Analytical Mechanics 2 Project

Cart and pendulum problem with Lagrange method, with numerical solution

Ali Ashtari

ID: 400100038

Professor: Dr. Mahmood Bahman-Abadi

Introduction

"Lagrangian Mechanics" can be used instead of "Newtonian Mechanics" to solve problems. Using lagrange equations is much simpler than newton equations and we can use them to treat complicated problems easier.

In this project, I am going to examine "Cart and Pendulum problem" thoroughly; in the absence of any drag force, and also in presence of them and in addition to a driving force.

First, I will obtain equations of motion, exploiting lagrange equations of motion. Then, I will use numerical methods to solve the equations and find the actual motion.

**Note* I use "Python programming language" for my numerical calculations.*

Case 1; Cart and pendulum problem in the absence of any drag force

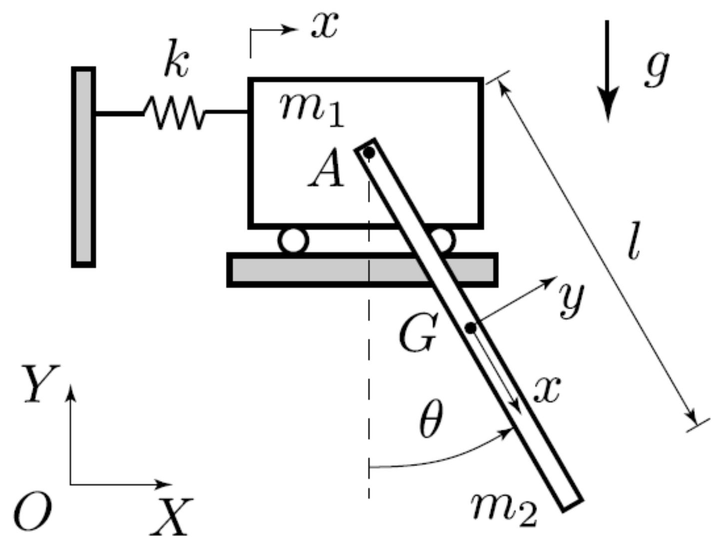


Figure 1 _ diagram of case 1

As is shown in figure 1, the problem consists of a cart of mass m_1 that is attached to a wall with a spring with spring constant k. A uniform rod of length l and mass m_2 is pivoting from point A. There is no frictional or drag forces.

According to Figure 1, potential energy is (Y is y_position of center of mass of rod measured from A, and zero of potential is at Y=0):

$$V = m_2 g Y + \frac{1}{2} k x^2 \qquad (1)$$

Kinetic energy is (X is x_position of center of mass of rod measured from the wall):

$$T = \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2} m_2 \left(\dot{X}^2 + \dot{Y}^2 \right) \qquad (2)$$

And we have (x is the x_position of A measured the wall):

$$X = x + \frac{l}{2} \sin\theta \qquad (3)$$

$$Y = -\frac{l}{2} \cos\theta \qquad (4)$$

Taking derivatives from equations (3) & (4) and using equations (1) & (2) we obtain:

$$V = -m_2 g \frac{l}{2} \cos\theta + \frac{1}{2} k x^2 \qquad (5)$$

We know for a rod of length l and mass m, I is $\frac{1}{12} m l^2$, so:

$$T = \frac{1}{2} \left(\frac{1}{12} m_2 l^2 \right) \dot{\theta}^2 + \frac{1}{2} m_2 \left(\dot{x}^2 + \frac{l^2 \dot{\theta}^2}{4} + l \dot{\theta} \dot{x} \cos\theta \right) + \frac{1}{2} m_1 \dot{x}^2 \qquad (6)$$

The lagrangian (L=T-V) is:

$$L = \frac{1}{6} m_2 l^2 \dot{\theta}^2 + \frac{1}{2} (m_1 + m_2) \dot{x}^2 + \frac{1}{2} m_2 l \dot{\theta} \dot{x} \cos\theta + m_2 g \frac{l}{2} \cos\theta - \frac{1}{2} k x^2 \qquad (7)$$

Using lagrange equations of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0 \quad \& \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0 \qquad (8)$$

We obtain:

$$(m_1 + m_2) \ddot{x} + \frac{1}{2} m_2 l \ddot{\theta} \cos\theta - \frac{1}{2} m_2 l \dot{\theta}^2 \sin\theta + k x = 0 \qquad (9)$$

$$\frac{1}{3} l \ddot{\theta} + \frac{1}{2} \ddot{x} \cos\theta + \frac{1}{2} g \sin\theta = 0 \qquad (10)$$

Equations (9) & (10) are the equations of motion.

So far is as much as we can proceed without any approximations or numerical methods. But to reach further, we need these ways.

Now, I introduce some numerical values, and then I use a method to solve them numerically, using python programming.

Numerical values and initial conditions for case 1

$m_1 = 5kg$

$m_2 = 1kg$

$g = 9.81m/s^2$

$l = 1m$

$k = 100N/m$

$x(0) = 0m\,, \dot{x}(0) = 0m/s \quad \theta(0) = 0.5rad \approx 28.65°\,, \dot{\theta}(0) = 0rad/s$

Numerical solution for case 1

Numerical solutions to equations (9) & (10) are plotted in figures 2 & 3. Because there is no damping force in this case, the motions for x and θ are periodic. By numerical methods, we can obtain the amplitude of x and θ:

$$x_{max} = 0.1069m$$

$$\theta_{max} = 0.5000rad$$

Figure 2 _ x(t) for case 1

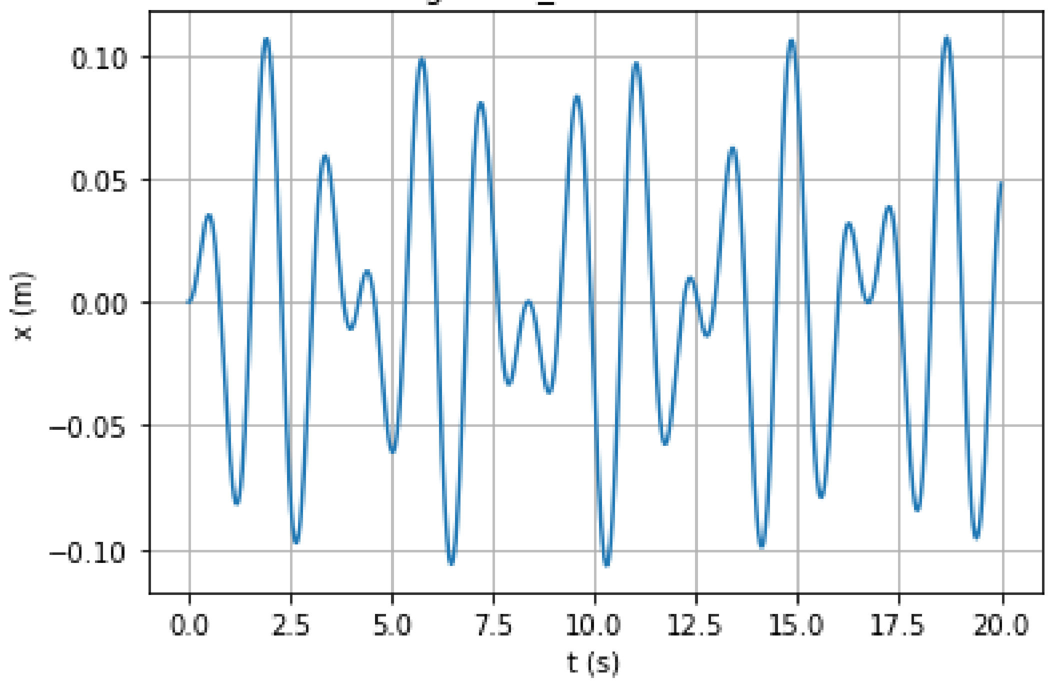
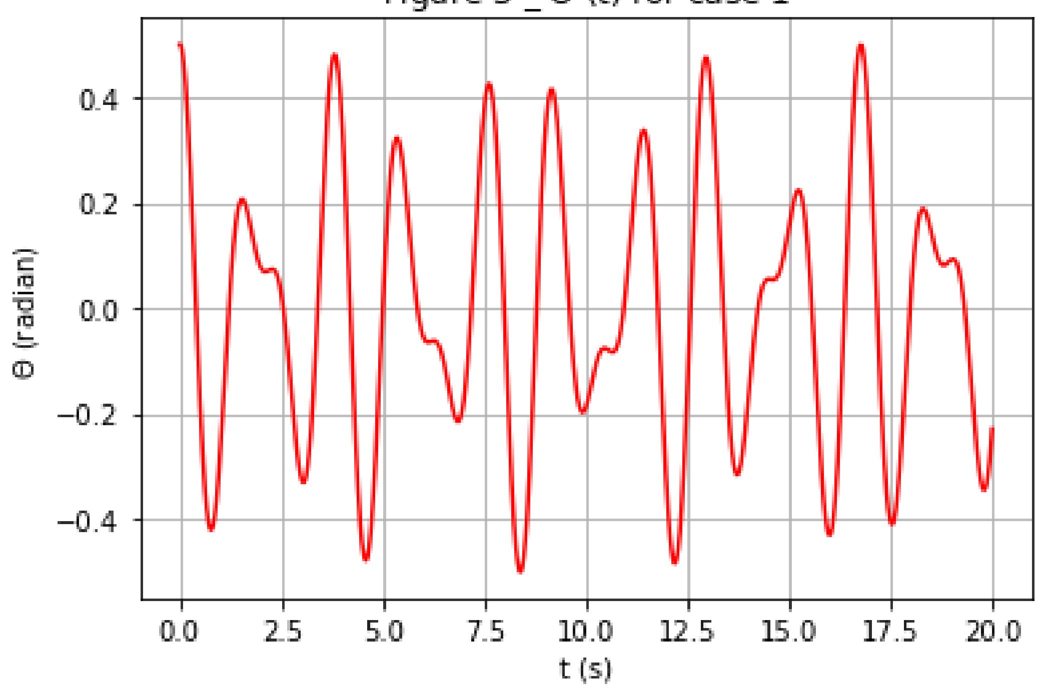


Figure 3 _ Θ (t) for case 1



Case 2; Cart and Pendulum problem in presence of drag force (proportional to velocity) and also sinusoidal driving force

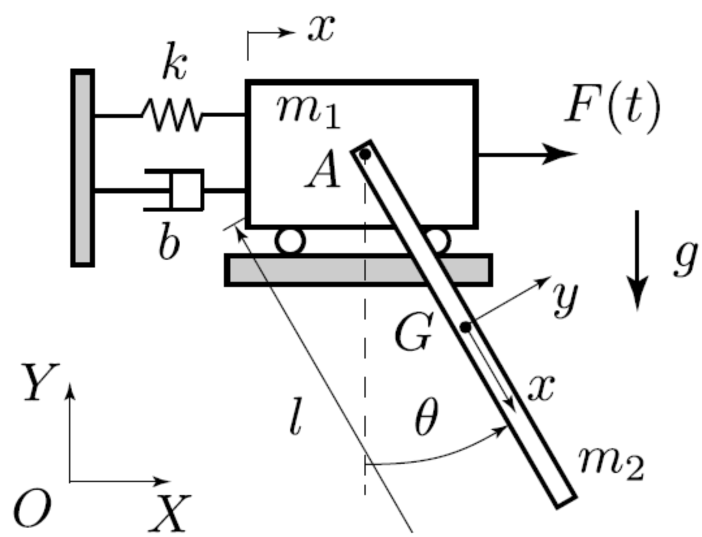


Figure 4 _ diagram of case 2

As is shown in figure 4, this case is similar to the last case, except the fact that there is a dashpot the exerts a drag force proportional to velocity $(-b\dot{x})$ and we apply a sinusoidal driving force $(F(t) = F_0 \sin \omega t)$.

In presence of this forces, we cannot apply equation (8). We have to take generalized forces into account. So, we use:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = Q_k \qquad (11)$$

I which Q_k is the generalized force, and q_k is generalized coordinate. Now, we can separate Q_k into two parts, first part related to conservative forces (which can be written as $-\frac{\partial V}{\partial q_k}$, and second part related to nonconservative forces:

$$Q_k = -\frac{\partial V}{\partial q_k} + Q'_k \qquad (12)$$

Recall the definition of lagrangian

$$L = T - V \qquad (13)$$

Using equations (11) & (12) & (13), we obtain:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = Q'_k \qquad (14)$$

Now I can write:

$$Q'_x = -b\dot{x} + F_0 \sin \omega t \qquad (15)$$

$$Q'_\theta = 0 \qquad (16)$$

Lagrangian is same as case one, because nothing about potential or kinetic energy has changed. So lagrangian can be written as equation (7), and by using equations (7), (14), (15) & (16) and taking derivatives, we obtain equations of motion as follow:

$$(m_1 + m_2)\ddot{x} + \frac{1}{2}m_2 l\ddot{\theta}\cos\theta - \frac{1}{2}m_2 l\dot{\theta}^2 \sin\theta + kx = -b\dot{x} + F_0 \sin \omega t \qquad (17)$$

$$\frac{1}{3}l\ddot{\theta} + \frac{1}{2}\ddot{x}\cos\theta + \frac{1}{2}g\sin\theta = 0 \qquad (18)$$

After introducing some numerical values, I will solve these couples second order differential equations numerically and provide the plots.

Numerical values and initial conditions for case 2

$m_1 = 5kg$

$m_2 = 1kg$

$g = 9.81m/s^2$

$l = 1m$

$k = 100N/m$

$b = 0.8kg/s$

$F_0 = 1N$

$\omega = 2rad/s$

$x(0) = 0m\,, \dot{x}(0) = 0m/s \quad \theta(0) = 0.5rad \approx 28.65°\,, \dot{\theta}(0) = 0rad/s$

Numerical solution for case 2

Numerical solutions to equations (17) & (18) are plotted in figures 5 & 6. Because there is a damping force, in addition to a sinusoidal driving force, we cannot define a well-defined amplitude for neither x nor θ. Bur from figures 5 & 6, we can see the damping effect of these forces.

Figure 5 _ x(t) for case 2

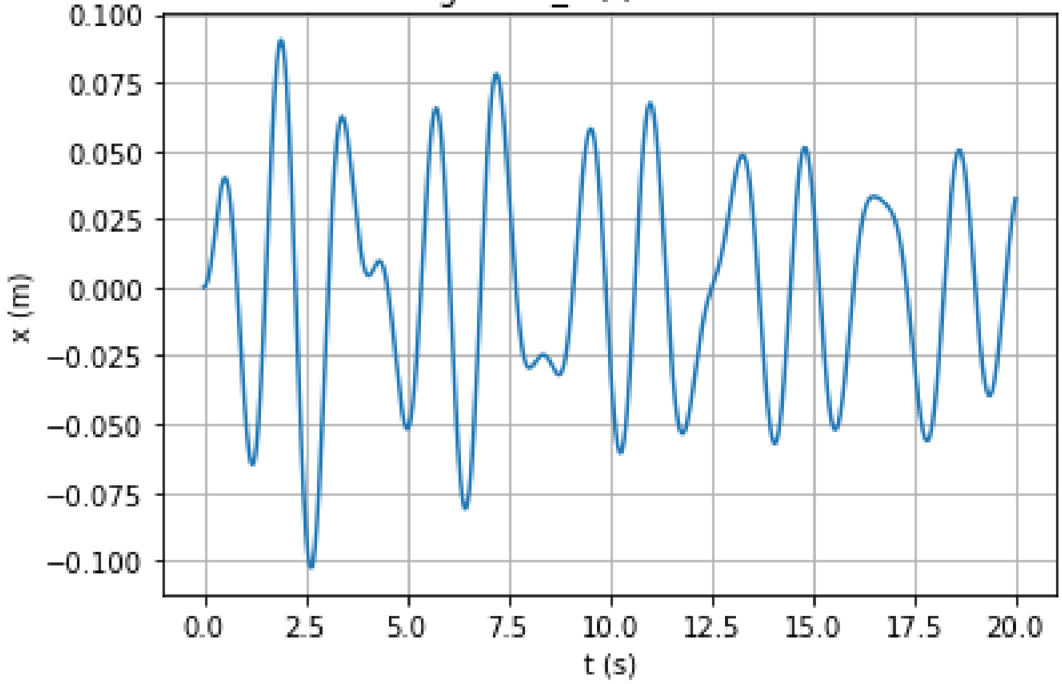
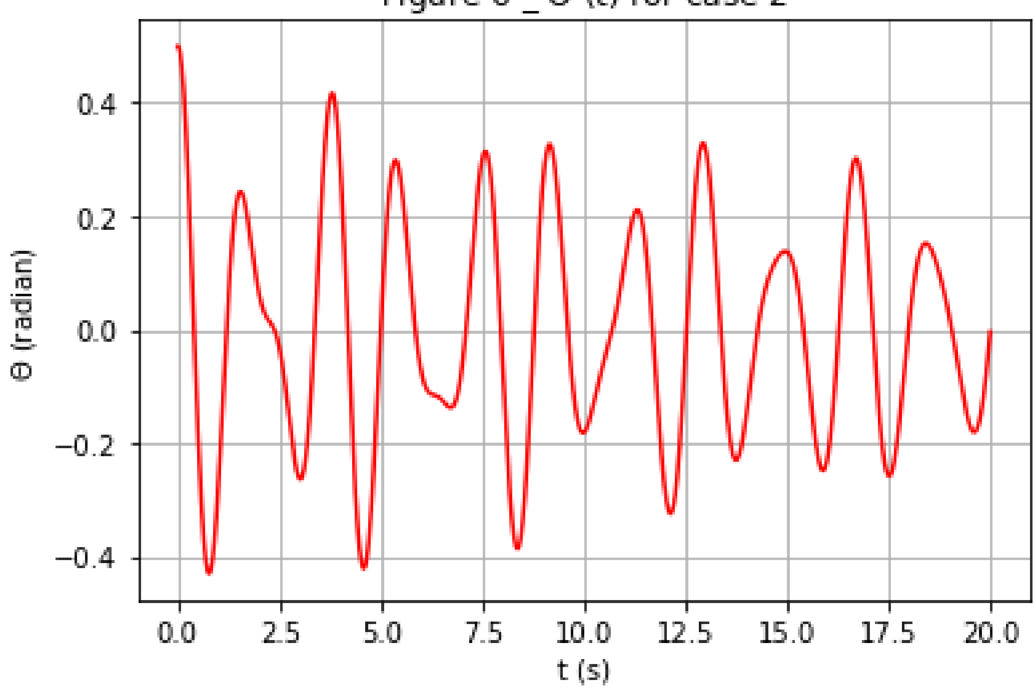
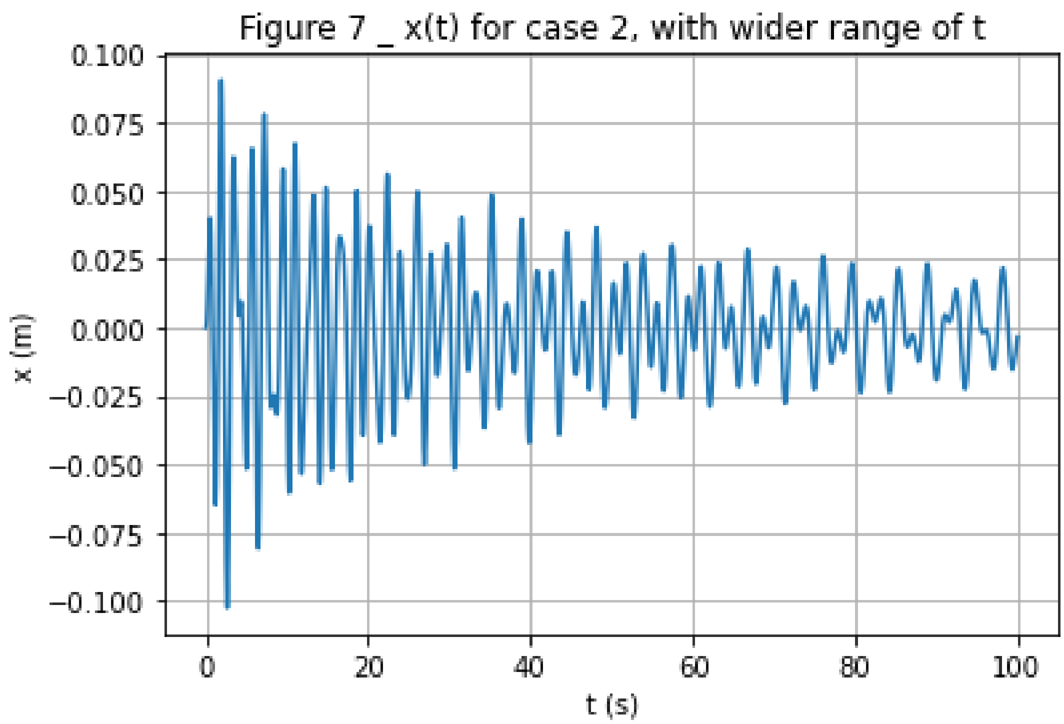
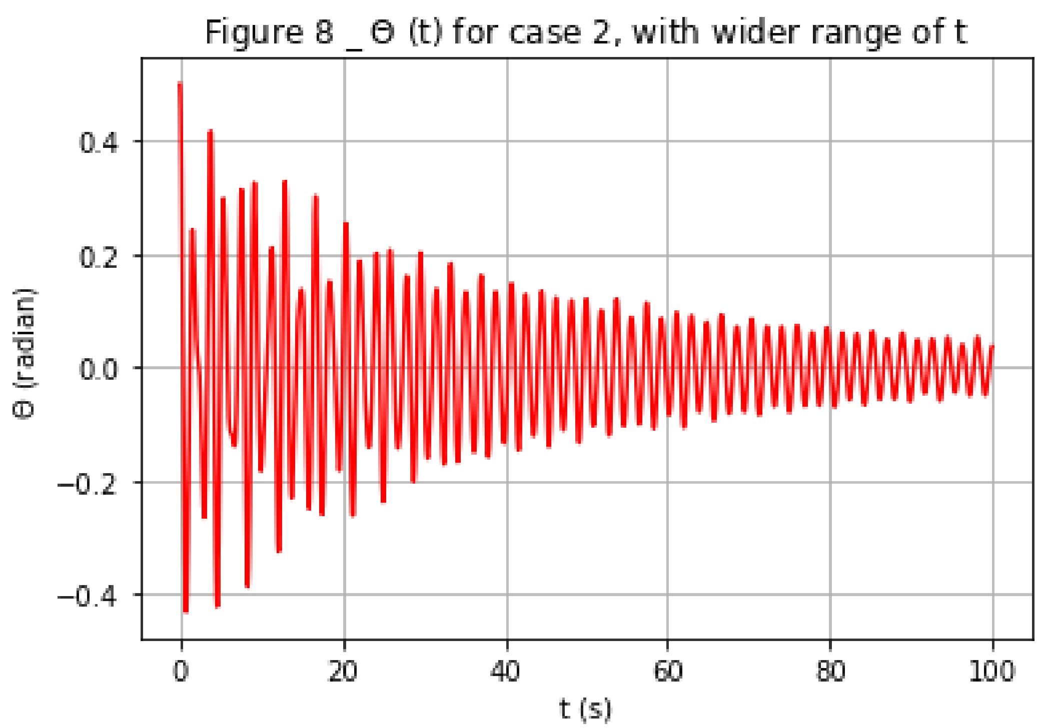


Figure 6 _ Θ (t) for case 2



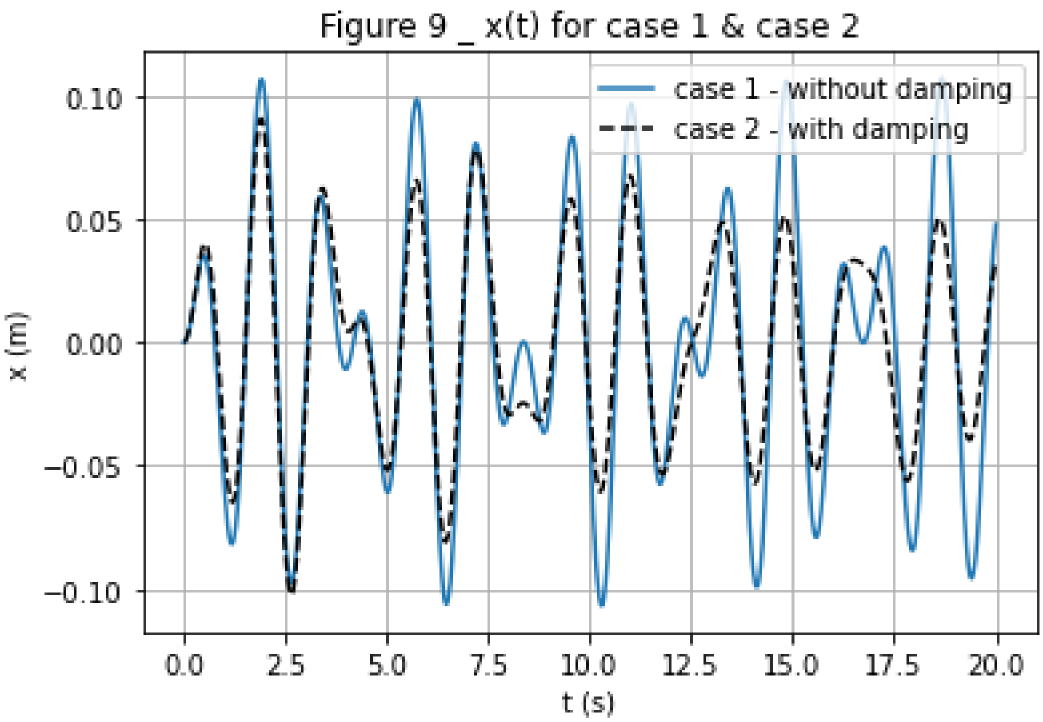
We can plot x(t) & θ(t) in wider range to see the damping effect. It has been done in figures 7 & 8.

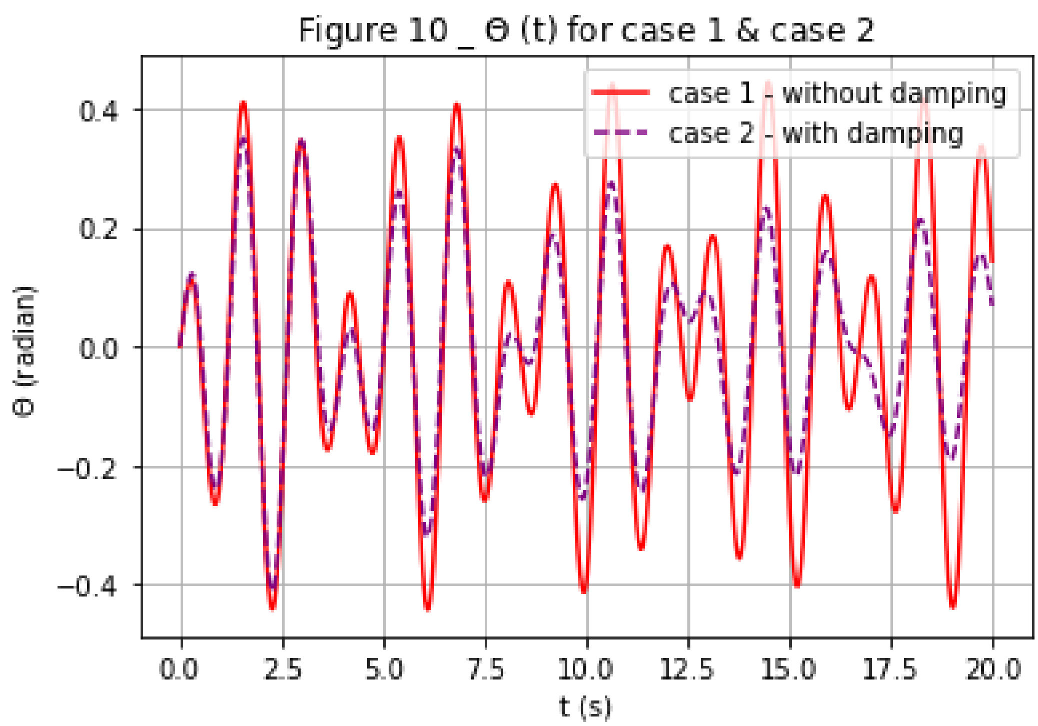




Comparison of case 1 and case 2

We can compare two cases more easily by plotting them in the same plot. This has been done in figures 9 & 10.





It is obvious from figures 9 & 10 that damping force reduce the amplitude of x and θ.

Conclusion

In this project, I use lagrange equations of motion to solve the problem of cart and pendulum, in two different cases, the former is in absence of any frictional or damping or driving force, the latter is in presence of them. We have seen how much these additional forced changed to solution, and ultimately damped the initial solution. If I wanted to examine the problem with Newton equations of motion, it would be much more complicated.

But these are not the only forces that can be included; there can be friction in pivot A (figure 1 or 4), friction of the railway or air resistance on the rod. We could also use a nonsymmetrical 3-D rigid body instead of a uniform rod, and then the problem would be much more complicated because there would be nutation for the rigid body.

References

1. Classical dynamics of particles and systems (2003) – Stephen T. Thornton, Jerry B. Marion
2. Mechanics (1971) – Symon K. R.
3. M.I.T Open Courseware – 2.003SC / 1.053J Engineering Dynamics – Recitation 8 notes