Analytical Mechanics 1 Project

Corrections and examining of a real projectile's trajectory

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Introduction

We can start to analyze a projectile's trajectory by writing its equations of motion; then trying to solve them. But solving them analytically may not always be possible. Even writing the "exact" equations of motion is not usually possible, there are lots of different factors that may affect the problem, and mainly for determining the motion analytically, we neglect them, or by not neglecting them, we choose to use approximation methods, or solving the problem numerically.

The problem of examining a projectile is a typical problem in all "Analytical Mechanics" textbooks and courses. Here I start from a basic problem that there is no air resistance, no rotation of Earth, no variation of gravitational field, no effects of being a rigid body, …; then I will focus on the air resistance and solve the problem(numerically) by assuming the air density is constant. At the end, I will discard the last simplifying assumption and take variations of air density with altitude into account.

Numerical Values

Before starting the calculations, I should say I am examining a World War I ball cannon, and all of the calculations and plots base on the following numerical values (I consider the cannon a uniform sphere):

Name: M114 Howitzer (projectile: HE M107)

Radius (r): 77.5 mm (155 mm caliber)

Mass of the projectile(m): 43 kg

Density of the air (ρ): 1.225 kg/m^3

Viscosity of the air at 20^{o}C (η): 1.8*10^{-5} kg/m.s

Maximum muzzle velocity (v): 564 m/s

Case 1 ; neglecting air resistance

I shall start with the basic problem. There is a projectile with initial velocities v_{0x}, v_{0y}, starts its motion from the origin, and we neglect the air resistance and rotation of the Earth; and also assume that the gravitational field (g) is constant. The equations of motion are:

$$\ddot{x} = 0 \qquad (1)$$

$$\ddot{y} = -g \qquad (2)$$

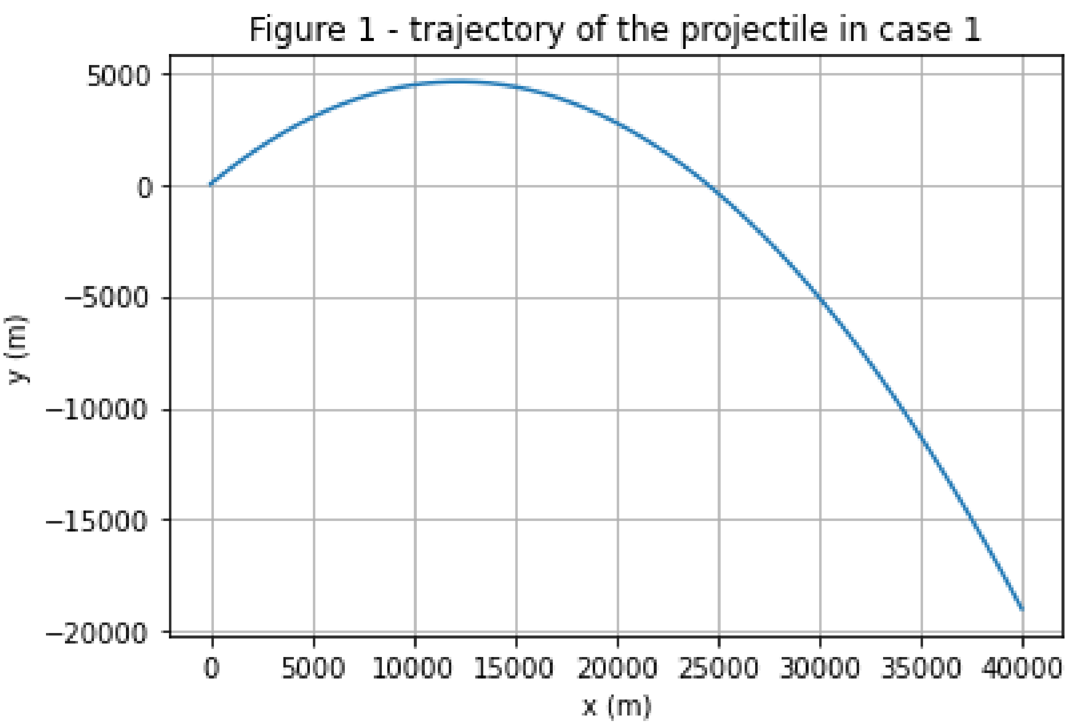
Solving these differential equations and applying initial conditions:

$$x = v_{0x} t \qquad (3)$$

$$y = -\frac{1}{2} g t^2 + v_{0y} t \qquad (4)$$

Also, we can calculate the equation of the path:

$$y(x) = \frac{v_{0y}}{v_{0x}} x - \frac{1}{2} \frac{g}{{v_{0x}}^2} x^2 \qquad (5)$$



Even though for this case the range can be evaluated analytically, I prefer to use the numerical method from now on. I use "Newton's method of finding roots" for evaluating the range and flight time (time elapsed before reaching the same altitude). The y(x) is plotted in figure 1. The range and flight time are:

$$R_1 = 24489.8\,m \quad \& \quad T_1 = 61.22\,s$$

Case 2 ; considering air resistance, assuming the air density to be constant

Now I want to consider the air resistance. The drag force acting on a projectile depends on its velocity, and can be proportional to the velocity, or its square. To determine this, we can use "Reynolds Number"

$$R = \frac{2r\rho v}{\eta} \qquad (6)$$

for a sphere of radius r, moving in a fluid with density ρ and viscosity η with velocity v. If $R<1$, then the drag force is proportional to first power of velocity, and if $R>1$, the drag force is proportional to second power of velocity. By using the numerical values mentioned earlier, $R \approx 5.95*10^6$. Hence, $R>>1$ and with good approximation, the drag force is proportional to the second power of velocity.

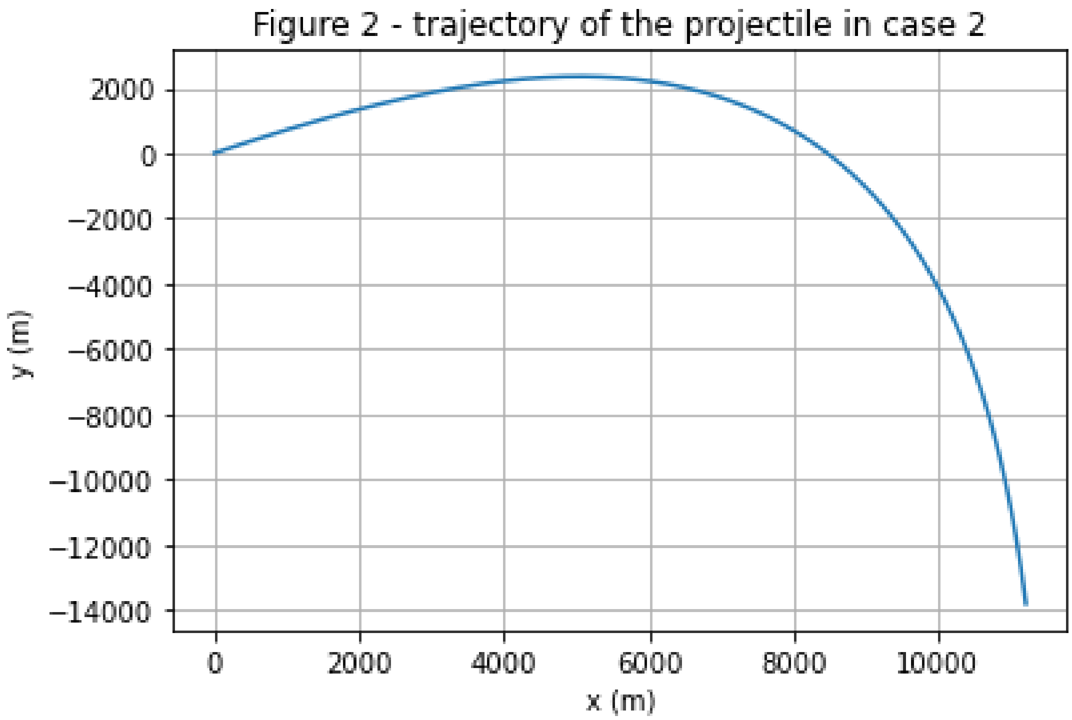
$$\overrightarrow{F_d} = -\frac{1}{4}\rho A v^2 \hat{v} = -\frac{1}{4}\pi r^2 \rho v \vec{v} \qquad (7)$$

Hence these are the equations of motion:

$$m\ddot{x} = -\frac{1}{4}\pi r^2 \rho \sqrt{\dot{x}^2 + \dot{y}^2}\,\dot{x} \qquad (8)$$

$$m\ddot{y} = -mg - \frac{1}{4}\pi r^2 \rho \sqrt{\dot{x}^2 + \dot{y}^2}\,\dot{y} \qquad (9)$$

These pare of equations (8 & 9) cannot be solved analytically. I use python to evaluate x, y at each time t and the following plots. The initial conditions are: v_{0x}=400m/s, v_{0y}=300m/s, and the projectile starts its motion from the origin. y(x) is plotted in figure 2.



The range and flight time can be evaluated by the "Newton's method of finding roots" and their numerical value are:

$$R_2 = 8465.9\,m \quad \& \quad T_2 = 43.18\,s$$

these are significantly smaller than the values when air resistance was absent!

Case 3 ; considering air resistance, assuming the air density is varying with altitude

Now I want to take variations of air density with altitude into account. Here I use references (4) & (5) to carry out the following results.

Inside troposphere, temperature varies linearly with altitude:

$$T = T_0 - Lh \qquad (10)$$

T_0 is sea level standard temperature, 288.15K, L is temperature lapse rate, 0.0065K/m and h is the altitude. Pressure can be evaluated as

$$P = P_0\left(1 - \frac{Lh}{T_0}\right)^{\frac{gM}{RL}} \qquad (11)$$

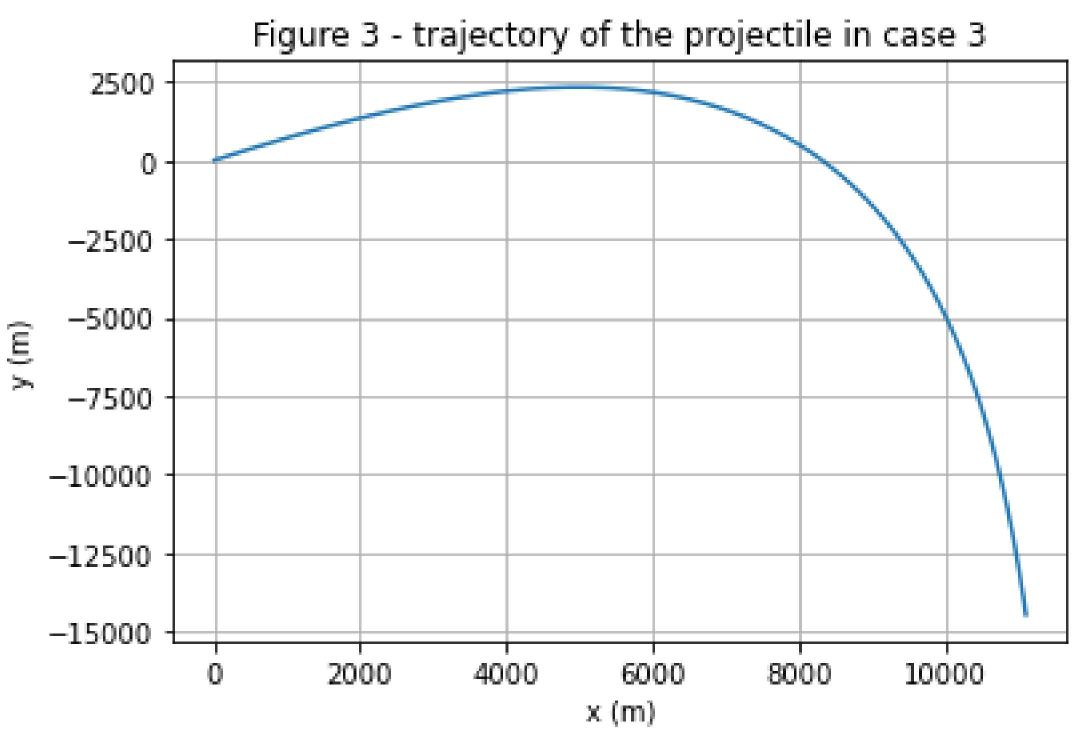
Where P_0 is sea level standard atmosphere pressure, 101325Pa, g is gravitational field on the Earth surface, 9.81m/s^2, M is molar mass of dry air, 0.0289652kg/mol. Using ideal gas law, we obtain air density as

$$\rho = \frac{PM}{RT} = \frac{P_0 M}{RT_0}\left(1 - \frac{Lh}{T_0}\right)^{\frac{gM}{RL}-1} \qquad (12)$$

Now we can use equations (8), (9), (12) and obtain the equations of motion:

$$m\ddot{x} = -\frac{1}{4}\pi r^2 \frac{P_0 M}{RT_0}\left(1 - \frac{Ly}{T_0}\right)^{\frac{gM}{RL}-1} \sqrt{\dot{x}^2 + \dot{y}^2}\,\dot{x} \qquad (13)$$

$$m\ddot{y} = -mg - \frac{1}{4}\pi r^2 \frac{P_0 M}{RT_0}\left(1 - \frac{Ly}{T_0}\right)^{\frac{gM}{RL}-1} \sqrt{\dot{x}^2 + \dot{y}^2}\,\dot{y} \qquad (14)$$



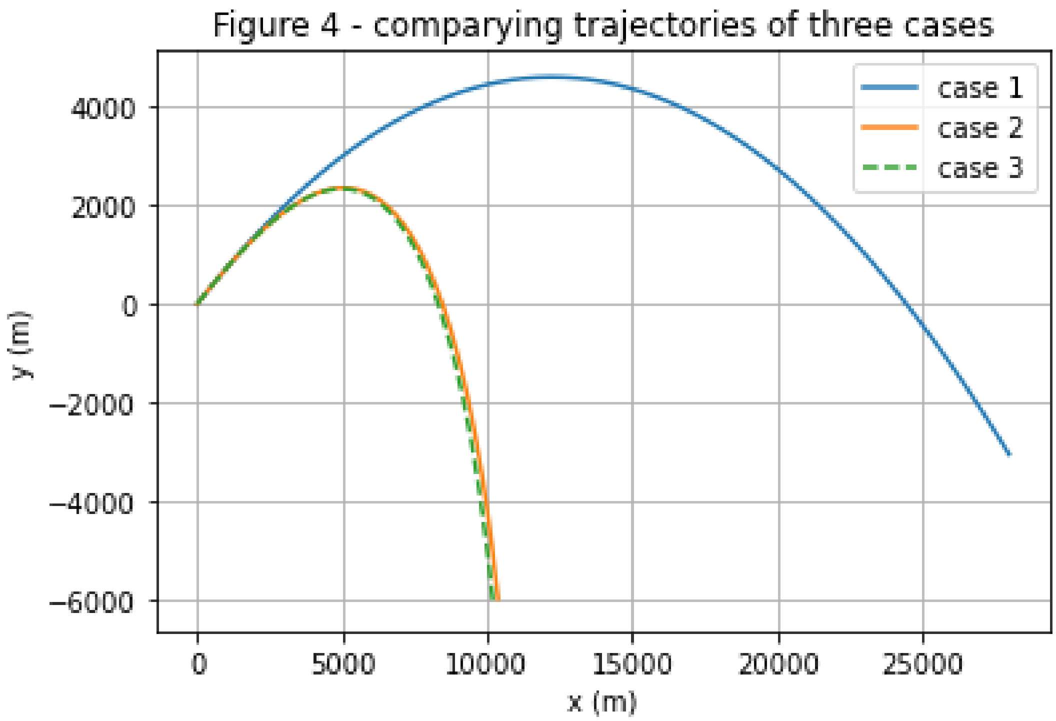
y(x) is plotted in figure 3. In this case the range and flight time are:

$$R_3 = 8315.4\,m \quad \& \quad T_3 = 42.97\,s$$

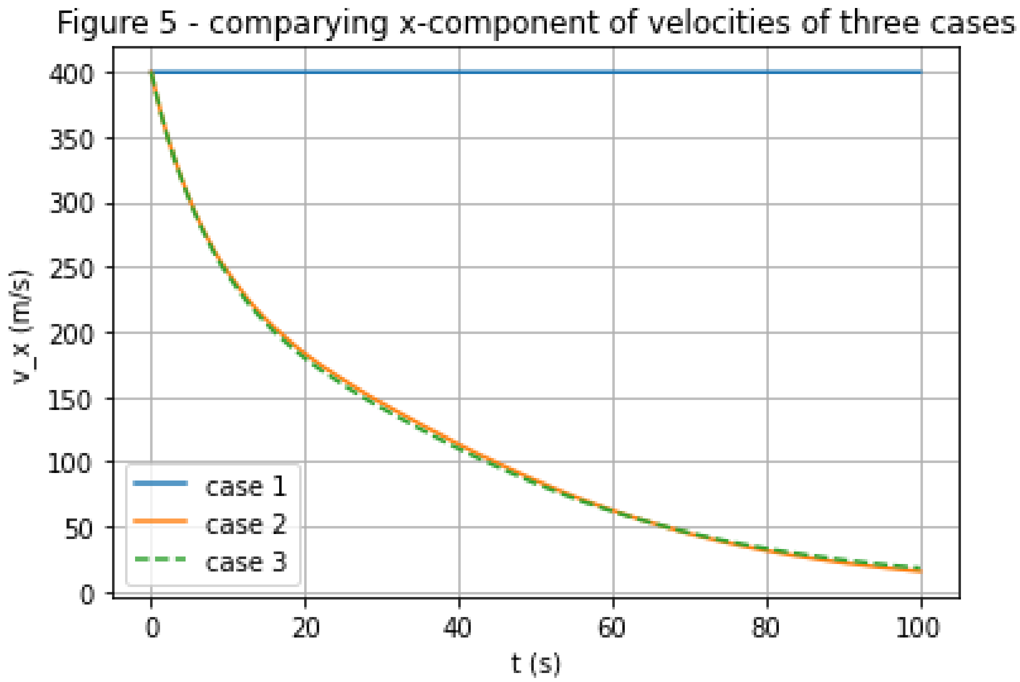
These are slightly smaller than the values when the air density was assumed to be constant.

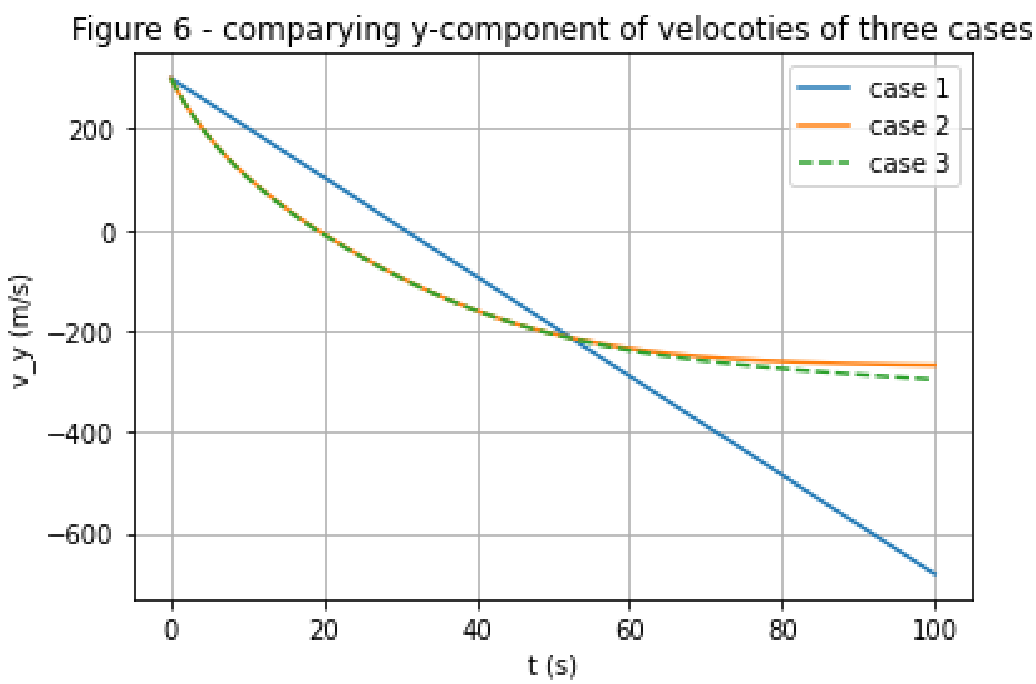
Comparing all three cases with each other

Now I can plot y(x) plots in one plot to compare them more easily. It's done it figure 4.



In addition, we can plot velocities (with respect to time) in both directions in all three cases to compare them too. It's done in figures 5 & 6.





By noting figure 6, we can conclude that the terminal velocity in “case 3” is more than in “case 2” (it’s more negative, but its magnitude is greater).

Conclusion

In each step, I discard more simplifying assumptions and the solution became more precise. We can observe that when we enter the effect of air resistance, the range became $R_2/R_1 \approx 0.35$ of the initial range! And the flight time became $T_2/T_1 \approx 0.71$ of the initial value. These are significant changes. In transmission from case 2 to case 3, the changes were not this much significant; the range reduced about 1.8% and flight time reduces about 0.5%.

There are lots of other effects that should be taken into account if we want to be more precise, such as rotation of the Earth, being a rigid body and exact shape of the projectile, turbulence in the air surrounding the projectile, … .

References

1- Classical dynamics of particles and systems (2003) – Stephen T. Thornton, Jerry B. Marion
2- Mechanics (1971) – Symon K. R.
3- Projectile motion with air resistance quadratic in speed (Article in American Journal of Physics – 1977) – G. W. Parker
4- Air density and density altitude (2009) – Shelquist, R. Equations
5- U.S Standard Atmosphere (1976) – page 12

**Notice* All of the codes that were used in the project will be attached to the email.*