Digital Signal Processing

Asst. Prof. Görkem SERBES

gserbes@yildiz.edu.tr

http://yarbis.yildiz.edu.tr/gserbes

http://www.yildiz.edu.tr/~naydin

Digital Signal Processing

Sample Questions For Final

(You are responsible from all the sample problems given in this document and also in the previous ones!)

Q1

The system function of a linear time-invariant filter is given by the formula

$$H(z) = (1 + z^{-1})(1 - e^{j2\pi/3}z^{-1})(1 - e^{-j2\pi/3}z^{-1})$$

(a) Write the difference equation that gives the relation between the input $x[n]$ and the output $y[n]$. Make sure that all the filter coefficients $\{b_k\}$ in your difference equation are purely real.

(b) Use multiplication of z-transform polynomials to find the output when the input is

$$x[n] = -\delta[n-2] - \delta[n-3] - \delta[n-4].$$

(c) Plot the poles and zeros of $H(z)$ in the z-plane.

(d) If the input is of the form $x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$, for what values of frequency $\hat{\omega}$ will the output signal be zero for all n (i.e., $y[n] = 0$) ? Find all possible frequencies in the range $-\pi \leq \hat{\omega} \leq \pi$. *Hint: Take a look at the locations of the zeros of $H(z)$ as plotted in part (c).*

Q1 Solution

(a) $H(z) = (1+z^{-1})(1-e^{j2\pi/3}z^{-1})(1-e^{-j2\pi/3}z^{-1})$

These are conjugates, a hint that when multiplied out, all the coefficients will be real.

$H(z) = (1+z^{-1})(1-e^{-j2\pi/3}z^{-1}-e^{j2\pi/3}z^{-1}+e^{0}z^{-2})$

$H(z) = (1+z^{-1})(1-z^{-1}(\cos\frac{-2\pi}{3}+j\sin\frac{-2\pi}{3})-z^{-1}(\cos\frac{2\pi}{3}+j\sin\frac{2\pi}{3})+z^{-2})$

$H(z) = (1+z^{-1})(1-z^{-1}(2\cos\frac{2\pi}{3}+j0)+z^{-2})$

$H(z) = (1+z^{-1})(1+z^{-1}+z^{-2}) = 1+2z^{-1}+2z^{-2}+z^{-3}$

(b) $H(z) = 1+2z^{-1}+2z^{-2}+z^{-3}$

$x[n] = -\delta[n-2]-\delta[n-3]-\delta[n-4]$

$X(z) = -z^{-2} \quad -z^{-3} \quad -z^{-4}$

$Y(z) = X(z)\cdot H(z)$

$Y(z) = (-z^{-2}-z^{-3}-z^{-4})(1+2z^{-1}+2z^{-2}+z^{-3})$

$$
\begin{array}{rrrrrr}
1 & +\ 2z^{-1} & +2z^{-2} & +z^{-3} & & \\
\times & -z^{-2} & -z^{-3} & -z^{-4} & & \\
\hline
-z^{-2} & -2z^{-3} & -2z^{-4} & -z^{-5} & & \\
 & -z^{-3} & -2z^{-4} & -2z^{-5} & -z^{-6} & \\
 & & -z^{-4} & -2z^{-5} & -2z^{-6} & -z^{-7} \\
\hline
\end{array}
$$

$Y(z) = -z^{-2}-3z^{-3}-5z^{-4}-5z^{-5}-3z^{-6}-z^{-7}$

$y[n] = -\delta[n-2]-3\delta[n-3]-5\delta[n-4]-5\delta[n-5]$
$\quad -3\delta[n-6]-\delta[n-7]$

Q1 Solution (Cont.)

(c) $H(z) = 1 + 2z^{-1} + 2z^{-2} + z^{-3}$

$H(z) = z^{-3}(z^3 + 2z^2 + 2z + 1)$

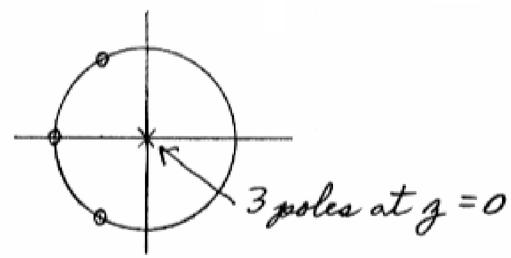
3 poles at $z = 0$

The zeros are roots of $z^3 + 2z^2 + 2z + 1 = 0$

$z_1 = -1 \qquad (z+1)(z^2 + z + 1) = 0$

Finally, the roots of $z^2 + z + 1 = 0$

$z_2, z_3 = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$



(d) $x[n] = Ae^{j\phi}e^{j\hat{\omega}n}$

Zero	$\hat{\omega}$ (where output will be zero)
$z_1 = -1$	$\pm\pi$
$z_2 = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$	$\frac{2\pi}{3}$
$z_3 = -\frac{1}{2} - j\frac{\sqrt{3}}{2}$	$\frac{-2\pi}{3}$

Q2

The diagram in Figure 1 depicts a *cascade connection* of two linear time-invariant systems; i.e., the output of the first system is the input to the second system, and the overall output is the output of the second system. In Figure 1, assume that both systems are first difference filters; i.e.,

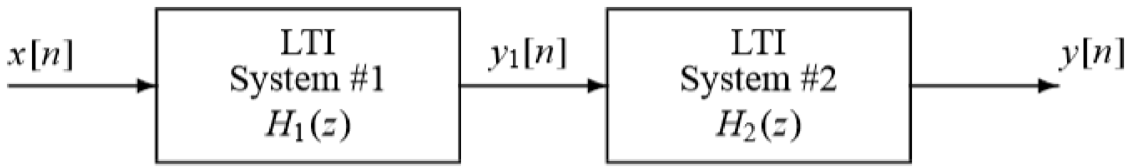


Figure 1: Cascade connection of two LTI systems.

$$y_1[n] = (x[n] - x[n-1]) \quad \text{and} \quad y[n] = (y_1[n] - y_1[n-1]).$$

(a) Determine the system function $H(z) = H_1(z)H_2(z)$ for the overall system.

(b) Plot the poles and zeros of $H(z)$ in the z-plane.

(c) From $H(z)$, determine the impulse response $h[n]$ of the overall system in Figure 1.

(d) From $H(z)$, obtain an expression for the frequency response $H(e^{j\hat{\omega}})$ of the overall cascade system.

(e) Use your result from (d) as an aid in sketching the frequency response (magnitude and phase) functions of the overall cascade system for $-\pi \le \hat{\omega} \le \pi$.

Q2 Solution

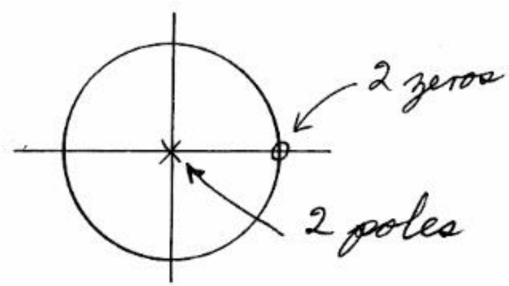
(a) $y_1[n] = x[n] - x[n-1]$ $\qquad$ $y[n] = y_1[n] - y_1[n-1]$

$$H_1(z) = 1 - z^{-1} \qquad H_2(z) = 1 - z^{-1}$$

$$H(z) = H_1(z)\,H_2(z) = (1-z^{-1})(1-z^{-1}) = 1 - 2z^{-1} + z^{-2}$$

(b) The poles and zeros can be found most conveniently by finding them for $H_1(z)$ and $H_2(z)$, since those system functions are simpler than $H(z)$. $H_1(z)$ and $H_2(z)$ are identical, so poles and zeros will overlap.

$$H_1(z) = 1 - z^{-1} = z^{-1}(z-1) = \frac{z-1}{z} \qquad p_1 = 0,\quad z_1 = 1$$



(c) $H(z) = 1 - 2z^{-1} + z^{-2}$

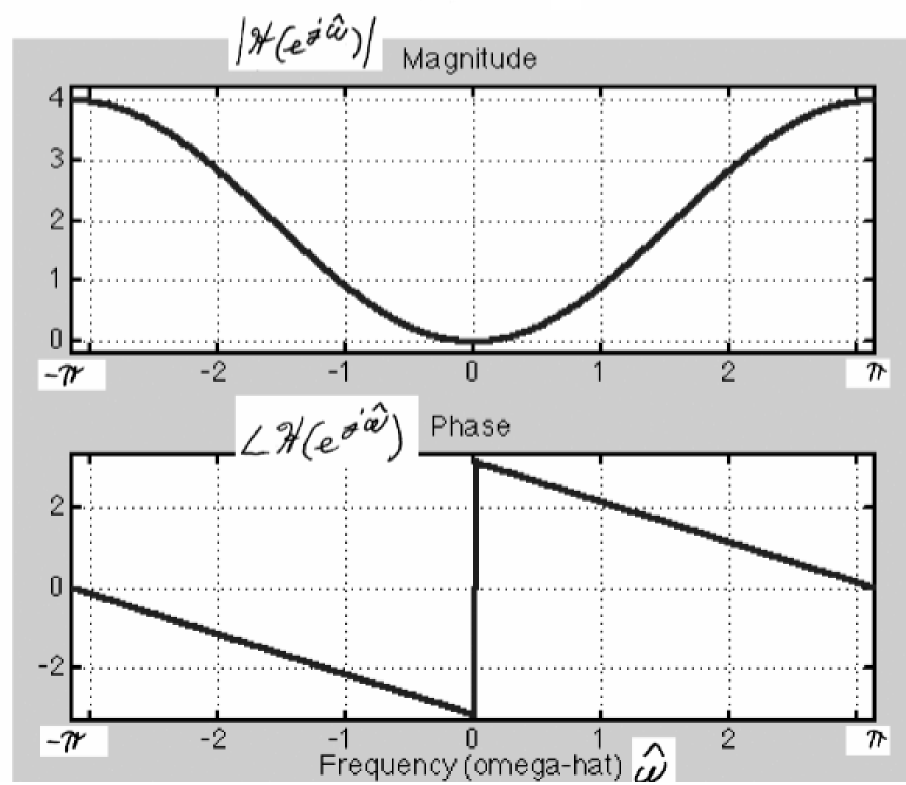
$$h[n] = \delta[n] - 2\delta[n-1] + \delta[n-2]$$

Q2 Solution (Cont.)

(d) To obtain the frequency response, we simply make the substitution $z = e^{j\hat{\omega}}$

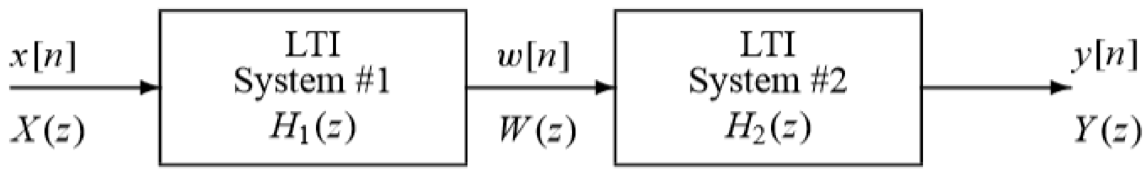
$$\mathcal{H}(e^{j\hat{\omega}}) = 1 - 2e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}$$
$$= e^{-j\hat{\omega}}\left(e^{j\hat{\omega}} - 2 + e^{-j\hat{\omega}}\right) = e^{-j\hat{\omega}}(2\cos\hat{\omega} - 2)$$
$$= 2e^{-j\hat{\omega}}(\cos\hat{\omega} - 1)$$

(e) $$H(e^{j\hat{\omega}}) = \underbrace{2e^{-j\hat{\omega}}}_{\text{phase}}\underbrace{(\cos\hat{\omega} - 1)}_{\text{signed magnitude}}$$



Q3

Consider the following cascade system:



The system function for the first system is

$$H_1(z) = \frac{(1 - 1.2z^{-1})}{(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})}$$

(a) We wish to find a System #2 such that $y[n] = x[n]$ for any input. How should $H_2(z)$ be chosen?

(b) Determine the difference equation that would be satisfied by the input $w[n]$ and the output $y[n]$ of the second system.

(c) Would there be any problem in implementing the system found above? Explain.

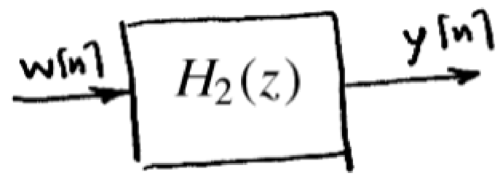
Q3 Solution

(a) $y[n] = x[n] \implies H_1(z)H_2(z) = 1$

$$\implies H_2(z) = \frac{1}{H_1(z)} = \frac{(1-.8e^{j\pi/4}z^{-1})(1-.8e^{-j\pi/4}z^{-1})}{1-1.2z^{-1}}$$

$$= \frac{1 - 1.6\cos(\pi/4)z^{-1} + .64z^{-2}}{1-1.2z^{-1}}$$

(b)

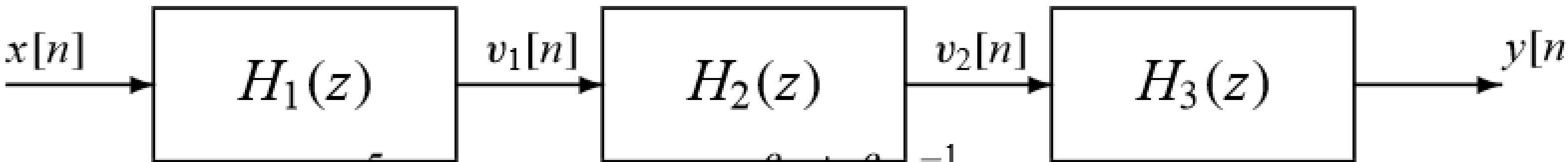


$$y[n] = 1.2y[n-1] + x[n] - \frac{1.6\sqrt{2}}{2}x[n-1] + 0.64x[n-2]$$

(c) Since the pole in $H_2(z)$ is at $z=1.2$, the response of System #2 to an input would contain $(1.2)^n$ which blows up. In other words, this system is UNSTABLE.

Q4

In the following cascade of systems, all systems are defined by their transfer functions.



$$H_1(z) = \frac{5}{1 - \frac{1}{2}z^{-1}} \qquad H_2(z) = \frac{\beta_0 + \beta_1 z^{-1}}{1 + \alpha z^{-1}} \qquad H_3(z) = 4z^{-1} - z^{-3}$$

(a) Determine the unknown coefficients $\{\ \beta_0,\ \beta_1,\ \alpha\ \}$ so that the impulse response of the overall system will be $h[n] = 2\delta[n-1] + 3\delta[n-2]$.

(b) Using part (a), determine the overall difference equation that relates $x[n]$ to $y[n]$.

Q4 Solution

(a) $H(z) = H_1(z)\,H_2(z)\,H_3(z)$

$$= \left(\frac{5}{1-\frac{1}{2}z^{-1}}\right)\left(\frac{\beta_0+\beta_1 z^{-1}}{1+\alpha z^{-1}}\right)\left(4z^{-1}-z^{-3}\right)$$

Want $H(z)$ to be $2z^{-1}+3z^{-2}$

$\Rightarrow$ we need pole-zero cancellations.

Note: $4z^{-1}-z^{-3} = 4z^{-1}\left(1-\frac{1}{4}z^{-2}\right) = 4z^{-1}\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)$

$$\Rightarrow H(z) = \frac{(5)(4z^{-1})\left(1+\frac{1}{2}z^{-1}\right)\left(\beta_0+\beta_1 z^{-1}\right)}{1+\alpha z^{-1}} \overset{?}{=} 2z^{-1}+3z^{-2}$$

Cross-multiply and compare terms

$$20\beta_0 z^{-1} + 20\left(\tfrac{1}{2}\beta_0+\beta_1\right)z^{-2} + 10\beta_1 z^{-3} = 2z^{-1} + (2\alpha+3)z^{-2} + 3\alpha z^{-3}$$

$\Rightarrow 20\beta_0 = 2 \Rightarrow \boxed{\beta_0 = 1/10}$

$$\left.\begin{aligned} 20\beta_1 + 10\beta_0 &= 2\alpha+3 \\ 10\beta_1 &= 3\alpha \end{aligned}\right\} \quad \begin{aligned} 2(3\alpha) + 10\left(\tfrac{1}{10}\right) &= 2\alpha+3 \\ 4\alpha = 2 &\Rightarrow \boxed{\alpha = \tfrac{1}{2}} \end{aligned}$$

$\Rightarrow \boxed{\beta_1 = \frac{3}{10}\alpha = \frac{3}{20}}$

(b) Since $H(z) = 2z^{-1}+3z^{-2}$, the filter is FIR.

$$y[n] = 2x[n-1] + 3x[n-2]$$

Q5

For each of the difference equations below, determine the poles and zeros of the corresponding system function, $H(z)$. Plot the poles **(X)** and zeros **(O)** in the complex z-plane.

$$\mathcal{S}_1: \quad y[n] = 0.8y[n-1] + x[n] + x[n-2]$$

$$\mathcal{S}_2: \quad y[n] = 0.8y[n-1] + 1.25x[n] - x[n-1]$$

$$\mathcal{S}_3: \quad y[n] = -0.64y[n-2] + x[n] + 0.64x[n-1]$$

$$\mathcal{S}_4: \quad y[n] = x[n] + \tfrac{3}{4}x[n-1] - \tfrac{1}{4}x[n-2]$$

Q5 Solution

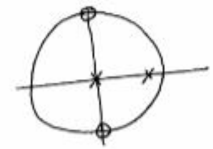
find poles and zeros

S_1: $Y(z) = 0.8z^{-1}Y(z) + X(z) + z^{-2}X(z)$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-2}}{1-0.8z^{-1}} \text{, multiply by } \frac{z^2}{z^2} \Rightarrow \frac{z^2+1}{(z)(z-0.8)}$$

poles @ $(z)(z-0.8)=0 \Rightarrow z = 0,\ 0.8$

zeros @ $z^2+1=0,\ z=\pm j$

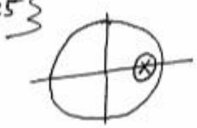


S_2: $H(z) = \frac{1.25-z^{-1}}{1-0.8z^{-1}} = \frac{1.25z-1}{z-0.8}$

zero @ $z=0.8$

pole @ $z=0.8$

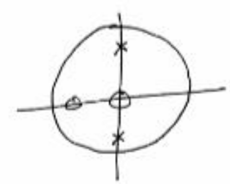
$\left\{ \text{note } H(z) = (1.25)\left(\frac{1.25-z^{-1}}{1.25-z^{-1}}\right) = 1.25 \right\}$



S_3: $H(z) = \frac{1+0.64z^{-1}}{1+0.64z^{-2}} = \frac{z^2+0.64z}{z^2+0.64} = \frac{(z)(z+0.64)}{(z+0.8j)(z-0.8j)}$

zeros @ $z = 0,\ -0.64$

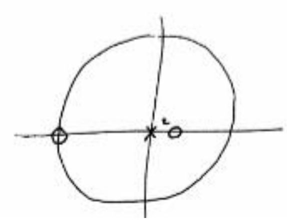
poles @ $z = \pm 0.8j$



S_4: $H(z) = \frac{1+\frac{3}{4}z^{-1}-\frac{1}{4}z^{-2}}{1} = \frac{z^2+\frac{3}{4}z-\frac{1}{4}}{z^2} = \frac{(z+1)(z-\frac{1}{4})}{(z)(z)}$

zeros @ $z=-1,\ z=\frac{1}{4}$

poles @ $z = 0, 0$



Q6

(a) "If $H(e^{j\hat{\omega}})$ is the frequency response of a digital filter, and the input is $x[n] = \cos(0.3\pi n)$, then the output is $y[n] = H(e^{j0.3\pi})x[n]$." This statement is

(a) Always true
(b) Sometimes true
(c) Never true

(b) "If the signal $x(t)$ is a sinusoid and its spectrum has frequency components at $f = \pm 55$ Hz, then the signal $y(t) = x^2(t)$ has frequency components at the same frequencies." This statement is:

(a) True
(b) False

(c) A causal IIR filter with system function $H(z) = \dfrac{1 - 2z^{-1}}{1 + 0.25z^{-1}}$ is:

(a) *not* stable.
(b) stable

(d) Evaluate the complex number $z = j^{-1} + j^{-2} + j^{-3}$.

(a) $z = 0$
(b) $z = j$
(c) $z = -j$
(d) $z = 1$
(e) $z = -1$

(e) Suppose that the discrete-time signal $x[n] = \cos(0.2\pi n)$ is the input to an FIR filter whose frequency response is $H(e^{j\hat{\omega}}) = 2e^{-j2\hat{\omega}}\cos(2\hat{\omega})$. Determine the output signal, $y[n]$.

(a) $y[n] = 2\cos(0.2\pi n - 0.2\pi)$
(b) $y[n] = 2\cos(0.2\pi n - 0.4\pi)$
(c) $y[n] = 0.618\cos(0.2\pi n - 0.4\pi)$
(d) $y[n] = 0.618\cos(0.2\pi n - 0.2\pi)$
(e) $y[n] = 0$

Q6 Solution

(a) "If $H(e^{j\hat{\omega}})$ is the frequency response of a digital filter, and the input is $x[n] = \cos(0.3\pi n)$, then the output is $y[n] = H(e^{j0.3\pi})x[n]$." This statement is

- (a) Always true
- (b) Sometimes true
- (c) Never true

If $H(e^{j0.3\pi})$ is real, it is true

No phase

(b) "If the signal $x(t)$ is a sinusoid and its spectrum has frequency components at $f = \pm 55$ Hz, then the signal $y(t) = x^2(t)$ has frequency components at the same frequencies." This statement is:

- (a) True
- (b) False

$$\left(e^{j2\pi(55)t} + e^{-j2\pi(55)t}\right)^2 \rightarrow e^{j2\pi(110)t} \text{ etc.}$$

(c) An IIR filter with system function $H(z) = \dfrac{1 - 2z^{-1}}{1 + 0.25z^{-1}}$ is:

- (a) *not* stable.
- (b) stable

pole at $z = -0.25$ is inside unit circle.

(d) Evaluate the complex number $z = j^{-1} + j^{-2} + j^{-3}$.

- (a) $z = 0$
- (b) $z = j$
- (c) $z = -j$
- (d) $z = 1$
- (e) $z = -1$

$$z = -j - 1 + j = -1$$

(e) Suppose that the discrete-time signal $x[n] = \cos(0.2\pi n)$ is the input to an FIR filter whose frequency response is $H(e^{j\hat{\omega}}) = 2e^{-j2\hat{\omega}}\cos(2\hat{\omega})$. Determine the output signal, $y[n]$.

- (a) $y[n] = 2\cos(0.2\pi n\cancel{)} - 0.2\pi)$
- (b) $y[n] = 2\cos(0.2\pi n - 0.4\pi)$
- (c) $y[n] = 0.618\cos(0.2\pi n - 0.4\pi)$
- (d) $y[n] = 0.618\cos(0.2\pi n\cancel{)} - 0.2\pi)$
- (e) $y[n] = 0$

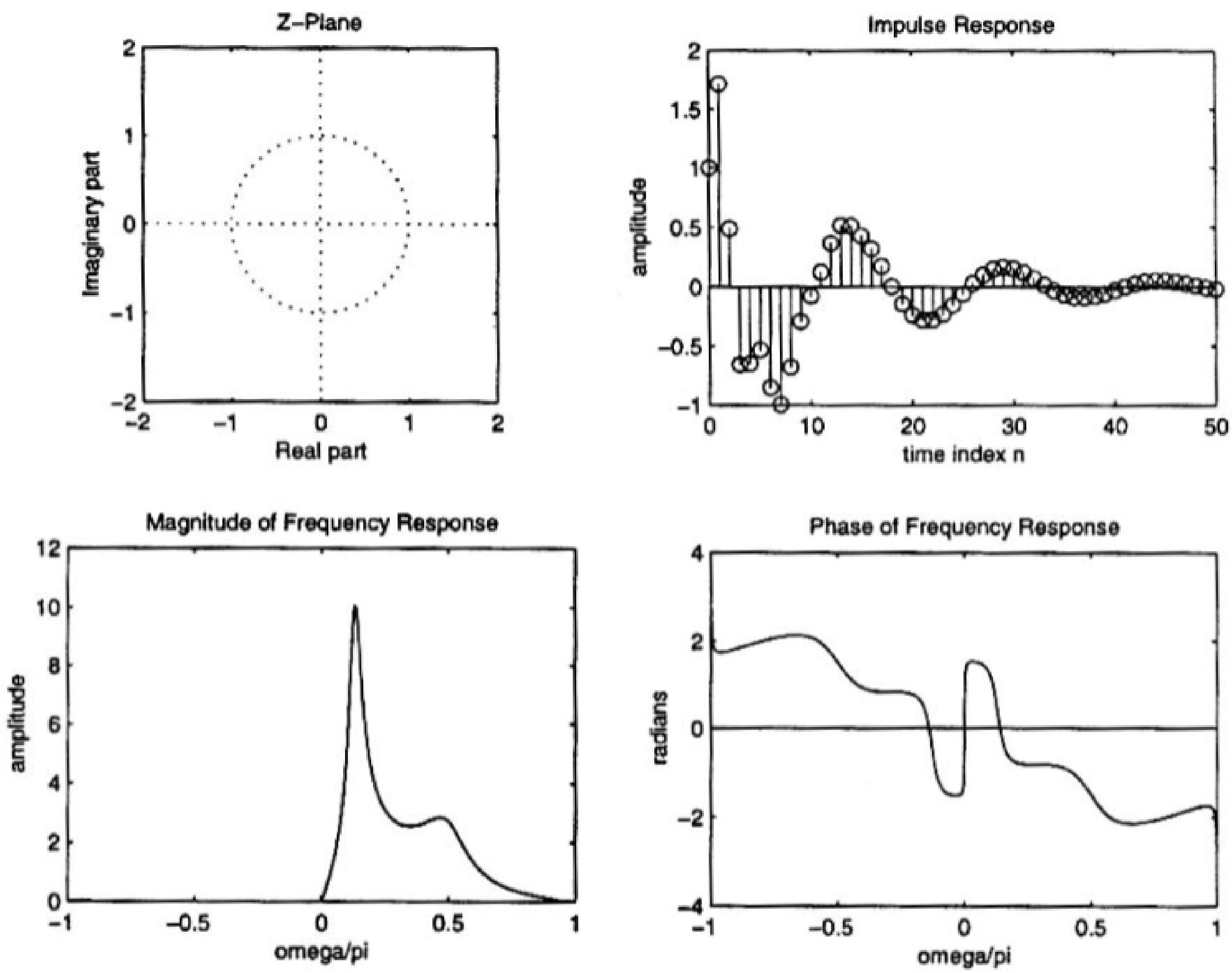
Mag phase

$$H(e^{j0.2\pi}) = 2e^{-j0.4\pi}\cos(0.4\pi) = 0.618\, e^{-j0.4\pi}$$

$$y[n] = 0.618\cos(0.2\pi n - 0.4\pi)$$

Consider the following output

Q7



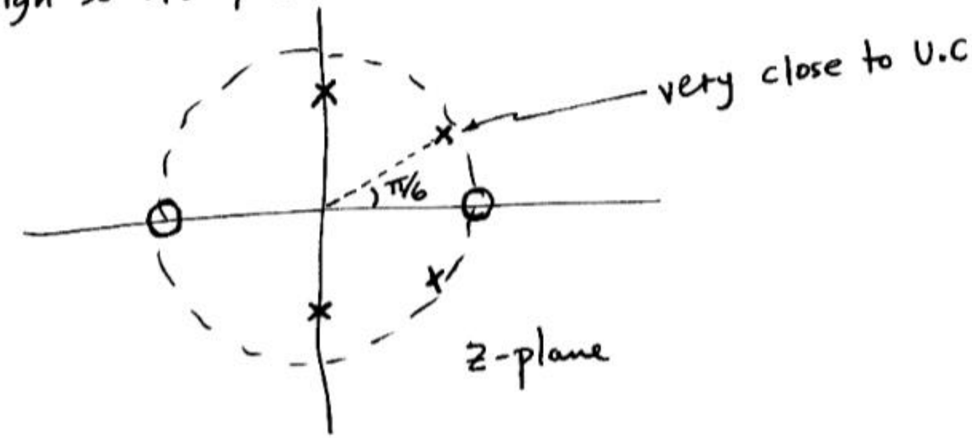
(a) Is this a FIR or IIR system? Explain your answer.

(b) Fill in an estimate of pole-zero plot corresponding to other plots in the figure. Place the poles and zeros to indicate which ones are on the unit circle, which are close to the unit circle, and which are relatively farther away. Also fill in the missing part of the magnitude of the frequency response; i.e., for $-\pi < \hat{\omega}/\pi < 0$.

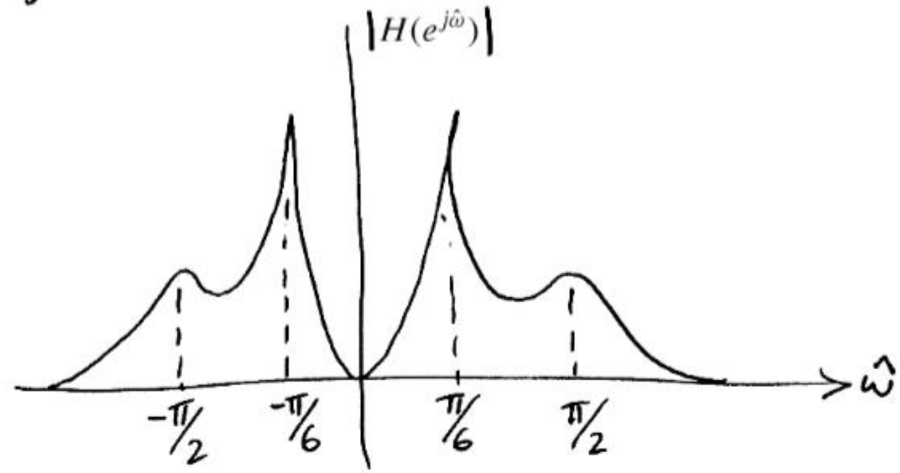
Q7 Solution

(a) The system is IIR because the impulse response is not finite length. It lasts at least 50 samples.

(b) The frequency response is zero at $\hat{\omega}=0$ & $\hat{\omega}=\pi$.
The two peaks are due to pole pairs at angles of $\pi/2$ and $\pi/6$. The peak at $\pi/6$ is narrow and high so its pole is closer to the unit circle.



Magnitude Response must be symmetric:



Q8

For the following systems, determine $H(e^{j\hat{\omega}})$ and make a sketch over the range $-\pi \le \hat{\omega} \le \pi$.

(a) For the following system

$$y[n] = -0.8y[n-1] + 2x[n-1]$$

derive a simple $\boxed{\text{real-valued}}$ expression for $|H(e^{j\hat{\omega}})|^2$, and sketch a plot of $|H(e^{j\hat{\omega}})|$.

(b) For the following system derive a simple expression for $|H(e^{j\hat{\omega}})|$, and sketch a plot of $|H(e^{j\hat{\omega}})|$.

$$y[n] = 3x[n] - 6x[n-1] + 3x[n-2]$$

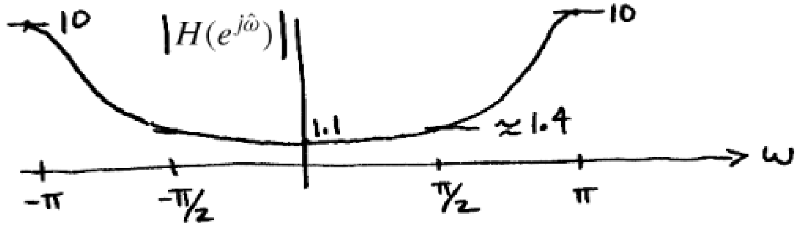
Q8 Solution

(a) $y[n] = -0.8y[n-1] + 2x[n-1]$.

$$H(z) = \frac{2z^{-1}}{1+0.8z^{-1}} \Longrightarrow H(e^{j\hat\omega}) = \frac{2e^{-j\hat\omega}}{1+0.8e^{-j\hat\omega}}$$

$$|H(e^{j\hat\omega})|^2 = H(e^{j\hat\omega})H^*(e^{j\hat\omega})$$

$$= \frac{2e^{-j\hat\omega}}{1+0.8e^{-j\hat\omega}} \frac{2e^{j\hat\omega}}{1+0.8e^{j\hat\omega}} = \frac{4}{1.64+1.6\cos\hat\omega}$$



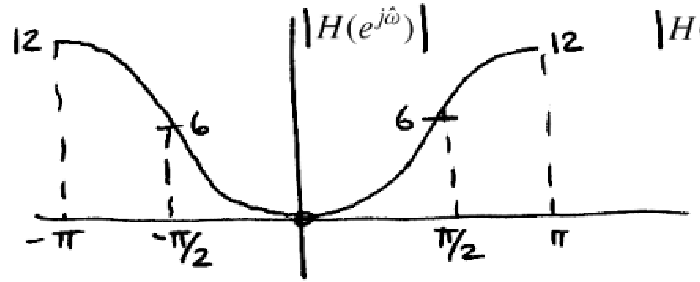
$\hat\omega$	$H(e^{j\hat\omega})$
0	1.1
π	-10
π/2	$\frac{-2j}{1-.8j}$

(b) $y[n] = 3x[n] - 6x[n-1] + 3x[n-2]$

$$H(z) = 3 - 6z^{-1} + 3z^{-2} \Longrightarrow H(e^{j\hat\omega}) = 3 - 6e^{-j\hat\omega} + 3e^{-j2\hat\omega}$$

$$H(e^{j\hat\omega}) = e^{-j\hat\omega}(3e^{j\hat\omega} - 6 + 3e^{-j\hat\omega})$$

$$= e^{-j\hat\omega}(\underbrace{6\cos\hat\omega - 6}_{\text{"MAG" part}}).$$



$|H(e^{j\hat\omega})| = |6\cos\hat\omega - 6|$

$\hat\omega$	$H(e^{j\hat\omega})$
0	0
π	12
π	6j