Introduction Into Probability Theory

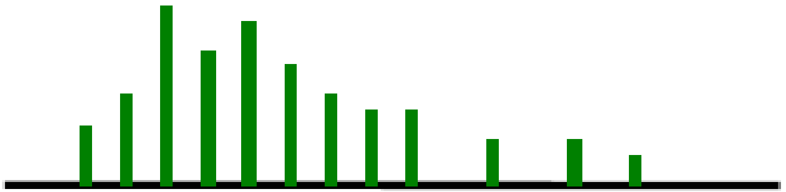
MTH 231
Lecture 7
Chapter V

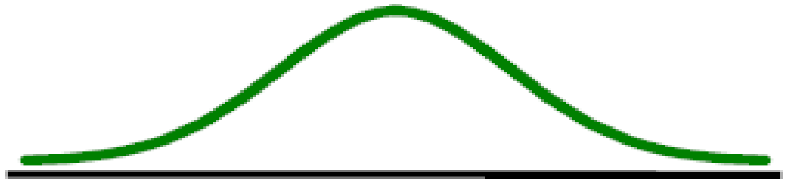
Some Useful Discrete Distributions

Overview

Probability Distributions

- **Discrete** Probability Distributions
  - Binomial
  - Poisson
- **Continuous** Probability Distributions
  - Normal
  - Uniform
  - Exponential





Today's lecture

- Some Important Discrete Distributions:
  - Bernoulli Distribution
  - Binomial Distribution
  - Poisson Distribution

Bernoulli Trial

- Trial with only two possible outcomes
  - Success (*S*)
  - Failure (*F*)
  - The trials are independent
- Examples
  - Toss of a coin (heads or tails)
  - Sex of a newborn (male or female)
  - Survival of an organism in a region (live or die)

Jacob Bernoulli (1654-1705)

Bernoulli Distribution

- The sample space of a Bernoulli trial is $\{S, F\}$.
- Defining a variable X in such way that

$$X(S) = 1 \quad and \quad X(F) = 0,$$

- Then X is a r.v. taking only two possible values: 0 and 1. This r.v. is called **Bernoulli random variable.**
- Denoting $P(X = 1) = p$, it is then called the **probability of success**.
- The PMF of a Bernoulli r.v., called Bernoulli Distribution, is seen to be

$$P(x) = p^x (1-p)^{n-x}, \; x = 0,1$$

Bernoulli Distribution

- The expectation: $E[X] = 1\times P(X=1) + 0\times P(X=0) = p.$
- The variance:

$$\text{Since} \quad E[X^2] = 1^2 \times P(X=1) + 0^2 \times P(X=0) = p,$$
$$\text{then} \quad Var(X) = E[X^2] - (E[X])^2 = p - p^2 = p(1-p)$$

Example: If in a throw of a fair die the event of obtaining 4 or 6 is called a success, and the event of obtaining 1, 2, 3, or 5 is called a failure, then

$$X = \begin{cases} 1 & \text{if 4 or 6 is obtained} \\ 0 & \text{otherwise,} \end{cases}$$

is a Bernoulli r.v. with parameter $p = 1/3$.

Binomial distribution

It is a discrete distribution that describes many experiments which require the probability of the number of successes or failures in a sample of repeated trials.

The following characteristics identify the binomial experiment:

1- The experiment consists of a fixed number of trials denoted by n.

2- The outcome of each trial can be classified as being either a "success " or a "failure".

3- The trials are independent.

4- The probability of **success**, denoted by **p**, remains the same from trial to trial. The probability of failure equals **q = 1-p**.

5- The random variable x being studied is the number of successes obtained in the n trials.

The Binomial Probability Function:

If X is a random variable having a binomial distribution then its probability function is given by

$$P[X=r]=\binom{n}{r} p^r q^{n-r} \qquad , r=0,1,2,\ldots,n \qquad 0<p<1 \; , q=1-p$$

Where

$\binom{n}{r}=C_r^n$ = The number of ways that r objects can be selected from n objects , n and p are called the parameters of the binomial distribution .

Note that :

If n = 1 the binomial distribution is called the Bernoulli distribution .

The mean and variance of the binomial distribution:

If X is binomial with parameters n and p , then X has

(a) mean $= E(X) = \mu = n\,p$ and

(b) variance $= Var(X) = \sigma^2 = n\,p\,q$

Possible Binomial Distribution Settings

A manufacturing plant labels items as either defective or acceptable

A firm bidding for contracts will either get a contract or not

A marketing research firm receives survey responses of "yes I will buy" or "no I will not"

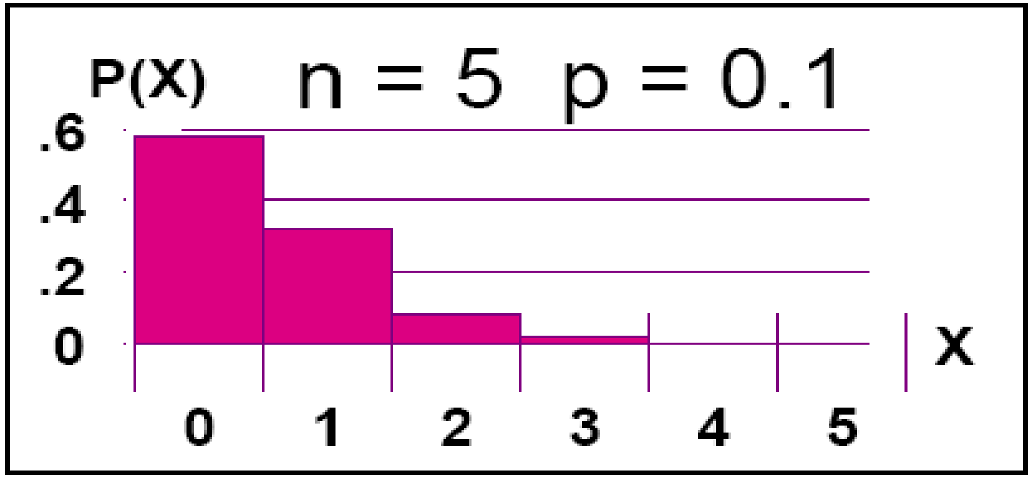
New job applicants either accept the offer or reject it

Binomial Distribution (Cont.)

- The shape of the binomial distribution depends on the values of p and n

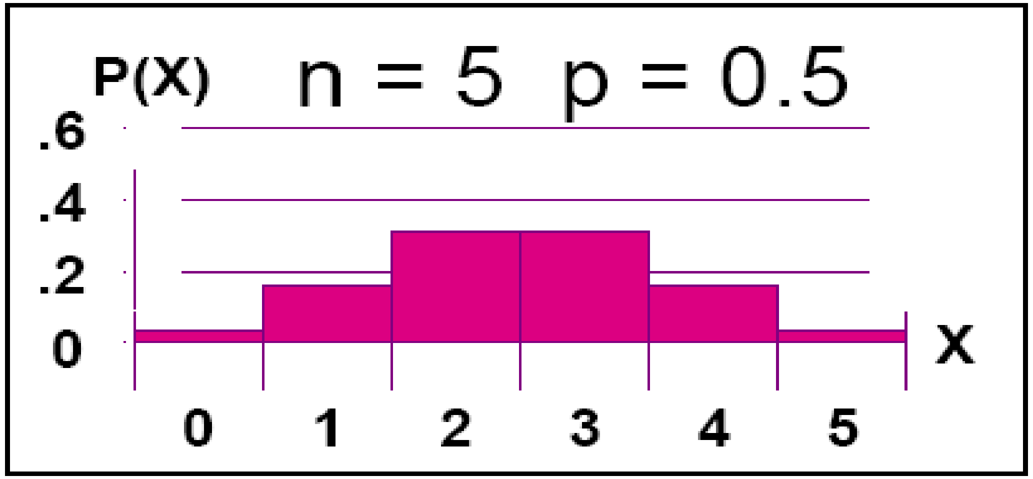
  - Here, n = 5 and p = .1

$$\sigma = \sqrt{np(1-p)} = \sqrt{(5)(.1)(1-.1)} = 0.6708$$



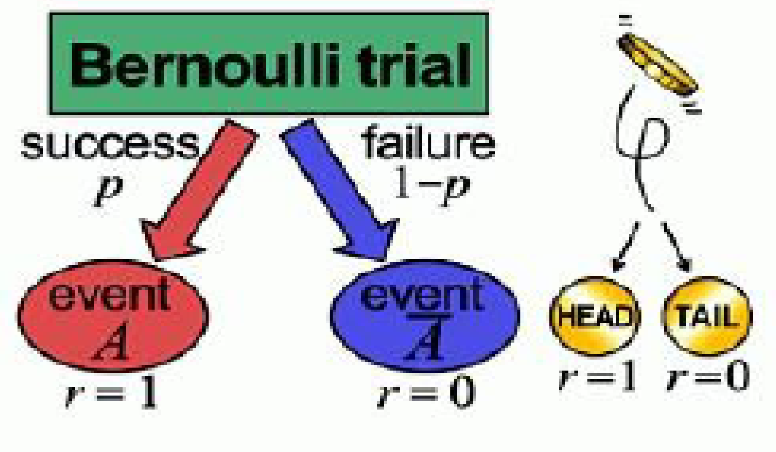
  - Here, n = 5 and p = .5

$$\sigma = \sqrt{np(1-p)} = \sqrt{(5)(.5)(1-.5)} = 1.118$$



Binomial Distribution

- A Binomial Random Variable
  - n identical trials
  - Two outcomes: **S**uccess or **F**ailure
  - $P(\boldsymbol{S}) = p;\ P(\boldsymbol{F}) = q = 1 - p$
  - Trials are independent
  - X is the number of Successes in n trials



- We say X has a binomial distribution with parameters n and p and write $X \sim Bin(n, p)$.

Binomial Distribution

- Example: You are taking a 10 question multiple choice test. If each question has four choices and you guess on each question, what is the probability of getting exactly 7 questions correct?

- Answer:

$$n = 10, \qquad k = 7, \qquad n - k = 3$$

p = 0.25 = probability of guessing the correct answer on a question

q = 0.75 = *probability of guessing the wrong answer on a question*

$$P(7\,correct\ guesses\ out\ of\ 10\,questions) = \binom{10}{7}(0.25)^7(0.75)^3 \approx 0.0031.$$

➢**Example** : A driving examiner finds that he passes 40% of the candidates. For a particular day on which he examines 9 people, find the probability that

(a) he passes exactly 6 people

(b) he passes at least one person

(c) he passes more than 7 people

(d) Find the mean and the standard deviation of the number of people who passes on that day, and then find E(2X-1) and Var(-2X+4).

Solution: Let X be the number of people he passes on that day, then X has the binomial random variable with parameters n = 9, p = 0.4 and q = 0.6, then

$$P[X=r]=\binom{9}{r}(0.4)^{r}(0.6)^{9-r} \qquad ,r=0,1,\ldots,9$$

(a) $P[\text{ he passes } 6 \text{ people}]=P[X=6]= \binom{9}{6}(0.4)^{6}(0.6)^{3}=0.0743$

(b) $P[\text{he passes at least one person}] = P[X \geq 1] = 1 - P[\text{he passes none}]$

$$= 1 - P[X=0] = 1 - \binom{9}{0}(0.4)^0(0.6)^9$$

$$= 0.9899$$

(c) $P[\text{he passes more than 7 people}] = P[X > 7] = P[X = 8] + P[X = 9]$

$$= \binom{9}{8}(0.4)^8(0.6)^1 + \binom{9}{9}(0.4)^9(0.6)^0$$

$$= 0.0038$$

(d) $E(X) = \mu = n\,p = 9(0.4) = 3.6$

$Var(X) = \sigma^2 = n\,p\,q = (9)(0.4)(0.6) = 2.16$,

$\sigma = \sqrt{Var\,(X)} = \sqrt{2.16} = 1.47$

Then,

$$E(2X-1) = 2\,E(X) - 1 = 2(np) - 1 = 2\,(3.6) - 1 = 6.2$$

$$Var(-2X+4) = (-2)^2\,Var(X) = 4\,npq = 4\,(2.16) = 8.64.$$

➢**Example** : If 20 % of the bolts produced by a machine are defective, 4 bolts are chosen at random from the production of these machine, what is the probability that

(a) one bolt is defective

(b) all 4 bolts will be good ,

(c) at most 2 bolts will be defective

(d) Find the mean and the standard deviation of the number of defective items in the sample

Solution: Let X be a binomial random variable with parameters $n = 4$, $p = 0.2$, then

$$P[X = r] = \binom{4}{r} (0.2)^{r} (0.8)^{4-r} \qquad , r = 0,1,2,3,4$$

(a) $P[X = 1] = \binom{4}{1}(0.2)^1(0.8)^3 = 0.4096$

(b) **P [all 4 bolts will be good]**

$= P[X = 0] = \binom{4}{0}(0.2)^0(0.8)^4 = 0.4096$

(c) $$P[X \le 2] = P[X = 0] + P[X = 1] + P[X = 2]$$

$$= \binom{4}{0}(0.2)^0(0.8)^4 + \binom{4}{1}(0.2)^1(0.8)^3 + \binom{4}{2}(0.2)^2(0.8)^2$$

$$= 0.9728$$

(d) $E(X) = \mu = n\,p = 4(0.2) = 0.8$

$$Var(X) = \sigma^2 = n\,p\,q = (4)(0.2)(0.8) = 0.64$$

$$\sigma = \sqrt{0.64} = 0.8$$

2- Poisson distribution

A random variable X, taking of the values 0, 1, 2… is said to be a Poisson random variable with parameter λ if for some $\lambda > 0$,

$$P(X = r) = \frac{e^{-\lambda}\,\lambda^{r}}{r!}, \qquad r = 0, 1, 2, \ldots$$

The Poisson random variable has a tremendous range of applications in several areas because it may be used as an approximation for a binomial random variable , with parameters (n , p) when n is large (≥ 30) and p is small enough so that **$\lambda = n\,p$ is fixed .**

where:

X = number of successes per unit

λ = expected number of successes per unit

e = base of the natural logarithm system (2.71828…)

The mean and the Variance of Poisson Distribution

- Mean $\boxed{\mu = \lambda}$
- Variance and Standard Deviation $\boxed{\sigma^2 = \lambda}$ $\boxed{\sigma = \sqrt{\lambda}}$

where λ = expected number of successes per unit

Some examples of random variables that usually obey the Poisson probability law:

1- The number of misprints on a page (or a group of pages) of a book
2- The number of people in a community living to 100 years of age
3- The number of wrong telephone numbers that are dialed in a day
4- The number of customers entering a post office on a given day
5- The number of α- particles discharged in a fixed period of time from some radioactive material.
6- The number of transistors that fail on their first day of use .

➢**Example** : Suppose that the number of typographical errors on a single page of a book has a Poisson distribution with parameter λ = 1/2. Calculate the probability that there is at least one error on a page.

Solution: Letting X denote the number of errors on a page, we have λ = 1/2

$$P(X = r) = \frac{e^{-\frac{1}{2}}\left(\frac{1}{2}\right)^r}{r!}, \qquad r = 0, 1, 2, \ldots$$

we have

$$P(X \geq 1) = 1 - P(X = 0) = 1 - e^{\frac{-1}{2}} \approx 0.393$$

The Relation between the Binomial and the Poisson distributions

▪ If a random variable X has a binomial distribution with parameters n and p, then when <u>n is very large</u> and <u>p is small</u> such that **λ = n p** is **fixed** then X has approximately the Poisson distribution with parameter λ = n p.

➢**Example** : Suppose that the probability that an item produced by a certain machine will be defective is 0.1. Find the probability that a sample of 10 items will contain at most 1 defective item by using the Binomial and Poisson distribution and compare the answer.

Solution

(a) Binomial distribution

n = 10 , p = 0.1 , q = 0.9

$$P[X = r] = \binom{10}{r} (0.1)^r (0.9)^{10-r} \qquad , r = 0,1,\ldots,10$$

The desired probability is

$$P(X \le 1) = P(X = 0) + P(X = 1) = \binom{10}{0}(0.1)^0 (0.9)^{10} + \binom{10}{1}(0.1)^1 (0.9)^9 = 0.7361$$

(b) Poisson distribution

$$\lambda = n\,p = (10)\,(0.1) = 1$$

$$P(X = r) = \frac{e^{-1}\,(1)^{r}}{r!}, \qquad r = 0, 1, 2, \ldots$$

$$P(X \leq 1) = P(X = 0) + P(X = 1)$$

$$= e^{-1}\,\frac{1^{0}}{0!} + e^{-1}\,\frac{1^{1}}{1!} \approx 0.7358$$

The probability obtained by binomial distribution equals approximately (to the first three decimals) the probability obtained by the Poisson distribution .

➢ **Example** :

If approximately 2% of the people in a room of 200 people are left-handed, find the probability that exactly 5 people there are left-handed.

Solution:

Since λ=n p, then λ=200(0.02)= 4. Hence,

$P(X=5)=\frac{e^{-4}4^{5}}{5!}$ =0.1563.

➢**Example** : At a certain manufacturing plant, accidents have been occurring at the rate of 1 every 2 months. Assuming the accidents occur independently:

(a) What is the expected number and the standard deviation of accidents per year?

(b) What is the probability that there will be no accidents in a given month?

Solution: Letting X denote the number accidents that have been occurring, the number of such accidents should be approximately Poisson distribution with λ = 1 for every 2 months

(a) $\lambda = 6$, then , $E(X) = \lambda = 6$, $\sigma = \sqrt{Var(X)} = \sqrt{\lambda} = \sqrt{6} = 2.45$

(b) $\lambda = ½$ in a given month , then

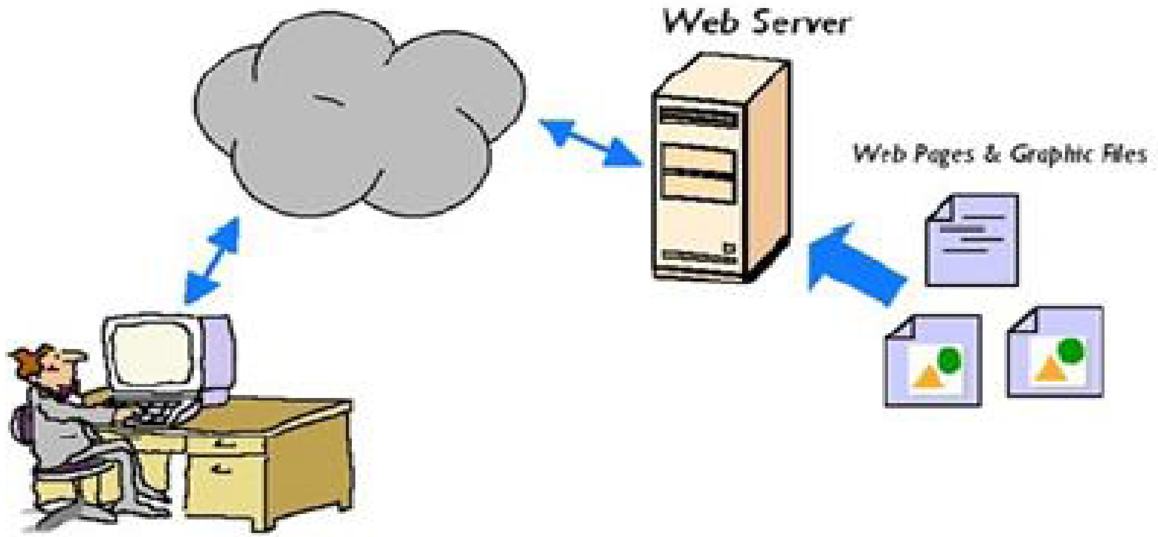
$$P(X=0) = \frac{e^{-\frac{1}{2}}\left(\frac{1}{2}\right)^0}{0!} = e^{-0.5} = 0.607$$

Problems

- Problem1 : The number of visitors to a webserver per minute follows a Poisson distribution. If the average number of visitors per minute is 4, what is the probability that:

  (i) There are two or fewer visitors in one minute?;

  (ii) There are exactly two visitors in 30 seconds?



Problems

- Solution: (i) There are two or fewer visitors in one minute?;

  we need the average number of visitors in a minute.

  In this case the parameter $\lambda = 4$.

- We wish to calculate

$$P(X=0) = \frac{e^{-4}4^0}{0!} = e^{-4}$$

$$P(X=1) = \frac{e^{-4}4^1}{1!} = 4e^{-4}$$

$$P(X=2) = \frac{e^{-4}4^2}{2!} = 8e^{-4}$$

So the probability of two or fewer visitors in a minute is

$$P(X=0) + P(X=1) + P(X=2).$$

$$= e^{-4} + 4e^{-4} + 8e^{-4} = 0.238.$$

Problems

Solution: (ii) There are exactly two visitors in 30 seconds?

- If the average number of visitors in 1 minute is 4, then the average in 30 seconds is 2.
- So for this point, our parameter $\lambda = 2$. So

$$P(X = 2) = \frac{e^{-2}2^2}{2!} = 2e^{-2} = 0.271.$$

Problems

- **Problem 2**: A fair coin is tossed 6 times. The probability of appearing heads on any toss is 30%. If X denote the number of heads that appeared.
  - Calculate:

a) $P(X=3)$

b) $P(X=4)$

c) $P(2 \leq X < 6)$

d) $E(X)$

e) $Var(X)$.

Questions!


THANK YOU


THINK
IDEA
TRY
DO
DO AGAIN
AND AGAIN
KEEP ON DOING
Success