Digital Signal Processor and Applications

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Introduction to the Fourier Transform

- Review
  - Frequency Response
  - Fourier Series
- Definition of Fourier transform

$$\boxed{X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt}$$

Relation to Fourier Series

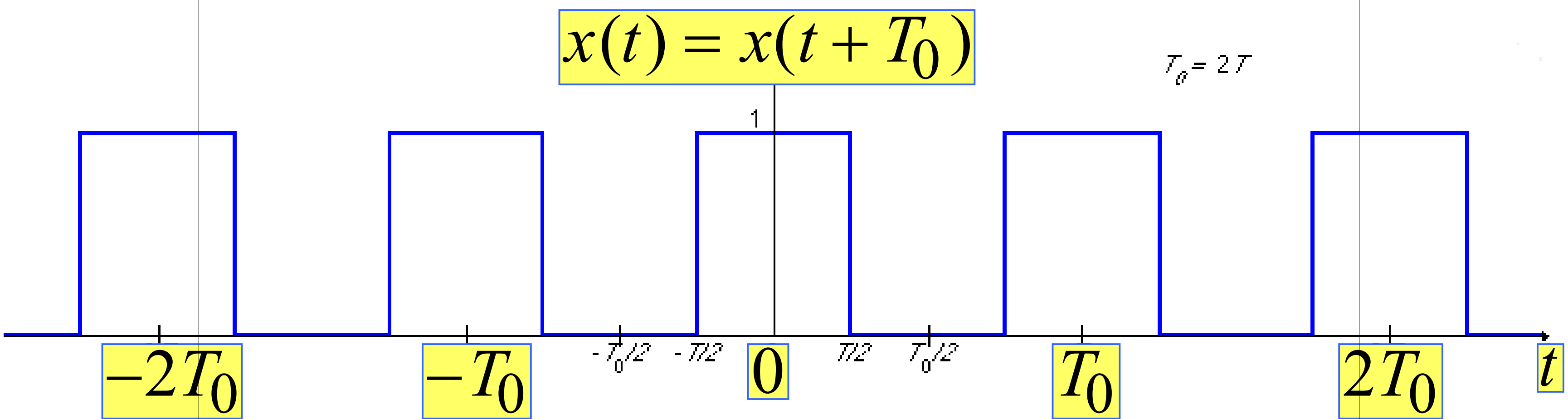
- Examples of Fourier transform pairs

Everything = Sum of Sinusoids

- One Square Pulse = Sum of Sinusoids
  - ???????????
- Finite Length
- Not Periodic

- Limit of Square Wave as Period → infinity
  - Intuitive Argument

Fourier Series: Periodic *x(t)*



$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t}$$

Fourier Synthesis

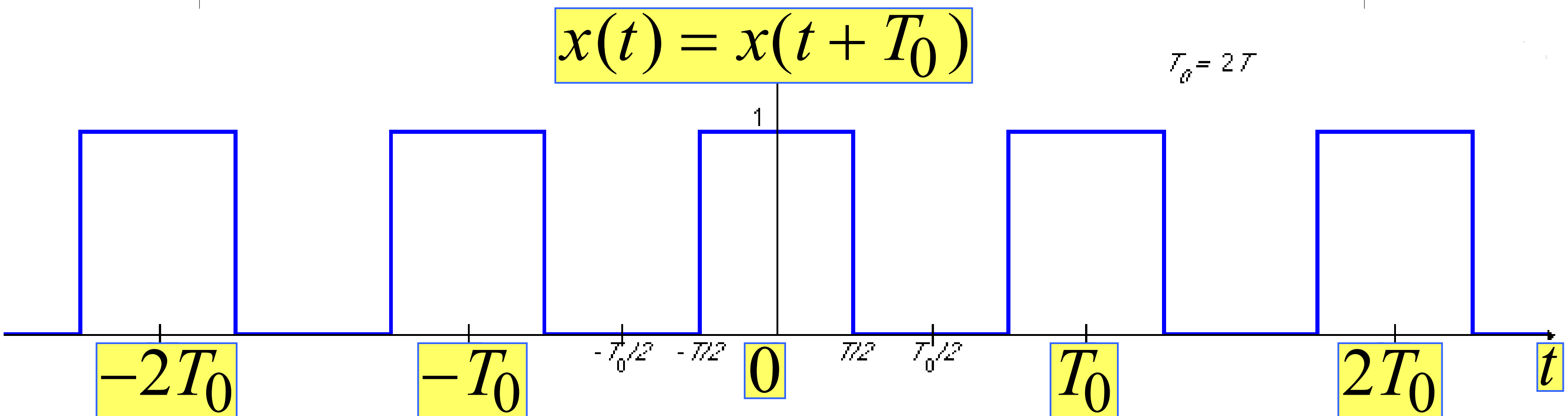
Fundamental Freq.

$$\omega_0 = 2\pi / T_0 = 2\pi f_0$$

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-j\omega_0 k t} dt$$

Fourier Analysis

Square Wave Signal

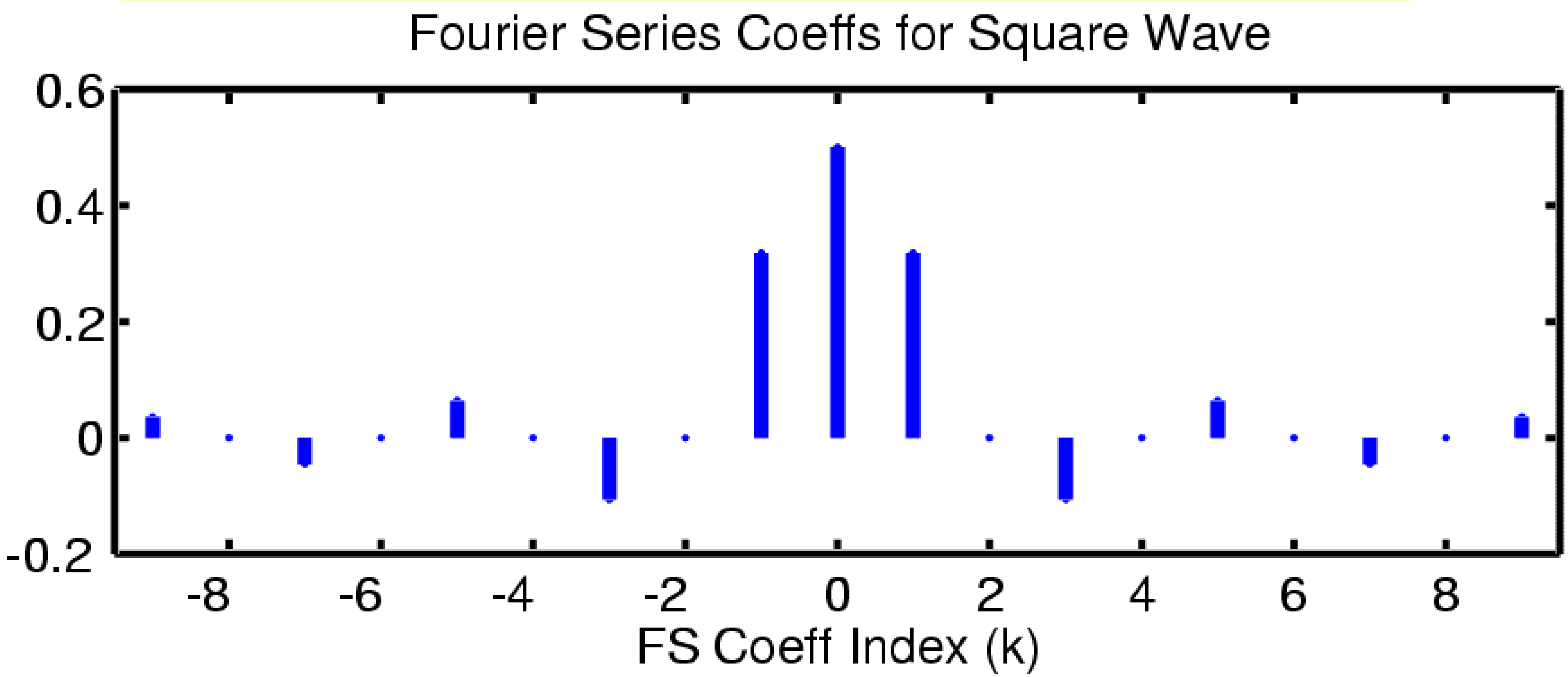


$$a_k = \frac{1}{T_0}\int_{-T_0/4}^{T_0/4}(1)e^{-j\omega_0 k t}\,dt$$

$$a_k = \left.\frac{e^{-j\omega_0 k t}}{-j\omega_0 k T_0}\right|_{-T_0/4}^{T_0/4} = \frac{e^{-j\pi k/2} - e^{j\pi k/2}}{-j2\pi k} = \frac{\sin(\pi k/2)}{\pi k}$$

Spectrum from Fourier Series

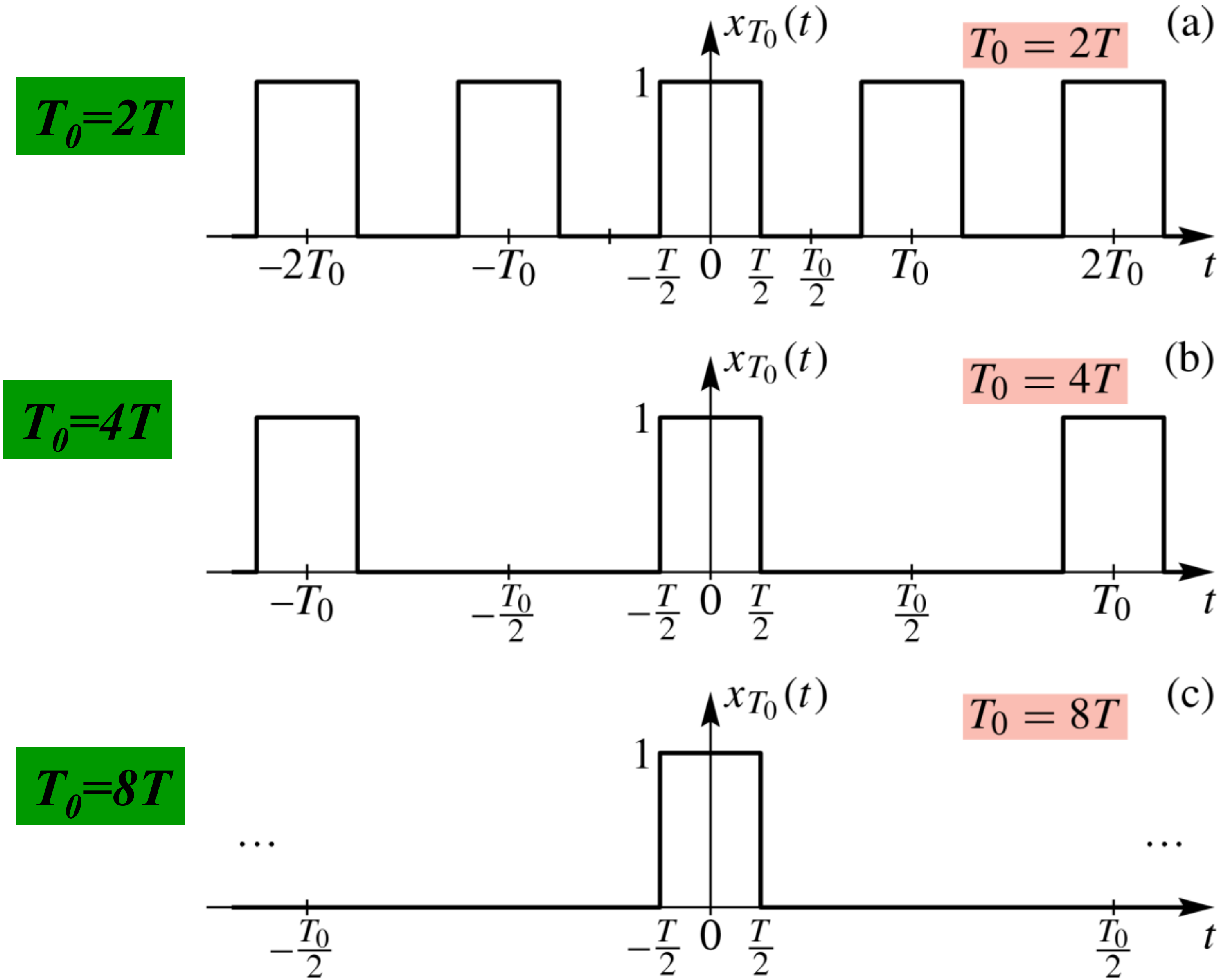
$$a_k = \frac{\sin(\pi k/2)}{\pi k} = \begin{cases} \neq 0 & k = 0, \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \end{cases}$$



What if *x(t)* is not periodic?

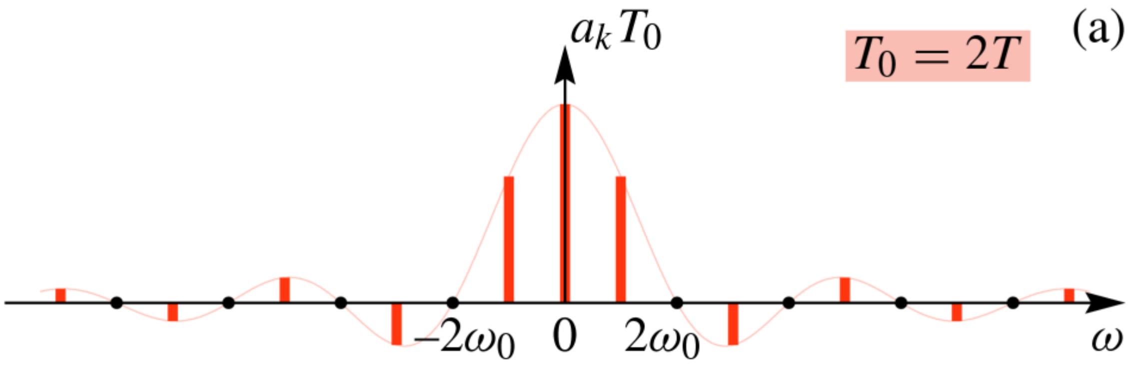
- Sum of Sinusoids?
  - Non-harmonically related sinusoids
  - Would not be periodic, but would probably be non-zero for all *t*.
- Fourier transform
  - gives a “sum” (actually an **integral**) that involves **ALL** frequencies
  - can represent signals that are identically zero for negative *t*. !!!!!!!!!

Limiting Behavior of FS

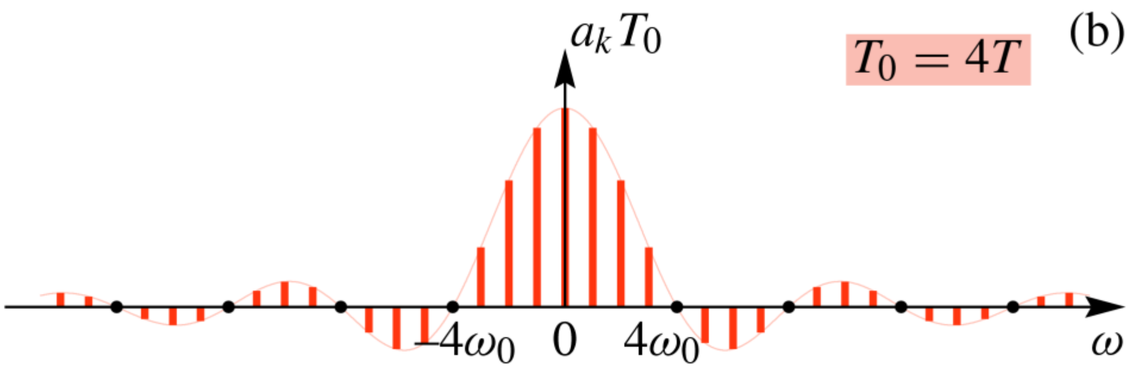


Limiting Behavior of Spectrum

T_0=2T

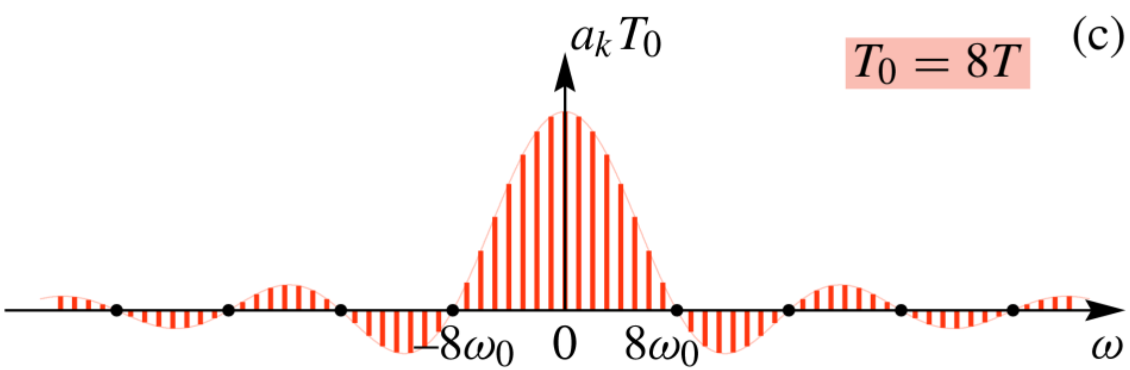


T_0=4T



Plot $(T_0 a_k)$

T_0=8T



FS in the LIMIT (long period)

$$x_{T_0}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} (T_0 a_k) e^{j\omega_0 kt} \left(\tfrac{2\pi}{T_0}\right) \mapsto x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier Synthesis

$$\lim_{T_0 \to \infty} \frac{2\pi}{T_0} = d\omega \qquad \lim_{T_0 \to \infty} \frac{2\pi}{T_0} k = \omega \qquad \lim_{T_0 \to \infty} T_0 a_k = X(j\omega)$$

$$T_0 a_k = \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-j\omega_0 kt} dt \mapsto X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Analysis

Fourier Transform Defined

- For non-periodic signals

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier Synthesis

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Analysis

Example 1: $\boxed{x(t) = e^{-at}u(t)}$

$$X(j\omega) = \int_0^\infty e^{-at} e^{-j\omega t} dt = \int_0^\infty e^{-(a+j\omega)t} dt$$

$\boxed{a > 0}$

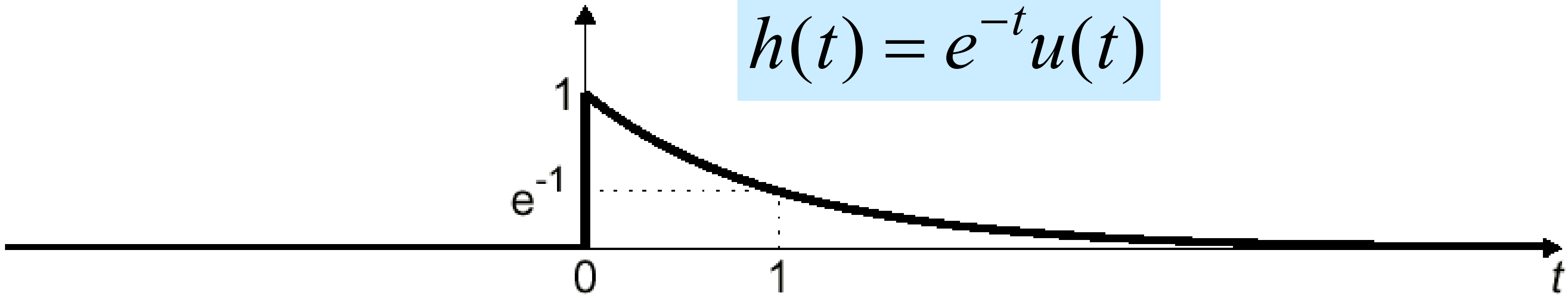
$$X(j\omega) = -\frac{e^{-at} e^{-j\omega t}}{a + j\omega}\Bigg|_0^\infty = \frac{1}{a + j\omega}$$

$$\boxed{X(j\omega) = \frac{1}{a + j\omega}}$$

Frequency Response

- Fourier Transform of $h(t)$ **is** the Frequency Response

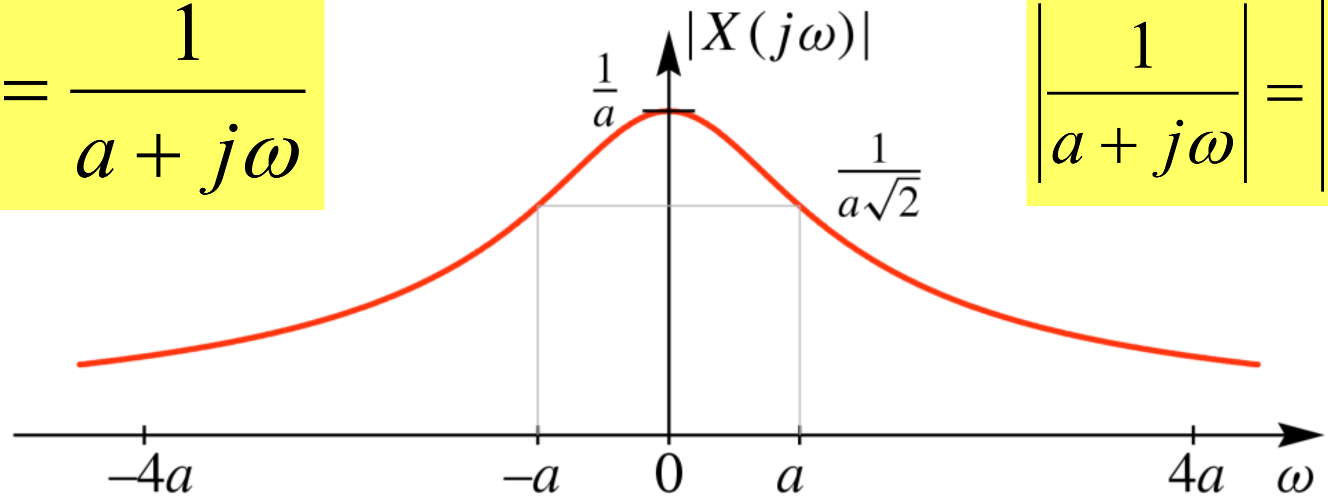
$$h(t) = e^{-t}u(t)$$



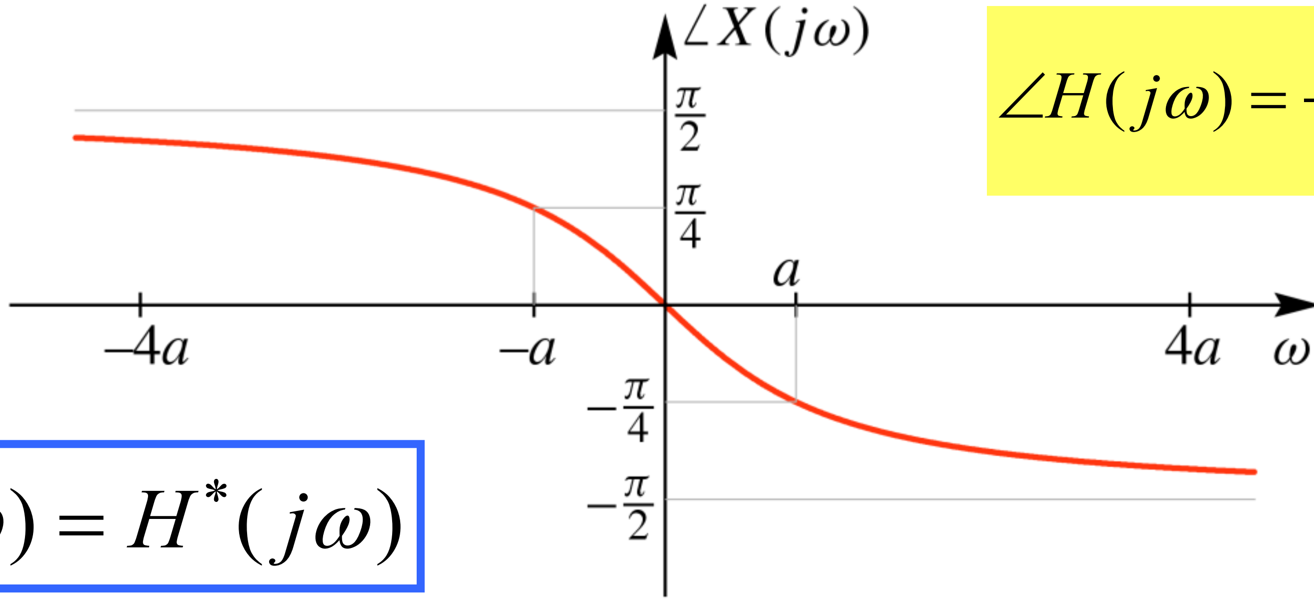
$$\boxed{h(t) = e^{-t}u(t) \Leftrightarrow H(j\omega) = \frac{1}{1+j\omega}}$$

Magnitude and Phase Plots

$$H(j\omega) = \frac{1}{a + j\omega}$$



$$\left|\frac{1}{a + j\omega}\right| = \left|\frac{1}{\sqrt{a^2 + \omega^2}}\right|$$



$$\angle H(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

$$\boxed{H(-j\omega) = H^*(j\omega)}$$

Example 2:

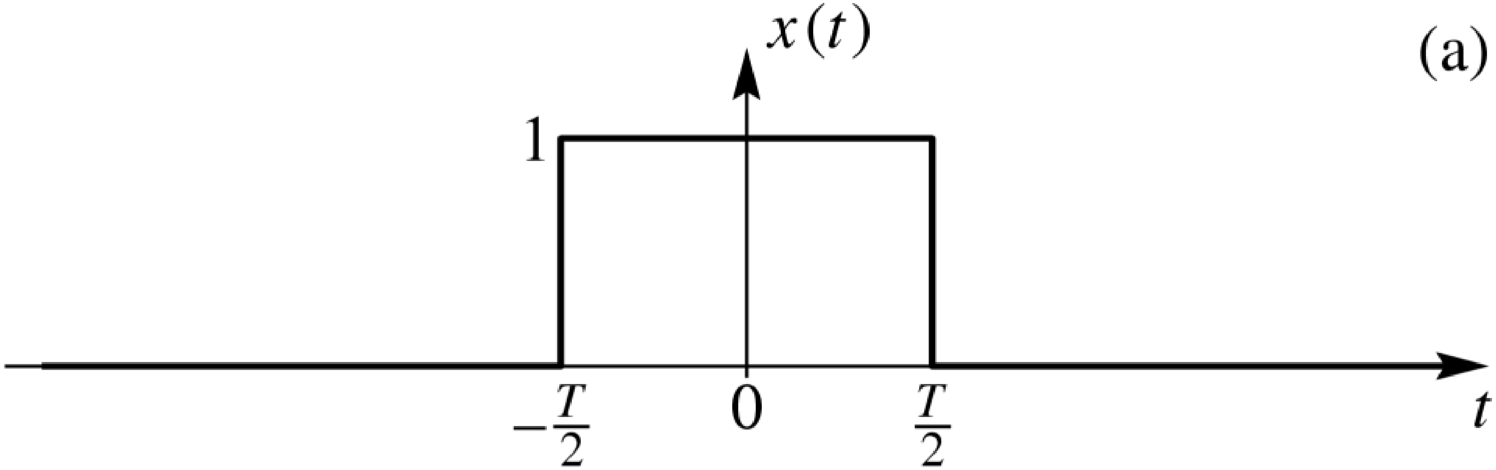
$$\boxed{x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases}}$$

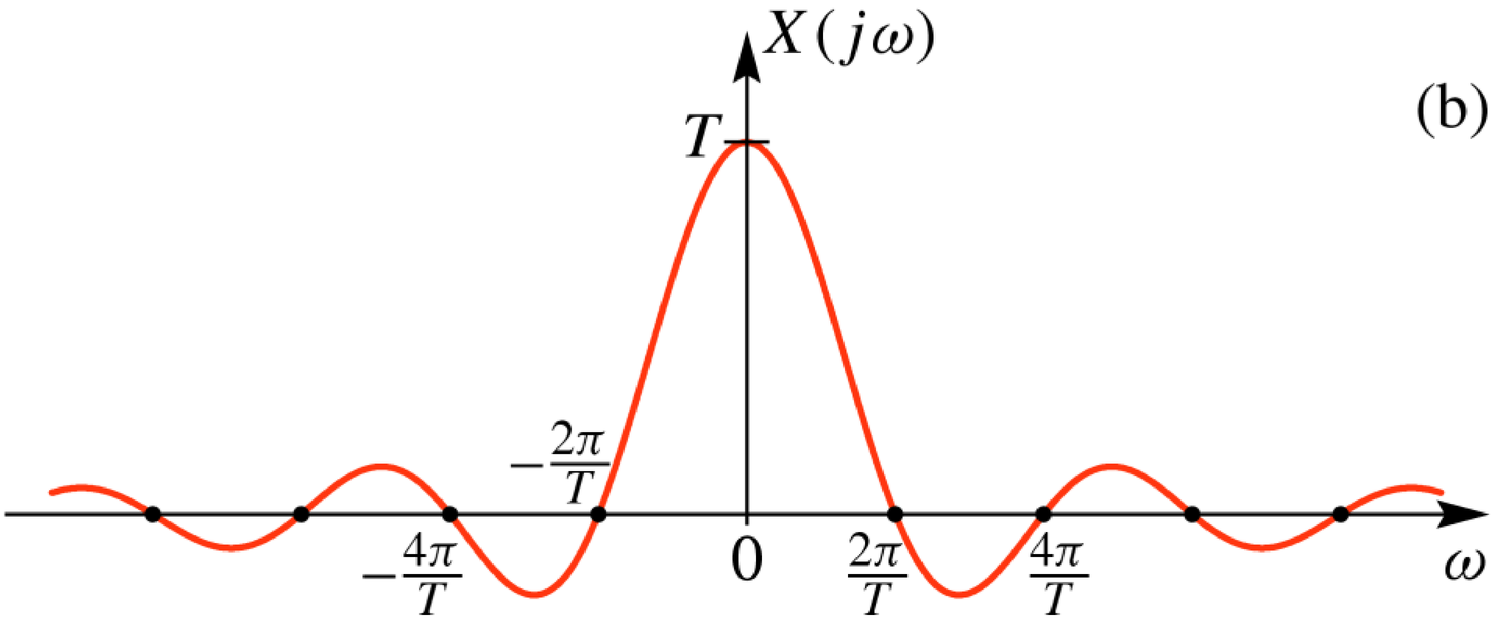
$$X(j\omega) = \int_{-T/2}^{T/2} (1) e^{-j\omega t} dt = \int_{-T/2}^{T/2} e^{-j\omega t} dt$$

$$X(j\omega) = \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{-T/2}^{T/2} = \frac{e^{-j\omega T/2} - e^{j\omega T/2}}{-j\omega}$$

$$\boxed{X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}}$$

$$x(t)=\begin{cases}1 & |t|<T/2\\ 0 & |t|>T/2\end{cases} \quad\Leftrightarrow\quad X(j\omega)=\frac{\sin(\omega T/2)}{(\omega/2)}$$





Example 3:

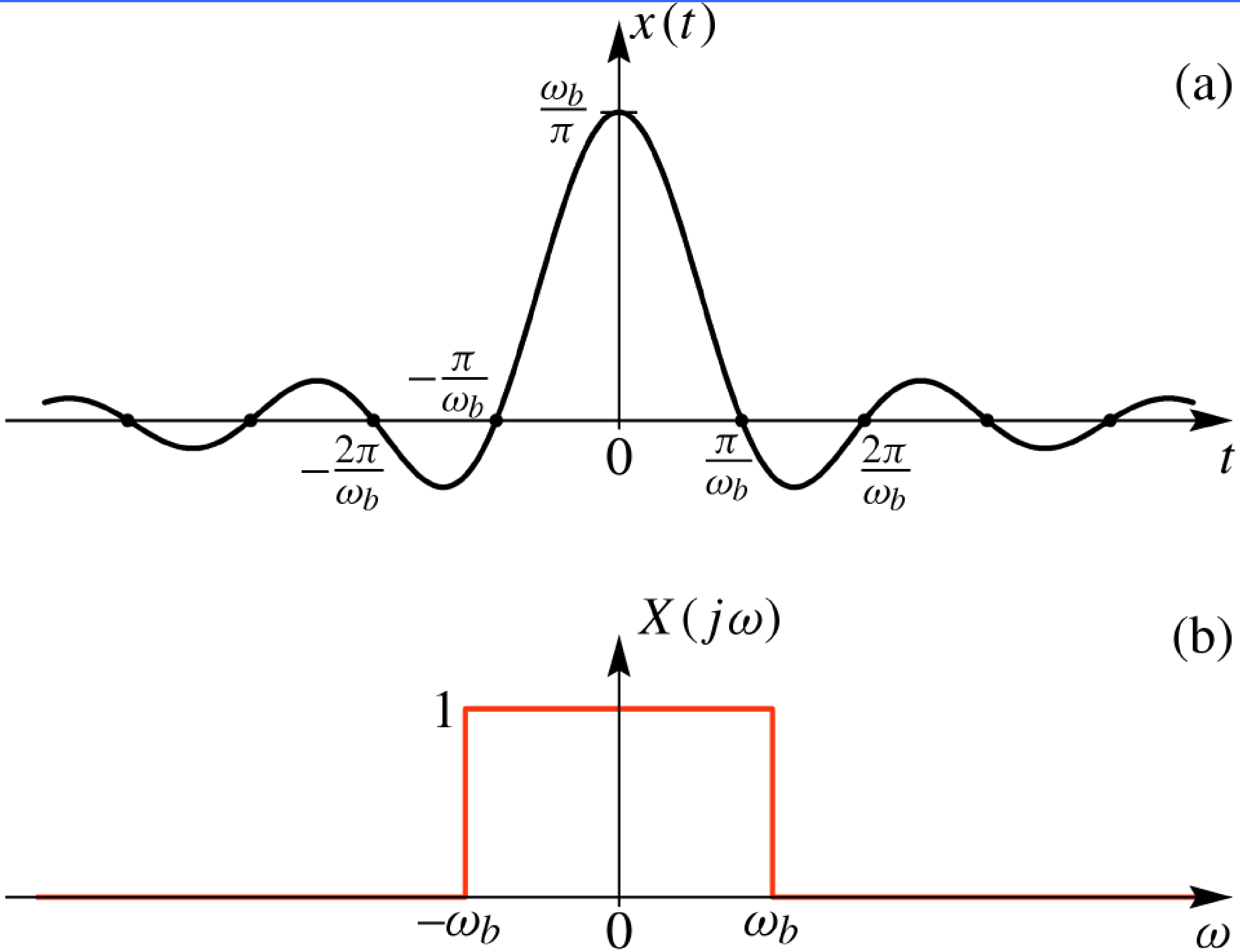
$$\boxed{X(j\omega) = \begin{cases} 1 & |\omega| < \omega_b \\ 0 & |\omega| > \omega_b \end{cases}}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\omega_b}^{\omega_b} 1e^{j\omega t}d\omega$$

$$x(t) = \frac{1}{2\pi}\frac{e^{j\omega t}}{jt}\Bigg|_{-\omega_b}^{\omega_b} = \frac{1}{2\pi}\frac{e^{j\omega_b t} - e^{-j\omega_b t}}{jt}$$

$$\boxed{x(t) = \frac{\sin(\omega_b t)}{\pi t}}$$

$$x(t) = \frac{\sin(\omega_b t)}{\pi t} \quad \Leftrightarrow \quad X(j\omega) = \begin{cases} 1 & |\omega| < \omega_b \\ 0 & |\omega| > \omega_b \end{cases}$$



Example 4:

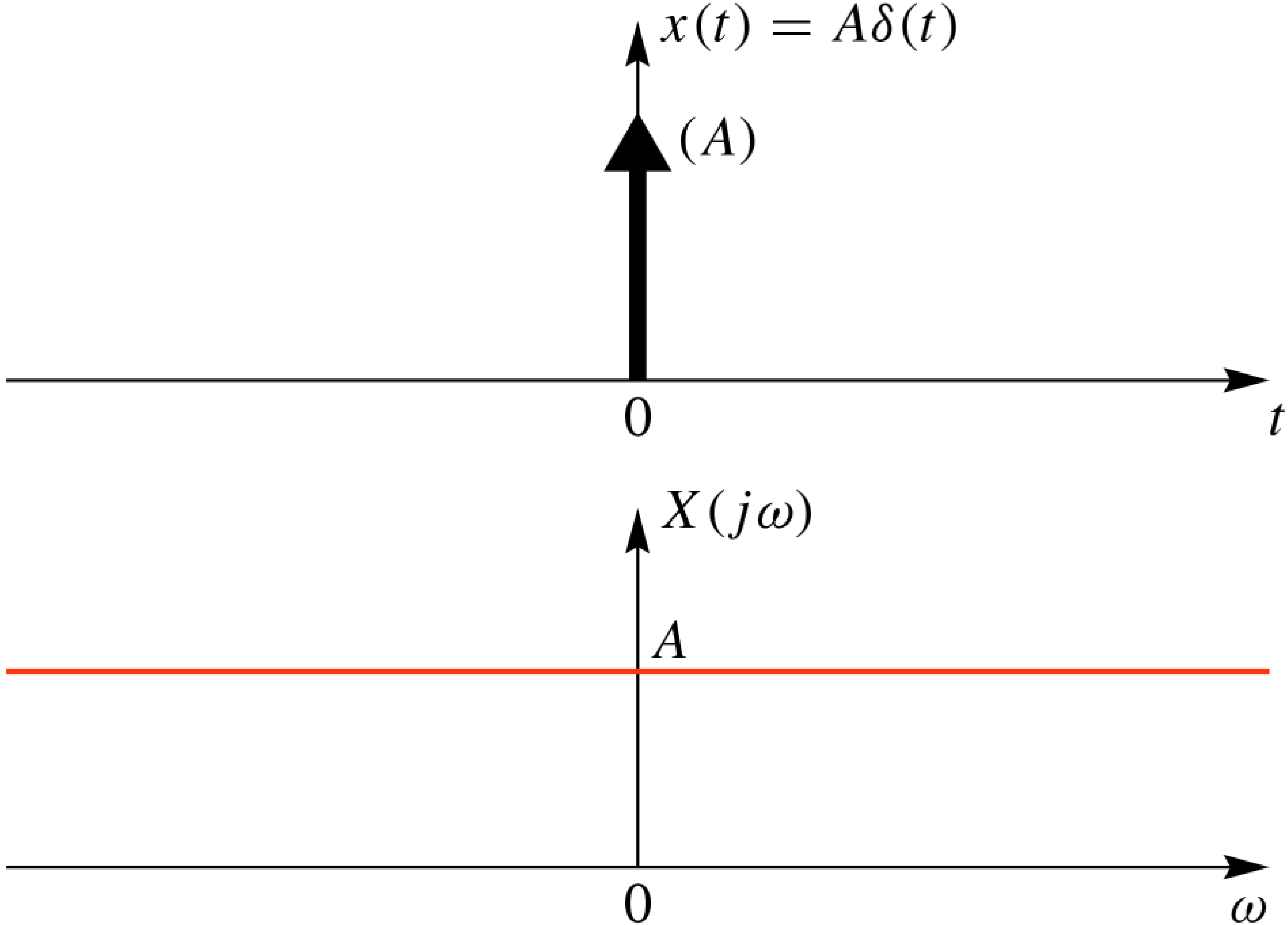
$$\boxed{x(t) = \delta(t - t_0)}$$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

Shifting Property of the Impulse

$$\boxed{X(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} dt = e^{-j\omega t_0}}$$

$$\boxed{x(t) = \delta(t) \Leftrightarrow X(j\omega) = 1}$$



Example 5: $\boxed{X(j\omega) = 2\pi\delta(\omega - \omega_0)}$

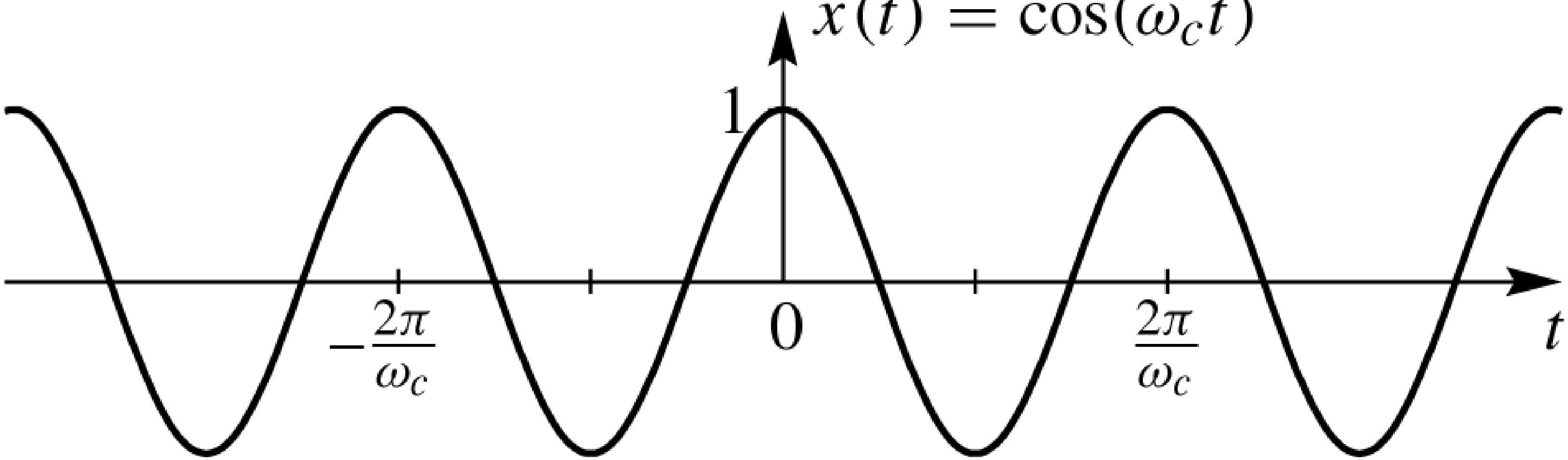
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}d\omega = e^{j\omega_0 t}$$

$$\boxed{x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)}$$

$$\boxed{x(t) = 1 \Leftrightarrow X(j\omega) = 2\pi\delta(\omega)}$$

$$\boxed{\begin{aligned} &x(t) = \cos(\omega_0 t) \Leftrightarrow \\ &\qquad X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0) \end{aligned}}$$

$$x(t)=\cos(\omega_0 t) \Leftrightarrow$$
$$X(j\omega)=\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$$



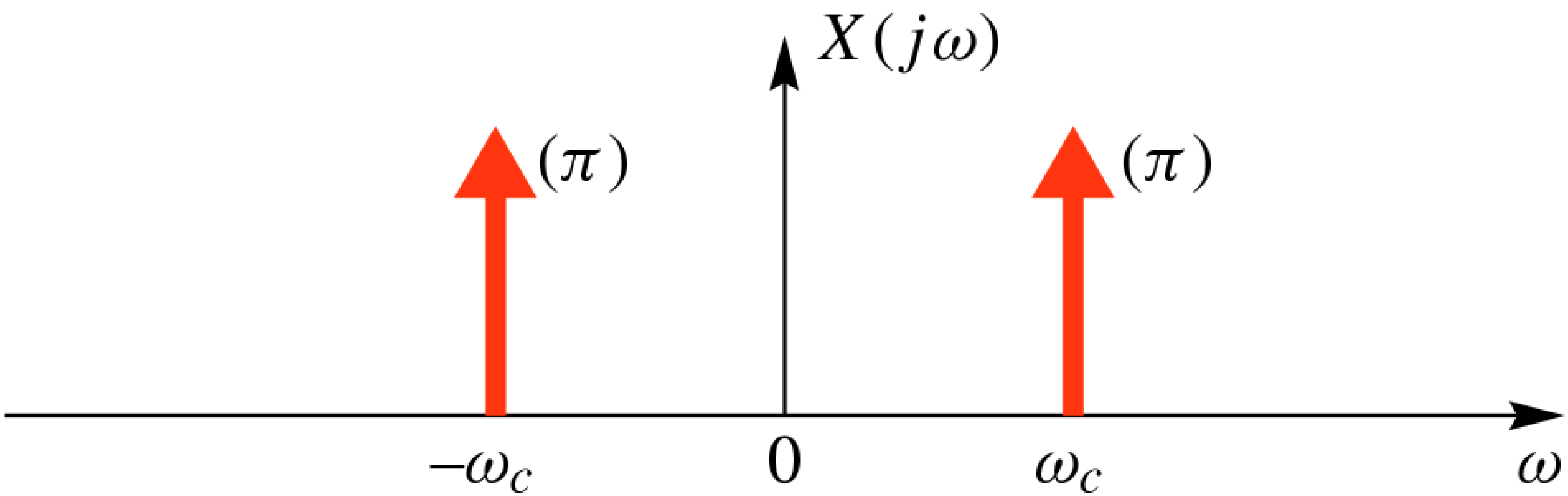


Table of Fourier Transforms

$$x(t) = e^{-at}u(t) \Leftrightarrow X(j\omega) = \frac{1}{a + j\omega}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \quad \Leftrightarrow \quad X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)} \quad \Leftrightarrow \quad X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

$$x(t) = \delta(t - t_0) \Leftrightarrow X(j\omega) = e^{-j\omega t_0}$$

$$x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

Fourier Transform Properties

- The Fourier transform

$$\boxed{X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt}$$

- More examples of Fourier transform pairs
- Basic properties of Fourier transforms
  - **Convolution** property
  - **Multiplication** property

Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier Synthesis (Inverse Transform)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Analysis (Forward Transform)

$$\text{Time - Domain} \Leftrightarrow \text{Frequency - Domain}$$

$$x(t) \Leftrightarrow X(j\omega)$$

Table of Easy FT Properties

Linearity Property

$$ax_1(t) + bx_2(t) \Leftrightarrow aX_1(j\omega) + bX_2(j\omega)$$

Delay Property

$$x(t - t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)$$

Frequency Shifting

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

Scaling

$$x(at) \Leftrightarrow \frac{1}{|a|} X(j(\frac{\omega}{a}))$$

Scaling Property

$$\boxed{x(at) \Leftrightarrow \frac{1}{|a|} X(j\frac{\omega}{a})}$$

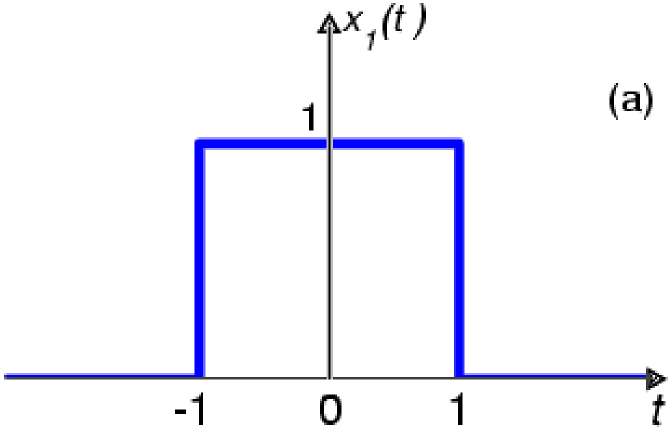
$$\int_{-\infty}^{\infty} x(at)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(\lambda)e^{-j\omega(\lambda/a)} \frac{d\lambda}{|a|}$$

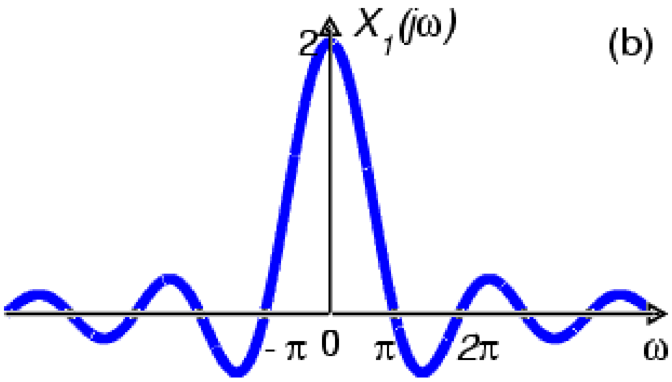
$$= \frac{1}{|a|} X(j\frac{\omega}{a})$$

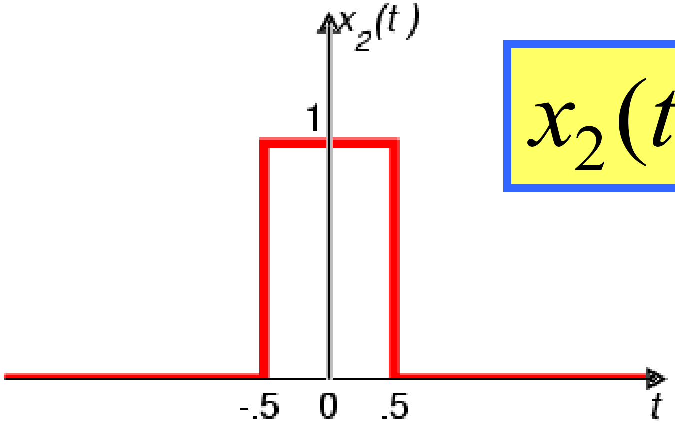
$x(2t)$ shrinks; $\quad \frac{1}{2} X(j\frac{\omega}{2})$ expands

Scaling Property

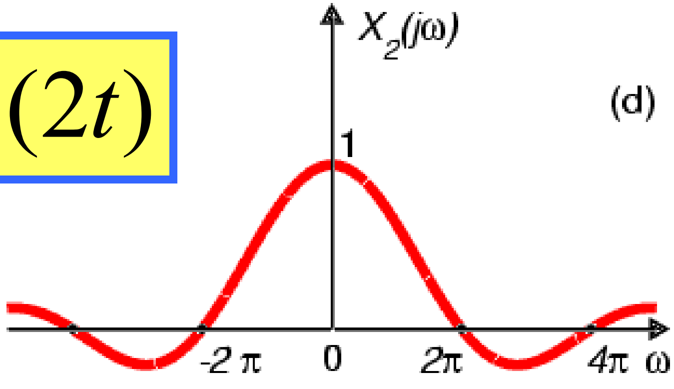
$$x(at) \Leftrightarrow \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$$







$$x_2(t) = x_1(2t)$$



Uncertainty Principle

- Try to make $x(t)$ shorter
  - Then $X(j\omega)$ will get wider
  - Narrow pulses have wide bandwidth
- Try to make $X(j\omega)$ narrower
  - Then $x(t)$ will have longer duration
- **Cannot simultaneously reduce time duration and bandwidth**

Significant FT Properties

$$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

$$x(t)p(t) \Leftrightarrow \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

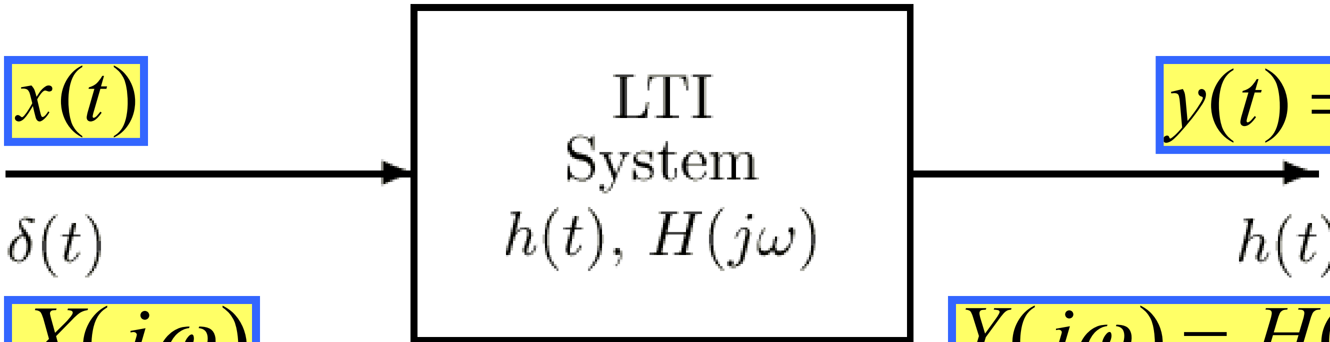
Differentiation Property

$$\frac{dx(t)}{dt} \Leftrightarrow (j\omega)X(j\omega)$$

Convolution Property

$x(t)$

$y(t) = h(t) * x(t)$



$X(j\omega)$

$Y(j\omega) = H(j\omega)X(j\omega)$

- Convolution in the time-domain

$$\boxed{y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau}$$

corresponds to **MULTIPLICATION** in the frequency-domain

$$\boxed{Y(j\omega) = H(j\omega)X(j\omega)}$$

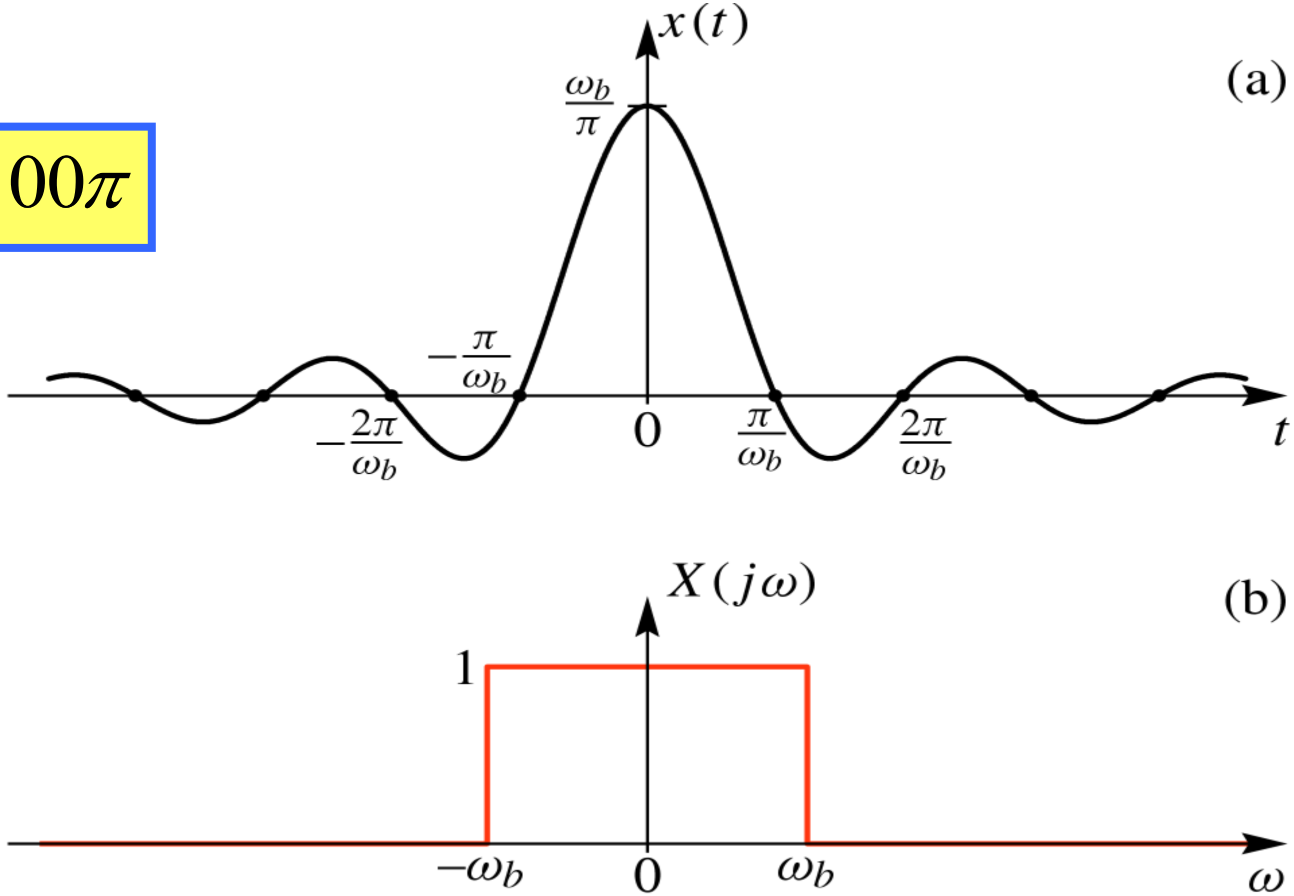
Convolution Example

- Bandlimited **Input** Signal
  - “sinc” function
- Ideal LPF (Lowpass Filter)
  - ***h*(*t*)** is a “sinc”
- **Output** is Bandlimited
  - Convolve “sincs”

Ideally Bandlimited Signal

$$x(t) = \frac{\sin(100\pi t)}{\pi t} \quad \Leftrightarrow \quad X(j\omega) = \begin{cases} 1 & |\omega| < 100\pi \\ 0 & |\omega| > 100\pi \end{cases}$$

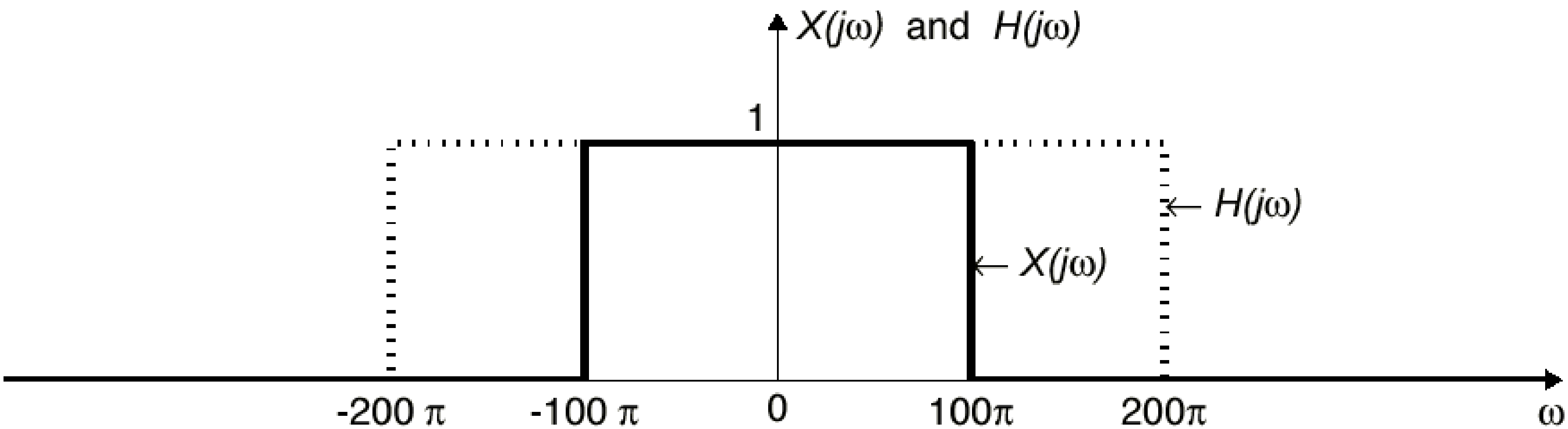
$$\omega_b = 100\pi$$



Convolution Example

$$\boxed{x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)}$$

$$\frac{\sin(100\pi t)}{\pi t} * \frac{\sin(200\pi t)}{\pi t} = \frac{\sin(100\pi t)}{\pi t}$$



Cosine Input to LTI System

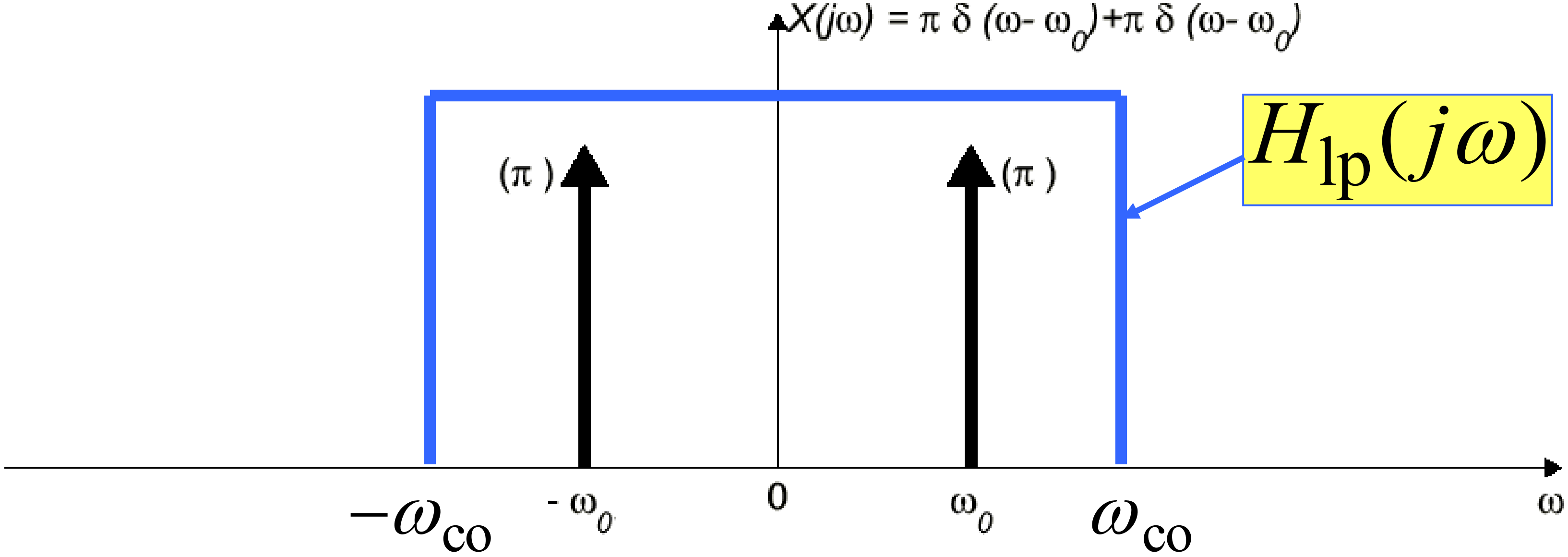
$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$= H(j\omega)[\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)]$$

$$= H(j\omega_0)\pi\delta(\omega-\omega_0)+H(-j\omega_0)\pi\delta(\omega+\omega_0)$$

$$\begin{aligned} y(t) &= H(j\omega_0)\tfrac{1}{2}e^{j\omega_0 t}+H(-j\omega_0)\tfrac{1}{2}e^{-j\omega_0 t} \\ &= H(j\omega_0)\tfrac{1}{2}e^{j\omega_0 t}+H^*(j\omega_0)\tfrac{1}{2}e^{-j\omega_0 t} \\ &= \left|H(j\omega_0)\right|\cos(\omega_0 t+\angle H(j\omega_0)) \end{aligned}$$

Ideal Lowpass Filter

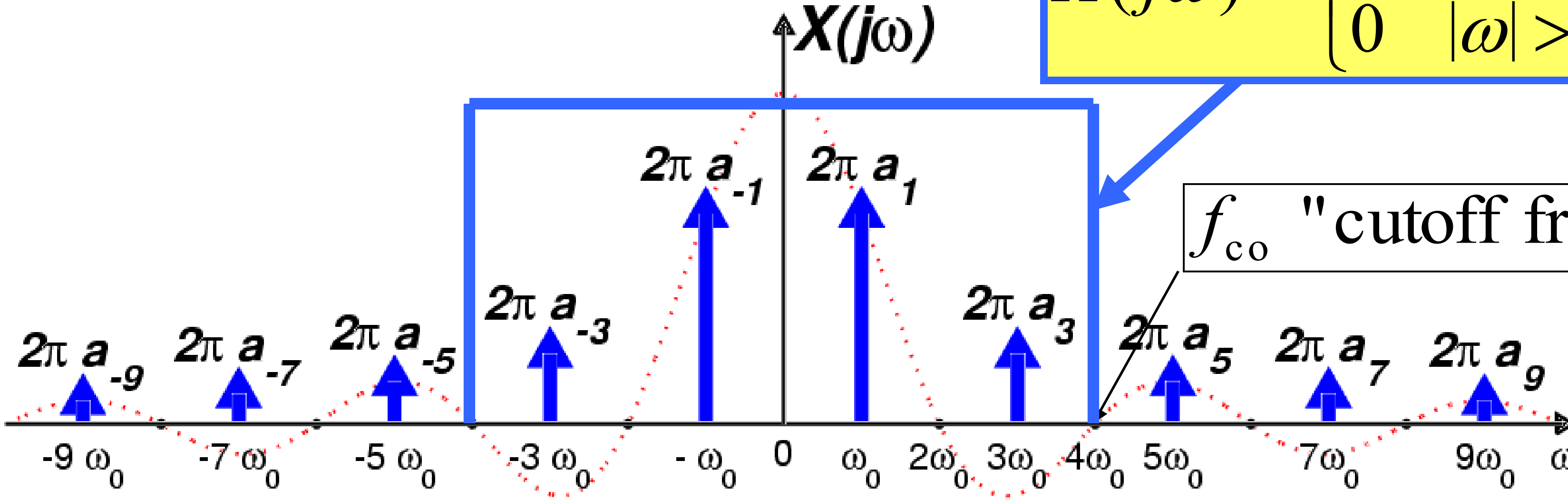


$$y(t) = x(t) \qquad \text{if } \omega_0 < \omega_{\mathrm{co}}$$

$$y(t) = 0 \qquad \text{if } \omega_0 > \omega_{\mathrm{co}}$$

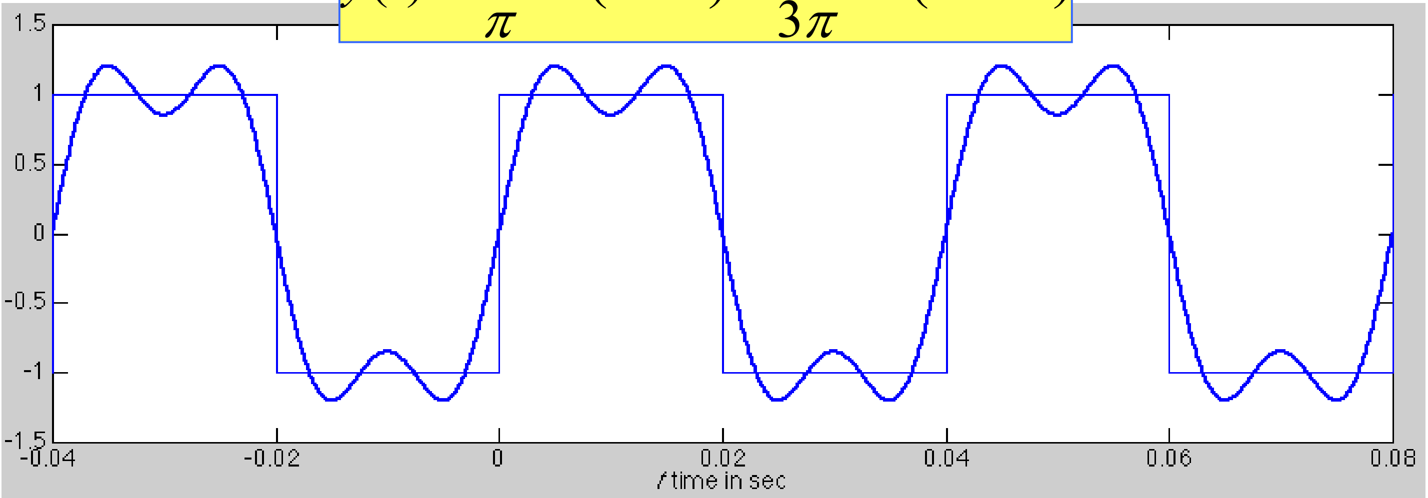
Ideal Lowpass Filter

$$H(j\omega) = \begin{cases} 1 & |\omega| < \omega_{co} \\ 0 & |\omega| > \omega_{co} \end{cases}$$

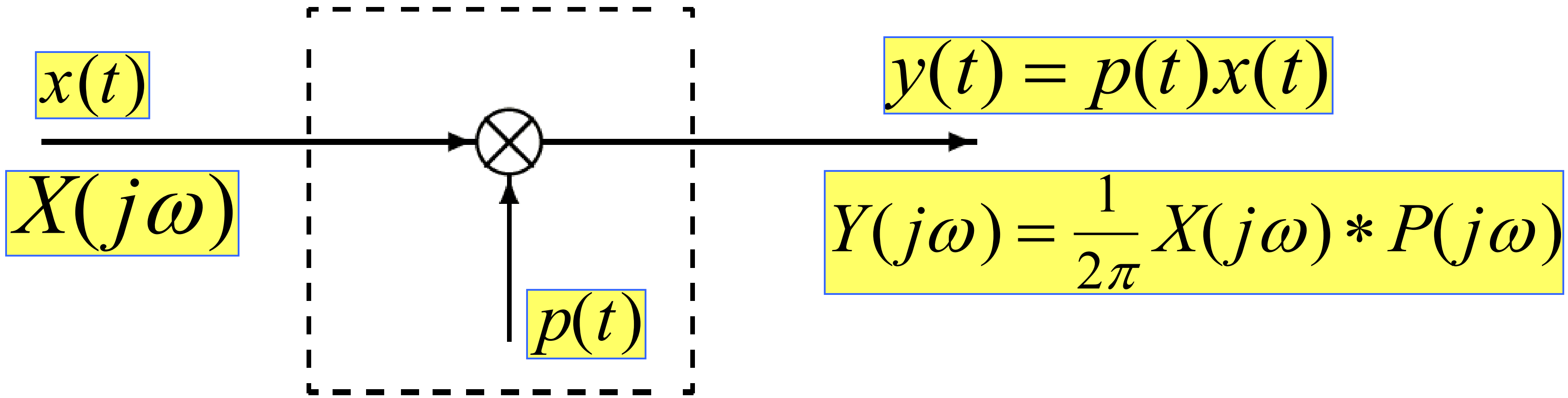


f_{co} "cutoff freq."

$$y(t) = \frac{4}{\pi}\sin(50\pi t) + \frac{4}{3\pi}\sin(150\pi t)$$



Signal Multiplier (Modulator)



- Multiplication in the time-domain corresponds to convolution in the frequency-domain.

$$Y(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta) P(j(\omega - \theta)) d\theta$$

Frequency Shifting Property

$$\boxed{x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))}$$

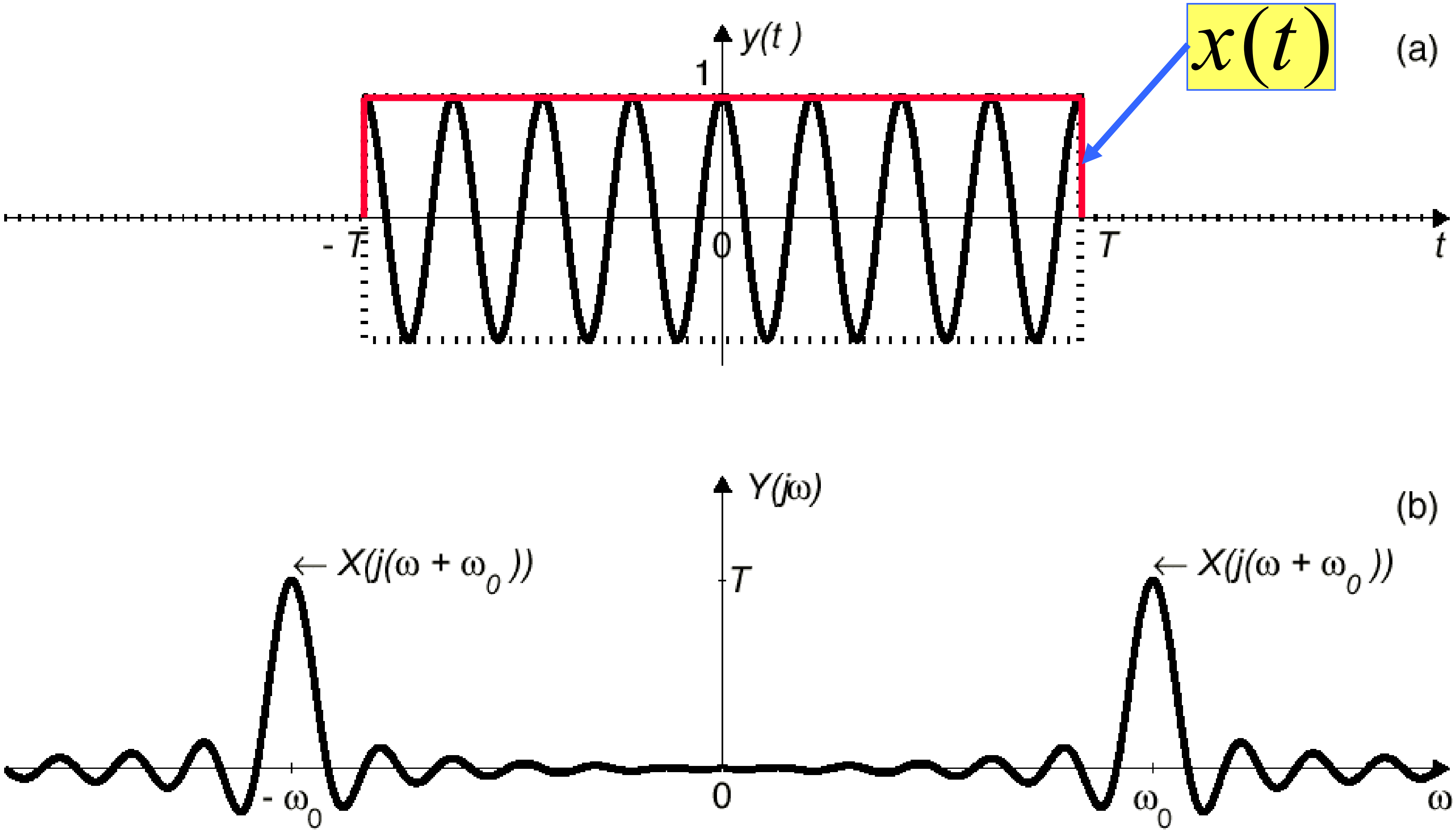
$$\int_{-\infty}^{\infty} e^{j\omega_0 t} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t) e^{-j(\omega-\omega_0)t} dt$$

$$= X(j(\omega - \omega_0))$$

$$\boxed{y(t) = \frac{\sin 7t}{\pi t} e^{j\omega_0 t} \Leftrightarrow Y(j\omega) = \begin{cases} 1 & \omega_0 - 7 < \omega < \omega_0 + 7 \\ 0 & elsewhere \end{cases}}$$

$$y(t) = x(t)\cos(\omega_0 t) \Leftrightarrow$$

$$Y(j\omega) = \frac{1}{2}X(j(\omega - \omega_0)) + \frac{1}{2}X(j(\omega + \omega_0))$$



Differentiation Property

$$\frac{dx(t)}{dt} = \frac{d}{dt}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} (j\omega)X(j\omega)e^{j\omega t}d\omega$$

Multiply by jω

$$\frac{d}{dt}\left(e^{-at}u(t)\right) = -ae^{-at}u(t) + e^{-at}\delta(t)$$

$$= \delta(t) - ae^{-at}u(t)$$

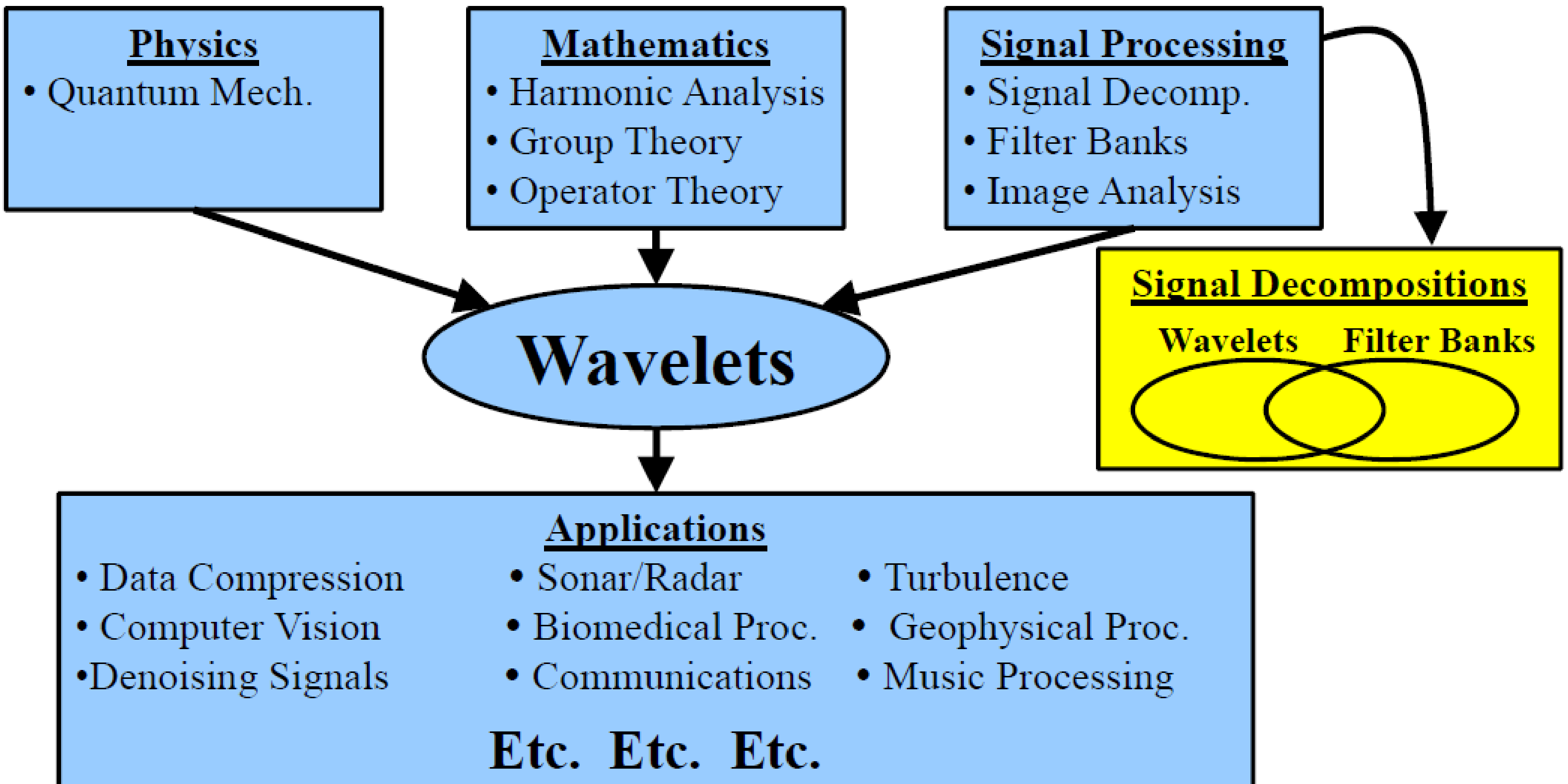
$$\Leftrightarrow \frac{j\omega}{a+j\omega}$$

Introduction to the Wavelet Transform

Origins and Applications

The Wavelet Transform (WT) is a signal processing tool that is replacing the Fourier Transform (FT) in many (but not all!) applications.

WT theory has its origins in ideas in three main areas and now is being applied in countless different areas of application.



So, What's Wrong With The FT?

First, recall the FT:

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$$

Weight @ f

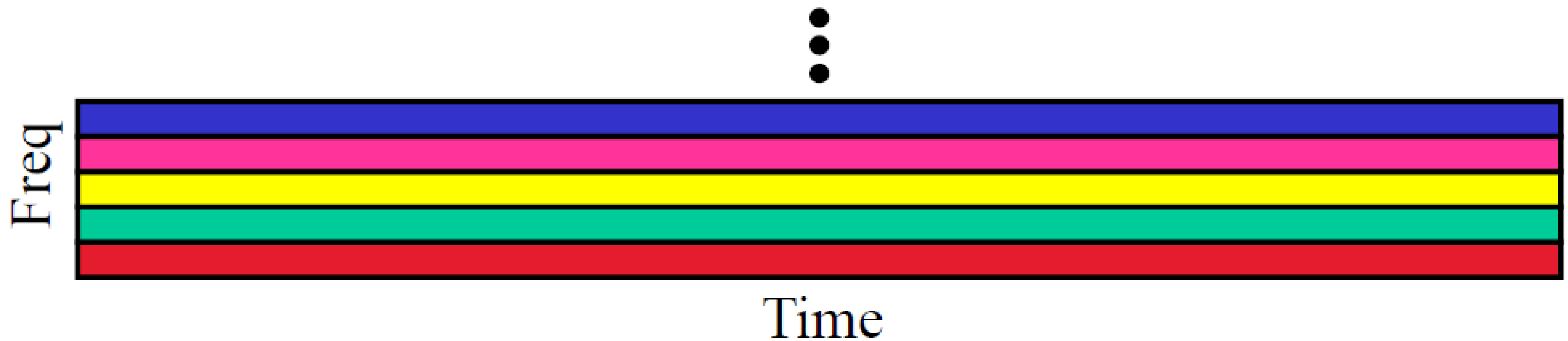
Component @ f

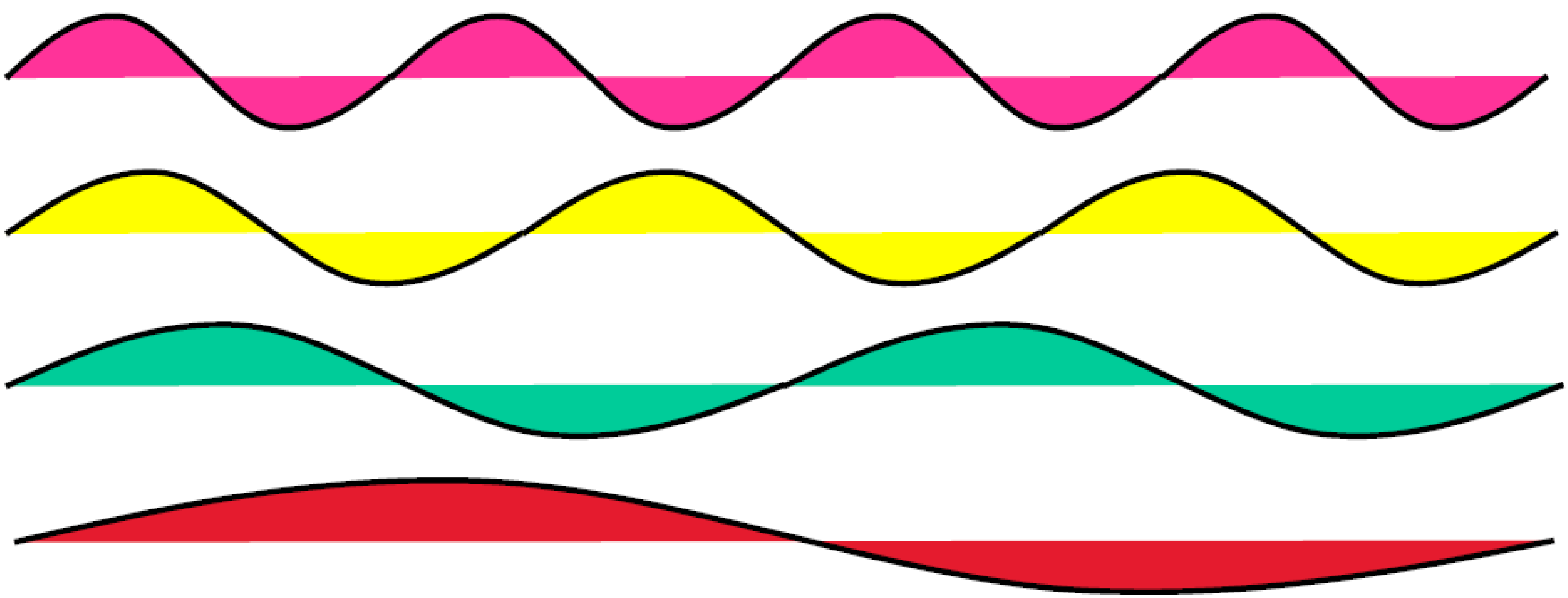
Remember: An integral is like a summation… So, the second equation says that we are decomposing $x(t)$ into a weighted "sum" of complex exponentials (sinusoids!)… The first equation tells what each weight should be.

Note: These components exist for ***ALL*** time!!!

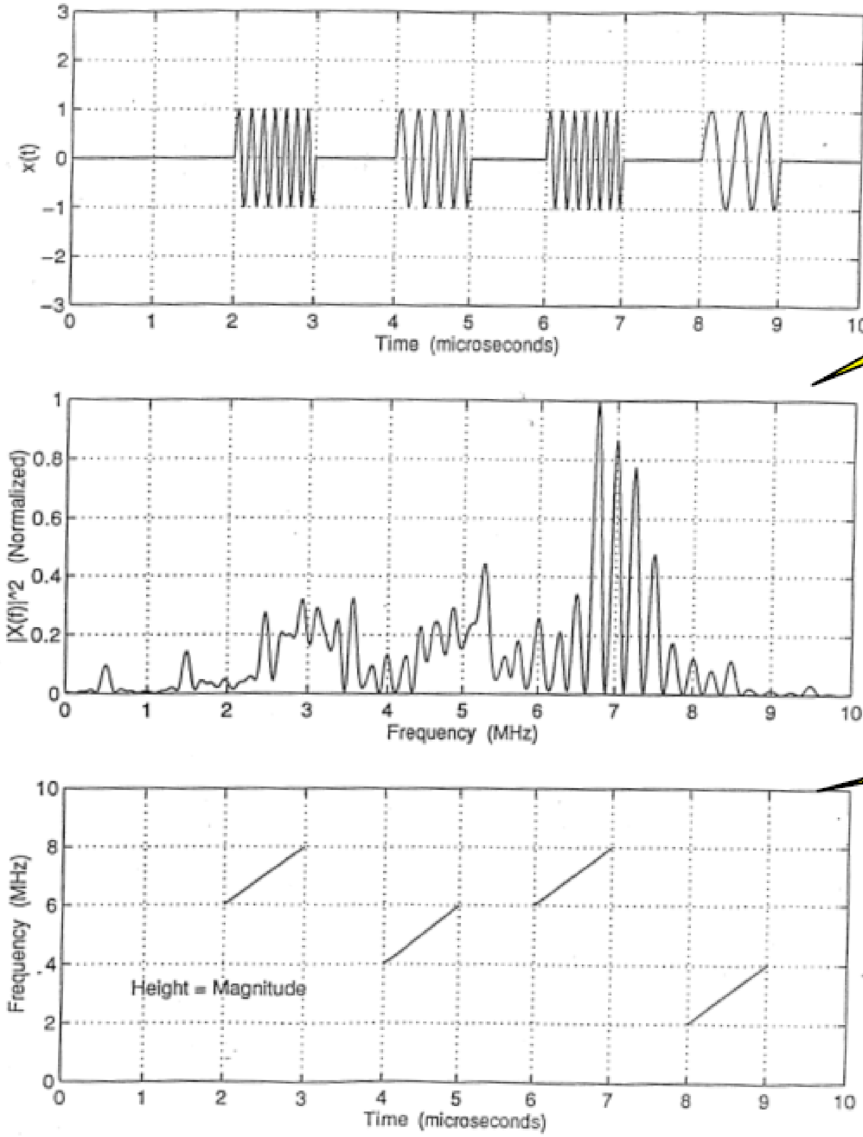
This is not necessarily a good model for real-life signals.

DFT Basis Functions… and "Time-Freq Tiles"





Example: Frequency-Hopping Chirped Pulses



DFT tries to build from freq components that last 10 μsec

The DFT representation is "correct" but does not show us a "joint representation" in time and frequency

This would be a more desirable time-frequency representation

It would show at each time exactly what frequencies were in existence at that time!

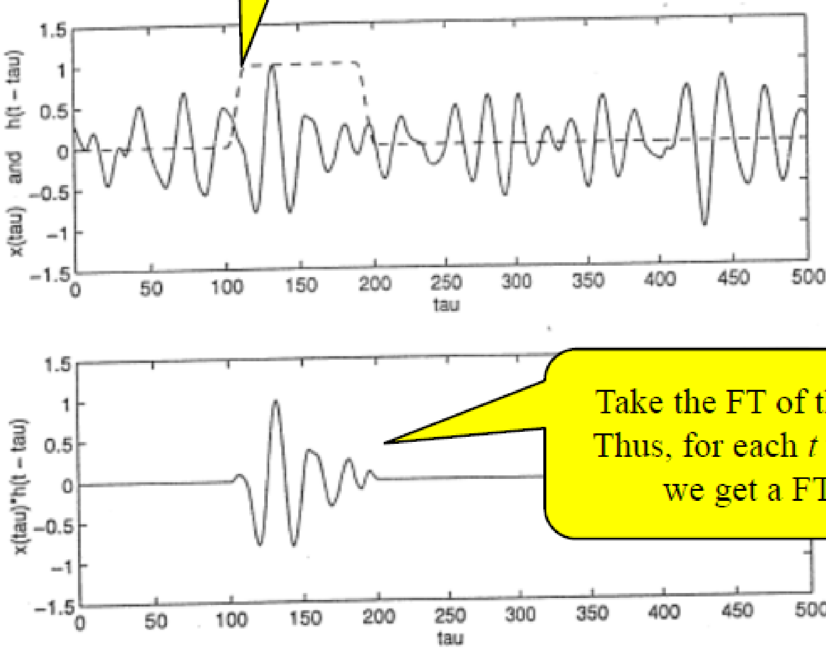
However… such a "perfect" t-f representation is not possible… Heisenberg Uncertainty Principle!

But What About The Short-Time FT (STFT)?

$$X(f,t) = \int_{-\infty}^{\infty} x(\tau) h(\tau - t) e^{-j2\pi f \tau} d\tau$$

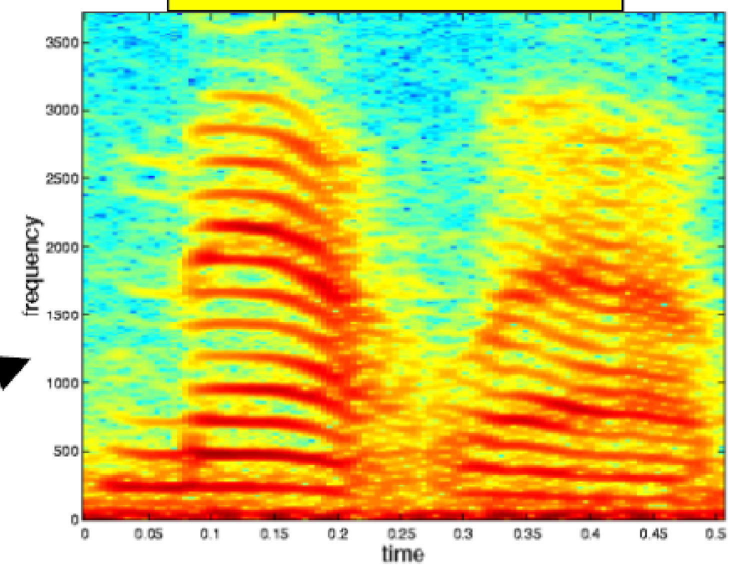
Dummy Variable!

$h(\tau - t)$



Take the FT of this…
Thus, for each t value
we get a FT

Spectrogram = $|X(f,t)|^2$



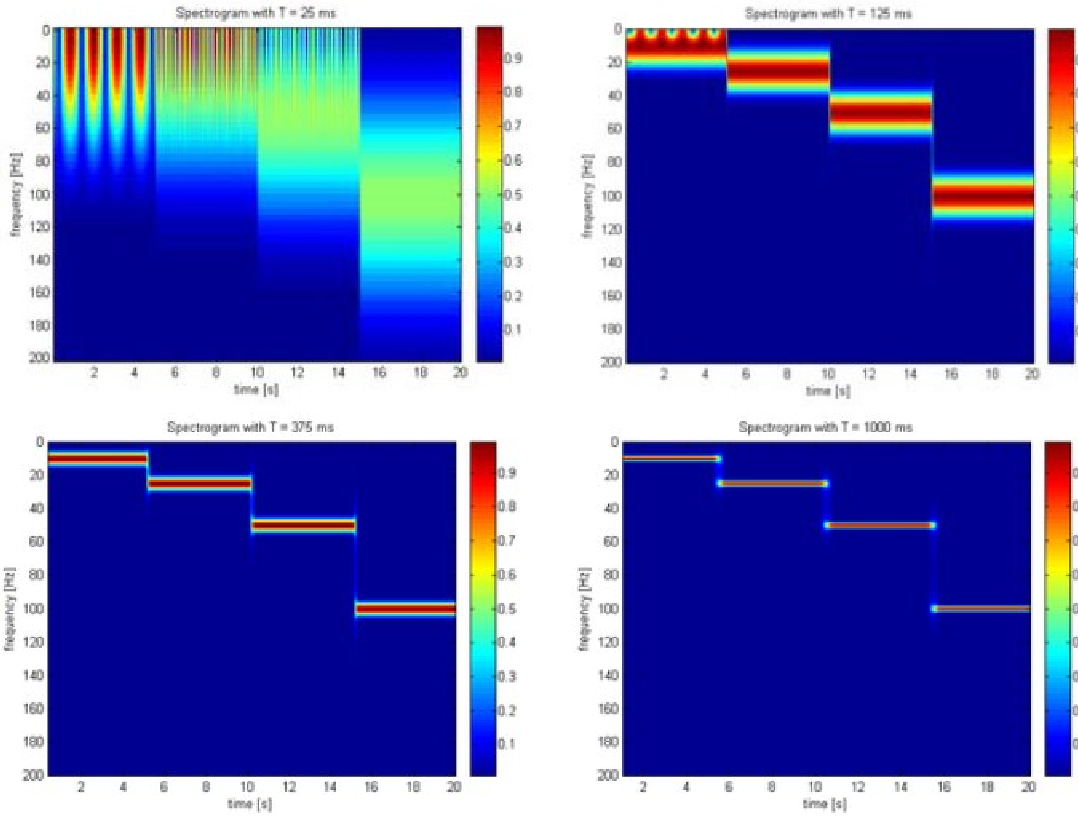
Selesnick, Ivan, "Short Time Fourier Transform," Connexions, August 9, 2005. http://cnx.org/content/m10570/2.4/.

STFT T-F Resolution

- The window function $h(t)$ sets the characteristic of how the STFT is able to "probe" the signal $x(t)$.
  - The narrower $h(t)$ is, the better you can resolve the time of occurrence of a feature
  - However… the narrower $h(t)$ is, the wider $H(f)$ is… and that means a reduction in the ability to resolve frequency occurance
    - Just like windowing of the DFT that you've probably studied!
- Each given $h(t)$ has a given time and frequency resolution
  - Δt describes the time resolution
  - Δf describes the frequency resolution
- The Heisenberg Uncertainty Principle states that
  - $$(\Delta t)(\Delta f) \geq \frac{1}{4\pi}$$
  - Improving Time Resolution…. Degrades Frequency Resolution
    - And vice versa

Illustration of Time-Frequency Resolution Trade-Off

$$x(t) = \begin{cases} \cos(2\pi 10t) & 0 \le t < 5 \\ \cos(2\pi 25t) & 5 \le t < 10 \\ \cos(2\pi 50t) & 10 \le t < 15 \\ \cos(2\pi 100t) & 15 \le t < 20 \end{cases}$$



Window Width 25 ms

Window Width 125 ms

Window Width 375 ms

Window Width 1000 ms

http://en.wikipedia.org/wiki/Short-time_Fourier_transform

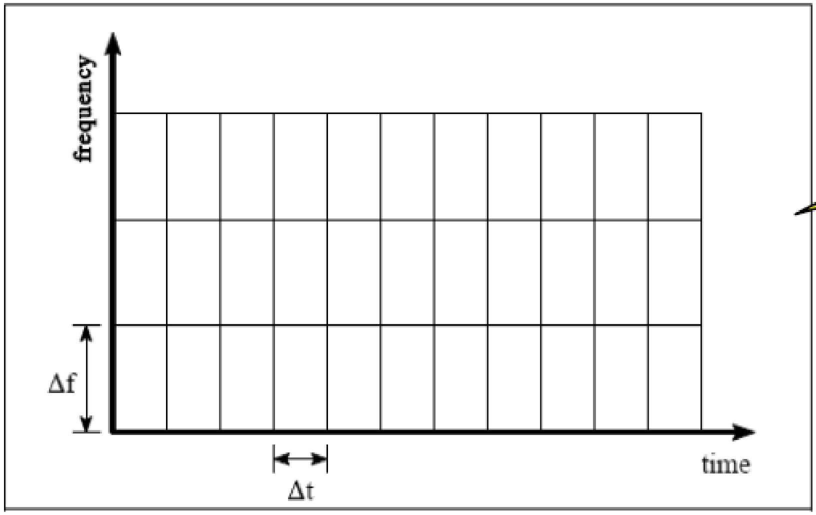
STFT View of Tiling the T-F Plane

Generally only compute the STFT for discrete values of t and f

$$X(f_m, t_n) = \int_{-\infty}^{\infty} x(\tau) h(\tau - nT) e^{-j2\pi(mF)\tau} d\tau$$

In some applications it is desirable to minimize the number of points in $X(f_m,t_n)$ and that means making T and F as large as possible… $T \approx \Delta t$ and $F \approx \Delta f$

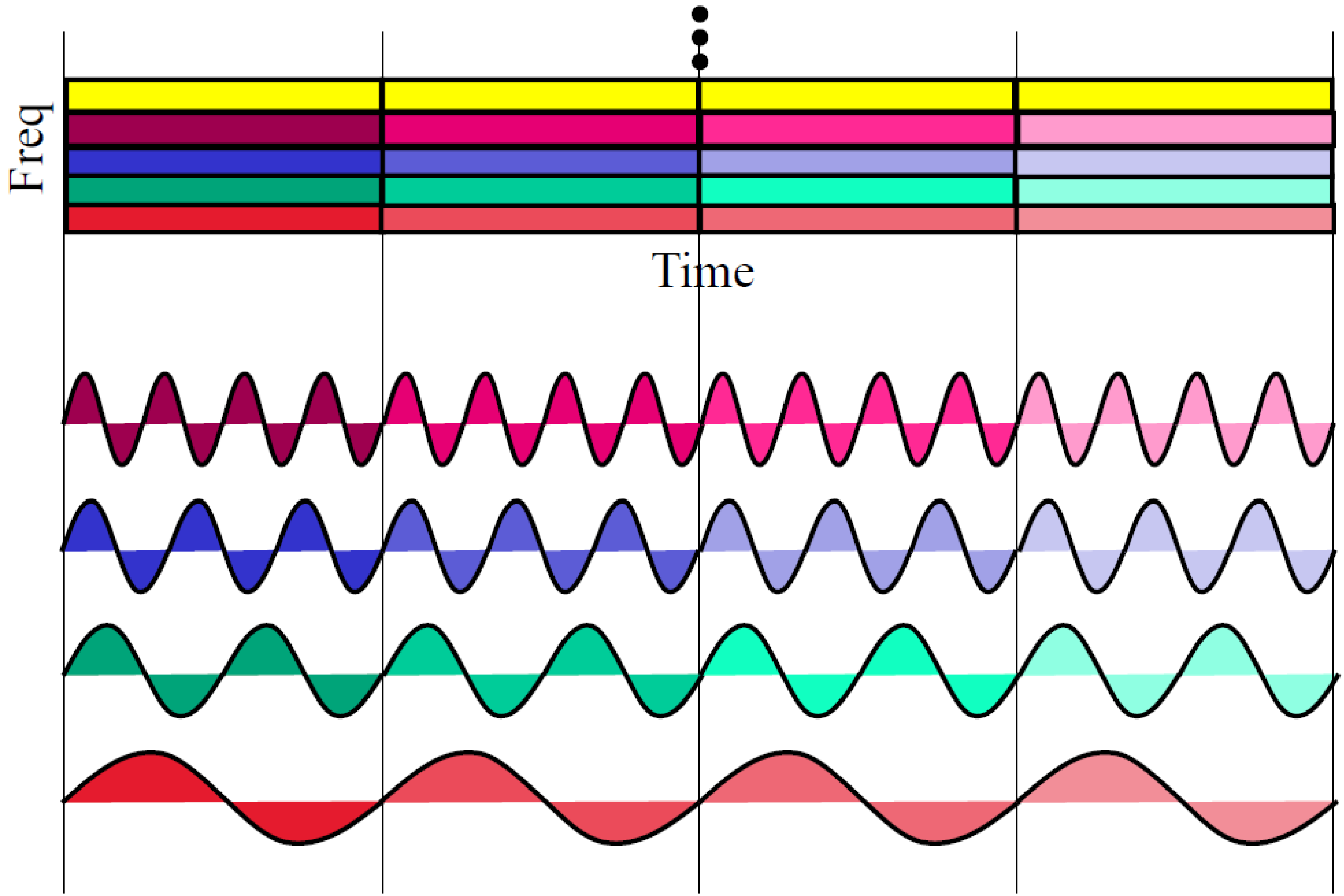
Then each $X(f_m,t_n)$ represents the "content" of the signal in a rectangular cell of dimension Δt by Δf



STFT Tiling of the T-F Plane

STFT tiling consists of a uniform tiling by fixed rectangles

STFT Basis Functions… and "Time-Freq Tiles"



STFT Disadvantages and Advantages

- The fact that the STFT tiles the plane with cells having the same Δt and Δf is a disadvantage in many application
  - Especially in the data compression!
- This characteristic leads to the following:
  - If you try to make the STFT be a "non-redundant" decomposition (e.g., ON… like is good for data compression…
  - You necessarily get very poor time-frequency resolution
- This is one of the main ways that the WT can help
  - It can provide ON decompositions while still giving good t-f resolution
- However, in applications that do not need a non-redundant decomposition the STFT is still VERY useful and popular
  - Good for applications where humans want to view results of t-f decomposition

<u>So… What *IS* the WT???</u>

Recall the STFT: $$X(f,t) = \int_{-\infty}^{\infty} x(\tau)\underbrace{h(\tau - t)e^{-j2\pi f\tau}}_{\textbf{Basis Functions}}d\tau$$

So… $X(f,t)$ is computing by "comparing" $x(t)$ to each of these basis functions

For the STFT the basis function are created by applying
<u>Time Shift</u> and <u>Frequency Shift</u> to prototype $h(t)$

This leads to the "uniform tiling" we saw before…

And it also causes the problems with the non-redundant form of the STFT

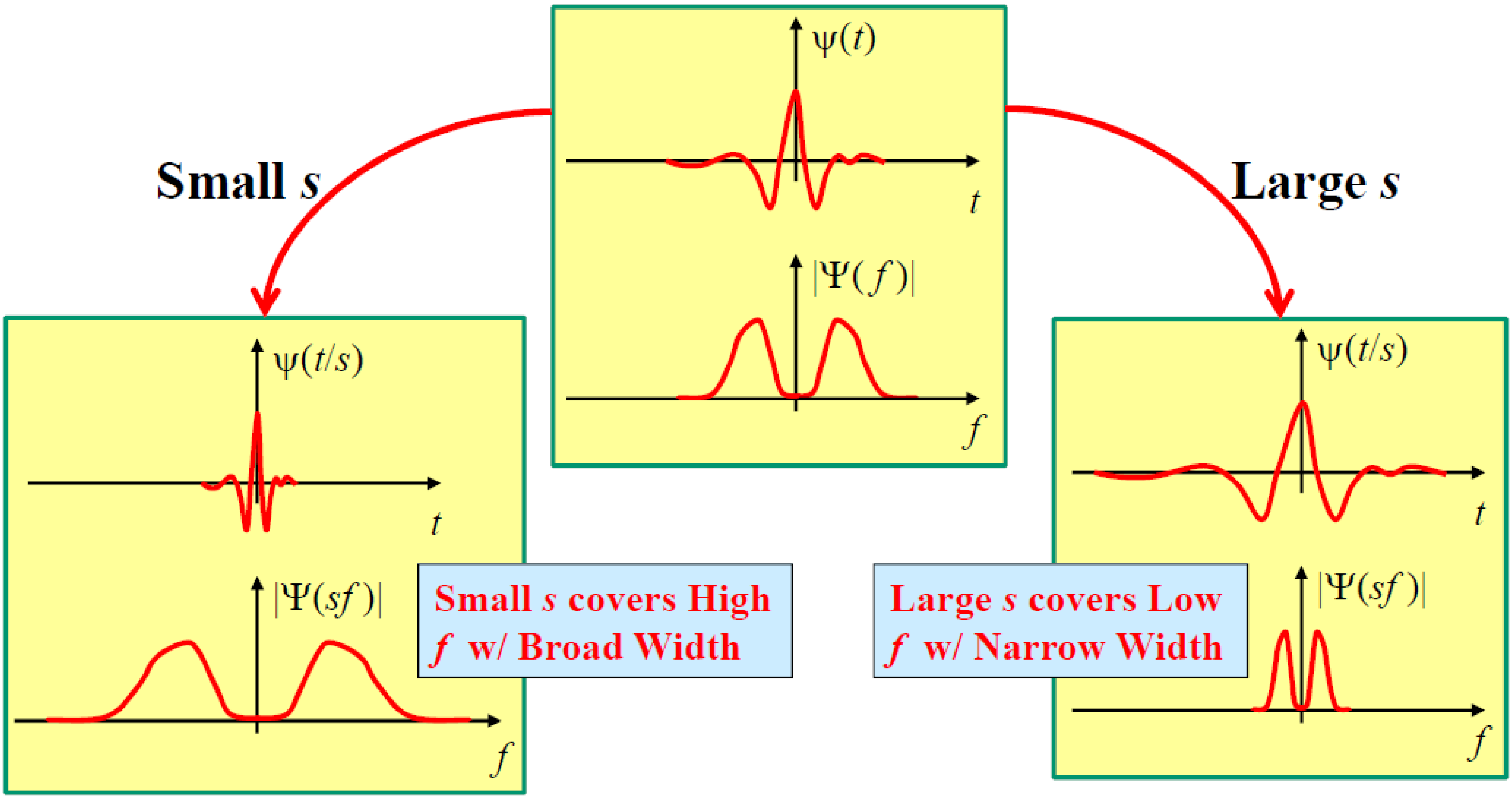
So… we need to find a new way to make T-F basis functions that don't have these problems!!!

The WT comes about from replacing frequency shifting by time scaling…

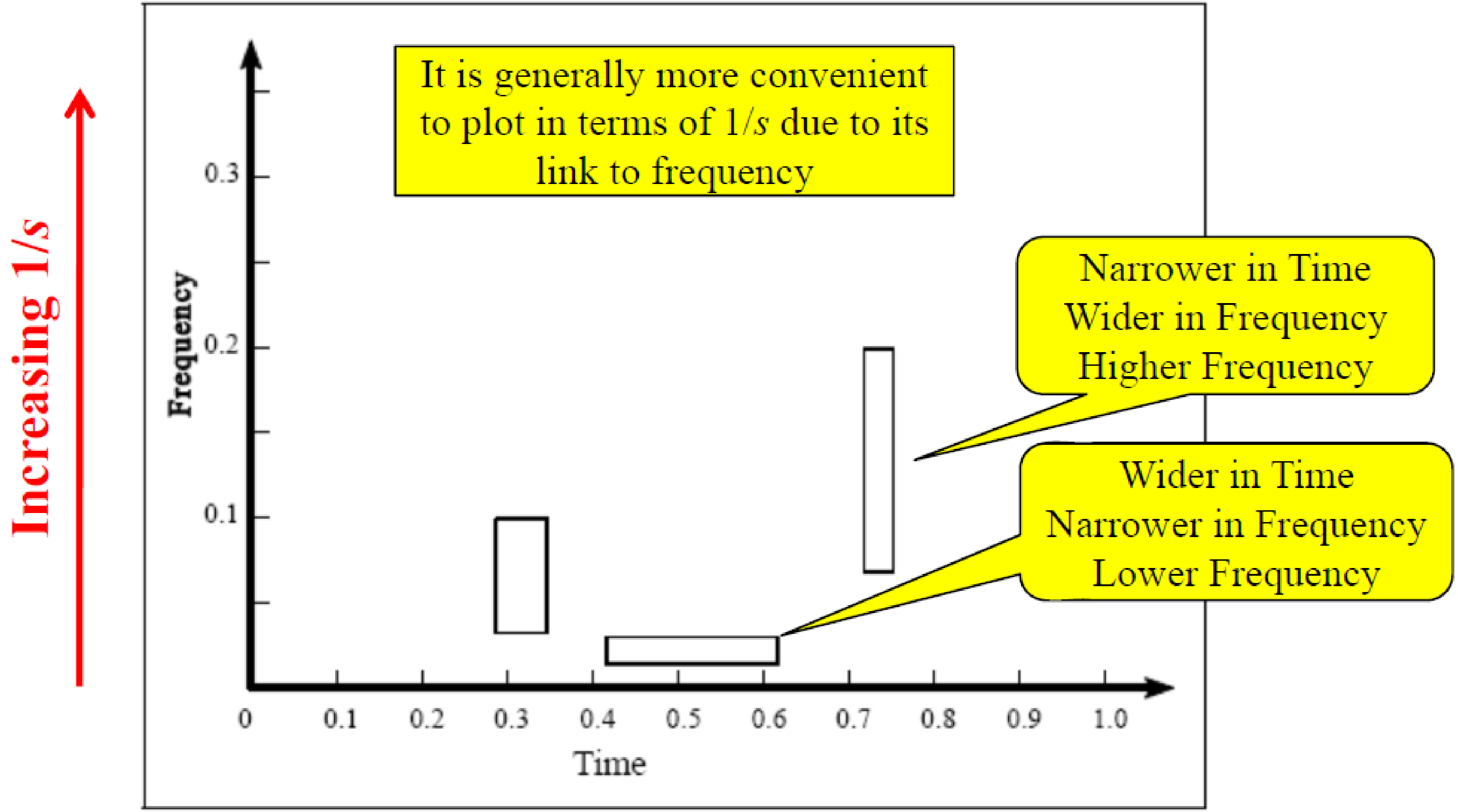
Start with a prototype signal $\psi(t)$ and time scale it:

$$\boxed{\psi(t/s) \quad \leftrightarrow \quad s\Psi(sf)}$$

Increasing s: Stretches the signal "Scrunches" the spectrum	**Decreasing s**: "Scrunches" the signal Stretches the spectrum



This shows some typical t-f cells for wavelets



We still need to satisfy the uncertainty principle: $(\Delta t)(\Delta f) \geq \dfrac{1}{4\pi}$

But now Δt and Δf are adjusted depending on what region of frequency is being "probed".

All this leads to… **The Wavelet Transform:**

$$X(s,t) = \int_{-\infty}^{\infty} x(\tau)\left[\tfrac{1}{\sqrt{s}}\psi\left(\frac{\tau - t}{s}\right)\right] d\tau, \quad s > 0$$

$\psi(t)$ is called the Mother Wavelet

The Inverse Wavelet Transform (Reconstruction Formula):

$$x(t) = \frac{1}{C_\psi}\int_{0}^{\infty}\int_{-\infty}^{\infty} X(s,\tau)\left[\tfrac{1}{\sqrt{s}}\psi\left(\frac{\tau - t}{s}\right)\right]\frac{ds\,d\tau}{s^2},$$

$$C_\psi = \int_0^\infty \frac{|\Psi(\omega)|^2}{\omega}\,d\omega$$

Requirements for a Mother Wavelet are:

Finite Energy: $\psi(t) \in L^2(\mathbb{R}) \quad \Rightarrow \quad \int_{-\infty}^{\infty} \psi^2(t)\,dt < \infty \quad \underset{Parseval}{\Rightarrow} \quad \int_{-\infty}^{\infty} |\Psi(\omega)|^2\,d\omega < \infty$

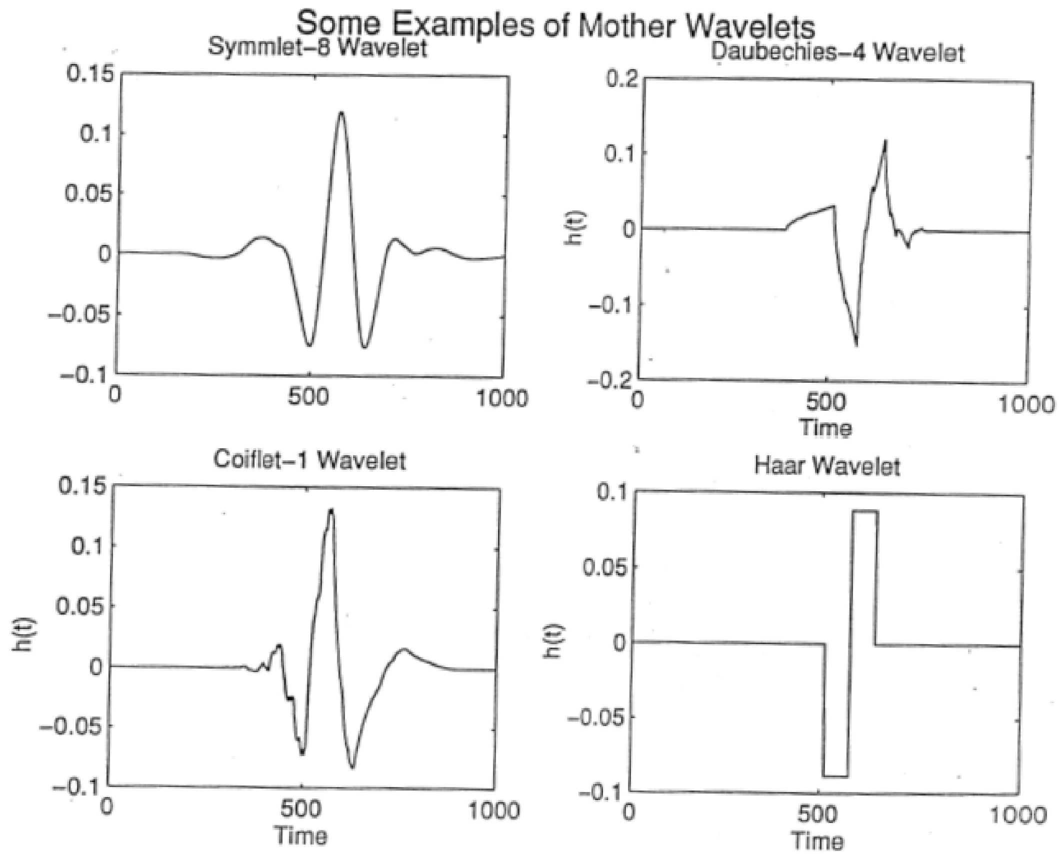
Admissibility Condition: $\int_0^\infty \frac{|\Psi(\omega)|^2}{|\omega|}\,d\omega < \infty$

- $|\Psi(\omega)|^2$ must go to zero fast enough as $\omega \to 0$
- $|\Psi(\omega)|^2$ must go to zero fast enough as $\omega \to \infty$

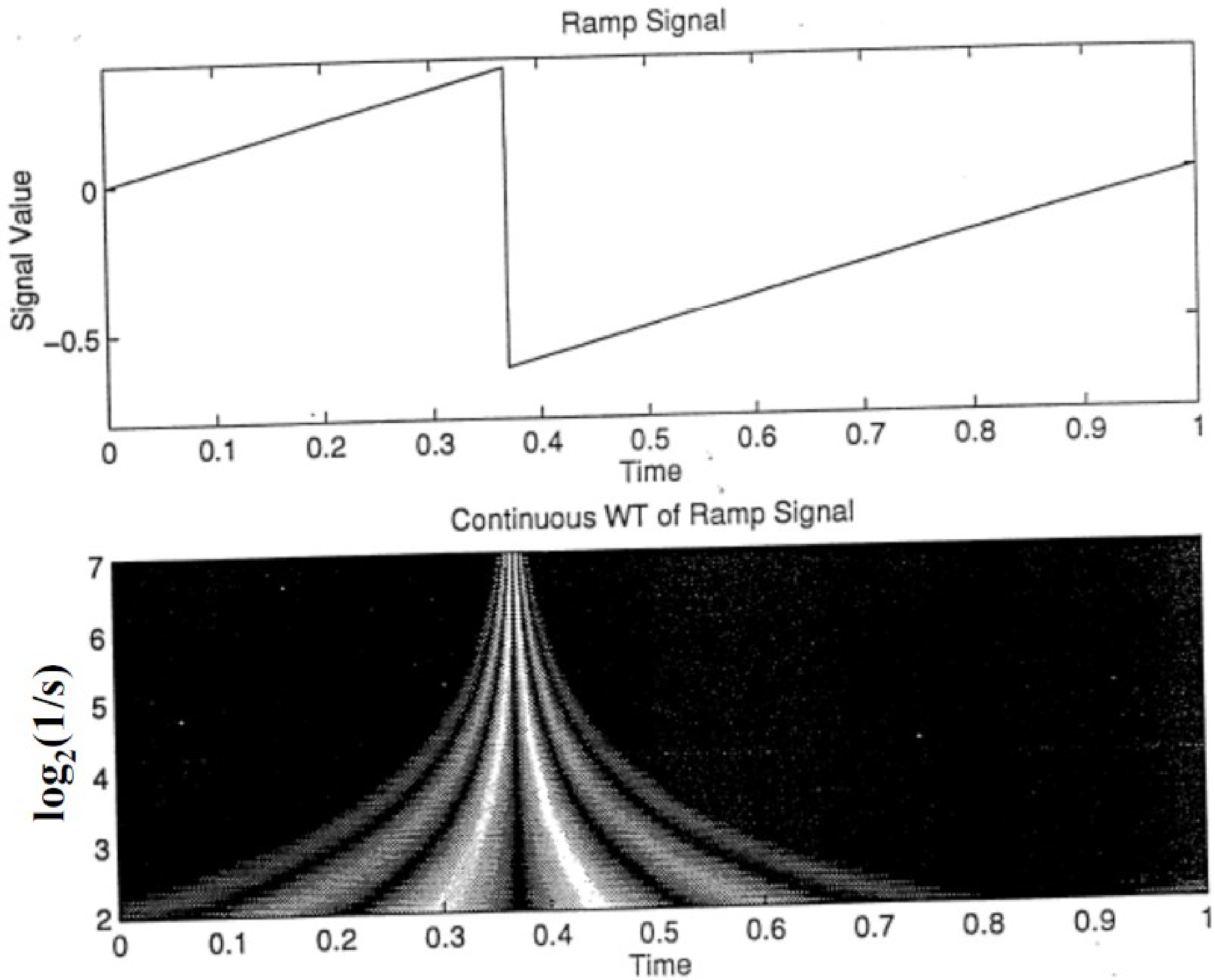
$\psi(t)$ must be a bandpass signal

The prototype basis function $\psi(t)$ is called the mother wavelet…
All the other basis functions come from scaling and shifting the mother wavelet.
There are many choices of mother wavelet:

- Each gives rise to a slightly different WT
- …with slightly different characteristics
- …suited to different applications.



Example of a WT



Non-Redundant Form of WT

It is often desirable to use a discrete form of the WT that is "non-redundant"… that is, we only need $X(s,t)$ on a discrete set of s and t values to reconstruct $x(t)$.

Under some conditions it is possible to do this with only s and t taking these values:

$$s = 2^m \quad t = n2^m \quad \text{for} \quad m = \ldots -3,-2,-1,0,1,2,3,\ldots \quad n = \ldots -3,-2,-1,0,1,2,3,\ldots$$

In practice you truncate the range of m and n

Lower values of m $\Rightarrow$ Smaller values of s $\Rightarrow$ Higher Frequency

Incrementing m doubles the scale value and doubles the time spacing

Then the WT becomes a countably infinite collection of numbers (recall the Fourier series vs. the Fourier transform):

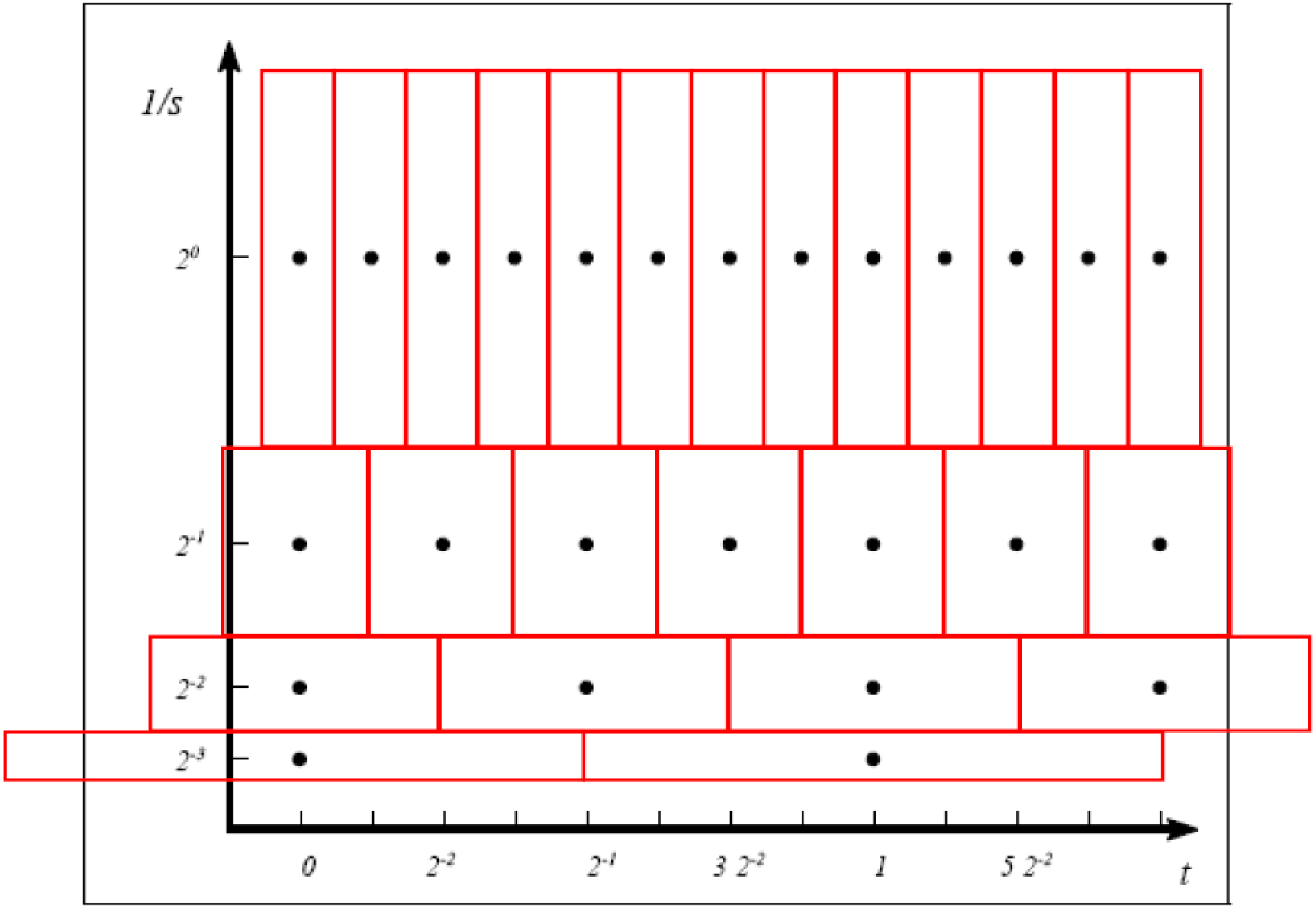
$$X(s,t) = \int_{-\infty}^{\infty} x(\tau)\left[\frac{1}{\sqrt{s}}\psi\left(\frac{\tau - t}{s}\right)\right] d\tau \Rightarrow X_{m,n} = \int_{-\infty}^{\infty} x(\tau)\underbrace{\left[2^{-m/2}\psi\left(2^{-m}\tau - n\right)\right]}_{\triangleq \psi_{m,n}(\tau)} d\tau, \quad m,n \in \mathbb{Z}$$

Advantages of This Form

- The $\psi_{m,n}(t)$ can be an ON basis for L^2
- Good for Data Compression
- Simple, numerically stable inverse
- Leads to efficient discrete-time form

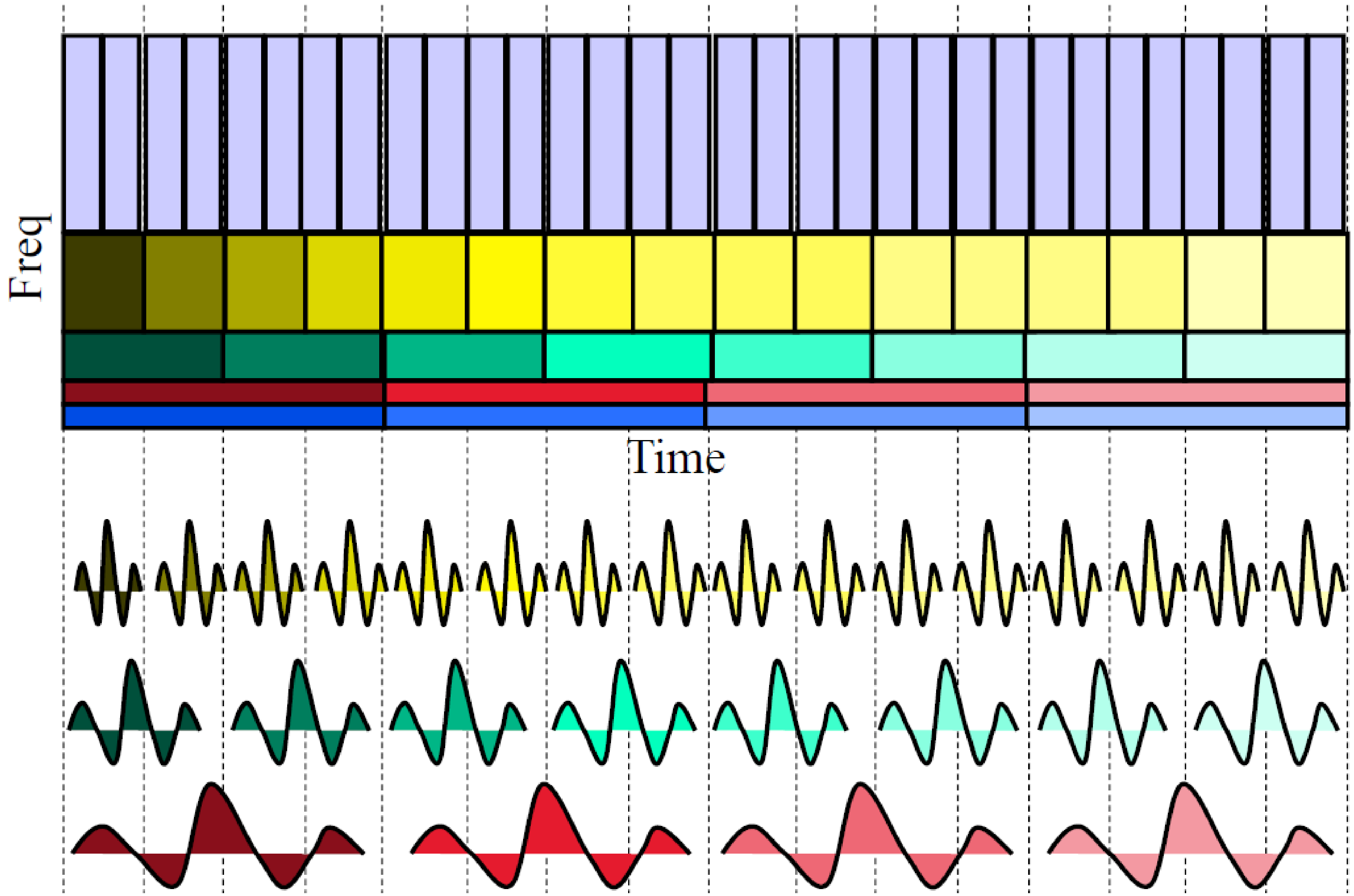
$$x(t) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} X_{m,n}\left[2^{-m/2}\psi\left(2^{-m}t - n\right)\right]$$

This leads to the sampling and the tiling of the t-$1/s$ plane as shown below:



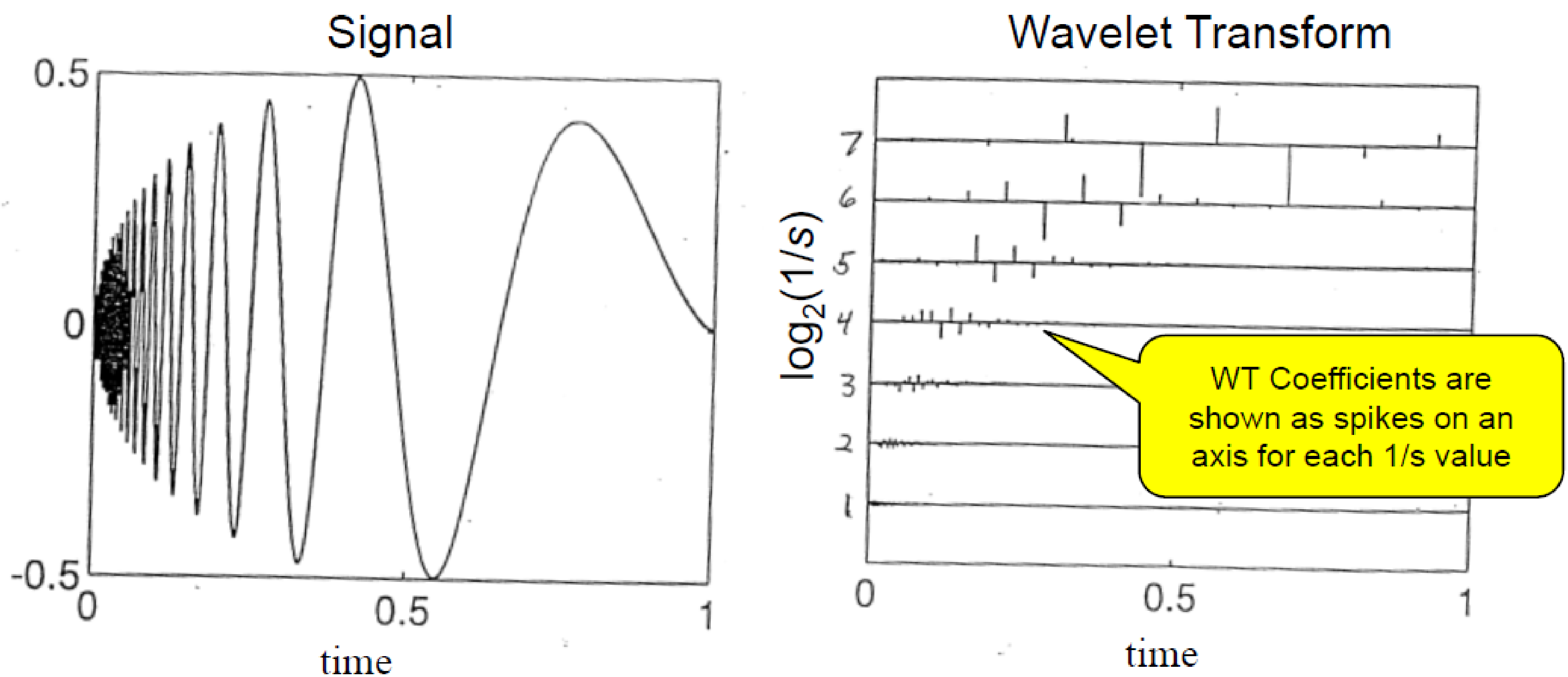
(Time)-(Inverse Scale) Sampling Grid for Wavelet Transform

WT Tiling and Basis Functions



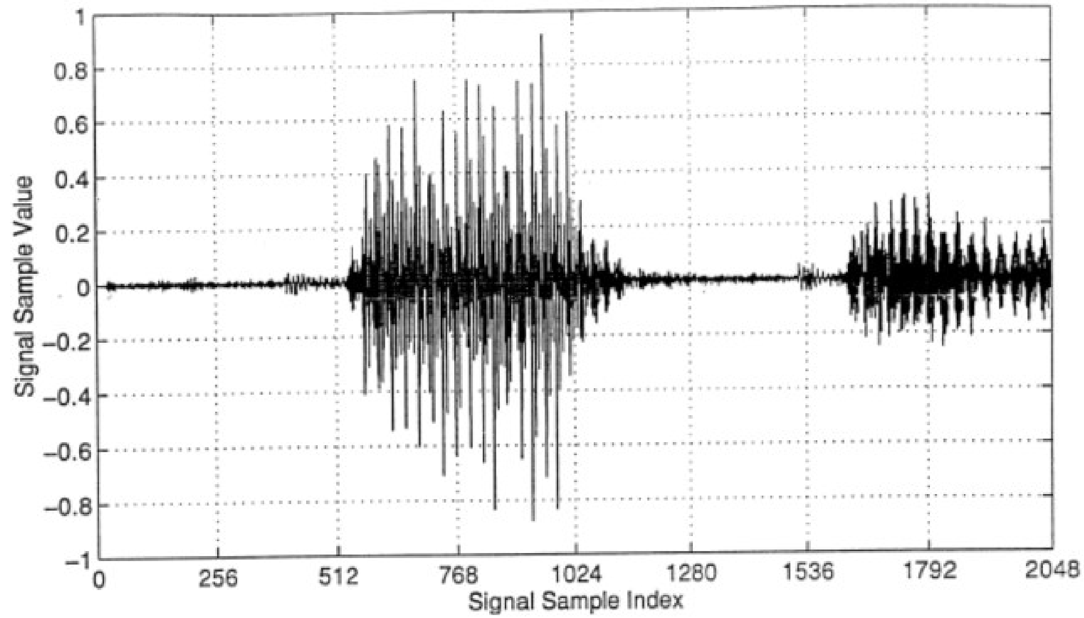
Example #1

- A synthetic Chirp Signal
  - Frequency decreases with time
  - Amplitude increases with time
- Notice that
  - High frequency components dominate early
  - Low frequency components dominate later
  - Low frequency components are stronger

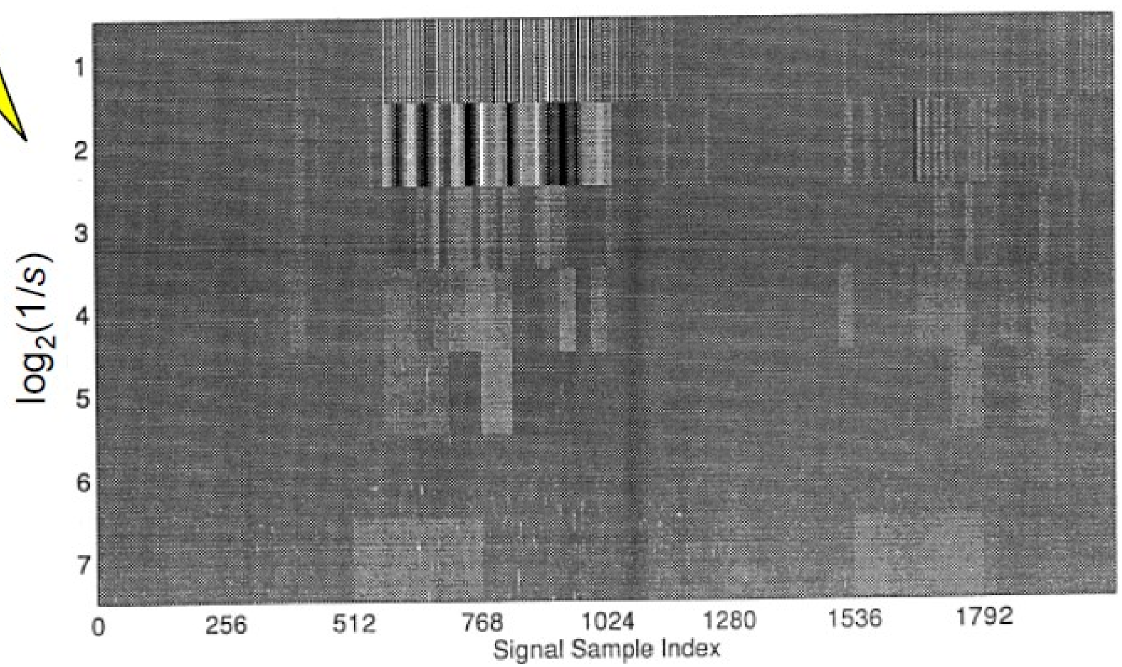


Example #2

Speech Signal



- WT coeff's are displayed as gray- scale blocks
- WT coeffs concentrated
- Blocks closely spaced at high f
- Blocks widely spaced at low f



Summary So Far: What is a Wavelet Transform?

- Note that there are many ways to decompose a signal. Some are:
  - Fourier series: basis functions are harmonic sinusoids;
  - Fourier transform (FT): basis functions are nonharmonic sinusoids;
  - Walsh decomposition: basis functions are "harmonic" square waves;
  - Karhunen-Loeve decomp: basis functions are eigenfunctions of covariance;
  - Short-Time FT (STFT): basis functions are windowed, nonharmonic sinusoids;
    - Provides a time-frequency viewpoint
  - **Wavelet Transform**: basis functions are time-shifted and time-scaled versions of a mother wavelet
    - Provides a time-scale viewpoint

$$\boxed{\psi_{m,n}(t) = 2^{-m/2}\psi\left(2^{-m}t - n\right)}$$

$$\boxed{x(t) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} X_{m,n}\psi_{m,n}(t)} \qquad \boxed{X_{m,n} = \int_{-\infty}^{\infty} x(\tau)\psi_{m,n}(\tau)d\tau\ ,}$$

- Wavelet transform also provides time-frequency view: 1/scale relates to f
  - Decomposes signal in terms of duration-limited, band-pass components
    - high-frequency components are short-duration, wide-band
    - low-frequency components are longer-duration, narrow-band
  - Can provide combo of good time-frequency localization and orthogonality
    - the STFT can't do this

Why are Wavelets Effective?

- Provide a good basis for a large signal class
  - wavelet coefficients drop-off rapidly…
  - thus, good for compression, denoising, detection/recognition
  - goal of any expansion is
    - have the coefficients provide more info about signal than time-domain
    - have most of the coefficients be very small (***sparse*** representation)
  - FT is not sparse for transients… WT is sparse for many signals
- Accurate local description and separation of signal characteristics
  - Fourier puts localization info in the phase in a complicated way
  - STFT can't give localization ***and*** orthogonality
- Wavelets can be adjusted or adapted to application
  - remaining degrees of freedom are used to achieve goals
- Computation of wavelet coefficient is well-suited to computer
  - no derivatives or integrals needed
  - turns out to be a digital filter bank… as we will see.