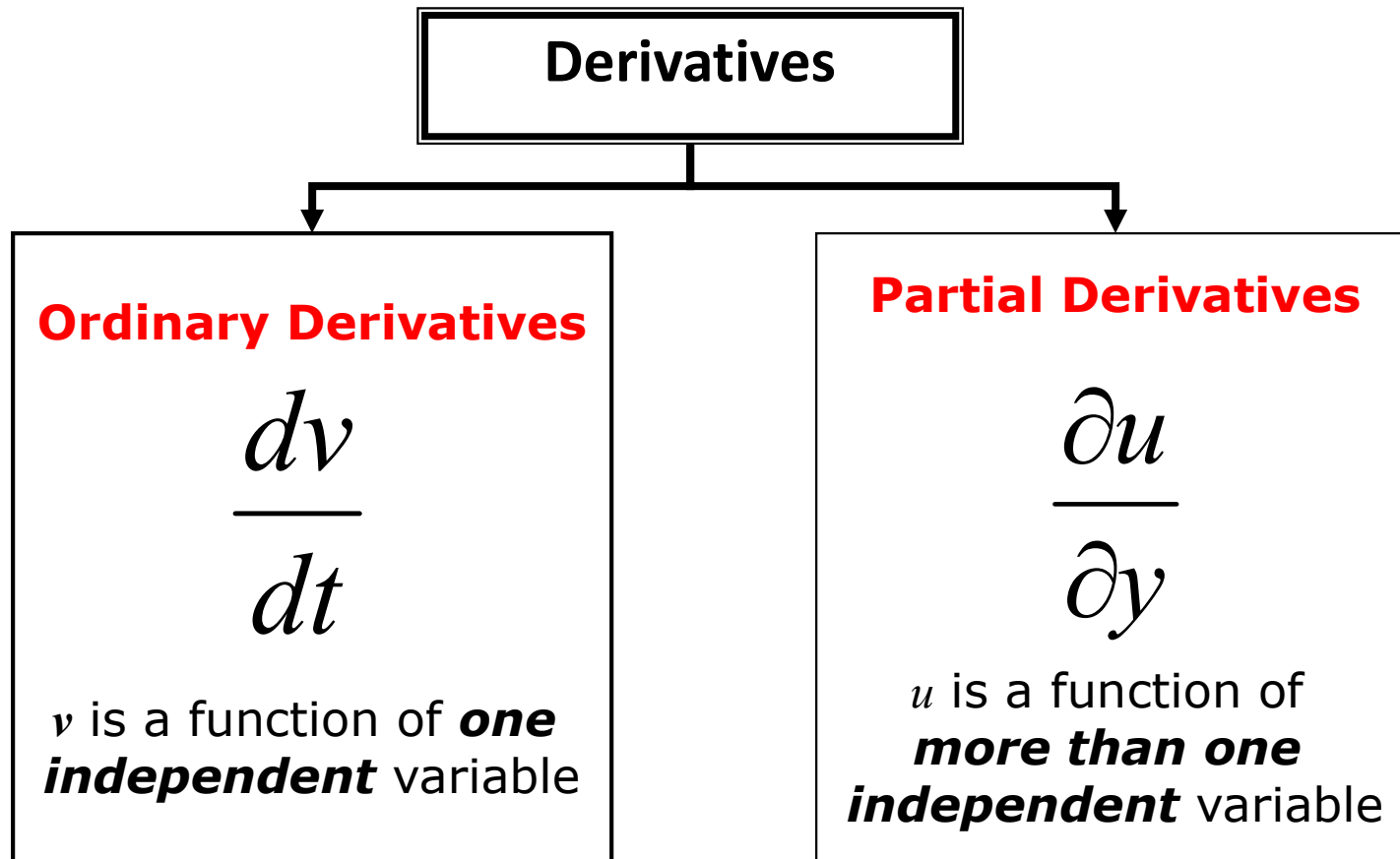


Ordinary Differential Equations

Derivatives



Differential Equations

Differential Equations

Ordinary Differential Equations

$$\frac{d^2 v}{dt^2} + 6tv = 1$$

involve one or more
Ordinary derivatives of
unknown functions

Partial Differential Equations

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

involve one or more
partial derivatives of
unknown functions

Ordinary Differential Equations

Ordinary Differential Equations (ODEs) involve one or more **ordinary derivatives of unknown functions with respect to one independent variable**

Examples :

$$\frac{dv(t)}{dt} - v(t) = e^t$$

x(t): unknown function

$$\frac{d^2x(t)}{dt^2} - 5\frac{dx(t)}{dt} + 2x(t) = \cos(t)$$

t: independent variable

Example of ODE: Model of Falling Parachutist

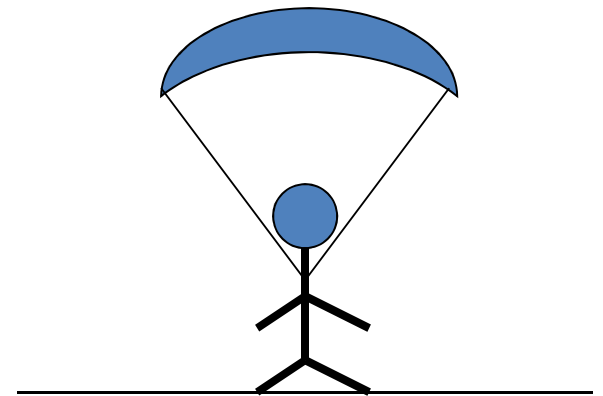
The velocity of a falling
parachutist is given by:

$$\frac{d v}{d t} = 9.8 - \frac{c}{M} v$$

M : *mass*

c : *drag coefficient*

v : *velocity*



Model of Falling Parachutist

Drag force : $F_d = cv$

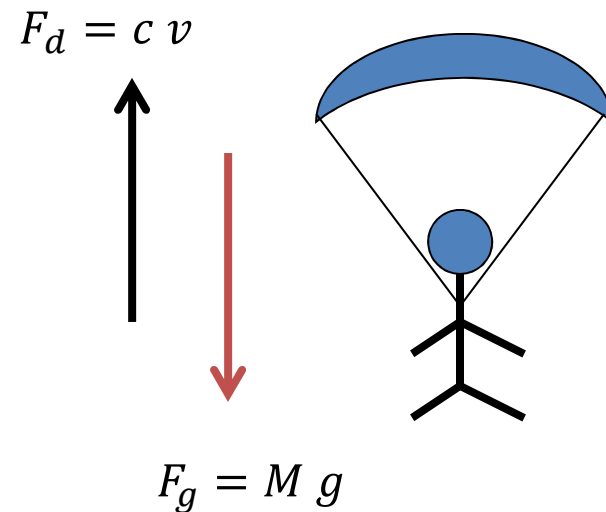
Gravity force: $F_g = Mg$

Acceleration: $a = \frac{dv}{dt}$

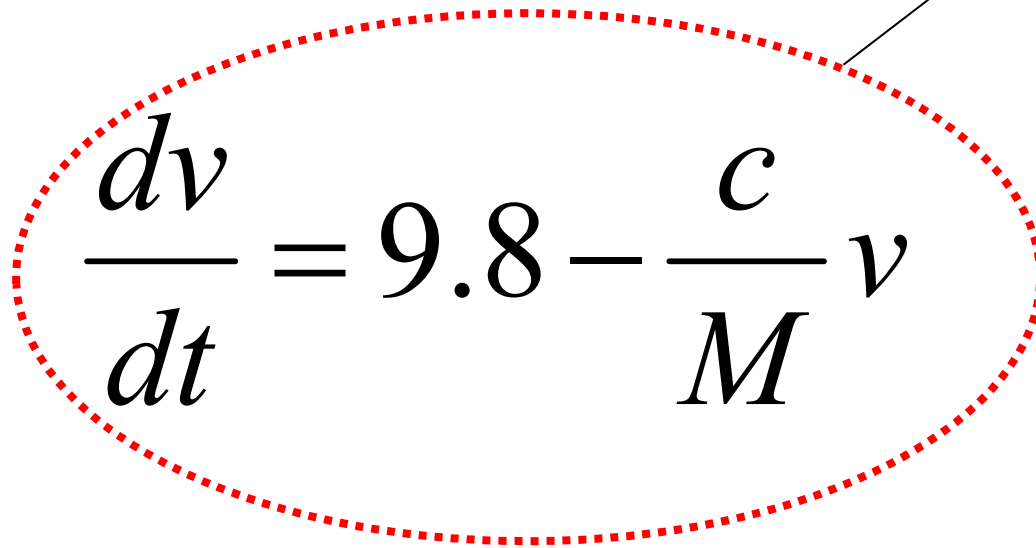
Net Force: $F = Ma$ or $F = F_g - F_d$

$$M \frac{dv}{dt} = Mg - cv$$

$$\frac{dv}{dt} = g - \frac{c}{M}v$$



Definitions


$$\frac{dv}{dt} = 9.8 - \frac{c}{M} v$$

Ordinary
differential
equation

Definitions (Cont.)

$$\frac{dv}{dt} = 9.8 - \frac{c}{M}v$$

(Dependent variable)
unknown
function to be
determined

Definitions (Cont.)

$$\frac{d v}{d t} = 9.8 - \frac{c}{M} v$$

(independent variable)
the variable with respect to which
other variables are differentiated

Order of a Differential Equation

- The number of the highest derivative in a differential equation. A differential equation of order 1 is called first order, order 2 second order, etc.

Example: The differential equation $y'' + xy' - x^3y = \sin x$ is second order since the highest derivative is y'' or the second derivative.

Order of a Differential Equation

The **order** of an ordinary differential equation is the order of the highest order derivative.

Examples :

$$\frac{dx(t)}{dt} - x(t) = e^t$$

First order ODE

$$\frac{d^2 x(t)}{dt^2} - 5 \frac{dx(t)}{dt} + 2x(t) = \cos(t)$$

Second order ODE

$$\left(\frac{d^2 x(t)}{dt^2} \right)^3 - \frac{dx(t)}{dt} + 2x^4(t) = 1$$

Second order ODE

Solution of a Differential Equation

A **solution** to a differential equation is a function that satisfies the equation.

Example :

$$\frac{dx(t)}{dt} + x(t) = 0$$

Solution $x(t) = e^{-t}$

Proof :

$$\frac{dx(t)}{dt} = -e^{-t}$$

$$\frac{dx(t)}{dt} + x(t) = -e^{-t} + e^{-t} = 0$$

Linear and non-linear

Both ordinary and partial differential equations are broadly classified as **linear** and **nonlinear**.

A [differential equation is linear](#) if the unknown function and its derivatives appear to the power 1 (products of the unknown function and its derivatives are not allowed) and [nonlinear](#) otherwise.

The characteristic property of linear equations is that their solutions form an affine subspace of an appropriate function space, which results in much more developed theory of linear differential equations. **Homogeneous** linear differential equations are a further subclass for which the space of solutions is a linear subspace i.e. the sum of any set of solutions or multiples of solutions is also a solution. The coefficients of the unknown function and its derivatives in a linear differential equation are allowed to be (known) functions of the independent variable or variables; if these coefficients are constants then one speaks of a **constant coefficient linear differential equation**.

Linear and Affine Function

- $f(x)=2x$ is linear and affine.
- $f(x)=2x+3$ is affine but not linear

Linear and Non-linear

There are very few methods of solving nonlinear differential equations exactly; those that are known typically depend on the equation having particular [symmetries](#).

Nonlinear differential equations can exhibit very complicated behavior over extended time intervals, characteristic of [chaos](#). Even the fundamental questions of existence, uniqueness, and extendability of solutions for nonlinear differential equations, and well-posedness of initial and boundary value problems for nonlinear PDEs are hard problems.

However, if the differential equation is a correctly formulated representation of a meaningful physical process, then one expects it to have a solution.

Linear differential equations frequently appear as [approximations](#) to nonlinear equations. These approximations are only valid under restricted conditions.

Linear and Non-linear

A **linear differential equation** is any differential equation that can be written in the following form:

$$a_n(t) y^{(n)}(t) + a_{n-1}(t) y^{(n-1)}(t) + \dots + a_1(t) y'(t) + a_0(t) y(t) = g(t)$$

If a differential equation cannot be written in this form, then it is called **a non-linear differential equation**.

Linear ODE

An ODE is linear if the unknown function and its derivatives appear to power one.

No product of the unknown function and/or its derivatives

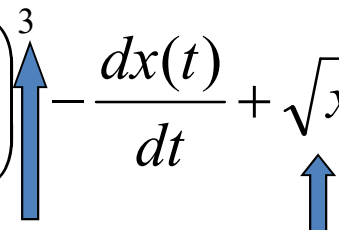
Examples :

$$\frac{dx(t)}{dt} - x(t) = e^t$$

Linear ODE

$$\frac{d^2x(t)}{dt^2} - 5\frac{dx(t)}{dt} + 2t^2x(t) = \cos(t)$$

Linear ODE

$$\left(\frac{d^2x(t)}{dt^2}\right)^3 - \frac{dx(t)}{dt} + \sqrt{x(t)} = 1$$


Non-linear ODE

Nonlinear ODE

Examples of nonlinear ODE :

$$\frac{dx(t)}{dt} - \cos(x(t)) = 1$$

$$\frac{d^2x(t)}{dt^2} - 5 \frac{dx(t)}{dt} x(t) = 2$$

$$\frac{d^2x(t)}{dt^2} - \left| \frac{dx(t)}{dt} \right| + x(t) = 1$$

Solutions of Ordinary Differential Equations

$$x(t) = \cos(2t)$$

is a solution to the ODE

$$\frac{d^2 x(t)}{dt^2} + 4x(t) = 0$$

Is it unique?

All functions of the form $x(t) = \cos(2t + c)$
(where c is a real constant) are solutions.

Uniqueness of a Solution

In order to uniquely specify a solution to an n^{th} order differential equation we need n conditions.

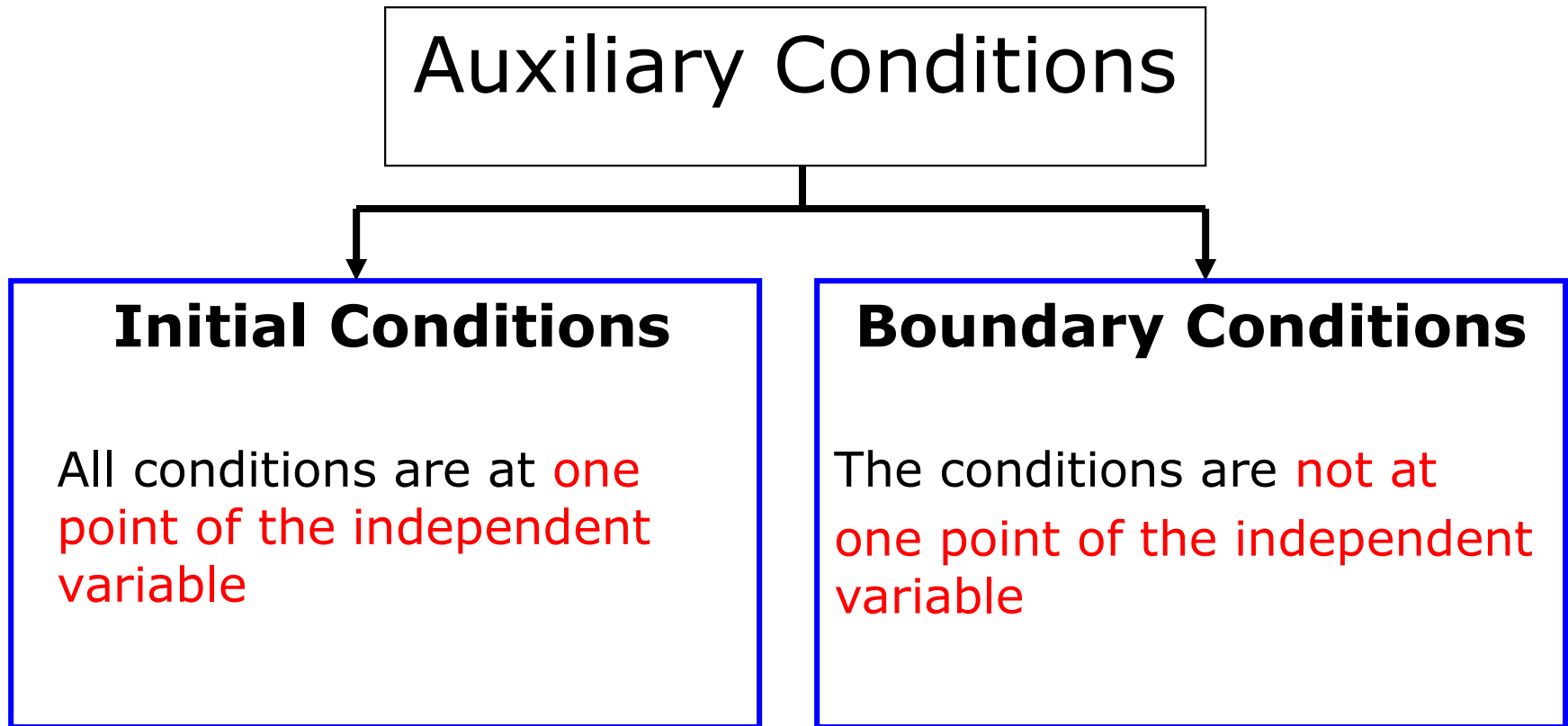
$$\frac{d^2 x(t)}{dt^2} + 4x(t) = 0$$

Second order ODE

$$\begin{aligned} x(0) &= a \\ \dot{x}(0) &= b \end{aligned}$$

Two conditions are needed to uniquely specify the solution

Auxiliary Conditions



Boundary-Value and Initial value Problems

Initial-Value Problems

- The auxiliary conditions are at **one point of the independent variable**

$$\ddot{x} + 2\dot{x} + x = e^{-2t}$$

$$x(0) = 1, \dot{x}(0) = 2.5$$

same

Boundary-Value Problems

The auxiliary conditions are **not at one point of the independent variable**

More difficult to solve than initial value problems

$$\ddot{x} + 2\dot{x} + x = e^{-2t}$$

$$x(0) = 1, x(2) = 1.5$$

different

Classification of ODEs

ODEs can be classified in different ways:

- Order
 - First order ODE
 - Second order ODE
 - N^{th} order ODE
- Linearity
 - Linear ODE
 - Nonlinear ODE
- Auxiliary conditions
 - Initial value problems
 - Boundary value problems

Analytical Solutions

- Analytical Solutions to ODEs are available for linear ODEs and special classes of nonlinear differential equations.

Numerical Solutions

- Numerical methods are used to obtain a graph or a table of the unknown function.
- Most of the Numerical methods used to solve ODEs are based directly (or indirectly) on the truncated Taylor series expansion.

Separable Differential Equations

A separable differential equation can be expressed as the product of a function of x and a function of y .

$$\frac{dy}{dx} = g(x) \cdot h(y) \quad h(y) \neq 0$$

Example:

$$\frac{dy}{dx} = 2xy^2$$

Multiply both sides by dx and divide both sides by y^2 to separate the variables. (Assume y^2 is never zero.)

$$\frac{dy}{y^2} = 2x \, dx$$

$$y^{-2} dy = 2x \, dx$$

Separable Differential Equations

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Example:

$$\frac{dy}{dx} = 2xy^2$$

$$\frac{dy}{y^2} = 2x \, dx$$

$$y^{-2} dy = 2x \, dx$$

$$\int y^{-2} dy = \int 2x \, dx$$

$$-y^{-1} + C_1 = x^2 + C_2$$

$$-\frac{1}{y} = x^2 + C$$

$$-\frac{1}{x^2 + C} = y$$

$$y = -\frac{1}{x^2 + C}$$

Combined
constants of
integration



Family of solutions (general solution) of a differential equation

Example

$$\frac{dy}{dx} = \frac{x}{y} \quad \int y dy = \int x dx$$
$$y^2 = x^2 + C$$

The picture on the right shows some solutions to the above differential equation.

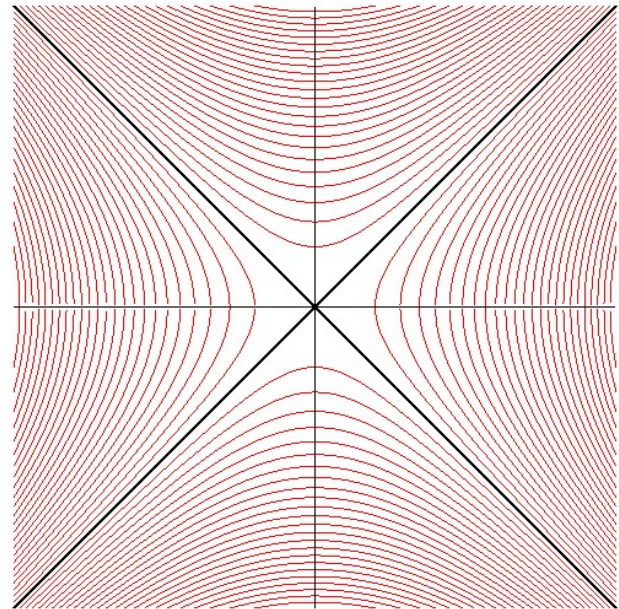
The straight lines

$$y = x \text{ and } y = -x$$

are special solutions.

A unique solution curve goes through any point of the plane different from the origin.

The special solutions $y = x$ and $y = -x$ go both through the origin.



Ordinary Differential Equations (Initial Value Problem)

Initial conditions

- In many physical problems we need to find the particular solution that satisfies a condition of the form $y(x_0)=y_0$. This is called an **initial condition**, and the problem of finding a solution of the differential equation that satisfies the initial condition is called an **initial-value problem**.
- *Example (cont.):* Find a solution to $y^2 = x^2 + C$ satisfying the initial condition $y(0) = 2$.

$$2^2 = 0^2 + C$$

$$C = 4$$

$$y^2 = x^2 + 4$$

Example:

$$\frac{dy}{dx} = 2x(1+y^2)e^{x^2} \leftarrow \text{Separable differential equation}$$

$$\frac{1}{1+y^2} dy = 2x e^{x^2} dx$$

$$\int \frac{1}{1+y^2} dy = \int 2x e^{x^2} dx$$

$$\begin{aligned} u &= x^2 \\ du &= 2x dx \end{aligned}$$

$$\int \frac{1}{1+y^2} dy = \int e^u du$$

$$\tan^{-1} y + C_1 = e^u + C_2$$

$$\tan^{-1} y + C_1 = e^{x^2} + C_2$$

$$\tan^{-1} y = e^{x^2} + C \leftarrow \text{Combined constants of integration}$$



Example (cont.):

$$\frac{dy}{dx} = 2x(1 + y^2)e^{x^2}$$

⋮

$$\tan^{-1} y = e^{x^2} + C \quad \leftarrow \text{We now have } y \text{ as an implicit function of } X.$$

$$\tan(\tan^{-1} y) = \tan(e^{x^2} + C) \quad \text{We can find } y \text{ as an explicit function of } X by taking the tangent of both sides.}$$

$$y = \tan(e^{x^2} + C)$$



Law of natural growth or decay

A population of living creatures normally increases at a rate that is proportional to the current level of the population. Other things that increase or decrease at a rate proportional to the amount present include radioactive material and money in an interest-bearing account.

If the rate of change is proportional to the amount present, the change can be modeled by:

$$\frac{dy}{dt} = ky$$



$$\frac{dy}{dt} = ky$$

Rate of change is proportional to the amount present. This equation is the **Analytical Model**.

$$\frac{1}{y} dy = k dt$$

Divide both sides by y .

$$\int \frac{1}{y} dy = \int k dt$$

Integrate both sides. $\longrightarrow \ln|y| = kt + C$

$$e^{\ln|y|} = e^{kt+C}$$

Exponentiate both sides.

$$|y| = e^C \cdot e^{kt} \longrightarrow y = \pm e^C e^{kt}$$

$\pm e^C$ is a constant value which can be represented as **A**

$$y = Ae^{kt}$$

Analytical Solution for the Model

Logistic Growth Model

Real-life populations do not increase forever. There is some limiting factor such as food or living space.

There is a maximum population, or carrying capacity, M .

A more realistic model is the **logistic growth model** where growth rate is proportional to both the size of the population (y) and the amount by which y falls short of the maximal size ($M-y$). Then we have the equation:

$$\frac{dy}{dt} = ky(M - y)$$

Analytical solution to this differential equation :

$$y = \frac{y_0 M}{y_0 + (M - y_0)e^{-kMt}}, \quad \text{where } y_0 = y(0)$$

Example - Mixing Problems

A tank has pure water flowing into it at 10 liter/min.

The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 liter/min. Initially, the tank contains 10 kg of salt in 100 liter of water.

How much salt will be in the tank

(a) after t minutes;

(b) after 30 minutes?

We are interested in the **amount of salt** at any **time** --

S – amount of salt - is the dependent variable – unknown function to be determined
 t - time - independent variable

To study such a question, we consider ***the rate of change of the amount of salt in the tank*** : $\frac{dS}{dt}$

If we can create an equation relating $\frac{dS}{dt}$ to S and t , then we will have a differential equation which we can, ideally, solve to determine the relationship between S and t .

To describe $\frac{dS}{dt}$, we can use the concept of concentration (the amount of salt per unit of volume of water).

In this example, **the inflow and outflow rates are the same**, so **the volume of liquid in the tank stays constant at 100 liter**.

Hence, we can describe ***the concentration of salt in the tank*** by :

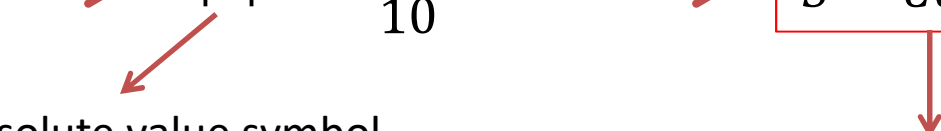
$$\text{Concentration of salt} = \frac{S}{100} \text{ kg/liter}$$

Since mixture **leaves the tank at the rate of 10 liter/min**, salt is leaving the tank at the rate of $\frac{S}{100} (10 \text{ liter/min}) = \frac{S}{10}$

Salt **leaves** the tank so; the rate of change is $k = -\frac{1}{10}$

$$\boxed{\frac{dS}{dt} = -\frac{S}{10}}$$

The model of decay in the amount of salt

$$\int \frac{dS}{S} = -\int \frac{1}{10} dt \longrightarrow \ln|S| = -\frac{1}{10}t + C \longrightarrow \boxed{S = Ce^{-\frac{1}{10}t}}$$


We eliminated the absolute value symbol due to the fact that $S \geq 0$

This is the implicit solution of the ODE

For an **explicit solution** we will use the **initial value** provided :

Since $S = 10$ when $t = 0$, we can find that $C = 10$

(a) After t minutes, the amount of salt in the tank will be:

$$S = 10e^{-\frac{1}{10}t}$$

Now, this is the **explicit solution** of the ODE

We can see from this that as t goes to infinity, the amount of salt in the tank goes to zero.

(b) After 30 minutes, the amount of salt in the tank will be:

$$S = 10e^{-\frac{1}{10}30} = 10e^{-3} = 0,497870 \text{ kg}$$

Exercise for Home-1

A tank has pure water flowing into it at 10 liter/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 l/min. Salt is added to the tank at the rate of 0.1 kg/min. Initially, the tank contains 10 kg of salt in 100 liter of water.

How much salt is in the tank after 30 minutes?

Clue : The setup is very similar the previous example. The only difference is the addition of 0.1 kg/min of salt to the tank. So modify your differential equation to take this into account.

Exercise for Home-2

A tank has pure water flowing into it at 12 liter/min. The contents of the tank are kept thoroughly mixed, and the contents flow out at 10 liter/min.

Initially, the tank contains 10 kg of salt in 100 liter of water.

How much salt will be in the tank

- (a) after t minutes;
- (b) after 30 minutes?

Clue: The inflow rate is greater than the outflow rate. As a result, the volume is not constant.

Using the initial conditions and the flow rates, we can say that the volume V of liquid in the tank is $V = 100 + 2t$

The concentration of salt after t minutes is $\frac{S}{V}$