Taylor Series Expansion

Taylor's Theorem : Suppose *f* is continuous on the closed interval [a, b] and has *n + 1* continuous derivatives on the open interval (a, b). If *x* and *c* are points in (a, b), then

$$\text{The Taylor series expansion of } f(x) \text{ about } c:$$

$$f(c) + f^{'}(c)(x-c) + \frac{f^{(2)}(c)}{2!}(x-c)^2 + \frac{f^{(3)}(c)}{3!}(x-c)^3 + \ldots$$

$$or$$

$$Taylor\ \ Series = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(c)\ (x-c)^k$$

$$\text{If the series converge, we can write:}$$

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(c)\ (x-c)^k$$

Maclaurin Series

Maclaurin series is a special case of Taylor series with the center of expansion $c = 0$.

$$\text{The Maclaurin series expansion of } f(x):$$

$$f(0) + f^{'}(0)\,x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \ldots$$

$$\text{If the series converge, we can write:}$$

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(0)\; x^k$$

What If we change the interval ?

If we change the interval so $x=x+h$ and $c=x$

The Taylor series expansion of $f(x+h)$ about $c=x$

$$f(x+h) = f(x) + f^{'}(x)h + \frac{f^{(2)}(x)}{2!}h^2 + \frac{f^{(3)}(x)}{3!}h^3 + \ldots$$

Finite Differences for Derivation

Taylor expansion for $F(x+h)$

$$F(x+h) = F(x) + hF'(x) + \frac{h^2}{2!}F''(x) + ...$$

$$F(x+h) - F(x) = hF'(x) + O(h^2)$$

Use only first two terms of the expansion

$$\frac{F(x+h) - F(x)}{h} = F'(x) + \frac{O(h^2)}{h}$$

$$\boxed{F'(x) = \frac{F(x+h) - F(x)}{h}}$$

Forward Difference Formula

Finite Differences for Derivation

Taylor expansion for $F(x-h)$

$$F(x-h) = F(x) - hF'(x) + \frac{h^2}{2!}F''(x) + ...$$

$$F(x) - F(x-h) = hF'(x) + O(h^2)$$

$$\boxed{F'(x) = \frac{F(x) - F(x-h)}{h}}$$

Backward Difference Formula

Finite Differences for Derivation

$$F(x+h) = F(x) + hF'(x) + \frac{h^2}{2!}F''(x) + \frac{h^3}{3!}F'''(x) + ...$$

$$F(x-h) = F(x) - hF'(x) + \frac{h^2}{2!}F''(x) - \frac{h^3}{3!}F'''(x) + ...$$

$-$

$$F(x+h) - F(x-h) = 2hF'(x) + 2\frac{h^3}{3!}F'''(x) + ...$$

$$\frac{F(x+h) - F(x-h)}{2h} = F'(x) + O(h^3)$$

$$\boxed{F'(x) = \frac{F(x+h) - F(x-h)}{2h}}$$

Central Difference Formula

Example : Derivation of Forward Difference Formula

$$F(x+h) = F(x) + hF'(x) + \frac{h^2}{2!}F''(x) + ...$$

$$F(x-h) = F(x) - hF'(x) + \frac{h^2}{2!}F''(x) + ...$$

$+$

$$F(x+h) + F(x-h) = 2F(x) + hF'(x) - hF'(x) + \frac{h^2}{2!}F''(x) + \frac{h^2}{2!}F''(x)$$

$$F(x+h) + F(x-h) = 2F(x) + \cancel{2}\frac{h^2}{\cancel{2}!}F''(x)$$

$$\boxed{F''(x) = \frac{F(x+h) - 2F(x) + F(x-h)}{h^2} + O(h^2)}$$

Better approximations

By using Taylor expansions for $F(x+2h)$ and $F(x-2h)$, better approximations

can be obtained:

$$F'(x) = \frac{-F(x+2h)+4F(x+h)-3F(x)}{2h} + O(h^2)$$

Forward Difference Formula

$$F'(x) = \frac{3F(x)-4F(x-h)+F(x-2h)}{2h} + O(h^2)$$

Backward Difference Formula

$$F'(x) = \frac{-F(x+2h)+8F(x+h)-8F(x-h)+F(x-2h)}{12h} + O(h^4)$$

Central Difference Formula

Example : Derivation of Forward Difference Formula

$$F(x+2h) = F(x) + 2hF'(x) + \frac{4h^2}{2!}F''(x) + ...$$

$$*\,(-4) \quad F(x+h) = F(x) + hF'(x) + \frac{h^2}{2!}F''(x) + ...$$

$+$

$$F(x+2h) - 4F(x+h) = F(x) - 4F(x) + 2hF'(x) - 4hF'(x) + \frac{4h^2}{2!}F''(x) - .\frac{4h^2}{2!}F''(x)$$

$$F(x+2h) - 4F(x+h) = -3F(x) + 2hF'(x)$$

$$\boxed{F'(x) = \frac{F(x+2h) - 4F(x+h) + 3F(x)}{2h}}$$

Second Derivatives

Similarly various approximations for second derivatives are possible :

$$F''(x) = \frac{F(x+2h) - 2F(x+h) + F(x)}{h^2} + O(h)$$

$$F''(x) = \frac{F(x+h) - 2F(x) + F(x-h)}{h^2} + O(h^2)$$

$$F''(x) = \frac{-F(x+2h) + 16F(x+h) - 30F(x) + 16F(x-h) - F(x-2h)}{12h^2} + O(h^4)$$

Derivatives of Bivariate & Multivariate Functions

First Order Partial Derivatives

For functions with more variables, the partial derivatives can be approximated by grouping together all of the same variables and applying the univariate approximation for that group.
For example, if *F(x, y)* is our function, then some first order partial derivative approximations are:

$$f_x(x,y) = \frac{F(x+h,y) - F(x-h,y)}{2h}$$

$$f_y(x,y) = \frac{F(x,y+k) - F(x,y-k)}{2k}$$

Second Partial Derivatives

Following formulas can be verified by taking the limits $h \to 0$ and $k \to 0$

$$f_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{F(x+h,y) - 2f(x,y) + F(x-h,y)}{h^2}$$

$$f_{yy}(x,y) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{F(x,y+k) - 2f(x,y) + F(x,y-k)}{k^2}$$

$$\boxed{f_{xy}(x,y) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{F(x+h,y+k) - F(x+h,y-k) - F(x-h,y+k) + F(x+h,y-k)}{4hk}}$$

The derivatives F_x, F_y, F_{xx} and F_{yy} just use the univariate approximation formulas.

The mixed derivative requires slightly more work. The important observation is that the approximation for F_{xy} is obtained by applying the x–derivative approximation for F_x, then applying the y–derivative approximation to the previous approximation.

1 - the x–derivative approximation for F_x :

$$f_x(x,y) = \frac{F(x+h,y) - F(x-h,y)}{2h}$$

2- the y–derivative approximation:

$$f_y(x,y) = \frac{F(x,y+k) - F(x,y-k)}{2k}$$

3- Apply the 2 nd formula to the 1st one:

$$f_{xy}(x,y) = \frac{F(x+h,y+k) - F(x+h,y-k) - F(x-h,y+k) + F(x+h,y-k)}{4hk}$$

Second Partial Derivatives

Following formulas can be verified by taking the limits $h \to 0$ and $k \to 0$

To find the F_{xy} partial derivative first use the formula for F_x

Then apply the approximation for (f_y)

$$f_{xx}(x,y) = \frac{F(x+h,y) - 2f(x,y) + F(x-h,y)}{h^2}$$

$$f_{yy}(x,y) = \frac{F(x,y+k) - 2f(x,y) + F(x,y-k)}{k^2}$$

$$f_{xy}(x,y) = \frac{F(x+h,y+k) - F(x+h,y-k) - F(x-h,y+k) + F(x+h,y-k)}{4hk}$$

Classification of the Methods

Numerical Methods for Solving ODE

Single-Step Methods

Estimates of the solution at a particular step are entirely based on information on the previous step

Multiple-Step Methods

Estimates of the solution at a particular step are based on information on more than one step

Taylor Series Method

The problem to be solved is a first order ODE:

$$\boxed{\frac{dy(x)}{dx} = f(x, y), \quad y(x_0) = y_0}$$

Estimates of the solution at different base points:

$$y(x_0 + h),\ y(x_0 + 2h),\ y(x_0 + 3h),\ \ldots.$$

are computed using the truncated Taylor series expansions.

Taylor Series Expansion

Truncated Taylor Series Expansion

$$y(x_0+h) \approx \sum_{k=0}^{n} \frac{h^k}{k!} \left(\left. \frac{d^k y}{dx^k} \right|_{x=x_0,\, y=y_0} \right)$$

$$\approx y(x_0) + h \left. \frac{dy}{dx} \right|_{\substack{x=x_0, \\ y=y_0}} + \frac{h^2}{2!} \left. \frac{d^2 y}{dx^2} \right|_{\substack{x=x_0, \\ y=y_0}} + \ldots + \frac{h^n}{n!} \left. \frac{d^n y}{dx^n} \right|_{\substack{x=x_0, \\ y=y_0}}$$

The n^{th} order Taylor series method uses the n^{th} order Truncated Taylor series expansion.

Euler Method

- First order Taylor series method is known as Euler Method.

- Only the constant term and linear term are used in the Euler method.

- The error due to the use of the truncated Taylor series is of order $O(h^2)$.

First Order Taylor Series Method

(Euler Method)

$$y(x_0+h)=y(x_0)+h\left.\frac{dy}{dx}\right|_{\substack{x=x_0,\\ y=y_0}}+O(h^2)$$

$$Notation:$$

$$x_n=x_0+nh, \qquad y_n=y(x_n),$$

$$\left.\frac{dy}{dx}\right|_{\substack{x=x_i,\\ y=y_i}}=f(x_i,y_i)$$

$$Euler\ \ Method$$

$$y_{i+1}=y_i+h\ f(x_i,y_i)$$

Euler Method

Problem:

Given the first order ODE: $\dot{y}(x) = f(x, y)$

with the initial condition: $y_0 = y(x_0)$

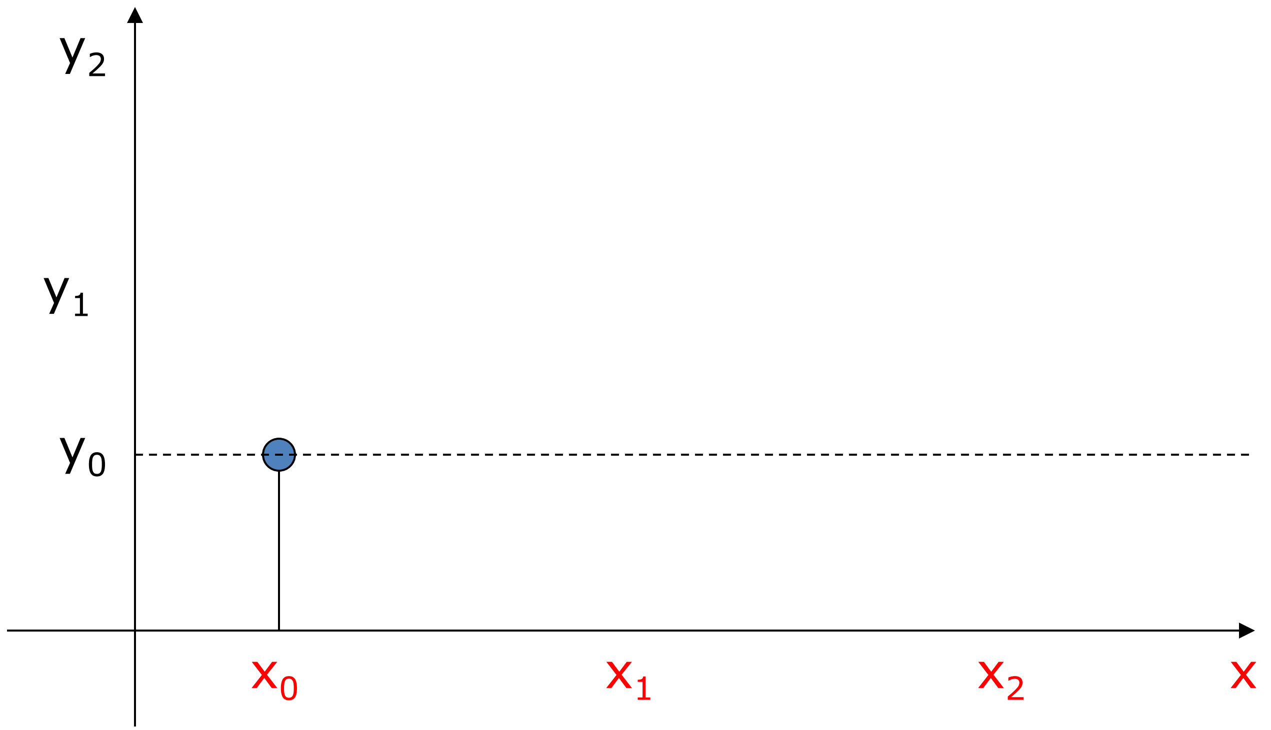
Determine: $y_i = y(x_0 + ih) \qquad for \ i = 1,2,...$

Euler Method:

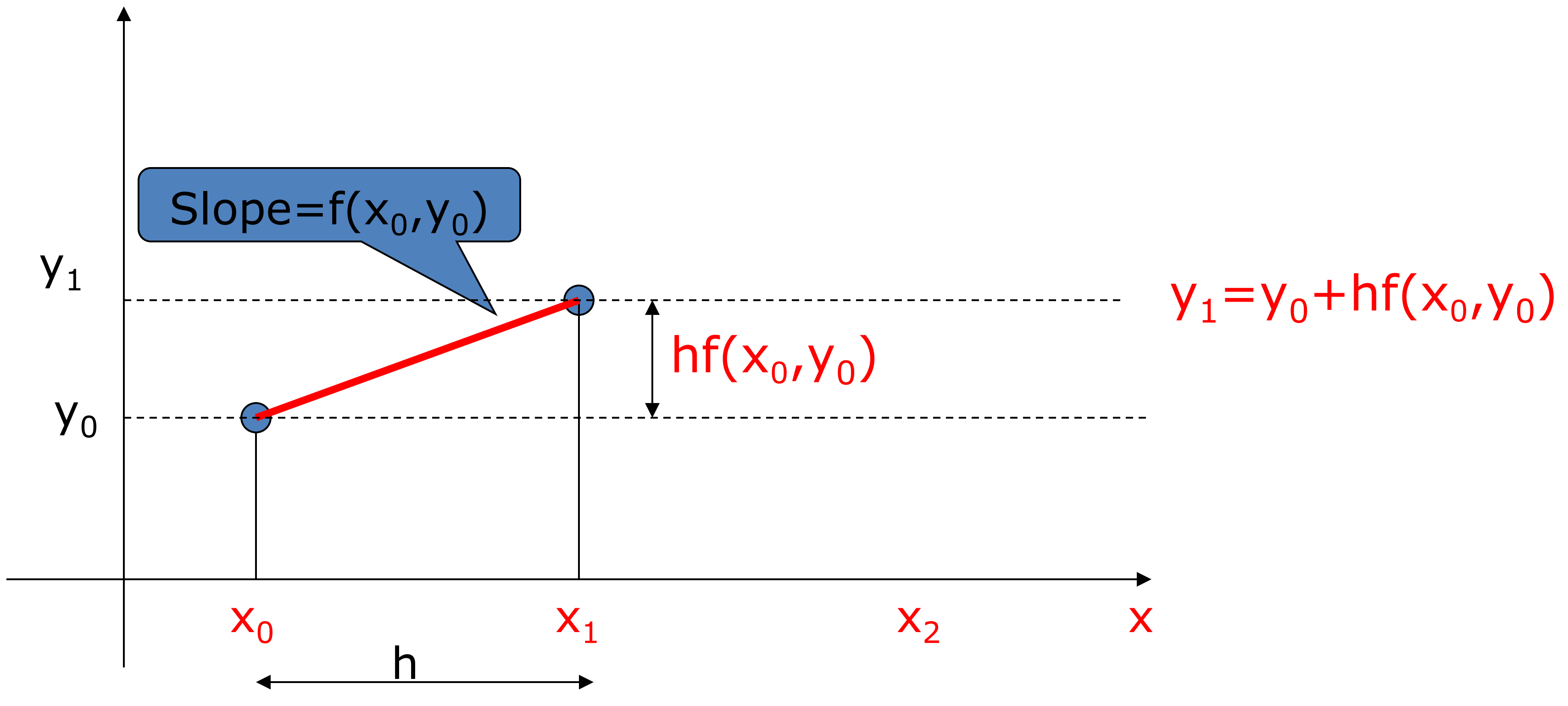
$$y_0 = y(x_0)$$

$$y_{i+1} = y_i + h\ f(x_i, y_i) \qquad for \ i = 1,2,...$$

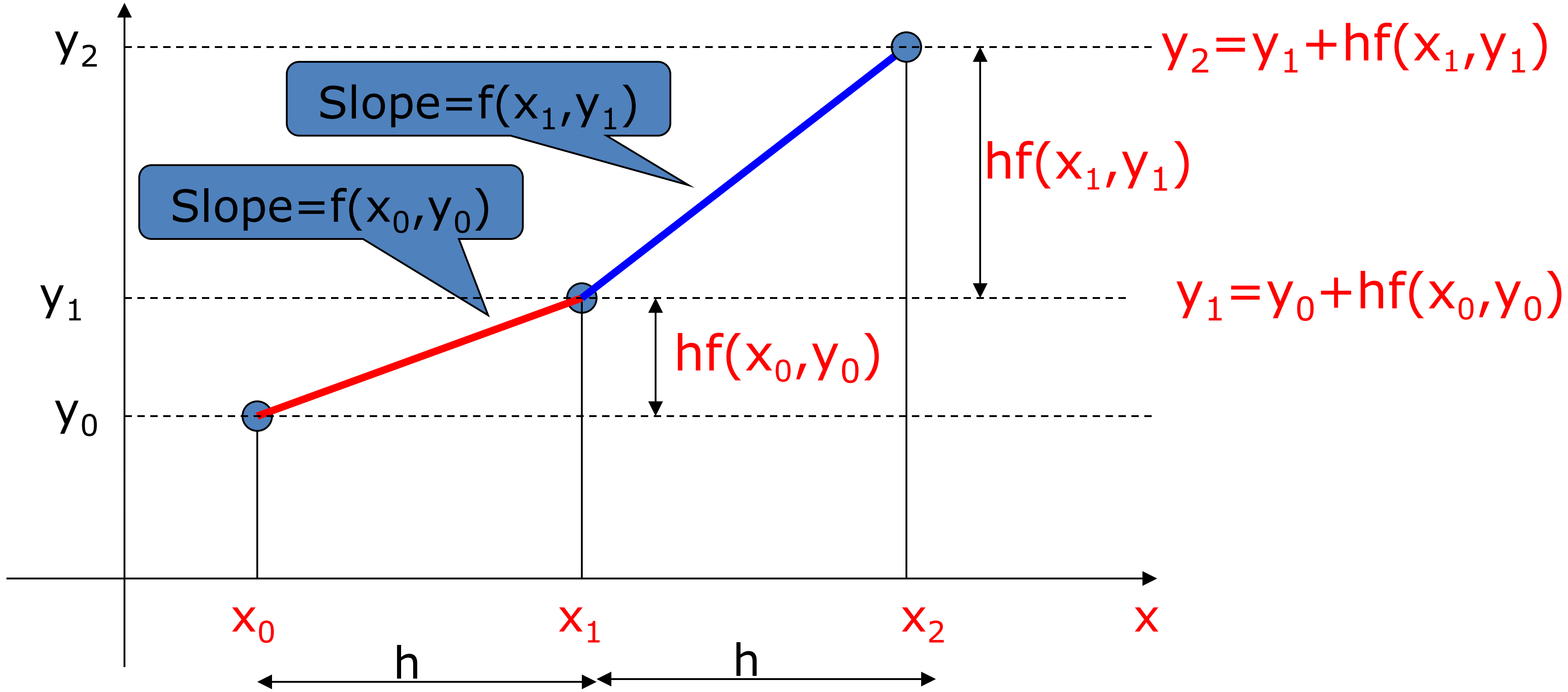
Interpretation of Euler Method



Interpretation of Euler Method



Interpretation of Euler Method



Example 1

Use Euler method to solve the ODE:

$$\frac{dy}{dx} = 1 + x^2, \qquad y(1) = -4$$

to determine y(1.01), y(1.02) and y(1.03).

Example 1

$$\boxed{f(x,y) = 1 + x^2, \quad x_0 = 1, \ y_0 = -4, \ h = 0.01}$$

Euler Method

$$y_{i+1} = y_i + h\,f(x_i, y_i)$$

$$Step1: \quad y_1 = y_0 + h\,f(x_0, y_0) = -4 + 0.01(1 + (1)^2) = -3.98$$

$$Step2: \quad y_2 = y_1 + h\,f(x_1, y_1) = -3.98 + 0.01\left(1 + (1.01)^2\right) = -3.9598$$

$$Step3: \quad y_3 = y_2 + h\,f(x_2, y_2) = -3.9598 + 0.01\left(1 + (1.02)^2\right) = -3.9394$$

Example 1

$$f(x,y) = 1 + x^2, \quad x_0 = 1, \ y_0 = -4, \ h = 0.01$$

Summary of the result:

i	xi	yi
0	1.00	-4.00
1	1.01	-3.98
2	1.02	-3.9595
3	1.03	-3.9394

Example 1

$$f(x, y) = 1 + x^2, \quad x_0 = 1, \; y_0 = -4, \; h = 0.01$$

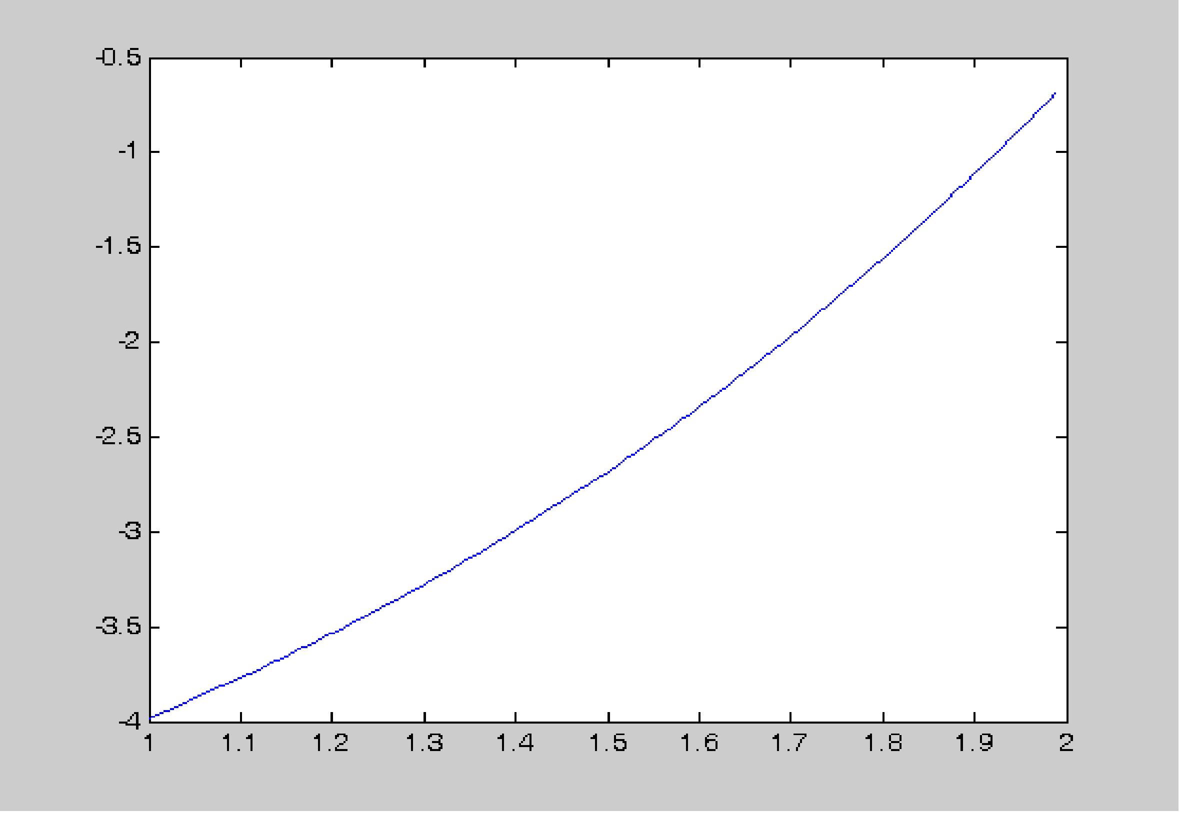
Comparison with true value:

i	xi	yi	True value of yi
0	1.00	-4.00	-4.00
1	1.01	-3.98	-3.97990
2	1.02	-3.9595	-3.95959
3	1.03	-3.9394	-3.93909

Example 1

$$f(x,y)=1+x^2, \quad x_0=1, \quad y_0=-4, \quad h=0.01$$

A graph of the solution of the ODE for 1<x<2



Types of Errors

- Local truncation error:

  Error due to the use of truncated Taylor series to compute x(t+h) in one step.

- Global Truncation error:

  Accumulated truncation over many steps.

- Round off error:

  Error due to finite number of bits used in representation of numbers. This error could be accumulated and magnified in succeeding steps.

Second Order Taylor Series Methods

$$\boxed{Given\ \ \frac{dy(x)}{dx} = f(y, x), \quad y(x_0) = y_0}$$

Second order Taylor Series method

$$y_{i+1} = y_i + h\frac{dy}{dx} + \frac{h^2}{2!}\frac{d^2y}{dx^2} + O(h^3)$$

$\dfrac{d^2y}{dx^2}$ needs to be derived analytically.

Third Order Taylor Series Methods

$$\boxed{Given \ \frac{dy(x)}{dx} = f(y,x), \quad y(x_0) = y_0}$$

Third order Taylor Series method

$$y_{i+1} = y_i + h\frac{dy}{dx} + \frac{h^2}{2!}\frac{d^2y}{dx^2} + \frac{h^3}{3!}\frac{d^3y}{dx^3} + O(h^4)$$

$\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$ need to be derived analytically.

High Order Taylor Series Methods

$$\boxed{Given \ \frac{dy(x)}{dx} = f(y,x), \quad y(x_0) = y_0}$$

n$^{\text{th}}$ order Taylor Series method

$$y_{i+1} = y_i + h\frac{dy}{dx} + \frac{h^2}{2!}\frac{d^2y}{dx^2} + \dots. + \frac{h^n}{n!}\frac{d^ny}{dx^n} + O(h^{n+1})$$

$\dfrac{d^2y}{dx^2}, \dfrac{d^3y}{dx^3}, \dots.., \dfrac{d^ny}{dx^n}$ need to be derived analytically.

Higher Order Taylor Series Methods

- High order Taylor series methods are more accurate than Euler method.

- But, the 2nd, 3rd, and higher order derivatives need to be derived analytically which may not be easy.

Example 2

Second order Taylor Series Method

Use Second order Taylor Series method to solve:

$$\frac{dx}{dt} + 2x^2 + t = 1, \; x(0) = 1, \qquad use \quad h = 0.01$$

What is: $\dfrac{d^2x(t)}{dt^2}$?

Example 2

$$\text{Use Second order Taylor Series method to solve:}$$

$$\frac{dx}{dt} + 2x^2 + t = 1,\ x(0) = 1, \qquad use \quad h = 0.01$$

$$\frac{dx}{dt} = 1 - 2x^2 - t$$

$$\frac{d^2x(t)}{dt^2} = 0 - 4x\ \frac{dx}{dt} - 1 = -4x(1 - 2x^2 - t) - 1$$

$$\boxed{x_{i+1} = x_i + h(1 - 2x_i^{\ 2} - t_i) + \frac{h^2}{2}(-1 - 4x_i(1 - 2x_i^{\ 2} - t_i))}$$

Example 2

$$\boxed{f(t,x) = 1 - 2x^2 - t, \quad t_0 = 0, \ \ x_0 = 1, \ \ h = 0.01}$$

$$x_{i+1} = x_i + h(1 - 2x_i^{\,2} - t_i) + \frac{h^2}{2}(-1 - 4x_i(1 - 2x_i^{\,2} - t_i))$$

$Step\,1:$

$$x_1 = 1 + 0.01(1 - 2(1)^2 - 0) + \frac{(0.01)^2}{2}(-1 - 4(1)(1 - 2 - 0)) = 0.9901$$

$Step\,2:$

$$x_2 = 0.9901 + 0.01(1 - 2(0.9901)^2 - 0.01) + \frac{(0.01)^2}{2}(-1 - 4(0.9901)(1 - 2(0.9901)^2 - 0.01)) = 0.9807$$

$Step\,3:$

$$x_3 = 0.9716$$

Example 2

$$f(t,x) = 1 - 2x^2 - t, \quad t_0 = 0, \; x_0 = 1, \; h = 0.01$$

Summary of the results:

i	t_i	x_i
0	0.00	1
1	0.01	0.9901
2	0.02	0.9807
3	0.03	0.9716

Programming Euler Method

Write a MATLAB program to implement Euler method to solve:

$$\frac{dv}{dt} = 1 - 2v^2 - t. \qquad v(0) = 1$$

$$for \quad t_i = 0.01i, \qquad i = 1,2,\ldots,100$$

Programming Euler Method

```
f=inline('1-2*v^2-t','t','v')
h=0.01
t=0
v=1
T(1)=t;
V(1)=v;
for i=1:100
    v=v+h*f(t,v)
    t=t+h;
    T(i+1)=t;
    V(i+1)=v;
end
```

Programming Euler Method

```
f=inline('1-2*v^2-t','t','v')
h=0.01
t=0
v=1
T(1)=t;
V(1)=v;
for i=1:100
    v=v+h*f(t,v)
    t=t+h;
    T(i+1)=t;
    V(i+1)=v;
end
```

Definition of the ODE

Initial condition

Main loop

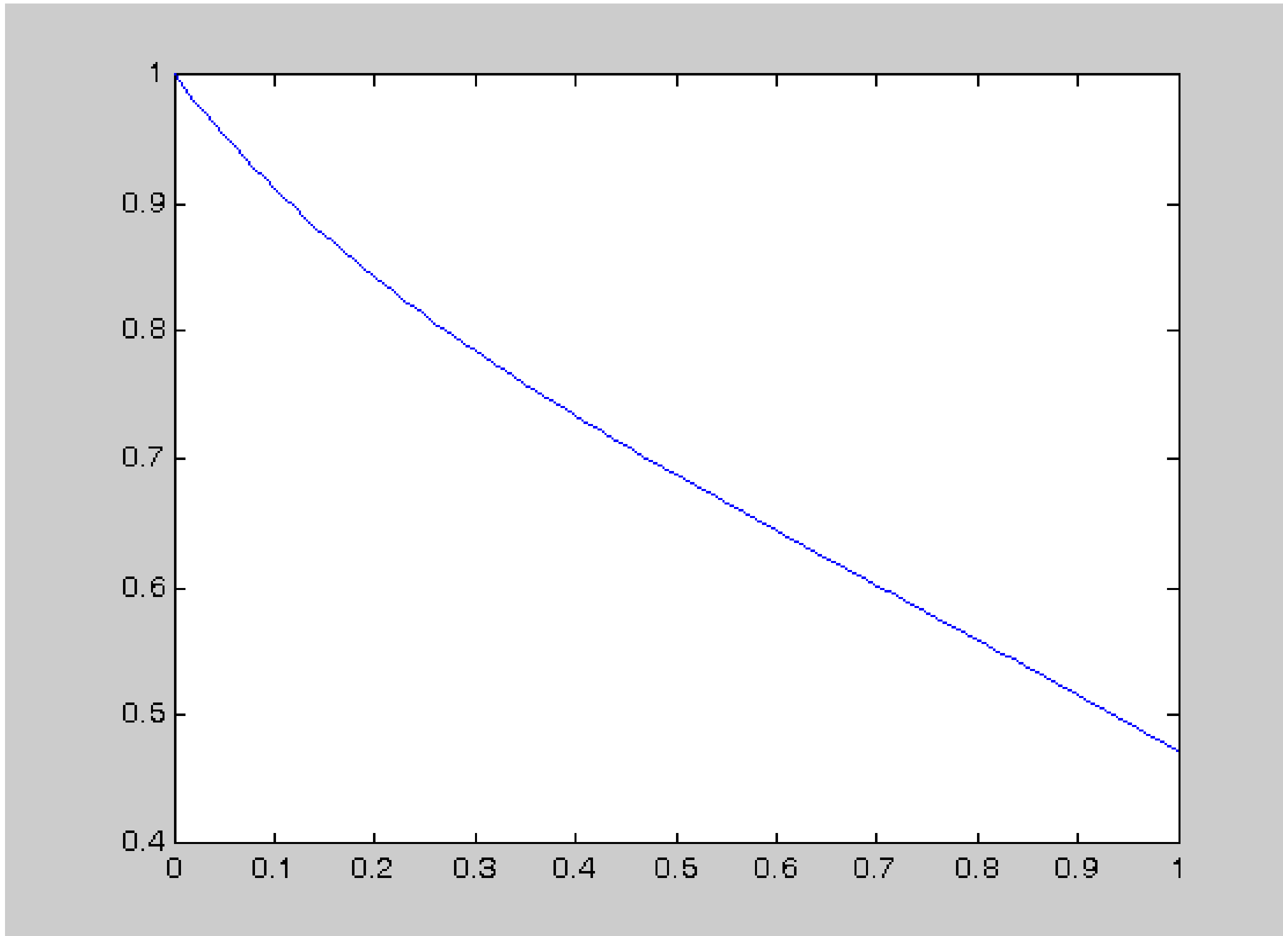
Euler method

Storing information

Programming Euler Method

Plot of the solution

plot(T,V)



Example 1:

Find **y(0.5)** if **y** is the solution of IVP **$y' = -2x-y$, $y(0) = -1$** using Euler's method with step length **0.1**. Also find the error in the approximation.

Solution: **$f(x, y) = -2x - y$**,

$y_1 = y_0 + h\, f(x_0, y_0) = -1 + 0.1*(-2*0 - (-1)) = -0.8999$

$y_2 = y_1 + h\, f(x_1, y_1) = -0.8999 + 0.1*(-2*0.1 - (-0.8999)) = -0.8299$

$y_3 = y_2 + h\, f(x_2, y_2) = -0.8299 + 0.1*(-2*0.2 - (-0.8299)) = -0.7869$

$y_4 = y_3 + h\, f(x_3, y_3) = -0.7869 + 0.1*(-2*0.3 - (-0.7869)) = -0.7683$

$y_5 = y_4 + h\, f(x_4, y_4) = -0.7683 + 0.1*(-2*0.4 - (-0.7683)) = -0.7715$

<u>Example 2</u>:

Use Eulers method to solve for **y[0.1]** from **y' = x + y + xy**, **y(0) = 1** with **h = 0.01** also estimate how small **h** would need to obtain four-decimal accuracy.

<u>Solution</u> : f(x, y) = x + y + xy,

$y_1 = y_0 + h\ f(x_0, y_0) = 1.0 + .01*(0 + 1 + 0*1) = 1.01$

$y_2 = y_1 + h\ f(x_1, y_1) = 1.01 + .01*(0.01 + 1.01 + 0.01*1.01) =1.02$

$y_3 = y_2 + h\ f(x_2, y_2) = 1.02 + .01*(0.02 + 1.02 + 0.02*1.02) =1.031$

$y_4 = y_3 + h\ f(x_3, y_3) = 1.031 + .01*(0.03 + 1.031 + 0.03*1.031) =1.042$

$y_5 = y_4 + h\ f(x_4, y_4) = 1.042 + .01*(0.04 + 1.042 + 0.04*1.042) = 1.053$

$y_6 = y_5 + h\ f(x_5, y_5) = 1.053 + .01*(0.05 + 1.053 + 0.05*1.053) = 1.065$

$y_7 = y_6 + h\ f(x_6, y_6) = 1.065 + .01*(0.06 + 1.065 + 0.06*1.065) = 1.076$

$y_8 = y_8 + h\ f(x_7, y_7) = 1.076 + .01*(0.07 + 1.076 + 0.07*1.076) = 1.089$

$y_9 = y_9 + h\ f(x_8, y_8) = 1.089 + .01*(0.08 + 1.089 + 0.08*1.089) = 1.101$

$y_{10} = y_{10} + h\ f(x_9, y_9) = 1.101 + .01*(0.09 + 1.101 + 0.09*1.101) = 1.114$

Example 3:

Solve the differential equation **$y' = x/y$**, **$y(0)=1$** by Euler's method to get **$y(1)$**. Use the step lengths **$h = 0.1$** and **0.2** and compare the results with the analytical solution ($y^2 = 1 + x^2$)

Solution: **$f(x, y) = x/y$**,

with h = 0.1

$y_1 = y_0 + h\, f(x_0, y_0) = 1.0 + 0.1*0.0/1.0 = 1.00$

$y_2 = y_1 + h\, f(x_1, y_1) = 1.0 + 0.1*0.1/1.0 = 1.01$

$y_3 = y_2 + h\, f(x_2, y_2) = 1.01 + 0.1*0.2/1.01 = 1.0298$

$y_4 = y_3 + h\, f(x_3, y_3) = 1.0298 + 0.1*0.3/1.0298 = 1.0589$

$y_5 = y_4 + h\, f(x_4, y_4) = 1.0589 + 0.1*0.4/1.0589 = 1.0967$

$y_6 = y_5 + h\, f(x_5, y_5) = 1.0967 + 0.1*0.5/1.0967 = 1.1423$

$y_7 = y_6 + h\, f(x_6, y_6) = 1.1423 + 0.1*0.6/1.1423 = 1.1948$

$y_8 = y_7 + h\, f(x_7, y_7) = 1.1948 + 0.1*0.7/1.1948 = 1.2534$

$y_9 = y_8 + h\, f(x_8, y_8) = 1.2534 + 0.1*0.8/1.2534 = 1.3172$

$y_{10} = y_9 + h\, f(x_9, y_9) = 1.3172 + 0.1*0.9/1.3172 = 1.3855$

with h = 0.2

$y_1 = y_0 + h\, f(x_0, y_0) = 1.0 + 0.2*0.0/1.0 = 1.0$
$y_2 = y_1 + h\, f(x_1, y_1) = 1.0 + 0.2*0.2/1.0 = 1.0400$
$y_3 = y_2 + h\, f(x_2, y_2) = 1.0400 + 0.2*0.4/1.0400 = 1.1169$
$y_4 = y_3 + h\, f(x_3, y_3) = 1.1169 + 0.2*0.6/1.1169 = 1.2243$
$y_5 = y_4 + h\, f(x_4, y_4) = 1.2243 + 0.2*0.8/1.2243 = 1.3550$

Comparison of numerical and analytical solutions

x	Numerical Solution h = 0.1	h = 0.2	Analytical solution
0.0	1.0	1.0	1.0
0.1	1.0		1.0050
0.2	1.01	1.0	1.0198
0.3	1.0298		1.0440
0.4	1.0589	1.0400	1.0770
0.5	1.0967		1.1180
0.6	1.1423	1.1169	1.1662
0.7	1.1948		1.2207
0.8	1.2534	1.2243	1.2806
0.9	1.3172		1.3454
1.0	1.3855	1.3550	1.4142