Digital Signal Processing

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Digital Signal Processing

Sample Questions For Midterm 2

Q1

The diagram in Fig. 1 depicts a *cascade connection* of two linear time-invariant systems; i.e., the output of the first system is the input to the second system, and the overall output is the output of the second system.

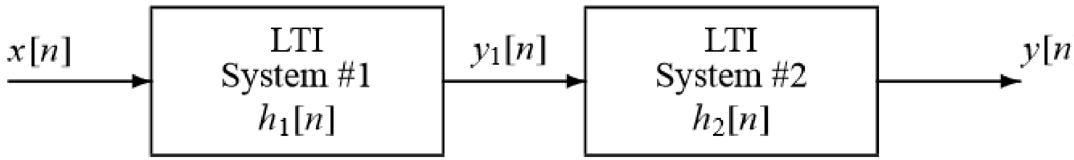


Figure 1: Cascade connection of two LTI systems.

Suppose that System #1 is an FIR filter described by the impulse response:

$$h_1[n] = \begin{cases} 0 & n < 0 \\ 2^n & n = 0, 1, 2, 3, 4, 5 \\ 0 & n > 5 \end{cases}$$

and System #2 is described by the difference equation

$$y_2[n] = y_1[n] - 2y_1[n-1]$$

(a) Determine the filter coefficients of System #1, and also for System #2.

(b) When the input signal $x[n]$ is an impulse, $\delta[n]$, determine the signal $y_1[n]$ and make a plot.

(c) Determine the impulse response of System #2.

(d) Determine the impulse response of the overall cascade system, i.e., find $y[n]$ when $x[n] = \delta[n]$.

Q1 Solution

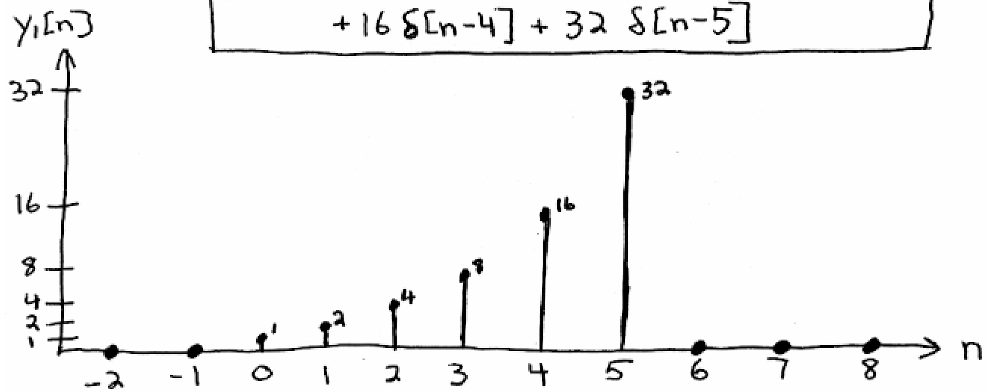
a) Filter coefficients for system #1: $a_0 = 2^0;\ a_1 = 2^1;\ a_2 = 2^2;\ a_3 = 2^3;\ a_4 = 2^4;\ a_5 = 2^5$

$\boxed{a_0 = 1,\ a_1 = 2,\ a_2 = 4,\ a_3 = 8,\ a_4 = 16,\ a_5 = 32}$

For system #2: $\boxed{b_0 = 1,\ b_1 = -2}$

b) When $x[n] = \delta[n]$, then $y_1[n] = h_1[n]$.

Thus, $\boxed{y_1[n] = \delta[n] + 2\delta[n-1] + 4\delta[n-2] + 8\delta[n-3] + 16\delta[n-4] + 32\delta[n-5]}$



c) $h_2[n] = b_0\delta[n] + b_1\delta[n-1] = \boxed{\delta[n] - 2\delta[n-1]}$

(simply plug in $y_1[n] = \delta[n]$)

d) $h[n]$ = impulse response of cascade system $= h_1[n] * h_2[n]$

$$= h_2[n] * h_1[n] = \sum_{k=-\infty}^{\infty} h_2[k]\, h_1[n-k]$$

$$= h_2[0]\, h_1[n] + h_2[1]\, h_1[n-1]$$

$$= h_1[n] - 2h_1[n-1]$$

$$= \left(\delta[n] + 2\delta[n-1] + 4\delta[n-2] + 8\delta[n-3] + 16\delta[n-4] + 32\delta[n-5]\right)$$
$$\underline{-2\left(\delta[n-1] + 2\delta[n-2] + 4\delta[n-3] + 8\delta[n-4] + 16\delta[n-5] + 32\delta[n-6]\right)}$$

$$= \boxed{\delta[n] - 64\delta[n-6]}$$

Q2

This problem is concerned with finding the output of an FIR filter for a given input signal. A linear time-invariant system is described by the difference equation

$$y[n] = \sum_{k=0}^{4}(k+1)x[n-k]$$

The input to this system is *unit step* signal, denoted by $u[n]$, i.e., $x[n] = u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$

(a) Determine the filter coefficients $\{b_k\}$ of this FIR filter.

(b) Determine the impulse response, $h[n]$, for this FIR filter. The impulse response is a discrete-time signal, so make a (stem) plot of $h[n]$ versus n.

(c) Use convolution to compute $y[n]$, over the range $-5 \leq n \leq \infty$, when the input is $u[n]$.

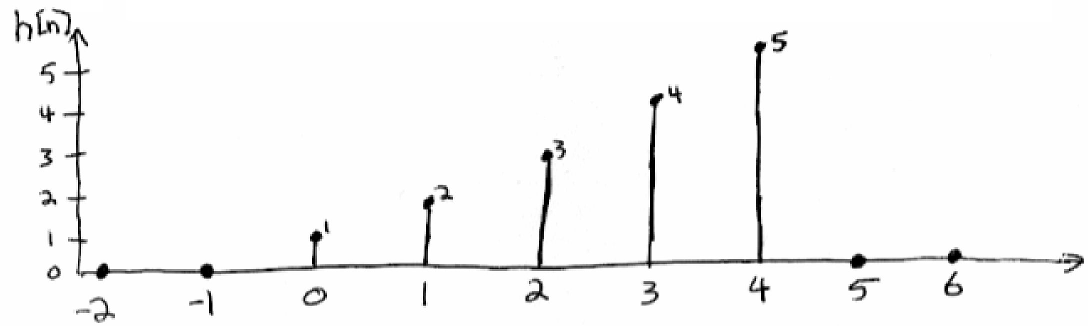
Q2 Solution

a) $y[n] = 1x[n] + 2x[n-1] + 3x[n-2] + 4x[n-3] + 5x[n-4]$

Filter coefficients $\boxed{b_0 = 1 \quad b_1 = 2 \quad b_2 = 3 \quad b_3 = 4 \quad b_4 = 5}$

($b_n = 0$ for $n < 0$ and $n > 4$)

b) $h[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 4\delta[n-3] + 5\delta[n-4]$



c) $y[n] = \sum_{k=0}^{4} h[k]\, u(n-k)$

n	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
u(n)	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1
h(n)	0	0	0	0	0	1	2	3	4	5	0	0	0	0	0	0
h(0) u(n)	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1
h(1) u(n-1)	0	0	0	0	0	0	2	2	2	2	2	2	2	2	2	2
h(2) u(n-2)	0	0	0	0	0	0	0	3	3	3	3	3	3	3	3	3
h(3) u(n-3)	0	0	0	0	0	0	0	0	4	4	4	4	4	4	4	4
h(4) u(n-4)	0	0	0	0	0	0	0	0	0	5	5	5	5	5	5	5
y[n]	0	0	0	0	0	1	3	6	10	15	15	15	15	15	15	15

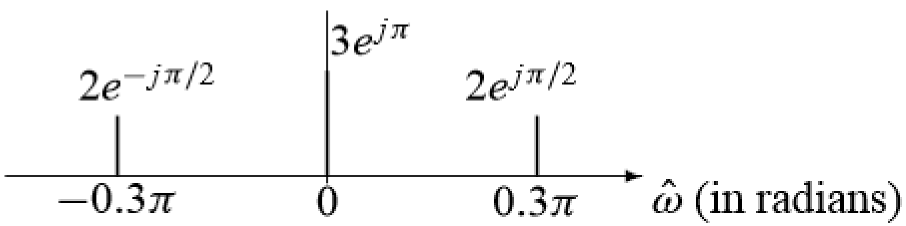
y[n] for n < 0

y[0] y[1] y[2] y[3]

y[n] for n ≥ 4

Q3

A discrete-time signal $x[n]$ has the two-sided spectrum representation shown below.



(a) Write an equation for $x[n]$. Make sure to express $x[n]$ as a real-valued signal.

(b) Determine the formula for the output signal $y[n]$.

Q3 Solution

Part A

$$\begin{aligned} x[n] &= 3e^{j\pi}e^{j0n} + 2e^{j\pi/2}e^{j0.3\pi n} + 2e^{-j\pi/2}e^{-j0.3\pi n} \\ &= \boxed{-3 + 4\cos(0.3\pi n + \pi/2)} \end{aligned}$$

Part B

Nine-point averaging filter implies that

$$y[n] = \frac{1}{9}\Big(x[n-4]+x[n-3]+x[n-2]+x[n-1]+x[n]+x[n+1]+x[n+2]+x[n+3]+x[n+4]\Big)$$

which means

$$h[n] = \frac{1}{9}\Big(\delta[n-4]+\delta[n-3]+\delta[n-2]+\delta[n-1]+\delta[n]+\delta[n+1]+\delta[n+2]+\delta[n+3]+\delta[n+4]\Big).$$

The corresponding frequency response is given by

$$\begin{aligned} \mathcal{H}(\hat{\omega}) &= \tfrac{1}{9}\Big(e^{-j4\hat{\omega}} + e^{-j3\hat{\omega}} + e^{-j2\hat{\omega}} + e^{-j\hat{\omega}} + 1 + e^{j\hat{\omega}} + e^{j2\hat{\omega}} + e^{j3\hat{\omega}} + e^{j4\hat{\omega}}\Big) \\ &= \tfrac{1}{9}\Big(1 + 2\cos(\hat{\omega}) + 2\cos(2\hat{\omega}) + 2\cos(3\hat{\omega}) + 2\cos(4\hat{\omega})\Big) \\ \mathcal{H}(0) &= \tfrac{1}{9}(1+2+2+2+2) = 1 \\ \mathcal{H}(0.3\pi) &= \tfrac{1}{9}\Big(1 + 2\cos(0.3\pi) + 2\cos(0.6\pi) + 2\cos(0.9\pi) + 2\cos(1.2\pi)\Big) \\ &= \tfrac{1}{9}\Big(1 + 1.1755 - 0.6180 - 1.9021 - 1.6180\Big) = -0.2181 \end{aligned}$$

$$y[n] = -3(1) + 4(-0.2181)\cos(0.3\pi n + \pi/2) = \boxed{-3 + 0.8724\cos(0.3\pi n - \pi/2)}$$

If the nine-point averaging filter is constrained to be *causal*:

$$y[n] = \frac{1}{9}\Big(x[n]+x[n-1]+x[n-2]+x[n-3]+x[n-4]+x[n-5]+x[n-6]+x[n-7]+x[n-8]\Big)$$

Then the frequency response contains an additional phase term:

$$\mathcal{H}(\hat{\omega}) = \tfrac{1}{9}\Big(1 + 2\cos(\hat{\omega}) + 2\cos(2\hat{\omega}) + 2\cos(3\hat{\omega}) + 2\cos(4\hat{\omega})\Big)e^{-j4\hat{\omega}}$$

and $y[n]$ will be delayed by 4, because the filter's impulse response is shifted right by 4.

$$y[n] = -3 + 0.8724\cos(0.3\pi(n-4) - 0.5\pi) = \boxed{-3 + 0.8724\cos(0.3\pi n + 0.3\pi)}$$

Q4

A linear time-invariant system is described by the difference equation

$$y[n] = 2x[n] + 4x[n-1] + 2x[n-2]$$

(a) When the input to this system is

$$x[n] = \begin{cases} 0 & n < 0 \\ 3 & n = 0, 1, 2 \\ 6 - n & n = 3, 4 \\ 1 & n \geq 5 \end{cases}$$

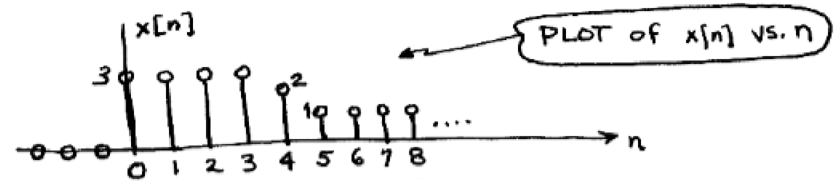
Compute the values of $y[n]$, over the range $0 \leq n \leq 10$.

(b) For the previous part, plot both $x[n]$ and $y[n]$.

(c) *Impulse Response:* Determine the response of this system to a unit impulse input; i.e., find the output $y[n] = h[n]$ when the input is $x[n] = \delta[n]$. Plot $h[n]$ as a function of n.

Q4 Solution

(a)



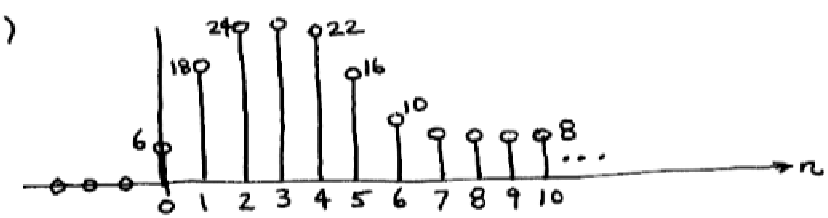
Make a table when computing y[n] from x[n].

n	$n<0$	0	1	2	3	4	5	6	7	8	$n \geq 9$
$x[n]$	0	3	3	3	3	2	1	1	1	1	1
$y[n]$	0	6	18	24	24	22	16	10	8	8	8

$$y[0] = 2x[0] + 4x[-1] + 2x[-2] = 2(3) + 4(0) + 2(0) = 6$$

$$y[5] = 2x[5] + 4x[4] + 2x[3] = 2(1) + 4(2) + 2(3) = 2 + 8 + 6 = 16$$

(b)



(c) When $x[n] = \delta[n]$, the output is denoted $h[n]$

$$y[n] = 2x[n] + 4x[n-1] + 2x[n-2]$$

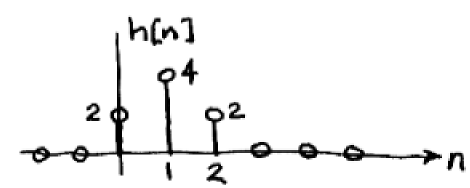
$$h[n] = 2\delta[n] + 4\delta[n-1] + 2\delta[n-2]$$

NON-ZERO WHEN n=0

NON-ZERO FOR n=1

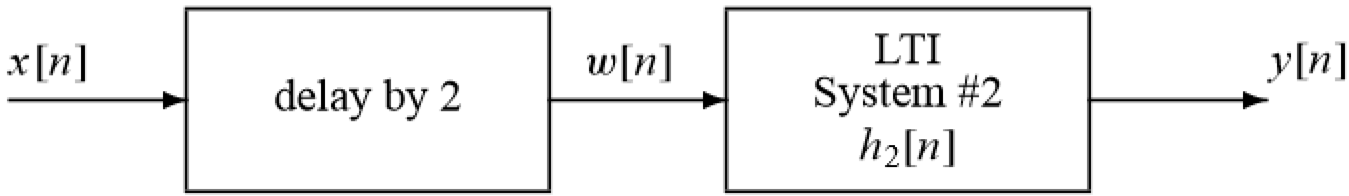
NON-ZERO WHEN n=2

$$\therefore h[n] = \begin{cases} 2, & \text{for } n=0 \\ 4, & n=1 \\ 2, & n=2 \\ 0, & \text{elsewhere} \end{cases}$$



Q5

Consider the following cascade system:



(a) Find and plot the magnitude of the frequency response of the first filter $|\mathcal{H}_1(\hat{\omega})|$.

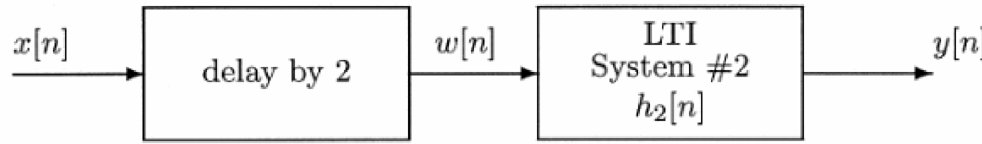
(b) If the overall impulse response of the cascade is

$$h_{eq}[n] = \delta[n-3] + \tfrac{1}{2}\delta[n-4]$$

determine the impulse response of the second filter $h_2[n]$.

Q5 Solution

Consider the following cascade system:

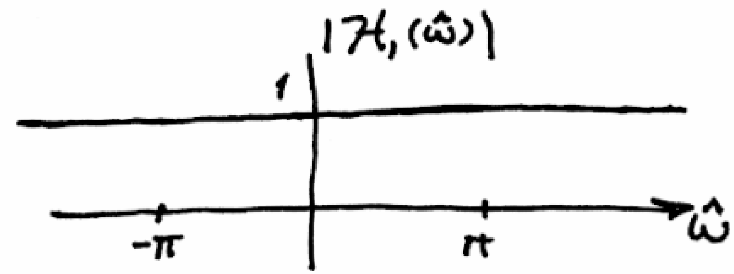


(a) Find and plot the magnitude of the frequency response of the first filter $|\mathcal{H}_1(\hat{\omega})|$.

$h_1[n] = \delta[n-2]$

$\mathcal{H}_1(\hat{\omega}) = e^{-j2\hat{\omega}}$

$\boxed{|\mathcal{H}_1(\hat{\omega})| = 1}$



(b) If the overall impulse response of the cascade is

$$h_{eq}[n] = \delta[n-3] + \tfrac{1}{2}\delta[n-4]$$

determine the impulse response of the second filter $h_2[n]$.

$$\delta[n-2] * h_2[n] = \delta[n-3] + \tfrac{1}{2}\delta[n-4]$$

$$\Longrightarrow \boxed{h_2[n] = \delta[n-1] + \tfrac{1}{2}\delta[n-2]}$$

Q6

A discrete-time system is defined by the input/output relation

$$y[n] = 2x[n+2] + 6x[n] + 2x[n-2]. \tag{1}$$

(a) Determine whether or not the system defined by Equation ($\boxed{1}$) is (i) linear; (ii) time-invariant; (iii) causal. Explain your answers.

(b) Obtain an expression for the frequency response of this system.

(c) Make a sketch of the frequency response (magnitude and phase) as a function of frequency. *Hint: Use symmetry to simplify your expression before determining the magnitude and phase.*

(d) For the system of Equation ($\boxed{1}$), determine the output $y_1[n]$ when the input is

$$x_1[n] = 10 - 10\cos(0.5\pi(n-1))$$

Hint: Use the frequency response and superposition to solve this problem.

Q6 Solution

a.i) The system is linear because:

$$x_1[n] \rightarrow y_1[n] = 2x_1[n+2] + 6x_1[n] + 2x_1[n-2]$$

$$x_2[n] \rightarrow y_2[n] = 2x_2[n+2] + 6x_2[n] + 2x_2[n-2]$$

$$c_1 x_1[n] + c_2 x_2[n] \longrightarrow y_3[n] = 2\left(c_1 x_1[n+2] + \right.$$
$$\left. + c_2 x_2[n+2]\right) + 6\left(c_1 x_1[n] + c_2 x_2[n]\right) +$$
$$+ 2\left(c_1 x_1[n-2] + c_2 x_2[n-2]\right) =$$
$$= c_1\left(2x_1[n+2] + 6x_1[n] + 2x_1[n-2]\right) +$$
$$+ c_2\left(2x_2[n+2] + 6x_2[n] + 2x_1[n-2]\right) =$$
$$= c_1 y_1[n] + c_2 y_2[n].$$

ii) The system is time-invariant because:

$$x[n-n_0] \rightarrow 2x[n+2-n_0] + 6x[n-n_0] +$$
$$+ 2x[n-2-n_0] = y[n-n_0].$$

iii) The system is <u>not</u> causal because $y[n]$ depends on $x[n+2]$.

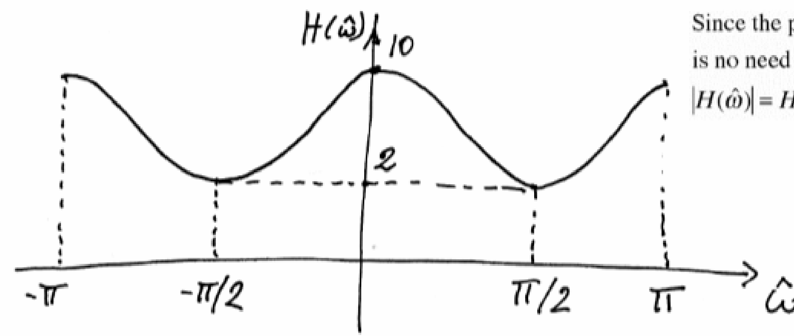
Q6 Solution

b) $h[n] = 2\delta[n+2] + 6\delta[n] + 2\delta[n-2]$

$$\Rightarrow H(\hat{\omega}) = 2e^{j2\hat{\omega}} + 6 + 2e^{-j2\hat{\omega}} =$$

$$= 6 + 4\cos(2\hat{\omega})$$

c)



Since the phase is zero, there is no need to plot the phase: $|H(\hat{\omega})| = H(\hat{\omega})$

d) $H(0) = 10 \qquad H(0.5\pi) = 2 = H(-0.5\pi)$

$$10 \longrightarrow H(0)\cdot 10 = 100$$

$$10\cos\left[\frac{\pi}{2}(n-1)\right] = 5e^{-j\frac{\pi}{2}}e^{j\frac{\pi}{2}n} + 5e^{j\frac{\pi}{2}}e^{-j\frac{\pi}{2}n}$$

$$\longrightarrow 5e^{-j\frac{\pi}{2}}H(0.5\pi)e^{j\frac{\pi}{2}n} + 5e^{j\frac{\pi}{2}}H(-0.5\pi)e^{-j\frac{\pi}{2}n} =$$

$$= 10e^{-j\frac{\pi}{2}}e^{j\frac{\pi}{2}n} + 10e^{j\frac{\pi}{2}}e^{-j\frac{\pi}{2}n} =$$

$$= 20\cos\left[\frac{\pi}{2}(n-1)\right]$$

$$y_1[n] = 100 - 20\cos\left[\frac{\pi}{2}(n-1)\right]$$

Q7

A linear time-invariant system is described by the FIR difference equation

$$y[n] = x[n] - 3x[n-1] + 9x[n-2] - 3x[n-3] + x[n-4]$$

(a) Write a simple formula for the magnitude of the frequency response $|H(e^{j\hat{\omega}})|$. Express your answer in terms of real-valued functions only.

(b) Derive a simple formula for the phase of the frequency response $\angle H(e^{j\hat{\omega}})$.

Q7 Solution

A linear time-invariant system is described by the FIR difference equation

$$y[n] = x[n] - 3x[n-1] + 9x[n-2] - 3x[n-3] + x[n-4]$$

(a) Write a simple formula for the magnitude of the frequency response $|H(e^{j\hat{\omega}})|$. Express your answer in terms of real-valued functions only.

$$H(e^{j\hat{\omega}}) = 1 - 3e^{-j\hat{\omega}} + 9e^{-j2\hat{\omega}} - 3e^{-j3\hat{\omega}} + e^{-j4\hat{\omega}}$$

$$= e^{-j2\hat{\omega}}\left(e^{j2\hat{\omega}} - 3e^{j\hat{\omega}} + 9 - 3e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}\right)$$

$$= e^{-j2\hat{\omega}}\left(2\cos 2\hat{\omega} - 6\cos\hat{\omega} + 9\right)$$

$\Rightarrow$

$$|H(e^{j\hat{\omega}})| = 2\cos 2\hat{\omega} - 6\cos\hat{\omega} + 9$$

(b) Derive a simple formula for the phase of the frequency response $\angle H(e^{j\hat{\omega}})$.

$\angle H(e^{j\hat{\omega}}) = -2\hat{\omega}$ from part (a)

Q8

Suppose that a LTI system has a frequency response function equal to

$$\mathcal{H}(\hat{\omega}) = 2 + 3e^{-j\hat{\omega}} + 3e^{-j3\hat{\omega}} + 2e^{-j4\hat{\omega}}$$

(a) Determine the difference equation that relates the output $y[n]$ of the system to the input $x[n]$.

(b) Determine and plot the *impulse response*.

(c) Determine the output when the input is a pulse:

$$p[n] = \begin{cases} 1 & \text{for } 0 \leq n \leq 3 \\ 0 & n < 0 \end{cases}$$

Use *convolution* for a quick solution.

Q8 Solution

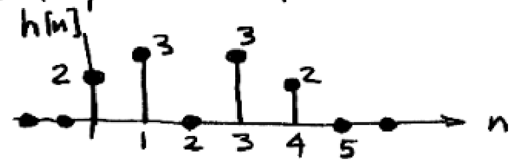
$$\mathcal{H}(\hat{\omega}) = \underset{b_0}{2} + \underset{b_1}{3}e^{-j\hat{\omega}} + \underset{b_3}{3}e^{-j3\hat{\omega}} + \underset{b_4}{2}e^{-j4\hat{\omega}} \qquad b_2 = 0$$

(a) $y[n] = 2x[n] + 3x[n-1] + 3x[n-3] + 2x[n-4]$

(b) $h[n] = 2\delta[n] + 3\delta[n-1] + 3\delta[n-3] + 2\delta[n-4]$

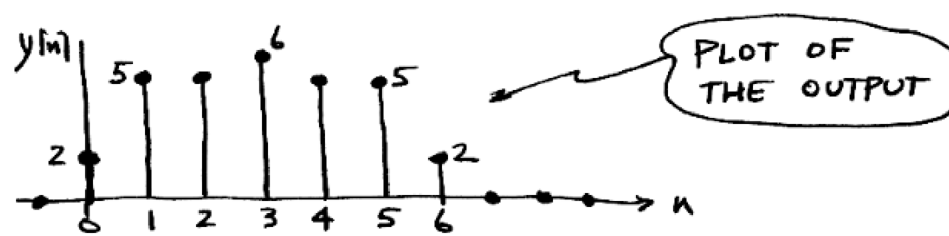
because we just use $x[n] = \delta[n]$

The impulse response reads out the filter coeffs.



(c) Use "convolution" which can be done with the following table:

2	3	0	3	2		
1	1	1				
2	3	0	3	2		
	2	3	0	3	2	
		2	3	0	3	2
2	5	5	6	5	5	2
$n=0$						$n=6$.

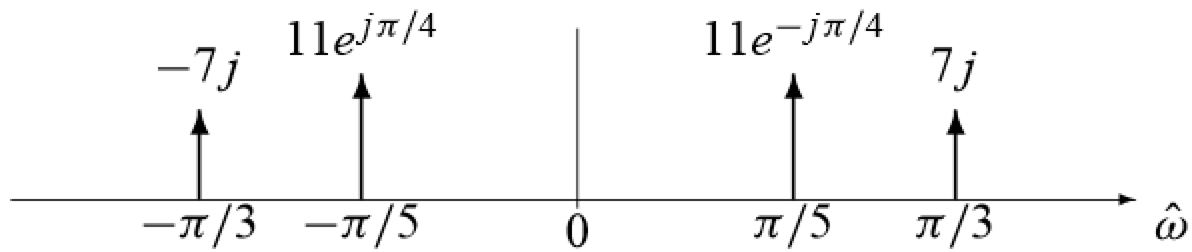


Q9

An FIR filter is characterized by the following frequency response:

$$H(e^{j\hat{\omega}}) = \frac{\sin(5\hat{\omega})}{\sin(\frac{1}{2}\hat{\omega})} e^{-j5\hat{\omega}}$$

(a) If the input to the filter is a signal with the following spectrum, determine a formula for the input signal, $x[n]$ for $-\infty < n < \infty$.



(b) Using the input signal from part (a), determine the output, $y[n]$ for $-\infty < n < \infty$.

Q9 Solution

(a) $x[n] = 22\cos\left(\frac{\pi}{5}n - \frac{\pi}{4}\right) + 14\cos\left(\frac{\pi}{3}n + \frac{\pi}{2}\right)$

by reading values off the spectrum.

(b) Evaluate $H(e^{j\hat{\omega}})$ at $\hat{\omega} = \frac{\pi}{5}$ and $\hat{\omega} = \frac{\pi}{3}$

$$H(e^{j\pi/5}) = \frac{\sin(5 \cdot \pi/5)}{\sin(\frac{1}{2} \cdot \pi/5)} e^{-j5 \cdot \pi/5} = 0$$

$-1 = e^{-j\pi}$

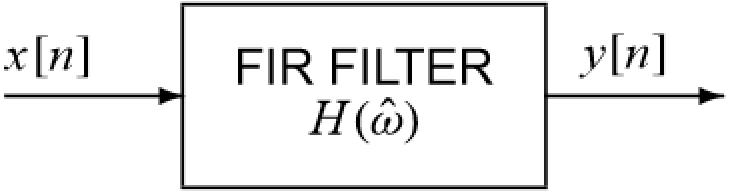
$$H(e^{j\pi/3}) = \frac{\sin(5\pi/3)}{\sin(\pi/6)} e^{-j5\pi/3} = \frac{-\frac{1}{2}\sqrt{3}}{1/2} e^{j\pi/3} = \sqrt{3}\, e^{-j2\pi/3}$$

Thus $y[n]$ has only the $\frac{\pi}{3}$ component

$$y[n] = 14\sqrt{3}\cos\left(\frac{\pi}{3}n + \frac{\pi}{2} - \frac{2\pi}{3}\right)$$

$$= 14\sqrt{3}\cos\left(\frac{\pi}{3}n - \frac{\pi}{6}\right)$$

Q10



The frequency response of the filter above is

$$\mathcal{H}(\hat{\omega}) = \cos(\tfrac{1}{2}\hat{\omega})e^{-j\hat{\omega}}$$

If the input signal is $x[n] = 7 + 2\cos(0.5\pi n + \pi)$ for $-\infty < n < \infty$, determine a simple mathematical expression for the output signal $y[n]$.

Q10 Solution

The frequency response of the filter above is

$$\mathcal{H}(e^{j\omega}) = \cos(\tfrac{1}{2}\hat{\omega})e^{-j\hat{\omega}}$$

If the input signal is $x[n] = 7 + 2\cos(0.5\pi n + \pi)$ for $-\infty < n < \infty$, determine a simple mathematical expression for the output signal $y[n]$.

$$\boxed{y[n] = 7 + \sqrt{2}\cos(0.5\pi n + \pi/2)}$$

Recall that

$$x[n] = \cos(n\hat{\omega}) \Rightarrow y[n] = |H(\hat{\omega})| \cos(n\hat{\omega} + \angle H(\hat{\omega}))$$

Therefore, with

- $H(\hat{\omega})\big|_{\omega=0} = 1$
- $H(\hat{\omega})\big|_{\hat{\omega}=\pi/2} = \cos(\pi/4)e^{-j\pi/2} = \frac{1}{\sqrt{2}}e^{-j\pi/2}$

then

$$y[n] = 7 + \frac{1}{\sqrt{2}}\, 2\cos(0.5\pi n + \pi - \pi/2)$$

$$= 7 + \sqrt{2}\cos(0.5\pi n + \pi/2)$$

Q11

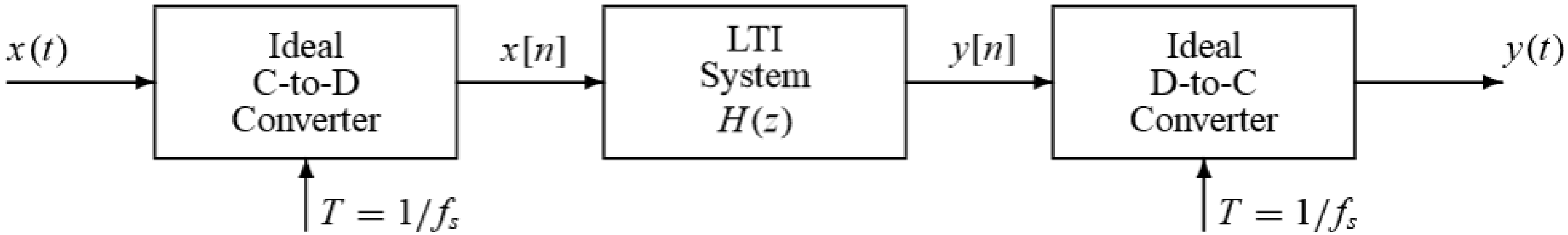
The input to the C-to-D converter in the figure below is

$$x(t) = 10 + 4\cos(4000\pi t - \pi/8) + 6\cos(11000\pi t - \pi/3)$$

The system function of the LTI system is

$$H(z) = (1 + z^{-2})$$

If $f_s = 8000$ samples/second, determine an expression for $y(t)$, the output of the D-to-C converter.



Q11 Solution

$$\mathcal{H}(\hat{\omega}) = 1 + e^{-j2\hat{\omega}}$$

Consider each term separately:

$$10 \xrightarrow{C/D} 10 \xrightarrow{LTI} 10 \cdot \mathcal{H}(0) = 10 \cdot 2 = 20 \xrightarrow{D/C} 20$$

$4\cos\left(4000\pi t - \frac{\pi}{8}\right)$:

$$\omega = \pm 4000\pi \xrightarrow{C/D} \hat{\omega} = \frac{\omega}{f_s} = \pm \frac{\pi}{2}$$

$$\mathcal{H}\left(\pm\frac{\pi}{2}\right) = 1 + e^{\mp j2\frac{\pi}{2}} = 1 + e^{\mp j\pi} = 0$$

Therefore this term is removed by the filter.

$$6\cos\left(11000\pi t - \frac{\pi}{3}\right) = 3e^{-j\frac{\pi}{3}}e^{j11000\pi t} + 3e^{j\frac{\pi}{3}}e^{-j11000\pi t}$$

Q11 Solution

$\omega = \pm 11000\pi \xrightarrow{C/D} \hat{\omega} = \pm \frac{11}{8}\pi$

(Warning : These values of $\hat{\omega}$ are outside the $[-\pi, \pi]$ interval, therefore aliasing occurs during D/C conversion).

$\mathcal{H}(\pm \frac{11}{8}\pi) = 1 + e^{\mp j2\frac{11}{8}\pi} = 1 + e^{\mp j\frac{11}{4}\pi} =$

$= 0.7654\, e^{\mp j 1.1781} = 0.7654\, e^{\mp j \frac{3\pi}{8}}$

$3e^{-j\frac{\pi}{3}} e^{j\frac{11}{8}\pi n} \xrightarrow{LTI} 2.296\, e^{-j\frac{17}{24}\pi} e^{j\frac{11}{8}\pi n} \xrightarrow{D/C}$

$\xrightarrow{D/C} 2.296\, e^{-j\frac{17}{24}\pi} e^{-j5000\pi t}$

(because of aliasing, $\omega = (\hat{\omega} - 2\pi) f_s = -5000\pi$)

$3e^{j\frac{\pi}{3}} e^{-j\frac{11}{8}\pi n} \xrightarrow{LTI} 2.296\, e^{j\frac{17}{24}\pi} e^{-j\frac{11}{8}\pi n} \xrightarrow{D/C}$

$\xrightarrow{D/C} 2.296\, e^{j\frac{17}{24}\pi} e^{j5000\pi t}$

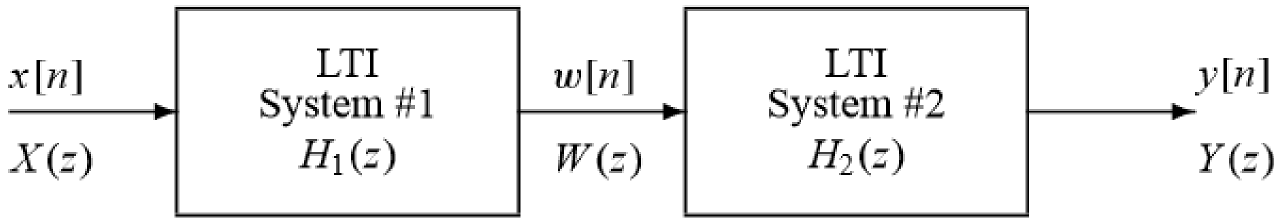
(because of aliasing, $\omega = (\hat{\omega} + 2\pi) f_s = 5000\pi$).

$y(t) = 20 + 2.296\, e^{j(5000\pi t + \frac{17}{24}\pi)} + 2.296\, e^{-j(5000\pi t + \frac{17}{24}\pi)} =$

$= 20 + 4.592 \cos(5000\pi t + \frac{17}{24}\pi).$

Q12

Consider the following cascade system:



The system functions for the two systems are

$$H_1(z) = 1 - z^{-1} + z^{-2}$$

and

$$H_2(z) = 1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4}$$

(a) Determine the system function $H(z)$ of the overall system from the input $x[n]$ to the output $y[n]$.

(b) Determine the corresponding impulse response of the overall system.

Q12 Solution

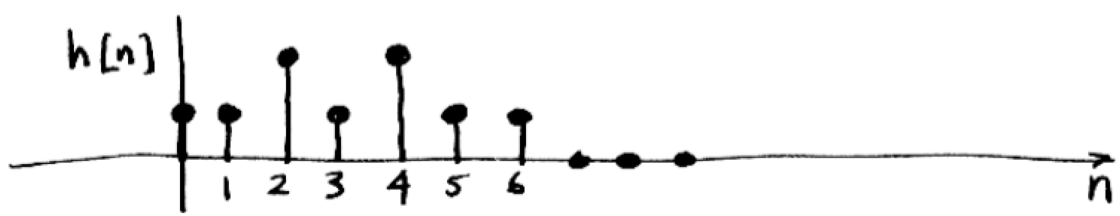
(a) $H(z) = H_1(z)H_2(z)$

$$= (1 - z^{-1} + z^{-2})(1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4})$$

$$= 1 + z^{-1} + 2z^{-2} + z^{-3} + 2z^{-4} + z^{-5} + z^{-6}$$

(b) Impulse Response is found by taking the coefficients of the polynomial $H(z)$

$$h[n] = \delta[n] + \delta[n-1] + 2\delta[n-2] + \delta[n-3] + 2\delta[n-4] + \delta[n-5] + \delta[n-6]$$



Q13

Suppose that a system is defined by the following operator

$$\hat{H}(z) = (1 - z^{-1})(1 + z^{-2})(1 + z^{-1})$$

(a) Write the time-domain description of this system—in the form of a difference equation.

(b) Write the formula for the frequency response of the system.

(c) Derive simple formulas for the magnitude response versus ω, and the phase response versus ω. These formulas must contain no complex terms and no square roots.

Q13 Solution

Suppose that a system is defined by the following operator

$$H(z) = (1 - z^{-1})(1 + z^{-2})(1 + z^{-1})$$

(a) Write the time-domain description of this system—in the form of a difference equation.

$$(1-z^{-1})(1+z^{-1})(1+z^{-2}) = (1-z^{-2})(1+z^{-2}) = 1 - z^{-4}$$

$$H(z) = 1 - z^{-4}$$

$$\Rightarrow \quad y[n] = x[n] - x[n-4]$$

(b) Write the formula for the frequency response of the system.

$$H(e^{j\hat{\omega}}) = 1 - e^{-j4\hat{\omega}}$$

(c) Derive simple formulas for the magnitude response versus $\hat{\omega}$, and the phase response versus $\hat{\omega}$. These formulas must contain no complex terms and no square roots.

$$H(e^{j\hat{\omega}}) = e^{-j2\hat{\omega}} \left(\frac{e^{j2\hat{\omega}} - e^{-j2\hat{\omega}}}{2j} \right) . \, 2j$$

$$= 2j\, e^{-j2\hat{\omega}} \sin 2\hat{\omega}$$

$$= e^{j\pi/2} e^{-j2\hat{\omega}}\, 2 \sin 2\hat{\omega}$$

$$M(\hat{\omega}) = 2 \sin 2\hat{\omega}$$

$$\varphi(\hat{\omega}) = -2\hat{\omega} + \pi/2$$

Q14

We now have four ways of describing an LTI system: the difference equation; the impulse response, $h[n]$; the frequency response, $H(e^{j\hat{\omega}})$; and the system function, $H(z)$. In the following, you are given one of these representations and you must find the other three.

(a) $y[n] = (x[n] + 2x[n-2] + x[n-4])$.

(b) $h[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4]$.

(c) $H(e^{j\hat{\omega}}) = [1 + \cos(2\hat{\omega})]e^{-j\hat{\omega}3}$. *Hint: Expand the cosine using Euler's formula.*

(d) $H(z) = 1 - 2z^{-2} + z^{-4} + z^{-7}$.

Q14 Solution

(a) Given: $y(n) = x(n) + 2x(n-2) + x(n-4)$ (diff. eq.)

Impulse response: $h(n) = \delta(n) + 2\delta(n-2) + \delta(n-4)$

Frequency response: $H(e^{j\hat{\omega}}) = 1 + 2e^{-j2\hat{\omega}} + e^{-j4\hat{\omega}}$

System function: $H(z) = 1 + 2z^{-2} + z^{-4}$

(b) Given: $h(n) = \delta(n) + \delta(n-1) + \delta(n-2) + \delta(n-3) + \delta(n-4)$ (Imp. resp.)

Difference Eq: $y(n) = x(n) + x(n-1) + x(n-2) + x(n-3) + x(n-4)$

Frequency resp: $H(e^{j\hat{\omega}}) = 1 + e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} + e^{-j3\hat{\omega}} + e^{-j4\hat{\omega}}$

System function: $H(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4}$

(c) Given: $H(e^{j\hat{\omega}}) = [1 + \cos(2\hat{\omega})]e^{-j\hat{\omega}3}$ (freq. resp.)

$$H(e^{j\hat{\omega}}) = \left(1 + \frac{e^{j2\hat{\omega}} + e^{-j2\hat{\omega}}}{2}\right)e^{-j3\hat{\omega}} =$$

$$= e^{-j3\hat{\omega}} + \frac{1}{2}e^{-j\hat{\omega}} + \frac{1}{2}e^{-j5\hat{\omega}} =$$

$$= \frac{1}{2}e^{-j\hat{\omega}} + e^{-j3\hat{\omega}} + \frac{1}{2}e^{-j5\hat{\omega}} \Rightarrow$$

System function: $H(z) = \frac{1}{2}z^{-1} + z^{-3} + \frac{1}{2}z^{-5}$

Impulse Response: $h(n) = \frac{1}{2}\delta(n-1) + \delta(n-3) + \frac{1}{2}\delta(n-5)$

Difference Eq: $y(n) = \frac{1}{2}x(n-1) + x(n-3) + \frac{1}{2}x(n-5)$

(d) Given: $H(z) = 1 - 2z^{-2} + z^{-4} + z^{-7}$

Frequency Response: $H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}} \Rightarrow$

$$H(e^{j\hat{\omega}}) = 1 - 2e^{-j2\hat{\omega}} + e^{-j4\hat{\omega}} + e^{-j7\hat{\omega}}$$

Impulse response: $h(n) = \delta(n) - 2\delta(n-2) + \delta(n-4) + \delta(n-7)$

Difference Eq: $y(n) = x(n) - 2x(n-2) + x(n-4) + x(n-7)$

Q15

Use the z-transform of

$$x[n] = \delta[n-1] - \delta[n-2] + \delta[n-3] - \delta[n-4]$$

and the system function $H(z) = 1 - z^{-1}$ to find the output of a first-difference filter when $x[n]$ is the input. Compute your answer by using polynomial multiplication and also by using the difference equation:

$$y[n] = x[n] - x[n-1]$$

What is the degree of the output z-transform polynomial that represents $y[n]$?

Q15 Solution

$$X(z) = \sum_{n=0}^{4} x[n] z^{-n} = 1z^{-1} - 1z^{-2} + z^{-3} - z^{-4}$$

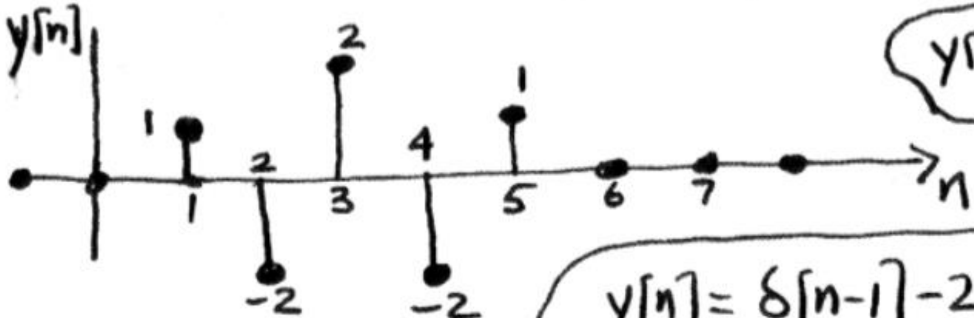
$$Y(z) = H(z)\, X(z)$$

POLYNOMIAL MULTIPLICATION

$$= (1 - z^{-1})(z^{-1} - z^{-2} + z^{-3} - z^{-4})$$

DEGREE IS FIVE

$$Y(z) = \underbrace{z^{-1}}_{y[1]} - \underbrace{2z^{-2}}_{y[2]} + \underbrace{2z^{-3}}_{y[3]} - \underbrace{2z^{-4}}_{y[4]} + \underbrace{z^{-5}}_{y[5].}$$



$y[n] = 0$ for $n < 1$ and for $n > 5$

$$\boxed{y[n] = \delta[n-1] - 2\delta[n-2] + 2\delta[n-3] - 2\delta[n-4] + \delta[n-5]}$$