Euler's Method

$$\frac{dy}{dx} = f(x, y), y(0) = y_0$$

$$\text{Slope } = \frac{Rise}{Run}$$

$$= \frac{y_1 - y_0}{x_1 - x_0}$$

$$= f(x_0, y_0)$$

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

$$= y_0 + f(x_0, y_0)h$$

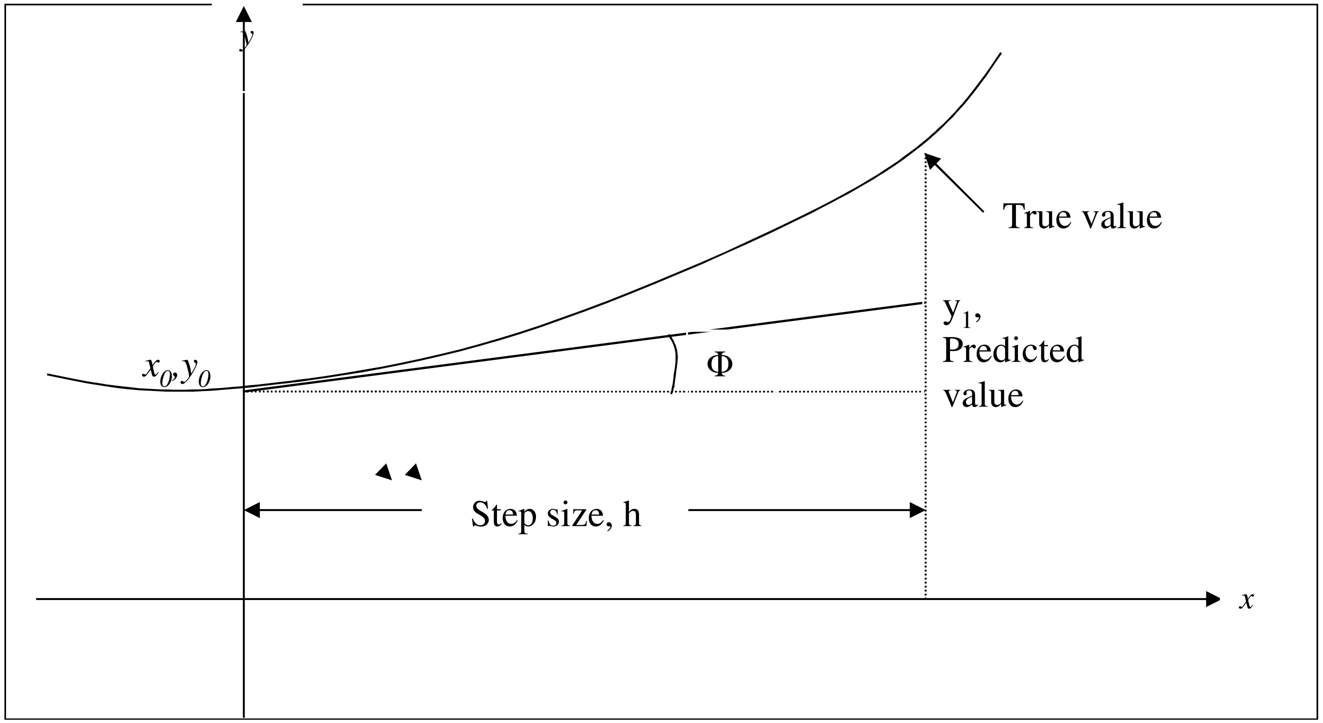


Figure 1 Graphical interpretation of the first step of Euler's method

Euler's Method

$$y_{i+1} = y_i + f\left(x_i, y_i\right)h$$

$$h = x_{i+1} - x_i$$

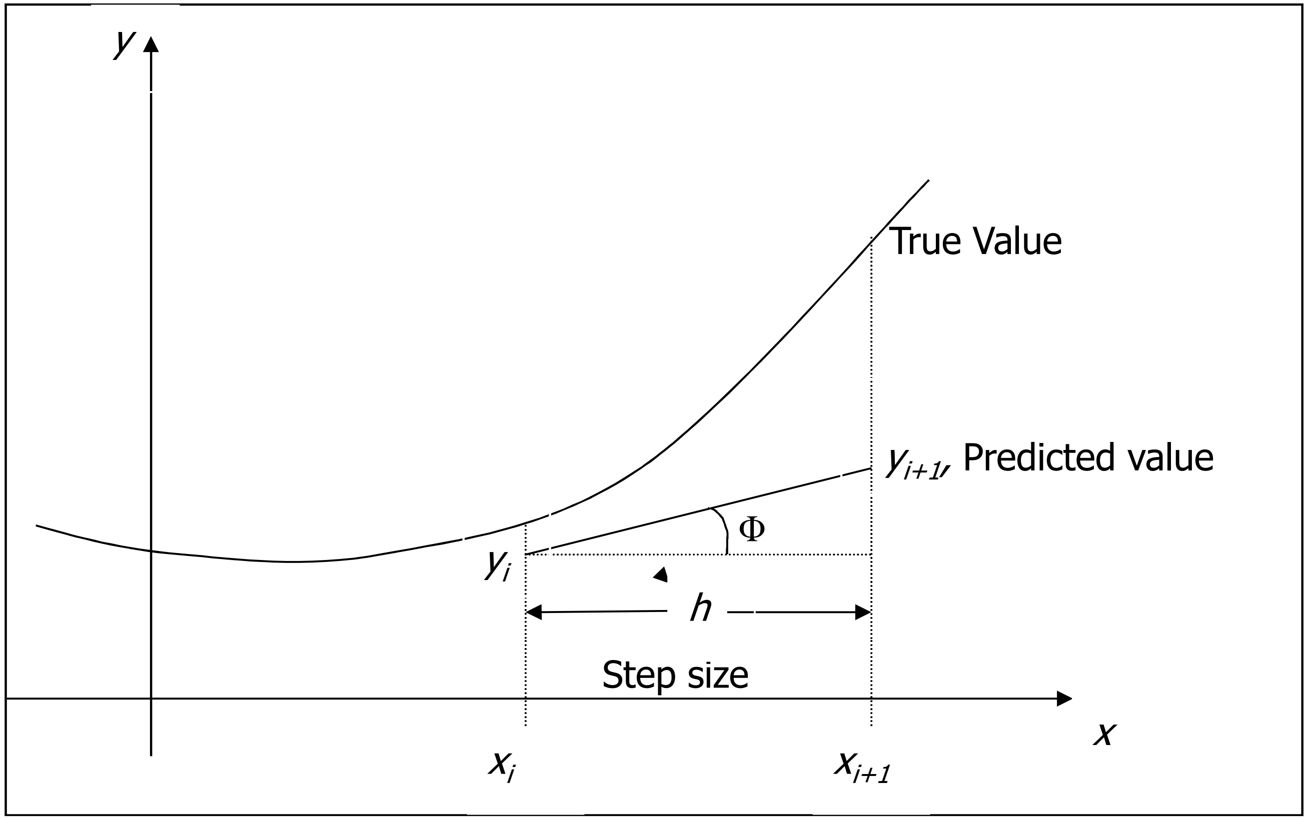


Figure 2 General graphical interpretation of Euler's method

Euler's Method

Write the first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

$$y_{i+1} = y_i + f(x_i, y_i)h$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x},\ y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y,\ y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right), \theta(0) = 1200\,K$$

Find the temperature at $t = 480$ seconds using Euler's method. Assume a step size of $h = 240$ seconds.

Solution

Step 1:

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$f(t,\theta) = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$\theta_{i+1} = \theta_i + f(t_i, \theta_i)h$$

$$\begin{aligned}
\theta_1 &= \theta_0 + f(t_0, \theta_0)h \\
&= 1200 + f(0,1200)240 \\
&= 1200 + \left(-2.2067\times10^{-12}\left(1200^4 - 81\times10^8\right)\right)240 \\
&= 1200 + (-4.5579)240 \\
&= 106.09K
\end{aligned}$$

θ_1 is the approximate temperature at $t = t_1 = t_0 + h = 0 + 240 = 240$

$$\theta(240) \approx \theta_1 = 106.09K$$

Solution Cont

Step 2: For $i = 1, \quad t_1 = 240, \quad \theta_1 = 106.09$

$$\begin{aligned}
\theta_2 &= \theta_1 + f(t_1, \theta_1)h \\
&= 106.09 + f(240, 106.09)240 \\
&= 106.09 + \left(-2.2067 \times 10^{-12}\left(106.09^4 - 81 \times 10^8\right)\right)240 \\
&= 106.09 + (0.017595)240 \\
&= 110.32K
\end{aligned}$$

θ_2 is the approximate temperature at $t = t_2 = t_1 + h = 240 + 240 = 480$

$$\theta(480) \approx \theta_2 = 110.32K$$

Solution Cont

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$0.92593\ln\frac{\theta-300}{\theta+300}-1.8519\tan^{-1}(0.00333\theta)=-0.22067\times10^{-3}t-2.9282$$

The solution to this nonlinear equation at t=480 seconds is

$$\theta(480)=647.57K$$

Comparison of Exact and Numerical Solutions

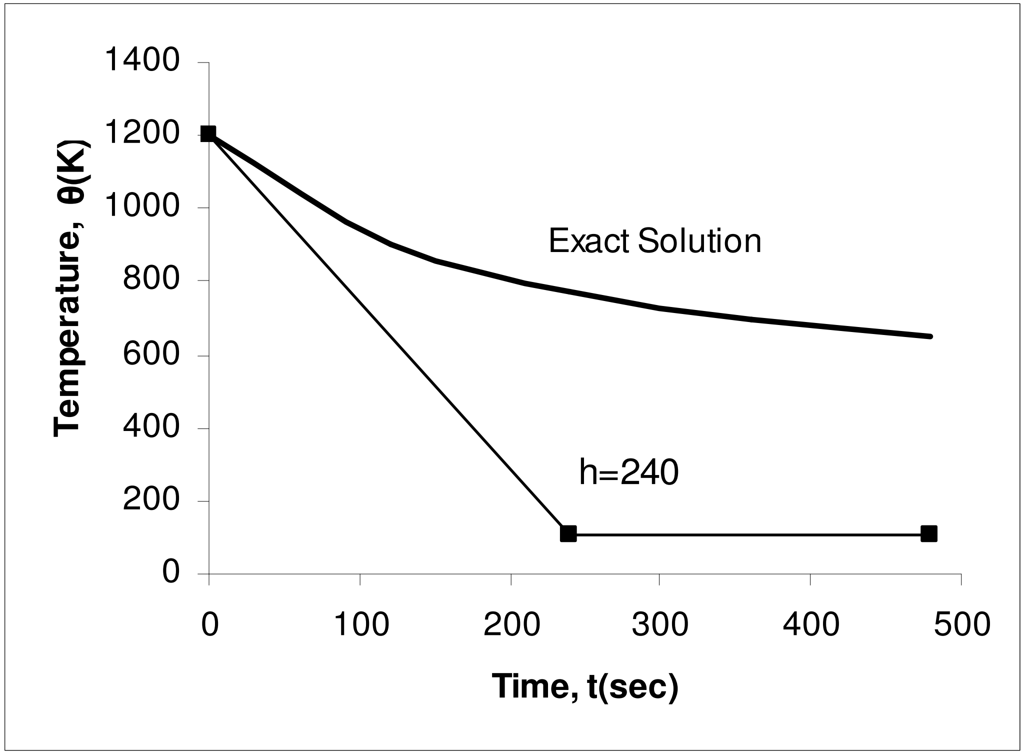


Figure 3. Comparing exact and Euler's method

Effect of step size

Table 1. Temperature at 480 seconds as a function of step size, h

Step, h	$\theta(480)$	E_t	$\lvert\epsilon_t\rvert\%$
480	−987.81	1635.4	252.54
240	110.32	537.26	82.964
120	546.77	100.80	15.566
60	614.97	32.607	5.0352
30	632.77	14.806	2.2864

$\theta(480) = 647.57K$ (exact)

Comparison with exact results

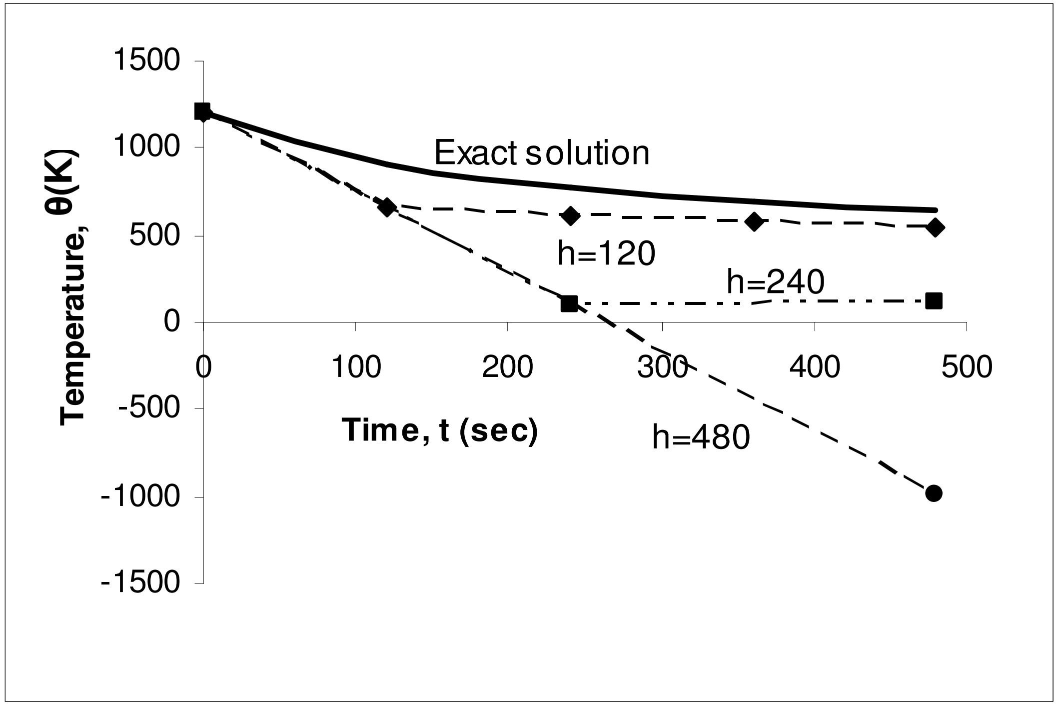


Figure 4. Comparison of Euler’s method with exact solution for different step sizes

Effects of step size on Euler's Method

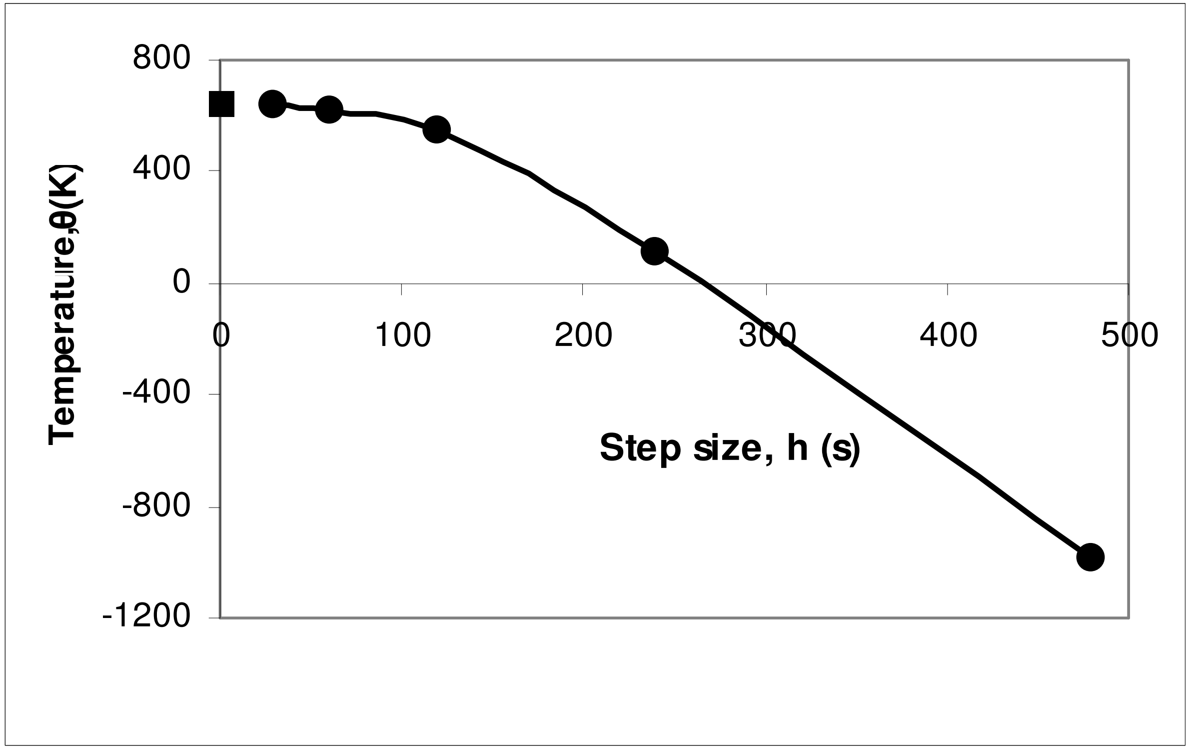


Figure 5. Effect of step size in Euler's method.

Errors in Euler's Method

It can be seen that Euler's method has large errors. This can be illustrated using Taylor series.

$$y_{i+1} = y_i + \left.\frac{dy}{dx}\right|_{x_i, y_i} (x_{i+1} - x_i) + \frac{1}{2!}\left.\frac{d^2 y}{dx^2}\right|_{x_i, y_i} (x_{i+1} - x_i)^2 + \frac{1}{3!}\left.\frac{d^3 y}{dx^3}\right|_{x_i, y_i} (x_{i+1} - x_i)^3 + \dots$$

$$y_{i+1} = y_i + f(x_i, y_i)(x_{i+1} - x_i) + \frac{1}{2!} f'(x_i, y_i)(x_{i+1} - x_i)^2 + \frac{1}{3!} f''(x_i, y_i)(x_{i+1} - x_i)^3 + \dots$$

As you can see the first two terms of the Taylor series

$y_{i+1} = y_i + f(x_i, y_i)h$ are the Euler's method.

The true error in the approximation is given by

$$E_t = \frac{f'(x_i, y_i)}{2!} h^2 + \frac{f''(x_i, y_i)}{3!} h^3 + \dots$$

Euler Method

Problem

$$\dot{y}(x) = f(x, y)$$

$$y(x_0) = y_0$$

Euler Method

$$y_0 = y(x_0)$$

$$y_{i+1} = y_i + h\ f(x_i, y_i)$$

$$for\ i = 1,2,\ldots$$

Local Truncation Error $O(h^2)$

Global Truncation Error $O(h)$

Introduction

Problem to be solved is a first order ODE:

$$\dot{y}(x) = f(x, y), \qquad y(x_0) = y_0$$

- The methods have the general form:

$$y_{i+1} = y_i + h\,\phi$$

- For the case of Euler: $\phi = f(x_i, y_i)$
- Different forms of ϕ will be used for the Midpoint and Heun's Methods.

Midpoint Method

Problem

$$\dot{y}(x) = f(x, y)$$

$$y(x_0) = y_0$$

Midpoint Method

$$y_0 = y(x_0)$$

$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(x_i, y_i)$$

$$y_{i+1} = y_i + h\ f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

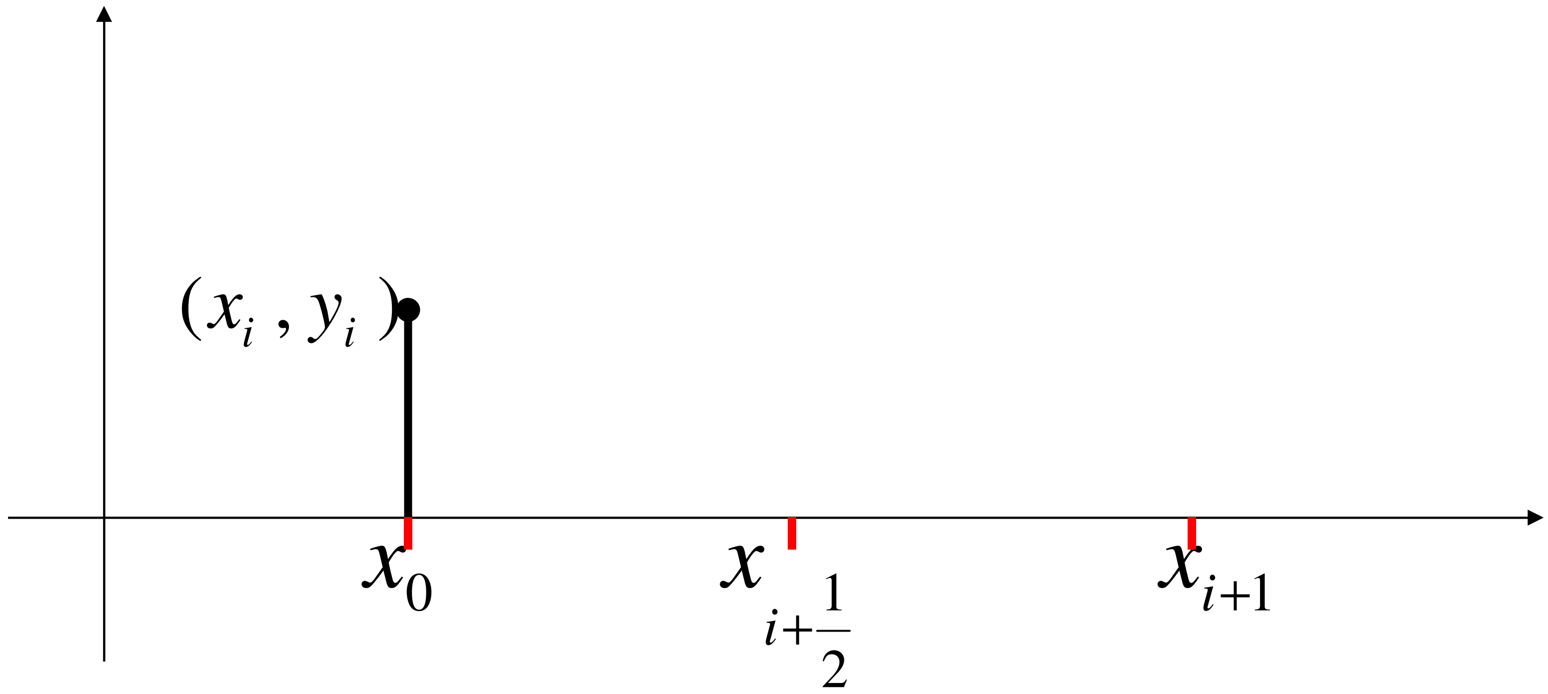
Local Truncation Error $O(h^3)$

Global Truncation Error $O(h^2)$

Motivation

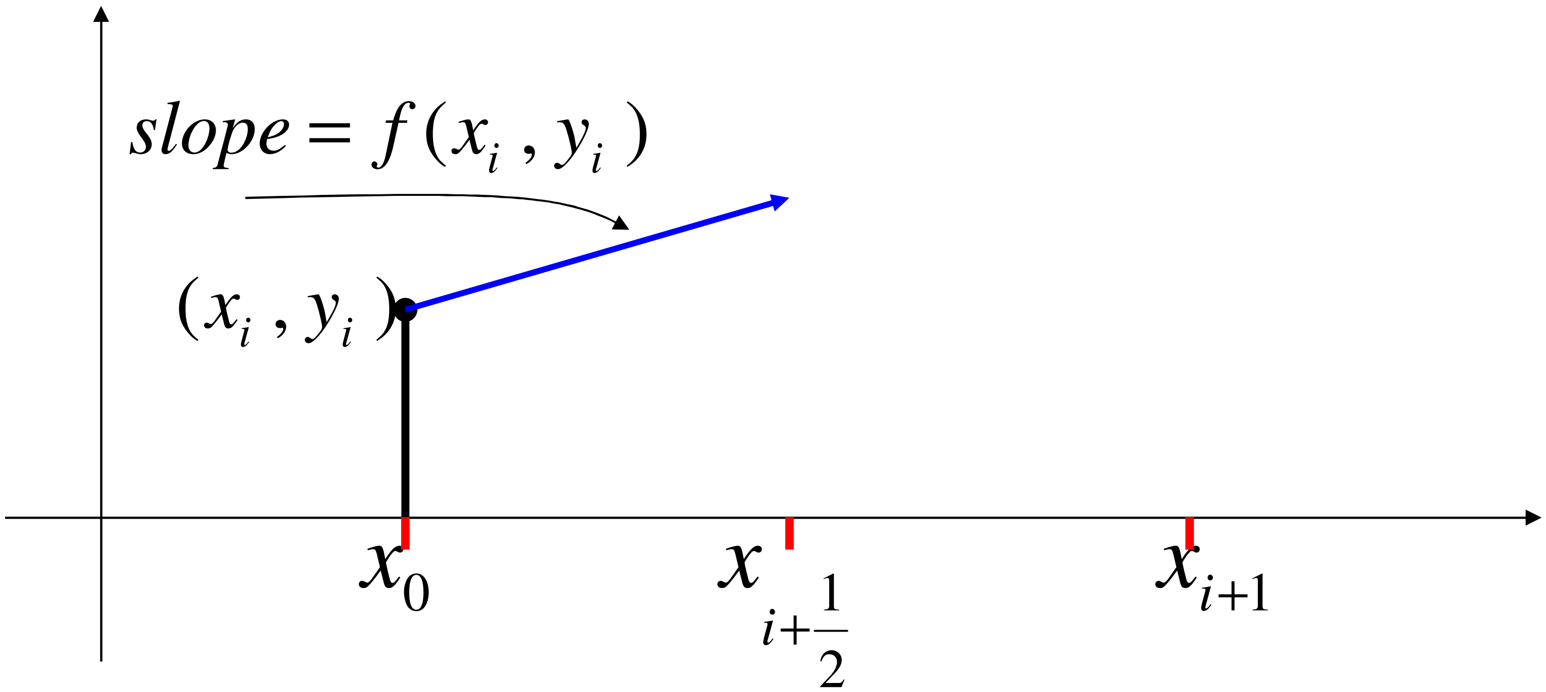
- The midpoint can be summarized as:
  - Euler method is used to estimate the solution at the midpoint.
  - The value of the rate function $f(x,y)$ at the mid point is calculated.
  - This value is used to estimate y_{i+1}.
- Local Truncation error of order $O(h^3)$.
- Comparable to Second order Taylor series method.

Midpoint Method



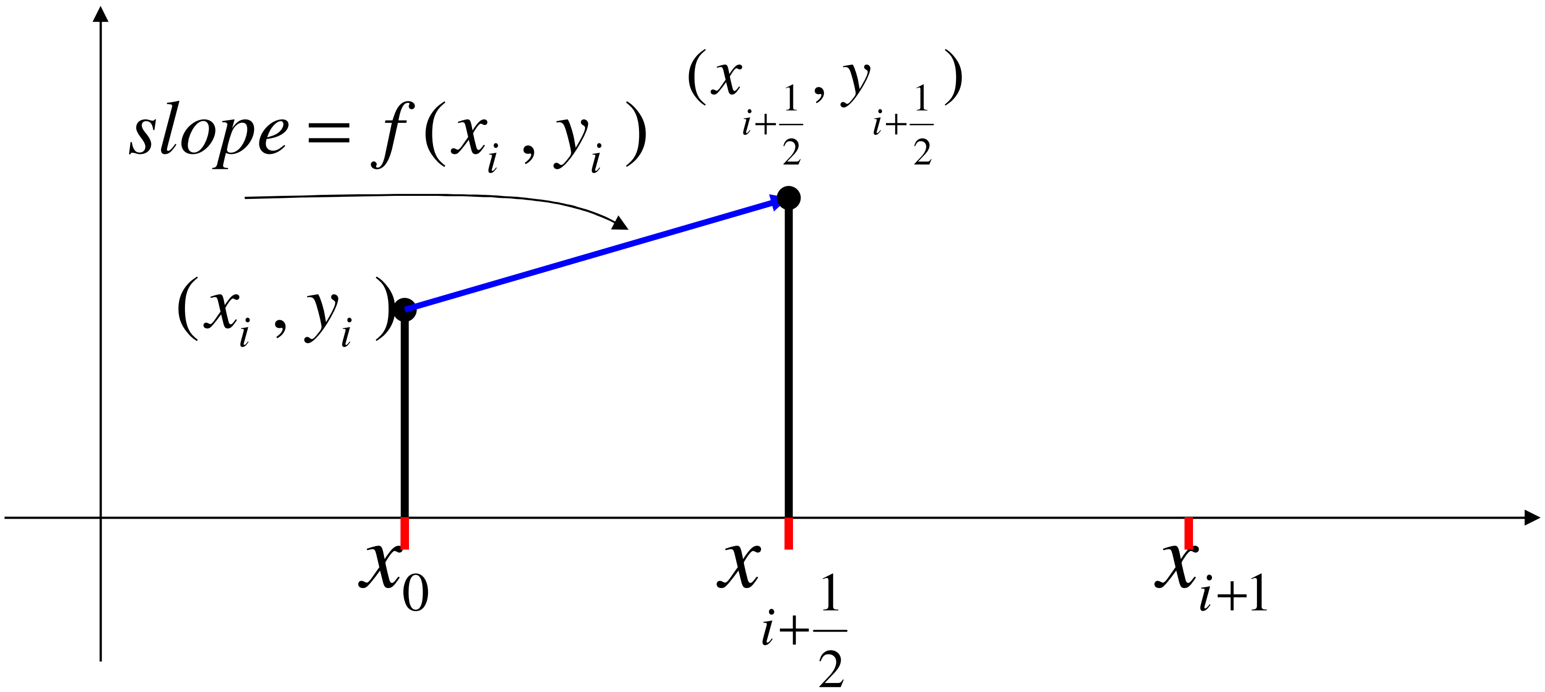
$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(x_i, y_i), \qquad y_{i+1} = y_i + h\, f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

Midpoint Method



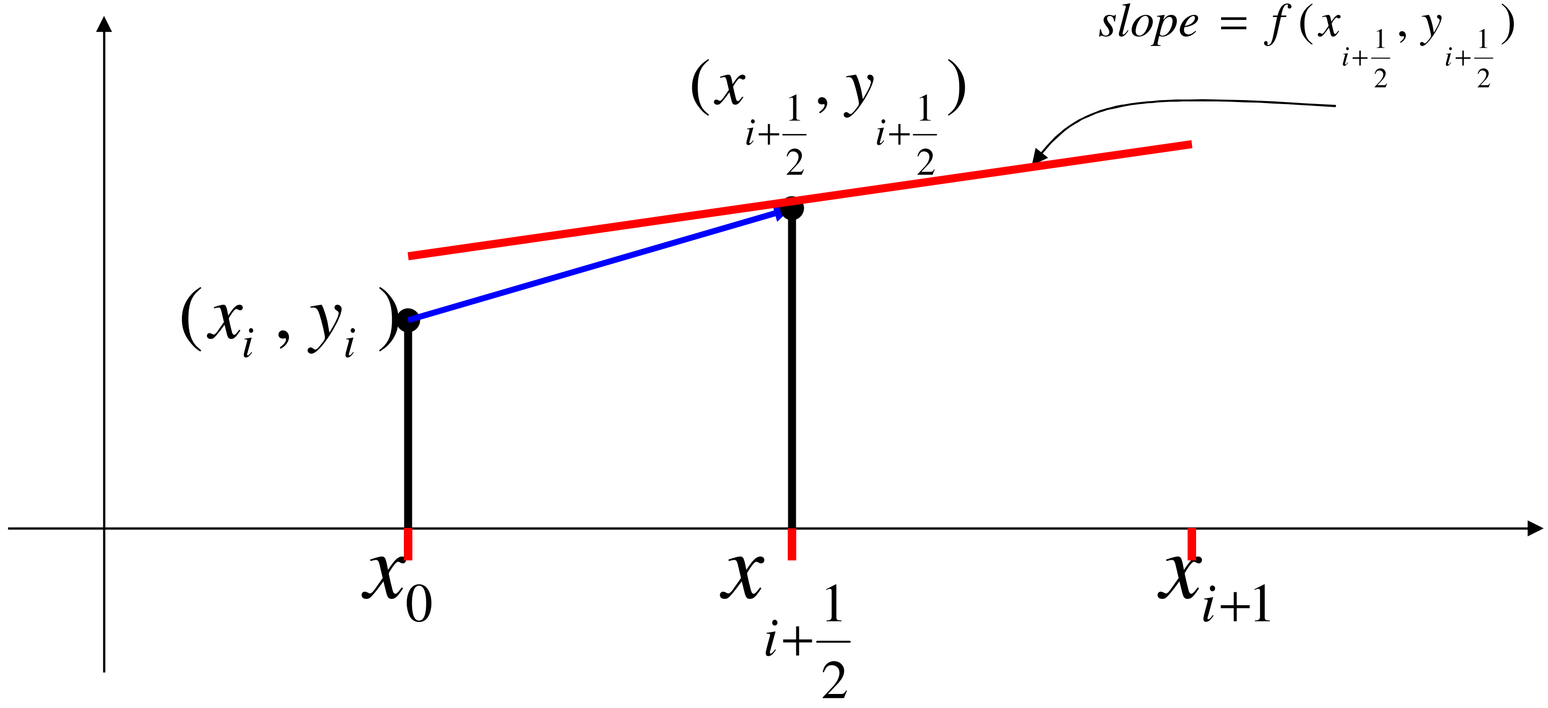
$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(x_i, y_i), \qquad y_{i+1} = y_i + h\, f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

Midpoint Method



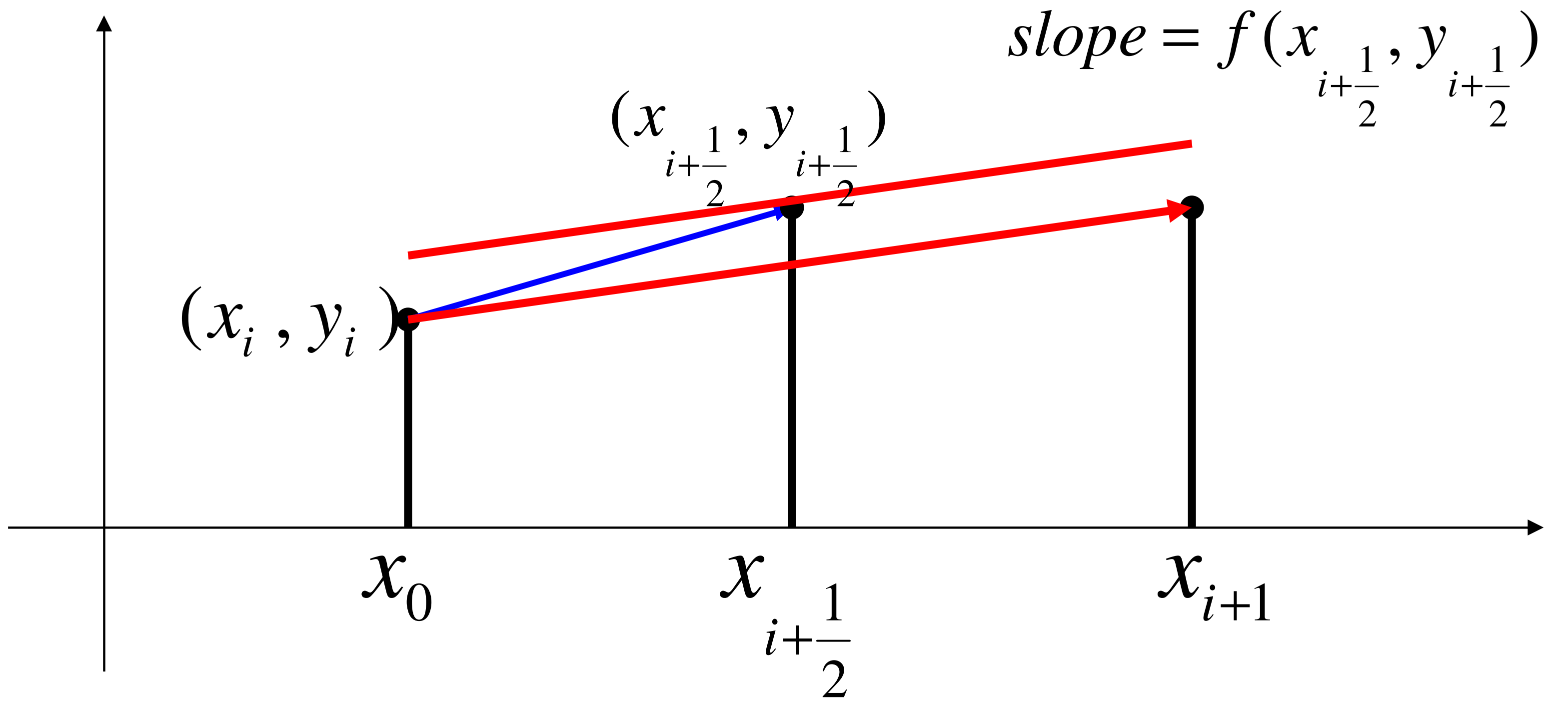
$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(x_i, y_i), \qquad y_{i+1} = y_i + h\, f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

Midpoint Method



$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(x_i, y_i), \qquad y_{i+1} = y_i + h\, f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

Midpoint Method



$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(x_i, y_i), \qquad y_{i+1} = y_i + h\, f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$$

Example 1

Use the Midpoint Method to solve the ODE

$$\dot{y}(x) = 1 + x^2 + y$$

$$y(0) = 1$$

Use $h = 0.1$. Determine y(0.1) and y(0.2)

Example 1

$$\text{Problem}: f(x,y) = 1 + x^2 + y,\ y_0 = y(0) = 1, h = 0.1$$

Step1:

$$y_{0+\frac{1}{2}} = y_0 + \frac{h}{2}\, f(x_0, y_0) = 1 + 0.05(1 + 0 + 1) = 1.1$$

$$y_1 = y_0 + h\ f(x_{0+\frac{1}{2}}, y_{0+\frac{1}{2}}) = 1 + 0.1(1 + 0.0025 + 1.1) = 1.2103$$

Step2:

$$y_{1+\frac{1}{2}} = y_1 + \frac{h}{2}\, f(x_1, y_1) = 1.2103 + .05(1 + 0.01 + 1.2103) = 1.3213$$

$$y_2 = y_1 + h\ f(x_{1+\frac{1}{2}}, y_{1+\frac{1}{2}}) = 1.2103 + 0.1(2.3438) = 1.4446$$

Heun's Predictor Corrector

Heun's Predictor Corrector Method

Problem

$$\dot{y}(x) = f(x, y)$$

$$y(x_0) = y_0$$

Heun's Method

$$y_0 = y(x_0)$$

$$\text{Predictor}: \quad y_{i+1}^0 = y_i + h\ f(x_i, y_i)$$

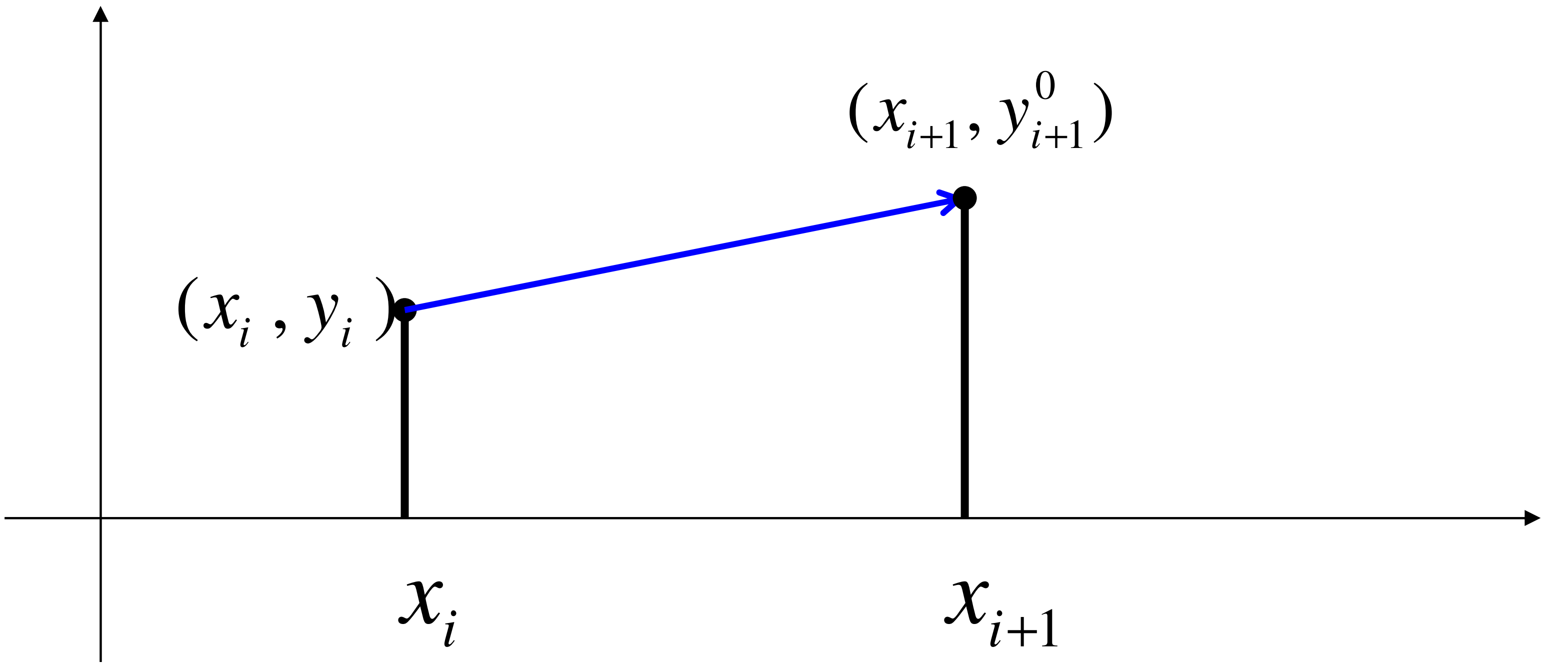
$$\text{Corrector}: \quad y_{i+1}^1 = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y_{i+1}^0)\right)$$

Local Truncation Error $O(h^3)$

Global Truncation Error $O(h^2)$

Heun's Predictor Corrector

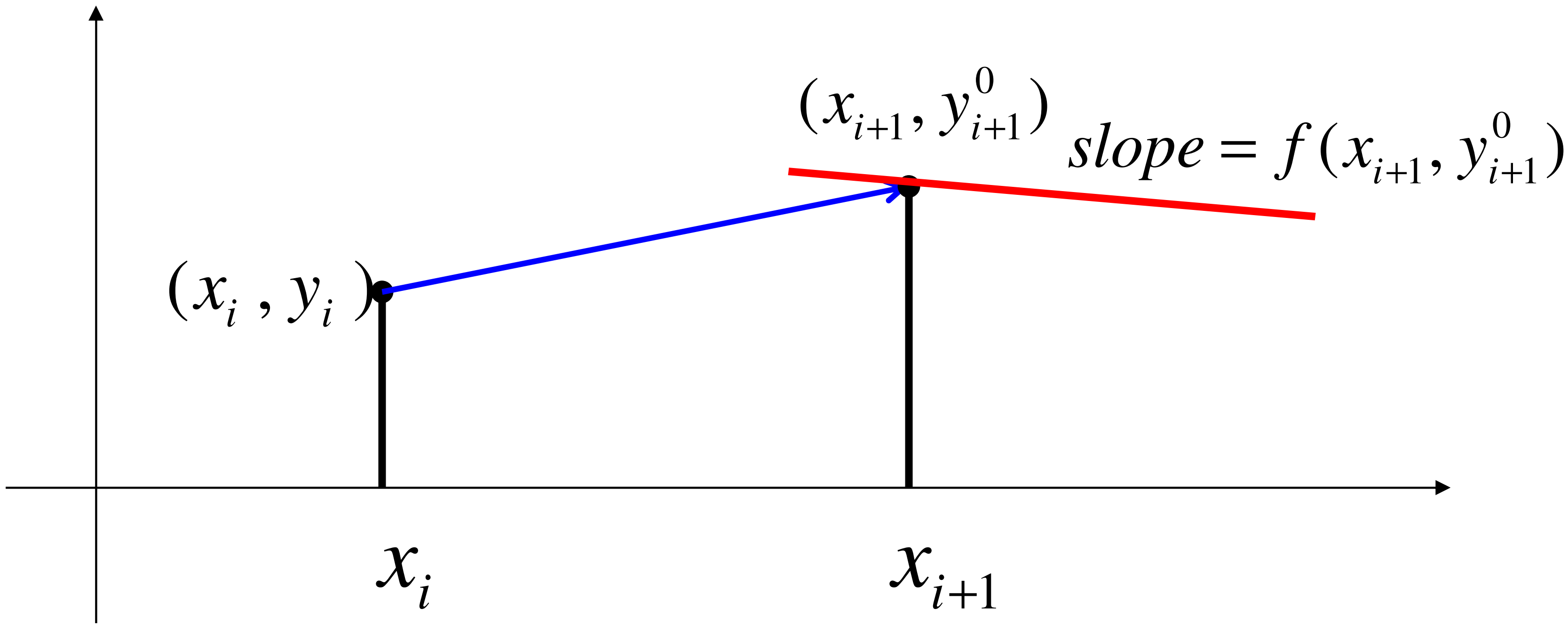
(Prediction)



Prediction $y_{i+1}^0 = y_i + h\, f(x_i, y_i)$

Heun's Predictor Corrector

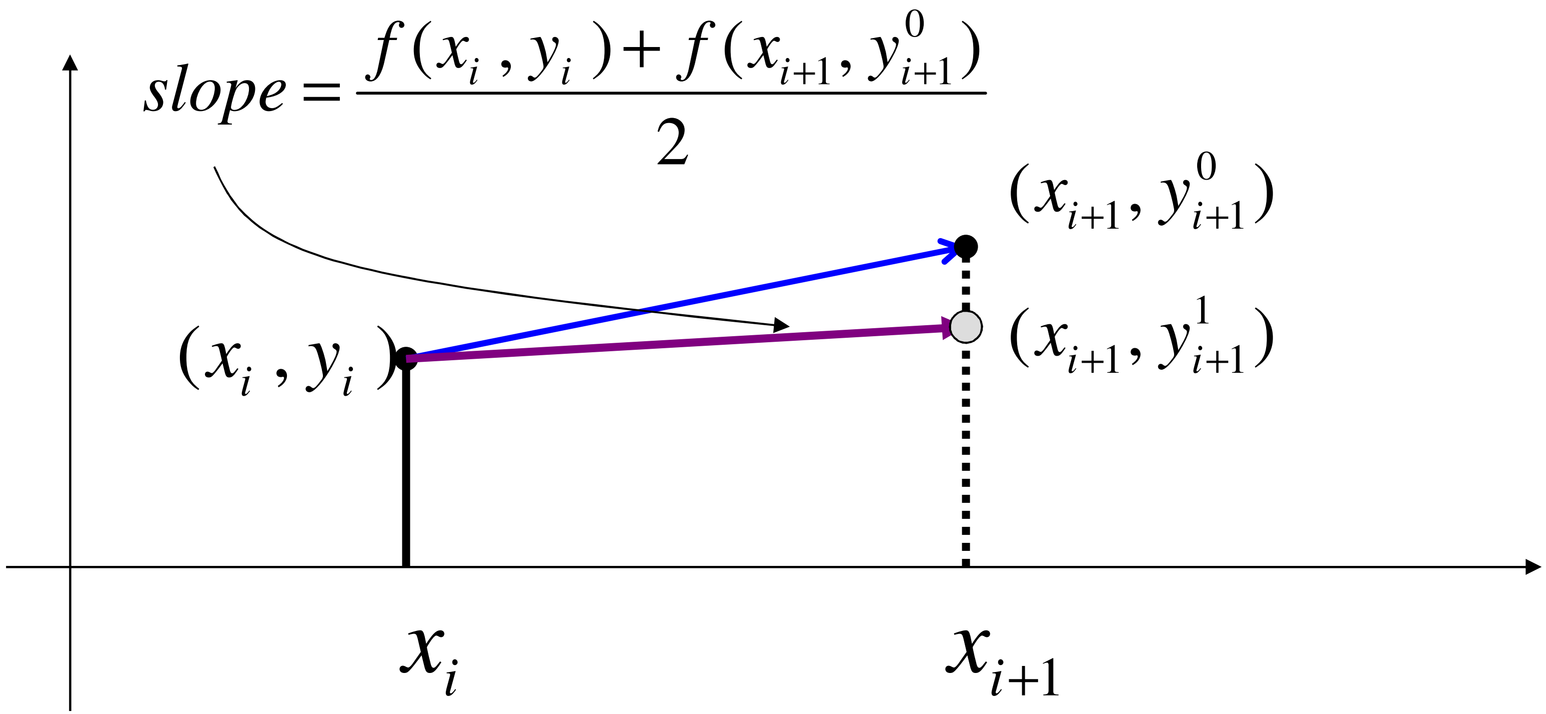
(Prediction)



Prediction $y_{i+1}^0 = y_i + h\ f(x_i, y_i)$

Heun's Predictor Corrector

(Correction)



$$y^1_{i+1} = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y^0_{i+1})\right)$$

Example 2

Use the Heun's Method to solve the ODE

$$\dot{y}(x) = 1 + x^2 + y$$

$$y(0) = 1$$

Use $h = 0.1$. One correction only

Determine y(0.1) and y(0.2)

Example 2

$$\text{Problem}: f(x, y) = 1 + y + x^2,\ y_0 = y(x_0) = 1, h = 0.1$$

$$\text{Step 1}:$$

$$\text{Predictor}:\ y_1^0 = y_0 + h\ f(x_0, y_0) = 1 + 0.1(2) = 1.2$$

$$\text{Corrector}:\ y_1^1 = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_1^0)\right) = 1.2105$$

$$\text{Step 2}:$$

$$\text{Predictor}:\ y_2^0 = y_1 + h\ f(x_1, y_1) = 1.4326$$

$$\text{Corrector}:\ y_2^1 = y_1 + \frac{h}{2}\left(f(x_1, y_1) + f(x_2, y_2^0)\right) = 1.4452$$

Summary

- Euler, Midpoint and Heun’s methods are similar in the following sense:

$$y_{i+1} = y_i + h \times slope$$

  - Different methods use different estimates of the slope.
- Both Midpoint and Heun’s methods are comparable in accuracy to the second order Taylor series method.

Comparison

Method	Local truncation error	Global truncation error
Euler Method $\quad y_{i+1} = y_i + h\ f(x_i, y_i)$	$O(h^2)$	$O(h)$
Heun's Method Predictor : $y_{i+1}^0 = y_i + h\ f(x_i, y_i)$ Corrector : $y_{i+1}^{k+1} = y_i + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y_{i+1}^k)\right)$	$O(h^3)$	$O(h^2)$
Midpoint $\quad y_{i+\frac{1}{2}} = y_i + \frac{h}{2}\ f(x_i, y_i)$ $y_{i+1} = y_i + h\ f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$	$O(h^3)$	$O(h^2)$

Runge-Kutta Methods

Learning Objectives

- To understand the motivation for using Runge Kutta method and the basic idea used in deriving them.
- To Familiarize with Taylor series for functions of two variables.
- Use Runge Kutta of order 2 to solve ODEs.

Motivation

- We seek accurate methods to solve ODEs that do not require calculating high order derivatives.
- The approach is to use a formula involving unknown coefficients then determine these coefficients to match as many terms of the Taylor series expansion.

Second Order Runge-Kutta Method

$$K_1 = h\ f(x_i, y_i)$$

$$K_2 = h\ f(x_i + \alpha h,\quad y_i + \beta K_1)$$

$$y_{i+1} = y_i + w_1 K_1 + w_2 K_2$$

Problem:

Find α, β, w_1, w_2

such that y_{i+1} is as accurate as possible.

Taylor Series in Two Variables

The Taylor Series is extended to the 2-independent variable case.

This is used to prove RK formula.

Taylor Series in One Variable

The n^{th} order Taylor Series expansion of $f(x)$

$$f(x+h) = \boxed{\sum_{i=0}^{n} \frac{h^i}{i!} f^{(i)}(x)} + \boxed{\frac{h^{n+1}}{(n+1)!} f^{(n+1)}(\bar{x})}$$

Approximation

Error

$$where\ \bar{x}\ is\ between\ x\ and\ x+h$$

Derivation of 2nd Order Runge-Kutta Methods – 1 of 5

Second Order Taylor Series Expansion

Used to solve ODE : $\dfrac{dy}{dx} = f(x, y)$

$$y_{i+1} = y_i + h\frac{dy}{dx} + \frac{h^2}{2}\frac{d^2 y}{dx^2} + O(h^3)$$

which is written as :

$$y_{i+1} = y_i + h\, f(x_i, y_i) + \frac{h^2}{2} f'(x_i, y_i) + O(h^3)$$

Derivation of 2nd Order Runge-Kutta Methods – 2 of 5

where $f'(x,y)$ is obtained by chain - rule differentiation

$$f'(x,y) = \frac{\partial f(x,y)}{\partial x} + \frac{\partial f(x,y)}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f(x,y)$$

Substituting :

$$y_{i+1} = y_i + f(x_i, y_i)h + \underbrace{\boxed{\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f(x_i, y_i)\right)}}_{f'(x,y)}\frac{h^2}{2} + O(h^3)$$

Eq. 1

The Multivariable Chain Rule

- Suppose that $z = f(x, y)$ where x and y themselves depend on one or more variables. Multivariable Chain Rules allow us to differentiate z with respect to any of the variables involved
- Let $x = x(t)$ and $y = y(t)$ be differentiable at t and suppose that $z = f(x, y)$ is differentiable athe point $\big(x(t), y(t)\big)$. Then $z = (x(t), y(t))$ is differentiable at t and
- $$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$
- *!! In our case only y depend on x and x is independent !!*

Taylor Series in Two Variables

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} \right) +$$

$$\frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + k^2\frac{\partial^2 f}{\partial y^2} + 2hk\frac{\partial^2 f}{\partial x\,\partial y} \right) + \ldots$$

$$= \boxed{\underbrace{\sum_{i=0}^{n} \frac{1}{i!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^i f(x, y)}_{approximation}} + \boxed{\underbrace{\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} \right)^{n+1} f(\bar{x}, \bar{y})}_{error}}$$

$(\bar{x}, \bar{y})$ is on the line joining between (x, y) and $(x+h, y+k)$

Derivation of 2nd Order Runge-Kutta Methods – 3 of 5

$$\text{Problem}: \textit{Find} \;\; \alpha, \beta, \; w_1, w_2 \;\; \text{such that}$$

$$K_1 = h\; f(x_i, y_i)$$

$$K_2 = h\; f(x_i + \alpha h, \;\; y_i + \beta K_1)$$

$$y_{i+1} = y_i + w_1 K_1 + w_2 K_2$$

$$\text{Substituting}:$$

$$y_{i+1} = y_i + w_1 h\, f(x_i, y_i) + w_2 h\, f(x_i + \alpha h, \; y_i + \beta K_1)$$

Derivation of 2nd Order Runge-Kutta Methods – 4 of 5

$$f(x_i + \alpha h,\ \ y_i + \beta K_1) = f(x_i, y_i) + \alpha h \frac{\partial f}{\partial x} + \beta K_1 \frac{\partial f}{\partial y} + \ldots$$

$Substituting:$

$$y_{i+1} = y_i + w_1 h\, f(x_i, y_i) + w_2 h \left(f(x_i, y_i) + \alpha h \frac{\partial f}{\partial x} + \beta K_1 \frac{\partial f}{\partial y} + \ldots \right)$$

$$y_{i+1} = y_i + (w_1 + w_2) h\, f(x_i, y_i) + w_2 h \left(\alpha h \frac{\partial f}{\partial x} + \beta K_1 \frac{\partial f}{\partial y} + \ldots \right)$$

$$y_{i+1} = y_i + (w_1 + w_2) h\, f(x_i, y_i) + w_2 \alpha h^2 \frac{\partial f}{\partial x} + w_2 \beta h^2 \frac{\partial f}{\partial y} f(x_i, y_i) + \ldots$$

Eq. 2

Derivation of 2nd Order Runge-Kutta Methods – 5 of 5

We derived two expansions for y_{i+1} :

$$y_{i+1} = y_i + (w_1 + w_2)h\, f(x_i, y_i) + w_2 \alpha h^2 \frac{\partial f}{\partial x} + w_2 \beta h^2 \frac{\partial f}{\partial y} f(x_i, y_i) + \ldots$$

$$y_{i+1} = y_i + f(x_i, y_i)h + \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f(x_i, y_i) \right) \frac{h^2}{2} + O(h^3)$$

Matching terms, we obtain the following three equations :

$$w_1 + w_2 = 1, \quad w_2 \alpha = \frac{1}{2}, \quad \text{and} \quad w_2 \beta = \frac{1}{2}$$

3 equations with 4 unknowns $\Rightarrow$ infinite solutions

One possible solution : $\alpha = \beta = 1, \quad w_1 = w_2 = \frac{1}{2}$

2^{nd} Order Runge-Kutta Methods

$$K_1 = h\ f(x_i, y_i)$$

$$K_2 = h\ f(x_i + \alpha h, \quad y_i + \beta K_1)$$

$$y_{i+1} = y_i + w_1 K_1 + w_2 K_2$$

$$\text{Choose} \quad \alpha, \beta,\ w_1, w_2 \quad \text{such that :}$$

$$w_1 + w_2 = 1,\ \ w_2 \alpha = \frac{1}{2},\ \text{and}\ \ w_2 \beta = \frac{1}{2}$$

Alternative Form

Second Order Runge Kutta

$$K_1 = h\ f(x_i, y_i)$$

$$K_2 = h\ f(x_i + \alpha h,\ \ y_i + \beta K_1)$$

$$y_{i+1} = y_i + w_1 K_1 + w_2 K_2$$

Alternative Form

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \alpha h,\ \ y_i + \beta h k_1)$$

$$y_{i+1} = y_i + h\left(w_1 k_1 + w_2 k_2\right)$$

Choosing α, β, w_1 and w_2

For example, choosing $\alpha = 1$, then $\beta = 1$, $w_1 = w_2 = \frac{1}{2}$

Second Order Runge - Kutta method becomes :

$$K_1 = h \ f(x_i, y_i)$$

$$K_2 = h \ f(x_i + h, y_i + K_1)$$

$$y_{i+1} = y_i + \frac{1}{2}\left(K_1 + K_2\right) = y_i + \frac{h}{2}\Big(f(x_i, y_i) + f(x_{i+1}, y_{i+1}^0)\Big)$$

This is *Heun's Method* with a Single Corrector

Choosing α, β, w_1 and w_2

Choosing $\alpha = \frac{1}{2}$ then $\beta = \frac{1}{2}$, $w_1 = 0$, $w_2 = 1$

Second Order Runge - Kutta method becomes :

$$K_1 = h\ f(x_i, y_i)$$

$$K_2 = h\ f(x_i + \frac{h}{2}, y_i + \frac{K_1}{2})$$

$$y_{i+1} = y_i + K_2 = y_i + h\ f(x_i + \frac{h}{2}, y_i + \frac{K_1}{2})$$

This is the Midpoint Method

2nd Order Runge-Kutta Methods

Alternative Formulas

$$\alpha w_2 = \frac{1}{2}, \quad \beta w_2 = \frac{1}{2}, \quad w_1 + w_2 = 1$$

$$\text{Pick any nonzero } \alpha \text{ number}: \ \beta = \alpha, \ w_2 = \frac{1}{2\alpha}, \quad w_1 = 1 - \frac{1}{2\alpha}$$

$$\text{Second Order Runge Kutta Formulas (select } \alpha \neq 0)$$

$$K_1 = h\, f(x_i, y_i)$$

$$K_2 = h\ f(x_i + \alpha h, \quad y_\mathrm{i} + \alpha K_1)$$

$$y_{i+1} = y_i + \left(1 - \frac{1}{2\alpha}\right) K_1 + \frac{1}{2\alpha} K_2$$

Second order Runge-Kutta Method

Example

Solve the following system to find $x(1.02)$ using RK2

$$\dot{x}(t) = 1 + x^2 + t^3, \qquad x(1) = -4,\ h = 0.01,\ \alpha = 1$$

STEP1:

$$K_1 = h\ f(t_0 = 1, x_0 = -4) = 0.01(1 + {x_0}^2 + {t_0}^3) = 0.18$$

$$K_2 = h\ f(t_0 + h, x_0 + K_1)$$

$$= 0.01(1 + (x_0 + 0.18)^2 + (t_0 + .01)^3) = 0.1662$$

$$x(1 + 0.01) = x(1) + \left(K_1 + K_2\right)/2$$

$$= -4 + (0.18 + 0.1662)/2 = -3.8269$$

Second order Runge-Kutta Method

Example

STEP 2

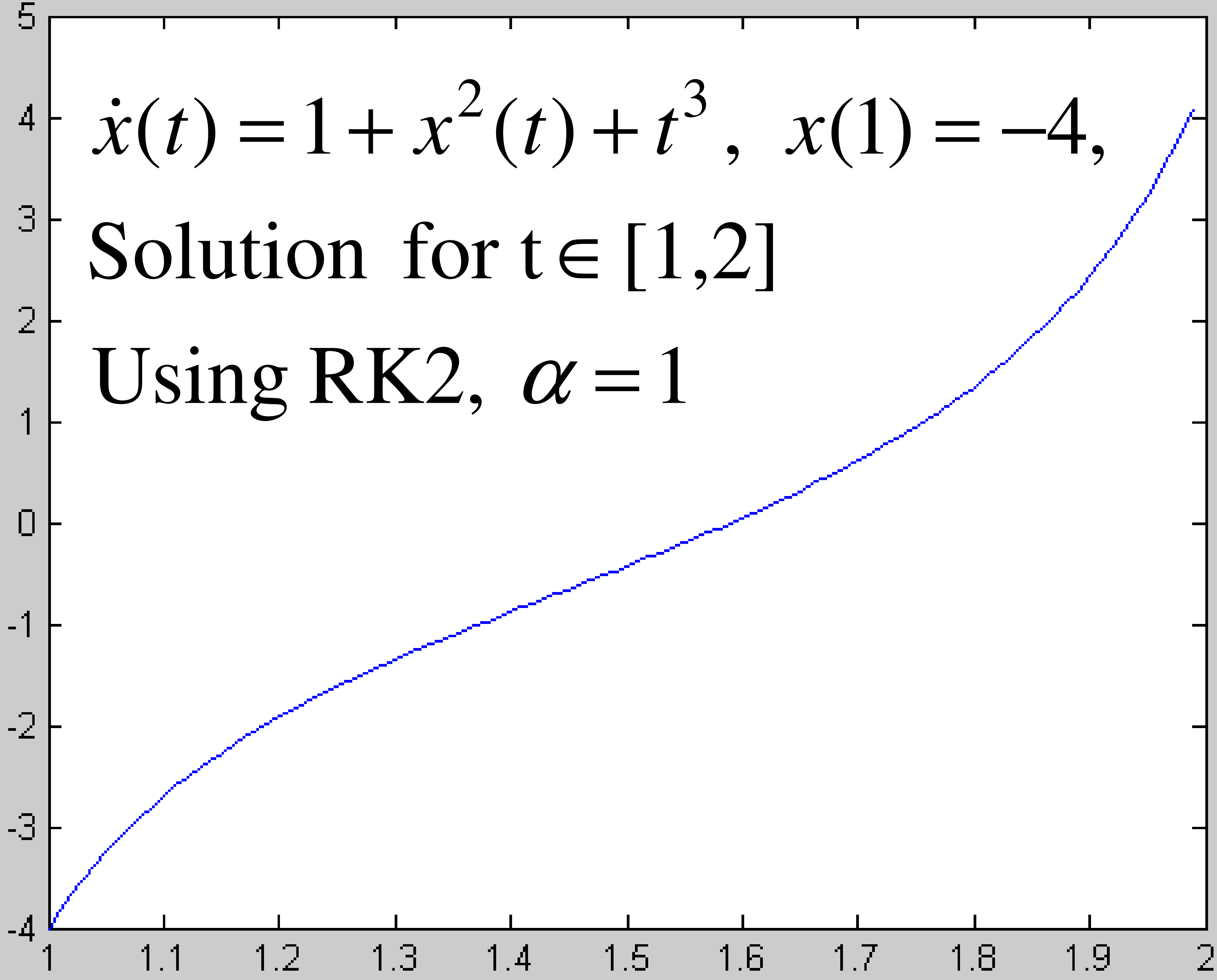
$$K_1 = h\ f(t_1 = 1.01, x_1 = -3.8269) = 0.01(1 + x_1^{\ 2} + t_1^{\ 3}) = 0.1668$$

$$K_2 = h\ f(t_1 + h, x_1 + K_1)$$

$$= 0.01(1 + (x_1 + 0.1668)^2 + (t_1 + .01)^3) = 0.1546$$

$$x(1.01 + 0.01) = x(1.01) + \frac{1}{2}\left(K_1 + K_2\right)$$

$$= -3.8269 + \frac{1}{2}(0.1668 + 0.1546) = -3.6662$$


$\dot{x}(t) = 1 + x^2(t) + t^3, \; x(1) = -4,$
Solution for $t \in [1,2]$
Using RK2, $\alpha = 1$
5
4
3
2
1
0
-1
-2
-3
-4
1
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2

Applications of Runge-Kutta Methods to Solve First Order ODEs

Using Runge-Kutta methods of different orders to solve first order ODEs

2^{nd} Order Runge-Kutta

RK2

Typical value of $\alpha = 1$, Know as RK2

Equivalent to Heun's method with a single corrector

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h, y_i + k_1 h)$$

$$y_{i+1} = y_i + \frac{h}{2}(k_1 + k_2)$$

Local error is $O(h^3)$ and global error is $O(h^2)$

Higher-Order Runge-Kutta

Higher order Runge-Kutta methods are available.

Derived similar to second-order Runge-Kutta.

Higher order methods are more accurate but require more calculations.

3rd Order Runge-Kutta

RK3

Known as RK3

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \frac{h}{2},\ \ y_i + \frac{1}{2}k_1 h)$$

$$k_3 = f(x_i + h,\ \ y_i - k_1 h + 2k_2 h)$$

$$y_{i+1} = y_i + \frac{h}{6}\left(k_1 + 4k_2 + k_3\right)$$

Local error is $O(h^4)$ and Global error is $O(h^3)$

4^{th} Order Runge-Kutta

RK4

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \frac{h}{2}, \quad y_i + \frac{1}{2}k_1 h)$$

$$k_3 = f(x_i + \frac{h}{2}, \quad y_\mathrm{i} + \frac{1}{2}k_2 h)$$

$$k_4 = f(x_i + h, \quad y_\mathrm{i} + k_3 h)$$

$$y_{i+1} = y_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

Local error is $O(h^5)$ and global error is $O(h^4)$

Higher-Order Runge-Kutta

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \frac{1}{4}h,\ y_i + \frac{1}{4}k_1 h)$$

$$k_3 = f(x_i + \frac{1}{4}h,\ y_i + \frac{1}{8}k_1 h + \frac{1}{8}k_2 h)$$

$$k_4 = f(x_i + \frac{1}{2}h,\ y_i - \frac{1}{2}k_2 h + k_3 h)$$

$$k_5 = f(x_i + \frac{3}{4}h,\ y_i + \frac{3}{16}k_1 h + \frac{9}{16}k_4 h)$$

$$k_6 = f(x_i + h,\ y_i - \frac{3}{7}k_1 h + \frac{2}{7}k_2 h + \frac{12}{7}k_3 h - \frac{12}{7}k_4 h + \frac{8}{7}k_5 h)$$

$$y_{i+1} = y_i + \frac{h}{90}\left(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6\right)$$

Example

4th-Order Runge-Kutta Method

RK4

$$\frac{dy}{dx} = 1 + y + x^2$$

$$y(0) = 0.5$$

$$h = 0.2$$

$$Use\ RK4\ to\ compute\ \ y(0.2)\ and\ \ y(0.4)$$

Example: RK4

Problem:

$$\frac{dy}{dx} = 1 + y + x^2, \qquad y(0) = 0.5$$

Use RK4 to find $y(0.2), y(0.4)$

4^{th} Order Runge-Kutta

RK4

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \frac{h}{2}, \quad y_i + \frac{1}{2}k_1 h)$$

$$k_3 = f(x_i + \frac{h}{2}, \quad y_\mathrm{i} + \frac{1}{2}k_2 h)$$

$$k_4 = f(x_i + h, \quad y_\mathrm{i} + k_3 h)$$

$$y_{i+1} = y_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

Local error is $O(h^5)$ and global error is $O(h^4)$

Example: RK4

See RK4 Formula

Problem :

$$\frac{dy}{dx} = 1 + y + x^2, \qquad y(0) = 0.5$$

Use RK4 to find $y(0.2), y(0.4)$

$$h = 0.2$$

$$f(x, y) = 1 + y + x^2$$

$$x_0 = 0, \quad y_0 = 0.5$$

Step 1

$$k_1 = f(x_0, y_0) = (1 + y_0 + x_0^{\ 2}) = 1.5$$

$$k_2 = f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1 h) = 1 + (y_0 + 0.15) + (x_0 + 0.1)^2 = 1.64$$

$$k_3 = f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2 h) = 1 + (y_0 + 0.164) + (x_0 + 0.1)^2 = 1.654$$

$$k_4 = f(x_0 + h, y_0 + k_3 h) = 1 + (y_0 + 0.16545) + (x_0 + 0.2)^2 = 1.7908$$

$$y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.8293$$

Example: RK4

Problem :

$$\frac{dy}{dx} = 1 + y + x^2, \qquad y(0) = 0.5$$

Use RK4 to find $y(0.2), y(0.4)$

$$h = 0.2$$

$$f(x, y) = 1 + y + x^2$$

$$x_1 = 0.2, \quad y_1 = 0.8293$$

Step 2

$$k_1 = f(x_1, y_1) = 1.7893$$

$$k_2 = f(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_1 h) = 1.9182$$

$$k_3 = f(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}k_2 h) = 1.9311$$

$$k_4 = f(x_1 + h, y_1 + k_3 h) = 2.0555$$

$$y_2 = y_1 + \frac{0.2}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = 1.2141$$

Example: RK4

Problem:

$$\frac{dy}{dx} = 1 + y + x^2, \qquad y(0) = 0.5$$

Use RK4 to find $y(0.2), y(0.4)$

Summary of the solution

x_i	y_i
0.0	0.5
0.2	0.8293
0.4	1.2141

Summary

- Runge Kutta methods generate an accurate solution without the need to calculate high order derivatives.
- Second order RK have local truncation error of order $O(h^3)$ and global truncation error of order $O(h^2)$.
- Higher order RK have better local and global truncation errors.
- N function evaluations are needed in the N^{th} order RK method.

Solving Systems of ODEs

- Convert a single (or a system of) high order ODE to a system of first order ODEs.

- Use the methods discussed earlier in this topic to solve systems of first order ODEs.

Outline

- Solution of a system of first order ODEs.
- Conversion of a high order ODE to a system of first order ODEs.
- Conversion of a system of high order ODEs to a system of first order ODEs.
- Use different methods to solve systems of first order ODEs.
- Use different methods to solve high order ODEs.
- Use different methods to solve systems of high order ODEs.

Solving a System of First Order ODEs

- Methods discussed earlier such as Euler, Runge-Kutta,… are used to solve first order ordinary differential equations.
- The same formulas will be used to solve a system of first order ODEs.
  - In this case, the differential equation is a vector equation and the dependent variable is a vector variable.

Euler Method for Solving a System of First Order ODEs

Recall Euler method for solving a first order ODE:

$$\boxed{Given\ \ \frac{dy(x)}{dx} = f(y, x), \quad y(a) = y_a}$$

$$Euler\,Method:$$

$$y(a+h) = y(a) + h\,f(y(a), a)$$

$$y(a+2h) = y(a+h) + h\,f(y(a+h), a+h)$$

$$y(a+3h) = y(a+2h) + h\,f(y(a+2h), a+2h)$$

Example - Euler Method

Euler method to solve a system of *n* first order ODEs.

$$\text{Given} \quad \frac{dY(x)}{dx} = F(Y,x) = \begin{bmatrix} f_1(Y,x) \\ f_2(Y,x) \\ \dots \\ f_n(Y,x) \end{bmatrix}, \quad Y(a) = \begin{bmatrix} y_1(a) \\ y_2(a) \\ \dots \\ y_n(a) \end{bmatrix}$$

$$\text{Euler Method}:$$

$$Y(a+h) = Y(a) + h \; F(Y(a), a)$$

$$Y(a+2h) = Y(a+h) + h\,F(Y(a+h), a+h)$$

$$Y(a+3h) = Y(a+2h) + h\,F(Y(a+2h), a+2h)$$

Solving a System of *n* First Order ODEs

- Exactly the same formula is used but the scalar variables and functions are replaced by vector variables and vector value functions.
- Y is a vector of length *n.*
- F(Y,x) is a vector valued function.

$$Y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \cdots \\ y_n(x) \end{bmatrix} \qquad Y \text{ is } n \times 1 \text{ vector}$$

$$\frac{d\,Y(x)}{dx} = \begin{bmatrix} \frac{d\,y_1}{dx} \\ \frac{d\,y_2}{dx} \\ \cdots \\ \frac{d\,y_n}{dx} \end{bmatrix} = \begin{bmatrix} f_1(Y,x) \\ f_2(Y,x) \\ \cdots \\ f_n(Y,x) \end{bmatrix} = F(Y,x)$$

$$Y(a+h) = Y(a) + h\;\; F(Y(a),a)$$

$$Y(a+2h) = Y(a+h) + h\,F(Y(a+h),a+h)$$

$$Y(a+3h) = Y(a+2h) + h\,F(Y(a+2h),a+2h)$$

Example :

Euler method for solving a system of first order ODEs.

$$\boxed{\begin{bmatrix} \dot{y}_1(x) \\ \dot{y}_2(x) \end{bmatrix} = \begin{bmatrix} y_2 \\ 1 - y_1 \end{bmatrix} = F(Y, x), \quad Y(0) = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

$Two\ steps\ of\ \ Euler\ Method\ \ with\ \ h = 0.1$

$STEP\ 1:$

$$Y(0+h) = Y(0) + h\ \ F(Y(0), 0)$$

$$\begin{bmatrix} y_1(0.1) \\ y_2(0.1) \end{bmatrix} = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} + 0.1 \begin{bmatrix} y_2(0) \\ 1 - y_1(0) \end{bmatrix} = \begin{bmatrix} -1 + 0.1 \\ 1 + 0.1(1+1) \end{bmatrix} = \begin{bmatrix} -0.9 \\ 1.2 \end{bmatrix}$$

$STEP\ 2:$

$$Y(0+2h) = Y(h) + h\, F(Y(h), h)$$

$$\begin{bmatrix} y_1(0.2) \\ y_2(0.2) \end{bmatrix} = \begin{bmatrix} y_1(0.1) \\ y_2(0.1) \end{bmatrix} + 0.1 \begin{bmatrix} y_2(0.1) \\ 1 - y_1(0.1) \end{bmatrix} = \begin{bmatrix} -0.9 + 0.12 \\ 1.2 + .1(1 + 0.9) \end{bmatrix} = \begin{bmatrix} -0.78 \\ 1.39 \end{bmatrix}$$

Example :

RK2 method for solving a system of first order ODEs

$$\begin{bmatrix} \dot{y}_1(x) \\ \dot{y}_2(x) \end{bmatrix} = \begin{bmatrix} y_2 \\ 1 - y_1 \end{bmatrix} = F(Y, x), \quad Y(0) = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$Two\ steps\ of$ second order Runge – Kutta Method $with\ \ h = 0.1$

$STEP\ 1:$

$$K1 = h\ \ F(Y(0),0) = 0.1 \begin{bmatrix} y_2(0) \\ 1 - y_1(0) \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}$$

$$K2 = h\ \ F(Y(0) + K1, 0 + h) = 0.1 \begin{bmatrix} y_2(0) + 0.2 \\ 1 - (y_1(0) + 0.1) \end{bmatrix} = \begin{bmatrix} 0.12 \\ 0.19 \end{bmatrix}$$

$$Y(0 + h) = Y(0) + 0.5(K1 + K2)$$

$$\begin{bmatrix} y_1(0.1) \\ y_2(0.1) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{1}{2}\left(\begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix} + \begin{bmatrix} 0.12 \\ 0.19 \end{bmatrix} \right) = \begin{bmatrix} -0.89 \\ 1.195 \end{bmatrix}$$

Example :

RK2 method for solving a system of first order ODEs

$$\begin{bmatrix} \dot{y}_1(x) \\ \dot{y}_2(x) \end{bmatrix} = \begin{bmatrix} y_2 \\ 1-y_1 \end{bmatrix} = F(Y,x), \quad Y(0) = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$STEP\ 2:$

$$K1 = h\ \ F(Y(0.1), 0.1) = 0.1 \begin{bmatrix} y_2(0.1) \\ 1-y_1(0.1) \end{bmatrix} = \begin{bmatrix} 0.1195 \\ 0.1890 \end{bmatrix}$$

$$K2 = h\ \ F(Y(0.1)+K1, 0.1+h) = 0.1 \begin{bmatrix} y_2(0.1)+0.189 \\ 1-(y_1(0.1)+0.1195) \end{bmatrix} = \begin{bmatrix} 0.1384 \\ 0.1771 \end{bmatrix}$$

$$Y(0.1+h) = Y(0.1) + 0.5(K1+K2)$$

$$\begin{bmatrix} y_1(0.2) \\ y_2(0.2) \end{bmatrix} = \begin{bmatrix} -0.89 \\ 1.195 \end{bmatrix} + \frac{1}{2}\left(\begin{bmatrix} 0.1195 \\ 0.1890 \end{bmatrix} + \begin{bmatrix} 0.1384 \\ 0.1771 \end{bmatrix} \right) = \begin{bmatrix} -0.7611 \\ 1.3780 \end{bmatrix}$$

Methods for Solving a System of First Order ODEs

- We have extended Euler and RK2 methods to solve systems of first order ODEs.
- Other methods used to solve first order ODE can be easily extended to solve systems of first order ODEs.

High Order ODEs

- How do solve a second order ODE?

$$\ddot{x} + 3\dot{x} + 6x = 1$$

- How do solve high order ODEs?

The General Approach to Solve ODEs

High order ODE → Convert → System of first order ODEs → Solve

$$\ddot{x} + 3\dot{x} + 6x = 1$$
$$\dot{x}(0) = 1; x(0) = 4$$

Second order ODE

Convert

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ 1 - 3z_2 - 6z_1 \end{bmatrix},$$
$$Z(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Two first order ODEs

Solve

Conversion Procedure

High order ODE | Convert | System of first order ODEs | Solve

1. **<u>Select the dependent variables</u>**

   One way is to take the original dependent variable and its derivatives up to one degree less than the highest order derivative.

2. **<u>Write the Differential Equations</u>** in terms of the new variables. The equations come from the way the new variables are defined or from the original equation.

3. **<u>Express the equations in a matrix form.</u>**

Remarks on the Conversion Procedure

High order ODE | Convert → | System of first order ODE | Solve →

1. Any n^{th} order ODE is converted to a system of n first order ODEs.
2. There are an infinite number of ways to select the new variables. As a result, for each high order ODE there are an infinite number of set of equivalent first order systems of ODEs.
3. Use a table to make the conversion easier.

Example of Converting a High Order ODE to First Order ODEs

Convert $\ddot{x} + 3\dot{x} + 6x = 1, \quad \dot{x}(0) = 1; \quad x(0) = 4$

to a system of first order ODEs

1. Select a new set of variables

(Second order ODE $\Rightarrow$ We need two variables)

$z_1 = x$

$z_2 = \dot{x}$

One degree less than the highest order derivative

Example of Converting a High Order ODE to First Order ODEs

old name	new name	Initial cond.	Equation
x	z_1	4	$\dot{z}_1 = z_2$
$\dot{x}$	z_2	1	$\dot{z}_2 = 1 - 3z_2 - 6z_1$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ 1 - 3z_2 - 6z_1 \end{bmatrix}, Z(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Example of Converting a High Order ODE to First Order ODEs

Convert

$$\dddot{x} + 2\ddot{x} + 7\dot{x} + 8x = 0$$

$$\ddot{x}(0) = 9,\ \dot{x}(0) = 1;\quad x(0) = 4$$

1. Select a new set of variables (3 of them)

$$z_1 = x$$

$$z_2 = \dot{x}$$

$$z_3 = \ddot{x}$$

One degree less than the highest order derivative

Example of Converting a High Order ODE to First Order ODEs

old name	new name	Initial cond.	Equation
x	z_1	4	$\dot{z}_1 = z_2$
$\dot{x}$	z_2	1	$\dot{z}_2 = z_3$
$\ddot{x}$	z_3	9	$\dot{z}_3 = -2z_3 - 7z_2 - 8z_1$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} z_2 \\ z_3 \\ -2z_3 - 7z_2 - 8z_1 \end{bmatrix}, Z(0) = \begin{bmatrix} 4 \\ 1 \\ 9 \end{bmatrix}$$

Conversion Procedure for Systems of High Order ODEs

System of high order ODEs → Convert → System of first order ODE → Solve

1. **Select the dependent variables**

   Take the original dependent variables and their derivatives up to one degree less than the highest order derivative for each variable.

2. **Write the Differential Equations** in terms of the new variables. The equations come from the way the new variables are defined or from the original equation.

3. **Express the equations in a matrix form.**

Example of Converting a High Order ODE to First Order ODEs

Convert

$$\dddot{x} + 5\ddot{x} + 2\dot{x} + 8y = 0$$

$$\ddot{y} + 2xy + \dot{x} = 2$$

$$x(0) = 4;\ \dot{x}(0) = 2;\ \ddot{x}(0) = 9;\ y(0) = 1;\ \dot{y}(0) = -3$$

1. Select a new set of variables ($(3+2)$ variables)

$$z_1 = x$$

$$z_2 = \dot{x}$$

$$z_3 = \ddot{x}$$

$$z_4 = y$$

$$z_5 = \dot{y}$$

One degree less than the highest order derivative

Example of Converting a High Order ODE to First Order ODEs

old name	new name	Initial cond.	Equation
x	z_1	4	$\dot{z}_1 = z_2$
$\dot{x}$	z_2	2	$\dot{z}_2 = z_3$
$\ddot{x}$	z_3	9	$\dot{z}_3 = -5z_3 - 2z_2 - 8z_4$
y	z_4	1	$\dot{z}_4 = z_5$
$\dot{y}$	z_5	-3	$\dot{z}_5 = 2 - z_2 - 2z_1 z_4$

Solution of a Second Order ODE

- Solve the equation using Euler method. Use h=0.1

$$\ddot{x} + 2\dot{x} + 8x = 2$$

$$x(0) = 1; \dot{x}(0) = -2$$

Select a new set of variables : $z_1 = x, z_2 = \dot{x}$

The second order equation is expressed as :

$$\dot{Z} = F(Z) = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ 2 - 2z_2 - 8z_1 \end{bmatrix}, Z(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Solution of a Second Order ODE

$$F(Z) = \begin{bmatrix} z_2 \\ 2 - 2z_2 - 8z_1 \end{bmatrix}, Z(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, h = 0.1$$

$$Z(0+0.1) = Z(0) + hF(Z(0))$$

$$= \begin{bmatrix} 1 \\ -2 \end{bmatrix} + 0.1 \begin{bmatrix} -2 \\ 2 - 2(-2) - 8(1) \end{bmatrix} = \begin{bmatrix} 0.8 \\ -2.2 \end{bmatrix}$$

$$Z(0.2) = Z(0.1) + hF(Z(0.1))$$

$$= \begin{bmatrix} 0.8 \\ -2.2 \end{bmatrix} + 0.1 \begin{bmatrix} -2.2 \\ 2 - 2(-2.2) - 8(0.8) \end{bmatrix} = \begin{bmatrix} 0.58 \\ -2.2 \end{bmatrix}$$

Summary

- Formulas used in solving a first order ODE are used to solve systems of first order ODEs.
  - Instead of scalar variables and functions, we have vector variables and vector functions.

- High order ODEs are converted to a set of first order ODEs.