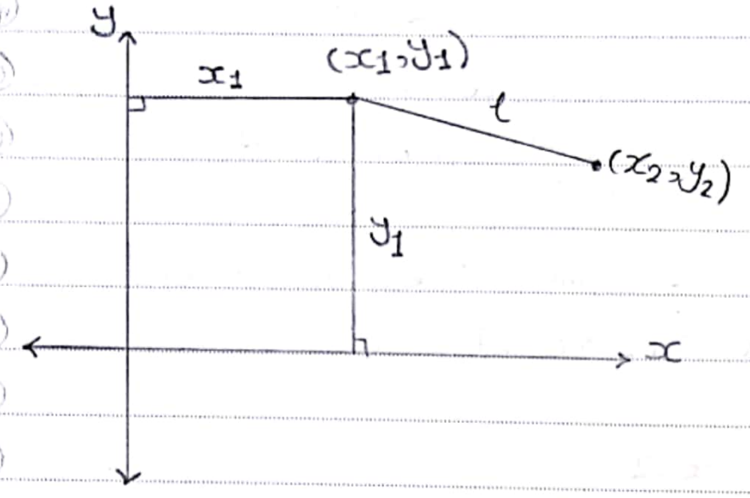
« Math Analysis » / /

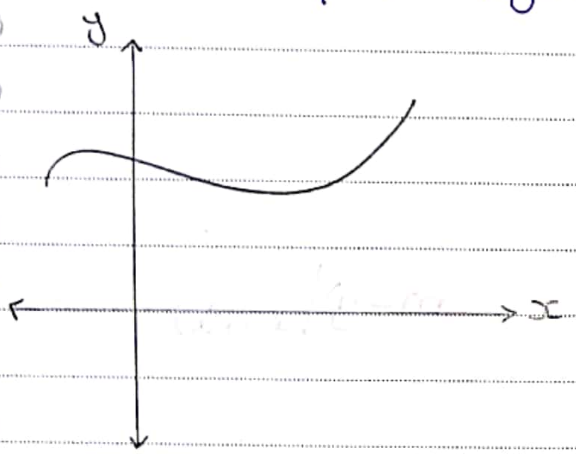
Lecture one :-

3D Geometry 3D

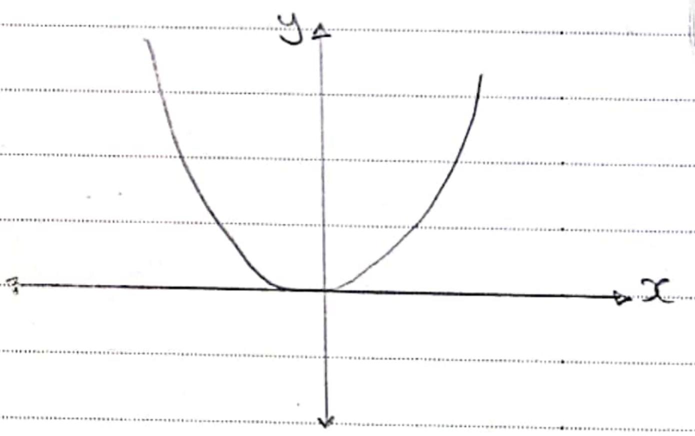


$$l = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

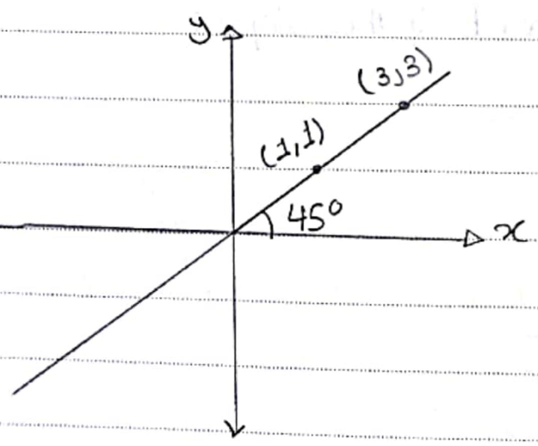
for unknown equation $\Rightarrow y = f(x)$

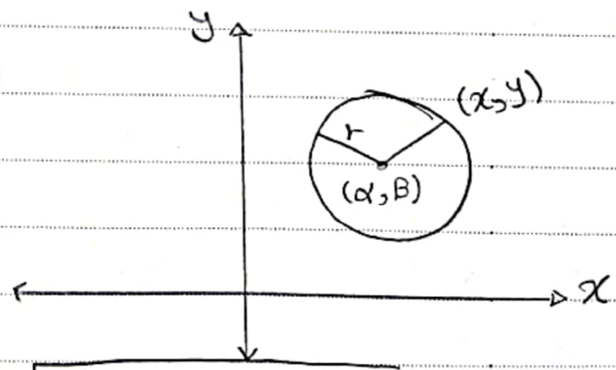


known ex: $y = x^2$



ex $y = x$

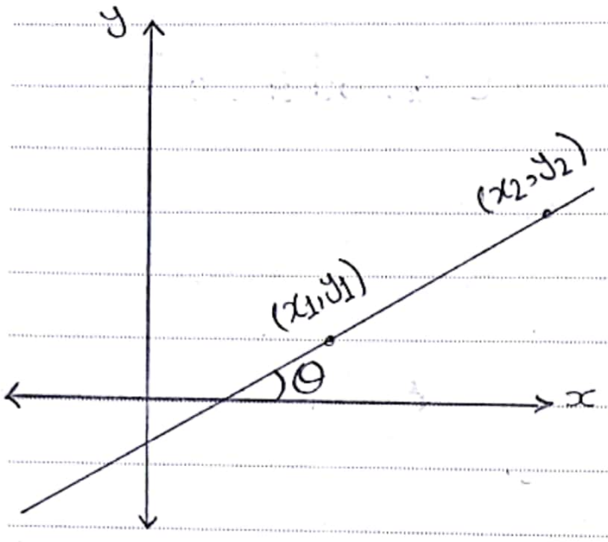




$$r = \sqrt{(x-\alpha)^2 + (y-\beta)^2}$$

$$r^2 = (x-\alpha)^2 + (y-\beta)^2$$

Slope of Striaght line:-



$$y = \underbrace{a}_{\text{Constant}}x + \underbrace{b}_{\text{Constant}}$$

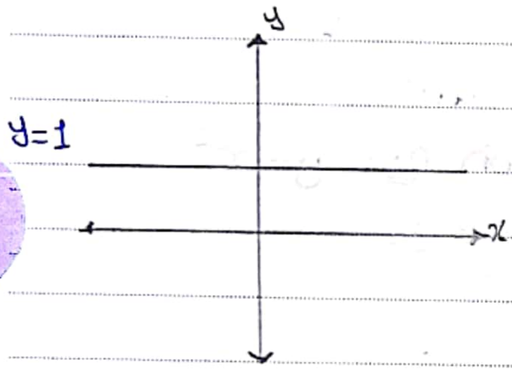
① if y alone in one Side
   Slope $= m = a$

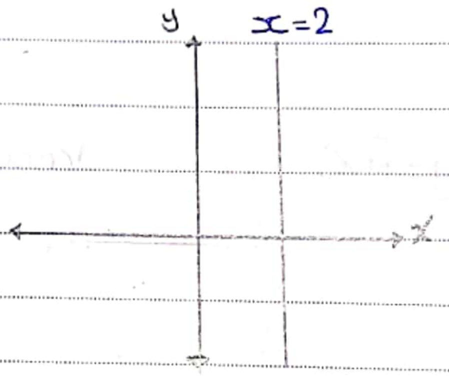
② given angle $\Rightarrow m = \tan\theta$

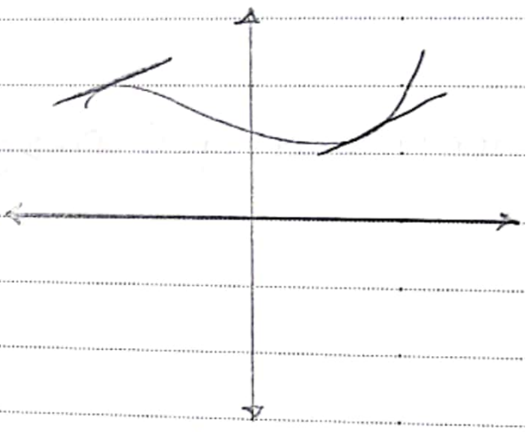
③ $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

④ $m = -$ Coeff of x / Coeff of y

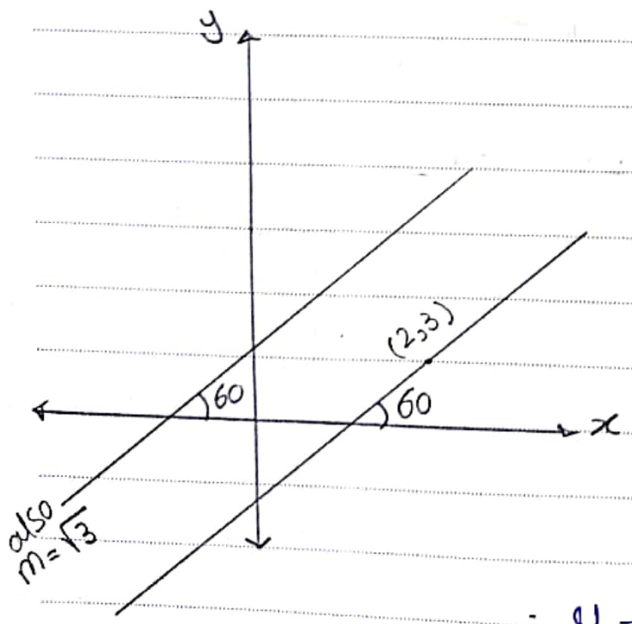
ex $2y - 3x = 7 \quad \therefore m = \dfrac{3}{2}$







(tangent) ميل المماس للمنحنى $m = y'|_{(x_0,y_0)}$



$m = \tan\theta = \tan 60 = \sqrt{3}$

Note: To get an equation we Should Know a known point & the slope:

ex $(2,3)$

$\because y = ax + b$, $a = \sqrt{3}$

from the point $(2,3) \Rightarrow 3 = \sqrt{3}\times 2 + b$

$\therefore b = 3 - 2\sqrt{3}$

$\therefore y = \sqrt{3}x + (3 - 2\sqrt{3})$ #

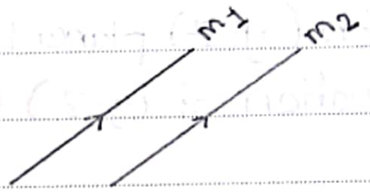
another answer :-

$$\frac{y-y_1}{x-x_1} = m \implies \frac{y-3}{x-2} = \sqrt{3} \quad \therefore y-3 = \sqrt{3}x - 2\sqrt{3}$$

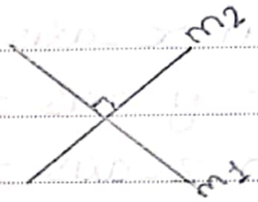
$$y = \sqrt{3}x + (3 - 2\sqrt{3})$$

Note

- In parallel $\implies m_1 = m_2$



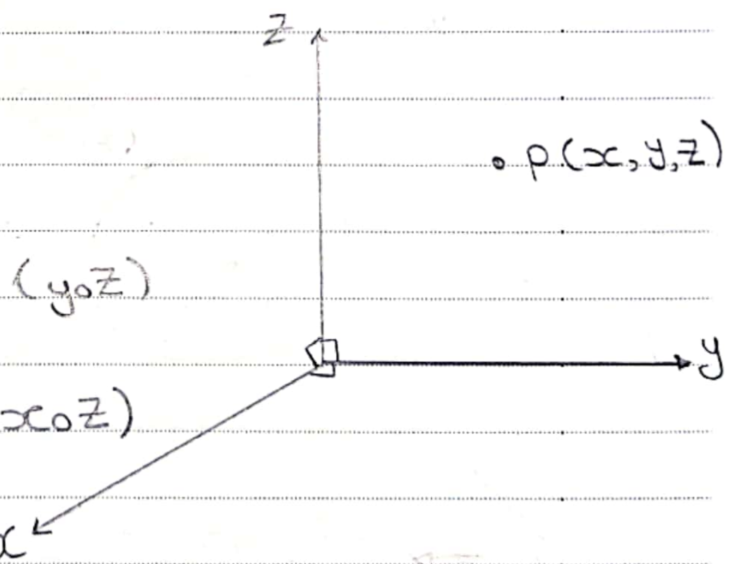
- perpendicular $\implies m_1 m_2 = -1$



On The plane :-

(x, y, z)

$2^3 = 8$ quads



$x \implies$ Distance between p & plane $(y \circ z)$

$y \implies$ Distance between p & plane $(x \circ z)$

$z \implies$ Distance between p & plane $(x \circ y)$

Notes:

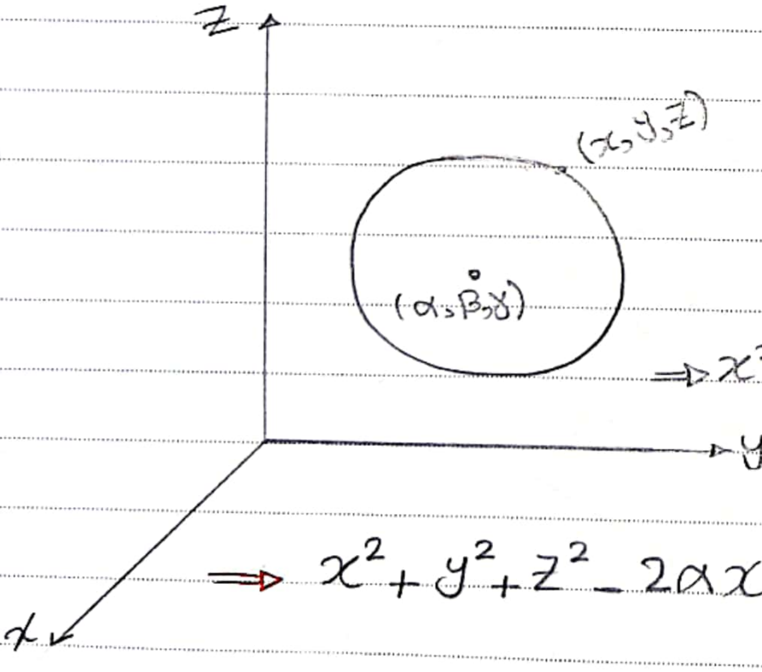
- any point on (xoy) plane has the coordinate $(x, y, 0)$
  $\therefore$ The equation of (xoy) plane. $z=0$

- any point on (xoz) plane has the Coordinate $(x, 0, z)$
  $\therefore$ The equation of $(x o z)$ plane $y=0$

- any point on (yoz) plane has the Coordinate $(0, y, z)$
  $\therefore$ The equation of (yoz) plane $x=0$

- equations on x-axis $y=0$, $z=0$ & its Coordinates $(x, 0, 0)$
- equations on y-axis $x=0$, $z=0$ & its Coordinates $(0, y, 0)$
- equations on z-axis $x=0$, $y=0$ & its Coordinates $(0, 0, z)$

Equation of Sphere:-



$$\sqrt{(x-\alpha)^2+(y-\beta)^2+(z+\gamma)^2} = r$$

by Squaring

$$(x-\alpha)^2+(y-\beta)^2+(z+\gamma)^2 = r^2$$

$$\Rightarrow x^2+y^2+z^2-2\alpha x-2\beta y-2\gamma z+\alpha^2+\beta^2+\gamma^2=r$$

at $C = \alpha^2+\beta^2+\gamma^2-r^2$

$$\Rightarrow x^2+y^2+z^2-2\alpha x-2\beta y-2\gamma z+C=0$$

centre $= (\alpha, \beta, \gamma)$

$$r = \sqrt{\alpha^2+\beta^2+\gamma^2-c}$$

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ex Find the radius & Centre of the Sphere

$$x^2+y^2+z^2+3x-2z-7=0$$

centre = $\left(\frac{-3}{2}, 0, 1\right)$

radius = $\sqrt{\left(\frac{-3}{2}\right)^2+(1)^2+7} = \frac{\sqrt{41}}{2}$

Book ⇒ Calculus 7th edition (Anton, Bivens, Davis)

Lecture two:-

The equation $x=0$ $\Rightarrow$ "in the plane" ⇝ (y-axis)

$\Rightarrow$ "in the Space" ⇝ (yoz plane)

equation of Sphere:-

$$x^2+y^2+z^2+2\alpha x+2\beta y+2\gamma z+c=0$$

centre = $(-\alpha,-\beta,-\gamma)$

radius = $\sqrt{\alpha^2+\beta^2+\gamma^2-c}$ at $c<\alpha^2+\beta^2+\gamma^2$

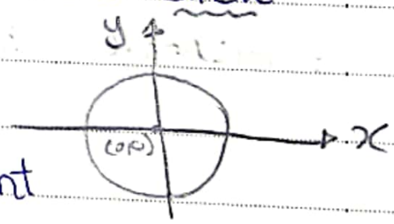
ex $x^2+y^2-z^2-2x+2y-z+10=0$

radius = $\sqrt{(1)^2+(1)^2+(\frac{1}{2})^2-10}$ =

⇝ Doesn't represent asphere

ex in the plane the equation $x^2+y^2=9$ represent circle

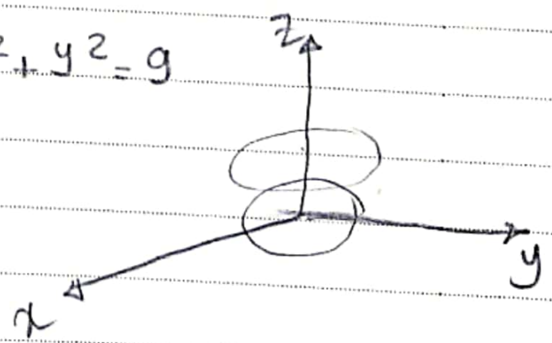
centre = $(0,0)$ radius = $\sqrt{(0)^2+(0)^2}$



in the Space the equation $x^2+y^2=9$ represent Circle Cylinder.

$z=0 \Rightarrow (xoy)$ plane ⇝ $x^2+y^2=9$

$z=1 \Rightarrow x^2+y^2=9$



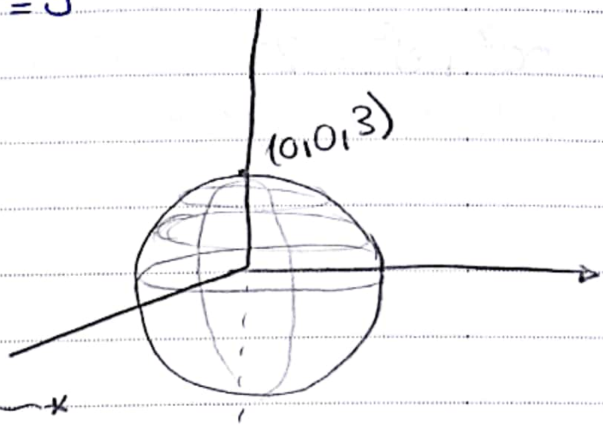
radius = 3 & with axis = z (oz)

Similary, $x^2+z^2=9$

- sketch the graph of $x^2+y^2+z^2=9$

centre $=(0,0,0)$

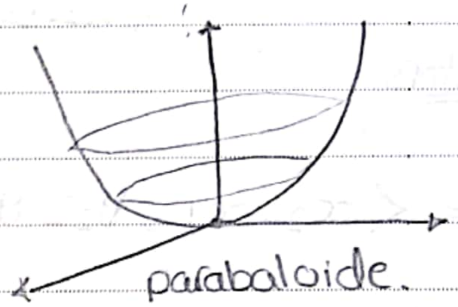
radius $=3$



- $x^2+y^2=z$

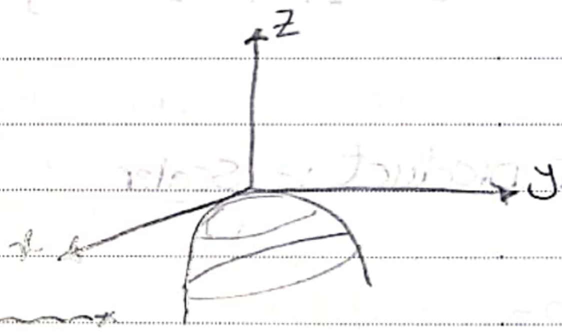
as z increase the radius increase

The z never be (-ve) so there is no graph in the (-ve) side



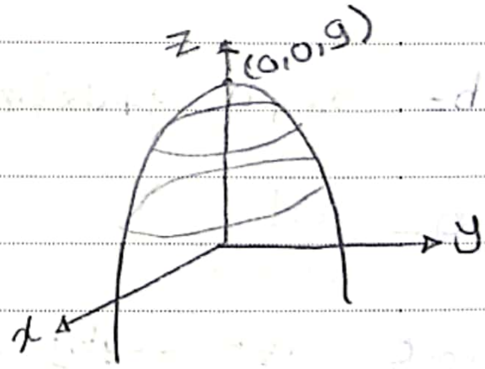
$x^2+y^2+z=0 \Rightarrow x^2+y^2=-z$

give the z (-ve) value

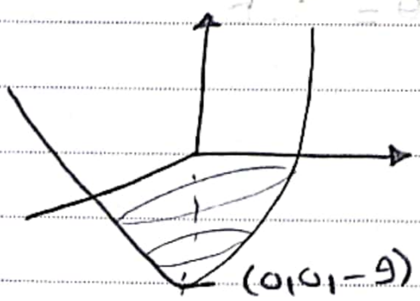


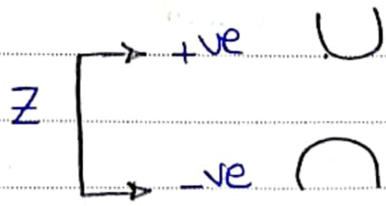
$x^2+y^2=9-z$

as z increase $\therefore$ radius decrease



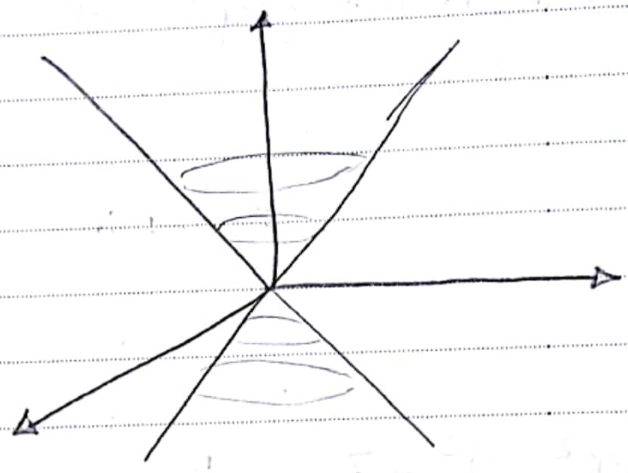
$x^2+y^2=9+z$



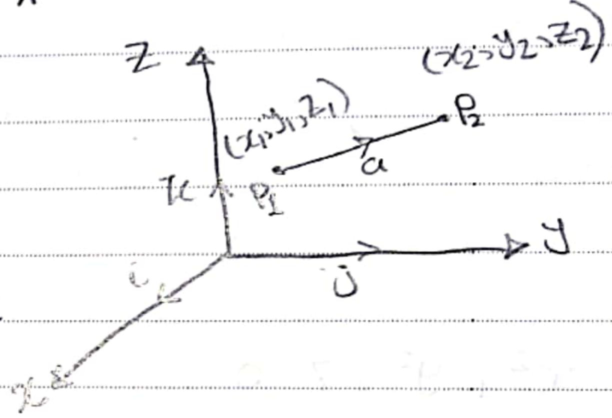


$x^2 + y^2 = z^2$

both (+ve) & (-ve) Sides



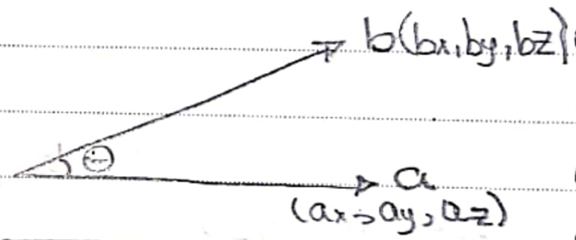
Vectors



$a = \langle x_2 - x_1 , y_2 - y_1 , z_2 - z_1 \rangle$

$a = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$

Dot product :- Scaler



$a \cdot b = \|a\| \|b\| \cos\theta$

or

$a \cdot b = a_x b_x + a_y b_y + a_z b_z$

$\|a\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$\cos\theta = \frac{a \cdot b}{\|a\| \|b\|}$

unit vector $= \hat{a} = \frac{a}{\|a\|}$ in direction of a

or $\cos\theta = \frac{a_x b_x + a_y b_y + a_z b_z}{\|a\| \|b\|}$ or $\cos\theta = \hat{a} \cdot \hat{b}$

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Cross product :- vector

$a \times b = (\|a\| \, \|b\| \sin\theta)\, \hat{n}$

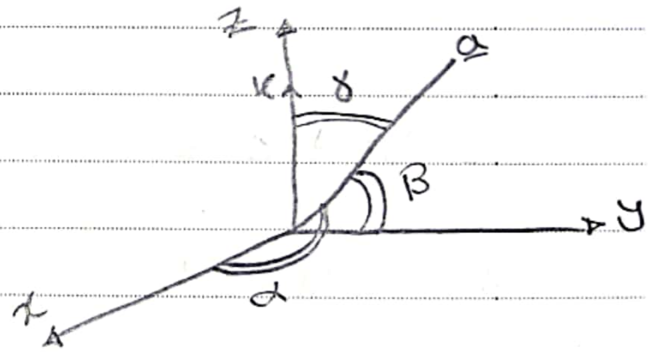
where $\hat{n}$ is a unit vector perpendicular on each of a & b

$$a \times b = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$\underline{a} = \langle a_x, a_y, a_z \rangle$
↳ direction ratios

Direction Cosine :-

⇒ The Cosine of the angles made by the vector with the 3 axes (ox, oY, oz)



$D.C = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$

$\cos\gamma = \underline{k} \cdot \langle a_x, a_y, a_z \rangle / 1 \cdot \sqrt{a_x^2 + a_y^2 + a_z^2}$

$$\cos\alpha = \frac{a_x}{\sqrt{a_x^2 + a_y^2 + a_z^2}}$$

$$\cos\beta = \frac{a_y}{\sqrt{a_x^2 + a_y^2 + a_z^2}}$$

$$\cos\gamma = \frac{a_z}{\sqrt{a_x^2 + a_y^2 + a_z^2}}$$

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

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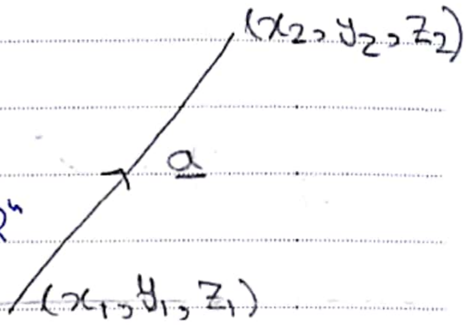
Ex find direction ratios & direction cosine of

$a = \langle 2, 3, -1 \rangle$

$D.R = \langle 2, 3, -1 \rangle$

$D.C = \left\langle \frac{2, 3, -1}{\sqrt{(2)^2 + (3)^2 + (-1)^2}} \right\rangle = \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}$

Lecture Three :-



$\bar{a} = (x_2 - x_1)\bar{i} + (y_2 - y_1)\bar{j} + (z_2 - z_1)\bar{k}$

$= \langle x_2 - x_1 , y_2 - y_1 , z_2 - z_1 \rangle$

$= \langle L , M , N \rangle \Rightarrow$ direction Ratio "D.R"

$k\bar{a} \parallel \bar{a}$

↳ difference in the length

$\| k\bar{a} \| = k \| \bar{a} \|$

Note

* Direction Ratios donot changed if it multiplied by constant

$\langle 1, 2, 3 \rangle \equiv \langle 3, 6, 15 \rangle$

Direction Cosines :-

$\langle \cos\alpha , \cos\beta , \cos\gamma \rangle \equiv \left\langle \frac{L , M , N}{\sqrt{L^2 + M^2 + N^2}} \right\rangle$

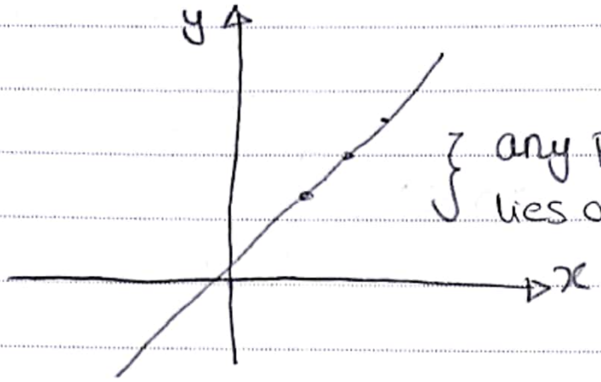
Special Case

The two vectors $a = \langle L_1 , M_1 , N_1 \rangle$, $b = \langle L_2 , M_2 , N_2 \rangle$

parallel $\rightsquigarrow$ $\frac{L_1}{L_2} = \frac{M_1}{M_2} = \frac{N_1}{N_2}$ $\rightsquigarrow$ same ratio.

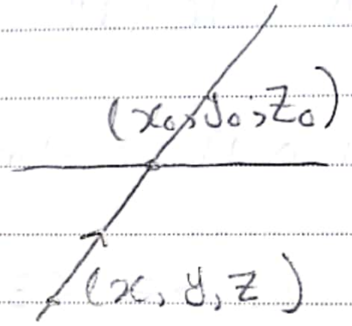
perpendicular $\rightsquigarrow$ $a \odot b = 0$ or $L_1L_2 + M_1M_2 + N_1N_2 = 0$

Equation of Striaght line :-



any point lies on the st. line $\rightarrow 3x + y = 5$

on plane :-



$\langle L, M, N \rangle \Rightarrow$ Direction vector

$\langle x - x_0, y - y_0, z - z_0 \rangle \parallel \langle L, M, N \rangle$

$$\frac{x - x_0}{L} = \frac{y - y_0}{M} = \frac{z - z_0}{N}$$ $\Rightarrow$ line equation "Symmetric form"

Ex find 3 points on the line $\frac{x-2}{3} = \frac{y-1}{4} = \frac{z+2}{5} = 3$

Sol $(2, 1, -2) \rightsquigarrow = 0$

or $= 3$

$\therefore (11, 13, 13)$

or $x = 0 \quad \therefore \frac{-2}{3} = \frac{y-1}{4} = \frac{z+2}{5}$

$\therefore y = \frac{-5}{3}, z = \frac{-16}{3}$

$\left(0, \frac{-5}{3}, \frac{-16}{3}\right)$

$$\frac{x-x_0}{L} = \frac{y-y_0}{M} = \frac{z-z_0}{N} = t$$

↳ parameter

$$\therefore \left.\begin{aligned} x &= x_0 + Lt \\ y &= y_0 + Mt \\ z &= z_0 + Nt \end{aligned}\right\} \text{parametric equation of astriaght line}$$

Notes

- as you change the parameter ⇒ different point appear.
- $\forall t \in \mathbb{R}$, we can get apoint on aline

* ~~~~~~~~ *

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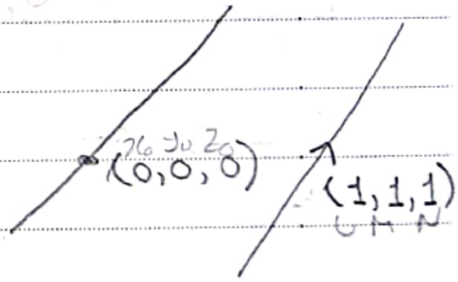
ex1

© passing through (0,0,0) & parallel <1,1,1>

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1} = t$$

$$x = y = z = t$$



Ex find The D.R & apoint on the line

$$\frac{2x}{1} = \frac{3-y}{2} = \frac{z+1}{-3} \Rightarrow \frac{x}{1/2} = \frac{y-3}{-2} = \frac{z-(-1)}{-3}$$

(annotations: $\times\frac{1}{2}$ on the first fraction, $\times -1$ on the second)

point = (0, 3, -1) & <L, M, N> = < $\frac{1}{2}$, -2, -3 >

or $\times -2$

= < -1, 4, 6 >

or $\because 2x = \frac{3-y}{2} = \frac{z+1}{-3} = t$

$\therefore x = \frac{t}{2}$ vectors

$y = 3 - 2t$ $\qquad \langle L, M, N \rangle = \langle \frac{1}{2}, -2, -3 \rangle$ ※

$z = -1 - 3t$

at $t=0 \Rightarrow (0, 3, -1)$

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Ex2 find parametric equation $P_1(2,4,-1)$ $P_2(5,0,7)$

(a)

(b) Does the line intersect the x-y plane

$$\frac{x-5}{3} = \frac{y-0}{-4} = \frac{z-7}{8}$$

$\therefore x = 5 + 3t$

$y = -4t$ $\quad$ parametric equation

$z = 7 + \underline{8t}$

↳ depend on t

on xy plane $z=0$ $\therefore 7+8t=0$ $\therefore$ at $t = \frac{-7}{8}$

$x = 5 + 3\left(\frac{-7}{8}\right) = \frac{19}{8}$

$y = -4\left(\frac{-7}{8}\right) = \frac{28}{8}$ $\qquad \Rightarrow \left(\frac{19}{8}, \frac{28}{8}, 0\right)$

Cannot multiply by any number $\Rightarrow$

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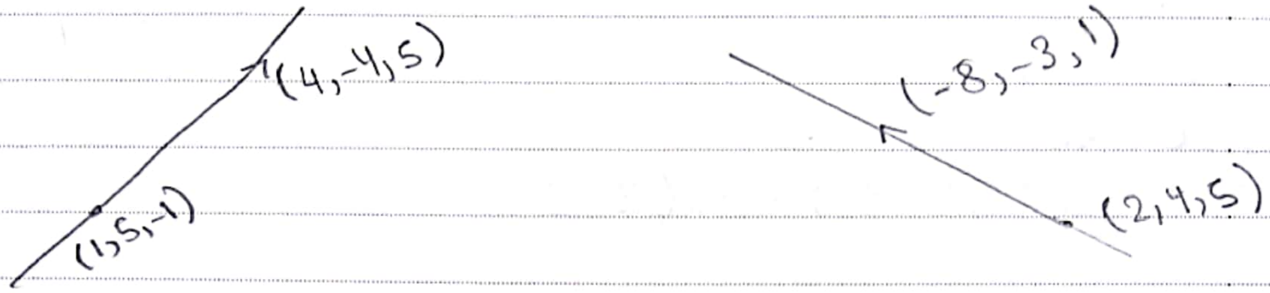
ex3

$$L_1 \begin{cases} x = 1 + 4t \\ y = 5 - 4t \\ z = -1 + 5t \end{cases} \qquad L_2 \begin{cases} x = 2 + 8t \\ y = 4 - 3t \\ z = 5 + t \end{cases}$$



$$\frac{L_2}{L_1} = \frac{-8}{4} \neq \frac{M_2}{M_1} = \frac{-3}{-4} \quad \therefore \text{" } L_1 \text{ not parallel to } L_2 \text{"}$$

- $1 + 4t = 2 - 8t \qquad t = \frac{1}{12}$
- $5 - 4t = 4 - 3t \qquad \therefore t = 1$

$\therefore t \neq t \quad \therefore$ The two lines are not intersected

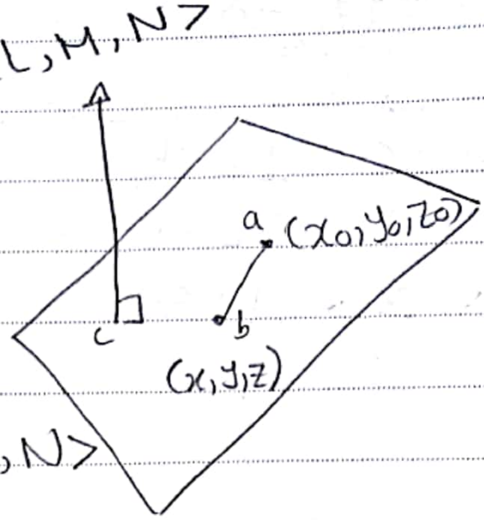
or Get the t & put it in the 6 equations if $y_1 \neq y_2$ & $z_1 \neq z_2$

$\therefore$ The two lines are not intersected

$\Rightarrow$ So the two lines are not parallel nor intersected

$\Rightarrow$ are not lie in the same plane $\Rightarrow$ Called skew lines

Equation of a plane :-



$C \perp ab$

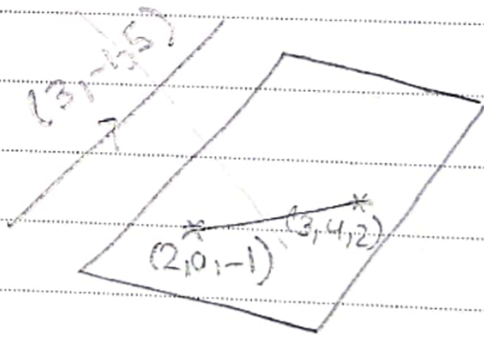
$\langle x - x_0 , y - y_0 , z - z_0 \rangle \perp \langle L, M, N \rangle$

$L(x - x_0) + M(y - y_0) + N(z - z_0)$

$Lx + My + Nz + C = 0$

as $C = Lx_0 - My_0 - Nz_0$

ex find the equation of the plane passing through $(2, 0, -1)$ $(3, 4, 2)$ & parallel $\langle 3, -1, 5 \rangle$



$\langle L, M, N \rangle \perp \langle 3, -1, 5 \rangle$

$\perp (3, 4, 2) - (2, 0, -1) = \langle 1, 4, 3 \rangle$

$3L - M + 5N = 0$

$L + 4M + 3N = 0$

$$\frac{L}{\begin{vmatrix} -1 & 5 \\ 4 & 3 \end{vmatrix}} = \frac{-M}{\begin{vmatrix} 3 & 5 \\ 1 & 3 \end{vmatrix}} = \frac{N}{\begin{vmatrix} 3 & -1 \\ 1 & 4 \end{vmatrix}}$$

$$\frac{L}{-23} = \frac{M}{-4} = \frac{N}{13}$$

$\therefore \langle L, M, N \rangle = \langle -23, -4, 13 \rangle$

$L(x - x_0) + M(y - y_0) + N(z - z_0)$

$-23(x - 2) - 4(y) + 13(z + 1) = 0$

$-23x - 4y + 13z + 59 = 0$ #

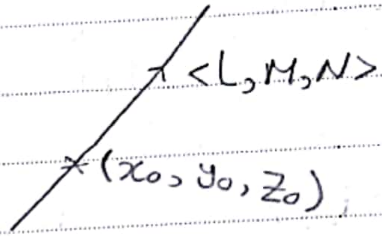
<u>or</u>

$\langle L, M, N \rangle = \langle 3, -1, 5 \rangle \times \langle 1, 4, 3 \rangle$

$$= \begin{vmatrix} i & j & k \\ 3 & -1 & 5 \\ 1 & 4 & 3 \end{vmatrix} = \langle -23, -4, 13 \rangle$$

Lecture Four:-

Striaght line equation

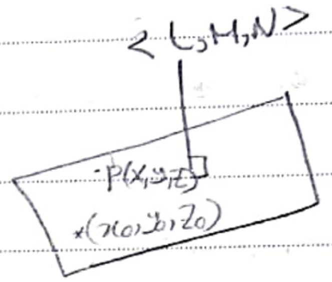


$$L = \frac{x - x_0}{L} = \frac{y - y_0}{M} = \frac{z - z_0}{N} = t$$

$x = Lt + x_0 \qquad y = Mt + y_0 \qquad z = Nt + z_0$

↳ parametric equation.

Plane:-



$$L(x - x_0) + M(y - y_0) + N(z - z_0) = 0$$

$\therefore Ax + By + Cz + D = 0$ Constants

$<A, B, C>$ are Directional Ratio of the normal

Special cases:-

① If $A_1x + B_1y + C_1z + D_1 = 0$ is parallel to $A_2x + B_2y + C_2z + D_2 = 0$

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$$

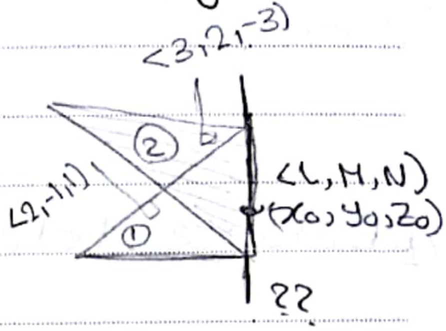
② The plane $A_1x + B_1y + C_1z + D_1 = 0$ is parallel to $A_1x + B_1y + C_1z + D_2 = 0$

ex Find aplane parallel to $x - y + z = 7$ & passing through $(2, 8, -1)$

$x - y + z = D$ at $(2, 8, -1) \Rightarrow 2 - 8 - 1 = D \quad \therefore D = -7$

$x - y + z = -7$

Ex2: Find the equation of the line $2x - y + z = 5$ ①, $3x + 2y - 3z = 8$ ②



Vector 1 ⊥ on the line

vector 2 also ⊥ on the line

- $\langle L, M, N \rangle = \langle 2,-1,1 \rangle \times \langle 3,2,-3 \rangle$

$$\begin{vmatrix} i & j & k \\ 2 & -1 & 1 \\ 3 & 2 & -3 \end{vmatrix} = \langle 1, 9, 7 \rangle \rightarrow (*)$$

At ① & ② Let $x = 0$

$-y + z = 5$ $\qquad -z = 18 \quad \therefore z = -18$

$2y - 3z = 8$ $\qquad \therefore y = -23$

Point $(0, -23, -18) \rightarrow (**)$

$x = t$

$y = -23 + 9t$

$z = -18 + 7t$

③ If the plane parallel to the axis ox Then the equation of the plane is $By + Cz + D = 0$

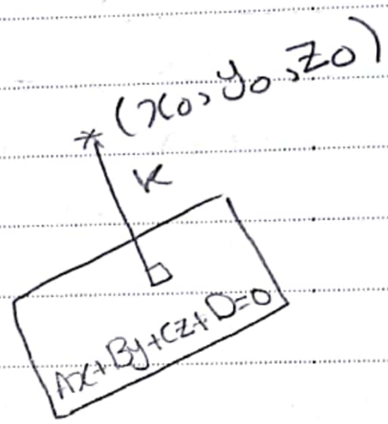
④ If the plane parallel to the axis OY

5

⑥ The plane Contains ox $\therefore By + CZ = 0$ $\because$ pass through the orgin $\therefore D = 0$

⑦ Distance between a point & plane

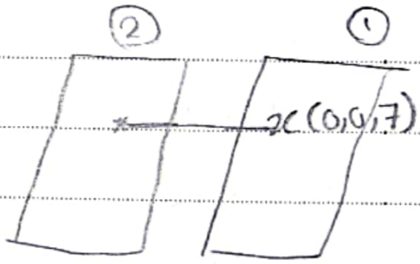
$$K = \left| \frac{Ax_0 + By_0 + CZ_0 + D}{\sqrt{A^2 + B^2 + C^2}} \right|$$



Skew $\rightarrow$ must be in different planes but not every 2 lines in different planes are Skew.

Ex ① Show that the two planes $2x - y + Z = 7$ ① $4x + 2Z = 2y + 14$ ② are parallel then find the distance between them

at $x, y = 0$ $\therefore Z = 7$ $(0, 0, 7)$



$$K = \left| \frac{4x - 2y + 2Z - 14}{\sqrt{16 + 4 + 4}} \right|$$

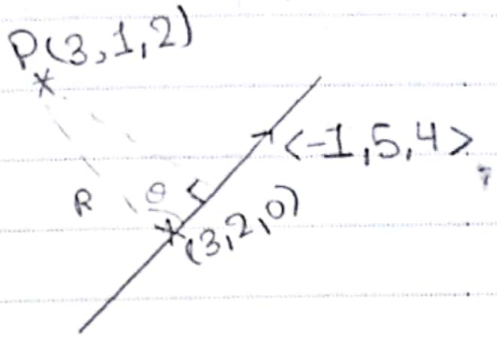
$$= \left| \frac{4\times 0 - 2\times 0 + 2\times 7 - 14}{\sqrt{16 + 4 + 14}} \right| = 0$$

$\because \frac{4}{2} = \frac{2}{1} = \frac{2}{1} \Rightarrow$ parallel

$\Rightarrow$ Same plane $\therefore$ distance $= 0$

② Find the distance from $P \equiv (3,1,2)$ to the line

$L: x = 3 - t, \quad y = 2 + 5t, \quad z = 4t$

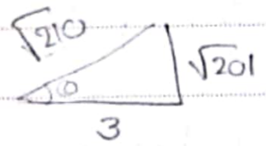


$R = \sqrt{(0)^2 + (1)^2 + (-2)^2} = \sqrt{5}$

$\cos\theta = \hat{n}_1 \cdot \hat{n}_2$

$\hat{n}_1 = Q - P = \dfrac{\langle 0, 1, -2 \rangle}{\sqrt{5}}, \quad \hat{n}_2 = \left\langle \dfrac{-1, 5, 4}{\sqrt{42}} \right\rangle$

$\cos\theta = \dfrac{5-8}{\sqrt{210}} = \dfrac{-3}{\sqrt{210}}$



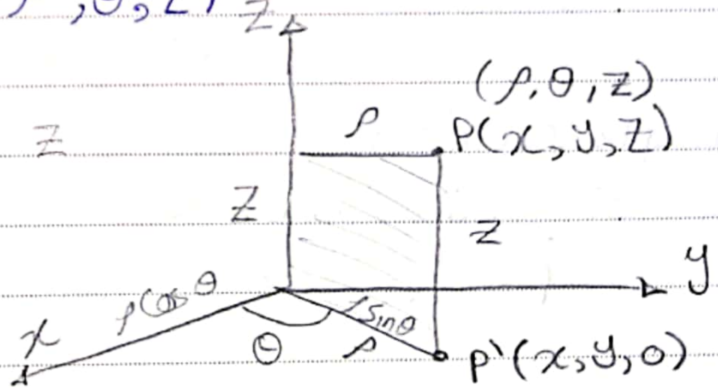
$k = R \sin\theta$

$= \sqrt{5} \cdot \left(\dfrac{\sqrt{201}}{\sqrt{210}} \right) =$

12.8

⊛ Cylinderical Coordinates (ρ, θ, z)

$\rho \longrightarrow$ distance between point & z



$x = \rho \cos\theta$

$y = \rho \sin\theta$

$z = z$

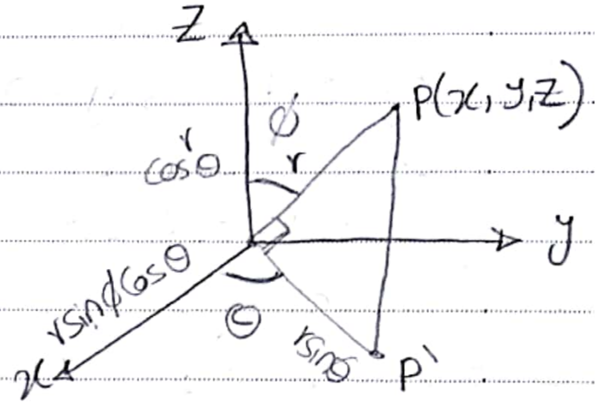
$\rho = \sqrt{x^2 + y^2}$

$\theta = \tan^{-1}\left(\dfrac{y}{x}\right) \qquad z = z$

/ /

Spherical Coordinates:-
(r, θ, ϕ)

$x = r \sin\phi \cos\theta$
$y = r \sin\phi \sin\theta$
$z = r \cos\phi$



$r = \sqrt{x^2 + y^2 + z^2}$

$\theta = \tan^{-1} \frac{y}{x}$

$\phi = \tan^{-1} \left(\frac{\sqrt{x^2+y^2}}{z} \right)$

ex find the equation of the sphere $x^2 + y^2 + z^2 = 9$ in
i) Cylinderical form
$\rho^2 + z^2 = 9$

ii) Spherical form
$r^2 = 9 \quad \therefore r = 3$

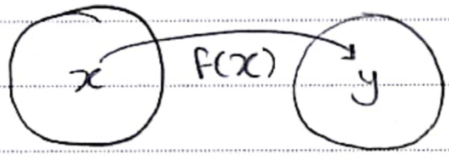
Lecture five:

Functions in more than one variable

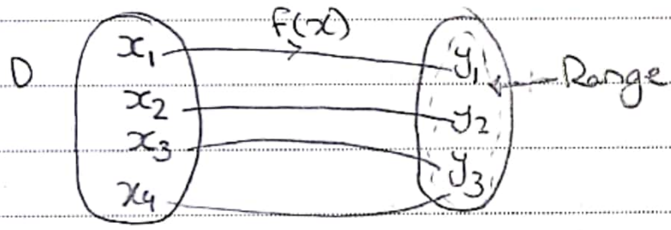
Function: $\overset{y}{f(x)} = 3x^2 - 2$

x ↳ independent variable

$f(x)$ ↳ dependent variable



domain Co-domain



Range ⊂ Codomain

$\forall x \in D \ \exists y \in R \ni y = f(x)$

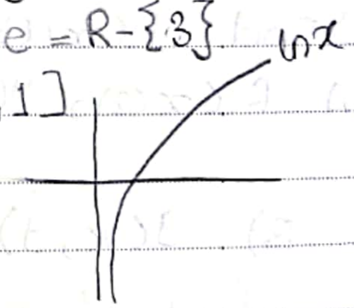
$u(x) = \ln x + 3$ Domain $=]0, \infty[$, Range $= \mathbb{R}$

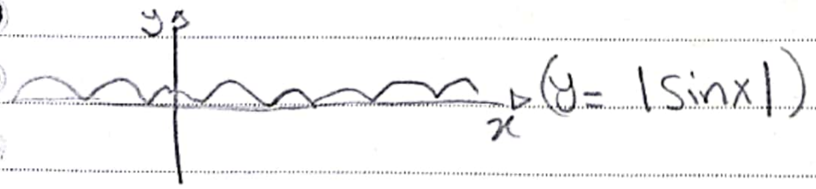
$h(x) = \frac{2}{x} + 3$ Domain $= R - \{0\}$, Range $= R - \{3\}$

$r(x) = \sin x$ Domain $= \mathbb{R}$, Range $= [-1, 1]$

$g(x) = \sqrt{x-1}$ Domain $= [1, \infty[$, Range $= [0, \infty[$

$f(x) = |\sin x|$ Domain $= R$, Range $= [0, 1]$





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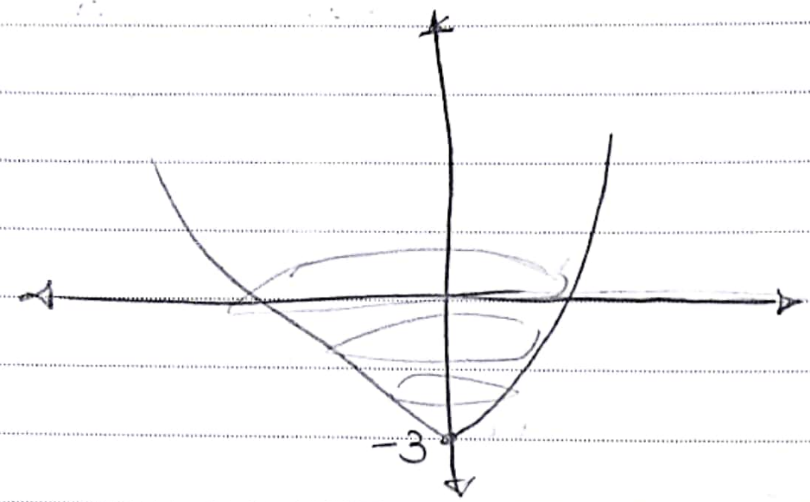
$z = f(x, y) = x^2 + y^2 - 3$

$x^2 + y^2 = (z+3) = 0 \qquad z = -3$



ex 2 p. 931

$f(x, y) = x^2 y + 1$

i) Find $f(2, 1) = 2^2 + 1 + 1 = 5$

(iii) $F(ab, a-b) = (ab)^2 (a-b) + 1$

7) $f(x, y) = x + 3x^2 y^2$

$x(t) = t^2 \; , \; y(t) = t^2$

i) Find $f(x(t), y(t)) = t^2 + 3(t^2)^2 (t^3)^2 = t^2 + 3t^{10} =$

iv) $f(x(2), y(2)) = 2^2 + 3(2)^{10}$

8) $g(x, y) = ye^{-3x} \qquad x(t) = \ln(t^2+1) \; , \; y(t) = \sqrt{t}$

Find $g(x(t), y(t)) = \sqrt{t} \; e^{-3\ln(t^2+1)}$

/ /

17) b) $f(x_1, x_2, x_3 \ldots x_n) = \sum_{k=1}^{n} k x_k = x_1 + 2x_2 + 3x_3 \ldots + n x_n$

Find $f(1,1,1,1 \ldots 1) = \sum_{k=1}^{n} k x_k = 1+2+3+4+ \ldots n$

$\frac{n(n+1)}{2}$ #

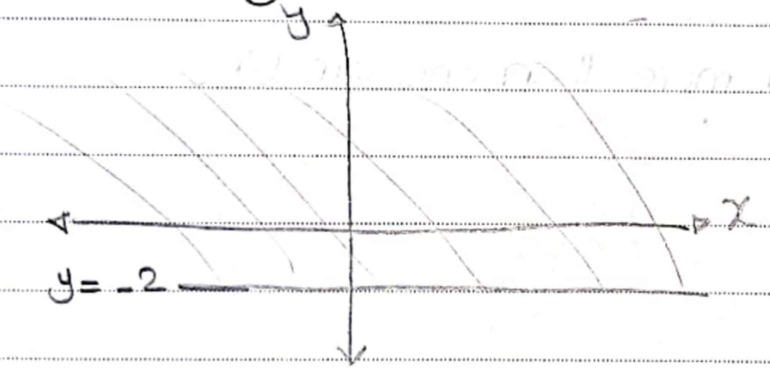
$1^2 + 2^2 + 3^2 + \ldots n^2 = \frac{n(n+1)(2n+1)}{6}$

23) $f(x,y) = x e^{-\sqrt{y+2}}$ Domain ?

$x = \mathbb{R}$ $y+2 \geq 0$

$y \geq -2 \rightsquigarrow y \in [-2, \infty[$

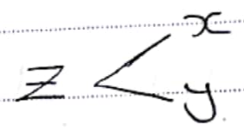
Domain $= \{ (x,y) \in \mathbb{R}^2 ; y \geq -2 \}$



Lecture six:

$$Z = f(x, y)$$

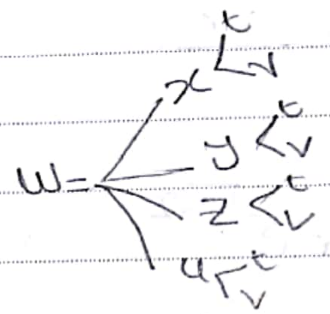
dependent variable ↑ (for Z) ; independent variable ↑ (for x, y)



$$W = F(x, y, z, u)$$

dependent ↑ (for W) ; independent ↑ (for x, y, z, u)

$$w = g(t, v)$$



Ex: $Z = (xy)^3 + 2\sin xy + 3xy + 7 \Leftarrow f(x, y)$

$Z = g(xy) = g(u) = u^3 + 2\sin u + 3u + 7$

Derivative of functions in more than one variable.

$y = f(x)$

$$y' = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

ex $f(x) = x^2 + 3$

$f(x + \Delta x) = (x + \Delta x)^2 + 3 = x^2 + 2x\Delta x + (\Delta x)^2 + 3$

$$y' = \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = \frac{\cancel{\Delta x}(2x + \Delta x)}{\cancel{\Delta x}} \quad \text{at } \Delta x = 0$$

$= 2x$

$$z = f(x, y)$$

$$\frac{\partial z}{\partial x} \quad \text{or} \quad \frac{\partial z}{\partial y}$$

$$z_x = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

or $$\lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

$$\frac{\partial^2 z}{\partial x^2} = z_{xx} = \frac{\partial z_x}{\partial x} \quad \& \quad \frac{\partial^2 z}{\partial y^2} = z_{yy} = \frac{\partial z_y}{\partial y}$$

$$\frac{\partial^2 z}{\partial x \partial y} = z_{xy} = \frac{\partial}{\partial x}(z_y)$$

$$\frac{\partial^2 z}{\partial y \partial x} = z_{yx} = \frac{\partial}{\partial y}(z_x)$$

Note $z_{xy} = z_{yx}$

Ex let $z = 2e^{xy} + 3x^y$ find z_x, z_y, z_{xy}, z_{yy} & show that $z_{xy} = z_{yx}$.

$$z_x = 2ye^{xy} + 3yx^{y-1}$$

$$z_y = 2xe^{xy} + 3x^y \ln x$$

$$z_{yx} = 2(yxe^{xy} + e^{xy}) + 3(y\,x^{y-1}\ln x + x^{y-1})$$

$$z_{xy} = 2(xye^{xy} + e^{xy}) + 3(x^y \cdot \frac{1}{x} + \ln x \; y x^{y-1})$$

$$\therefore z_{yx} = z_{xy}$$

$$z_{yy} = 2x^2e^{xy} + 3(x^y(\ln x)^2)$$

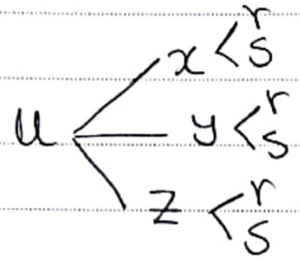
$$= 2x^2e^{xy} + 3(\ln x)^2 x^y$$

Chain Rule

①

if $u = f(x, y, z)$

$x = f_1(r, s)$, $y = f_2(r, s)$ & $z = f_3(r, s)$

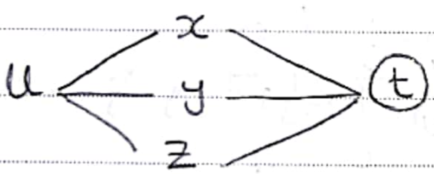


$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \Rightarrow \text{same as } y \,\&\, z$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial s} \quad \#$$

② $u = f(x, y, z)$

$x = f_1(t)$, $y = f_2(t)$ & $z = f_3(t)$



$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

Ex Let $z = x^3 + y^3$ $\quad x = u \ln v$, $y = v \ln u$

Find z_u , z_v

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$3x^2 \cdot \ln v + 3y^2 \cdot \frac{v}{u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$= 3x^2 \cdot \frac{u}{v} + 3y^2 \cdot \ln u$$

EX2

Let $Z = x^3 + y^3$ $x = u^3$, $y = \ln u$

$$\frac{\partial Z}{\partial u} = \frac{\partial Z}{\partial x} \cdot \frac{dx}{du} + \frac{\partial Z}{\partial y} \cdot \frac{dy}{du}$$

$$= 3x^2 \cdot 3u^2 + 3y^2 \cdot \frac{1}{u}$$

Explicit function :-

$$y = 3x^2 + 5\sin x + \frac{3}{\sqrt{1+x^2}}$$

at $x=2$ $\therefore 12 + 5\sin 2 + \frac{3}{\sqrt{5}}$

Implicit function :-

$$xy + 3\ln x + 5\sqrt{xy} = 7$$

at $x=2$ $\Rightarrow 2y + 3\ln 2 + 5\sqrt{2} = 7$

$$y = 7 - 3\ln 2 - 5\sqrt{2} \,/\, 2$$

- $Z = f(x, y) = 0$

$$dZ = \frac{\partial Z}{\partial x} \cdot dx + \frac{\partial Z}{\partial y} dy = 0$$

$$Z_y \, dy = - Z_x \, dx \quad \div dx \quad = \frac{dy}{dx} = \frac{-Z_x}{Z_y}$$

EX Find y' at $x \ln y = x^y - \sin(xy) = x^y - \sin(xy) - x\ln y$

$$Z_x = y x^{y-1} - y\sin(xy) - \ln y$$

$$Z_y = x^y \ln x - x\sin(xy) - \frac{x}{y}$$

$$\frac{dy}{dx} = \frac{-yx^{y-1} + y\cos(xy) + \ln y}{x^y \ln x - \cancel{x}\cos(xy) - x/y}$$

another answer

① $x\ln y = x^y - \sin(xy) = 0$

$x\left(\frac{y'}{y}\right) + \ln y = x^y\left[\frac{y}{x} + y'\ln x\right] - \cos(xy)[xy' + y]$

$= x\left(\frac{y'}{y}\right) + \ln y = x^y\left[\frac{y}{x} + y'\ln x\right] - (xy' + y)\cos(xy)$

$u = x^y$

$\ln u = y\ln x$

$\frac{u'}{u} = \frac{y}{x} + y'\ln x$

$u' = x^y\left[\frac{y}{x} + y'\ln x\right]$

/ /

Lecture 7:

Differentian of integrals

EX let $f(x) = \int_0^{x^2} \sin t \, dt$

find $f'(x)$

Sol

$f(x) = -\cos t \Big|_0^{x^2} = 1 - \cos x^2$

$\therefore f'(x) = 2x \sin x^2$

EX2 let $F(x) = \int_{\cos x}^{2x} e^{t^2} \, dt$

$\Rightarrow$ Impossible integration

$F(x) = \int_{f_1(x)}^{f_2(x)} f(x_1(t)) \, dt$ $\int_a^b f(x)\,dx = \int_a^b f(u)\,du$

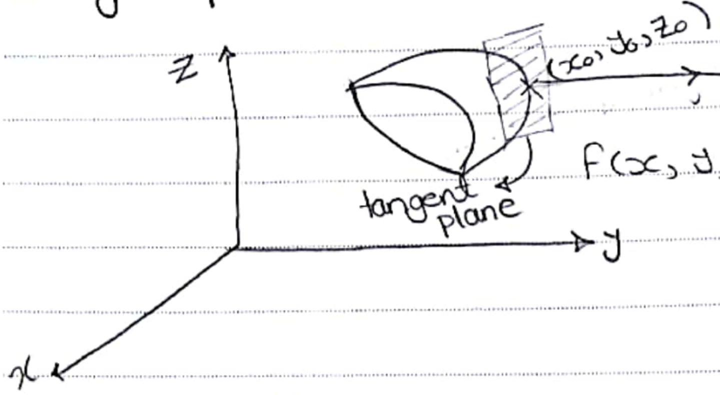
$F'(x) = \left[f(x, f_2(x)) \cdot \frac{d f_2(x)}{dx} \right] - \left[f(x, f_1(x)) \cdot \frac{d f_1(x)}{dx} \right] +$

$\int_{f_1(x)}^{f_2(x)} f_x(x,t) \, dt$

on EX2:

$F'(x) = (e^{(2x)^2} (2)) - (e^{\cos^2 x} (-\sin x)) + 0 = 2e^{4x^2} + \sin x \, e^{\cos^2 x}$

Tangent plane , Normal line :-



↳ Normal line

$f(x, y, z) = 0$ "Any surface 's equation"

Direction Ratio $\underline{a} = \langle f_x , f_y , f_z \rangle = \langle L, M, N \rangle$

$\underline{\underline{ex}}$ $f(x, y, z) = x^2 + y^2 - z + 5 = 0$

$\therefore \langle f_x , f_y , f_z \rangle = \langle 2x , 2y , -1 \rangle$

Tangent plane ⇒ "Changable"

Equation of Normal line :-

$$\frac{x - x_0}{L} = \frac{y - y_0}{M} = \frac{z - z_0}{N} = t$$

$x = x_0 + Lt$

$y = y_0 + Mt$

$z = z_0 + Nt$

Equation of tangent plane :-

$L(x - x_0) + M(y - y_0) + N(z - z_0) = 0$

/ /

Ex Find the normal line & tangent plane for the surface $xy + z^2 - 9 = 0$ at point $(0,1,3)$

$\langle L, M, N \rangle = \langle y, x, 2z \rangle = \langle 1, 0, 6 \rangle$

Normal line equation $= \dfrac{x}{1} + \dfrac{y-1}{0} + \dfrac{z-3}{6}$

Tangent equation $= 1(x) + 0(y-1) + 6(z-3) = 0$

$= x + 6z = 18$

14.7

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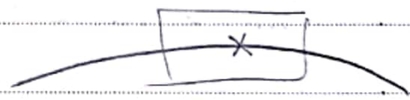
9) The tangent plane is horizontal, find all points on the surface $z = x^3 y^2$ such that

normal line $\langle L, M, N \rangle = \langle 0, 0, 1 \rangle$ (oz axis)



Let point $= (x_0, y_0, z_0)$

$\langle L, M, N \rangle = \langle f_x, f_y, f_z \rangle$

$x^3 y^2 - z = 0$

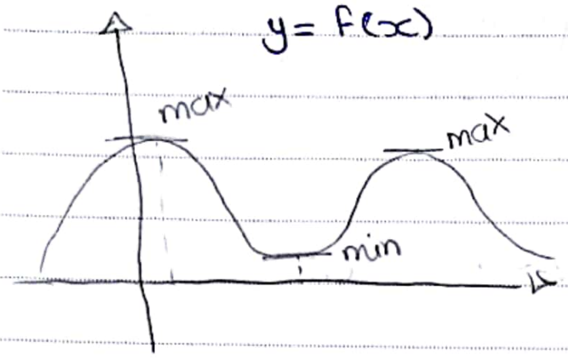
$f(x) = 3x_0^2 y_0^2$ $\quad f(y) = 2x_0^3 y_0$ $\quad f(z) = -1 \Rightarrow \langle 3x^2y^2, 2x^3y, -1 \rangle$

$3x_0^2 y_0^2 = 0$ $\quad 2x_0^3 y_0 = 0$ $\quad \Rightarrow x_0$ or $y_0 = 0$

$x_0 = 0$	$y_0 = 0$
$z_0 = 0$	$z_0 = 0$
$(0, a, 0)$	$(b, 0, 0)$

$\forall\, a, b \in \mathbb{R}$

Local Maximum & local minimum (Extreme points)



$$f''(x)\Big|_{(x_0,y_0)} \begin{cases} \to \oplus & \text{Min} \\ \to \ominus & \text{Max} \end{cases}$$

$f'(x)=0 \Rightarrow$ Critical points

let $z = f(x,y)$

to find the maximum & minimum of z

① $z_x = 0 \to ①$, $z_y = 0 \to ②$

② find them & then get the points $(x_0,y_0),(x_1,y_1)$ "Critical points"

$$\Delta = Z_{xx} Z_{yy} - (Z_{xy})^2 \to +ve < \begin{matrix} \text{max } (Z_{xx} < 0) \\ \text{min } (Z_{xx} > 0) \end{matrix}$$

$\to -ve \ \& \ 0 \to$ Saddle point

page 1002

Ex locate all relative Max, relative min & Saddle point if any (12) $f(x,y) = xy - x^3 - y^3$

$Z_x = y - 3x^2 = 0$ ①

$Z_y = x - 3y^2 = 0$ ②

$Z_{xx} = -6x$

$Z_{yy} = -6y$

$Z_{xy} = 1$

$(0,0) = (-6\times 0)(-6\times 0) - (1)^2 = -1 \ \ominus$ Saddle point

$(\frac{1}{3},\frac{1}{3}) = (-6\times\frac{1}{3})(-6\times\frac{1}{3}) - (1) = 3 \ +ve$

$Z_{xx} < 0 \ \therefore$ max.

from ① $y = 3x^2$

in ② $x - 27x^4 = 0$

$x(1-27x^3) = 0$

$x(1-3x)(1+3x+9x^2) = 0$

$x = 0$, $x = \frac{1}{3}$

$y = 0$ | $y = \frac{1}{3}$

$(0,0)$ | $(\frac{1}{3},\frac{1}{3})$

$F(x,y) = \frac{1}{3}\times\frac{1}{3} - \frac{1}{27} - \frac{1}{27}$

Lecture 8

Conditional Max & Min:-

To find the max & min of $F(x,y)$ such that (x,y) satisfy the constraint (condition) $g(x,y)=0$

In this case we introduce the equation : $\psi(x,y)$

$$\psi(x,y) = f(x,y) + \lambda g(x,y)$$

λ → Lagrange multiplier

$\psi_x = 0$ $\quad$ $\psi_y = 0$ $\quad$ $\psi_\lambda = 0$

Solve & get the critical points $(x_0, y_0, z_0), (x_1, y_1, z_1)$ ---

A, B, C --

F_A, F_B, F_C --- → Max "largest"

→ Min "smallest"

14.9

page 1010

7) $F(x,y) = 4x^3 + y^2$ $\quad$ Condition: $g(x,y) = 2x^2 + y^2 = 1$

$\psi(x,y) = f(x,y) + \lambda g(x,y)$

$(4x^3 + y^2) + \lambda(2x^2 + y^2 - 1)$

$\psi_x = 12x^2 + 4\lambda x = 0 \rightarrow ①$

$\psi_y = 2y + 2y\lambda = 0 \rightarrow ②$

$\psi_\lambda = (2x^2 + y^2 - 1) = 0 \rightarrow ③$ (Condition)

from 2 $\quad 2y(1+\lambda) = 0 \rightarrow y = 0 \quad \lambda = -1$

at 3 $\quad 2x^2 + 0 - 1 \quad \therefore 2x^2 = 1 \quad \therefore x = \sqrt{\frac{1}{2}} = \pm \frac{1}{\sqrt{2}}$

A $\left(\frac{1}{\sqrt{2}}, 0\right)$ B $\left(\frac{-1}{\sqrt{2}}, 0\right)$

at $\lambda = -1$ in 1

$12x^2 - 4x = 0$ $\quad 4x(3x-1) = 0$

$\therefore x = 0$ or $3x = 1$ $\quad \therefore x = \frac{1}{3}$

$\downarrow$ $\quad y = \pm 1$ $\quad \downarrow$ $\quad y = \pm \frac{\sqrt{7}}{3}$

$(0, 1)\ (0, -1)$ $\quad \left(\frac{1}{3}, \frac{\sqrt{7}}{3}\right) \left(\frac{1}{3}, -\frac{\sqrt{7}}{3}\right)$

C D E F

by substitution at $f(x, y) = 4x^3 + y^2$

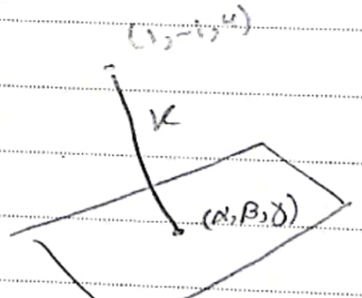
$A = \sqrt{2}$ (1.4) $\quad B = -\sqrt{2}$ (-1.4) $\quad$ c) 1 $\quad$ d) 1

$E = \frac{25}{27}$ 0.92 $\quad F = \frac{25}{27} = 0.92$

f Max $\Rightarrow A\left(\frac{1}{\sqrt{2}}, 0\right)$ $\quad f$ Min $\Rightarrow B\left(\frac{-1}{\sqrt{2}}, 0\right)$

16) Find the point on the plane $4x + 3y + z = 2$ that is closest to the point $(1, -1, 4)$

a point (α, β, γ)



مربع البعد

$K = (\alpha - 1)^2 + (\beta + 1)^2 + (\gamma - 4)^2$ min

Condition (α, β, γ) satisfy the $4x + 3y + z = 2$

$4\alpha + 3\beta + \gamma = 2$

$(\alpha-1)^2 + (\beta+1)^2 + (\gamma-4)^2 + \lambda(4\alpha + 3\beta + \gamma - 2)$

$\psi_\alpha = 2(\alpha - 1) + 4\lambda = 0$

$\psi_\beta = 2(\beta + 1) + 3\lambda = 0$ $\quad = 2\beta + 2$

$\psi_\gamma = 2(\gamma - 4) + \lambda = 0$

$\psi_\lambda = 4\alpha + 3\beta + \gamma - 2 = 0$

ALADIN

$$2\alpha - 2 + 4\lambda = 0$$

from 1 $\alpha = \frac{1}{2}(2 - 4\lambda) = \frac{7}{13}$

(2) $\beta = \frac{1}{2}(-2-3\lambda) = \frac{-35}{26}$

$\gamma = \frac{1}{2}(+8-\lambda) = \frac{101}{26}$

$x = 2(2-4\lambda) + \frac{3}{2}(-2-3\lambda) + \frac{1}{2}(-8-\lambda) - 2 = 0$

$4 - 8\lambda - 3 - \frac{9}{2}\lambda + 4 - \frac{1}{2}\lambda - 2 = 0$

$-13\lambda = -3 \quad \therefore \lambda = \frac{3}{13}$

Lecture 9

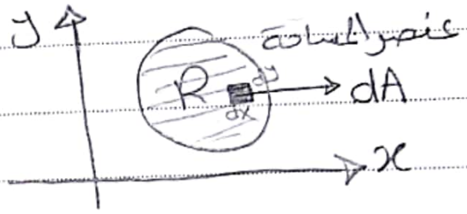
((Ch. 15))

Multiple Integral

1) $\iint$ 2) $\iiint$

Ex $\int_1^x \underset{\uparrow \text{constant}}{x}\, y^2\, dy = x\int_1^x y^2\, dy = x\left(\frac{y^3}{3}\right)_1^x = x\left[\frac{x^3}{3} - \frac{1}{3}\right]$

Double Integral: $\iint_R f(x,y)\, dA$



$dA = dx\, dy$, $dA = dy\, dx$

Ex: (2b) Page 1019

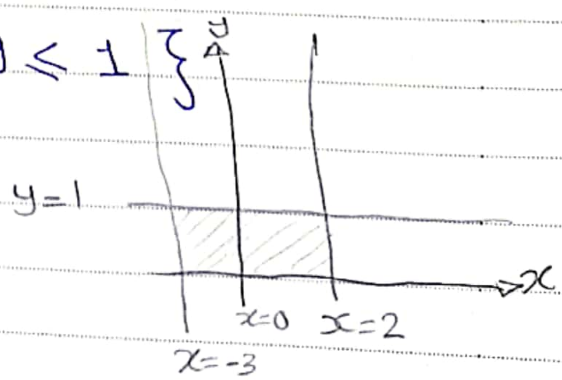
Find $\int_1^2 \int_0^3 \underbrace{(1+8xy)}_{F(x,y)} \underbrace{dx\, dy}_{dA}$

$\int_0^3 1+8xy\, dx = x + 4x^2 y \Big]_0^3 = 3 + 36y$

$\int_1^2 3+36y\, dy = 3y + 18y^2 \Big]_1^2 = (6+72)-(3+18) = 57$

EX3 Evaluate $\iint_R y^2 x\, dA$ where R is the rectangular

$\{(x,y) : -3 \leq x \leq 2 \;,\; 0 \leq y \leq 1\}$



$$\int_0^1 \int_{-3}^{2} y^2 x \, dx \, dy \quad = \frac{1}{2} x^2 y^2 \Big]_{-3}^{2} = 2y^2 - \frac{9}{2} y^2$$

$$\int_0^1 \frac{-5}{2} y^2 \, dy = \frac{-5}{2} \left(\frac{y^3}{3} \right)_0^1 = \frac{-5}{6}$$

or

$$I = \int_{-3}^{2} \int_0^1 x y^2 \, dy \, dx$$

$$= \int_{-3}^{2} x \cdot \int_0^1 y^2 \, dx \Rightarrow \int_0^1 y^2 \, dy \cdot \int_{-3}^{2} x \, dx$$

$$= \frac{1}{3} \left(\frac{-5}{2} \right) = \frac{-5}{6}$$

―――――――――――

$$I = \int_c^d \int_a^b f_1(x) \cdot f_2(y) \, dx \, dy$$

$$= \int_a^b f_1(x) dx \cdot \int_c^d f_2(y) dy \quad \text{where } a, b, c, d \text{ constants}$$

―――――――――――

$$\iint_R x y \, dA$$

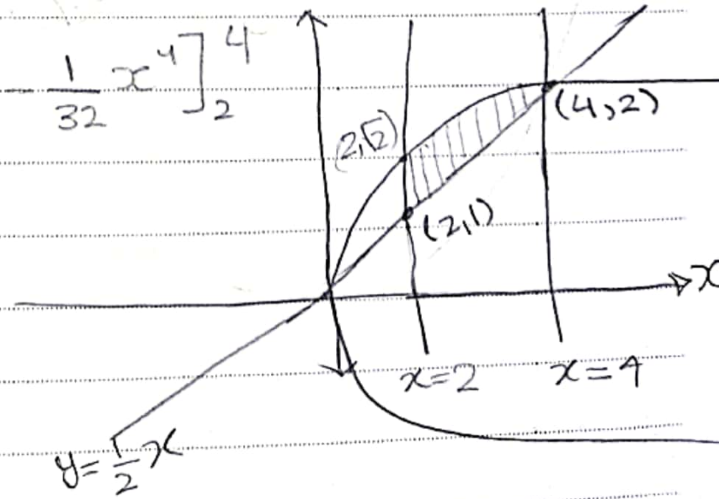
خط مستقيم $\quad y^2 = x$

$$y = \frac{1}{2} x, \; y = \sqrt{x}, \; x = 2, \; x = 4$$

$$\int_2^4 \int_{\frac{1}{2}x}^{\sqrt{x}} x y \, dy \, dx \Rightarrow \frac{1}{2} x y^2 \Big]_{\frac{1}{2}x}^{\sqrt{x}} = \frac{1}{2} x \left[x - \frac{1}{4} x^2 \right]$$

$$\int_2^4 \frac{1}{2} x^2 - \frac{1}{8} x^3 \, dx = \frac{1}{6} x^3 - \frac{1}{32} x^4 \Big]_2^4$$

$$\frac{56}{6} - \frac{240}{32} = \frac{11}{6}$$



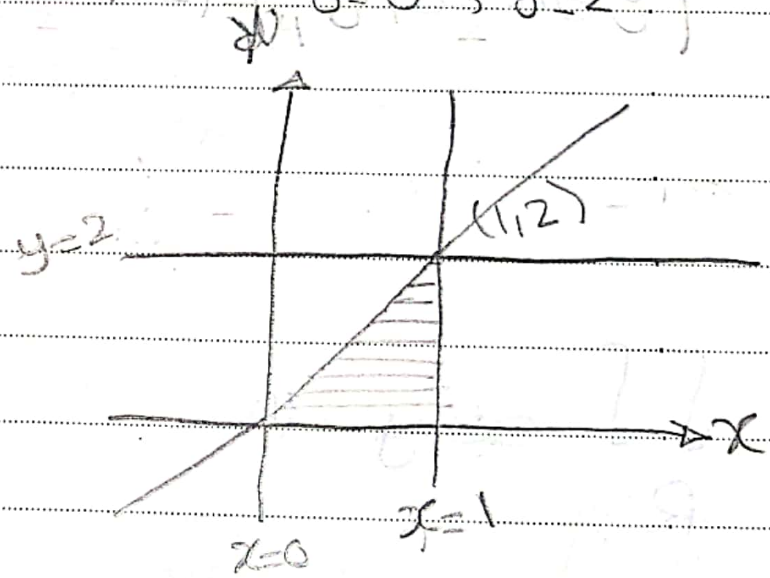
Ex (48) page 1030

$$\int_0^2 \int_{\frac{y}{2}}^1 \cos x^2 \, dx \, dy$$

$$= \int_0^1 \int_0^{2x} \cos x^2 \, dy \, dx \Rightarrow$$

$x = \frac{y}{2}$, $x = 1$

$y = 0$, $y = 2$



$$y \cos x^2 \Big]_0^{2x} = 2x \cos x^2$$

$$\int_0^1 2x \cos x^2 \, dx = \sin x^2 \Big]_0^1$$

$$\Rightarrow \sin 1$$

lecture 10 :

لازم ارقام → ○   ○ — ممكن value of y

$$\int_{\bigcirc}^{\bigcirc}\int^{\bigcirc} f(x,y)\,dx\,dy$$

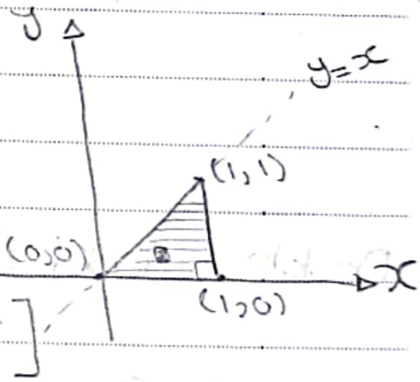
Mean value (average) of a function $f(x,y)$ on the region R.

$$\boxed{\iint_R f(x,y)\,dA} \quad dA = dx\,dy \;/\; dy\,dx$$

Mean value (average) $= \dfrac{1}{A} \boxed{}$ (Area)

$$\text{Mean value} = \frac{\iint_R f(x,y)\,dA}{\iint_R dA}$$

ex find the mean value of $f(x,y) = y\sqrt{x^2-y^2}$ over the triangle with vertices at $(0,0),(1,0),(1,1)$



$$\text{average} = \frac{\iint y\sqrt{x^2-y^2}\,dA}{\iint dA} \quad dA = dy\,dx$$

$\iint dA = \frac{1}{2}$ "area of R "triangle""

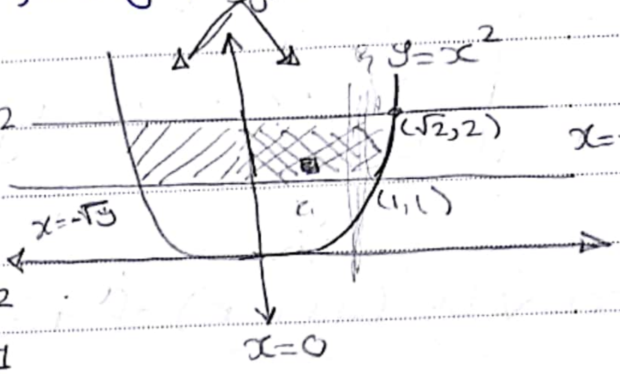
$$\frac{-1}{2}\int_0^1\int_0^x -2y\sqrt{x^2-y^2}\,dy\,dx = \frac{-1}{2}\times\frac{2}{3}(x^2-y^2)^{3/2}\Big]$$

$$= \frac{-1}{3}(\cancel{x^2-x^2})^{3/2} + \int \frac{1}{3}(x)^3\,dx = \frac{1}{12}x^4 = 1/12$$

$$\int_0^1 (y)_0^x\,dx = \int_0^1 x = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$\therefore \text{average} = \frac{1/12}{1/2} = \frac{1}{6}$$

Ex find the area bounded by $y=x^2$, $1 \leq y \leq 2$ symmetric



$$A = 2\iint dA = 2\int_1^2\int_0^{\sqrt{y}} dx\,dy$$

$$x \Big|_0^{\sqrt{y}} \qquad = \int_1^2 \sqrt{y}\,dy = \frac{2}{3}(y)^{3/2}\Big|_1^2$$

$$= \frac{2}{3}(2\sqrt{2}-1) \quad \therefore \text{Area} = 2\left[\frac{2}{3}(2\sqrt{2}-1)\right]$$

another answer

$$A = \iint_1^2 dy\,dx = \iint_{R_1} + \iint_{R_2}$$

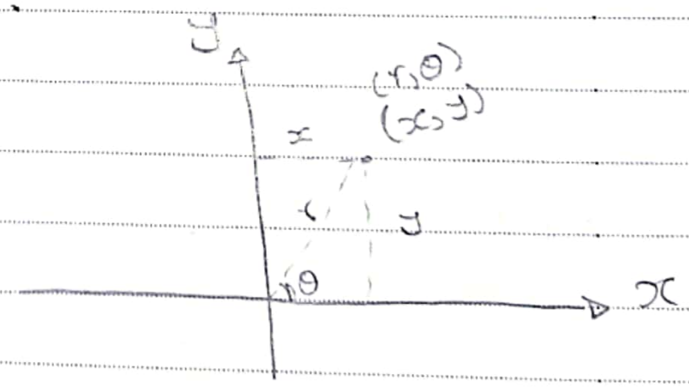
$$= \int_0^1\int_1^2 dy\,dx + \int_1^{\sqrt{2}}\int_{x^2}^2 dy\,dx$$

$$= 1 + \left(2\sqrt{2} - \frac{2\sqrt{2}}{3} - 2 + \frac{1}{3}\right) = \frac{2}{3}\left(2\sqrt{2} - \frac{1}{3}\right) \times 2$$

Double integral by polar Cordinates.

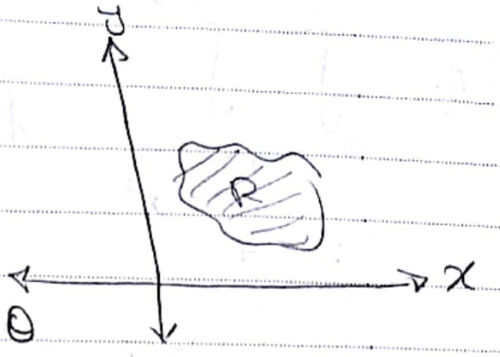
$x = r\cos\theta$

$y = r\sin\theta$



$r = \sqrt{x^2+y^2}$ $\qquad \theta = \tan^{-1}\frac{y}{x}$

To find: $\iint_R f(x,y)\, dA$



using polar Coordinates.

$$I = \iint f(x,y)\, dA = \iint \phi(r,\theta)\, r\, dr\, d\theta$$

Ex: 4 page 1036

use polar Coordinates: $\int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\, dy\, dx$ $y=0$

$x = \pm 1$

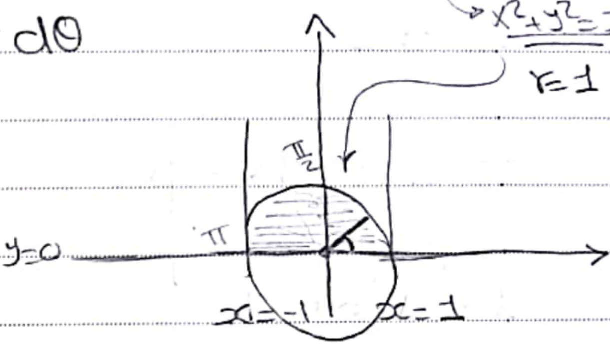
$y = \sqrt{1-x^2} \to x^2+y^2=1$

$r=1$

$$\iint (r^2)^{3/2}\, r\, dr\, d\theta = \int \int_0^1 r^4\, dr\, d\theta$$

$$= \int_0^{\pi} \frac{1}{5} r^5 \Big]_0^1 \qquad = \int_0^{\pi} \frac{1}{5}\, d\theta$$

$$= \frac{1}{5} \theta \Big]_0^{\pi} = \frac{\pi}{5}$$



$\frac{1}{2} y^2 \ln x$

EX: $\int_2^4 \int_2^4 \frac{x^2+y^2}{xy}\, dx\, dy = \int_2^4 \int_2^4 \frac{x}{y} + \frac{y}{x}\, dx\, dy$

$$\int_2^4 \int_2^4 \frac{x}{y}\, dx\, dy + \int_2^4 \int_2^4 \frac{y}{x}\, dx\, dy = 2 \int_2^4 \int_2^4 \frac{x}{y}\, dx\, dy$$

$$= 2 \int_2^4 x\, dx \cdot \int_2^4 \frac{1}{y}\, dy = 2(6 \ln 2)$$

Ex $\displaystyle\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{dy\,dx}{\sqrt{a^2-x^2-y^2}}$

$x = r\cos\theta$

$y = r\sin\theta$

$dy\,dx = r\,dr\,d\theta$

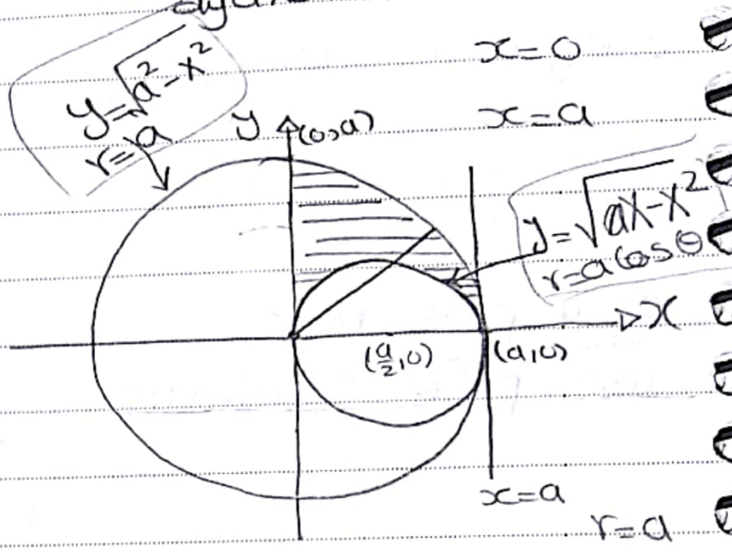
$x = 0$

$x = a$

$$I = -\frac{1}{2}\int_0^{\pi/2}\int_{a\cos\theta}^{a} \frac{-2r\,dr\,d\theta}{\sqrt{a^2-r^2}}$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} 2\sqrt{a^2-r^2}\,\Big|_{a\cos\theta}^{a}$$

$$= 0 - \sqrt{a^2-a^2\cos^2\theta}$$



$y=\sqrt{a^2-x^2} \therefore y^2+x^2=a^2$

$y=\sqrt{ax-x^2} \therefore y^2+x^2-ax=0$

centre $= (\frac{a}{2}, 0)$

$r = \sqrt{(\frac{a}{2})^2+(0)^2-0} = \frac{a}{2}$

$x^2+y^2-ax=0$

$r^2 = a\,r\cos\theta$

$r = a\cos\theta$

$\sqrt{1-\cos^2\theta}$

$$\int_0^{\frac{\pi}{2}} \sqrt{a^2-a^2\cos^2\theta}\;d\theta = a\int_0^{\frac{\pi}{2}} \sin\theta\,d\theta$$

$$= a(-\cos\theta)\Big|_0^{\pi/2} = a$$

another answer

$$\int_{0}^{a}\int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{dy\,dx}{\sqrt{a^2-x^2-y^2}}$$

$$= \int_{0}^{a} \sin^{-1}\left(\frac{y}{\sqrt{a^2-x^2}}\right)\Bigg]_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} dx$$

$$= \int_{0}^{a} \sin^{-1}(1) - \sin^{-1}\left(\frac{\sqrt{ax-x^2}}{\sqrt{a^2-x^2}}\right) dx$$

$$= \int_{0}^{a} \frac{\pi}{2}\, dx - I_1$$

$$I = \frac{a\pi}{2} - I_1$$

$$I_1 = \int_{0}^{a} \sin^{-1}\left(\frac{\sqrt{ax-x^2}}{\sqrt{a^2-x^2}}\right) dx$$ [assignment]

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