

Definition of the Laplace Transform

Let f be a function defined for $t \geq 0$. Then the integral

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

is said to be the Laplace Transform of f , provided the integral converges

Evaluate $L\{1\}$

from laplace transform

$$\begin{aligned} L\{1\} &= \int_0^{\infty} e^{-st} (1) dt = \lim_{b \rightarrow \infty} \int_0^b e^{-st} dt \\ &= \lim_{b \rightarrow \infty} \left. \frac{-e^{-st}}{s} \right|_0^b = \lim_{b \rightarrow \infty} \frac{-e^{-sb} + 1}{s} = \frac{1}{s} \end{aligned}$$

provided $s > 0$

In other words,

when $s > 0$ the exponent $-sb$ is negative

and $e^{-sb} \rightarrow 0$ as $b \rightarrow \infty$.

The integral diverges for $s < 0$

Evaluate $L\{t\}$

from laplace transform

$$L\{t\} = \int_0^{\infty} e^{-st} t dt = \lim_{b \rightarrow \infty} \int_0^b e^{-st} t dt$$

integrating ..

$$\lim_{t \rightarrow \infty} t e^{-st} = 0, s > 0$$

$$L\{t\} = \frac{-te^{-st}}{s} \Big|_0^{\infty} + \frac{1}{s} \int_0^{\infty} e^{-st} dt = \frac{1}{s} L\{1\} = \frac{1}{s} \left(\frac{1}{s} \right) = \frac{1}{s^2}$$

$$\int f(x)g(x) dx = f(x) \int g(x) dx - \int \left(\int g(x) dx \right) d(f(x))$$

Evaluate $L\{e^{-3t}\}$

from laplace transform

$$L\{e^{-3t}\} = \int_0^{\infty} e^{-st} e^{-3t} dt = \int_0^{\infty} e^{-(s+3)t} dt$$

$$= \frac{-e^{-(s+3)t}}{s+3} \Big|_0^{\infty} = \frac{1}{s+3}, s > -3$$

Evaluate $L\{\sin 2t\}$

from laplace transform

$$L\{\sin 2t\} = \int_0^{\infty} e^{-st} \sin 2t dt = \frac{-e^{-st} \sin 2t}{s} \Big|_0^{\infty} + \frac{2}{s} \int_0^{\infty} e^{-st} \cos 2t dt$$

$$= \frac{2}{s} \int_0^{\infty} e^{-st} \cos 2t dt, s > 0$$

$$\Rightarrow \frac{2}{s} \left[\frac{-e^{-st} \cos 2t}{s} \Big|_0^{\infty} - \frac{2}{s} \int_0^{\infty} e^{-st} \sin 2t dt \right]$$

$$= \frac{2}{s^2} - \frac{4}{s^2} L\{\sin 2t\}$$

$$\therefore L\{\sin 2t\} = \frac{2}{s^2 + 4}$$

Laplace Transforms of Some Basic Functions

$$(a) L\{1\} = \frac{1}{s}$$

$$(b) L\{t^n\} = \frac{n!}{s^{n+1}}, n = 1, 2, 3, \dots$$

$$(c) L\{e^{at}\} = \frac{1}{s - a}$$

$$(d) L\{\sin kt\} = \frac{k}{s^2 + k^2}$$

$$(e) L\{\cos kt\} = \frac{s}{s^2 + k^2}$$

$$(f) L\{\sinh kt\} = \frac{k}{s^2 - k^2}$$

$$(g) L\{\cosh kt\} = \frac{s}{s^2 - k^2}$$

Laplace Transforms of Some Basic Functions

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
2	t	$1/s^2$	8	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\frac{s}{s^2 - a^2}$
4	t^n ($n = 0, 1, \dots$)	$\frac{n!}{s^{n+1}}$	10	$\sinh at$	$\frac{a}{s^2 - a^2}$
5	t^a (a positive)	$\frac{\Gamma(a + 1)}{s^{a+1}}$	11	$e^{at} \cos \omega t$	$\frac{s - a}{(s - a)^2 + \omega^2}$
6	e^{at}	$\frac{1}{s - a}$	12	$e^{at} \sin \omega t$	$\frac{\omega}{(s - a)^2 + \omega^2}$

Some Inverse Transforms

$$(a) 1 = L^{-1} \left\{ \frac{1}{s} \right\}$$

$$(b) t^n = L^{-1} \left\{ \frac{n!}{s^{n+1}} \right\}, n = 1, 2, 3, \dots$$

$$(c) e^{at} = L^{-1} \left\{ \frac{1}{s-a} \right\}$$

$$(d) \sin kt = L^{-1} \left\{ \frac{k}{s^2 + k^2} \right\}$$

$$(e) \cos kt = L^{-1} \left\{ \frac{s}{s^2 + k^2} \right\}$$

$$(f) \sinh kt = L^{-1} \left\{ \frac{k}{s^2 - k^2} \right\}$$

$$(g) \cosh kt = L^{-1} \left\{ \frac{s}{s^2 - k^2} \right\}$$

Both L and L^{-1} are linear

Examples

(a) Evaluate $L^{-1}\left\{\frac{1}{s^5}\right\}$

$$L^{-1}\left\{\frac{1}{s^5}\right\} = \frac{1}{4!} L^{-1}\left\{\frac{4!}{s^5}\right\} = \frac{1}{24} t^4$$

(b) Evaluate $L^{-1}\left\{\frac{1}{s^2 + 7}\right\}$

$$L^{-1}\left\{\frac{1}{s^2 + 7}\right\} = \frac{1}{\sqrt{7}} L^{-1}\left\{\frac{\sqrt{7}}{s^2 + 7}\right\} = \frac{1}{\sqrt{7}} \sin \sqrt{7}t$$

Termwise Division and Linearity

Evaluate $L^{-1}\left\{\frac{-2s+6}{s^2+4}\right\}$

$$L^{-1}\left\{\frac{-2s+6}{s^2+4}\right\} = L^{-1}\left\{\frac{-2s}{s^2+4} + \frac{6}{s^2+4}\right\} = -2L^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{6}{2}L^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= -2 \cos 2t + 3 \sin 2t$$

Partial Fraction Expansions

$$\frac{s+1}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$$

$$\frac{s+1}{(s+2)(s+3)} = \frac{A(s+3) + B(s+2)}{(s+2)(s+3)}$$

$$A + B = 1 \qquad 3A + 2B = 1$$

$$\frac{s+1}{(s+2)(s+3)} = \frac{-1}{s+2} + \frac{2}{s+3}$$

- Expand into a term for each factor in the denominator.
- Recombine RHS
- Equate terms in s and constant terms. Solve.
- Each term is in a form so that inverse Laplace transforms can be applied.

Partial Fractions and Linearity

$$\text{Evaluate } L^{-1} \left\{ \frac{s^2 + 6s + 9}{(s-1)(s-2)(s+4)} \right\}$$

$$\frac{s^2 + 6s + 9}{(s-1)(s-2)(s+4)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+4}$$

$$= \frac{A(s-2)(s+4) + B(s-1)(s+4) + C(s-1)(s-2)}{(s-1)(s-2)(s+4)}$$

$$\rightarrow s^2 + 6s + 9 = A(s-2)(s+4) + B(s-1)(s+4) + C(s-1)(s-2)$$

$$\text{set } s = 1, 2, -4$$

$$16 = A(-1)(5), \quad 25 = B(1)(6), \quad 1 = C(-5)(-6)$$

$$\therefore A = -\frac{16}{5}, \quad B = \frac{25}{6}, \quad C = \frac{1}{30}$$

$$L^{-1} \left\{ \frac{s^2 + 6s + 9}{(s-1)(s-2)(s+4)} \right\} = L^{-1} \left\{ -\frac{16}{5} \frac{1}{s-1} + \frac{25}{6} \frac{1}{s-2} + \frac{1}{30} \frac{1}{s+4} \right\}$$

$$= -\frac{16}{5} e^t + \frac{25}{6} e^{2t} + \frac{1}{30} e^{-4t}$$

Transform of a Derivative

If $f, f', \dots, f^{(n-1)}$ are continuous on $[0, \infty)$ and are of exponential order and if $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$, then

$$L\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0)$$

where $F(s) = L\{f(t)\}$

Laplace Transform of Derivatives

The transforms of the first and second derivatives of $f(t)$ satisfy

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$$

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$$

Laplace Transform of the Derivative $f^{(n)}$ of Any Order

Let $f, f', \dots, f^{(n-1)}$ be continuous for all $t \geq 0$ and satisfy the growth restriction.

Furthermore, let $f^{(n)}$ be piecewise continuous on every finite interval on the semi-axis $t \geq 0$. Then the transform of $f^{(n)}$ satisfies

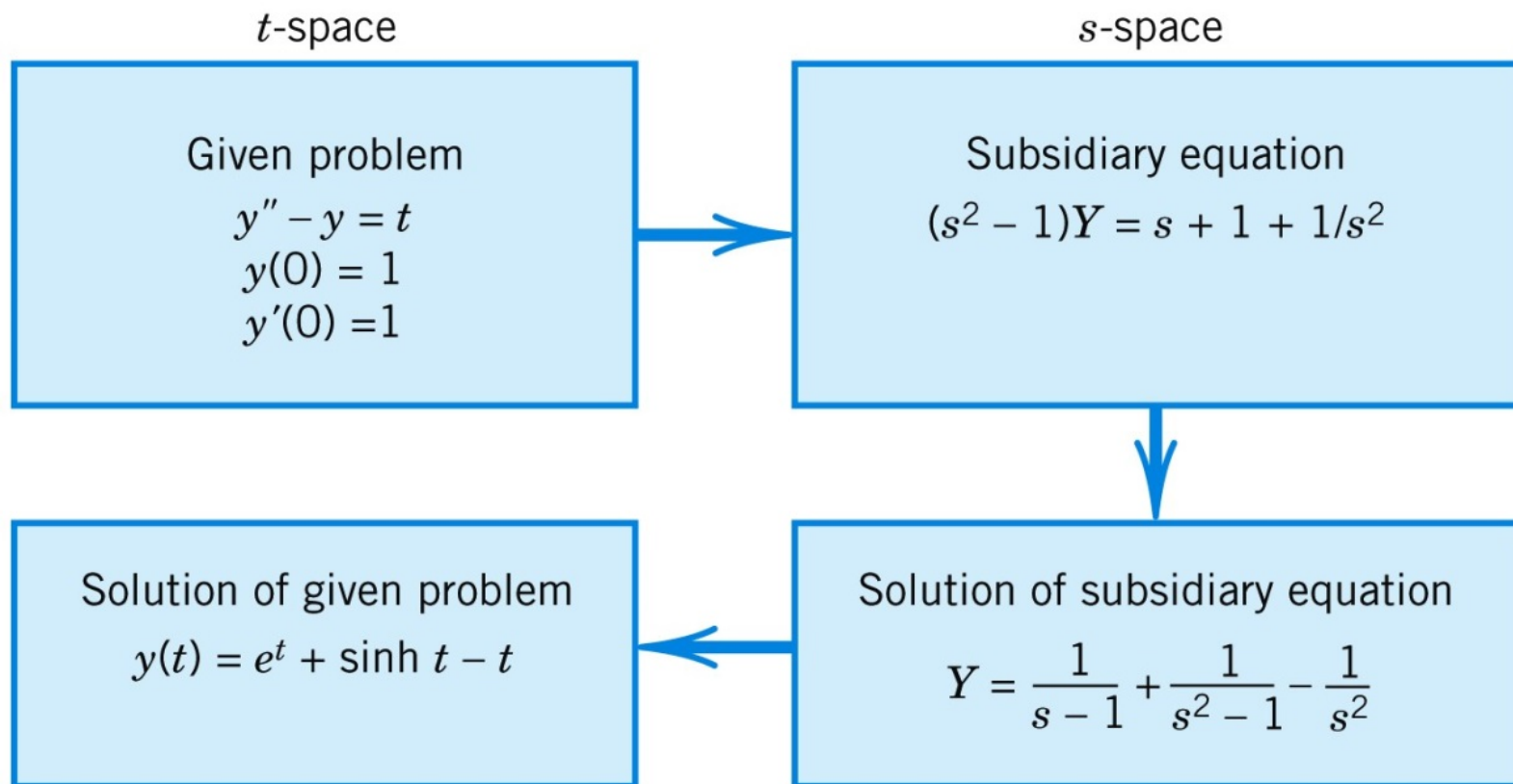
$$\mathcal{L}(f^{(n)}) = s^n \mathcal{L}(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0).$$

Laplace Transform of Integral

- *Let $F(s)$ denote the transform of a function $f(t)$ which is piecewise continuous for $t \geq 0$ and satisfies a growth restriction (2), Sec. 6.1. Then, for $s > 0$, $s > k$, and $t > 0$,*

$$\mathcal{L} \left\{ \int_0^t f(\tau) d\tau \right\} = \frac{1}{s} F(s), \quad \text{thus} \quad \int_0^t f(\tau) d\tau = \mathcal{L}^{-1} \left\{ \frac{1}{s} F(s) \right\}.$$

Initial Value Problem: The Basic Laplace Steps



How Laplace Transforms Turn Initial Value Problems Into Algebraic Equations

Time Domain (t)

Transform domain

Original
DE & IVP

$\xrightarrow{\mathcal{L}}$

Algebraic equation
the Laplace transform

Algebraic solution,
partial fractions

↓

Solution

$\xleftarrow{\mathcal{L}^{-1}}$

Laplace transform

solve the IVP) $\frac{dy}{dt} + 3y = 13 \sin 2t$, $y(0) = 6$

$$L\left\{\frac{dy}{dt}\right\} + 3L\{y\} = 13L\{\sin 2t\}$$

$$L\left\{\frac{dy}{dt}\right\} = sY(s) - y(0) = sY(s) - 6$$

$$\text{Since } L\{\sin 2t\} = 2/(s^2 + 4)$$

$$sY(s) - 6 + 3Y(s) = \frac{26}{s^2 + 4}, (s + 3)Y(s) = 6 + \frac{26}{s^2 + 4}$$

$$Y(s) = \frac{6}{s + 3} + \frac{26}{(s + 3)(s^2 + 4)} = \frac{6s^2 + 50}{(s + 3)(s^2 + 4)}$$

$$\frac{6s^2 + 50}{(s + 3)(s^2 + 4)} = \frac{A}{s + 3} + \frac{Bs + C}{s^2 + 4}$$

$$\text{set } s = -3, A = 8$$

$$6 = A + B, 0 = 3B + C, \rightarrow B = -2, C = 6$$

$$\frac{6s^2 + 50}{(s + 3)(s^2 + 4)} = \frac{8}{s + 3} + \frac{-2s + 6}{s^2 + 4}$$

$$y(t) = 8L^{-1}\left\{\frac{1}{s + 3}\right\} - 2L^{-1}\left\{\frac{s}{s^2 + 4}\right\} + 3L^{-1}\left\{\frac{2}{s^2 + 4}\right\}$$

$$= 8e^{-3t} - 2\cos 2t + 3\sin 2t$$

How Laplace Transforms Turn Initial Value Problems Into Algebraic Equations

1. The first key property of the Laplace transform is derivatives are transformed.
 - 1.1 $\mathcal{L}\{y\}(s) =: Y(s)$ (This is just notation.)
 - 1.2 $\mathcal{L}\{y'\}(s) = sY(s) - y(0)$
 - 1.3 $\mathcal{L}\{y''\}(s) = s^2Y(s) - sy(0) - y'(0)$
 - 1.4 $\mathcal{L}\{y^{(n)}(t)\}(s) = s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) \dots$
2. The right sides above do not involve derivatives of whatever Y is.
3. The other key property is that constants and sums

Solve the Initial Value Problem

$$y'' + 7y' + 12y = 0, y(0) = 1, y'(0) = 2$$

Finding the Laplace transform of the solution.

$$y'' + 7y' + 12y = 0, y(0) = 1,$$

$$s^2 Y - s - 2 + 7sY - 7 + 12Y = 0$$

$$(s^2 + 7s + 12) Y = s + 9$$

$$\begin{aligned} Y &= \frac{s + 9}{s^2 + 7s + 12} \\ &= \frac{s + 9}{(s + 3)(s + 4)} \end{aligned}$$

Solve the Initial Value Problem

$$y'' + 7y' + 12y = 0, y(0) = 1, y'(0) = 2$$

Partial fraction decomposition.

$$\begin{aligned} Y &= \frac{s+9}{(s+3)(s+4)} = \frac{A}{s+3} + \frac{B}{s+4} \\ \frac{s+9}{(s+3)(s+4)} &= \frac{A(s+4) + B(s+3)}{(s+3)(s+4)} \\ s+9 &= A(s+4) + B(s+3) \\ &\text{Heaviside's Method} \\ s = -3 \quad A &= 6 \end{aligned}$$

Solve the Initial Value Problem

$$y'' + 7y' + 12y = 0, y(0) = 1, y'(0) = 2$$

Inverting the Laplace Transform.

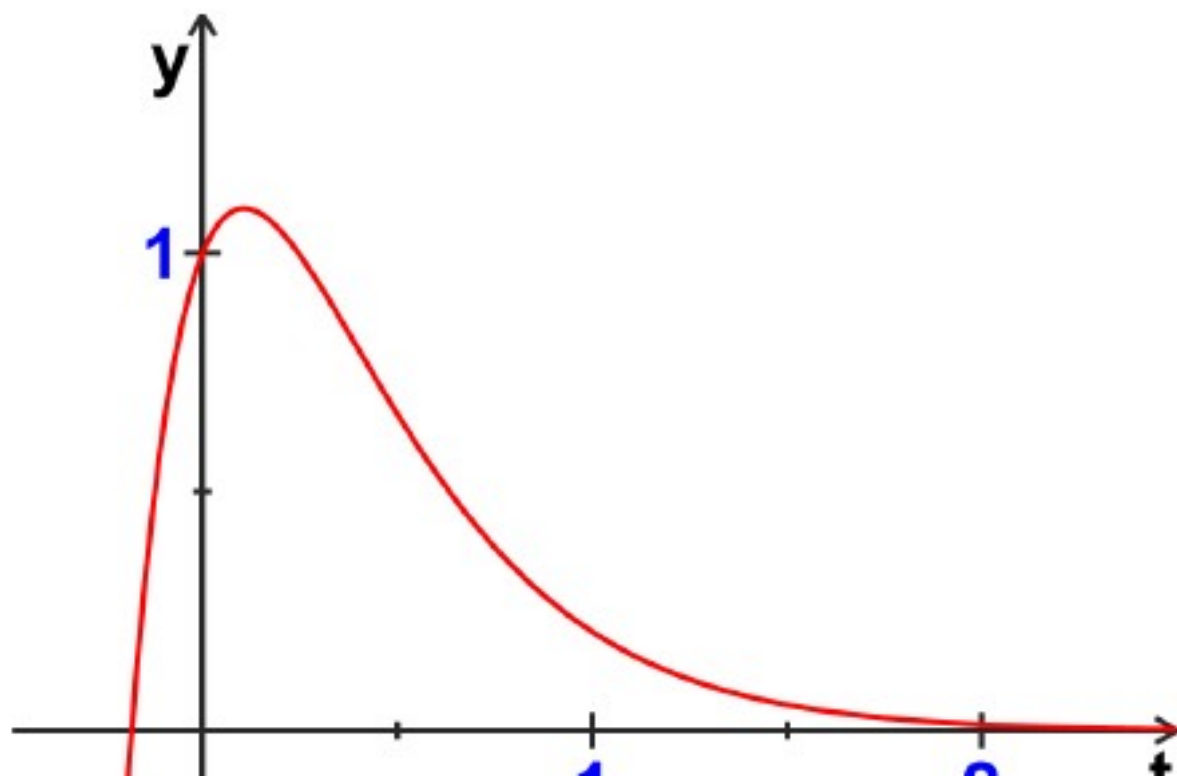
$$Y = \frac{6}{s+3} - \frac{5}{s+4}$$

Use the transform table.

$$\begin{aligned}\mathcal{L}\{e^{at}\}(s) &= \frac{1}{s-a} \\ &= 6\frac{1}{s-(-3)} - 5\frac{1}{s-(-4)}\end{aligned}$$

Solve the Initial Value Problem

$$y'' + 7y' + 12y = 0, y(0) = 1, y'(0) =$$



Does $y = 6e^{-3t} - 5e^{-4t}$ Really Solve the Value Problem $y'' + 7y' + 12y = 0$, $y(0) : y'(0) = 2$?

Checking the differential equation.

$$\begin{aligned} & \left(54e^{-3t} - 80e^{-4t} \right) + 7 \left(-18e^{-3t} + 20e^{-4t} \right) + 12 \left(6e^{-3t} - 5e^{-4t} \right) \\ & \quad (54 - 126 + 72)e^{-3t} + (-80 + 140 - 60)e^{-4t} \end{aligned}$$

Does $y = 6e^{-3t} - 5e^{-4t}$ Really Solve the
Value Problem $y'' + 7y' + 12y = 0$, $y(0) = 1$,
 $y'(0) = 2$?

Checking the initial values.

$$\begin{aligned}y &= 6e^{-3t} - 5e^{-4t} \\y(0) &= 6 - 5 = 1 \quad \checkmark \\y' &= -18e^{-3t} + 20e^{-4t}\end{aligned}$$

Laplace Transform: General Formulas

Formula	Name, Comments
$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$ $f(t) = \mathcal{L}^{-1}\{F(s)\}$	<p>Definition of Transform</p> <p>Inverse Transform</p>
$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	Linearity
$\mathcal{L}\{e^{at}f(t)\} = F(s - a)$ $\mathcal{L}^{-1}\{F(s - a)\} = e^{at}f(t)$	<p>s-Shifting (First Shifting Theorem)</p>

Laplace Transform: General Formulas

$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ $\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$ $\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{(n-1)}f(0) - \dots$ $\dots - f^{(n-1)}(0)$	<p>Differentiation of Function</p>
$\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{1}{s} \mathcal{L}(f)$	<p>Integration of Function</p>
$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$ $= \int_0^t f(t - \tau)g(\tau) d\tau$ $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$	<p>Convolution</p>

Laplace Transform: General Formulas

$\mathcal{L}\{f(t - a)u(t - a)\} = e^{-as}F(s)$ $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t - a)u(t - a)$	t -Shifting (Second Shifting Theorem)
$\mathcal{L}\{tf(t)\} = -F'(s)$ $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\tilde{s}) d\tilde{s}$	Differentiation of Transform Integration of Transform
$\mathcal{L}(f) = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t) dt$	f Periodic with Period p

Laplace Transform: General Formulas

$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ $\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$ $\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{(n-1)}f(0) - \dots$ $\dots - f^{(n-1)}(0)$ $\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{1}{s} \mathcal{L}(f)$	<p>Differentiation of Function</p> <p>Integration of Function</p>
$(f * g)(t) = \int_0^t f(\tau)g(t - \tau) d\tau$ $= \int_0^t f(t - \tau)g(\tau) d\tau$ $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$	<p>Convolution</p>

Laplace Transform: General Formulas

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Table of Laplace Transforms

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$
1	$1/s$	1
2	$1/s^2$	t
3	$1/s^n \quad (n = 1, 2, \dots)$	$t^{n-1}/(n-1)!$
4	$1/\sqrt{s}$	$1/\sqrt{\pi t}$
5	$1/s^{3/2}$	$2\sqrt{t/\pi}$
6	$1/s^a \quad (a > 0)$	$t^{a-1}/\Gamma(a)$
7	$\frac{1}{s-a}$	e^{at}
8	$\frac{1}{(s-a)^2}$	te^{at}
9	$\frac{1}{(s-a)^n} \quad (n = 1, 2, \dots)$	$\frac{1}{(n-1)!} t^{n-1} e^{at}$
10	$\frac{1}{(s-a)^k} \quad (k > 0)$	$\frac{1}{\Gamma(k)} t^{k-1} e^{at}$

Table of Laplace Transforms

11	$\frac{1}{(s-a)(s-b)} \quad (a \neq b)$	$\frac{1}{a-b} (e^{at} - e^{bt})$
12	$\frac{s}{(s-a)(s-b)} \quad (a \neq b)$	$\frac{1}{a-b} (ae^{at} - be^{bt})$
13	$\frac{1}{s^2 + \omega^2}$	$\frac{1}{\omega} \sin \omega t$
14	$\frac{s}{s^2 + \omega^2}$	$\cos \omega t$
15	$\frac{1}{s^2 - a^2}$	$\frac{1}{a} \sinh at$
16	$\frac{s}{s^2 - a^2}$	$\cosh at$
17	$\frac{1}{(s-a)^2 + \omega^2}$	$\frac{1}{\omega} e^{at} \sinh \omega t$
18	$\frac{s-a}{(s-a)^2 + \omega^2}$	$e^{at} \cos \omega t$

Table of Laplace Transforms

19	$\frac{1}{s(s^2 + \omega^2)}$	$\frac{1}{\omega^2}(1 - \cos \omega t)$
20	$\frac{1}{s^2(s^2 + \omega^2)}$	$\frac{1}{\omega^3}(\omega t - \sin \omega t)$
21	$\frac{1}{(s^2 + \omega^2)^2}$	$\frac{1}{2\omega^3}(\sin \omega t - \omega t \cos \omega t)$
22	$\frac{s}{(s^2 + \omega^2)^2}$	$\frac{t}{2\omega} \sin \omega t$
23	$\frac{s^2}{(s^2 + \omega^2)^2}$	$\frac{1}{2\omega}(\sin \omega t + \omega t \cos \omega t)$
24	$\frac{s}{(s^2 + a^2)(s^2 + b^2)} \quad (a^2 \neq b^2)$	$\frac{1}{b^2 - a^2}(\cos at - \cos bt)$

Table of Laplace Transforms

25	$\frac{1}{s^4 + 4k^4}$	$\frac{1}{4k^3}(\sin kt \cos kt - \cos kt \sinh kt)$
26	$\frac{s}{s^4 + 4k^4}$	$\frac{1}{2k^2} \sin kt \sinh kt$
27	$\frac{1}{s^4 - k^4}$	$\frac{1}{2k^3}(\sinh kt - \sin kt)$
28	$\frac{s}{s^4 - k^4}$	$\frac{1}{2k^2}(\cosh kt - \cos kt)$
29	$\sqrt{s-a} - \sqrt{s-b}$	$\frac{1}{2\sqrt{\pi t^3}}(e^{bt} - e^{at})$
30	$\frac{1}{\sqrt{s+a} \sqrt{s+b}}$	$e^{-(a+b)t/2} I_0\left(\frac{a-b}{2}t\right)$
31	$\frac{1}{\sqrt{s^2 + a^2}}$	$J_0(at)$

Table of Laplace Transforms

32	$\frac{s}{(s-a)^{3/2}}$	$\frac{1}{\sqrt{\pi t}} e^{at}(1+2at)$
33	$\frac{1}{(s^2-a^2)^k} \quad (k > 0)$	$\frac{\sqrt{\pi}}{\Gamma(k)} \left(\frac{t}{2a}\right)^{k-1/2} I_{k-1/2}(at)$
34	e^{-as}/s	$u(t-a)$
35	e^{-as}	$\delta(t-a)$
36	$\frac{1}{s} e^{-k/s}$	$J_0(2\sqrt{kt})$
37	$\frac{1}{\sqrt{s}} e^{-k/s}$	$\frac{1}{\sqrt{\pi t}} \cos 2\sqrt{kt}$
38	$\frac{1}{s^{3/2}} e^{k/s}$	$\frac{1}{\sqrt{\pi k}} \sinh 2\sqrt{kt}$
39	$e^{-k\sqrt{s}} \quad (k > 0)$	$\frac{k}{2\sqrt{\pi t^3}} e^{-k^2/4t}$

Table of Laplace Transforms

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$
40	$\frac{1}{s} \ln s$	$-\ln t - \gamma \quad (\gamma \approx 0.5772)$
41	$\ln \frac{s-a}{s-b}$	$\frac{1}{t}(e^{bt} - e^{at})$
42	$\ln \frac{s^2 + \omega^2}{s^2}$	$\frac{2}{t}(1 - \cos \omega t)$
43	$\ln \frac{s^2 - a^2}{s^2}$	$\frac{2}{t}(1 - \cosh at)$
44	$\arctan \frac{\omega}{s}$	$\frac{1}{t} \sin \omega t$
45	$\frac{1}{s} \operatorname{arccot} s$	$\operatorname{Si}(t)$