45. $\displaystyle\int_0^1 \int_{\sin^{-1} y}^{\pi/2} f(x, y)\, dx\, dy$

46. $\displaystyle\int_0^1 \int_{y^2}^{\sqrt{y}} f(x, y)\, dx\, dy$

In Exercises 47–50, evaluate the integral by first reversing the order of integration.

47. $\displaystyle\int_0^1 \int_{4x}^{4} e^{-y^2}\, dy\, dx$

48. $\displaystyle\int_0^2 \int_{y/2}^{1} \cos(x^2)\, dx\, dy$

49. $\displaystyle\int_0^4 \int_{\sqrt{y}}^{2} e^{x^3}\, dx\, dy$

50. $\displaystyle\int_1^3 \int_0^{\ln x} x\, dy\, dx$

51. Evaluate $\displaystyle\iint_R \sin(y^3)\, dA$, where R is the region bounded by $y = \sqrt{x}$, $y = 2$, and $x = 0$. [*Hint:* Choose the order of integration carefully.]

52. Evaluate $\displaystyle\iint_R x\, dA$, where R is the region bounded by $x = \ln y$, $x = 0$, and $y = e$.

53. Try to evaluate the integral with a CAS using the stated order of integration, and then by reversing the order of integration.

(a) $\displaystyle\int_0^4 \int_{\sqrt{x}}^{2} \sin \pi y^3\, dy\, dx$

(b) $\displaystyle\int_0^1 \int_{\sin^{-1} y}^{\pi/2} \sec^2(\cos x)\, dx\, dy$

54. Use the appropriate Wallis formula (see Exercise Set 8.3) to find the volume of the solid enclosed between the circular paraboloid $z = x^2 + y^2$, the right circular cylinder $x^2 + y^2 = 4$, and the xy-plane (see the accompanying figure for cut view).

55. Evaluate $\displaystyle\iint_R xy^2\, dA$ over the region R shown in the accompanying figure.

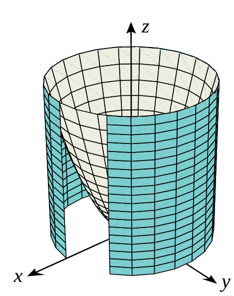


Figure Ex-54

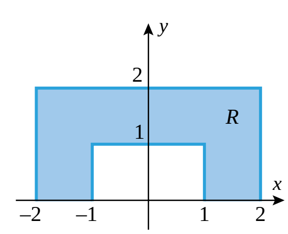


Figure Ex-55

56. Give a geometric argument to show that

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \sqrt{1 - x^2 - y^2}\, dx\, dy = \frac{\pi}{6}$$

The ***average value*** or ***mean value*** of a continuous function $f(x, y)$ over a region R in the xy-plane is defined as

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x, y)\, dA$$

where $A(R)$ is the area of the region R (compare to the definition preceding Exercise 27 of Section 15.1). Use this definition in Exercises 57 and 58.

57. Find the average value of $1/(1 + x^2)$ over the triangular region with vertices $(0, 0)$, $(1, 1)$, and $(0, 1)$.

58. Find the average value of $f(x, y) = x^2 - xy$ over the region enclosed by $y = x$ and $y = 3x - x^2$.

59. Suppose that the temperature in degrees Celsius at a point (x, y) on a flat metal plate is $T(x, y) = 5xy + x^2$, where x and y are in meters. Find the average temperature of the diamond-shaped portion of the plate for which $|2x + y| \le 4$ and $|2x - y| \le 4$.

60. A circular lens of radius 2 inches has thickness $1 - (r^2/4)$ inches at all points r inches from the center of the lens. Find the average thickness of the lens.

C **61.** Use a CAS to approximate the intersections of the curves $y = \sin x$ and $y = x/2$, and then approximate the volume of the solid in the first octant that is below the surface $z = \sqrt{1 + x + y}$ and above the region in the xy-plane that is enclosed by the curves.

15.3 DOUBLE INTEGRALS IN POLAR COORDINATES

In this section we will study double integrals in which the integrand and the region of integration are expressed in polar coordinates. Such integrals are important for two reasons: first, they arise naturally in many applications, and second, many double integrals in rectangular coordinates can be evaluated more easily if they are converted to polar coordinates.

SIMPLE POLAR REGIONS

Some double integrals are easier to evaluate if the region of integration is expressed in polar coordinates. This is usually true if the region is bounded by a cardioid, a rose curve, a spiral, or, more generally, by any curve whose equation is simpler in polar coordinates than in rectangular coordinates. Moreover, double integrals whose integrands involve $x^2 + y^2$ also tend to be easier to evaluate in polar coordinates because this sum simplifies to r^2 when the conversion formulas $x = r\cos\theta$ and $y = r\sin\theta$ are applied.

Figure 15.3.1*a* shows a region R in a polar coordinate system that is enclosed between two rays, $\theta = \alpha$ and $\theta = \beta$, and two polar curves, $r = r_1(\theta)$ and $r = r_2(\theta)$. If, as shown in that figure, the functions $r_1(\theta)$ and $r_2(\theta)$ are continuous and their graphs do not cross, then the region R is called a *simple polar region*. If $r_1(\theta)$ is identically zero, then the boundary $r = r_1(\theta)$ reduces to a point (the origin), and the region has the general shape shown in Figure 15.3.1*b*. If, in addition, $\beta = \alpha + 2\pi$, then the rays coincide, and the region has the general shape shown in Figure 15.3.1*c*. The following definition expresses these geometric ideas algebraically.

15.3.1 DEFINITION. A ***simple polar region*** in a polar coordinate system is a region that is enclosed between two rays, $\theta = \alpha$ and $\theta = \beta$, and two continuous polar curves, $r = r_1(\theta)$ and $r = r_2(\theta)$, where the equations of the rays and the polar curves satisfy the following conditions:

(i) $\alpha \le \beta$ (ii) $\beta - \alpha \le 2\pi$ (iii) $0 \le r_1(\theta) \le r_2(\theta)$

REMARK. Conditions (i) and (ii) together imply that the ray $\theta = \beta$ can be obtained by rotating the ray $\theta = \alpha$ counterclockwise through an angle that is at most 2π radians. This is consistent with Figure 15.3.1. Condition (iii) implies that the boundary curves $r = r_1(\theta)$ and $r = r_2(\theta)$ can touch but cannot actually cross over one another (why?). Thus, in keeping with Figure 15.3.1, it is appropriate to describe $r = r_1(\theta)$ as the ***inner boundary*** of the region and $r = r_2(\theta)$ as the ***outer boundary***.

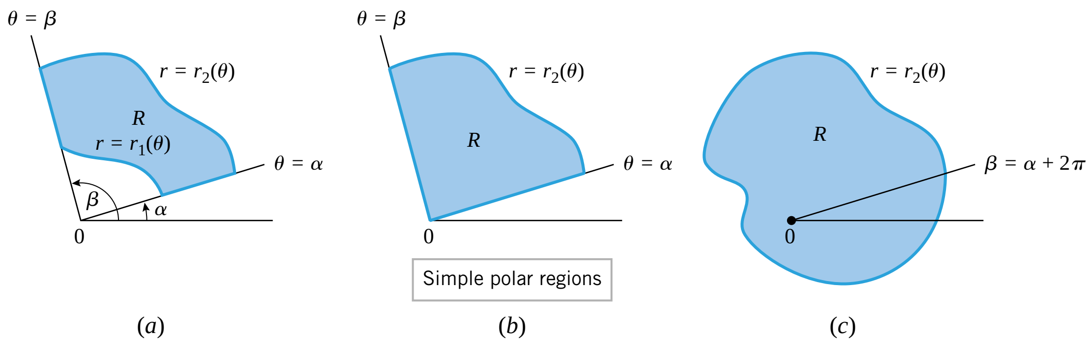


Figure 15.3.1

DOUBLE INTEGRALS IN POLAR COORDINATES

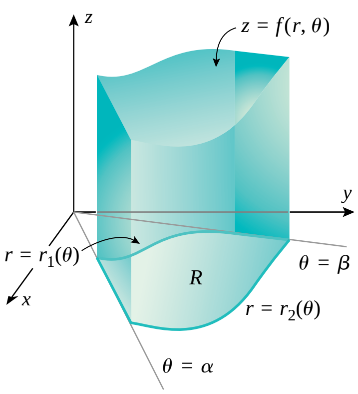


Figure 15.3.2

Next, we will consider the polar version of Problem 15.1.1.

15.3.2 THE VOLUME PROBLEM IN POLAR COORDINATES. Given a function $f(r, \theta)$ that is continuous and nonnegative on a simple polar region R, find the volume of the solid that is enclosed between the region R and the surface whose equation in cylindrical coordinates is $z = f(r, \theta)$ (Figure 15.3.2).

To motivate a formula for the volume V of the solid in Figure 15.3.2, we will use a limit process similar to that used to obtain Formula (2) of Section 15.1, except that here we will use circular arcs and rays to subdivide the region R into blocks, called ***polar rectangles***. As shown in Figure 15.3.3, we will exclude from consideration all polar rectangles that contain any points outside of R, leaving only polar rectangles that are subsets of R. Assume that there are n such polar rectangles, and denote the area of the kth polar rectangle by ΔA_k. Let (r_k^*, θ_k^*) be any point in this polar rectangle. As shown in Figure 15.3.4, the product $f(r_k^*, \theta_k^*)\Delta A_k$ is the volume of a solid with base area ΔA_k and height $f(r_k^*, \theta_k^*)$, so the sum

$$\sum_{k=1}^{n} f(r_k^*, \theta_k^*)\Delta A_k$$

can be viewed as an approximation to the volume V of the entire solid.

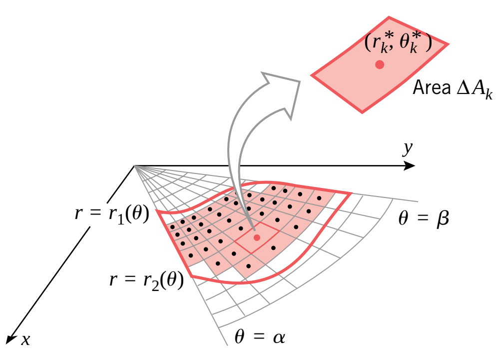


Figure 15.3.3

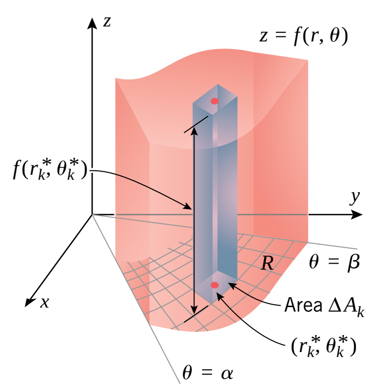


Figure 15.3.4

If we now increase the number of subdivisions in such a way that the dimensions of the polar rectangles approach zero, then it seems plausible that the errors in the approximations approach zero, and the exact volume of the solid is

$$V = \lim_{n \to +\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*)\Delta A_k \tag{1}$$

If $f(r, \theta)$ is continuous on R and has both positive and negative values, then the limit

$$\lim_{n \to +\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*) \Delta A_k \tag{2}$$

represents the net signed volume between the region R and the surface $z = f(r, \theta)$ (as with double integrals in rectangular coordinates). The sums in (2) are called ***polar Riemann sums***, and the limit of the polar Riemann sums is denoted by

$$\iint_R f(r, \theta)\, dA = \lim_{n \to +\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*) \Delta A_k \tag{3}$$

which is called the ***polar double integral*** of $f(r, \theta)$ over R. If $f(r, \theta)$ is continuous and nonnegative on R, then the volume formula (1) can be expressed as

$$V = \iint_R f(r, \theta)\, dA \tag{4}$$

REMARK. Polar double integrals are also called ***double integrals in polar coordinates*** to distinguish them from double integrals over regions in the xy-plane, which are called ***double integrals in rectangular coordinates***. Because double integrals in polar coordinates are defined as limits, they have the usual integral properties, such as those stated in Formulas (6), (7), and (8) of Section 15.1.

EVALUATING POLAR DOUBLE INTEGRALS

In Sections 15.1 and 15.2 we evaluated double integrals in rectangular coordinates by expressing them as iterated integrals. Polar double integrals are evaluated the same way. To motivate the formula that expresses a double polar integral as an iterated integral, we will assume that $f(r, \theta)$ is nonnegative so that we can interpret (3) as a volume. However, the results that we will obtain will also be applicable if f has negative values. To begin, let us choose the arbitrary point (r_k^*, θ_k^*) in (3) to be at the "center" of the kth polar rectangle as shown in Figure 15.3.5. Suppose also that this polar rectangle has a central angle $\Delta\theta_k$ and a "radial thickness" Δr_k. Thus, the inner radius of this polar rectangle is $r_k^* - \frac{1}{2}\Delta r_k$ and the outer radius is $r_k^* + \frac{1}{2}\Delta r_k$. Treating the area ΔA_k of this polar rectangle as the difference in area of two sectors, we obtain

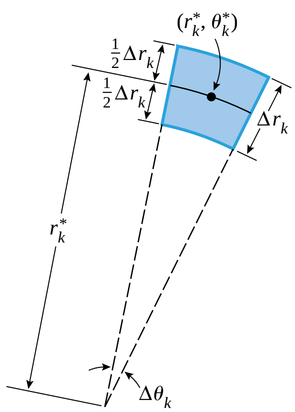


Figure 15.3.5

$$\Delta A_k = \tfrac{1}{2}\left(r_k^* + \tfrac{1}{2}\Delta r_k\right)^2 \Delta\theta_k - \tfrac{1}{2}\left(r_k^* - \tfrac{1}{2}\Delta r_k\right)^2 \Delta\theta_k$$

which simplifies to

$$\Delta A_k = r_k^* \Delta r_k \Delta\theta_k \tag{5}$$

Thus, from (3) and (4)

$$V = \iint_R f(r, \theta)\, dA = \lim_{n \to +\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*) r_k^* \Delta r_k \Delta\theta_k$$

which suggests that the volume V can be expressed as the iterated integral

$$V = \iint_R f(r, \theta)\, dA = \int_{\alpha}^{\beta} \int_{r_1(\theta)}^{r_2(\theta)} f(r, \theta) r\, dr\, d\theta \tag{6}$$

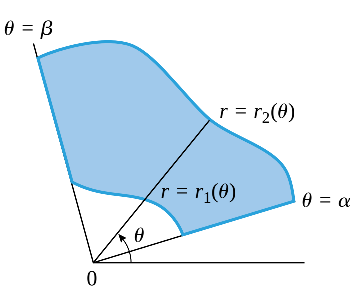


Figure 15.3.6

in which the limits of integration are chosen to cover the region R; that is, with θ fixed between α and β, the value of r varies from $r_1(\theta)$ to $r_2(\theta)$ (Figure 15.3.6).

Although we assumed $f(r, \theta)$ to be nonnegative in deriving Formula (6), it can be proved that the relationship between the polar double integral and the iterated integral in this formula also holds if f has negative values. Accepting this to be so, we obtain the following theorem, which we state without formal proof.

15.3.3 THEOREM. *If R is a simple polar region whose boundaries are the rays $\theta = \alpha$ and $\theta = \beta$ and the curves $r = r_1(\theta)$ and $r = r_2(\theta)$ shown in Figure 15.3.6, and if $f(r, \theta)$ is continuous on R, then*

$$\iint_R f(r,\theta)\, dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta) r\, dr\, d\theta \tag{7}$$

To apply this theorem, you will need to be able to find the rays and the curves that form the boundary of the region R, since these determine the limits of integration in the iterated integral. This can be done as follows:

Step 1. Since θ is held fixed for the first integration, draw a radial line from the origin through the region R at a fixed angle θ (Figure 15.3.7*a*). This line crosses the boundary of R at most twice. The innermost point of intersection is on the inner boundary curve $r = r_1(\theta)$ and the outermost point is on the outer boundary curve $r = r_2(\theta)$. These intersections determine the r-limits of integration in (7).

Step 2. Imagine rotating a ray along the polar x-axis one revolution counterclockwise about the origin. The smallest angle at which this ray intersects the region R is $\theta = \alpha$ and the largest angle is $\theta = \beta$ (Figure 15.3.7*b*). This determines the θ-limits of integration.

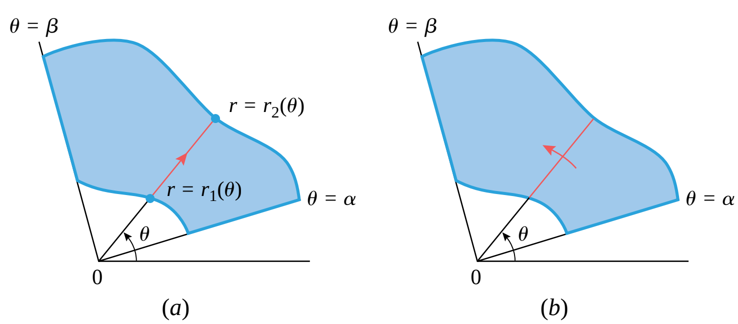


Figure 15.3.7

Example 1 Evaluate

$$\iint_R \sin\theta\, dA$$

where R is the region in the first quadrant that is outside the circle $r = 2$ and inside the cardioid $r = 2(1 + \cos\theta)$.

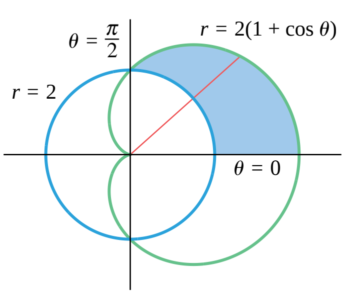


Figure 15.3.8

Solution. The region R is sketched in Figure 15.3.8. Following the two steps outlined above we obtain

$$\iint_R \sin\theta\, dA = \int_0^{\pi/2} \int_2^{2(1+\cos\theta)} (\sin\theta) r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{1}{2} r^2 \sin\theta\right]_{r=2}^{2(1+\cos\theta)} d\theta$$

$$= 2\int_0^{\pi/2} [(1+\cos\theta)^2 \sin\theta - \sin\theta]\, d\theta$$

$$= 2\left[-\frac{1}{3}(1+\cos\theta)^3 + \cos\theta\right]_0^{\pi/2}$$

$$= 2\left[-\frac{1}{3} - \left(-\frac{5}{3}\right)\right] = \frac{8}{3}$$

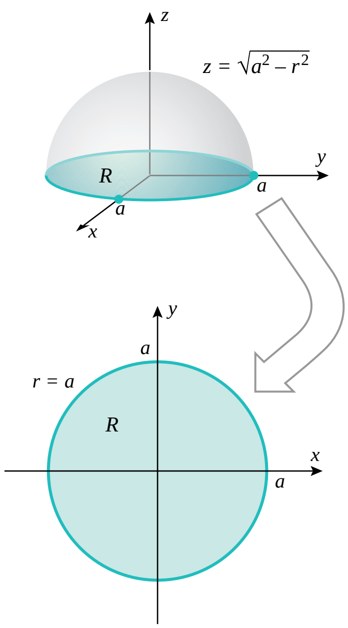


Figure 15.3.9

Example 2 The sphere of radius a centered at the origin is expressed in rectangular coordinates as $x^2 + y^2 + z^2 = a^2$, and hence its equation in cylindrical coordinates is $r^2 + z^2 = a^2$. Use this equation and a polar double integral to find the volume of the sphere.

Solution. In cylindrical coordinates the upper hemisphere is given by the equation

$$z = \sqrt{a^2 - r^2}$$

so the volume enclosed by the entire sphere is

$$V = 2\iint_R \sqrt{a^2 - r^2}\, dA$$

where R is the circular region shown in Figure 15.3.9. Thus,

$$V = 2\iint_R \sqrt{a^2 - r^2}\, dA = \int_0^{2\pi}\int_0^a \sqrt{a^2 - r^2}(2r)\, dr\, d\theta$$

$$= \int_0^{2\pi}\left[-\frac{2}{3}(a^2 - r^2)^{3/2}\right]_{r=0}^{a} d\theta = \int_0^{2\pi} \frac{2}{3}a^3\, d\theta$$

$$= \left[\frac{2}{3}a^3\theta\right]_0^{2\pi} = \frac{4}{3}\pi a^3$$

FINDING AREAS USING POLAR DOUBLE INTEGRALS

Recall from Formula (7) of Section 15.2 that the area of a region R in the xy-plane can be expressed as

$$\text{area of } R = \iint_R 1\, dA = \iint_R dA \tag{8}$$

The argument used to derive this result can also be used to show that the formula applies to polar double integrals over regions in polar coordinates.

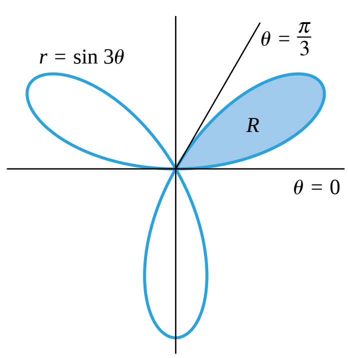


Figure 15.3.10

Example 3 Use a polar double integral to find the area enclosed by the three-petaled rose $r = \sin 3\theta$.

Solution. The rose is sketched in Figure 15.3.10. We will use Formula (8) to calculate the area of the petal R in the first quadrant and multiply by three.

$$A = 3\iint_R dA = 3\int_0^{\pi/3}\int_0^{\sin 3\theta} r\, dr\, d\theta$$

$$= \frac{3}{2}\int_0^{\pi/3} \sin^2 3\theta\, d\theta = \frac{3}{4}\int_0^{\pi/3} (1 - \cos 6\theta)\, d\theta$$

$$= \frac{3}{4}\left[\theta - \frac{\sin 6\theta}{6}\right]_0^{\pi/3} = \frac{1}{4}\pi$$

CONVERTING DOUBLE INTEGRALS FROM RECTANGULAR TO POLAR COORDINATES

Sometimes a double integral that is difficult to evaluate in rectangular coordinates can be evaluated more easily in polar coordinates by making the substitution $x = r\cos\theta$, $y = r\sin\theta$ and expressing the region of integration in polar form; that is, we rewrite the double integral in rectangular coordinates as

$$\iint_R f(x, y)\, dA = \iint_R f(r\cos\theta, r\sin\theta)\, dA = \iint_{\substack{\text{appropriate}\\ \text{limits}}} f(r\cos\theta, r\sin\theta) r\, dr\, d\theta \qquad (9)$$

Example 4 Use polar coordinates to evaluate $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\, dy\, dx$.

Solution. In this problem we are starting with an iterated integral in rectangular coordinates rather than a double integral, so before we can make the conversion to polar coordinates we will have to identify the region of integration. To do this, we observe that for fixed x the y-integration runs from $y = 0$ to $y = \sqrt{1-x^2}$, which tells us that the lower boundary of the region is the x-axis and the upper boundary is a semicircle of radius 1 centered at the origin. From the x-integration we see that x varies from -1 to 1, so we conclude that the region of integration is as shown in Figure 15.3.11. In polar coordinates, this is the region swept out as r varies between 0 and 1 and θ varies between 0 and π. Thus,

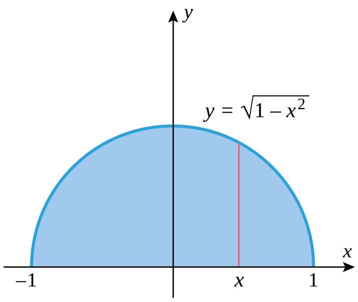


Figure 15.3.11

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\, dy\, dx = \iint_R (x^2+y^2)^{3/2}\, dA$$

$$= \int_0^{\pi}\int_0^1 (r^3) r\, dr\, d\theta = \int_0^{\pi} \frac{1}{5}\, d\theta = \frac{\pi}{5} \qquad \blacktriangleleft$$

REMARK. The conversion to polar coordinates worked so nicely in this example because the substitution $x = r\cos\theta,\ y = r\sin\theta$ collapsed the sum $x^2 + y^2$ into the single term r^2, thereby simplifying the integrand. Whenever you see an expression involving $x^2 + y^2$ in the integrand, you should consider the possibility of converting to polar coordinates.

EXERCISE SET 15.3 C CAS

In Exercises 1–6, evaluate the iterated integral.

1. $\displaystyle\int_0^{\pi/2}\int_0^{\sin\theta} r\cos\theta\, dr\, d\theta$

2. $\displaystyle\int_0^{\pi}\int_0^{1+\cos\theta} r\, dr\, d\theta$

3. $\displaystyle\int_0^{\pi/2}\int_0^{a\sin\theta} r^2\, dr\, d\theta$

4. $\displaystyle\int_0^{\pi/6}\int_0^{\cos 3\theta} r\, dr\, d\theta$

5. $\displaystyle\int_0^{\pi}\int_0^{1-\sin\theta} r^2\cos\theta\, dr\, d\theta$

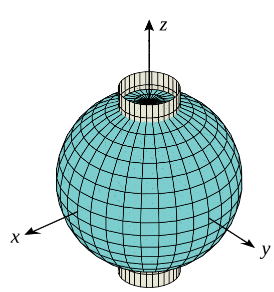
6. $\displaystyle\int_0^{\pi/2}\int_0^{\cos\theta} r^3\, dr\, d\theta$

In Exercises 7–12, use a double integral in polar coordinates to find the area of the region described.

7. The region enclosed by the cardioid $r = 1 - \cos\theta$.
8. The region enclosed by the rose $r = \sin 2\theta$.
9. The region in the first quadrant bounded by $r = 1$ and $r = \sin 2\theta$, with $\pi/4 \le \theta \le \pi/2$.
10. The region inside the circle $x^2 + y^2 = 4$ and to the right of the line $x = 1$.
11. The region inside the circle $r = 4\sin\theta$ and outside the circle $r = 2$.
12. The region inside the circle $r = 1$ and outside the cardioid $r = 1 + \cos\theta$.

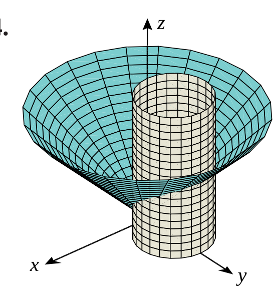
In Exercises 13–18, use a double integral in polar coordinates to find the volume of the solid that is described.

13.



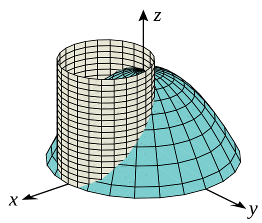
Inside of $x^2 + y^2 + z^2 = 9$
Outside of $x^2 + y^2 = 1$

14.



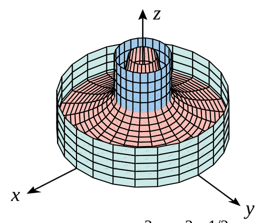
Below $z = \sqrt{x^2 + y^2}$
Inside of $x^2 + y^2 = 2y$
Above $z = 0$

15.



Below $z = 1 - x^2 - y^2$
Inside of $x^2 + y^2 - x = 0$
Above $z = 0$

16.



Below $z = (x^2 + y^2)^{-1/2}$
Outside of $x^2 + y^2 = 1$
Inside of $x^2 + y^2 = 9$
Above $z = 0$

17. The solid in the first octant bounded above by the surface $z = r\sin\theta$, below by the xy-plane, and laterally by the plane $x = 0$ and the surface $r = 3\sin\theta$.

18. The solid inside of the surface $r^2 + z^2 = 4$ and outside of the surface $r = 2\cos\theta$.

In Exercises 19–22, use polar coordinates to evaluate the double integral.

19. $\displaystyle\iint_R e^{-(x^2+y^2)}\,dA$, where R is the region enclosed by the circle $x^2 + y^2 = 1$.

20. $\displaystyle\iint_R \sqrt{9 - x^2 - y^2}\,dA$, where R is the region in the first quadrant within the circle $x^2 + y^2 = 9$.

21. $\displaystyle\iint_R \frac{1}{1 + x^2 + y^2}\,dA$, where R is the sector in the first quadrant bounded by $y = 0$, $y = x$, and $x^2 + y^2 = 4$.

22. $\displaystyle\iint_R 2y\,dA$, where R is the region in the first quadrant bounded above by the circle $(x - 1)^2 + y^2 = 1$ and below by the line $y = x$.

In Exercises 23–30, evaluate the iterated integral by converting to polar coordinates.

23. $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\,dy\,dx$

24. $\displaystyle\int_{-2}^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} e^{-(x^2+y^2)}\,dx\,dy$

25. $\displaystyle\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2 + y^2}\,dy\,dx$

26. $\displaystyle\int_0^1 \int_0^{\sqrt{1-y^2}} \cos(x^2 + y^2)\,dx\,dy$

27. $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{dy\,dx}{(1 + x^2 + y^2)^{3/2}} \quad (a > 0)$

28. $\displaystyle\int_0^1 \int_y^{\sqrt{y}} \sqrt{x^2 + y^2}\,dx\,dy$

29. $\displaystyle\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \frac{1}{\sqrt{1 + x^2 + y^2}}\,dx\,dy$

30. $\displaystyle\int_0^4 \int_3^{\sqrt{25-x^2}} dy\,dx$

31. Use a double integral in polar coordinates to find the volume of a cylinder of radius a and height h.

32. (a) Use a double integral in polar coordinates to find the volume of the oblate spheroid

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1 \quad (0 < c < a)$$

(b) Use the result in part (a) and the World Geodetic System of 1984 (WGS-84) discussed in Exercise 50 of Section 12.7 to find the volume of the Earth in cubic meters.

33. Use polar coordinates to find the volume of the solid that is inside of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1$$

above the xy-plane, and inside of the cylinder $x^2 + y^2 - ay = 0$.

34. Find the area of the region enclosed by the lemniscate $r^2 = 2a^2\cos 2\theta$.

35. Find the area in the first quadrant that is inside of the circle $r = 4\sin\theta$ and outside of the lemniscate $r^2 = 8\cos 2\theta$.

36. Show that the shaded area in the accompanying figure is $a^2\phi - \frac{1}{2}a^2\sin 2\phi$.

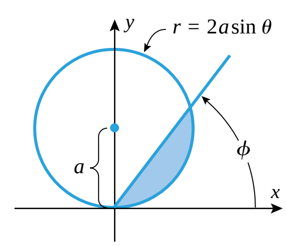


Figure Ex-36

37. The integral $\displaystyle\int_0^{+\infty} e^{-x^2}\,dx$, which arises in probability theory, can be evaluated using the following method. Let the value of the integral be I. Thus,

$$I = \int_0^{+\infty} e^{-x^2}\,dx = \int_0^{+\infty} e^{-y^2}\,dy$$

since the letter used for the variable of integration in a definite integral does not matter.

(a) Give a reasonable argument to show that

$$I^2 = \int_0^{+\infty} \int_0^{+\infty} e^{-(x^2+y^2)}\,dx\,dy$$

(b) Evaluate the iterated integral in part (a) by converting to polar coordinates.

(c) Use the result in part (b) to show that $I = \sqrt{\pi}/2$.

C **38.** (a) Use the numerical integration capability of a CAS to approximate the value of the double integral

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} e^{-(x^2+y^2)^2}\,dy\,dx$$

(b) Compare the approximation obtained in part (a) to the approximation that results if the integral is first converted to polar coordinates.

39. Suppose that a geyser, centered at the origin of a polar coordinate system, sprays water in a circular pattern in such a way that the depth D of water that reaches a point at a distance of r feet from the origin in 1 hour is $D = ke^{-r}$. Find the total volume of water that the geyser sprays inside a circle of radius R centered at the origin.

40. Evaluate $\iint_R x^2\,dA$ over the region R shown in the accompanying figure.

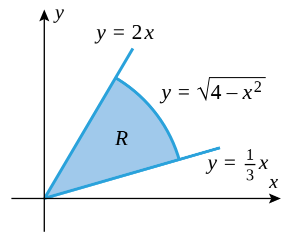


Figure Ex-40

15.4 PARAMETRIC SURFACES; SURFACE AREA

In previous sections we considered parametric curves in 2-space and 3-space. In this section we will discuss parametric surfaces in 3-space. As we will see, parametric representations of surfaces are not only important in computer graphics but also allow us to study more general kinds of surfaces than those encountered so far. In Section 6.5 we showed how to find the surface area of a surface of revolution. Our work on parametric surfaces will enable us to derive area formulas for more general kinds of surfaces.

PARAMETRIC REPRESENTATION OF SURFACES

We have seen that curves in 3-space can be represented by three equations involving one parameter, say

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

Surfaces in 3-space can be represented parametrically by three equations involving two parameters, say

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v) \tag{1}$$

To visualize why such equations represent a surface, think of (u, v) as a point that varies over some region in a uv-plane. If u is held constant, then v is the only varying parameter in (1), and hence these equations represent a curve in 3-space. We call this a ***constant u-curve*** (Figure 15.4.1). Similarly, if v is held constant, then u is the only varying parameter in (1), so again these equations represent a curve in 3-space. We call this a ***constant v-curve***. By varying the constants we generate a family of u-curves and a family of v-curves that together form a surface.

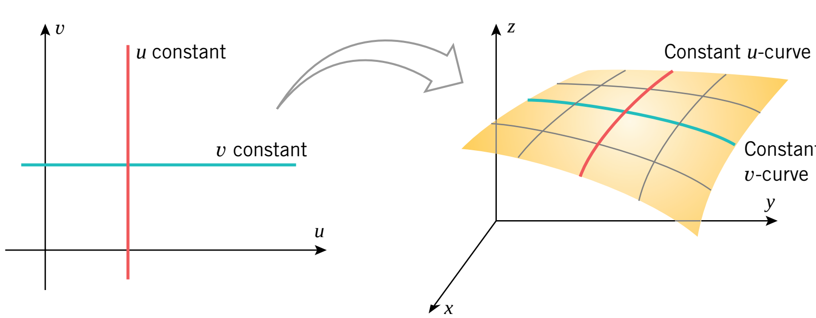


Figure 15.4.1

Example 1 Consider the paraboloid $z = 4 - x^2 - y^2$. One way to parametrize this surface is to take $x = u$ and $y = v$ as the parameters, in which case the surface is represented by the parametric equations

$$x = u, \quad y = v, \quad z = 4 - u^2 - v^2 \tag{2}$$

Figure 15.4.2*a* shows a computer-generated graph of this surface. The constant u-curves correspond to constant x-values and hence appear on the surface as traces parallel to the yz-plane. Similarly, the constant v-curves correspond to constant y-values and hence appear on the surface as traces parallel to the xz-plane. ◀

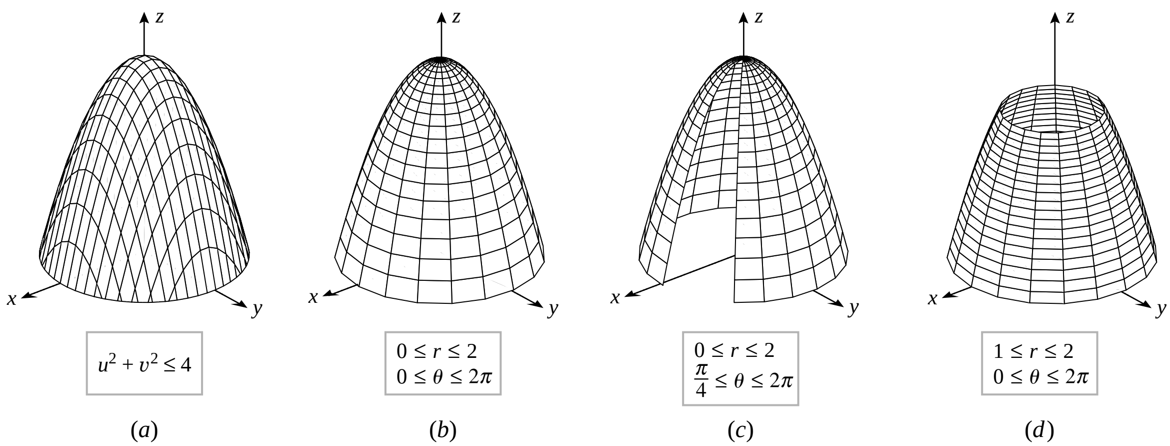


Figure 15.4.2

Example 2 The paraboloid $z = 4 - x^2 - y^2$ that was considered in Example 1 can also be parametrized by first expressing the equation in cylindrical coordinates. For this purpose, we make the substitution $x = r\cos\theta$, $y = r\sin\theta$, which yields $z = 4 - r^2$. Thus, the paraboloid can be represented parametrically in terms of r and θ as

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = 4 - r^2 \tag{3}$$

Figure 15.4.2*b* shows a computer-generated graph of this surface for $0 \le r \le 2$ and $0 \le \theta \le 2\pi$. The constant r-curves correspond to constant z-values and hence appear on the surface as traces parallel to the xy-plane. The constant θ-curves appear on the surface as traces from vertical planes through the origin at varying angles with the x-axis. Parts (*c*) and (*d*) of Figure 15.4.2 show the effect of restrictions on the parameters r and θ. ◀

FOR THE READER. If you have a graphing utility that can generate parametric surfaces, read the relevant documentation and then try to make reasonable duplicates of the surfaces in Figure 15.4.2.

Example 3 One way to generate the sphere $x^2 + y^2 + z^2 = 1$ with a graphing utility is to graph the upper and lower hemispheres

$$z = \sqrt{1 - x^2 - y^2} \quad \text{and} \quad z = -\sqrt{1 - x^2 - y^2}$$

on the same screen. However, this usually produces a fragmented sphere (Figure 15.4.3*a*) because roundoff error sporadically produces negative values inside the radical when $1 - x^2 - y^2$ is near zero. A better graph can be generated by first expressing the sphere in spherical coordinates as $\rho = 1$ and then using the spherical-to-rectangular conversion formulas in Table 12.8.1 to obtain the parametric equations

$$x = \sin\phi\cos\theta, \quad y = \sin\phi\sin\theta, \quad z = \cos\phi$$

with parameters θ and ϕ. Figure 15.4.3*b* shows the graph of this parametric surface for

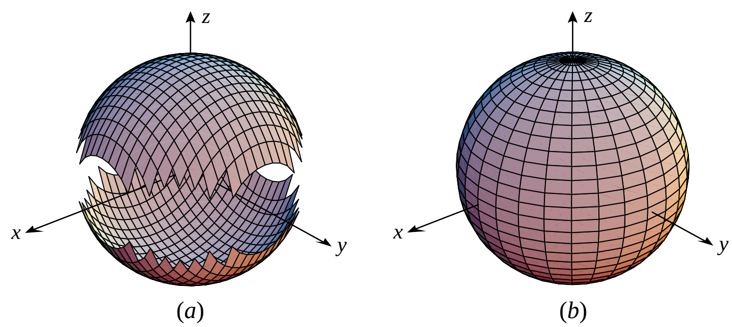


Figure 15.4.3

$0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$. In the language of cartographers, the constant ϕ-curves are the ***lines of latitude*** and the constant θ-curves are the ***lines of longitude***. ◀

Example 4 Find parametric equations for the portion of the right circular cylinder $x^2 + z^2 = 9$ for which $0 \le y \le 5$ in terms of the parameters u and v shown in Figure 15.4.4*a*. The parameter u is the y-coordinate of a point $P(x, y, z)$ on the surface, and v is the angle shown in the figure.

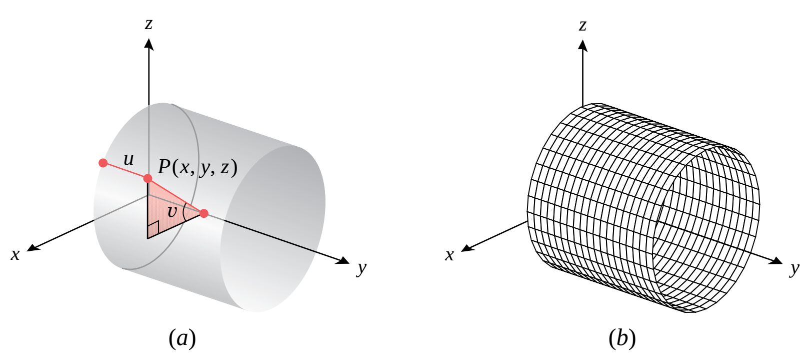


Figure 15.4.4

Solution. The radius of the cylinder is 3, so it is evident from the figure that $y = u$, $x = 3\cos v$, and $z = 3\sin v$. Thus, the surface can be represented parametrically as

$$x = 3\cos v, \quad y = u, \quad z = 3\sin v$$

To obtain the portion of the surface from $y = 0$ to $y = 5$, we let the parameter u vary over the interval $0 \le u \le 5$, and to ensure that the entire lateral surface is covered, we let the parameter v vary over the interval $0 \le v \le 2\pi$. Figure 15.4.4*b* shows a computer-generated graph of the surface in which u and v vary over these intervals. Constant u-curves appear as circular traces parallel to the xz-plane, and constant v-curves appear as lines parallel to the y-axis. ◀

REPRESENTING SURFACES OF REVOLUTION PARAMETRICALLY

The basic idea of Example 4 can be adapted to obtain parametric equations for surfaces of revolution. For example, suppose that we want to find parametric equations for the surface generated by revolving the plane curve $y = f(x)$ about the x-axis. Figure 15.4.5 suggests that the surface can be represented parametrically as

$$x = u, \quad y = f(u)\cos v, \quad z = f(u)\sin v \tag{4}$$

where v is the angle shown. In the exercises we will discuss analogous formulas for surfaces of revolution about other axes.

Example 5 Find parametric equations for the surface generated by revolving the curve $y = 1/x$ about the x-axis.

Solution. From (4) this surface can be represented parametrically as

$$x = u, \quad y = \frac{1}{u}\cos v, \quad z = \frac{1}{u}\sin v$$

Figure 15.4.6 shows a computer-generated graph of the surface for $0.7 \le u \le 5$ and $0 \le v \le 2\pi$. This surface is a portion of Gabriel's horn, which was discussed in Exercise 49 of Section 8.8. ◀

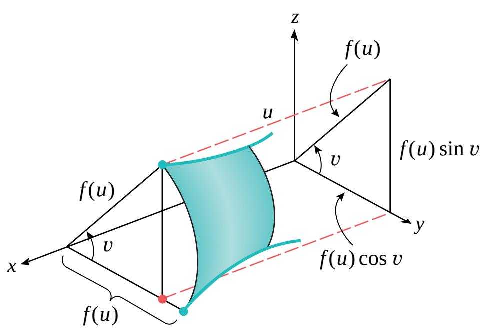


Figure 15.4.5

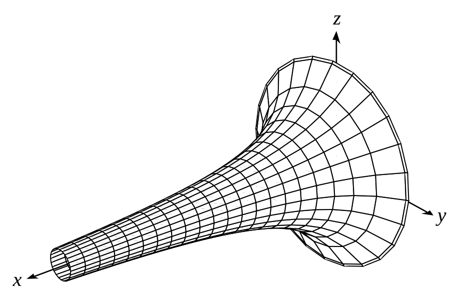


Figure 15.4.6

VECTOR-VALUED FUNCTIONS OF TWO VARIABLES

Recall that the parametric equations

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

can be expressed in vector form as

$$\mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the radius vector and $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is a vector-valued function of one variable. Similarly, the parametric equations

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v)$$

can be expressed in vector form as

$$\mathbf{r} = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

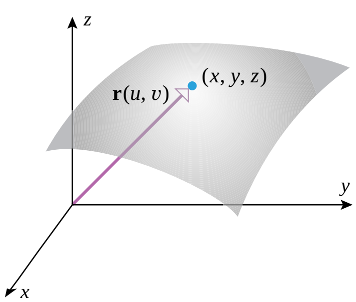


Figure 15.4.7

Here the function $\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$ is a ***vector-valued function of two variables***. We define the ***graph*** of $\mathbf{r}(u, v)$ to be the graph of the corresponding parametric equations. Geometrically, we can view $\mathbf{r}$ as a vector from the origin to a point (x, y, z) that moves over the surface $\mathbf{r} = \mathbf{r}(u, v)$ as u and v vary (Figure 15.4.7). As with vector-valued functions of one variable, we say that $\mathbf{r}(u, v)$ is ***continuous*** if each component is continuous.

Example 6 The paraboloid in Example 1 was expressed parametrically as

$$x = u, \quad y = v, \quad z = 4 - u^2 - v^2$$

These equations can be expressed in vector form as

$$\mathbf{r} = u\mathbf{i} + v\mathbf{j} + (4 - u^2 - v^2)\mathbf{k}$$ ◀

PARTIAL DERIVATIVES OF VECTOR-VALUED FUNCTIONS

Partial derivatives of vector-valued functions of two variables are obtained by taking partial derivatives of the components. For example, if

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

then

$$\frac{\partial \mathbf{r}}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial v} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

These derivatives can also be written as $\mathbf{r}_u$ and $\mathbf{r}_v$ or $\mathbf{r}_u(u, v)$ and $\mathbf{r}_v(u, v)$ and can be expressed as the limits

$$\frac{\partial \mathbf{r}}{\partial u} = \lim_{w \to u} \frac{\mathbf{r}(w, v) - \mathbf{r}(u, v)}{w - u} = \lim_{\Delta u \to 0} \frac{\mathbf{r}(u + \Delta u, v) - \mathbf{r}(u, v)}{\Delta u} \tag{5}$$

$$\frac{\partial \mathbf{r}}{\partial v} = \lim_{w \to v} \frac{\mathbf{r}(u, w) - \mathbf{r}(u, v)}{w - v} = \lim_{\Delta v \to 0} \frac{\mathbf{r}(u, v + \Delta v) - \mathbf{r}(u, v)}{\Delta v} \tag{6}$$

Example 7 Find the partial derivatives of the vector-valued function $\mathbf{r}$ in Example 6.

Solution. For the vector-valued function in Example 6, we have

$$\frac{\partial \mathbf{r}}{\partial u} = \frac{\partial}{\partial u}[u\mathbf{i} + v\mathbf{j} + (4 - u^2 - v^2)\mathbf{k}] = \mathbf{i} - 2u\mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial v} = \frac{\partial}{\partial v}[u\mathbf{i} + v\mathbf{j} + (4 - u^2 - v^2)\mathbf{k}] = \mathbf{j} - 2v\mathbf{k}$$ ◀

TANGENT PLANES TO PARAMETRIC SURFACES

Our next objective is to show how to find tangent planes to parametric surfaces. Let σ denote a parametric surface in 3-space, with P_0 a point on σ. We will say that a plane is ***tangent*** to σ at P_0 provided a line through P_0 lies in the plane if and only if it is a tangent line at P_0 to a curve on σ. We showed in Section 14.7 that if $z = f(x, y)$, then the graph of f has a tangent plane at a point if f is differentiable at that point. It is beyond the scope of this text to obtain precise conditions under which a parametric surface has a tangent plane at a point, so we will simply assume the existence of tangent planes at points of interest and focus on finding their equations.

Suppose that the parametric surface σ is the graph of the vector-valued function $\mathbf{r}(u, v)$ and that we are interested in the tangent plane at the point (x_0, y_0, z_0) on the surface that corresponds to the parameter values $u = u_0$ and $v = v_0$; that is,

$$\mathbf{r}(u_0, v_0) = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$$

If $v = v_0$ is kept fixed and u is allowed to vary, then $\mathbf{r}(u, v_0)$ is a vector-valued function of one variable whose graph is called the ***constant v-curve*** through the point (u_0, v_0); similarly, if $u = u_0$ is kept fixed and v is allowed to vary, then $\mathbf{r}(u_0, v)$ is a vector-valued function of one variable whose graph is called the ***constant u-curve*** through the point (u_0, v_0). Moreover, it follows from 13.2.5 that if $\partial \mathbf{r}/\partial u \neq \mathbf{0}$ at (u_0, v_0), then this vector is tangent to the constant v-curve through (u_0, v_0); and if $\partial \mathbf{r}/\partial v \neq \mathbf{0}$ at (u_0, v_0), then this vector is tangent to the constant u-curve through (u_0, v_0) (Figure 15.4.8). Thus, if $\partial \mathbf{r}/\partial u \times \partial \mathbf{r}/\partial v \neq \mathbf{0}$ at (u_0, v_0), then the vector

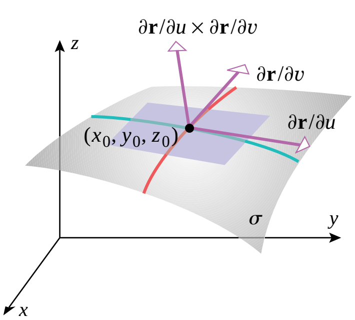


Figure 15.4.8

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix} \tag{7}$$

is orthogonal to both tangent vectors at the point (u_0, v_0) and hence is normal to the tangent plane and the surface at this point (Figure 15.4.8). Accordingly, we make the following definition.

15.4.1 DEFINITION. If a parametric surface σ is the graph of $\mathbf{r} = \mathbf{r}(u, v)$, and if $\partial\mathbf{r}/\partial u \times \partial\mathbf{r}/\partial v \neq \mathbf{0}$ at a point on the surface, then the ***principal unit normal vector*** to the surface at that point is denoted by $\mathbf{n}$ or $\mathbf{n}(u, v)$ and is defined as

$$\mathbf{n} = \frac{\dfrac{\partial\mathbf{r}}{\partial u} \times \dfrac{\partial\mathbf{r}}{\partial v}}{\left\|\dfrac{\partial\mathbf{r}}{\partial u} \times \dfrac{\partial\mathbf{r}}{\partial v}\right\|} \tag{8}$$

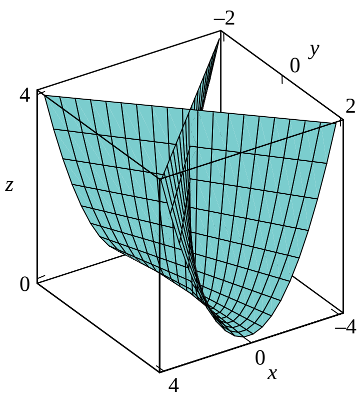


Figure 15.4.9

Example 8 Find an equation of the tangent plane to the parametric surface

$$x = uv, \quad y = u, \quad z = v^2$$

at the point where $u = 2$ and $v = -1$. This surface, called *Whitney's umbrella*, is an example of a self-intersecting parametric surface (Figure 15.4.9).

Solution. We start by writing the equations in the vector form

$$\mathbf{r} = uv\mathbf{i} + u\mathbf{j} + v^2\mathbf{k}$$

The partial derivatives of $\mathbf{r}$ are

$$\frac{\partial\mathbf{r}}{\partial u}(u, v) = v\mathbf{i} + \mathbf{j}$$

$$\frac{\partial\mathbf{r}}{\partial v}(u, v) = u\mathbf{i} + 2v\mathbf{k}$$

and at $u = 2$ and $v = -1$ these partial derivatives are

$$\frac{\partial\mathbf{r}}{\partial u}(2, -1) = -\mathbf{i} + \mathbf{j}$$

$$\frac{\partial\mathbf{r}}{\partial v}(2, -1) = 2\mathbf{i} - 2\mathbf{k}$$

Thus, from (7) and (8) a normal to the surface at this point is

$$\frac{\partial\mathbf{r}}{\partial u}(2, -1) \times \frac{\partial\mathbf{r}}{\partial v}(2, -1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ 2 & 0 & -2 \end{vmatrix} = -2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$$

Since any normal will suffice to find the tangent plane, it makes sense to multiply this vector by $-\frac{1}{2}$ and use the simpler normal $\mathbf{i} + \mathbf{j} + \mathbf{k}$. It follows from the given parametric equations that the point on the surface corresponding to $u = 2$ and $v = -1$ is $(-2, 2, 1)$, so the tangent plane at this point can be expressed in point-normal form as

$$(x + 2) + (y - 2) + (z - 1) = 0 \quad \text{or} \quad x + y + z = 1$$ ◀

FOR THE READER. Convince yourself that the result obtained in this example is consistent with Figure 15.4.9.

Example 9 The sphere $x^2 + y^2 + z^2 = a^2$ can be expressed in spherical coordinates as $\rho = a$, and the spherical-to-rectangular conversion formulas in Table 12.8.1 can then be used to express the sphere as the graph of the vector-valued function

$$\mathbf{r}(\phi, \theta) = a\sin\phi\cos\theta\,\mathbf{i} + a\sin\phi\sin\theta\,\mathbf{j} + a\cos\phi\,\mathbf{k}$$

where $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$ (verify). Use this function to show that at each point on the sphere the tangent plane is perpendicular to the radius vector.

Solution. We will show that at each point of the sphere the unit normal vector $\mathbf{n}$ is a scalar multiple of $\mathbf{r}$ (and hence is parallel to $\mathbf{r}$). We have

$$\frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial \phi} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial z}{\partial \phi} \\ \dfrac{\partial x}{\partial \theta} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= a^2\sin^2\phi\cos\theta\,\mathbf{i} + a^2\sin^2\phi\sin\theta\,\mathbf{j} + a^2\sin\phi\cos\phi\,\mathbf{k}$$

and hence

$$\left\| \frac{\partial \mathbf{r}}{\partial \phi} \times \frac{\partial \mathbf{r}}{\partial \theta} \right\| = \sqrt{a^4\sin^4\phi\cos^2\theta + a^4\sin^4\phi\sin^2\theta + a^4\sin^2\phi\cos^2\phi}$$

$$= \sqrt{a^4\sin^4\phi + a^4\sin^2\phi\cos^2\phi}$$

$$= a^2\sqrt{\sin^2\phi} = a^2|\sin\phi| = a^2\sin\phi$$

Thus, it follows from (8) that

$$\mathbf{n} = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k} = \frac{1}{a}\mathbf{r}$$ ◀

SURFACE AREA OF PARAMETRIC SURFACES

In Section 6.5 we obtained formulas for the surface area of a surface of revolution [see Formulas (4) and (5) and the discussion preceding Exercise 18 in that section]. We will now obtain a formula for the surface area S of a parametric surface σ and from that formula we will then derive a formula for the surface area of a surface of the form $z = f(x, y)$.

Let σ be a parametric surface whose vector equation is

$$\mathbf{r} = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

We will say that σ is a ***smooth parametric surface*** on a region R of the uv-plane if $\partial\mathbf{r}/\partial u$ and $\partial\mathbf{r}/\partial v$ are continuous on R and $\partial\mathbf{r}/\partial u \times \partial\mathbf{r}/\partial v \neq \mathbf{0}$ on R. Geometrically, this means that σ has a principal unit normal vector (and hence a tangent plane) for all (u, v) in R and $\mathbf{n} = \mathbf{n}(u, v)$ is a continuous function on R. Thus, on a smooth parametric surface the unit normal vector $\mathbf{n}$ varies continuously and has no abrupt changes in direction. We will derive a surface area formula for smooth surfaces that have no self-intersections.

We will begin by subdividing R into rectangular regions by lines parallel to the u- and v-axes and discarding any nonrectangular portions that contain points of the boundary. Assume that there are n rectangles, and let R_k denote the kth rectangle. Let (u_k, v_k) be the lower left corner of R_k, and assume that R_k has area $\Delta A_k = \Delta u_k \Delta v_k$, where Δu_k and Δv_k are the dimensions of R_k (Figure 15.4.10*a*). The image of R_k will be some *curvilinear patch* σ_k on the surface σ that has a corner at $\mathbf{r}(u_k, v_k)$; denote the area of this patch by ΔS_k (Figure 15.4.10*b*).

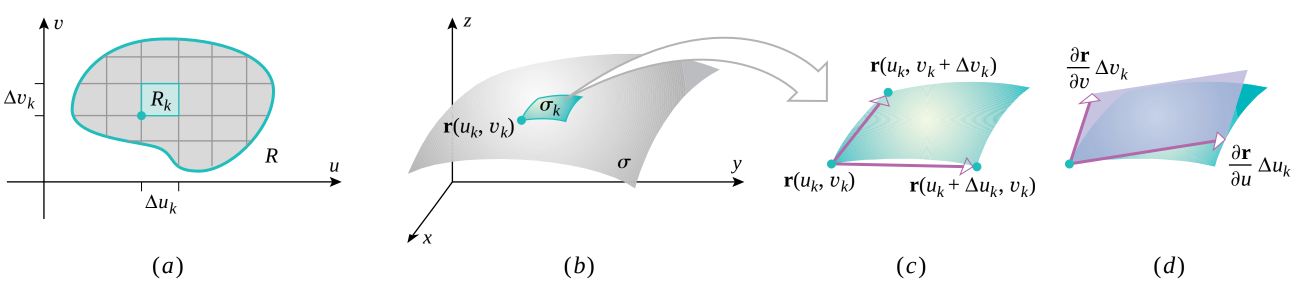


Figure 15.4.10

As suggested by Figure 15.4.10*c*, the two edges of the patch that meet at $\mathbf{r}(u_k, v_k)$ can be approximated by the "secant" vectors

$$\mathbf{r}(u_k + \Delta u_k, v_k) - \mathbf{r}(u_k, v_k)$$
$$\mathbf{r}(u_k, v_k + \Delta v_k) - \mathbf{r}(u_k, v_k)$$

and hence the area of σ_k can be approximated by the area of the parallelogram determined by these vectors. However, it follows from Formulas (5) and (6) that if Δu_k and Δv_k are small, then these secant vectors can in turn be approximated by the tangent vectors

$$\frac{\partial \mathbf{r}}{\partial u}\Delta u_k \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial v}\Delta v_k$$

where the partial derivatives are evaluated at (u_k, v_k). Thus, the area of the patch σ_k can be approximated by the area of the parallelogram determined by these vectors (Figure 15.4.10*d*); that is,

$$\Delta S_k \approx \left\| \frac{\partial \mathbf{r}}{\partial u}\Delta u_k \times \frac{\partial \mathbf{r}}{\partial v}\Delta v_k \right\| = \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \Delta u_k \Delta v_k = \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \Delta A_k \tag{9}$$

It follows that the surface area S of the entire surface σ can be approximated as

$$S \approx \sum_{k=1}^{n} \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \Delta A_k$$

Thus, if we assume that the errors in the approximations approach zero as n increases in such a way that the dimensions of the rectangles approach zero, then it is plausible that the exact value of S is

$$S = \lim_{n\to+\infty} \sum_{k=1}^{n} \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \Delta A_k$$

or, equivalently,

$$S = \iint_R \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dA \tag{10}$$

Example 10 It follows from (4) that the parametric equations

$$x = u, \quad y = u\cos v, \quad z = u\sin v$$

represent the cone that results when the line $y = x$ in the xy-plane is revolved about the x-axis. Use Formula (10) to find the surface area of that portion of the cone for which $0 \le u \le 2$ and $0 \le v \le 2\pi$ (Figure 15.4.11).

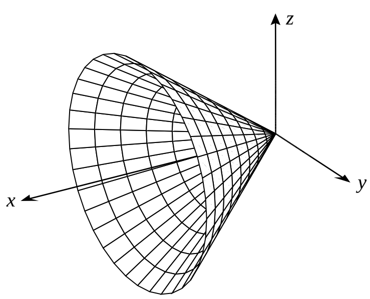


Figure 15.4.11

Solution. The surface can be expressed in vector form as

$$\mathbf{r} = u\mathbf{i} + u\cos v\mathbf{j} + u\sin v\mathbf{k} \quad (0 \le u \le 2,\ \ 0 \le v \le 2\pi)$$

Thus,

$$\frac{\partial \mathbf{r}}{\partial u} = \mathbf{i} + \cos v\mathbf{j} + \sin v\mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial v} = -u\sin v\mathbf{j} + u\cos v\mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & \cos v & \sin v \\ 0 & -u\sin v & u\cos v \end{vmatrix} = u\mathbf{i} - u\cos v\mathbf{j} - u\sin v\mathbf{k}$$

$$\left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| = \sqrt{u^2 + (-u\cos v)^2 + (-u\sin v)^2} = |u|\sqrt{2} = u\sqrt{2}$$

Thus, from (10)

$$S = \iint_R \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| dA = \int_0^{2\pi}\int_0^2 \sqrt{2}u\,du\,dv = 2\sqrt{2}\int_0^{2\pi} dv = 4\pi\sqrt{2}$$ ◀

SURFACE AREA OF SURFACES OF THE FORM $z = f(x, y)$

In the case where σ is a surface of the form $z = f(x, y)$, we can take $x = u$ and $y = v$ as parameters and express the surface parametrically as

$$x = u, \quad y = v, \quad z = f(u, v)$$

or in vector form as

$$\mathbf{r} = u\mathbf{i} + v\mathbf{j} + f(u, v)\mathbf{k}$$

Thus,

$$\frac{\partial \mathbf{r}}{\partial u} = \mathbf{i} + \frac{\partial f}{\partial u}\mathbf{k} = \mathbf{i} + \frac{\partial z}{\partial x}\mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial v} = \mathbf{j} + \frac{\partial f}{\partial v}\mathbf{k} = \mathbf{j} + \frac{\partial z}{\partial y}\mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \dfrac{\partial z}{\partial x} \\ 0 & 1 & \dfrac{\partial z}{\partial y} \end{vmatrix} = -\frac{\partial z}{\partial x}\mathbf{i} - \frac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}$$

$$\left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

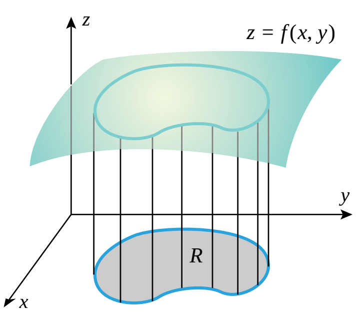


Figure 15.4.12

Thus, it follows from (10) that

$$S = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA \tag{11}$$

REMARK. In this formula the region R lies in the xy-plane because the parameters are x and y. Geometrically, this region is the projection on the xy-plane of that portion of the surface $z = f(x, y)$ whose area is being determined by the formula (Figure 15.4.12).

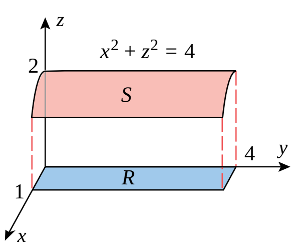


Figure 15.4.13

Example 11 Find the surface area of that portion of the surface $z = \sqrt{4 - x^2}$ that lies above the rectangle R in the xy-plane whose coordinates satisfy $0 \le x \le 1$ and $0 \le y \le 4$.

Solution. As shown in Figure 15.4.13, the surface is a portion of the cylinder $x^2 + z^2 = 4$. It follows from (11) that the surface area is

$$\begin{aligned} S &= \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA \\ &= \iint_R \sqrt{\left(-\frac{x}{\sqrt{4 - x^2}}\right)^2 + 0 + 1}\, dA = \int_0^4 \int_0^1 \frac{2}{\sqrt{4 - x^2}}\, dx\, dy \\ &= 2\int_0^4 \left[\sin^{-1}\left(\frac{1}{2}x\right)\right]_{x=0}^1 dy = 2\int_0^4 \frac{\pi}{6}\, dy = \frac{4}{3}\pi \end{aligned}$$ ◀

Formula (21) of Section 8.1

Example 12 Find the surface area of the portion of the paraboloid $z = x^2 + y^2$ below the plane $z = 1$.

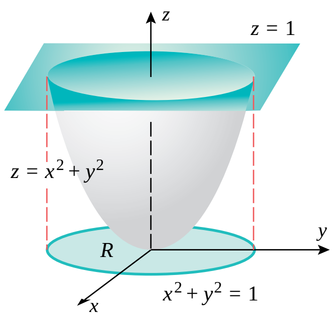


Figure 15.4.14

Solution. The surface is shown in Figure 15.4.14. The trace of the paraboloid $z = x^2 + y^2$ in the plane $z = 1$ projects onto the circle $x^2 + y^2 = 1$ in the xy-plane, and the portion of the paraboloid that lies below the plane $z = 1$ projects onto the region R that is enclosed by this circle. Thus, it follows from (11) that the surface area is

$$S = \iint_R \sqrt{4x^2 + 4y^2 + 1}\, dA$$

The expression $4x^2+4y^2+1 = 4(x^2+y^2)+1$ in the integrand suggests that we evaluate the integral in polar coordinates. In accordance with Formula (9) of Section 15.3, we substitute $x = r\cos\theta$ and $y = r\sin\theta$ in the integrand, replace dA by $r\,dr\,d\theta$, and find the limits of integration by expressing the region R in polar coordinates. This yields

$$S = \int_0^{2\pi}\int_0^1 \sqrt{4r^2+1}\, r\, dr\, d\theta = \int_0^{2\pi}\left[\frac{1}{12}(4r^2+1)^{3/2}\right]_{r=0}^{1} d\theta$$

$$= \int_0^{2\pi} \frac{1}{12}(5\sqrt{5}-1)\, d\theta = \frac{1}{6}\pi(5\sqrt{5}-1) \qquad ◀$$

EXERCISE SET 15.4 Graphing Utility C CAS

In Exercises 1 and 2, sketch the parametric surface.

1. (a) $x = u,\ y = v,\ z = \sqrt{u^2+v^2}$
(b) $x = u,\ y = \sqrt{u^2+v^2},\ z = v$
(c) $x = \sqrt{u^2+v^2},\ y = u,\ z = v$

2. (a) $x = u,\ y = v,\ z = u^2+v^2$
(b) $x = u,\ y = u^2+v^2,\ z = v$
(c) $x = u^2+v^2,\ y = u,\ z = v$

In Exercises 3 and 4, find a parametric representation of the surface in terms of the parameters $u = x$ and $v = y$.

3. (a) $2z - 3x + 4y = 5$ (b) $z = x^2$

4. (a) $z + zx^2 - y = 0$ (b) $y^2 - 3z = 5$

5. (a) Find parametric equations for the portion of the cylinder $x^2 + y^2 = 5$ that extends between the planes $z = 0$ and $z = 1$.
(b) Find parametric equations for the portion of the cylinder $x^2 + z^2 = 4$ that extends between the planes $y = 1$ and $y = 3$.

6. (a) Find parametric equations for the portion of the plane $x + y = 1$ that extends between the planes $z = -1$ and $z = 1$.
(b) Find parametric equations for the portion of the plane $y - 2z = 5$ that extends between the planes $x = 0$ and $x = 3$.

7. Find parametric equations for the surface generated by revolving the curve $y = \sin x$ about the x-axis.

8. Find parametric equations for the surface generated by revolving the curve $y - e^x = 0$ about the x-axis.

In Exercises 9–14, find a parametric representation of the surface in terms of the parameters r and θ, where (r, θ, z) are the cylindrical coordinates of a point on the surface.

9. $z = \dfrac{1}{1 + x^2 + y^2}$

10. $z = e^{-(x^2+y^2)}$

11. $z = 2xy$

12. $z = x^2 - y^2$

13. The portion of the sphere $x^2 + y^2 + z^2 = 9$ on or above the plane $z = 2$.

14. The portion of the cone $z = \sqrt{x^2 + y^2}$ on or below the plane $z = 3$.

15. Find a parametric representation of the cone

$$z = \sqrt{3x^2 + 3y^2}$$

in terms of parameters ρ and θ, where (ρ, θ, ϕ) are spherical coordinates of a point on the surface.

16. Find a parametric representation of the cylinder $x^2+y^2 = 9$ in terms of parameters θ and ϕ, where (ρ, θ, ϕ) are spherical coordinates of a point on the surface.

In Exercises 17–22, eliminate the parameters to obtain an equation in rectangular coordinates, and describe the surface.

17. $x = 2u + v,\ y = u - v,\ z = 3v$ for $-\infty < u < +\infty$ and $-\infty < v < +\infty$.

18. $x = u\cos v,\ y = u^2,\ z = u\sin v$ for $0 \le u \le 2$ and $0 \le v < 2\pi$.

19. $x = 3\sin u$, $y = 2\cos u$, $z = 2v$ for $0 \le u < 2\pi$ and $1 \le v \le 2$.

20. $x = \sqrt{u}\cos v$, $y = \sqrt{u}\sin v$, $z = u$ for $0 \le u \le 4$ and $0 \le v < 2\pi$.

21. $\mathbf{r}(u, v) = 3u\cos v\mathbf{i} + 4u\sin v\mathbf{j} + u\mathbf{k}$ for $0 \le u \le 1$ and $0 \le v < 2\pi$.

22. $r(u, v) = \sin u\cos v\mathbf{i} + 2\sin u\sin v\mathbf{j} + 3\cos u\mathbf{k}$ for $0 \le u \le \pi$ and $0 \le v < 2\pi$.

23. The accompanying figure shows the graphs of two parametric representations of the cone $z = \sqrt{x^2 + y^2}$ for $0 \le z \le 2$.
(a) Find parametric equations that produce reasonable facsimiles of these surfaces.
(b) Use a graphing utility to check your answer to part (a).

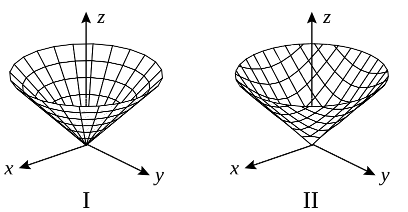


Figure Ex-23

24. The accompanying figure shows the graphs of two parametric representations of the paraboloid $z = x^2 + y^2$ for $0 \le z \le 2$.
(a) Find parametric equations that produce reasonable facsimiles of these surfaces.
(b) Use a graphing utility to check your answer to part (a).

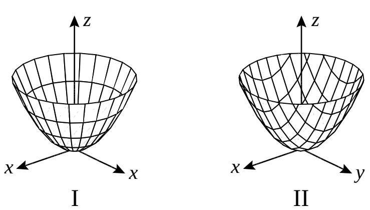
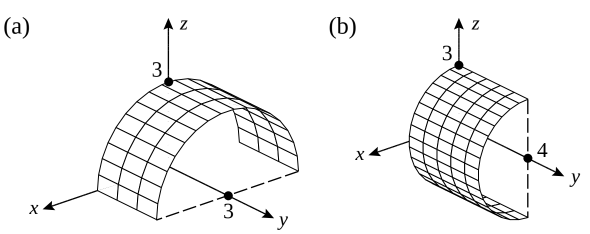
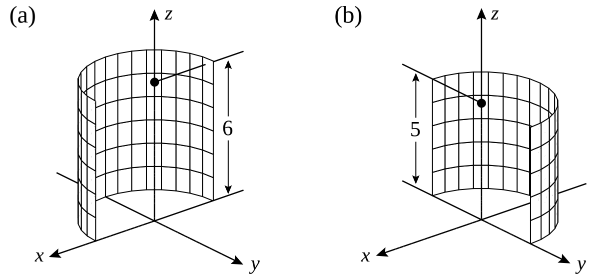


Figure Ex-24

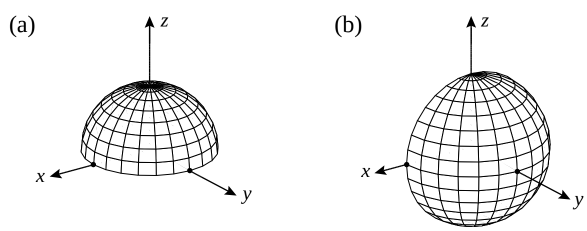
25. In each part, the figure shows a portion of the parametric surface $x = 3\cos v$, $y = u$, $z = 3\sin v$. Find restrictions on u and v that produce the surface, and check your answer with a graphing utility.



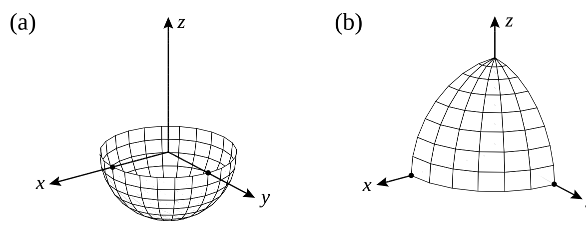
26. In each part, the figure shows a portion of the parametric surface $x = 3\cos v$, $y = 3\sin v$, $z = u$. Find restrictions on u and v that produce the surface, and check your answer with a graphing utility.



27. In each part, the figure shows a hemisphere that is a portion of the sphere $x = \sin\phi\cos\theta$, $y = \sin\phi\sin\theta$, $z = \cos\phi$. Find restrictions on ϕ and θ that produce the hemisphere, and check your answer with a graphing utility.



28. Each figure shows a portion of the sphere $x = \sin\phi\cos\theta$, $y = \sin\phi\sin\theta$, $z = \cos\phi$. Find restrictions on ϕ and θ that produce the surface, and check your answer with a graphing utility.



In Exercises 29–34, find an equation of the tangent plane to the parametric surface at the stated point.

29. $x = u$, $y = v$, $z = u^2 + v^2$; $(1, 2, 5)$

30. $x = u^2$, $y = v^2$, $z = u + v$; $(1, 4, 3)$

31. $x = 3v\sin u$, $y = 2v\cos u$, $z = u^2$; $(0, 2, 0)$

32. $\mathbf{r} = uv\mathbf{i} + (u - v)\mathbf{j} + (u + v)\mathbf{k}$; $u = 1$, $v = 2$

33. $\mathbf{r} = u\cos v\mathbf{i} + u\sin v\mathbf{j} + v\mathbf{k}$; $u = 1/2$, $v = \pi/4$

34. $\mathbf{r} = uv\mathbf{i} + ue^v\mathbf{j} + ve^u\mathbf{k}$; $u = \ln 2$, $v = 0$

In Exercises 35–46, find the area of the given surface.

35. The portion of the cylinder $y^2 + z^2 = 9$ that is above the rectangle $R = \{(x, y) : 0 \le x \le 2, -3 \le y \le 3\}$.

36. The portion of the plane $2x + 2y + z = 8$ in the first octant.

37. The portion of the cone $z^2 = 4x^2 + 4y^2$ that is above the region in the first quadrant bounded by the line $y = x$ and the parabola $y = x^2$.

38. The portion of the cone $z = \sqrt{x^2 + y^2}$ that lies inside the cylinder $x^2 + y^2 = 2x$.

39. The portion of the paraboloid $z = 1 - x^2 - y^2$ that is above the xy-plane.

40. The portion of the surface $z = 2x + y^2$ that is above the triangular region with vertices (0, 0), (0, 1), and (1, 1).

41. The portion of the paraboloid

$$\mathbf{r}(u, v) = u \cos v\,\mathbf{i} + u \sin v\,\mathbf{j} + u^2\mathbf{k}$$

for which $1 \le u \le 2$, $0 \le v \le 2\pi$.

42. The portion of the cone

$$\mathbf{r}(u, v) = u \cos v\,\mathbf{i} + u \sin v\,\mathbf{j} + u\mathbf{k}$$

for which $0 \le u \le 2v$, $0 \le v \le \pi/2$.

43. The portion of the surface $z = xy$ that is above the sector in the first quadrant bounded by the lines $y = x/\sqrt{3}$, $y = 0$, and the circle $x^2 + y^2 = 9$.

44. The portion of the paraboloid $2z = x^2 + y^2$ that is inside the cylinder $x^2 + y^2 = 8$.

45. The portion of the sphere $x^2 + y^2 + z^2 = 16$ between the planes $z = 1$ and $z = 2$.

46. The portion of the sphere $x^2 + y^2 + z^2 = 8$ that is inside of the cone $z = \sqrt{x^2 + y^2}$.

47. Use parametric equations to derive the formula for the surface area of a sphere of radius a.

48. Use parametric equations to derive the formula for the lateral surface area of a right circular cylinder of radius r and height h.

49. The portion of the surface

$$z = \frac{h}{a}\sqrt{x^2 + y^2} \quad (a, h > 0)$$

between the xy-plane and the plane $z = h$ is a right circular cone of height h and radius a. Use a double integral to show that the lateral surface area of this cone is $S = \pi a\sqrt{a^2 + h^2}$.

50. The accompanying figure shows the ***torus*** that is generated by revolving the circle

$$(x - a)^2 + z^2 = b^2 \quad (0 < b < a)$$

in the xz-plane about the z-axis.

(a) Show that this torus can be expressed parametrically as

$$x = (a + b \cos v) \cos u$$
$$y = (a + b \cos v) \sin u$$
$$z = b \sin v$$

where u and v are the parameters shown in the figure and $0 \le u \le 2\pi$, $0 \le v \le 2\pi$.

(b) Use a graphing utility to generate a torus.

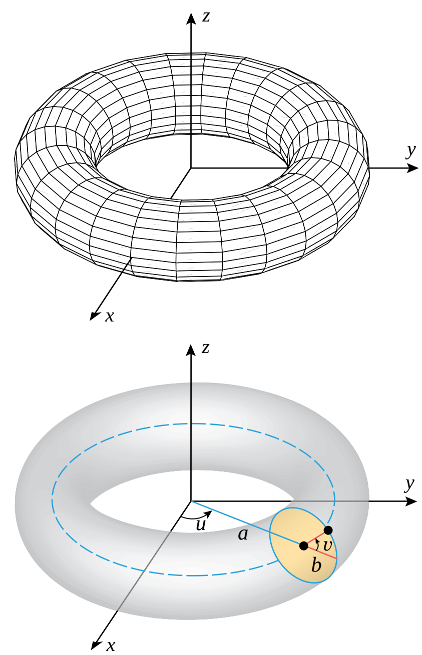


Figure Ex-50

51. Find the surface area of the torus in Exercise 50(a).

C **52.** Use a CAS to graph the ***helicoid***

$$x = u \cos v, \quad y = u \sin v, \quad z = v$$

for $0 \le u \le 5$ and $0 \le v \le 4\pi$ (see the accompanying figure), and then use the numerical double integration operation of the CAS to approximate the surface area.

C **53.** Use a CAS to graph the ***pseudosphere***

$$x = \cos u \sin v$$
$$y = \sin u \sin v$$
$$z = \cos v + \ln\left(\tan\frac{v}{2}\right)$$

for $0 \le u \le 2\pi$, $0 < v < \pi$ (see the accompanying figure), and then use the numerical double integration operation of the CAS to approximate the surface area between the planes $z = -1$ and $z = 1$.

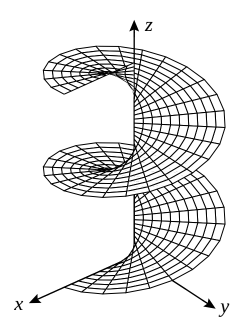


Figure Ex-52

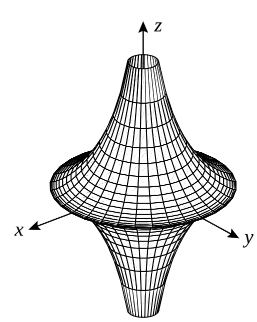


Figure Ex-53

54. (a) Find parametric equations for the surface of revolution that is generated by revolving the curve $z = f(x)$ in the xz-plane about the z-axis.

(b) Use the result obtained in part (a) to find parametric equations for the surface of revolution that is generated by revolving the curve $z = 1/x^2$ in the xz-plane about the z-axis.

(c) Use a graphing utility to check your work by graphing the parametric surface.

In Exercises 55–57, the parametric equations represent a quadric surface for positive values of a, b, and c. Identify the type of surface by eliminating the parameters u and v. Check your conclusion by choosing specific values for the constants and generating the surface with a graphing utility.

55. $x = a\cos u \cos v,\ \ y = b\sin u \cos v,\ \ z = c\sin v$

56. $x = a\cos u \cosh v,\ \ y = b\sin u \cosh v,\ \ z = c\sinh v$

57. $x = a\sinh v,\ \ y = b\sinh u \cosh v,\ \ z = c\cosh u \cosh v$

15.5 TRIPLE INTEGRALS

In the preceding sections we defined and discussed properties of double integrals for functions of two variables. In this section we will define triple integrals for functions of three variables.

DEFINITION OF A TRIPLE INTEGRAL

A single integral of a function $f(x)$ is defined over a finite closed interval on the x-axis, and a double integral of a function $f(x, y)$ is defined over a finite closed region R in the xy-plane. Our first goal in this section is to define what is meant by a *triple integral* of $f(x, y, z)$ over a closed solid region G in an xyz-coordinate system. To ensure that G does not extend indefinitely in some direction, we will assume that it can be enclosed in a suitably large box whose sides are parallel to the coordinate planes (Figure 15.5.1). In this case we say that G is a ***finite solid***.

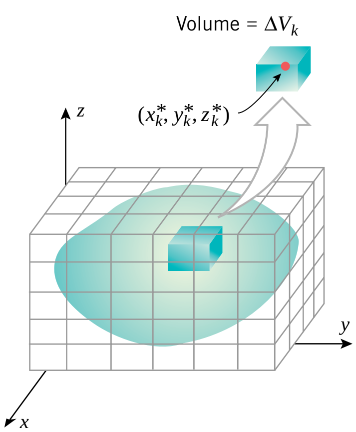


Figure 15.5.1

To define the triple integral of $f(x, y, z)$ over G, we first divide the box into n "subboxes" by planes parallel to the coordinate planes. We then discard those subboxes that contain any points outside of G and choose an arbitrary point in each of the remaining subboxes. As shown in Figure 15.5.1, we denote the volume of the kth remaining subbox by ΔV_k and the point selected in the kth subbox by (x_k^*, y_k^*, z_k^*). Next we form the product

$$f(x_k^*, y_k^*, z_k^*)\Delta V_k$$

for each subbox, then add the products for all of the subboxes to obtain the ***Riemann sum***

$$\sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k$$

Finally, we repeat this process with more and more subdivisions in such a way that the length, width, and height of each subbox approach zero, and n approaches $+\infty$. The limit

$$\iiint_G f(x, y, z)\,dV = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k \qquad (1)$$

is called the ***triple integral*** of $f(x, y, z)$ over the region G. Conditions under which the triple integral exists are studied in advanced calculus. However, for our purposes it suffices to say that existence is ensured when f is continuous on G and the region G is not too "complicated."

PROPERTIES OF TRIPLE INTEGRALS

Triple integrals enjoy many properties of single and double integrals:

$$\iiint_G cf(x, y, z)\, dV = c \iiint_G f(x, y, z)\, dV \quad (c \text{ a constant})$$

$$\iiint_G [f(x, y, z) + g(x, y, z)]\, dV = \iiint_G f(x, y, z)\, dV + \iiint_G g(x, y, z)\, dV$$

$$\iiint_G [f(x, y, z) - g(x, y, z)]\, dV = \iiint_G f(x, y, z)\, dV - \iiint_G g(x, y, z)\, dV$$

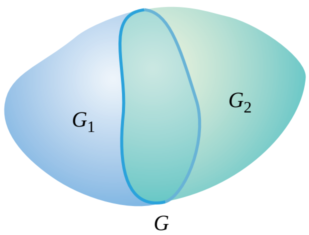


Figure 15.5.2

Moreover, if the region G is subdivided into two subregions G_1 and G_2 (Figure 15.5.2), then

$$\iiint_G f(x, y, z)\, dV = \iiint_{G_1} f(x, y, z)\, dV + \iiint_{G_2} f(x, y, z)\, dV$$

We omit the proofs.

EVALUATING TRIPLE INTEGRALS OVER RECTANGULAR BOXES

Just as a double integral can be evaluated by two successive single integrations, so a triple integral can be evaluated by three successive integrations. The following theorem, which we state without proof, is the analog of Theorem 15.1.3.

15.5.1 THEOREM. *Let G be the rectangular box defined by the inequalities*

$$a \le x \le b, \quad c \le y \le d, \quad k \le z \le \ell$$

If f is continuous on the region G, then

$$\iiint_G f(x, y, z)\, dV = \int_a^b \int_c^d \int_k^\ell f(x, y, z)\, dz\, dy\, dx \tag{2}$$

Moreover, the iterated integral on the right can be replaced with any of the five other iterated integrals that result by altering the order of integration.

Example 1 Evaluate the triple integral

$$\iiint_G 12xy^2z^3\, dV$$

over the rectangular box G defined by the inequalities $-1 \le x \le 2$, $0 \le y \le 3$, $0 \le z \le 2$.

Solution. Of the six possible iterated integrals we might use, we will choose the one in (2). Thus, we will first integrate with respect to z, holding x and y fixed, then with respect to y, holding x fixed, and finally with respect to x.

$$\begin{aligned}
\iiint_G 12xy^2z^3\, dV &= \int_{-1}^{2} \int_0^3 \int_0^2 12xy^2z^3\, dz\, dy\, dx \\
&= \int_{-1}^{2} \int_0^3 \left[3xy^2z^4\right]_{z=0}^{2} dy\, dx = \int_{-1}^{2} \int_0^3 48xy^2\, dy\, dx \\
&= \int_{-1}^{2} \left[16xy^3\right]_{y=0}^{3} dx = \int_{-1}^{2} 432x\, dx \\
&= 216x^2\Big]_{-1}^{2} = 648
\end{aligned}$$

◀

EVALUATING TRIPLE INTEGRALS OVER MORE GENERAL REGIONS

Next we will consider how triple integrals can be evaluated over solids that are not rectangular boxes. For the moment we will limit our discussion to solids of the type shown in Figure 15.5.3. Specifically, we will assume that the solid G is bounded above by a surface $z = g_2(x, y)$ and below by a surface $z = g_1(x, y)$ and that the projection of the solid on the xy-plane is a type I or type II region R (see Definition 15.2.1). In addition, we will assume that $g_1(x, y)$ and $g_2(x, y)$ are continuous on R and that $g_1(x, y) \leq g_2(x, y)$ on R. Geometrically, this means that the surfaces may touch but cannot cross. We call a solid of this type a ***simple xy-solid***.

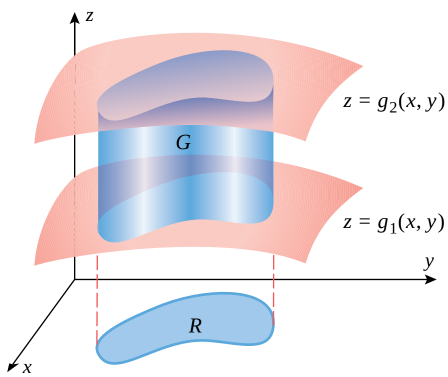


Figure 15.5.3

The following theorem, which we state without proof, will enable us to evaluate triple integrals over simple xy-solids.

15.5.2 THEOREM. *Let G be a simple xy-solid with upper surface $z = g_2(x, y)$ and lower surface $z = g_1(x, y)$, and let R be the projection of G on the xy-plane. If $f(x, y, z)$ is continuous on G, then*

$$\iiint_G f(x, y, z)\, dV = \iint_R \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x, y, z)\, dz\right] dA \tag{3}$$

In (3), the first integration is with respect to z, after which a function of x and y remains. This function of x and y is then integrated over the region R in the xy-plane. To apply (3), it is helpful to begin with a three-dimensional sketch of the solid G. The limits of integration can be obtained from the sketch as follows:

Step 1. Find an equation $z = g_2(x, y)$ for the upper surface and an equation $z = g_1(x, y)$ for the lower surface of G. The functions $g_1(x, y)$ and $g_2(x, y)$ determine the lower and upper z-limits of integration.

Step 2. Make a two-dimensional sketch of the projection R of the solid on the xy-plane. From this sketch determine the limits of integration for the double integral over R in (3).

Example 2 Let G be the wedge in the first octant cut from the cylindrical solid $y^2+z^2 \leq 1$ by the planes $y = x$ and $x = 0$. Evaluate

$$\iiint_G z\, dV$$

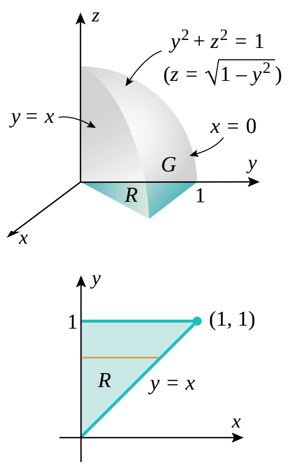


Figure 15.5.4

Solution. The solid G and its projection R on the xy-plane are shown in Figure 15.5.4. The upper surface of the solid is formed by the cylinder and the lower surface by the xy-plane. Since the portion of the cylinder $y^2 + z^2 = 1$ that lies above the xy-plane has the equation $z = \sqrt{1 - y^2}$, and the xy-plane has the equation $z = 0$, it follows from (3) that

$$\iiint_G z\,dV = \iint_R \left[\int_0^{\sqrt{1-y^2}} z\,dz\right] dA \tag{4}$$

For the double integral over R, the x- and y-integrations can be performed in either order, since R is both a type I and type II region. We will integrate with respect to x first. With this choice, (4) yields

$$\iiint_G z\,dV = \int_0^1 \int_0^y \int_0^{\sqrt{1-y^2}} z\,dz\,dx\,dy = \int_0^1 \int_0^y \frac{1}{2}z^2\bigg]_{z=0}^{\sqrt{1-y^2}} dx\,dy$$

$$= \int_0^1 \int_0^y \frac{1}{2}(1 - y^2)\,dx\,dy = \frac{1}{2}\int_0^1 (1 - y^2)x\bigg]_{x=0}^{y} dy$$

$$= \frac{1}{2}\int_0^1 (y - y^3)\,dy = \frac{1}{2}\left[\frac{1}{2}y^2 - \frac{1}{4}y^4\right]_0^1 = \frac{1}{8} \quad \blacktriangleleft$$

FOR THE READER. Most computer algebra systems have a built-in capability for computing iterated triple integrals. If you have a CAS, read the relevant documentation and use the CAS to check Examples 1 and 2.

VOLUME CALCULATED AS A TRIPLE INTEGRAL

Triple integrals have many physical interpretations, some of which we will consider in the next section. However, in the special case where $f(x, y, z) = 1$, Formula (1) yields

$$\iiint_G dV = \lim_{n\to+\infty} \sum_{k=1}^{n} \Delta V_k$$

which Figure 15.5.1 suggests is the volume of G; that is,

$$\text{volume of } G = \iiint_G dV \tag{5}$$

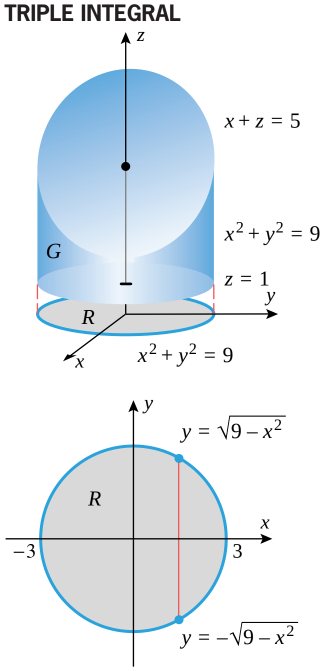


Figure 15.5.5

Example 3 Use a triple integral to find the volume of the solid within the cylinder $x^2 + y^2 = 9$ and between the planes $z = 1$ and $x + z = 5$.

Solution. The solid G and its projection R on the xy-plane are shown in Figure 15.5.5. The lower surface of the solid is the plane $z = 1$ and the upper surface is the plane $x + z = 5$ or, equivalently, $z = 5 - x$. Thus, from (3) and (5)

$$\text{volume of } G = \iiint_G dV = \iint_R \left[\int_1^{5-x} dz\right] dA \tag{6}$$

For the double integral over R, we will integrate with respect to y first. Thus, (6) yields

$$\begin{aligned}
\text{volume of } G &= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{1}^{5-x} dz\,dy\,dx = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} z\,\Big]_{z=1}^{5-x} dy\,dx\\
&= \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}(4-x)\,dy\,dx = \int_{-3}^{3}(8-2x)\sqrt{9-x^2}\,dx\\
&= 8\int_{-3}^{3}\sqrt{9-x^2}\,dx - \int_{-3}^{3}2x\sqrt{9-x^2}\,dx\\
&= 8\left(\frac{9}{2}\pi\right) - \int_{-3}^{3}2x\sqrt{9-x^2}\,dx\\
&= 8\left(\frac{9}{2}\pi\right) - 0 = 36\pi
\end{aligned}$$

For the first integral, see Formula (3) of Section 8.4.

The second integral is 0 because the integrand is an odd function. See Exercise 35 of Section 5.6.

◀

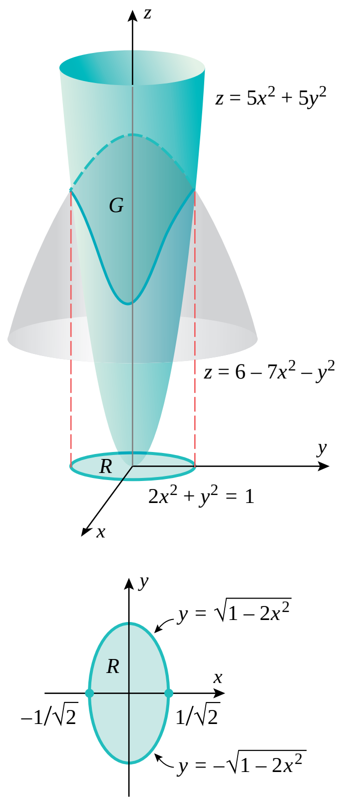


Figure 15.5.6

Example 4 Find the volume of the solid enclosed between the paraboloids

$$z = 5x^2 + 5y^2 \quad\text{and}\quad z = 6 - 7x^2 - y^2$$

Solution. The solid G and its projection R on the xy-plane are shown in Figure 15.5.6. The projection R is obtained by solving the given equations simultaneously to determine where the paraboloids intersect. We obtain

$$5x^2 + 5y^2 = 6 - 7x^2 - y^2$$

or

$$2x^2 + y^2 = 1 \tag{7}$$

which tells us that the paraboloids intersect in a curve on the elliptic cylinder given by (7).

The projection of this intersection on the xy-plane is an ellipse with this same equation. Therefore,

$$\begin{aligned}
\text{volume of } G &= \iiint_G dV = \iint_R\left[\int_{5x^2+5y^2}^{6-7x^2-y^2} dz\right]dA\\
&= \int_{-1/\sqrt{2}}^{1/\sqrt{2}}\int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}}\int_{5x^2+5y^2}^{6-7x^2-y^2} dz\,dy\,dx\\
&= \int_{-1/\sqrt{2}}^{1/\sqrt{2}}\int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}}(6-12x^2-6y^2)\,dy\,dx\\
&= \int_{-1/\sqrt{2}}^{1/\sqrt{2}}\left[6(1-2x^2)y - 2y^3\right]_{y=-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} dx\\
&= 8\int_{-1/\sqrt{2}}^{1/\sqrt{2}}(1-2x^2)^{3/2}\,dx = \frac{8}{\sqrt{2}}\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta = \frac{3\pi}{\sqrt{2}}
\end{aligned}$$

Let $x = \frac{1}{\sqrt{2}}\sin\theta$.

Use the Wallis cosine formula in Exercise 66 of Section 8.3.

◀

INTEGRATION IN OTHER ORDERS

In Formula (3) for integrating over a simple xy-solid, the z-integration was performed first. However, there are situations in which it is preferable to integrate in a different order. For example, Figure 15.5.7a shows a ***simple xz-solid***, and Figure 15.5.7b shows a ***simple yz-***

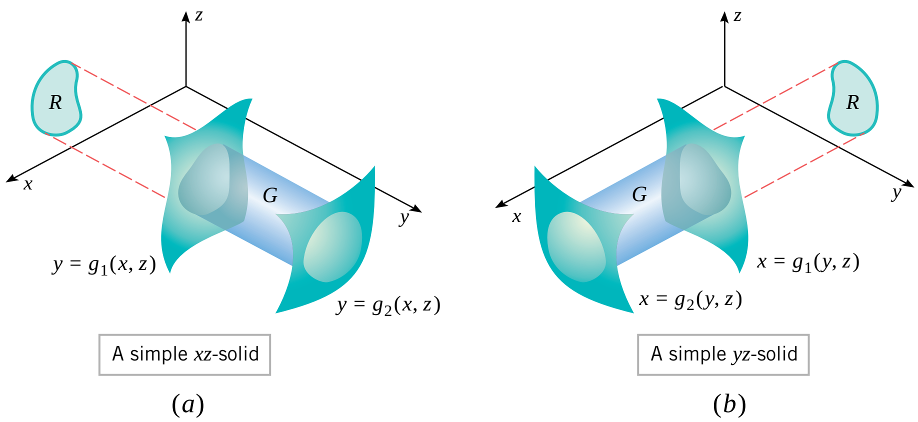


Figure 15.5.7

solid. For a simple *xz*-solid it is usually best to integrate with respect to y first, and for a simple *yz*-solid it is usually best to integrate with respect to x first:

$$\underset{\substack{G\\ \text{simple } xz\text{-solid}}}{\iiint} f(x, y, z)\, dV = \underset{R}{\iint} \left[\int_{g_1(x,z)}^{g_2(x,z)} f(x, y, z)\, dy\right] dA \tag{8}$$

$$\underset{\substack{G\\ \text{simple } yz\text{-solid}}}{\iiint} f(x, y, z)\, dV = \underset{R}{\iint} \left[\int_{g_1(y,z)}^{g_2(y,z)} f(x, y, z)\, dx\right] dA \tag{9}$$

Sometimes a solid G can be viewed as a simple *xy*-solid, a simple *xz*-solid, and a simple *yz*-solid, in which case the order of integration can be chosen to simplify the computations.

Example 5 In Example 2, we evaluated

$$\underset{G}{\iiint} z\, dV$$

over the wedge in Figure 15.5.4 by integrating first with respect to z. Evaluate this integral by integrating first with respect to x.

Solution. The solid is bounded in the back by the plane $x = 0$ and in the front by the plane $x = y$, so

$$\underset{G}{\iiint} z\, dV = \underset{R}{\iint} \left[\int_0^y z\, dx\right] dA$$

where R is the projection of G on the *yz*-plane (Figure 15.5.8). The integration over R can be performed first with respect to z and then y or vice versa. Performing the z-integration first yields

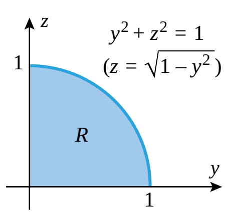


Figure 15.5.8

$$\underset{G}{\iiint} z\, dV = \int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^y z\, dx\, dz\, dy = \int_0^1 \int_0^{\sqrt{1-y^2}} zx\Big]_{x=0}^{y} dz\, dy$$

$$= \int_0^1 \int_0^{\sqrt{1-y^2}} zy\, dz\, dy = \int_0^1 \frac{1}{2}z^2 y\Big]_{z=0}^{\sqrt{1-y^2}} dy = \int_0^1 \frac{1}{2}(1 - y^2)y\, dy = \frac{1}{8}$$

which agrees with the result in Example 2. ◀

EXERCISE SET 15.5 C CAS

In Exercises 1–8, evaluate the iterated integral.

1. $\int_{-1}^{1}\int_{0}^{2}\int_{0}^{1}(x^2+y^2+z^2)\,dx\,dy\,dz$

2. $\int_{1/3}^{1/2}\int_{0}^{\pi}\int_{0}^{1} zx\sin xy\,dz\,dy\,dx$

3. $\int_{0}^{2}\int_{-1}^{y^2}\int_{-1}^{z} yz\,dx\,dz\,dy$

4. $\int_{0}^{\pi/4}\int_{0}^{1}\int_{0}^{x^2} x\cos y\,dz\,dx\,dy$

5. $\int_{0}^{3}\int_{0}^{\sqrt{9-z^2}}\int_{0}^{x} xy\,dy\,dx\,dz$

6. $\int_{1}^{3}\int_{x}^{x^2}\int_{0}^{\ln z} xe^y\,dy\,dz\,dx$

7. $\int_{0}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{-5+x^2+y^2}^{3-x^2-y^2} x\,dz\,dy\,dx$

8. $\int_{1}^{2}\int_{z}^{2}\int_{0}^{\sqrt{3}y}\frac{y}{x^2+y^2}\,dx\,dy\,dz$

In Exercises 9–12, evaluate the triple integral.

9. $\iiint\limits_G xy\sin yz\,dV$, where G is the rectangular box defined by the inequalities $0\le x\le\pi$, $0\le y\le 1$, $0\le z\le\pi/6$.

10. $\iiint\limits_G y\,dV$, where G is the solid enclosed by the plane $z=y$, the xy-plane, and the parabolic cylinder $y=1-x^2$.

11. $\iiint\limits_G xyz\,dV$, where G is the solid in the first octant that is bounded by the parabolic cylinder $z=2-x^2$ and the planes $z=0$, $y=x$, and $y=0$.

12. $\iiint\limits_G \cos(z/y)\,dV$, where G is the solid defined by the inequalities $\pi/6\le y\le\pi/2$, $y\le x\le\pi/2$, $0\le z\le xy$.

C 13. Use the numerical triple integral operation of a CAS to approximate

$$\iiint\limits_G \frac{\sqrt{x+z^2}}{y}\,dV$$

where G is the rectangular box defined by the inequalities $0\le x\le 3$, $1\le y\le 2$, $-2\le z\le 1$.

C 14. Use the numerical triple integral operation of a CAS to approximate

$$\iiint\limits_G e^{-x^2-y^2-z^2}\,dV$$

where G is the spherical region $x^2+y^2+z^2\le 1$.

In Exercises 15–18, use a triple integral to find the volume of the solid.

15. The solid in the first octant bounded by the coordinate planes and the plane $3x+6y+4z=12$.

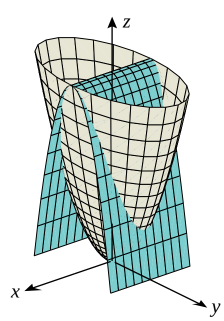
16. The solid bounded by the surface $z=\sqrt{y}$ and the planes $x+y=1$, $x=0$, and $z=0$.

17. The solid bounded by the surface $y=x^2$ and the planes $y+z=4$ and $z=0$.

18. The wedge in the first octant that is cut from the solid cylinder $y^2+z^2\le 1$ by the planes $y=x$ and $x=0$.

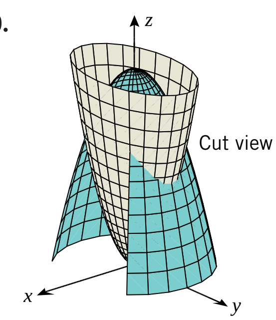
In Exercises 19–22, set up (but do not evaluate) an iterated triple integral for the volume of the solid enclosed between the given surfaces.

19.



$z=4x^2+y^2$
$z=4-3y^2$

20.



$z=3x^2+y^2$
$z=8-x^2-y^2$

21. The elliptic cylinder $x^2+9y^2=9$ and the planes $z=0$ and $z=x+3$.

22. The cylinders $x^2+y^2=1$ and $x^2+z^2=1$.

In Exercises 23 and 24, sketch the solid whose volume is given by the integral.

23. (a) $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{y+1} dz\,dy\,dx$

(b) $\int_{0}^{9}\int_{0}^{y/3}\int_{0}^{\sqrt{y^2-9x^2}} dz\,dx\,dy$

(c) $\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{2} dy\,dz\,dx$

24. (a) $\int_0^3 \int_{x^2}^9 \int_0^2 dz\,dy\,dx$

(b) $\int_0^2 \int_0^{2-y} \int_0^{2-x-y} dz\,dx\,dy$

(c) $\int_{-2}^2 \int_0^{4-y^2} \int_0^2 dx\,dz\,dy$

The ***average value*** or ***mean value*** of a continuous function $f(x, y, z)$ over a solid G is defined as

$$f_{\text{ave}} = \frac{1}{V(G)} \iiint_G f(x, y, z)\,dV$$

where $V(G)$ is the volume of the solid (compare to the definition preceding Exercise 57 of Section 15.2). Use this definition in Exercises 25–28.

25. Find the average value of $f(x, y, z) = x + y + z$ over the tetrahedron shown in the accompanying figure.

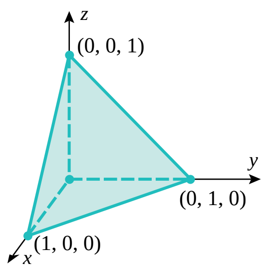


Figure Ex-25

26. Find the average value of $f(x, y, z) = xyz$ over the spherical region $x^2 + y^2 + z^2 \le 1$.

C **27.** Use the numerical triple integral operation of a CAS to approximate the average distance from the origin to a point in the solid of Example 4.

C **28.** Let $d(x, y, z)$ be the distance from the point (z, z, z) to the point $(x, y, 0)$. Use the numerical triple integral operation of a CAS to approximate the average value of d for $0 \le x \le 1$, $0 \le y \le 1$, and $0 \le z \le 1$. Write a short explanation as to why this value may be considered to be the average distance between a point on the diagonal from $(0, 0, 0)$ to $(1, 1, 1)$ and a point on the face in the xy-plane for the unit cube $0 \le x \le 1$, $0 \le y \le 1$, and $0 \le z \le 1$.

29. Let G be the tetrahedron in the first octant bounded by the coordinate planes and the plane

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad (a > 0, b > 0, c > 0)$$

(a) List six different iterated integrals that represent the volume of G.

(b) Evaluate any one of the six to show that the volume of G is $\frac{1}{6}abc$.

30. Use a triple integral to derive the formula for the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

In Exercises 31 and 32, express each integral as an equivalent integral in which the z-integration is performed first, the y-integration second, and the x-integration last.

31. (a) $\int_0^5 \int_0^2 \int_0^{\sqrt{4-y^2}} f(x, y, z)\,dx\,dy\,dz$

(b) $\int_0^9 \int_0^{3-\sqrt{x}} \int_0^z f(x, y, z)\,dy\,dz\,dx$

(c) $\int_0^4 \int_y^{8-y} \int_0^{\sqrt{4-y}} f(x, y, z)\,dx\,dz\,dy$

32. (a) $\int_0^3 \int_0^{\sqrt{9-z^2}} \int_0^{\sqrt{9-y^2-z^2}} f(x, y, z)\,dx\,dy\,dz$

(b) $\int_0^4 \int_0^2 \int_0^{x/2} f(x, y, z)\,dy\,dz\,dx$

(c) $\int_0^4 \int_0^{4-y} \int_0^{\sqrt{z}} f(x, y, z)\,dx\,dz\,dy$

C **33.** (a) Find the region G over which the triple integral

$$\iiint_G (1 - x^2 - y^2 - z^2)\,dV$$

has its maximum value.

(b) Use the numerical triple integral operation of a CAS to approximate the maximum value.

(c) Find the exact maximum value.

34. Let G be the rectangular box defined by the inequalities $a \le x \le b$, $c \le y \le d$, $k \le z \le \ell$. Show that

$$\iiint_G f(x)g(y)h(z)\,dV = \left[\int_a^b f(x)\,dx\right]\left[\int_c^d g(y)\,dy\right]\left[\int_k^\ell h(z)\,dz\right]$$

35. Use the result of Exercise 34 to evaluate

(a) $\iiint_G xy^2 \sin z\,dV$, where G is the set of points satisfying $-1 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le \pi/2$;

(b) $\iiint_G e^{2x+y-z}\,dV$, where G is the set of points satisfying $0 \le x \le 1$, $0 \le y \le \ln 3$, $0 \le z \le \ln 2$.

15.6 CENTROID, CENTER OF GRAVITY, THEOREM OF PAPPUS

Suppose that a rigid physical body is acted on by a gravitational field. Because the body is composed of many particles, each of which is affected by gravity, the action of a constant gravitational field on the body consists of a large number of forces distributed over the entire body. However, these individual forces can be replaced by a single force acting at a point called the ***center of gravity*** *of the body. In this section we will show how double and triple integrals can be used to locate centers of gravity.*

DENSITY OF A LAMINA

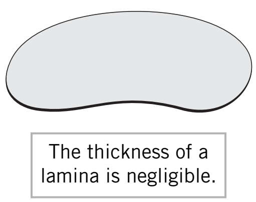


Figure 15.6.1

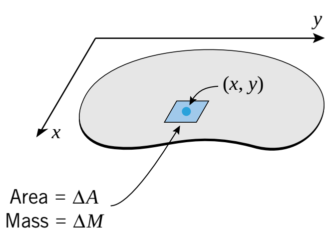


Figure 15.6.2

Let us consider an idealized flat object that is thin enough to be viewed as a two-dimensional plane region (Figure 15.6.1). Such an object is called a ***lamina***. A lamina is called ***homogeneous*** if its composition is uniform throughout and ***inhomogeneous*** otherwise. The ***density*** of a *homogeneous* lamina is defined to be its mass per unit area. Thus, the density δ of a homogeneous lamina of mass M and area A is given by $\delta = M/A$.

For an inhomogeneous lamina the composition may vary from point to point, and hence an appropriate definition of "density" must reflect this. To motivate such a definition, suppose that the lamina is placed in an *xy*-plane. The density at a point (x, y) can be specified by a function $\delta(x, y)$, called the ***density function***, which can be interpreted as follows. Construct a small rectangle centered at (x, y) and let ΔM and ΔA be the mass and area of the portion of the lamina enclosed by this rectangle (Figure 15.6.2). If the ratio $\Delta M/\Delta A$ approaches a limiting value as the dimensions (and hence the area) of the rectangle approach zero, then this limit is considered to be the density of the lamina at (x, y). Symbolically,

$$\delta(x, y) = \lim_{\Delta A \to 0} \frac{\Delta M}{\Delta A} \tag{1}$$

From this relationship we obtain the approximation

$$\Delta M \approx \delta(x, y)\Delta A \tag{2}$$

which relates the mass and area of a small rectangular portion of the lamina centered at (x, y). It is assumed that as the dimensions of the rectangle tend to zero, the error in this approximation also tends to zero.

MASS OF A LAMINA

The following result shows how to find the mass of a lamina from its density function.

15.6.1 MASS OF A LAMINA. If a lamina with a continuous density function $\delta(x, y)$ occupies a region R in the *xy*-plane, then its total mass M is given by

$$M = \iint_R \delta(x, y)\, dA \tag{3}$$

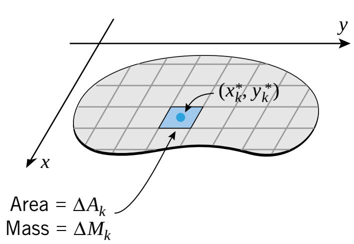


Figure 15.6.3

This formula can be motivated by a familiar limiting process that can be outlined as follows: Imagine the lamina to be subdivided into rectangular pieces using lines parallel to the coordinate axes and excluding from consideration any nonrectangular parts at the boundary (Figure 15.6.3). Assume that there are n such rectangular pieces, and suppose that the kth piece has area ΔA_k. If we let (x_k^*, y_k^*) denote the center of the kth piece, then from Formula (2), the mass ΔM_k of this piece can be approximated by

$$\Delta M_k \approx \delta(x_k^*, y_k^*)\Delta A_k \tag{4}$$

and hence the mass M of the entire lamina can be approximated by

$$M \approx \sum_{k=1}^{n} \delta(x_k^*, y_k^*)\Delta A_k$$

If we now increase n in such a way that the dimensions of the rectangles tend to zero, then it is plausible that the errors in our approximations will approach zero, so

$$M = \lim_{n\to+\infty} \sum_{k=1}^{n} \delta(x_k^*, y_k^*)\Delta A_k = \iint_R \delta(x, y)\, dA$$

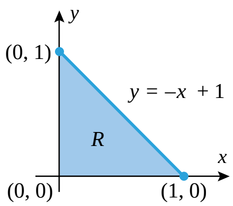


Figure 15.6.4

Example 1 A triangular lamina with vertices (0, 0), (0, 1), and (1, 0) has density function $\delta(x, y) = xy$. Find its total mass.

Solution. Referring to (3) and Figure 15.6.4, the mass M of the lamina is

$$M = \iint_R \delta(x, y)\, dA = \iint_R xy\, dA = \int_0^1 \int_0^{-x+1} xy\, dy\, dx$$

$$= \int_0^1 \left[\frac{1}{2}xy^2\right]_{y=0}^{-x+1} dx = \int_0^1 \left[\frac{1}{2}x^3 - x^2 + \frac{1}{2}x\right] dx = \frac{1}{24} \text{ (unit of mass)}$$ ◀

CENTER OF GRAVITY OF A LAMINA

Assume that the acceleration due to the force of gravity is constant and acts downward, and suppose that a lamina occupies a region R in a horizontal *xy*-plane. It can be shown that there exists a unique point $(\bar{x}, \bar{y})$ (which may or may not belong to R) such that the effect of gravity on the lamina is "equivalent" to that of a single force acting at the point $(\bar{x}, \bar{y})$. This point is called the ***center of gravity*** of the lamina, and if it is in R then the lamina will balance horizontally on the point of a support placed at $(\bar{x}, \bar{y})$. For example, the center of gravity for a disk of uniform density is at the center of the disk and the center of gravity for a rectangular region of uniform density is at the center of the rectangle. For less symmetric lamina or for lamina in which the density varies from point to point, locating the center of gravity requires calculus.

> **15.6.2 PROBLEM.** Suppose that a lamina with a continuous density function $\delta(x, y)$ occupies a region R in a horizontal *xy*-plane. Find the coordinates $(\bar{x}, \bar{y})$ of the center of gravity.

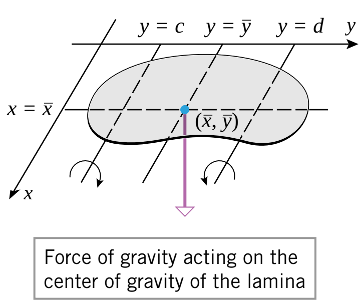


Figure 15.6.5

To motivate the solution, consider what happens if we try to balance the lamina on a knife-edge parallel to the x-axis. Suppose the lamina in Figure 15.6.5 is placed on a knife-edge along a line $y = c$ that does not pass through the center of gravity. Because the lamina behaves as if its entire mass is concentrated at the center of gravity $(\bar{x}, \bar{y})$, the lamina will be rotationally unstable and the force of gravity will cause a rotation about $y = c$. Similarly, the lamina will undergo a rotation if placed on a knife-edge along $y = d$. However, if the knife-edge runs along the line $y = \bar{y}$ through the center of gravity, the lamina will be in perfect balance. Similarly, the lamina will be in perfect balance on a knife-edge along the line $x = \bar{x}$ through the center of gravity. This suggests that the center of gravity of a lamina can be determined as the intersection of two lines of balance, one parallel to the x-axis and the other parallel to the y-axis. In order to find these lines of balance, we will need some preliminary results about rotations.

Children on a seesaw learn by experience that a lighter child can balance a heavier one by sitting farther from the fulcrum or pivot point. This is because the tendency for an object to produce rotation is proportional not only to its mass but also to the distance between the object and the fulcrum. To make this more precise, consider an x-axis, which we view as a weightless beam. If a point-mass m is located on the axis at x, then the tendency for that mass to produce a rotation of the beam about a point a on the axis is measured by the following quantity, called the ***moment of m about x = a***:

$$\begin{bmatrix}\text{moment of } m \\ \text{about } a\end{bmatrix} = m(x - a)$$

The number $x - a$ is called the ***lever arm***. Depending on whether the mass is to the right or

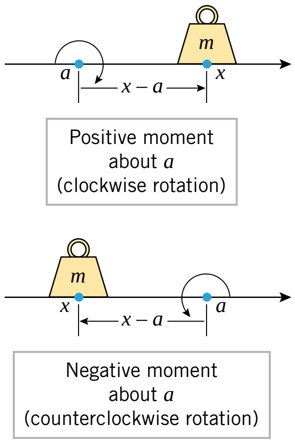


Figure 15.6.6

left of a, the lever arm is either the distance between x and a or the negative of this distance (Figure 15.6.6). Positive lever arms result in positive moments and clockwise rotations, and negative lever arms result in negative moments and counterclockwise rotations.

Suppose that masses $m_1, m_2, \ldots, m_n$ are located at $x_1, x_2, \ldots, x_n$ on a coordinate axis and a fulcrum is positioned at the point a (Figure 15.6.7). Depending on whether the sum of the moments about a,

$$\sum_{k=1}^{n} m_k(x_k - a) = m_1(x_1 - a) + m_2(x_2 - a) + \cdots + m_n(x_n - a)$$

is positive, negative, or zero, a weightless beam along the axis will rotate clockwise about a, rotate counterclockwise about a, or balance perfectly. In the last case, the system of masses is said to be in ***equilibrium***.

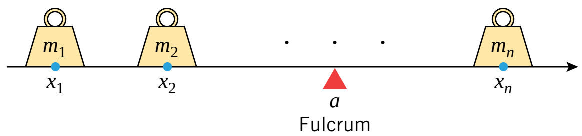


Figure 15.6.7

The preceding ideas can be extended to masses distributed in two-dimensional space. If we imagine the xy-plane to be a weightless sheet supporting a point-mass m located at a point (x, y), then the tendency for the mass to produce a rotation of the sheet about the line $x = a$ is $m(x - a)$, called the ***moment of m about x = a***, and the tendency for the mass to produce a rotation about the line $y = c$ is $m(y - c)$, called the ***moment of m about y = c*** (Figure 15.6.8). In summary,

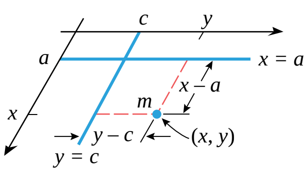


Figure 15.6.8

$$\begin{bmatrix} \text{moment of } m \\ \text{about the} \\ \text{line } x = a \end{bmatrix} = m(x - a) \quad \text{and} \quad \begin{bmatrix} \text{moment of } m \\ \text{about the} \\ \text{line } y = c \end{bmatrix} = m(y - c) \tag{5–6}$$

If a number of masses are distributed throughout the xy-plane, then the plane (viewed as a weightless sheet) will balance on a knife-edge along the line $x = a$ if the sum of the moments about the line is zero. Similarly for the line $y = c$.

We are now ready to solve Problem 15.6.2. We imagine the lamina to be subdivided into rectangular pieces using lines parallel to the coordinate axes and excluding from consideration any nonrectangular pieces at the boundary (Figure 15.6.3). We assume that there are n such rectangular pieces and that the kth piece has area ΔA_k and mass ΔM_k. We will let (x_k^*, y_k^*) be the center of the kth piece, and we will assume that the entire mass of the kth piece is concentrated at its center. From (4), the mass of the kth piece can be approximated by

$$\Delta M_k \approx \delta(x_k^*, y_k^*)\Delta A_k$$

Since the lamina balances on the lines $x = \bar{x}$ and $y = \bar{y}$, the sum of the moments of the rectangular pieces about those lines should be close to zero; that is,

$$\sum_{k=1}^{n}(x_k^* - \bar{x})\Delta M_k = \sum_{k=1}^{n}(x_k^* - \bar{x})\delta(x_k^*, y_k^*)\Delta A_k \approx 0$$

$$\sum_{k=1}^{n}(y_k^* - \bar{y})\Delta M_k = \sum_{k=1}^{n}(y_k^* - \bar{y})\delta(x_k^*, y_k^*)\Delta A_k \approx 0$$

If we now increase n in such a way that the dimensions of the rectangles tend to zero, then it is plausible that the errors in our approximations will approach zero, so that

$$\lim_{n \to +\infty} \sum_{k=1}^{n}(x_k^* - \bar{x})\delta(x_k^*, y_k^*)\Delta A_k = 0$$

$$\lim_{n \to +\infty} \sum_{k=1}^{n}(y_k^* - \bar{y})\delta(x_k^*, y_k^*)\Delta A_k = 0$$

from which we obtain

$$\iint_R (x-\bar{x})\delta(x,y)\,dA = 0$$

$$\iint_R (y-\bar{y})\delta(x,y)\,dA = 0$$

Since $\bar{x}$ and $\bar{y}$ are constant, these equations can be rewritten as

$$\iint_R x\delta(x,y)\,dA = \bar{x}\iint_R \delta(x,y)\,dA$$

$$\iint_R y\delta(x,y)\,dA = \bar{y}\iint_R \delta(x,y)\,dA$$

from which we obtain the following formulas for the center of gravity of the lamina:

Center of Gravity $(\bar{x}, \bar{y})$ ***of a Lamina***

$$\bar{x} = \frac{\displaystyle\iint_R x\delta(x,y)\,dA}{\displaystyle\iint_R \delta(x,y)\,dA}, \qquad \bar{y} = \frac{\displaystyle\iint_R y\delta(x,y)\,dA}{\displaystyle\iint_R \delta(x,y)\,dA} \tag{7–8}$$

Observe that in both formulas the denominator is the mass M of the lamina [see (3)]. The numerator in the formula for $\bar{x}$ is denoted by M_y and is called the ***first moment of the lamina about the y-axis***; the numerator of the formula for $\bar{y}$ is denoted by M_x and is called the ***first moment of the lamina about the x-axis***. Thus, Formulas (7) and (8) can be expressed as

$$\bar{x} = \frac{M_y}{M} = \frac{1}{\text{mass of } R}\iint_R x\delta(x,y)\,dA \tag{9}$$

$$\bar{y} = \frac{M_x}{M} = \frac{1}{\text{mass of } R}\iint_R y\delta(x,y)\,dA \tag{10}$$

Example 2 Find the center of gravity of the triangular lamina with vertices (0, 0), (0, 1), and (1, 0) and density function $\delta(x,y) = xy$.

Solution. The lamina is shown in Figure 15.6.4. In Example 1 we found the mass of the lamina to be

$$M = \iint_R \delta(x,y)\,dA = \iint_R xy\,dA = \frac{1}{24}$$

The moment of the lamina about the y-axis is

$$M_y = \iint_R x\delta(x,y)\,dA = \iint_R x^2y\,dA = \int_0^1\int_0^{-x+1} x^2y\,dy\,dx$$

$$= \int_0^1 \left[\frac{1}{2}x^2y^2\right]_{y=0}^{-x+1} dx = \int_0^1 \left(\frac{1}{2}x^4 - x^3 + \frac{1}{2}x^2\right) dx = \frac{1}{60}$$

and the moment about the x-axis is

$$M_x = \iint_R y\delta(x,y)\,dA = \iint_R xy^2\,dA = \int_0^1\int_0^{-x+1} xy^2\,dy\,dx$$

$$= \int_0^1 \left[\frac{1}{3}xy^3\right]_{y=0}^{-x+1} dx = \int_0^1 \left(-\frac{1}{3}x^4 + x^3 - x^2 + \frac{1}{3}x\right) dx = \frac{1}{60}$$

From (9) and (10),

$$\bar{x} = \frac{M_y}{M} = \frac{1/60}{1/24} = \frac{2}{5}, \quad \bar{y} = \frac{M_x}{M} = \frac{1/60}{1/24} = \frac{2}{5}$$

so the center of gravity is $\left(\frac{2}{5}, \frac{2}{5}\right)$. ◀

CENTROIDS

In the special case of a *homogeneous* lamina, the center of gravity is called the ***centroid of the lamina*** or sometimes the ***centroid of the region R***. Because the density function δ is constant for a homogeneous lamina, the factor δ may be moved through the integral signs in (7) and (8) and canceled. Thus, the centroid $(\bar{x}, \bar{y})$ is a geometric property of the region R and is given by the following formulas:

Centroid of a Region R

$$\bar{x} = \frac{\displaystyle\iint_R x\,dA}{\displaystyle\iint_R dA} = \frac{1}{\text{area of } R}\iint_R x\,dA \tag{11}$$

$$\bar{y} = \frac{\displaystyle\iint_R y\,dA}{\displaystyle\iint_R dA} = \frac{1}{\text{area of } R}\iint_R y\,dA \tag{12}$$

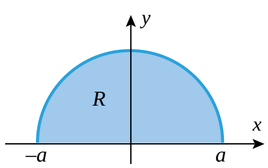


Figure 15.6.9

Example 3 Find the centroid of the semicircular region in Figure 15.6.9.

Solution. By symmetry, $\bar{x} = 0$ since the y-axis is obviously a line of balance. From (12),

$$\bar{y} = \frac{1}{\text{area of } R}\iint_R y\,dA = \frac{1}{\frac{1}{2}\pi a^2}\iint_R y\,dA$$

$$= \frac{1}{\frac{1}{2}\pi a^2}\int_0^{\pi}\int_0^{a}(r\sin\theta)r\,dr\,d\theta \qquad \boxed{\text{Evaluating in polar coordinates}}$$

$$= \frac{1}{\frac{1}{2}\pi a^2}\int_0^{\pi}\left[\frac{1}{3}r^3\sin\theta\right]_{r=0}^{a}d\theta$$

$$= \frac{1}{\frac{1}{2}\pi a^2}\left(\frac{1}{3}a^3\right)\int_0^{\pi}\sin\theta\,d\theta = \frac{1}{\frac{1}{2}\pi a^2}\left(\frac{2}{3}a^3\right) = \frac{4a}{3\pi}$$

so the centroid is $\left(0, \dfrac{4a}{3\pi}\right)$. ◀

CENTER OF GRAVITY AND CENTROID OF A SOLID

For a three-dimensional solid G, the formulas for moments, center of gravity, and centroid are similar to those for laminas. If G is *homogeneous*, then its ***density*** is defined to be its mass per unit volume. Thus, if G is a homogeneous solid of mass M and volume V, then its density δ is given by $\delta = M/V$. If G is inhomogeneous and is in an xyz-coordinate system, then its density at a general point (x, y, z) is specified by a ***density function*** $\delta(x, y, z)$ whose value at a point can be viewed as a limit:

$$\delta(x, y, z) = \lim_{\Delta V \to 0}\frac{\Delta M}{\Delta V}$$

where ΔM and ΔV represent the mass and volume of a rectangular parallelepiped, centered at (x, y, z), whose dimensions tend to zero (Figure 15.6.10).

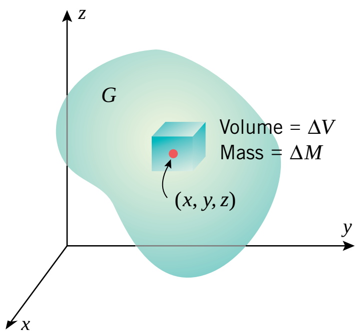


Figure 15.6.10

Using the discussion of laminas as a model, you should be able to show that the mass M of a solid with a continuous density function $\delta(x, y, z)$ is

$$M = \text{mass of } G = \iiint_G \delta(x, y, z)\, dV \tag{13}$$

The formulas for center of gravity and centroid are

Center of Gravity $(\bar{x}, \bar{y}, \bar{z})$ ***of a Solid G***

$$\bar{x} = \frac{1}{M}\iiint_G x\delta(x, y, z)\, dV$$
$$\bar{y} = \frac{1}{M}\iiint_G y\delta(x, y, z)\, dV$$
$$\bar{z} = \frac{1}{M}\iiint_G z\delta(x, y, z)\, dV$$

Centroid $(\bar{x}, \bar{y}, \bar{z})$ ***of a Solid G***

$$\bar{x} = \frac{1}{V}\iiint_G x\, dV$$
$$\bar{y} = \frac{1}{V}\iiint_G y\, dV$$
$$\bar{z} = \frac{1}{V}\iiint_G z\, dV$$

(14–15)

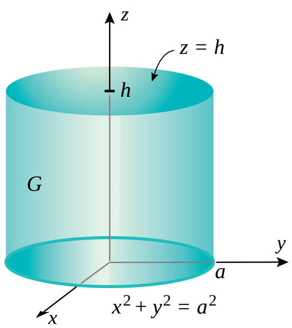


Figure 15.6.11

Example 4 Find the mass and the center of gravity of a cylindrical solid of height h and radius a (Figure 15.6.11), assuming that the density at each point is proportional to the distance between the point and the base of the solid.

Solution. Since the density is proportional to the distance z from the base, the density function has the form $\delta(x, y, z) = kz$, where k is some (unknown) positive constant of proportionality. From (13) the mass of the solid is

$$\begin{aligned}
M = \iiint_G \delta(x, y, z)\, dV &= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_0^h kz\, dz\, dy\, dx \\
&= k\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{1}{2}h^2\, dy\, dx \\
&= kh^2\int_{-a}^{a}\sqrt{a^2-x^2}\, dx \\
&= \tfrac{1}{2}kh^2\pi a^2
\end{aligned}$$

Interpret the integral as the area of a semicircle.

Without additional information, the constant k cannot be determined. However, as we will now see, the value of k does not affect the center of gravity.

From (14),

$$\begin{aligned}
\bar{z} = \frac{1}{M}\iiint_G z\delta(x, y, z)\, dV &= \frac{1}{\frac{1}{2}kh^2\pi a^2}\iiint_G z\delta(x, y, z)\, dV \\
&= \frac{1}{\frac{1}{2}kh^2\pi a^2}\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_0^h z(kz)\, dz\, dy\, dx \\
&= \frac{k}{\frac{1}{2}kh^2\pi a^2}\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{1}{3}h^3\, dy\, dx \\
&= \frac{\frac{1}{3}kh^3}{\frac{1}{2}kh^2\pi a^2}\int_{-a}^{a} 2\sqrt{a^2-x^2}\, dx \\
&= \frac{\frac{1}{3}kh^3\pi a^2}{\frac{1}{2}kh^2\pi a^2} = \frac{2}{3}h
\end{aligned}$$

Similar calculations using (14) will yield $\bar{x} = \bar{y} = 0$. However, this is evident by inspection, since it follows from the symmetry of the solid and the form of its density function that the center of gravity is on the z-axis. Thus, the center of gravity is $\left(0, 0, \frac{2}{3}h\right)$. ◀

THEOREM OF PAPPUS

The following theorem, due to the Greek mathematician Pappus,* gives an important relationship between the centroid of a plane region R and the volume of the solid generated when the region is revolved about a line.

15.6.3 THEOREM. *If R is a bounded plane region and L is a line that lies in the plane of R but is entirely on one side of R, then the volume of the solid formed by revolving R about L is given by*

$$\textit{volume} = (\textit{area of } R) \cdot \begin{pmatrix}\textit{distance traveled}\\ \textit{by the centroid}\end{pmatrix}$$

Proof. Introduce an xy-coordinate system so that L is along the y-axis and the region R is in the first quadrant (Figure 15.6.12). Let R be partitioned into subregions in the usual way and let R_k be a typical rectangle interior to R. If (x_k^*, y_k^*) is the center of R_k, and if the area of R_k is $\Delta A_k = \Delta x_k \Delta y_k$, then from Formula (1) of Section 6.3 the volume generated by R_k as it revolves about L is

$$2\pi x_k^* \Delta x_k \Delta y_k = 2\pi x_k^* \Delta A_k$$

Therefore, the total volume of the solid is approximately

$$V \approx \sum_{k=1}^{n} 2\pi x_k^* \Delta A_k$$

from which it follows that the exact volume is

$$V = \iint_R 2\pi x \, dA = 2\pi \iint_R x \, dA$$

Thus, it follows from (11) that

$$V = 2\pi \cdot \bar{x} \cdot [\text{area of } R]$$

This completes the proof since $2\pi\bar{x}$ is the distance traveled by the centroid when R is revolved about the y-axis. ■

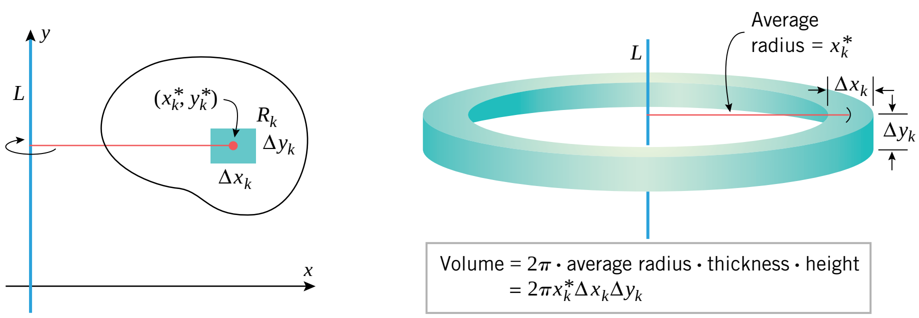


Figure 15.6.12

*PAPPUS OF ALEXANDRIA (4th century A.D.). Greek mathematician. Pappus lived during the early Christian era when mathematical activity was in a period of decline. His main contributions to mathematics appeared in a series of eight books called *The Collection* (written about 340 A.D.). This work, which survives only partially, contained some original results but was devoted mostly to statements, refinements, and proofs of results by earlier mathematicians. Pappus' Theorem, stated without proof in Book VII of *The Collection*, was probably known and proved in earlier times. This result is sometimes called Guldin's Theorem in recognition of the Swiss mathematician, Paul Guldin (1577–1643), who rediscovered it independently.

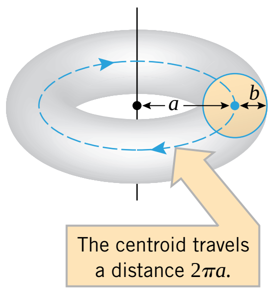


Figure 15.6.13

Example 5 Use Pappus' Theorem to find the volume V of the torus generated by revolving a circular region of radius b about a line at a distance a (greater than b) from the center of the circle (Figure 15.6.13).

Solution. By symmetry, the centroid of a circular region is its center. Thus, the distance traveled by the centroid is $2\pi a$. Since the area of a circle of radius b is πb^2, it follows from Pappus' Theorem that the volume of the torus is

$$V = (2\pi a)(\pi b^2) = 2\pi^2 ab^2$$

◀

EXERCISE SET 15.6 Graphing Utility CAS

1. Where should the fulcrum be placed so that the beam in the accompanying figure is in equilibrium?

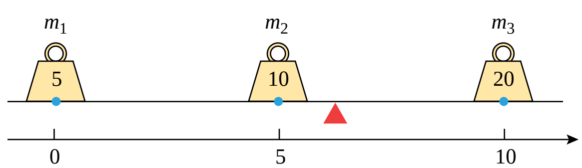


Figure Ex-1

2. Given that the beam in the accompanying figure is in equilibrium, what is the mass m?

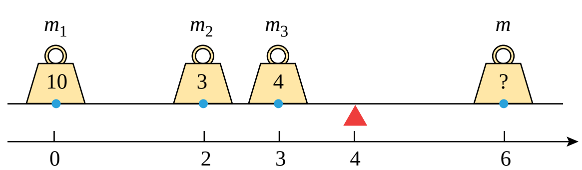
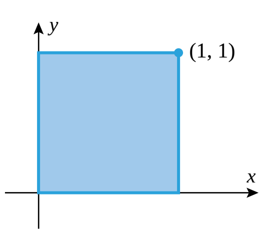


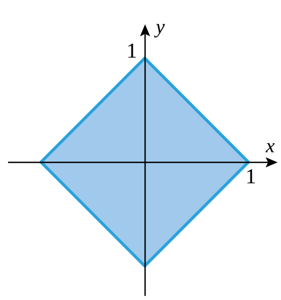
Figure Ex-2

For the regions in Exercises 3 and 4, make a conjecture about the coordinates of the centroid, and confirm your conjecture by integrating.

3.

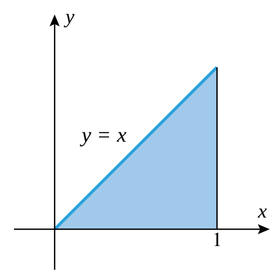


4.



In Exercises 5–10, find the centroid of the region.

5.



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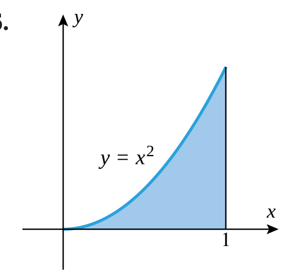

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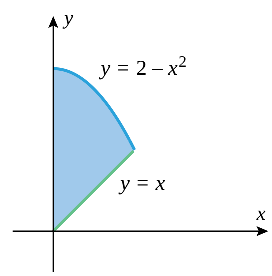
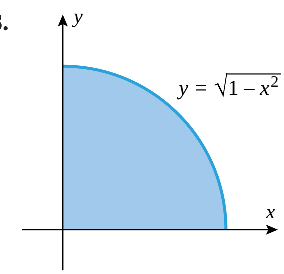


8.



9. The region above the x-axis and between the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ $(a < b)$.

10. The region enclosed between the y-axis and the right half of the circle $x^2 + y^2 = a^2$.

In Exercises 11 and 12, make a conjecture about the coordinates of the center of gravity, and confirm your conjecture by integrating.

11. The lamina of Exercise 3 with density function $\delta(x, y) = |x + y - 1|$.

12. The lamina of Exercise 4 with density function $\delta(x, y) = 1 + x^2 + y^2$.

In Exercises 13–16, find the mass and center of gravity of the lamina.

13. A lamina with density $\delta(x, y) = x + y$ is bounded by the x-axis, the line $x = 1$, and the curve $y = \sqrt{x}$.

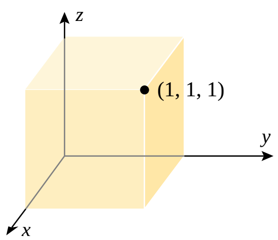
14. A lamina with density $\delta(x, y) = y$ is bounded by $y = \sin x$, $y = 0$, $x = 0$, and $x = \pi$.

15. A lamina with density $\delta(x, y) = xy$ is in the first quadrant and is bounded by the circle $x^2+y^2 = a^2$ and the coordinate axes.

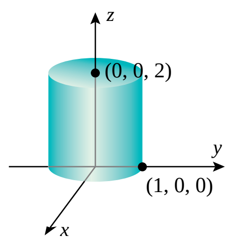
16. A lamina with density $\delta(x, y) = x^2 + y^2$ is bounded by the x-axis and the upper half of the circle $x^2 + y^2 = 1$.

For the solids in Exercises 17 and 18, make a conjecture about the coordinates of the centroid, and confirm your conjecture by integrating.

17.



18.



In Exercises 19–24, find the centroid of the solid.

19. The tetrahedron in the first octant enclosed by the coordinate planes and the plane $x + y + z = 1$.

20. The solid bounded by the parabolic cylinder $z = 1-y^2$ and the planes $x + z = 1$, $x = 0$, and $z = 0$.

21. The solid bounded by the surface $z = y^2$ and the planes $x = 0$, $x = 1$, and $z = 1$.

22. The solid in the first octant bounded by the surface $z = xy$ and the planes $z = 0$, $x = 2$, and $y = 2$.

23. The solid in the first octant that is bounded by the sphere $x^2 + y^2 + z^2 = a^2$ and the coordinate planes.

24. The solid enclosed by the xy-plane and the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$.

In Exercises 25–28, find the mass and center of gravity of the solid.

25. The cube that has density $\delta(x, y, z) = a - x$ and is defined by the inequalities $0 \le x \le a$, $0 \le y \le a$, and $0 \le z \le a$.

26. The cylindrical solid that has density $\delta(x, y, z) = h-z$ and is enclosed by $x^2 + y^2 = a^2$, $z = 0$, $y = 0$, and $z = h$.

27. The solid that has density $\delta(x, y, z) = yz$ and is enclosed by $z = 1 - y^2$ (for $y \ge 0$), $z = 0$, $x = -1$, and $x = 1$.

28. The solid that has density $\delta(x, y, z) = xz$ and is enclosed by $y = 9-x^2$ (for $x \ge 0$), $x = 0$, $y = 0$, $z = 0$, and $z = 1$.

29. Find the center of gravity of the square lamina with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$ if
(a) the density is proportional to the square of the distance from the origin
(b) the density is proportional to the distance from the y-axis.

30. Find the center of gravity of the cube that is determined by the inequalities $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$ if
(a) the density is proportional to the square of the distance to the origin
(b) the density is proportional to the sum of the distances to the faces that lie in the coordinate planes.

C **31.** Use the numerical triple integral capability of a CAS to approximate the location of the centroid of the solid that is bounded above by the surface $z = 1/(1 + x^2 + y^2)$, below by the xy-plane, and laterally by the plane $y = 0$ and the surface $y = \sin x$ for $0 \le x \le \pi$ (see the accompanying figure).

32. The accompanying figure shows the solid that is bounded above by the surface $z = 1/(x^2 + y^2 + 1)$, below by the xy-plane, and laterally by the surface $x^2 + y^2 = a^2$.
(a) By symmetry, the centroid of the solid lies on the z-axis. Make a conjecture about the behavior of the z-coordinate of the centroid as $a \to 0^+$ and as $a \to +\infty$.
(b) Find the z-coordinate of the centroid, and check your conjecture by calculating the appropriate limits.
(c) Use a graphing utility to plot the z-coordinate of the centroid versus a, and use the graph to estimate the value of a for which the centroid is $(0, 0, 0.25)$.

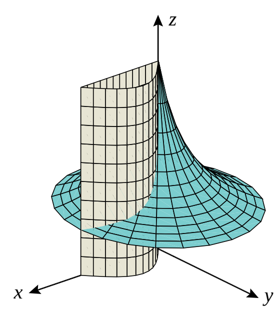


Figure Ex-31

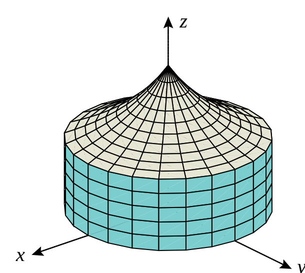


Figure Ex-32

33. Show that in polar coordinates the formulas for the centroid $(\bar{x}, \bar{y})$ of a region R are

$$\bar{x} = \frac{1}{\text{area of } R} \iint_R r^2 \cos\theta \, dr \, d\theta$$

$$\bar{y} = \frac{1}{\text{area of } R} \iint_R r^2 \sin\theta \, dr \, d\theta$$

34. Use the result of Exercise 33 to find the centroid $(\bar{x}, \bar{y})$ of the region enclosed by the cardioid $r = a(1 + \sin\theta)$.

35. Use the result of Exercise 33 to find the centroid $(\bar{x}, \bar{y})$ of the petal of the rose $r = \sin 2\theta$ in the first quadrant.

36. Let R be the rectangle bounded by the lines $x = 0$, $x = 3$, $y = 0$, and $y = 2$. By inspection, find the centroid of R and use it to evaluate

$$\iint_R x \, dA \quad \text{and} \quad \iint_R y \, dA$$

37. Use the Theorem of Pappus and the fact that the volume of a sphere of radius a is $V = \frac{4}{3}\pi a^3$ to show that the centroid of the lamina that is bounded by the x-axis and the semicircle $y = \sqrt{a^2 - x^2}$ is $(0, 4a/(3\pi))$. (This problem was solved directly in Example 3.)

38. Use the Theorem of Pappus and the result of Exercise 37 to find the volume of the solid generated when the region bounded by the x-axis and the semicircle $y = \sqrt{a^2 - x^2}$ is revolved about
(a) the line $y = -a$ (b) the line $y = x - a$.

39. Use the Theorem of Pappus and the fact that the area of an ellipse with semiaxes a and b is πab to find the volume of the elliptical torus generated by revolving the ellipse

$$\frac{(x-k)^2}{a^2} + \frac{y^2}{b^2} = 1$$

about the y-axis. Assume that $k > a$.

40. Use the Theorem of Pappus to find the volume of the solid that is generated when the region enclosed by $y = x^2$ and $y = 8 - x^2$ is revolved about the x-axis.

41. Use the Theorem of Pappus to find the centroid of the triangular region with vertices $(0, 0)$, $(a, 0)$, and $(0, b)$, where $a > 0$ and $b > 0$. [*Hint:* Revolve the region about the x-axis to obtain $\bar{y}$ and about the y-axis to obtain $\bar{x}$.]

The tendency of a lamina to resist a change in rotational motion about an axis is measured by its ***moment of inertia*** about that axis. If the lamina occupies a region R of the xy-plane, and if its density function $\delta(x, y)$ is continuous on R, then the moments of inertia about the x-axis, the y-axis, and the z-axis are denoted by I_x, I_y, and I_z, respectively, and are defined by

$$I_x = \iint_R y^2\,\delta(x, y)\,dA, \quad I_y = \iint_R x^2\,\delta(x, y)\,dA,$$

$$I_z = \iint_R (x^2 + y^2)\,\delta(x, y)\,dA$$

These definitions will be used in Exercises 42 and 43.

42. Consider the rectangular lamina that occupies the region described by the inequalities $0 \le x \le a$ and $0 \le y \le b$. Assuming that the lamina has constant density δ, show that

$$I_x = \frac{\delta ab^3}{3}, \quad I_y = \frac{\delta a^3 b}{3}, \quad I_z = \frac{\delta ab(a^2 + b^2)}{3}$$

43. Consider the circular lamina that occupies the region described by the inequalities $0 \le x^2 + y^2 \le a^2$. Assuming that the lamina has constant density δ, show that

$$I_x = I_y = \frac{\delta\pi a^4}{4}, \quad I_z = \frac{\delta\pi a^4}{2}$$

15.7 TRIPLE INTEGRALS IN CYLINDRICAL AND SPHERICAL COORDINATES

Earlier we saw that some double integrals are easier to evaluate in polar coordinates than in rectangular coordinates. Similarly, some triple integrals are easier to evaluate in cylindrical or spherical coordinates than in rectangular coordinates. In this section we will study triple integrals in these coordinate systems.

TRIPLE INTEGRALS IN CYLINDRICAL COORDINATES

Recall that in rectangular coordinates the triple integral of a continuous function f over a solid region G is defined as

$$\iiint_G f(x, y, z)\,dV = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\Delta V_k$$

where ΔV_k denotes the volume of a rectangular parallelepiped interior to G and (x_k^*, y_k^*, z_k^*) is a point in this parallelepiped (see Figure 15.5.1). Triple integrals in cylindrical and spherical coordinates are defined similarly, except that the region G is divided not into rectangular parallelepipeds but into regions more appropriate to these coordinate systems.

In cylindrical coordinates, the simplest equations are of the form

$$r = \text{constant}, \quad \theta = \text{constant}, \quad z = \text{constant}$$

As indicated in Figure 12.8.2b, the first equation represents a right circular cylinder centered on the z-axis, the second a vertical half-plane hinged on the z-axis, and the third a horizontal

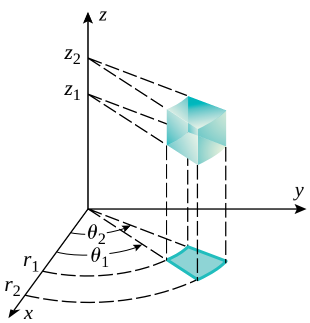


Figure 15.7.1

plane. These surfaces can be paired up to determine solids called ***cylindrical wedges*** or ***cylindrical elements of volume***. To be precise, a cylindrical wedge is a solid enclosed between six surfaces of the following form:

two cylinders	$r = r_1$,	$r = r_2$	$(r_1 < r_2)$
two half-planes	$\theta = \theta_1$,	$\theta = \theta_2$	$(\theta_1 < \theta_2)$
two planes	$z = z_1$,	$z = z_2$	$(z_1 < z_2)$

(Figure 15.7.1). The dimensions $\theta_2 - \theta_1$, $r_2 - r_1$, and $z_2 - z_1$ are called the ***central angle***, ***thickness***, and ***height*** of the wedge.

To define the triple integral over G of a function $f(r, \theta, z)$ in cylindrical coordinates we proceed as follows:

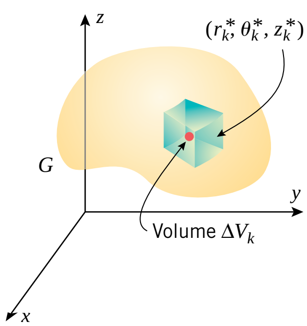


Figure 15.7.2

- Subdivide G into pieces by a three-dimensional grid consisting of concentric circular cylinders centered on the z-axis, half-planes hinged on the z-axis, and horizontal planes. Exclude from consideration all pieces that contain any points outside of G, thereby leaving only cylindrical wedges that are subsets of G.
- Assume that there are n such cylindrical wedges, and denote the volume of the kth cylindrical wedge by ΔV_k. As indicated in Figure 15.7.2, let $(r_k^*, \theta_k^*, z_k^*)$ be any point in the kth cylindrical wedge.
- Repeat this process with more and more subdivisions so that as n increases, the height, thickness, and central angle of the cylindrical wedges approach zero. Define

$$\iiint\limits_G f(r, \theta, z)\, dV = \lim_{n \to +\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*, z_k^*) \Delta V_k \tag{1}$$

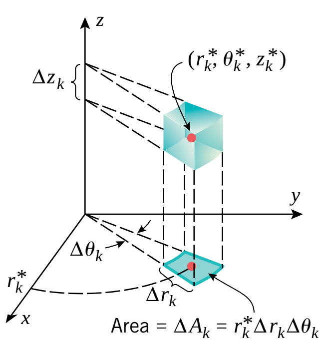


Figure 15.7.3

For computational purposes, it will be helpful to express (1) as an iterated integral. Toward this end we note that the volume ΔV_k of the kth cylindrical wedge can be expressed as

$$\Delta V_k = [\text{area of base}] \cdot [\text{height}] \tag{2}$$

If we denote the thickness, central angle, and height of this wedge by Δr_k, $\Delta\theta_k$, and Δz_k, and if we choose the arbitrary point $(r_k^*, \theta_k^*, z_k^*)$ to lie above the "center" of the base (Figures 15.3.5 and 15.7.3), then it follows from (5) of Section 15.3 that the base has area $\Delta A_k = r_k^* \Delta r_k \Delta\theta_k$. Thus, (2) can be written as

$$\Delta V_k = r_k^* \Delta r_k \Delta\theta_k \Delta z_k = r_k^* \Delta z_k \Delta r_k \Delta\theta_k$$

Substituting this expression in (1) yields

$$\iiint\limits_G f(r, \theta, z)\, dV = \lim_{n \to +\infty} \sum_{k=1}^{n} f(r_k^*, \theta_k^*, z_k^*) r_k^* \Delta z_k \Delta r_k \Delta\theta_k$$

which suggests that a triple integral in cylindrical coordinates can be evaluated as an iterated integral of the form

$$\iiint\limits_G f(r, \theta, z)\, dV = \iiint\limits_{\substack{\text{appropriate}\\ \text{limits}}} f(r, \theta, z) r\, dz\, dr\, d\theta \tag{3}$$

REMARK. Note the extra factor of r that appears in the integrand on converting from the triple integral to the iterated integral. In this formula the integration with respect to z is done first, then with respect to r, and then with respect to θ, but any order of integration is allowable.

The following theorem, which we state without proof, makes the preceding ideas more precise.

15.7.1 THEOREM. *Let G be a solid whose upper surface has the equation $z = g_2(r, \theta)$ and whose lower surface has the equation $z = g_1(r, \theta)$ in cylindrical coordinates. If the projection of the solid on the xy-plane is a simple polar region R, and if $f(r, \theta, z)$ is continuous on G, then*

$$\iiint_G f(r, \theta, z)\, dV = \iint_R \left[\int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r, \theta, z)\, dz \right] dA \tag{4}$$

where the double integral over R is evaluated in polar coordinates. In particular, if the projection R is as shown in Figure 15.7.4, then (4) can be written as

$$\iiint_G f(r, \theta, z)\, dV = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} \int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r, \theta, z) r\, dz\, dr\, d\theta \tag{5}$$

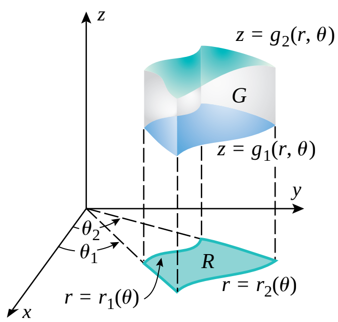


Figure 15.7.4

The type of solid to which Formula (5) applies is illustrated in Figure 15.7.4. To apply (4) and (5) it is best to begin with a three-dimensional sketch of the solid G, from which the limits of integration can be obtained as follows:

Step 1. Identify the upper surface $z = g_2(r, \theta)$ and the lower surface $z = g_1(r, \theta)$ of the solid. The functions $g_1(r, \theta)$ and $g_2(r, \theta)$ determine the z-limits of integration. (If the upper and lower surfaces are given in rectangular coordinates, convert them to cylindrical coordinates.)

Step 2. Make a two-dimensional sketch of the projection R of the solid on the xy-plane. From this sketch the r- and θ-limits of integration may be obtained exactly as with double integrals in polar coordinates.

Example 1 Use triple integration in cylindrical coordinates to find the volume and the centroid of the solid G that is bounded above by the hemisphere $z = \sqrt{25 - x^2 - y^2}$, below by the xy-plane, and laterally by the cylinder $x^2 + y^2 = 9$.

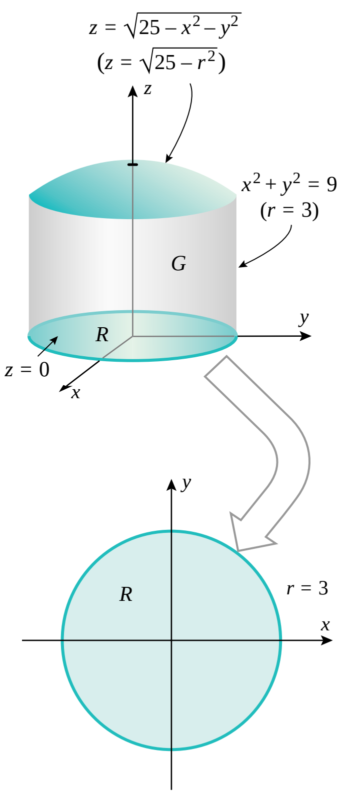


Figure 15.7.5

Solution. The solid G and its projection R on the xy-plane are shown in Figure 15.7.5. In cylindrical coordinates, the upper surface of G is the hemisphere $z = \sqrt{25 - r^2}$ and the lower surface is the plane $z = 0$. Thus, from (4), the volume of G is

$$V = \iiint_G dV = \iint_R \left[\int_0^{\sqrt{25-r^2}} dz \right] dA$$

For the double integral over R, we use polar coordinates:

$$\begin{aligned}
V &= \int_0^{2\pi} \int_0^3 \int_0^{\sqrt{25-r^2}} r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^3 \left[rz \right]_{z=0}^{\sqrt{25-r^2}} dr\, d\theta \\
&= \int_0^{2\pi} \int_0^3 r\sqrt{25 - r^2}\, dr\, d\theta = \int_0^{2\pi} \left[-\frac{1}{3}(25 - r^2)^{3/2} \right]_{r=0}^3 d\theta \\
&= \int_0^{2\pi} \frac{61}{3}\, d\theta = \frac{122}{3}\pi
\end{aligned}$$

$u = 25 - r^2$
$du = -2r\, dr$

From this result and (15) of Section 15.6,

$$\bar{z} = \frac{1}{V}\iiint_G z\,dV = \frac{3}{122\pi}\iiint_G z\,dV = \frac{3}{122\pi}\iint_R \left[\int_0^{\sqrt{25-r^2}} z\,dz\right] dA$$

$$= \frac{3}{122\pi}\int_0^{2\pi}\int_0^3\int_0^{\sqrt{25-r^2}} zr\,dz\,dr\,d\theta = \frac{3}{122\pi}\int_0^{2\pi}\int_0^3\left[\frac{1}{2}rz^2\right]_{z=0}^{\sqrt{25-r^2}} dr\,d\theta$$

$$= \frac{3}{244\pi}\int_0^{2\pi}\int_0^3 (25r - r^3)\,dr\,d\theta = \frac{3}{244\pi}\int_0^{2\pi}\frac{369}{4}\,d\theta = \frac{1107}{488}$$

By symmetry, the centroid $(\bar{x}, \bar{y}, \bar{z})$ of G lies on the z-axis, so $\bar{x} = \bar{y} = 0$. Thus, the centroid is at the point $(0, 0, 1107/488)$. ◀

CONVERTING TRIPLE INTEGRALS FROM RECTANGULAR TO CYLINDRICAL COORDINATES

Sometimes a triple integral that is difficult to integrate in rectangular coordinates can be evaluated more easily by making the substitution $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ to convert it to an integral in cylindrical coordinates. Under such a substitution, a rectangular triple integral can be expressed as an iterated integral in cylindrical coordinates as

$$\iiint_G f(x, y, z)\,dV = \iiint_{\substack{\text{appropriate}\\ \text{limits}}} f(r\cos\theta, r\sin\theta, z)r\,dz\,dr\,d\theta \tag{6}$$

REMARK. In (6), the order of integration is first with respect to z, then r, and then θ. However, the order of integration can be changed, provided the limits of integration are adjusted accordingly.

Example 2 Use cylindrical coordinates to evaluate

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2} x^2\,dz\,dy\,dx$$

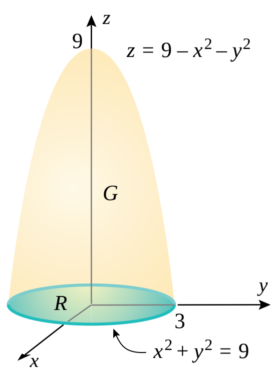


Figure 15.7.6

Solution. In problems of this type, it is helpful to sketch the region of integration G and its projection R on the xy-plane. From the z-limits of integration, the upper surface of G is the paraboloid $z = 9 - x^2 - y^2$ and the lower surface is the xy-plane $z = 0$. From the x- and y-limits of integration, the projection R is the region in the xy-plane enclosed by the circle $x^2 + y^2 = 9$ (Figure 15.7.6). Thus,

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^{9-x^2-y^2} x^2\,dz\,dy\,dx = \iiint_G x^2\,dV$$

$$= \iint_R \left[\int_0^{9-r^2} r^2\cos^2\theta\,dz\right] dA = \int_0^{2\pi}\int_0^3\int_0^{9-r^2} (r^2\cos^2\theta)\,r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3\int_0^{9-r^2} r^3\cos^2\theta\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^3 \left[zr^3\cos^2\theta\right]_{z=0}^{9-r^2} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 (9r^3 - r^5)\cos^2\theta\,dr\,d\theta = \int_0^{2\pi}\left[\left(\frac{9r^4}{4} - \frac{r^6}{6}\right)\cos^2\theta\right]_{r=0}^{3} d\theta$$

$$= \frac{243}{4}\int_0^{2\pi}\cos^2\theta\,d\theta = \frac{243}{4}\int_0^{2\pi}\frac{1}{2}(1 + \cos 2\theta)\,d\theta = \frac{243\pi}{4}$$ ◀

TRIPLE INTEGRALS IN SPHERICAL COORDINATES

In spherical coordinates, the simplest equations are of the form

$$\rho = \text{constant}, \quad \theta = \text{constant}, \quad \phi = \text{constant}$$

As indicated in Figure 12.8.2*c*, the first equation represents a sphere centered at the origin and the second a half-plane hinged on the z-axis. The third a right circular cone with its vertex at the origin and its line of symmetry along the z-axis for $\phi \neq \pi/2$, and is the xy-plane if $\phi \neq \pi/2$. By a ***spherical wedge*** or ***spherical element of volume*** we mean a solid enclosed between six surfaces of the following form:

two spheres	$\rho = \rho_1, \quad \rho = \rho_2$	$(\rho_1 < \rho_2)$
two half-planes	$\theta = \theta_1, \quad \theta = \theta_2$	$(\theta_1 < \theta_2)$
nappes of two right circular cones	$\phi = \phi_1, \quad \phi = \phi_2$	$(\phi_1 < \phi_2)$

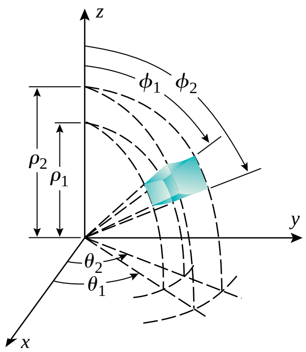


Figure 15.7.7

(Figure 15.7.7). We will refer to the numbers $\rho_2 - \rho_1$, $\theta_2 - \theta_1$, and $\phi_2 - \phi_1$ as the ***dimensions*** of a spherical wedge.

If G is a solid region in three-dimensional space, then the triple integral over G of a continuous function $f(\rho, \theta, \phi)$ in spherical coordinates is similar in definition to the triple integral in cylindrical coordinates, except that the solid G is partitioned into *spherical* wedges by a three-dimensional grid consisting of spheres centered at the origin, half-planes hinged on the z-axis, and nappes of right circular cones with vertices at the origin and lines of symmetry along the z-axis (Figure 15.7.8).

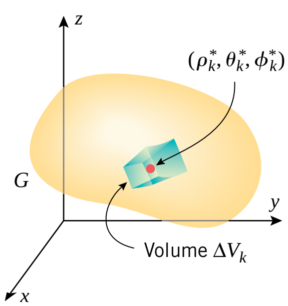


Figure 15.7.8

The defining equation of a triple integral in spherical coordinates is

$$\iiint_G f(\rho, \theta, \phi)\, dV = \lim_{n \to +\infty} \sum_{k=1}^{n} f(\rho_k^*, \theta_k^*, \phi_k^*) \Delta V_k \tag{7}$$

where ΔV_k is the volume of the kth spherical wedge that is interior to G, $(\rho_k^*, \theta_k^*, \phi_k^*)$ is an arbitrary point in this wedge, and n increases in such a way that the dimensions of each interior spherical wedge tend to zero.

For computational purposes, it will be desirable to express (7) as an iterated integral. In the exercises we will help you to show that if the point $(\rho_k^*, \theta_k^*, \phi_k^*)$ is suitably chosen, then the volume ΔV_k in (7) can be written as

$$\Delta V_k = \rho_k^{*2} \sin \phi_k^* \Delta\rho_k \Delta\phi_k \Delta\theta_k \tag{8}$$

where $\Delta\rho_k$, $\Delta\phi_k$, and $\Delta\theta_k$ are the dimensions of the wedge (Exercise 38). Substituting this in (7) we obtain

$$\iiint_G f(\rho, \theta, \phi)\, dV = \lim_{n \to +\infty} \sum_{k=1}^{n} f(\rho_k^*, \theta_k^*, \phi_k^*) \rho_k^{*2} \sin \phi_k^* \Delta\rho_k \Delta\phi_k \Delta\theta_k$$

which suggests that a triple integral in spherical coordinates can be evaluated as an iterated integral of the form

$$\boxed{\iiint_G f(\rho, \theta, \phi)\, dV = \iiint_{\substack{\text{appropriate}\\ \text{limits}}} f(\rho, \theta, \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta} \tag{9}$$

REMARK. Note the extra factor of $\rho^2 \sin \phi$ that appears in the integrand of the iterated integral. This is analogous to the extra factor of r that appeared when we integrated in cylindrical coordinates.

The analog of Theorem 15.7.1 for triple integrals in spherical coordinates is tedious to state, so instead we will give some examples that illustrate techniques for obtaining the limits of integration. In all of our examples we will use the same order of integration—first with respect to ρ, then ϕ, and then θ. Once you have mastered the basic ideas, there should be no trouble using other orders of integration.

Suppose that we want to integrate $f(\rho, \theta, \phi)$ over the spherical solid G enclosed by the sphere $\rho = \rho_0$. The basic idea is to choose the limits of integration so that every point of the solid is accounted for in the integration process. Figure 15.7.9 illustrates one way of doing this. Holding θ and ϕ fixed for the first integration, we let ρ vary from 0 to ρ_0. This covers a radial line from the origin to the surface of the sphere. Next, keeping θ fixed, we let ϕ vary from 0 to π so that the radial line sweeps out a fan-shaped region. Finally, we let θ vary from 0 to 2π so that the fan-shaped region makes a complete revolution, thereby sweeping out the entire sphere. Thus, the triple integral of $f(\rho, \theta, \phi)$ over the spherical solid G may be evaluated by writing

$$\iiint_G f(\rho, \theta, \phi)\, dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{\rho_0} f(\rho, \theta, \phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

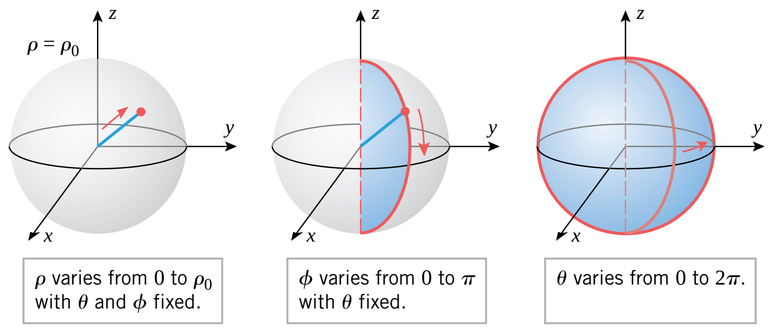


Figure 15.7.9

Table 15.7.1 suggests how the limits of integration in spherical coordinates can be obtained for some other common solids.

Example 3 Use spherical coordinates to find the volume and the centroid of the solid G bounded above by the sphere $x^2 + y^2 + z^2 = 16$ and below by the cone $z = \sqrt{x^2 + y^2}$.

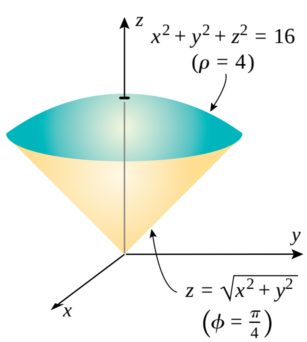


Figure 15.7.10

Solution. The solid G is sketched in Figure 15.7.10.

In spherical coordinates, the equation of the sphere $x^2 + y^2 + z^2 = 16$ is $\rho = 4$ and the equation of the cone $z = \sqrt{x^2 + y^2}$ is

$$\rho \cos\phi = \sqrt{\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta}$$

which simplifies to

$$\rho \cos\phi = \rho \sin\phi$$

or, on dividing both sides by $\rho \cos\phi$,

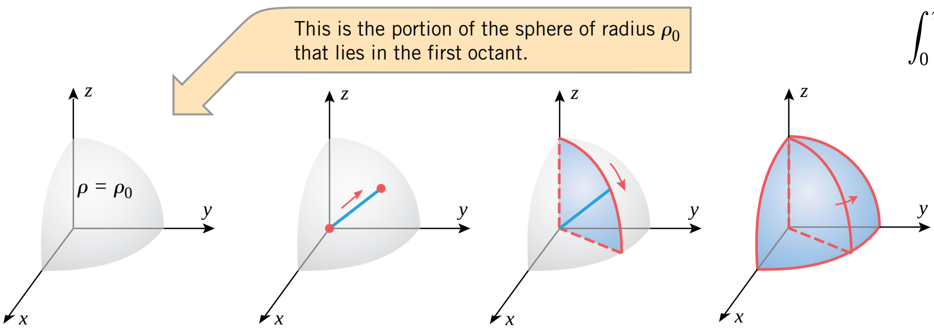
$$\tan\phi = 1$$

Thus $\phi = \pi/4$, and using the second entry in Table 15.7.1, the volume of G is

$$\begin{aligned}
V = \iiint_G dV &= \int_0^{2\pi} \int_0^{\pi/4} \int_0^{4} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\
&= \int_0^{2\pi} \int_0^{\pi/4} \left[\frac{\rho^3}{3} \sin\phi\right]_{\rho=0}^{4} d\phi \, d\theta \\
&= \int_0^{2\pi} \int_0^{\pi/4} \frac{64}{3} \sin\phi \, d\phi \, d\theta \\
&= \frac{64}{3} \int_0^{2\pi} [-\cos\phi]_{\phi=0}^{\pi/4} \, d\theta = \frac{64}{3} \int_0^{2\pi} \left(1 - \frac{\sqrt{2}}{2}\right) d\theta \\
&= \frac{64\pi}{3}(2 - \sqrt{2})
\end{aligned}$$

Table 15.7.1

DETERMINATION OF LIMITS | INTEGRAL

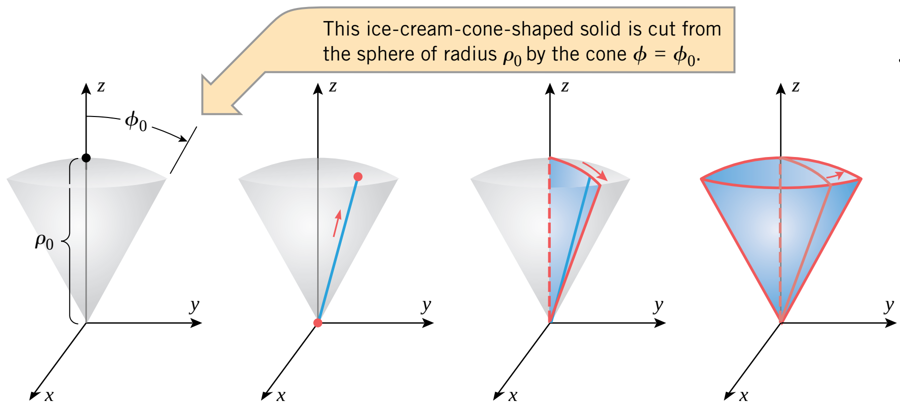


ρ varies from 0 to ρ_0 with θ and ϕ held fixed.

ϕ varies from 0 to $\pi/2$ with θ held fixed.

θ varies from 0 to $\pi/2$.

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\rho_0} f(\rho,\theta,\phi)\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

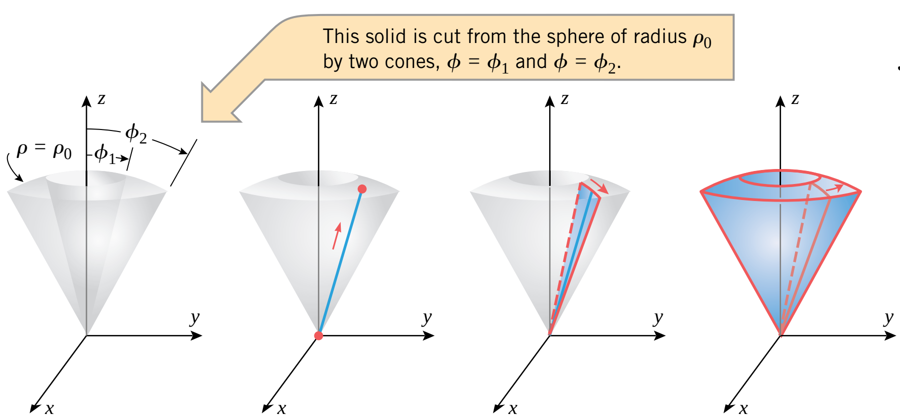


ρ varies from 0 to ρ_0 with θ and ϕ held fixed.

ϕ varies from 0 to ϕ_0 with θ held fixed.

θ varies from 0 to 2π.

$$\int_0^{2\pi}\int_0^{\phi_0}\int_0^{\rho_0} f(\rho,\theta,\phi)\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$



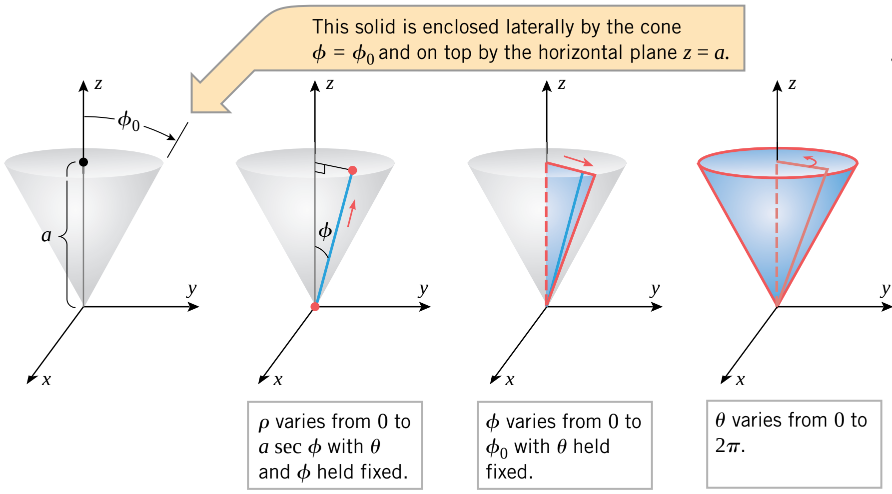
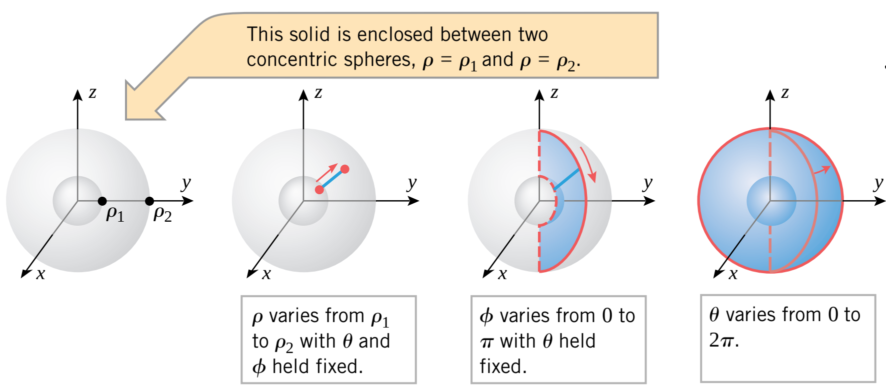
ρ varies from 0 to ρ_0 with θ and ϕ held fixed.

ϕ varies from ϕ_1 to ϕ_2 with θ held fixed.

θ varies from 0 to 2π.

$$\int_0^{2\pi}\int_{\phi_1}^{\phi_2}\int_0^{\rho_0} f(\rho,\theta,\phi)\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

Table 15.7.1 ***(continued)***

DETERMINATION OF LIMITS	INTEGRAL
	$\int_0^{2\pi}\int_0^{\phi_0}\int_0^{a\sec\phi} f(\rho,\theta,\phi)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$
	$\int_0^{2\pi}\int_0^{\pi}\int_{\rho_1}^{\rho_2} f(\rho,\theta,\phi)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

By symmetry, the centroid $(\bar{x}, \bar{y}, \bar{z})$ is on the z-axis, so $\bar{x} = \bar{y} = 0$. From (15) of Section 15.6 and the volume calculated above,

$$
\begin{aligned}
\bar{z} &= \frac{1}{V}\iiint_G z\, dV = \frac{1}{V}\int_0^{2\pi}\int_0^{\pi/4}\int_0^4 (\rho\cos\phi)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta \\
&= \frac{1}{V}\int_0^{2\pi}\int_0^{\pi/4}\left[\frac{\rho^4}{4}\cos\phi\sin\phi\right]_{\rho=0}^{4} d\phi\, d\theta \\
&= \frac{64}{V}\int_0^{2\pi}\int_0^{\pi/4}\sin\phi\cos\phi\, d\phi\, d\theta = \frac{64}{V}\int_0^{2\pi}\left[\frac{1}{2}\sin^2\phi\right]_{\phi=0}^{\pi/4} d\theta \\
&= \frac{16}{V}\int_0^{2\pi} d\theta = \frac{32\pi}{V} = \frac{3}{2(2-\sqrt{2})}
\end{aligned}
$$

With the help of a calculator, $\bar{z} \approx 2.56$ (to two decimal places), so the approximate location of the centroid in the xyz-coordinate system is (0, 0, 2.56). ◀

CONVERTING TRIPLE INTEGRALS FROM RECTANGULAR TO SPHERICAL COORDINATES

Referring to Table 12.8.1, triple integrals can be converted from rectangular coordinates to spherical coordinates by making the substitution $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$,

$z = \rho\cos\phi$. The two integrals are related by the equation

$$\iiint\limits_{G} f(x, y, z)\, dV = \iiint\limits_{\substack{\text{appropriate}\\ \text{limits}}} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta \tag{10}$$

Example 4 Use spherical coordinates to evaluate

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\, dz\, dy\, dx$$

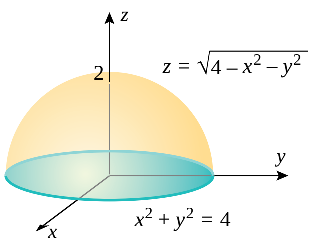


Figure 15.7.11

Solution. In problems like this, it is helpful to begin (when possible) with a sketch of the region G of integration. From the z-limits of integration, the upper surface of G is the hemisphere $z = \sqrt{4 - x^2 - y^2}$ and the lower surface is the xy-plane $z = 0$. From the x- and y-limits of integration, the projection of the solid G on the xy-plane is the region enclosed by the circle $x^2 + y^2 = 4$. From this information we obtain the sketch of G in Figure 15.7.11. Thus,

$$\begin{aligned}
&\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} z^2\sqrt{x^2+y^2+z^2}\, dz\, dy\, dx \\
&\quad = \iiint\limits_{G} z^2\sqrt{x^2+y^2+z^2}\, dV \\
&\quad = \int_{0}^{2\pi}\int_{0}^{\pi/2}\int_{0}^{2} \rho^5\cos^2\phi\sin\phi\, d\rho\, d\phi\, d\theta \\
&\quad = \int_{0}^{2\pi}\int_{0}^{\pi/2} \frac{32}{3}\cos^2\phi\sin\phi\, d\phi\, d\theta \\
&\quad = \frac{32}{3}\int_{0}^{2\pi}\left[-\frac{1}{3}\cos^3\phi\right]_{\phi=0}^{\pi/2} d\theta = \frac{32}{9}\int_{0}^{2\pi} d\theta = \frac{64}{9}\pi
\end{aligned}$$

◀

EXERCISE SET 15.7 C CAS

In Exercises 1–4, evaluate the iterated integral.

1. $\displaystyle\int_{0}^{2\pi}\int_{0}^{1}\int_{0}^{\sqrt{1-r^2}} zr\, dz\, dr\, d\theta$

2. $\displaystyle\int_{0}^{\pi/2}\int_{0}^{\cos\theta}\int_{0}^{r^2} r\sin\theta\, dz\, dr\, d\theta$

3. $\displaystyle\int_{0}^{\pi/2}\int_{0}^{\pi/2}\int_{0}^{1} \rho^3\sin\phi\cos\phi\, d\rho\, d\phi\, d\theta$

4. $\displaystyle\int_{0}^{2\pi}\int_{0}^{\pi/4}\int_{0}^{a\sec\phi} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta \quad (a > 0)$

In Exercises 5–8, use cylindrical coordinates to find the volume of the solid.

5. The solid enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 9$.

6. The solid that is bounded above and below by the sphere $x^2 + y^2 + z^2 = 9$ and inside the cylinder $x^2 + y^2 = 4$.

7. The solid that is inside the surface $r^2 + z^2 = 20$ and below the surface $z = r^2$.

8. The solid enclosed between the cone $z = (hr)/a$ and the plane $z = h$.

In Exercises 9–12, use spherical coordinates to find the volume of the solid.

9. The solid bounded above by the sphere $\rho = 4$ and below by the cone $\phi = \pi/3$.

10. The solid within the cone $\phi = \pi/4$ and between the spheres $\rho = 1$ and $\rho = 2$.

11. The solid enclosed by the sphere $x^2 + y^2 + z^2 = 4a^2$ and the planes $z = 0$ and $z = a$.

12. The solid within the sphere $x^2 + y^2 + z^2 = 9$, outside the cone $z = \sqrt{x^2 + y^2}$, and above the xy-plane.

In Exercises 13–16, use cylindrical or spherical coordinates to evaluate the integral.

13. $\displaystyle\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{a^2-x^2-y^2} x^2\, dz\, dy\, dx \quad (a > 0)$

14. $\displaystyle\int_{-1}^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} e^{-(x^2+y^2+z^2)^{3/2}}\, dz\, dy\, dx$

15. $\displaystyle\int_0^2 \int_0^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} z^2\, dz\, dx\, dy$

16. $\displaystyle\int_{-3}^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_{-\sqrt{9-x^2-y^2}}^{\sqrt{9-x^2-y^2}} \sqrt{x^2 + y^2 + z^2}\, dz\, dx\, dy$

C **17.** (a) Use a CAS to evaluate

$$\int_{-2}^2 \int_1^4 \int_{\pi/6}^{\pi/3} \frac{r \tan^3 \theta}{\sqrt{1 + z^2}}\, dz\, dr\, d\theta$$

(b) Find a function $f(x, y, z)$ and sketch a region G in 3-space so that the triple integral in rectangular coordinates

$$\iiint_G f(x, y, z)\, dV$$

matches the iterated integral in cylindrical coordinates given in part (a).

C **18.** Use a CAS to evaluate

$$\int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\cos\theta} \rho^{17} \cos\phi \cos^{19}\theta\, d\rho\, d\phi\, d\theta$$

19. Find the volume enclosed by $x^2 + y^2 + z^2 = a^2$ using
(a) cylindrical coordinates
(b) spherical coordinates.

20. Let G be the solid in the first octant bounded by the sphere $x^2 + y^2 + z^2 = 4$ and the coordinate planes. Evaluate

$$\iiint_G xyz\, dV$$

(a) using rectangular coordinates
(b) using cylindrical coordinates
(c) using spherical coordinates.

In Exercises 21 and 22, use cylindrical coordinates.

21. Find the mass of the solid with density $\delta(x, y, z) = 3 - z$ that is bounded by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 3$.

22. Find the mass of a right circular cylinder of radius a and height h if the density is proportional to the distance from the base. (Let k be the constant of proportionality.)

In Exercises 23 and 24, use spherical coordinates.

23. Find the mass of a spherical solid of radius a if the density is proportional to the distance from the center. (Let k be the constant of proportionality.)

24. Find the mass of the solid enclosed between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ if the density is $\delta(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$.

In Exercises 25 and 26, use cylindrical coordinates to find the centroid of the solid.

25. The solid that is bounded above by the sphere

$$x^2 + y^2 + z^2 = 2$$

and below by the paraboloid $z = x^2 + y^2$.

26. The solid that is bounded by the cone $z = \sqrt{x^2 + y^2}$ and the plane $z = 2$.

In Exercises 27 and 28, use spherical coordinates to find the centroid of the solid.

27. The solid in the first octant bounded by the coordinate planes and the sphere $x^2 + y^2 + z^2 = a^2$.

28. The solid bounded above by the sphere $\rho = 4$ and below by the cone $\phi = \pi/3$.

In Exercises 29 and 30, use the Wallis formulas in Exercises 64 and 66 of Section 8.3.

29. Find the centroid of the solid bounded above by the paraboloid $z = x^2 + y^2$, below by the plane $z = 0$, and laterally by the cylinder $(x - 1)^2 + y^2 = 1$.

30. Find the mass of the solid in the first octant bounded above by the paraboloid $z = 4 - x^2 - y^2$, below by the plane $z = 0$, and laterally by the cylinder $x^2 + y^2 = 2x$ and the plane $y = 0$, assuming the density to be $\delta(x, y, z) = z$.

In Exercises 31–36, solve the problem using either cylindrical or spherical coordinates (whichever seems appropriate).

31. Find the volume of the solid in the first octant bounded by the sphere $\rho = 2$, the coordinate planes, and the cones $\phi = \pi/6$ and $\phi = \pi/3$.

32. Find the mass of the solid that is enclosed by the sphere $x^2 + y^2 + z^2 = 1$ and lies within the cone $z = \sqrt{x^2 + y^2}$ if the density is $\delta(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.

33. Find the center of gravity of the solid bounded by the paraboloid $z = 1 - x^2 - y^2$ and the xy-plane, assuming the density to be $\delta(x, y, z) = x^2 + y^2 + z^2$.

34. Find the center of gravity of the solid that is bounded by the cylinder $x^2 + y^2 = 1$, the cone $z = \sqrt{x^2 + y^2}$, and the xy-plane if the density is $\delta(x, y, z) = z$.

35. Find the center of gravity of the solid hemisphere bounded by $z = \sqrt{a^2 - x^2 - y^2}$ and $z = 0$ if the density is proportional to the distance from the origin.

36. Find the centroid of the solid that is enclosed by the hemispheres $y = \sqrt{9 - x^2 - z^2}$, $y = \sqrt{4 - x^2 - z^2}$, and the plane $y = 0$.

37. Suppose that the density at a point in a gaseous spherical star is modeled by the formula

$$\delta = \delta_0 e^{-(\rho/R)^3}$$

where δ_0 is a positive constant, R is the radius of the star, and ρ is the distance from the point to the star's center. Find the mass of the star.

38. In this exercise we will obtain a formula for the volume of the spherical wedge in Figure 15.7.7.

(a) Use a triple integral in cylindrical coordinates to show that the volume of the solid bounded above by a sphere $\rho = \rho_0$, below by a cone $\phi = \phi_0$, and on the sides by $\theta = \theta_1$ and $\theta = \theta_2$ $(\theta_1 < \theta_2)$ is

$$V = \tfrac{1}{3}\rho_0^3(1 - \cos\phi_0)(\theta_2 - \theta_1)$$

[*Hint:* In cylindrical coordinates, the sphere has the equation $r^2 + z^2 = \rho_0^2$ and the cone has the equation $z = r\cot\phi_0$. For simplicity, consider only the case $0 < \phi_0 < \pi/2$.]

(b) Subtract appropriate volumes and use the result in part (a) to deduce that the volume ΔV of the spherical wedge is

$$\Delta V = \frac{\rho_2^3 - \rho_1^3}{3}(\cos\phi_1 - \cos\phi_2)(\theta_2 - \theta_1)$$

(c) Apply the Mean-Value Theorem to the functions $\cos\phi$ and ρ^3 to deduce that the formula in part (b) can be written as

$$\Delta V = \rho^{*2}\sin\phi^*\,\Delta\rho\,\Delta\phi\,\Delta\theta$$

where ρ^* is between ρ_1 and ρ_2, ϕ^* is between ϕ_1 and ϕ_2, and $\Delta\rho = \rho_2 - \rho_1$, $\Delta\phi = \phi_2 - \phi_1$, $\Delta\theta = \theta_2 - \theta_1$.

The tendency of a solid to resist a change in rotational motion about an axis is measured by its ***moment of inertia*** about that axis. If the solid occupies a region G in an xyz-coordinate system, and if its density function $\delta(x, y, z)$ is continuous on G, then the moments of inertia about the x-axis, the y-axis, and the z-axis are denoted by I_x, I_y, and I_z, respectively, and are defined by

$$I_x = \iiint_G (y^2 + z^2)\,\delta(x, y, z)\,dV$$

$$I_y = \iiint_G (x^2 + z^2)\,\delta(x, y, z)\,dV$$

$$I_z = \iiint_G (x^2 + y^2)\,\delta(x, y, z)\,dV$$

(compare with the discussion preceding Exercises 42 and 43 of Section 15.6). In Exercises 39–42, find the indicated moment of inertia of the solid, assuming that it has constant density δ.

39. I_z for the solid cylinder $x^2 + y^2 \le a^2$, $0 \le z \le h$.

40. I_y for the solid cylinder $x^2 + y^2 \le a^2$, $0 \le z \le h$.

41. I_z for the hollow cylinder $a_1^2 \le x^2 + y^2 \le a_2^2$, $0 \le z \le h$.

42. I_z for the solid sphere $x^2 + y^2 + z^2 \le a^2$.

15.8 CHANGE OF VARIABLES IN MULTIPLE INTEGRALS; JACOBIANS

In this section we will discuss a general method for evaluating double and triple integrals by substitution. Most of the results in this section are very difficult to prove, so our approach will be informal and motivational. Our goal is to provide a geometric understanding of the basic principles and an exposure to computational techniques.

CHANGE OF VARIABLE IN A SINGLE INTEGRAL

To motivate techniques for evaluating double and triple integrals by substitution, it will be helpful to consider the effect of a substitution $x = g(u)$ on a single integral over an interval $[a, b]$. If g is differentiable and either increasing or decreasing, then g is one-to-one and

$$\int_a^b f(x)\,dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))g'(u)\,du$$

In this relationship $f(x)$ and dx are expressed in terms of u, and the u-limits of integration result from solving the equations

$$a = g(u) \quad \text{and} \quad b = g(u)$$

In the case where g is decreasing we have $g^{-1}(b) < g^{-1}(a)$, which is contrary to our usual convention of writing definite integrals with the larger limit of integration at the top. We can remedy this by reversing the limits of integration and writing

$$\int_a^b f(x)\,dx = -\int_{g^{-1}(b)}^{g^{-1}(a)} f(g(u))g'(u)\,du = \int_{g^{-1}(b)}^{g^{-1}(a)} f(g(u))|g'(u)|\,du$$

where the absolute value results from the fact that $g'(u)$ is negative. Thus, regardless of whether g is increasing or decreasing we can write

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(g(u))|g'(u)|\,du \tag{1}$$

where α and β are the u-limits of integration and $\alpha < \beta$.

The expression $g'(u)$ that appears in (1) is called the ***Jacobian*** of the change of variable $x = g(u)$ in honor of C. G. J. Jacobi,* who made the first serious study of change of variables in multiple integrals in the mid 1800s. Formula (1) reveals three effects of the change of variable $x = g(u)$:

- The new integrand becomes $f(g(u))$ times the absolute value of the Jacobian.
- dx becomes du.
- The x-interval of integration is transformed into a u-interval of integration.

Our goal in this section is to show that analogous results hold for changing variables in double and triple integrals.

TRANSFORMATIONS OF THE PLANE

In earlier sections we considered parametric equations of three kinds:

$x = x(t), \quad y = y(t)$	A curve in the plane
$x = x(t), \quad y = y(t), \quad z = z(t)$	A curve in 3-space
$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v)$	A surface in 3-space

Now, we will consider parametric equations of the form

$$x = x(u, v), \quad y = y(u, v) \tag{2}$$

*CARL GUSTAV JACOB JACOBI (1804–1851). German mathematician, Jacobi, the son of a banker, grew up in a background of wealth and culture and showed brilliance in mathematics early. He resisted studying mathematics by rote, preferring instead to learn general principles from the works of the masters, Euler and Lagrange. He entered the University of Berlin at age 16 as a student of mathematics and classical studies. However, he soon realized that he could not do both and turned fully to mathematics with a blazing intensity that he would maintain throughout his life. He received his Ph.D. in 1825 and was able to secure a position as a lecturer at the University of Berlin by giving up Judaism and becoming a Christian. However, his promotion opportunities remained limited and he moved on to the University of Königsberg. Jacobi was born to teach—he had a dynamic personality and delivered his lectures with a clarity and enthusiasm that frequently left his audience spellbound. In spite of extensive teaching commitments, he was able to publish volumes of revolutionary mathematical research that eventually made him the leading European mathematician after Gauss. His main body of research was in the area of elliptic functions, a branch of mathematics with important applications in astronomy and physics as well as in other fields of mathematics. Because of his family wealth, Jacobi was not dependent on his teaching salary in his early years. However, his comfortable world eventually collapsed. In 1840 his family went bankrupt and he was personally wiped out financially. In 1842 he had a nervous breakdown from overwork. In 1843 he became seriously ill with diabetes and moved to Berlin with the help of a government grant to defray his medical expenses. In 1848 he made an injudicious political speech that caused the government to withdraw the grant, eventually resulting in the loss of his home. His health continued to decline and in 1851 he finally succumbed to successive bouts of influenza and smallpox. In spite of all his problems, Jacobi was a tireless worker to the end. When a friend expressed concern about the effect of the hard work on his health, Jacobi replied, “Certainly, I have sometimes endangered my health by overwork, but what of it? Only cabbages have no nerves, no worries. And what do they get out of their perfect well-being?”

Parametric equations of this type associate points in the *xy*-plane with points in the uv-plane. These equations can be written in vector form as

$$\mathbf{r} = \mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j}$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ is a position vector in the *xy*-plane and $\mathbf{r}(u, v)$ is a vector-valued function of the variables u and v.

It will also be useful in this section to think of the parametric equations in (2) in terms of inputs and outputs. If we think of the pair of numbers (u, v) as an input, then the two equations, in combination, produce a unique output (x, y), and hence define a function T that associates points in the *xy*-plane with points in the uv-plane. This function is described by the formula

$$T(u, v) = (x(u, v), y(u, v))$$

We call T a ***transformation*** from the uv-plane to the *xy*-plane and (x, y) the ***image*** of (u, v) under the transformation T. We also say that T ***maps*** (u, v) into (x, y). The set R of all images in the *xy*-plane of a set S in the uv-plane is called the ***image of S under T***. If distinct points in the uv-plane have distinct images in the *xy*-plane, then T is said to be ***one-to-one***. In this case the equations in (2) define u and v as functions of x and y, say

$$u = u(x, y), \quad v = v(x, y)$$

These equations, which can often be obtained by solving (2) for u and v in terms of x and y, define a transformation from the *xy*-plane to the uv-plane that maps the image of (u, v) under T back into (u, v). This transformation is denoted by T^{-1} and is called the ***inverse of T*** (Figure 15.8.1).

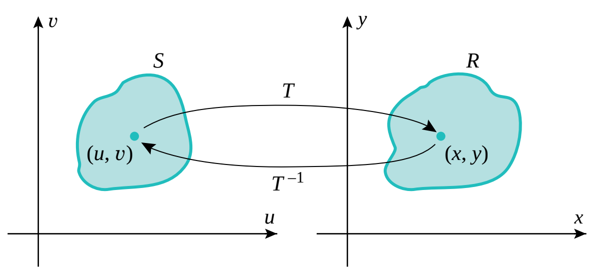


Figure 15.8.1

One way to visualize the geometric effect of a transformation T is to determine the images in the *xy*-plane of the vertical and horizontal lines in the uv-plane. Following the discussion on page XXX in Section 15.4, sets of points in the *xy*-plane that are images of horizontal lines (v constant) are called ***constant v-curves***, and sets of points that are images of vertical lines (u constant) are called ***constant u-curves*** (Figure 15.8.2).

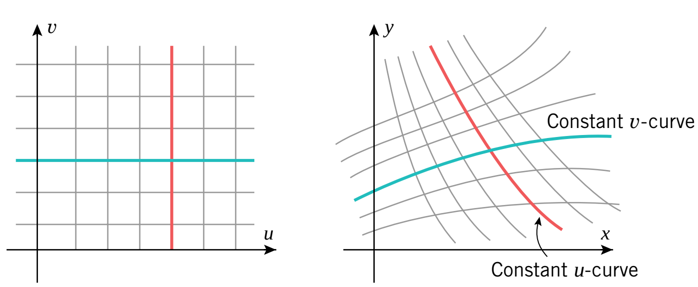


Figure 15.8.2

Example 1 Let T be the transformation from the uv-plane to the xy-plane defined by the equations

$$x = \tfrac{1}{4}(u + v), \quad y = \tfrac{1}{2}(u - v) \tag{3}$$

(a) Find $T(1, 3)$.

(b) Sketch the constant v-curves corresponding to $v = -2, -1, 0, 1, 2$.

(c) Sketch the constant u-curves corresponding to $u = -2, -1, 0, 1, 2$.

(d) Sketch the image under T of the square region in the uv-plane bounded by the lines $u = -2$, $u = 2$, $v = -2$, and $v = 2$.

Solution (a). Substituting $u = 1$ and $v = 3$ in (3) yields $T(1, 3) = (1, -1)$.

Solutions (b and c). In these parts it will be convenient to express the transformation equations with u and v as functions of x and y. We leave it for you to show that

$$u = 2x + y, \quad v = 2x - y$$

Thus, the constant v-curves corresponding to $v = -2, -1, 0, 1$, and 2 are

$$2x - y = -2, \quad 2x - y = -1, \quad 2x - y = 0, \quad 2x - y = 1, \quad 2x - y = 2$$

and the constant u-curves corresponding to $u = -2, -1, 0, 1$, and 2 are

$$2x + y = -2, \quad 2x + y = -1, \quad 2x + y = 0, \quad 2x + y = 1, \quad 2x + y = 2$$

In Figure 15.8.3 the constant v-curves are shown in green and the constant u-curves in red.

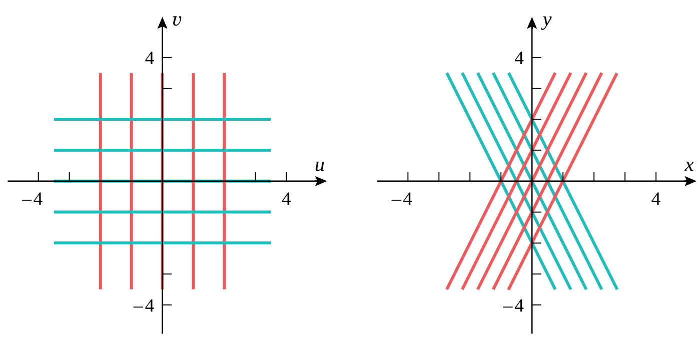


Figure 15.8.3

Solution (d). The image of a region can often be found by finding the image of its boundary. In this case the images of the boundary lines $u = -2$, $u = 2$, $v = -2$, and $v = 2$ enclose the diamond-shaped region in the xy-plane shown in Figure 15.8.4. ◀

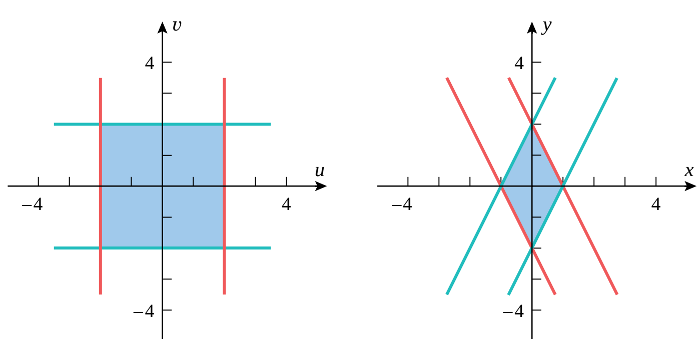


Figure 15.8.4

JACOBIANS IN TWO VARIABLES

To derive the change-of-variables formula for double integrals, we will need to understand the relationship between the area of a *small* rectangular region in the uv-plane and the area of its image in the *xy*-plane under a transformation T given by the equations

$$x = x(u, v), \quad y = y(u, v)$$

For this purpose, suppose that Δu and Δv are positive, and consider a rectangular region S in the uv-plane enclosed by the lines

$$u = u_0, \quad u = u_0 + \Delta u, \quad v = v_0, \quad v = v_0 + \Delta v$$

If the functions $x(u, v)$ and $y(u, v)$ are continuous, and if Δu and Δv are not too large, then the image of S in the *xy*-plane will be a region R that looks like a slightly distorted parallelogram (Figure 15.8.5). The sides of R are the constant u-curves and v-curves that correspond to the sides of S.

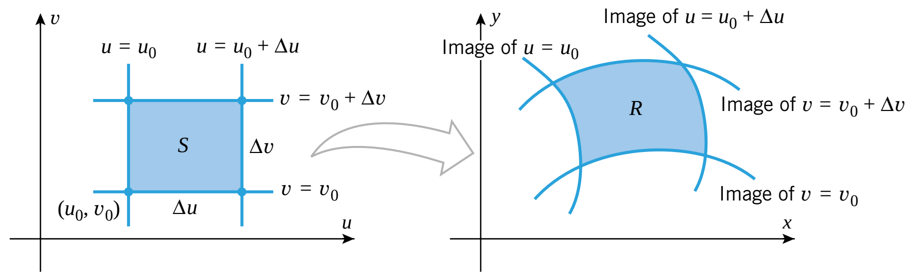


Figure 15.8.5

If we let

$$\mathbf{r} = \mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j}$$

be the position vector of the point in the *xy*-plane that corresponds to the point (u, v) in the uv-plane, then the constant v-curve corresponding to $v = v_0$ and the constant u-curve corresponding to $u = u_0$ can be represented in vector form as

$$\mathbf{r}(u, v_0) = x(u, v_0)\mathbf{i} + y(u, v_0)\mathbf{j} \qquad \boxed{\text{Constant } v\text{-curve}}$$

$$\mathbf{r}(u_0, v) = x(u_0, v)\mathbf{i} + y(u_0, v)\mathbf{j} \qquad \boxed{\text{Constant } u\text{-curve}}$$

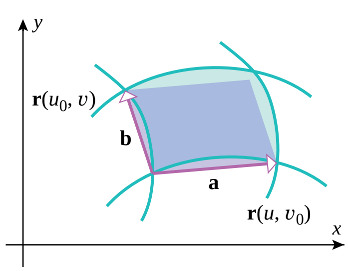


Figure 15.8.6

Since we are assuming Δu and Δv to be small, the region R can be approximated by a parallelogram determined by the "secant vectors"

$$\mathbf{a} = \mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \tag{4}$$

$$\mathbf{b} = \mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0) \tag{5}$$

shown in Figure 15.8.6. A more useful approximation of R can be obtained by using Formulas (5) and (6) of Section 15.4 to approximate these secant vectors by tangent vectors as follows:

$$\mathbf{a} = \frac{\mathbf{r}(u_0 + \Delta u, v_0) - \mathbf{r}(u_0, v_0)}{\Delta u}\Delta u$$

$$\approx \frac{\partial \mathbf{r}}{\partial u}\Delta u = \left(\frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j}\right)\Delta u$$

$$\mathbf{b} = \frac{\mathbf{r}(u_0, v_0 + \Delta v) - \mathbf{r}(u_0, v_0)}{\Delta v}\Delta v$$

$$\approx \frac{\partial \mathbf{r}}{\partial v}\Delta v = \left(\frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j}\right)\Delta v$$

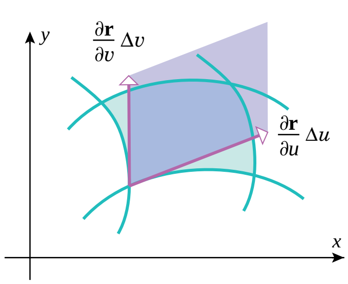


Figure 15.8.7

where the partial derivatives are evaluated at (u_0, v_0) (Figure 15.8.7). Hence, it follows that the area of the region R, which we will denote by ΔA, can be approximated by the area of the parallelogram determined by these vectors. Thus, from Formula (8) of Section 12.4 we have

$$\Delta A \approx \left\| \frac{\partial \mathbf{r}}{\partial u}\Delta u \times \frac{\partial \mathbf{r}}{\partial v}\Delta v \right\| = \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| \Delta u\, \Delta v \tag{6}$$

where the derivatives are evaluated at (u_0, v_0). Computing the cross product, we obtain

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & 0 \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} \mathbf{k} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \mathbf{k} \tag{7}$$

The determinant in (7) is sufficiently important that it has its own terminology and notation.

15.8.1 DEFINITION. If T is the transformation from the uv-plane to the xy-plane defined by the equations $x = x(u, v)$, $y = y(u, v)$, then the ***Jacobian of T*** is denoted by $J(u, v)$ or by $\partial(x, y)/\partial(u, v)$ and is defined by

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

Using the notation in this definition, it follows from (6) and (7) that

$$\Delta A \approx \left\| \frac{\partial(x, y)}{\partial(u, v)}\mathbf{k} \right\| \Delta u\, \Delta v$$

or, since $\mathbf{k}$ is a unit vector,

$$\Delta A \approx \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u\, \Delta v \tag{8}$$

At the point (u_0, v_0) this important formula relates the areas of the regions R and S in Figure 15.8.5: it tells us that *for small values of Δu and Δv, the area of R is approximately the absolute value of the Jacobian times the area of S*. Moreover, it is proved in advanced calculus courses that the relative error in the approximation approaches zero as $\Delta u \to 0$ and $\Delta v \to 0$.

CHANGE OF VARIABLES IN DOUBLE INTEGRALS

Our next objective is to provide a geometric motivation for the following result.

15.8.2 CHANGE-OF-VARIABLES FORMULA FOR DOUBLE INTEGRALS. If the transformation $x = x(u, v)$, $y = y(u, v)$ maps the region S in the uv-plane into the region R in the xy-plane, and if the Jacobian $\partial(x, y)/\partial(u, v)$ is nonzero and does not change sign on S, then with appropriate restrictions on the transformation and the regions it follows that

$$\iint_R f(x, y)\, dA_{xy} = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dA_{uv} \tag{9}$$

where we have attached subscripts to the dA's to help identify the associated variables.

REMARK. A precise statement of conditions under which Formula (9) holds would take us beyond the scope of this course. Suffice it to say that the formula holds if T is a one-to-one transformation, $f(x, y)$ is continuous on R, the partial derivatives of $x(u, v)$ and $y(u, v)$ exist and are continuous on S, and the regions R and S are not too complicated.

To motivate Formula (9), we proceed as follows:

- Subdivide the region S in the uv-plane into pieces by lines parallel to the coordinate axes, and exclude from consideration any pieces that contain points outside of S. This leaves only rectangular regions that are subsets of S. Assume that there are n such regions and denote the kth such region by S_k. Assume that S_k has dimensions Δu_k by Δv_k and, as shown in Figure 15.8.8*a*, let (u_k^*, v_k^*) be its "lower left corner."
- As shown in Figure 15.8.8*b*, the transformation T defined by the equations $x = x(u, v)$, $y = y(u, v)$ maps S_k into a curvilinear parallelogram R_k in the xy-plane and maps the point (u_k^*, v_k^*) into the point $(x_k^*, y_k^*) = (x(u_k^*, v_k^*), y(u_k^*, v_k^*))$ in R_k. Denote the area of R_k by ΔA_k.
- In rectangular coordinates the double integral of $f(x, y)$ over a region R is defined as a limit of Riemann sums in which R is subdivided into *rectangular* subregions. It is proved in advanced calculus courses that under appropriate conditions subdivisions into *curvilinear* parallelograms can be used instead. Accepting this to be so, we can approximate the double integral of $f(x, y)$ over R as

$$\begin{aligned}\iint_R f(x, y)\, dA_{xy} &\approx \sum_{k=1}^{n} f(x_k^*, y_k^*)\, \Delta A_k \\ &\approx \sum_{k=1}^{n} f(x(u_k^*, v_k^*), y(u_k^*, v_k^*)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u_k\, \Delta v_k\end{aligned}$$

where the Jacobian is evaluated at (u_k^*, v_k^*). But the last expression is a Riemann sum for the integral

$$\iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dA_{uv}$$

so Formula (9) follows if we assume that the errors in the approximations approach zero as $n \to +\infty$.

Example 2 Evaluate

$$\iint_R \frac{x - y}{x + y}\, dA$$

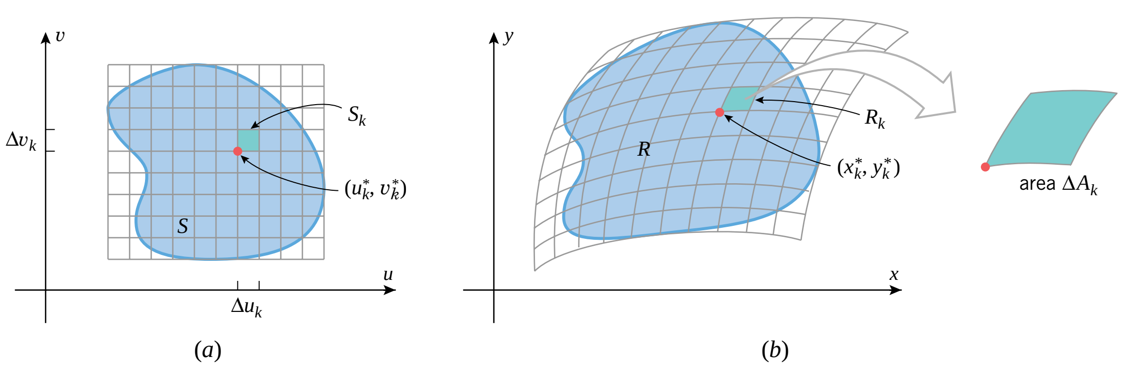


Figure 15.8.8

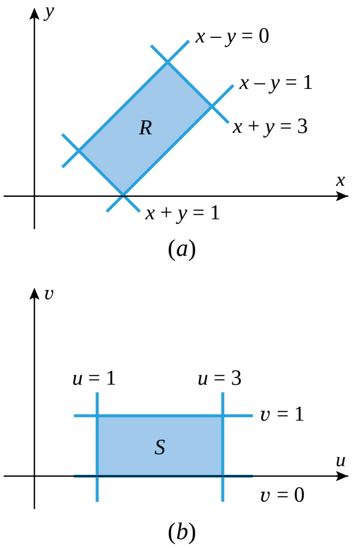


Figure 15.8.9

where R is the region enclosed by the lines $x-y=0$, $x-y=1$, $x+y=1$, and $x+y=3$ (Figure 15.8.9*a*).

Solution. This integral would be tedious to evaluate directly because the region R is oriented in such a way that we would have to subdivide it and integrate over each part separately. However, the occurrence of the expressions $x-y$ and $x+y$ in the equations of the boundary suggests that the transformation

$$u=x+y, \quad v=x-y \tag{10}$$

would be helpful, since with this transformation the boundary lines

$$x+y=1, \quad x+y=3, \quad x-y=0, \quad x-y=1$$

are constant u-curves and constant v-curves corresponding to the lines

$$u=1, \quad u=3, \quad v=0, \quad v=1$$

in the uv-plane. These lines enclose the rectangular region S shown in Figure 15.8.9*b*. To find the Jacobian $\partial(x,y)/\partial(u,v)$ of this transformation, we first solve (10) for x and y in terms of u and v. This yields

$$x=\tfrac{1}{2}(u+v), \quad y=\tfrac{1}{2}(u-v)$$

from which we obtain

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2}\end{vmatrix}=-\tfrac{1}{4}-\tfrac{1}{4}=-\tfrac{1}{2}$$

Thus, from Formula (9), but with the notation dA rather than dA_{xy},

$$\begin{aligned}\iint_R \frac{x-y}{x+y}\,dA &= \iint_S \frac{v}{u}\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA_{uv}\\ &= \iint_S \frac{v}{u}\left|-\frac{1}{2}\right| dA_{uv} = \frac{1}{2}\int_0^1\int_1^3 \frac{v}{u}\,du\,dv\\ &= \frac{1}{2}\int_0^1 v\ln|u|\bigg]_{u=1}^{3} dv\\ &= \frac{1}{2}\ln 3\int_0^1 v\,dv = \frac{1}{4}\ln 3\end{aligned}$$

◀

REMARK. In retrospect, the underlying idea illustrated in this example is to find a one-to-one transformation that maps a rectangle S in the uv-plane into the region R of integration, and then use that transformation as a substitution in the integral to produce an equivalent integral over S.

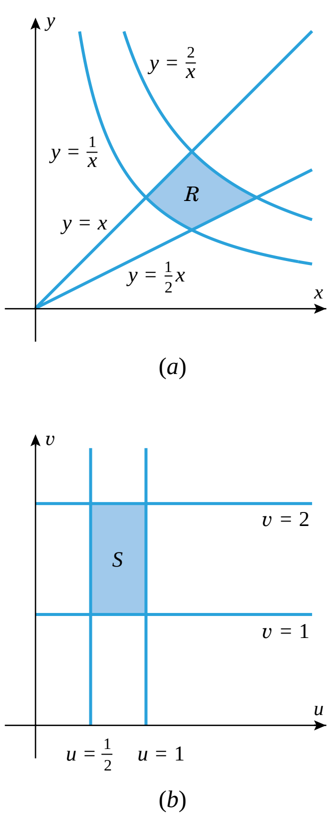


Figure 15.8.10

Example 3 Evaluate

$$\iint_R e^{xy}\,dA$$

where R is the region enclosed by the lines $y=\frac{1}{2}x$ and $y=x$ and the hyperbolas $y=1/x$ and $y=2/x$ (Figure 15.8.10*a*).

Solution. As in the last example, we look for a transformation in which the boundary curves in the xy-plane become constant v-curves and constant u-curves. For this purpose we rewrite the four boundary curves as

$$\frac{y}{x}=\frac{1}{2}, \quad \frac{y}{x}=1, \quad xy=1, \quad xy=2$$

which suggests the transformation

$$u = \frac{y}{x}, \quad v = xy \tag{11}$$

With this transformation the boundary curves in the *xy*-plane are constant u-curves and constant v-curves corresponding to the lines

$$u = \tfrac{1}{2}, \quad u = 1, \quad v = 1, \quad v = 2$$

in the uv-plane. These lines enclose the region S shown in Figure 15.8.10*b*. To find the Jacobian $\partial(x, y)/\partial(u, v)$ of this transformation, we first solve (11) for x and y in terms of u and v. This yields

$$x = \sqrt{v/u}, \quad y = \sqrt{uv}$$

from which we obtain

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\dfrac{1}{2u}\sqrt{\dfrac{v}{u}} & \dfrac{1}{2\sqrt{uv}} \\[2mm] \dfrac{1}{2}\sqrt{\dfrac{v}{u}} & \dfrac{1}{2}\sqrt{\dfrac{u}{v}} \end{vmatrix} = -\frac{1}{4u} - \frac{1}{4u} = -\frac{1}{2u}$$

Thus, from Formula (9), but with the notation dA rather than dA_{xy},

$$\begin{aligned} \iint_R e^{xy}\, dA &= \iint_S e^v \left|-\frac{1}{2u}\right| dA_{uv} = \frac{1}{2}\iint_S \frac{1}{u} e^v\, dA_{uv} \\ &= \frac{1}{2}\int_1^2 \int_{1/2}^1 \frac{1}{u} e^v\, du\, dv = \frac{1}{2}\int_1^2 e^v \ln|u| \Big]_{u=1/2}^{1} dv \\ &= \frac{1}{2}\ln 2 \int_1^2 e^v\, dv = \frac{1}{2}(e^2 - e)\ln 2 \end{aligned}$$

◀

CHANGE OF VARIABLES IN TRIPLE INTEGRALS

Equations of the form

$$x = x(u, v, w), \quad y = y(u, v, w), \quad z = z(u, v, w) \tag{12}$$

define a ***transformation*** T from uvw-space to *xyz*-space. Just as a transformation $x = x(u, v)$, $y = y(u, v)$ in two variables maps small rectangles in the uv-plane into curvilinear parallelograms in the *xy*-plane, so (12) maps small rectangular parallelepipeds in uvw-space into curvilinear parallelepipeds in *xyz*-space (Figure 15.8.11). The definition of the Jacobian of (12) is similar to Definition 15.8.1.

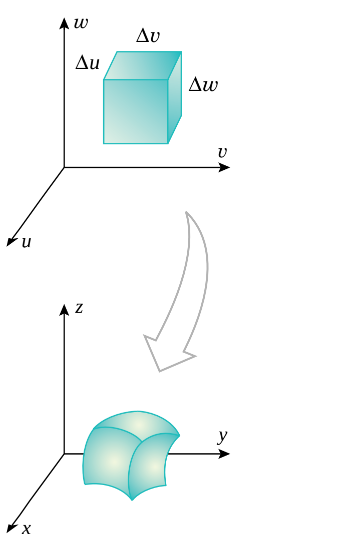


Figure 15.8.11

15.8.3 DEFINITION. If T is the transformation from uvw-space to *xyz*-space defined by the equations $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$, then the ***Jacobian of T*** is denoted by $J(u, v, w)$ or $\partial(x, y, z)/\partial(u, v, w)$ and is defined by

$$J(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\[2mm] \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}$$

For small values of Δu, Δv, and Δw, the volume ΔV of the curvilinear parallelepiped in Figure 15.8.11 is related to the volume $\Delta u\, \Delta v\, \Delta w$ of the rectangular parallelepiped by

$$\Delta V \approx \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| \Delta u\, \Delta v\, \Delta w \tag{13}$$

which is the analog of Formula (8). Using this relationship and an argument similar to the one that led to Formula (9), we can obtain the following result.

15.8.4 CHANGE-OF-VARIABLES FORMULA FOR TRIPLE INTEGRALS. If the transformation $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$ maps the region S in uvw-space into the region R in xyz-space, and if the Jacobian $\partial(x, y, z)/\partial(u, v, w)$ is nonzero and does not change sign on S, then with appropriate restrictions on the transformation and the regions it follows that

$$\iiint_R f(x, y, z)\, dV_{xyz} = \iiint_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| dV_{uvw} \tag{14}$$

Example 4 Find the volume of the region G enclosed by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Solution. The volume V is given by the triple integral

$$V = \iiint_G dV$$

To evaluate this integral, we make the change of variables

$$x = au, \quad y = bv, \quad z = cw \tag{15}$$

which maps the region S in uvw-space enclosed by a sphere of radius 1 into the region G in xyz-space. This can be seen from (15) by noting that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{becomes} \quad u^2 + v^2 + w^2 = 1$$

The Jacobian of (15) is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

Thus, from Formula (14), but with the notation dV rather than dV_{xyz},

$$V = \iiint_G dV = \iiint_S \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| dV_{uvw} = abc \iiint_S dV_{uvw}$$

The last integral is the volume enclosed by a sphere of radius 1, which we know to be $\frac{4}{3}\pi$. Thus, the volume enclosed by the ellipsoid is $V = \frac{4}{3}\pi abc$. ◀

Jacobians also arise in converting triple integrals in rectangular coordinates to iterated integrals in cylindrical and spherical coordinates. For example, we will ask you to show in Exercise 46 that the Jacobian of the transformation

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

is

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = r$$

and the Jacobian of the transformation

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$$

is

$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \rho^2 \sin\phi$$

Thus, Formulas (6) and (10) of Section 15.7 can be expressed in terms of Jacobians as

$$\iiint\limits_G f(x, y, z)\, dV = \iiint\limits_{\substack{\text{appropriate}\\ \text{limits}}} f(r\cos\theta, r\sin\theta, z)\frac{\partial(x, y, z)}{\partial(r, \theta, z)} dz\, dr\, d\theta \tag{16}$$

$$\iiint\limits_G f(x, y, z)\, dV = \iiint\limits_{\substack{\text{appropriate}\\ \text{limits}}} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} d\rho\, d\phi\, d\theta \tag{17}$$

REMARK. The absolute value signs are omitted in these formulas because the Jacobians are nonnegative (see the restrictions in Table 12.8.1).

Exercise Set 15.8

In Exercises 1–4, find the Jacobian $\partial(x, y)/\partial(u, v)$.

1. $x = u + 4v,\ y = 3u - 5v$

2. $x = u + 2v^2,\ y = 2u^2 - v$

3. $x = \sin u + \cos v,\ y = -\cos u + \sin v$

4. $x = \dfrac{2u}{u^2 + v^2},\ y = -\dfrac{2v}{u^2 + v^2}$

In Exercises 5–8, solve for x and y in terms of u and v, and then find the Jacobian $\partial(x, y)/\partial(u, v)$.

5. $u = 2x - 5y,\ v = x + 2y$

6. $u = e^x,\ v = ye^{-x}$

7. $u = x^2 - y^2,\ v = x^2 + y^2 \quad (x > 0, y > 0)$

8. $u = xy,\ v = xy^3 \quad (x > 0, y > 0)$

In Exercises 9–12, find the Jacobian $\partial(x, y, z)/\partial(u, v, w)$.

9. $x = 3u + v,\ y = u - 2w,\ z = v + w$

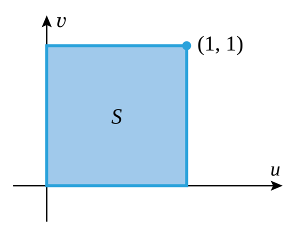
10. $x = u - uv,\ y = uv - uvw,\ z = uvw$

11. $u = xy,\ v = y,\ w = x + z$

12. $u = x + y + z,\ v = x + y - z,\ w = x - y + z$

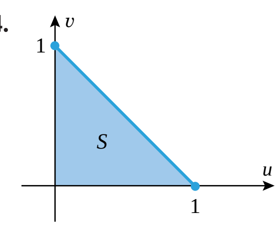
In Exercises 13–16, sketch the image in the xy-plane of the set S under the given transformation.

13.



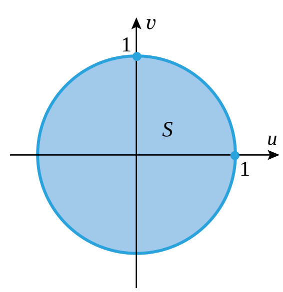
$x = u^2 - v^2$
$y = 2uv$

14.



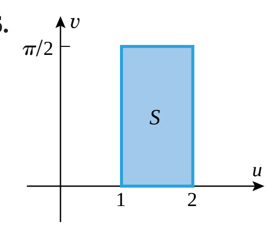
$x = 3u + 4v$
$y = 4u$

15.



$x = 2u$
$y = 3v$

16.



$x = u\cos v$
$y = u\sin v$

17. Use the transformation $u = x - 2y$, $v = 2x + y$ to find

$$\iint\limits_R \frac{x - 2y}{2x + y}\, dA$$

where R is the rectangular region enclosed by the lines $x - 2y = 1$, $x - 2y = 4$, $2x + y = 1$, $2x + y = 3$.

18. Use the transformation $u = x + y$, $v = x - y$ to find
$$\iint_R (x - y)e^{x^2 - y^2}\, dA$$
over the rectangular region R enclosed by the lines $x + y = 0$, $x + y = 1$, $x - y = 1$, $x - y = 4$.

19. Use the transformation $u = \frac{1}{2}(x + y)$, $v = \frac{1}{2}(x - y)$ to find
$$\iint_R \sin \tfrac{1}{2}(x + y) \cos \tfrac{1}{2}(x - y)\, dA$$
over the triangular region R with vertices $(0, 0)$, $(2, 0)$, $(1, 1)$.

20. Use the transformation $u = y/x$, $v = xy$ to find
$$\iint_R xy^3\, dA$$
over the region R in the first quadrant enclosed by $y = x$, $y = 3x$, $xy = 1$, $xy = 4$.

The transformation $x = au$, $y = bv$ $(a > 0, b > 0)$ can be rewritten as $x/a = u$, $y/b = v$, and hence it maps the circular region
$$u^2 + v^2 \le 1$$
into the elliptical region
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$$
In Exercises 21–24, perform the integration by transforming the elliptical region of integration into a circular region of integration and then evaluating the transformed integral in polar coordinates.

21. $\displaystyle\iint_R \sqrt{16x^2 + 9y^2}\, dA$, where R is the region enclosed by the ellipse $(x^2/9) + (y^2/16) = 1$.

22. $\displaystyle\iint_R e^{-(x^2+4y^2)}\, dA$, where R is the region enclosed by the ellipse $(x^2/4) + y^2 = 1$.

23. $\displaystyle\iint_R \sin(4x^2 + 9y^2)\, dA$, where R is the region in the first quadrant enclosed by the ellipse $4x^2 + 9y^2 = 1$ and the coordinate axes.

24. Show that the area of the ellipse
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
is πab.

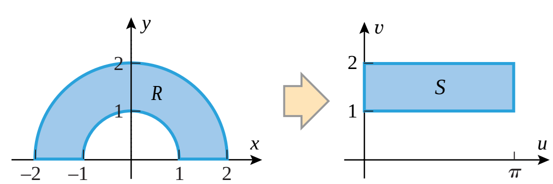
If a, b, and c are positive constants, then the transformation $x = au$, $y = bv$, $z = cw$ can be rewritten as $x/a = u$, $y/b = v$, $z/c = w$, and hence it maps the spherical region
$$u^2 + v^2 + w^2 \le 1$$
into the ellipsoidal region
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$$
In Exercises 25 and 26, perform the integration by transforming the ellipsoidal region of integration into a spherical region of integration and then evaluating the transformed integral in spherical coordinates.

25. $\displaystyle\iiint_G x^2\, dV$, where G is the region enclosed by the ellipsoid $9x^2 + 4y^2 + z^2 = 36$.

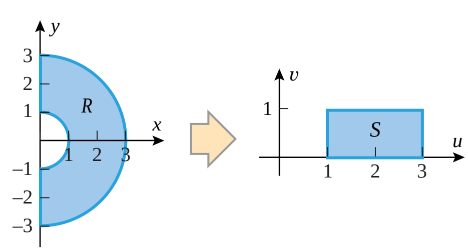
26. Find the moment of inertia about the x-axis of the solid ellipsoid bounded by
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$
given that $\delta(x, y, z) = 1$. [See the definition preceding Exercise 39 of Section 15.7.]

In Exercises 27–30, find a transformation
$$u = f(x, y),\ v = g(x, y)$$
that when applied to the region R in the xy-plane has as its image the region S in the uv-plane.

27.



28.



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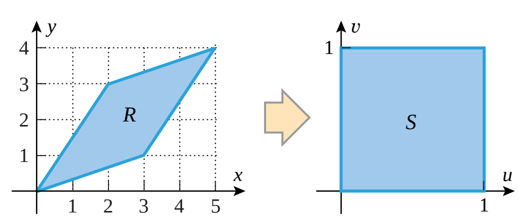
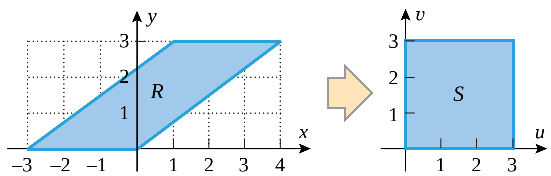


30.



In Exercises 31–34, evaluate the integral by making an appropriate change of variables.

31. $\displaystyle\iint_R \frac{y-4x}{y+4x}\,dA$, where R is the region enclosed by the lines $y = 4x$, $y = 4x + 2$, $y = 2 - 4x$, $y = 5 - 4x$.

32. $\displaystyle\iint_R (x^2 - y^2)\,dA$, where R is the rectangular region enclosed by the lines $y = -x$, $y = 1 - x$, $y = x$, $y = x + 2$.

33. $\displaystyle\iint_R \frac{\sin(x-y)}{\cos(x+y)}\,dA$, where R is the triangular region enclosed by the lines $y = 0$, $y = x$, $x + y = \pi/4$.

34. $\displaystyle\iint_R e^{(y-x)/(y+x)}\,dA$, where R is the region in the first quadrant enclosed by the trapezoid with vertices $(0, 1)$, $(1, 0)$, $(0, 4)$, $(4, 0)$.

35. Use an appropriate change of variables to find the area of the region in the first quadrant enclosed by the curves $y = x$, $y = 2x$, $x = y^2$, $x = 4y^2$.

36. Use an appropriate change of variables to find the volume of the solid bounded above by the plane $x + y + z = 9$, below by the xy-plane, and laterally by the elliptic cylinder $4x^2 + 9y^2 = 36$. [*Hint:* Express the volume as a double integral in xy-coordinates, then use polar coordinates to evaluate the transformed integral.]

37. Use the transformation $u = x$, $v = z - y$, $w = xy$ to find

$$\iiint_G (z-y)^2 xy\,dV$$

where G is the region enclosed by the surfaces $x = 1$, $x = 3$, $z = y$, $z = y + 1$, $xy = 2$, $xy = 4$.

38. Use the transformation $u = xy$, $v = yz$, $w = xz$ to find the volume of the region in the first octant that is enclosed by the hyperbolic cylinders $xy = 1$, $xy = 2$, $yz = 1$, $yz = 3$, $xz = 1$, $xz = 4$.

39. (a) Verify that

$$\begin{vmatrix} a_1 & b_1 \\ c_1 & d_1 \end{vmatrix}\begin{vmatrix} a_2 & b_2 \\ c_2 & d_2 \end{vmatrix} = \begin{vmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{vmatrix}$$

(b) If $x = x(u, v)$, $y = y(u, v)$ is a one-to-one transformation, then $u = u(x, y)$, $v = v(x, y)$. Assuming the necessary differentiability, use the result in part (a) and the chain rule to show that

$$\frac{\partial(x, y)}{\partial(u, v)} \cdot \frac{\partial(u, v)}{\partial(x, y)} = 1$$

40. In each part, confirm that the formula obtained in part (b) of Exercise 39 holds for the given transformation.
(a) $x = u - uv$, $y = uv$
(b) $x = uv$, $y = v^2$ $(v > 0)$
(c) $x = \frac{1}{2}(u^2 + v^2)$, $y = \frac{1}{2}(u^2 - v^2)$ $(u > 0, v > 0)$

The formula obtained in part (b) of Exercise 39 is useful in integration problems where it is inconvenient or impossible to solve the transformation equations $u = f(x, y)$, $y = g(x, y)$ explicitly for x and y in terms of u and v. In Exercises 41–43, use the relationship

$$\frac{\partial(x, y)}{\partial(u, v)} = 1 \Big/ \frac{\partial(u, v)}{\partial(x, y)}$$

to avoid solving for x and y in terms of u and v.

41. Use the transformation $u = xy$, $v = xy^4$ to find

$$\iint_R \sin(xy)\,dA$$

where R is the region enclosed by the curves $xy = \pi$, $xy = 2\pi$, $xy^4 = 1$, $xy^4 = 2$.

42. Use the transformation $u = x^2 - y^2$, $v = x^2 + y^2$ to find

$$\iint_R xy\,dA$$

where R is the region in the first quadrant that is enclosed by the hyperbolas $x^2 - y^2 = 1$, $x^2 - y^2 = 4$ and the circles $x^2 + y^2 = 9$, $x^2 + y^2 = 16$.

43. Use the transformation $u = xy$, $v = x^2 - y^2$ to find

$$\iint_R (x^4 - y^4)e^{xy}\,dA$$

where R is the region in the first quadrant enclosed by the hyperbolas $xy = 1$, $xy = 3$, $x^2 - y^2 = 3$, $x^2 - y^2 = 4$.

44. The three-variable analog of the formula derived in part (b) of Exercise 39 is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} \cdot \frac{\partial(u, v, w)}{\partial(x, y, z)} = 1$$

Use this result to show that the volume V of the oblique parallelepiped that is bounded by the planes $x + y + 2z = \pm 3$, $x - 2y + z = \pm 2$, $4x + y + z = \pm 6$ is $V = 16$.

45. (a) Show that if R is the triangular region with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$, then

$$\iint_R f(x+y)\,dA = \int_0^1 uf(u)\,du$$

(b) Use the result in part (a) to evaluate the integral

$$\iint_R e^{x+y}\,dA$$

46. (a) Consider the transformation $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ from cylindrical to rectangular coordinates, where $r \geq 0$. Show that

$$\frac{\partial(x, y, z)}{\partial(r, \theta, z)} = r$$

(b) Consider the transformation

$$x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi$$

from spherical to rectangular coordinates, where $0 \leq \phi \leq \pi$. Show that

$$\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \rho^2\sin\phi$$

SUPPLEMENTARY EXERCISES

C CAS

1. The double integral over a region R in the xy-plane is defined as

$$\iint_R f(x, y)\,dA = \lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*, y_k^*)\,\Delta A_k$$

Describe the procedure on which this definition is based.

2. The triple integral over a solid G in an xyz-coordinate system is defined as

$$\iiint_G f(x, y, z)\,dV = \lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\,\Delta V_k$$

Describe the procedure on which this definition is based.

3. (a) Express the area of a region R in the xy-plane as a double integral.
(b) Express the volume of a region G in an xyz-coordinate system as a triple integral.
(c) Express the area of the portion of the surface $z = f(x, y)$ that lies above the region R in the xy-plane as a double integral.

4. (a) Write down parametric equations for a sphere of radius a centered at the origin.
(b) Write down parametric equations for the right circular cylinder of radius a and height h that is centered on the z-axis, has its base in the xy-plane, and extends in the positive z-direction.

5. (a) In physical terms, what is meant by the center of gravity of a lamina?
(b) What is meant by the centroid of a lamina?
(c) Write down formulas for the coordinates of the center of gravity of a lamina in the xy-plane.
(d) Write down formulas for the coordinates of the centroid of a lamina in the xy-plane.

6. Suppose that you have a double integral over a region R in the xy-plane and you want to transform that integral into an equivalent double integral over a region S in the uv-plane. Describe the procedure you would use.

7. Let R be the region in the accompanying figure. Fill in the missing limits of integration in the iterated integral

$$\int_{\square}^{\square}\int_{\square}^{\square} f(x, y)\,dx\,dy$$

over R.

8. Let R be the region shown in the accompanying figure. Fill in the missing limits of integration in the sum of the iterated integrals

$$\int_0^2\int_{\square}^{\square} f(x, y)\,dy\,dx + \int_2^3\int_{\square}^{\square} f(x, y)\,dy\,dx$$

over R.

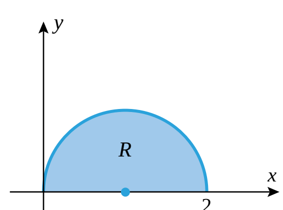


Figure Ex-7

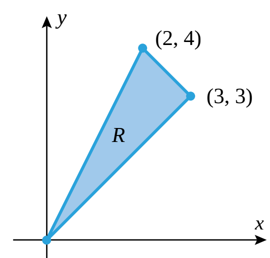


Figure Ex-8

9. (a) Find constants a, b, c, and d such that the transformation $x = au + bv$, $y = cu + dv$ maps the region S in the accompanying figure into the region R.
(b) Find the area of the parallelogram R by integrating over the region S, and check your answer using a formula from geometry.

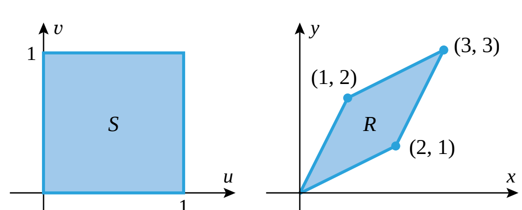


Figure Ex-9

10. Give a geometric argument to show that

$$0 < \int_0^{\pi}\int_0^{\pi} \sin\sqrt{xy}\,dy\,dx < \pi^2$$

In Exercises 11 and 12, evaluate the iterated integral.

11. $\displaystyle\int_{1/2}^{1}\int_0^{2x} \cos(\pi x^2)\,dy\,dx$

12. $\displaystyle\int_0^{2}\int_{-y}^{2y} xe^{y^3}\,dx\,dy$

In Exercises 13 and 14, express the iterated integral as an equivalent iterated integral with the order of integration reversed.

13. $\displaystyle\int_0^{2}\int_0^{x/2} e^x e^y\,dy\,dx$

14. $\displaystyle\int_0^{\pi}\int_y^{\pi} \frac{\sin x}{x}\,dx\,dy$

In Exercises 15 and 16, sketch the region whose area is represented by the iterated integral.

15. $\displaystyle\int_0^{\pi/2}\int_{\tan(x/2)}^{\sin x} dy\,dx$

16. $\displaystyle\int_{\pi/6}^{\pi/2}\int_a^{a(1+\cos\theta)} r\,dr\,d\theta \quad (a > 0)$

In Exercises 17 and 18, evaluate the double integral.

17. $\displaystyle\iint_R x^2\sin y^2\,dA$; R is the region that is bounded by $y = x^3$, $y = -x^3$, and $y = 8$.

18. $\displaystyle\iint_R (4 - x^2 - y^2)\,dA$; R is the sector in the first quadrant bounded by the circle $x^2 + y^2 = 4$ and the coordinate axes.

19. Convert to rectangular coordinates and evaluate:

$$\int_0^{\pi/2}\int_0^{2a\sin\theta} r\sin 2\theta\,dr\,d\theta$$

20. Convert to polar coordinates and evaluate:

$$\int_0^{\sqrt{2}}\int_x^{\sqrt{4-x^2}} 4xy\,dy\,dx$$

21. Convert to cylindrical coordinates and evaluate:

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{(x^2+y^2)^2}^{16} x^2\,dz\,dy\,dx$$

22. Convert to spherical coordinates and evaluate:

$$\int_0^{1}\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{1}{1 + x^2 + y^2 + z^2}\,dz\,dy\,dx$$

23. Let G be the region bounded above by the sphere $\rho = a$ and below by the cone $\phi = \pi/3$. Express

$$\iiint_G (x^2 + y^2)\,dV$$

as an iterated integral in
(a) spherical coordinates (b) cylindrical coordinates
(c) rectangular coordinates.

24. Let $G = \{(x, y, z) : x^2 + y^2 \le z \le 4x\}$. Express the volume of G as an iterated integral in
(a) rectangular coordinates (b) cylindrical coordinates.

In Exercises 25 and 26, find the area of the region using a double integral.

25. The region bounded by $y = 2x^3$, $2x + y = 4$, and the x-axis.

26. The region enclosed by the rose $r = \cos 3\theta$.

In Exercises 27 and 28, find the volume of the solid using a triple integral.

27. The solid bounded below by the cone $\phi = \pi/6$ and above by the plane $z = a$.

28. The solid enclosed between the surfaces $x = y^2 + z^2$ and $x = 1 - y^2$.

29. Find the surface area of the portion of the hyperbolic paraboloid

$$\mathbf{r}(u, v) = (u + v)\mathbf{i} + (u - v)\mathbf{j} + uv\mathbf{k}$$

for which $u^2 + v^2 \le 4$.

30. Find the surface area of the portion of the spiral ramp

$$\mathbf{r}(u, v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + v\mathbf{k}$$

for which $0 \le u \le 2$, $0 \le v \le 3u$.

In Exercises 31 and 32, find the equation of the tangent plane to the surface at the specified point.

31. $\mathbf{r} = u\mathbf{i} + v\mathbf{j} + (u^2 + v^2)\mathbf{k}$; $u = 1, v = 2$

32. $x = u\cosh v$, $y = u\sinh v$, $z = u^2$; $(-3, 0, 9)$

In Exercises 33 and 34, find the centroid of the region.

33. The region bounded by $y^2 = 4x$ and $y^2 = 8(x - 2)$.

34. The upper half of the ellipse $(x/a)^2 + (y/b)^2 = 1$.

In Exercises 35 and 36, find the centroid of the solid.

35. The solid cone with vertex $(0, 0, h)$ and with base $x^2 + y^2 \le a^2$ in the xy-plane.

36. The solid bounded by $y = x^2$, $z = 0$, and $y + z = 4$.

37. Show that

$$\int_0^{+\infty}\int_0^{+\infty} \frac{1}{(1+x^2+y^2)^2}\,dx\,dy = \frac{\pi}{4}$$

[*Hint:* See Exercise 37 of Section 15.3.]

38. It can be proved that if a bounded plane region slides along a helix in such a way that the region is always orthogonal to the helix (i.e., orthogonal to the unit tangent vector to the helix), then the volume swept out by the region is equal to the area of the region times the distance traveled by its centroid. Use this result to find the volume of the "tube" in the accompanying figure that is swept out by sliding a circle of radius $\frac{1}{2}$ along the helix .

$$x = \cos t, \quad y = \sin t, \quad z = \frac{t}{4} \qquad (0 \le t \le 4\pi)$$

in such a way that the circle is always centered on the helix and lies in the plane perpendicular to the helix.

C **39.** The accompanying figure shows the graph of an ***astroidal sphere***

$$x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$$

(a) Show that this surface can be represented parametrically as

$$\begin{aligned} x &= a(\sin\phi\cos\theta)^3 \\ y &= a(\sin\phi\cos\theta)^3 \qquad (0 \le \phi \le \pi,\ \ 0 \le \theta \le 2\pi) \\ z &= a(\cos\phi)^3 \end{aligned}$$

(b) Use a CAS to approximate the surface area in the case where $a = 1$.

(c) Use a triple integrand and the transformation

$$\begin{aligned} x &= \rho(\sin\phi\cos\theta)^3 \\ y &= \rho(\sin\phi\cos\theta)^3 \\ z &= \rho(\cos\phi)^3 \end{aligned}$$

for which $0 \le \rho \le a$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$ to find the volume of the astroidal sphere.

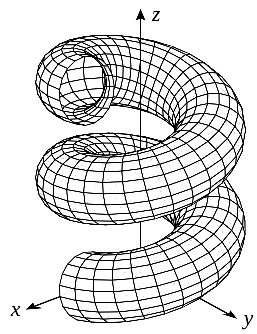


Figure Ex-38

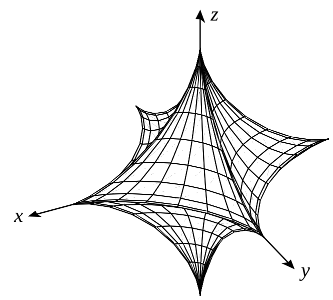


Figure Ex-39

40. Find the average distance from a point inside a sphere of radius a to the center. [See the definition preceding Exercise 25 of Section 15.5.]

C **41.** (a) Describe the surface that is represented by the parametric equations

$$\begin{aligned} x &= a\sin\phi\cos\theta \\ y &= b\sin\phi\sin\theta \qquad (0 \le \phi \le \pi,\ \ 0 \le \theta \le 2\pi) \\ z &= c\cos\phi \end{aligned}$$

where $a > 0$, $b > 0$, and $c > 0$.

(b) Use a CAS to approximate the area of the surface for $a = 2$, $b = 3$, $c = 4$.