Ordinary Differential Equations

Derivatives

Derivatives

Ordinary Derivatives

$$\frac{dv}{dt}$$

v is a function of ***one independent*** variable

Partial Derivatives

$$\frac{\partial u}{\partial y}$$

u is a function of ***more than one independent*** variable

Differential Equations

Differential Equations

Ordinary Differential Equations

$$\frac{d^2v}{dt^2} + 6tv = 1$$

involve one or more <u>Ordinary derivatives</u> of unknown functions

Partial Differential Equations

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

involve one or more <u>partial derivatives</u> of unknown functions

Ordinary Differential Equations

Ordinary Differential Equations (ODEs) involve one or more **ordinary *derivatives of unknown functions with respect to one independent variable***

$$Examples:$$

$$\frac{dv(t)}{dt} - v(t) = e^t$$

$$\frac{d^2x(t)}{dt^2} - 5\frac{dx(t)}{dt} + 2x(t) = \cos(t)$$

x(t): unknown function

t: independent variable

Example of ODE:
Model of Falling Parachutist

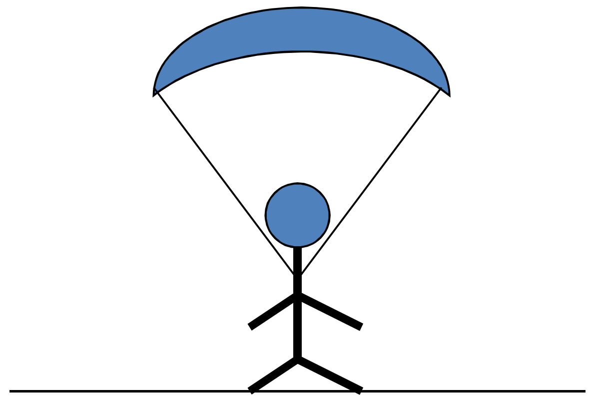
The velocity of a falling parachutist is given by:

$$\frac{dv}{dt} = 9.8 - \frac{c}{M} v$$

M : *mass*

c : *drag coefficient*

v : *velocity*



Definitions

$$\frac{dv}{dt} = 9.8 - \frac{c}{M} v$$

Ordinary differential equation

Definitions (Cont.)

$$\frac{dv}{dt} = 9.8 - \frac{c}{M} v$$

(Dependent variable) unknown function to be determined

Definitions (Cont.)

$$\frac{dv}{dt} = 9.8 - \frac{c}{M} v$$

(independent variable)
the variable with respect to which other variables are differentiated

Order of a Differential Equation

The **order** of an ordinary differential equation is the order of the highest order derivative.

$$Examples:$$

$$\frac{dx(t)}{dt} - x(t) = e^t$$ First order ODE

$$\frac{d^2x(t)}{dt^2} - 5\frac{dx(t)}{dt} + 2x(t) = \cos(t)$$ Second order ODE

$$\left(\frac{d^2x(t)}{dt^2}\right)^3 - \frac{dx(t)}{dt} + 2x^4(t) = 1$$ Second order ODE

Solution of a Differential Equation

A **solution** to a differential equation is a function that satisfies the equation.

$$Example:$$

$$\frac{dx(t)}{dt} + x(t) = 0$$

$$Solution \quad x(t) = e^{-t}$$

$$\text{Proof}:$$

$$\frac{dx(t)}{dt} = -e^{-t}$$

$$\frac{dx(t)}{dt} + x(t) = -e^{-t} + e^{-t} = 0$$

Linear ODE

An ODE is linear if

The unknown function and its derivatives appear to power one

No product of the unknown function and/or its derivatives

Examples:

$$\frac{dx(t)}{dt} - x(t) = e^t$$

Linear ODE

$$\frac{d^2x(t)}{dt^2} - 5\frac{dx(t)}{dt} + 2t^2x(t) = \cos(t)$$

Non-linear ODE

$$\left(\frac{d^2x(t)}{dt^2}\right)^3 - \frac{dx(t)}{dt} + \sqrt{x(t)} = 1$$

Non-linear ODE

Nonlinear ODE

Examples of nonlinear ODE :

$$\frac{dx(t)}{dt} - \cos(x(t)) = 1$$

$$\frac{d^2x(t)}{dt^2} - 5\frac{dx(t)}{dt}x(t) = 2$$

$$\frac{d^2x(t)}{dt^2} - \left|\frac{dx(t)}{dt}\right| + x(t) = 1$$

Solutions of Ordinary Differential Equations

$$x(t) = \cos(2t)$$

is a solution to the ODE

$$\frac{d^2x(t)}{dt^2} + 4x(t) = 0$$

Is it unique?

$$\boxed{\text{All functions of the form } x(t) = \cos(2t + c) \text{ (where } c \text{ is a real constant) are solutions.}}$$

Uniqueness of a Solution

In order to uniquely specify a solution to an n^{th} order differential equation we need n conditions.

$$\frac{d^2x(t)}{dt^2} + 4x(t) = 0$$

Second order ODE

$$\boxed{\begin{aligned} x(0) &= a \\ \dot{x}(0) &= b \end{aligned}}$$

Two conditions are needed to uniquely specify the solution

Auxiliary Conditions

Auxiliary Conditions

Initial Conditions

All conditions are at one point of the independent variable

Boundary Conditions

The conditions are not at one point of the independent variable

Boundary-Value and Initial value Problems

Initial-Value Problems

- The auxiliary conditions are at one point of the independent variable

$$\ddot{x} + 2\dot{x} + x = e^{-2t}$$

$$x(0) = 1, \quad \dot{x}(0) = 2.5$$

same

Boundary-Value Problems

The auxiliary conditions are not at one point of the independent variable

More difficult to solve than initial value problems

$$\ddot{x} + 2\dot{x} + x = e^{-2t}$$

$$x(0) = 1, \quad x(2) = 1.5$$

different

Classification of ODEs

ODEs can be classified in different ways:

- Order
  - First order ODE
  - Second order ODE
  - N^{th} order ODE
- Linearity
  - Linear ODE
  - Nonlinear ODE
- Auxiliary conditions
  - Initial value problems
  - Boundary value problems

Analytical Solutions

- Analytical Solutions to ODEs are available for linear ODEs and special classes of nonlinear differential equations.

Numerical Solutions

- Numerical methods are used to obtain a graph or a table of the unknown function.
- Most of the Numerical methods used to solve ODEs are based directly (or indirectly) on the truncated Taylor series expansion.

Separable Differential Equations

A separable differential equation can be expressed as the product of a function of x and a function of y.

$$\frac{dy}{dx} = g(x) \cdot h(y) \qquad h(y) \neq 0$$

Example:

$$\frac{dy}{dx} = 2xy^2$$

Multiply both sides by dx and divide both sides by y^2 to separate the variables. (Assume y^2 is never zero.)

$$\frac{dy}{y^2} = 2x\ dx$$

$$y^{-2} dy = 2x\ dx$$

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$$\frac{dy}{y^2} = 2x\ dx$$

$$y^{-2}dy = 2x\ dx$$

$$\int y^{-2}dy = \int 2x\ dx$$

$$-y^{-1} + C_1 = x^2 + C_2$$

$$-\frac{1}{y} = x^2 + C$$

Combined constants of integration

$$-\frac{1}{x^2 + C} = y$$

$$\boxed{y = -\frac{1}{x^2 + C}}$$

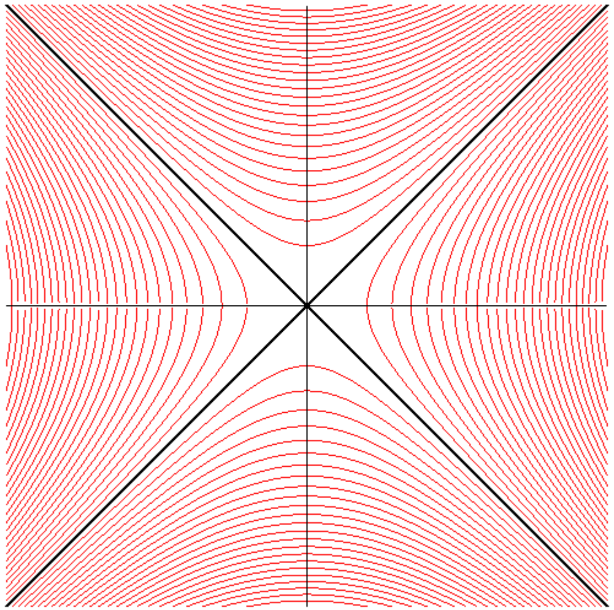
$\rightarrow$

Family of solutions (general solution) of a differential equation

Example

$$\frac{dy}{dx} = \frac{x}{y} \qquad \int y\,dy = \int x\,dx$$

$$y^2 = x^2 + C$$



The picture on the right shows some solutions to the above differential equation.

The straight lines

$y = x$ and $y = -x$

are special solutions.

A unique solution curve goes through any point of the plane different from the origin.

The special solutions $y = x$ and $y = -x$ go both through the origin.

Ordinary Differential Equations
(Initial Value Problem)

Initial conditions

- In many physical problems we need to find the particular solution that satisfies a condition of the form ***$y(x_0)=y_0$***. This is called an **initial condition**, and the problem of finding a solution of the differential equation that satisfies the initial condition is called an **initial-value problem**.
- *Example (cont.):* Find a solution to $y^2 = x^2 + C$ satisfying the initial condition $y(0) = 2$.

$$2^2 = 0^2 + C$$

$$C = 4$$

$$y^2 = x^2 + 4$$

Example:

$$\frac{dy}{dx} = 2x\left(1+y^2\right)e^{x^2}$$ Separable differential equation

$$\frac{1}{1+y^2}\,dy = 2x\,e^{x^2}\,dx$$

$$\int \frac{1}{1+y^2}\,dy = \int 2x\,e^{x^2}\,dx$$

$$u = x^2$$
$$du = 2x\,dx$$

$$\int \frac{1}{1+y^2}\,dy = \int e^u\,du$$

$$\tan^{-1} y + C_1 = e^u + C_2$$

$$\tan^{-1} y + C_1 = e^{x^2} + C_2$$

$$\tan^{-1} y = e^{x^2} + C$$ Combined constants of integration

→

Example (cont.):

$$\frac{dy}{dx} = 2x\left(1+y^2\right)e^{x^2}$$

$$\vdots$$

$$\tan^{-1} y = e^{x^2} + C \longleftarrow$$

We now have y as an implicit function of x.

$$\tan\left(\tan^{-1} y\right) = \tan\left(e^{x^2} + C\right)$$

We can find y as an explicit function of x by taking the tangent of both sides.

$$\boxed{y = \tan\left(e^{x^2} + C\right)}$$

$\rightarrow$

Law of natural growth or decay

A population of living creatures normally increases at a rate that is proportional to the current level of the population. Other things that increase or decrease at a rate proportional to the amount present include radioactive material and money in an interest-bearing account.

If the rate of change is proportional to the amount present, the change can be modeled by:

$$\boxed{\frac{dy}{dt} = ky}$$

→

$$\boxed{\frac{dy}{dt} = ky}$$

Rate of change is proportional to the amount present. This equation is the Analytical Model.

$$\frac{1}{y}dy = k\ dt$$

Divide both sides by ***y***.

$$\int \frac{1}{y}dy = \int k\ dt$$

Integrate both sides. $\longrightarrow$ $\ln|y| = kt + C$

$$e^{\ln|y|} = e^{kt+C}$$

Exponentiate both sides.

$$|y| = e^{C} \cdot e^{kt} \longrightarrow y = \pm e^{C} e^{kt}$$

$\pm e^{c}$ is a constant value which can be represented as ***A***

$$\boxed{y = Ae^{kt}}$$

Analytical Solutuion for the Model

Logistic Growth Model

Real-life populations do not increase forever. There is some limiting factor such as food or living space.

There is a maximum population, or carrying capacity, *M*.

A more realistic model is the **logistic growth model** where growth rate is proportional to both the size of the population (y) and the amount by which y falls short of the maximal size (M-y). Then we have the equation:

$$\boxed{\frac{dy}{dt} = ky(M - y)}$$

Analytical solution to this differential equation :

$$\boxed{y = \frac{y_0 M}{y_0 + (M - y_0)e^{-kMt}}, \quad \text{where } y_0 = y(0)}$$

Example - Mixing Problems

A tank has pure water flowing into it at 10 liter/min.
The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 liter/min.
Initially, the tank contains 10 kg of salt in 100 liter of water.
How much salt will be in the tank
(a)after *t* minutes;
(b)after 30 minutes?

We are interested in the amount of salt at any time --

S – amount of salt - is the dependent variable – unknown function to be determined
t - time - independent variable

To study such a question, we consider *the rate of change of the amount of salt in the tank* : $\frac{\partial S}{\partial t}$

If we can create an equation relating $\frac{\partial S}{\partial t}$ to S and t, then we will have a differential equation which we can, ideally, solve to determine the relationship between S and t.

To describe $\frac{\partial S}{\partial t}$, we can use the concept of concentration (the amaount of salt per unit of volume of water).

In this example, the inflow and outflow rates are the same, so the volume of liquid in the tank stays constant at 100 liter.

Hence, we can describe *the concentration of salt in the tank* by :

Concentration of salt = $\frac{S}{100}\ kg/liter$

Since mixture leaves the tank at the rate of 10 liter/min, salt is leaving the tank at the rate of $\frac{S}{100}(10\ liter/min) = \frac{S}{10}$

Salt leaves the tank so; the rte of cahnge is k= - $\frac{1}{10}$

$\boxed{\frac{dS}{dt} = -\frac{S}{10}}$ The model of decay in the amount of slat

$$\int \frac{dS}{S} = -\int \frac{1}{10}\,dt \longrightarrow \ln|S| = -\frac{1}{10}t + C \longrightarrow \boxed{S = Ce^{-\frac{1}{10}t}}$$

We eleminated the absolute value symbol due to the fact that $S \geq 0$

This is the implicit solution of the ODE

For an explicit solution we will use the initial value provided :

Since $S = 10$ when $t = 0$, we can find that $C = 10$

(a) After *t* minutes, the amount of salt in the tank will be:

$\boxed{S = 10e^{-\frac{1}{10}t}}$ Now, this is the explicit solution of the ODE

We can see from this that as t goes to infinity, the amount of salt in the tank goes to zero.

(b) After 30 minutes, the amount of salt in the tank will be:

$$\boxed{S = 10e^{-\frac{1}{10}30} = 10e^{-3} = 0{,}497870\ kg}$$

Exercise for Home-1

A tank has pure water flowing into it at 10 liter/min. The contents of the tank are kept
thoroughly mixed, and the contents flow out at 10 l/min. Salt is added to the tank at the rate of 0.1 kg/min. Initially, the tank contains 10 kg of salt in 100 liter of water.
How much salt is in the tank after 30 minutes?

Clue : The setup is very similar the previous example. The only difference is the addition of 0.1 kg/min of salt to the tank. So modify your differential equation to take this into account.

Exercise for Home-2

A tank has pure water flowing into it at 12 liter/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 liter/min.

Initially, the tank contains 10 kg of salt in 100 liter of water.

How much salt will be in the tank
(a) after t minutes;
(b) after 30 minutes?

Clue: The inflow rate is greater than the outflow rate. As a result, the volume is not constant.
Using the initial conditions and the flow rates, we can say that the volume ***V*** of liquid in the tank is $V = 100 + 2t$
The concentration of salt after t minutes is $\frac{S}{V}$

Model of Falling Parachutist

Drag force : $F_d = cv$

Gravity force: $F_d = M\text{g}$

Acceleration: $a = \frac{dv}{dt}$

Net Force: $F = Ma \quad or\, F = F_g - F_d$

$$M\frac{dv}{dt} = Mg - cv$$

$$\frac{dv}{dt} = g - \frac{c}{M}v$$