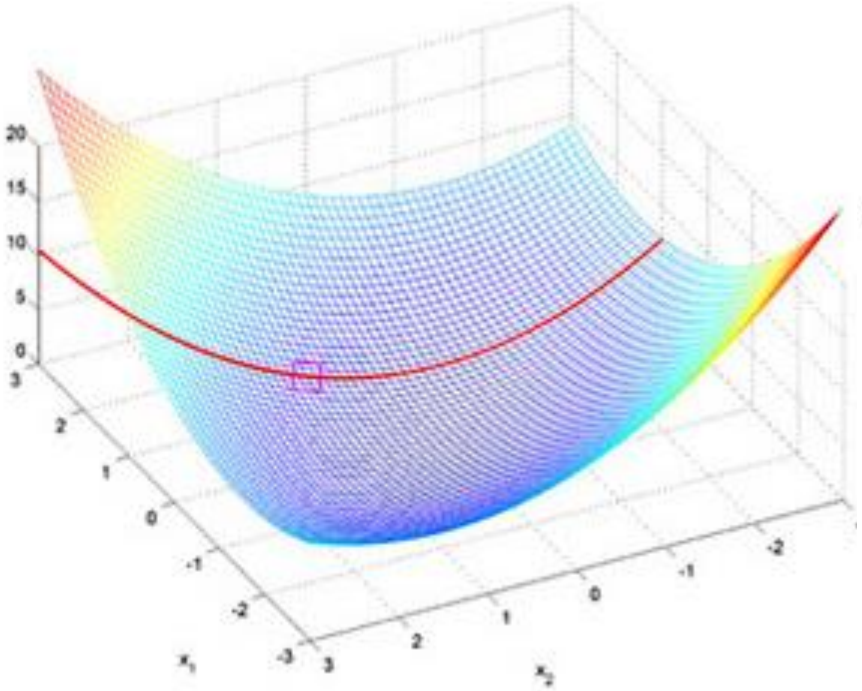
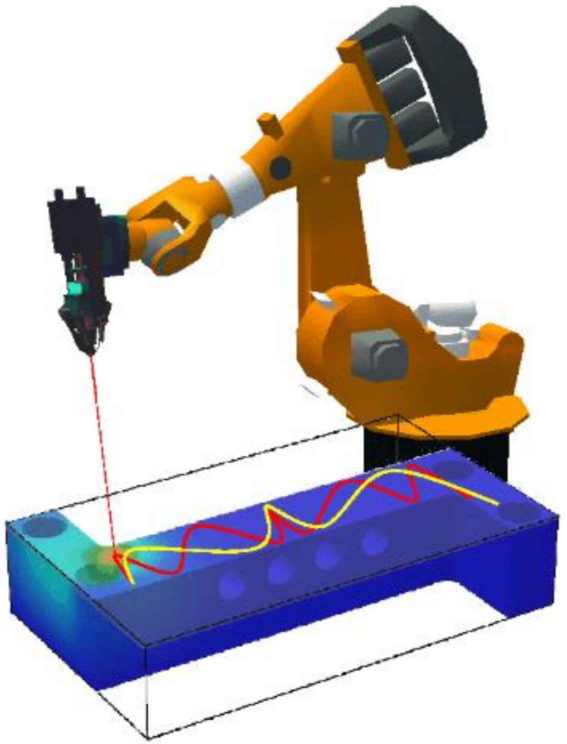
# MEGR 3090/7090/8090: Advanced Optimal Control



$$\boxed{V_n(\mathbf{x}_n)=\min_{\{\mathbf{u}_n,\mathbf{u}_{n+1},\cdots,\mathbf{u}_{N-1}\}}\left[\frac{1}{2}\sum_{k=n}^{N-1}\left(\mathbf{x}_k^T\mathbf{Q}_k\mathbf{x}_k+\mathbf{u}_k^T\mathbf{R}\mathbf{u}_k\right)+\frac{1}{2}\mathbf{x}_N^T\mathbf{Q}_N\mathbf{x}_N\right]}$$

$$\begin{aligned}
V_n(\mathbf{x}_n) &= \min_{\{\mathbf{u}_n,\mathbf{u}_{n+1},\cdots,\mathbf{u}_{N-1}\}}\left[\frac{1}{2}\sum_{k=n}^{N-1}\left(\mathbf{x}_k^T\mathbf{Q}_k\mathbf{x}_k+\mathbf{u}_k^T\mathbf{R}\mathbf{u}_k\right)+\frac{1}{2}\mathbf{x}_N^T\mathbf{Q}_N\mathbf{x}_N\right] \\
&= \min_{\mathbf{u}_n}\left[\frac{1}{2}\left(\mathbf{x}_n^T\mathbf{Q}_n\mathbf{x}_n+\mathbf{u}_n^T\mathbf{R}\mathbf{u}_n\right)+\underbrace{\min_{\{\mathbf{u}_{n+1},\cdots,\mathbf{u}_{N-1}\}}\left[\frac{1}{2}\sum_{k=n+1}^{N-1}\left(\mathbf{x}_k^T\mathbf{Q}_k\mathbf{x}_k+\mathbf{u}_k^T\mathbf{R}\mathbf{u}_k\right)+\frac{1}{2}\mathbf{x}_N^T\mathbf{Q}_N\mathbf{x}_N\right]}_{V_{n+1}(\mathbf{x}_{n+1})}\right] \\
&= \min_{\mathbf{u}_n}\left[\frac{1}{2}\left(\mathbf{x}_n^T\mathbf{Q}_n\mathbf{x}_n+\mathbf{u}_n^T\mathbf{R}\mathbf{u}_n\right)+V_{n+1}(\mathbf{x}_{n+1})\right]
\end{aligned}$$

$$\boxed{V_n(\mathbf{x}_n)=\min_{\mathbf{u}_n}\left[\frac{1}{2}\left(\mathbf{x}_n^T\mathbf{Q}_n\mathbf{x}_n+\mathbf{u}_n^T\mathbf{R}\mathbf{u}_n\right)+V_{n+1}(\mathbf{x}_{n+1})\right]}$$



**Lecture 2**

**August 24, 2017**

# Optimal Control – General Continuous Time Framework

Whether the control trajectory is optimized offline or online, every continuous time optimal control problem will involve the following general framework:

$$u^*(t) = \arg\min_{u(t)} J(u(t); \mathbf{x}(0)) \quad \text{where} \quad \underbrace{J\big(u(t); \mathbf{x}(0)\big)}_{\text{Total cost}} = \int_0^{t_f} \underbrace{g(\mathbf{x}(t), u(t))dt}_{\text{Instantaneous cost}} + \underbrace{h\big(\mathbf{x}(t_f)\big)}_{\text{Terminal cost}})$$

**Subject to:**

The dynamic model: $\dot{\mathbf{x}} = f(\mathbf{x}, u)$

Point-wise control constraints: $u(t) \in U, \forall t: 0 \le t \le t_f$

Point-wise state constraints: $\mathbf{x}(t) \in X, \forall t: 0 \le t \le t_f$

Terminal state constraints: $\mathbf{x}(t_f) \in X_f$

# Optimal Control – General Continuous Time Framework

$$u(t) \in U \quad \leftarrow \text{a constraint set}$$

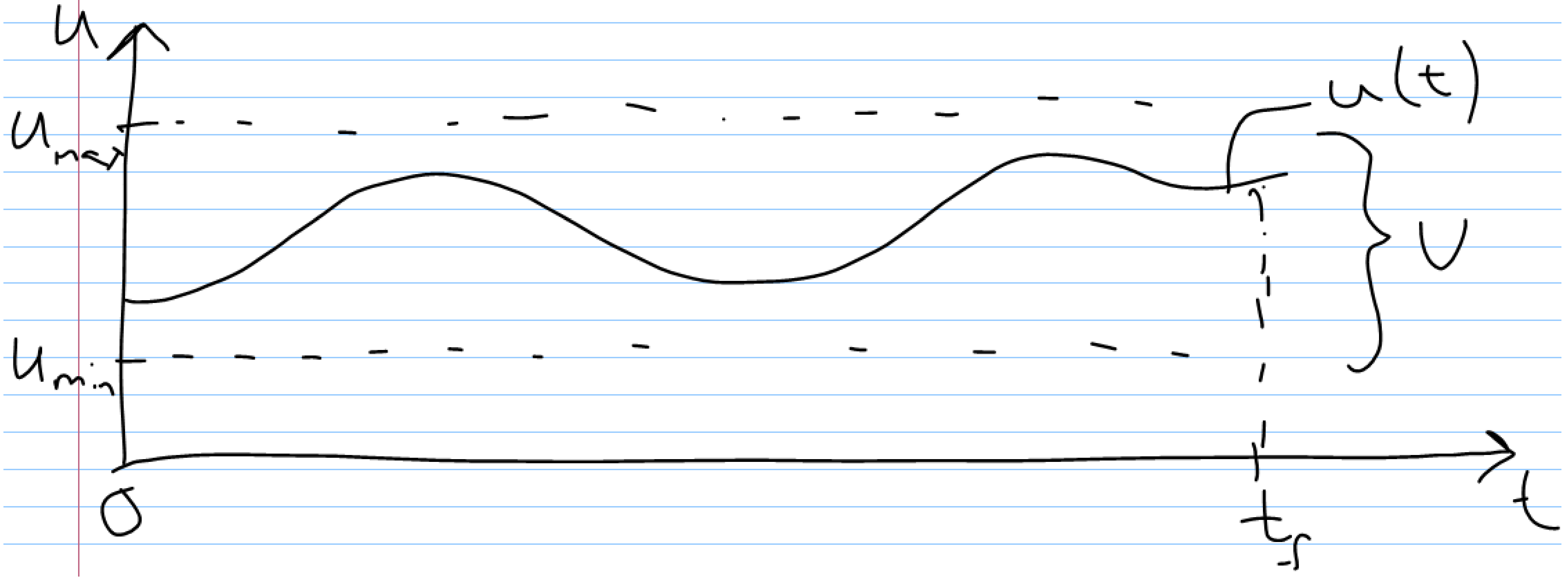
$$\forall t: 0 \le t \le t_f$$

control trajectory

contained in

# Optimal Control – General Continuous Time Framework

$$u_{min} \leq u(t) \leq u_{max}, \ \forall t: 0 \leq t \leq t_f$$



# Optimal Control – General Discrete Time Framework

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Whether the control trajectory is optimized offline or online, every continuous time optimal control problem will involve the following general framework:

$$\mathbf{u}^* = \arg\min_{\mathbf{u}} J(\mathbf{u};\mathbf{x}(0)) \quad \text{where} \quad \underbrace{J\big(\mathbf{u};\mathbf{x}(0)\big)}_{\text{Total cost}} = \sum_{i=0}^{N-1} \underbrace{g(\mathbf{x}(i), u(i))}_{\text{Stage cost}} + \underbrace{h(\mathbf{x}(N))}_{\text{Terminal cost}}$$

**Subject to:**

The dynamic model: $\mathbf{x}(i+1) = f(\mathbf{x}(i), u(i))$

Point-wise control constraints: $u(i) \in U, i = 0 \ldots N-1$

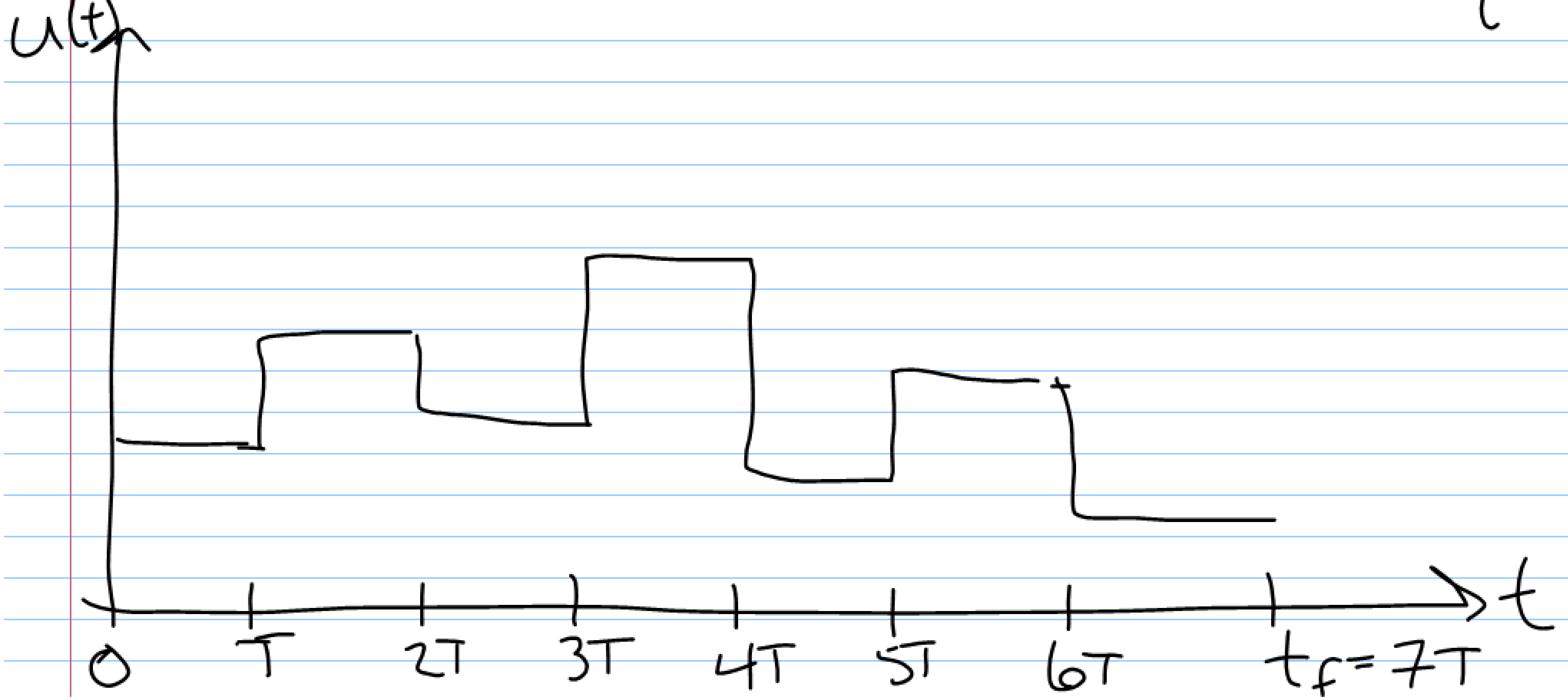
Point-wise state constraints: $\mathbf{x}(i) \in X, i = 0 \ldots N-1$

Terminal state constraints: $\mathbf{x}(N) \in X_f$

**Note:** $\mathbf{u} = [u(0) \quad \ldots \quad u(N-1)]^T$…Key point: $\mathbf{u}$ consists of a **finite number of design variables!**

# Optimal Control – General Discrete Time Framework

Idea of a discrete time control sequence:



# Common Ingredients in Optimal Control

**Decision variable/control input trajectory** – Specified by $\mathbf{u}$…note that $\mathbf{u}$ comprises the ***entire sequence*** of $u(t)$ between $t = 0$ and $t = t_f$.

**Objective function –** A function that is to be **minimized** or **maximized** over some **time window** (from 0 to $t_f$)

- When the objective function is to be minimized, it is called a *cost function*; when it is to be maximized, it's called a *fitness function* or *reward function* (or just "objective function")

**Initial condition –** The value of the system states at the beginning of the time window over which the objective function is evaluated (given by $\mathbf{x}(0)$)

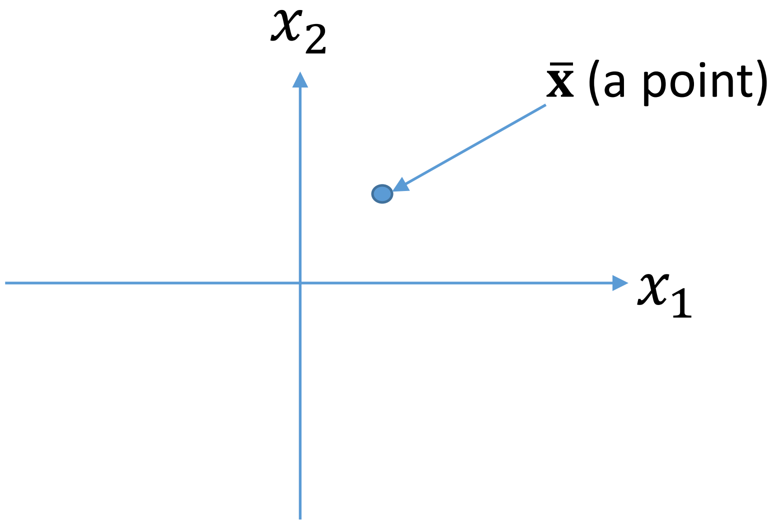
**Dynamic model –** A state space model that describes how the system evolves

**Constraints –** Limits on the states and control signals over the prescribed time window (when constraints are only imposed at time $t_f$, they are referred to as ***terminal constraints***

# Points, Sets, Equality Constraints, and Inequality Constraints

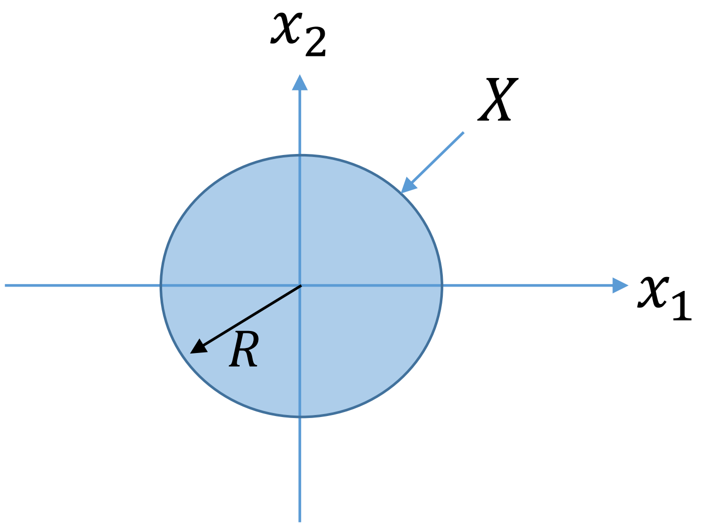
**Equality** constraints are equivalent to **point constraints**

**Example:** $\mathbf{x}(t) = \bar{\mathbf{x}} \Leftrightarrow \mathbf{x}(t) \in X$, where $X$ is shown at right




**Inequality** constraints are equivalent to **set constraints**, where **closed sets** are almost always used to describe the constraint (you'll see why soon)

**Example:** $\|\mathbf{x}(t)\| \leq R \Leftrightarrow \mathbf{x}(t) \in X$, where $X$ is shown at right



# Points, Sets, Equality Constraints, and Inequality Constraints

Example set constraints:

$-2 \le u \le 2$


$u \in U$

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$\|\underline{x}\|$ denotes the norm of the vector $\underline{x}$

... It is a measure of the size of $\underline{x}$

2 norm assumed: $\|\underline{x}\| = \left[x_1^2 + x_2^2 + \dots + x_n^2\right]^{1/2}$

# Admissibility and Feasibility

A control trajectory ($\mathbf{u}$ in discrete time or $u(t)$ in continuous time) is said to be ***admissible*** if it satisfies all constraints. Mathematically, the trajectory is admissible if $u(t) \in U, \forall t: 0 \leq t \leq t_f$.

Similarly, a state trajectory ($\mathbf{x}$ in discrete time or $\mathbf{x}(t)$ in continuous time) is said to be ***admissible*** if it satisfies all constraints. Mathematically, the trajectory is admissible if $\mathbf{x}(t) \in X, \forall t: 0 \leq t \leq t_f$.

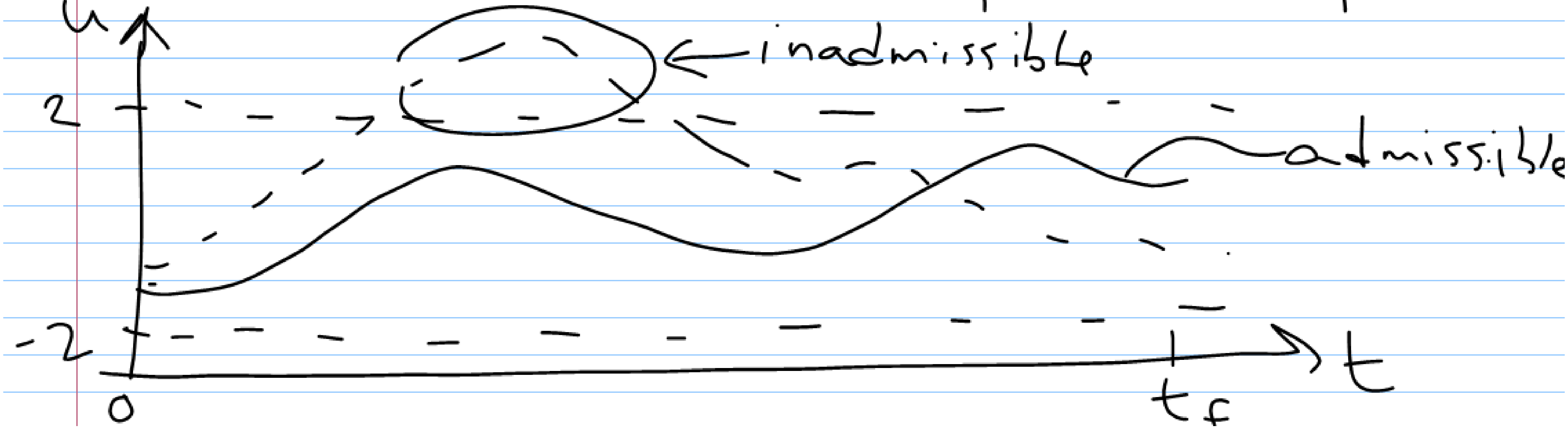
An optimization problem is said to be ***feasible*** if there exists at least one admissible control trajectory that leads to satisfaction of all constraints.

# Admissibility and Feasibility

Constraint: $-2 \le u(t) \le 2, \quad 0 \le t \le t_f$

$$U = \{u : -2 \le u(t) \le 2\}$$

$$\Rightarrow u(t) \in U, \quad 0 \le t \le t_f$$



# Setting up Optimal Control Problems – Example

Suppose that a ballistic missile is modeled by: $\dot{\mathbf{x}} = f(\mathbf{x}, u)$

where $\mathbf{x} \in \mathbb{R}^3$ is the missile position and $u \in \mathbb{R}$ is the thrust input.

Suppose that we want the missile to reach a target position denoted by $\mathbf{r}$ at time $t_f$. In particular, we want to minimize the distance between the missile and $\mathbf{r}$ at that time. Furthermore, the missile is useless if its position at time $t_f$ is outside a sphere of radius $R$ around point $\mathbf{r}$. We don't care about the missile's position at any other time.

Choose a cost function and constraints that capture the requirements above.

# Setting up Optimal Control Problems – Example

$\underline{x} \in \mathbb{R}^n$ : $\underline{x}$ is an n-dimensional vector of real numbers

Ex. 1: $\underline{x}$ = missile pos., $u$ = thrust

$\underline{r}$ = target pos. at $t = t_f$

$$\dot{\underline{x}} = f(\underline{x}, u)$$

Generic cost functional: $J(u(t), \underline{x}(0)) = \cancelto{0}{\int_0^{t_f} g(\underline{x}(t), u(t))\,dt} + h(\underline{x}(t_f))$

# Setting up Optimal Control Problems – Example

Minimize $J(u(t), \underline{x}(0)) = \|\underline{x}(t_f) - \underline{r}\|$

subj. to: $\dot{\underline{x}} = f(\underline{x}, u)$

$\|\underline{x}(t_f) - \underline{r}\| \le R$,

$\Uparrow$

$\Rightarrow \underline{x}(t_f) \in X$, where $X = \{\underline{x} : \|\underline{x} - \underline{r}\| \le R\}$

Suppose that $\exists$, $u(t)$ s.t. $\underline{x}(t_f) \in X$

$\uparrow$

there exists

# Setting up Optimal Control Problems – Example

$\Rightarrow$ The value of $\underline{x}(t_f)$ obtained by minimizing $\|\underline{x}(t_f) - \underline{r}\|$ will lie in $X$

$\Rightarrow$ $\underline{x}(t_f) \in X$ is unnecessary

# Setting up Optimal Control Problems – Example (Kirk Ex. 2.2-1, modified)

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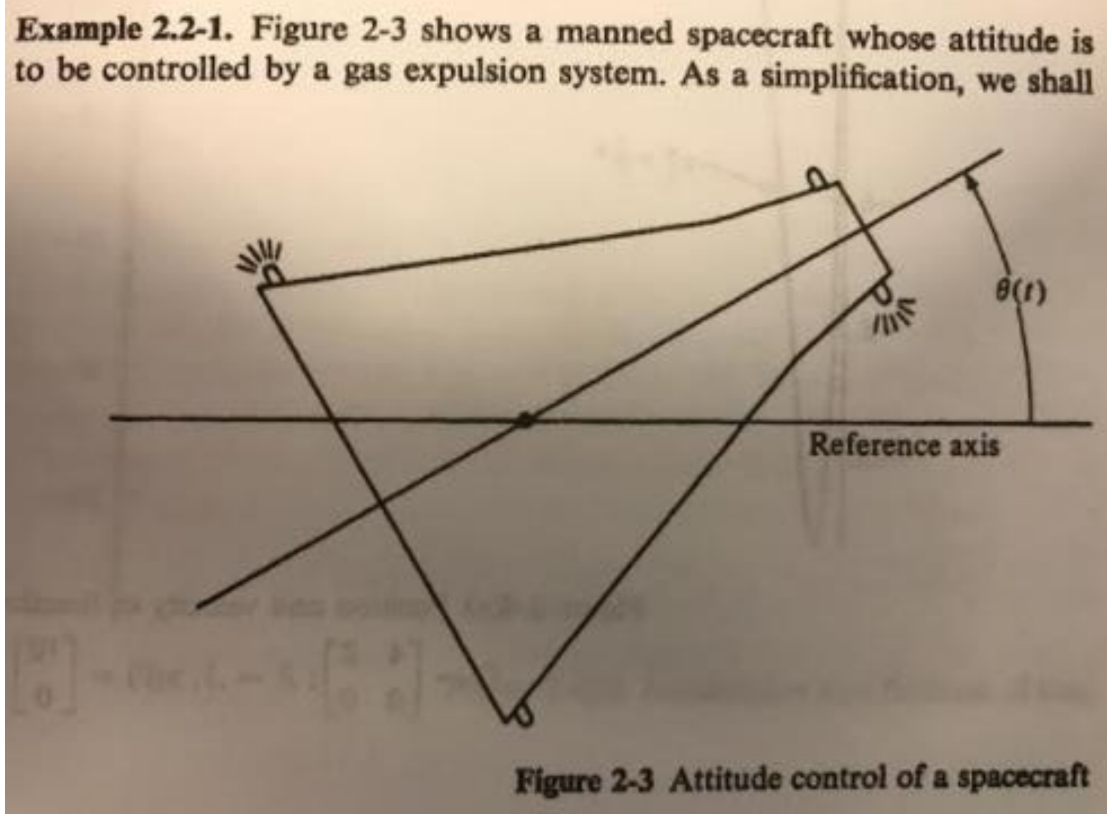
Suppose that a satellite's rotational position ($\theta$) and velocity ($\omega$) are modeled by:

$$\dot{\theta} = \omega$$

$$\dot{\omega} = \frac{1}{J} u$$

$u$ = control torque



**Example 2.2-1.** Figure 2-3 shows a manned spacecraft whose attitude is to be controlled by a gas expulsion system. As a simplification, we shall



**Figure 2-3** Attitude control of a spacecraft

Suppose the control objective is to keep the rotational position, velocity, and control signal as close to zero as possible, from time $t = 0$ to $t = \infty$ (i.e., for all time), while respecting the saturation limit on the controller, which requires that $|u(t)| \leq u_{max}, \forall t \geq 0$. Choose a cost function and constraints that capture these requirements.

# Setting up Optimal Control Problems – Example (Kirk Ex. 2.2-1, modified)

Ex. 2 : $\dot{\theta} = \omega$, $\dot{\omega} = \frac{1}{J} u$, $u$ = control torque

want $\theta$, $\omega$, $u$ small $\forall t \geq 0$

$$\left.\begin{aligned} \underline{x} &= \begin{bmatrix} \theta & \omega \end{bmatrix}^T \\ \dot{\underline{x}} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix} u \end{aligned}\right\} \text{model}$$

# Setting up Optimal Control Problems – Example (Kirk Ex. 2.2-1, modified)

Minimize: $J(u(t); \underline{x}(0)) = \int_0^\infty \left[\underline{x}^T(t) Q \underline{x}(t) + R u(t)^2\right] dt$

pos. def.

Subj. to: model

$|u(t)| \leq u_{max} \quad \forall\, t \geq 0$

# Setting up Optimal Control Problems – Example

Suppose that a system's dynamics are given by $\dot{\mathbf{x}} = f(\mathbf{x}, u)$, and our goal is to find a control trajectory, $u(t)$, that transfers the system from an initial state, $\mathbf{x_0}$, to a target state, $\mathbf{x_t}$, as quickly as possible, under the condition that $\|u(t)\| \leq u_{max}$ at all times. Choose a cost function and constraints that capture these requirements.

# Setting up Optimal Control Problems – Example

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Ex. 3 : $\dot{\underline{x}} = f(\underline{x}, u) \leftarrow$ model

Total time given by $\int_0^{t_f} 1\, dt \; (= t_f)$

Minimize $J(u(t); \underline{x}_0) = \int_0^{t_f} dt$

Subj to : model

$|u(t)| \leq u_{max}, \; 0 \leq t \leq t_f$

# Ill-Posed, Poorly Posed, and Well-Posed Optimal Control Problems

An **ill-posed** optimization problem is one for which **no optimal solution exists** (i.e., there does not exist an admissible control trajectory, $u^*(t)$, that minimizes the chosen cost function). This can arise for two reasons:

- Strict inequality constraints (e.g., $|u(t)| < 5$, rather than $|u(t)| \leq 5$)
- Infeasible constraints

A **poorly posed** optimization problem is one for which **the optimal solution is a trivial one that makes no sense in a real engineering application.**

A **well-posed** optimization problem is one that is **not poorly posed or ill-posed.**

# Example Ill-Posed Static Optimization

Minimize the following static objective function (where the decision variable is $u$):

$$J(u) = u^2$$

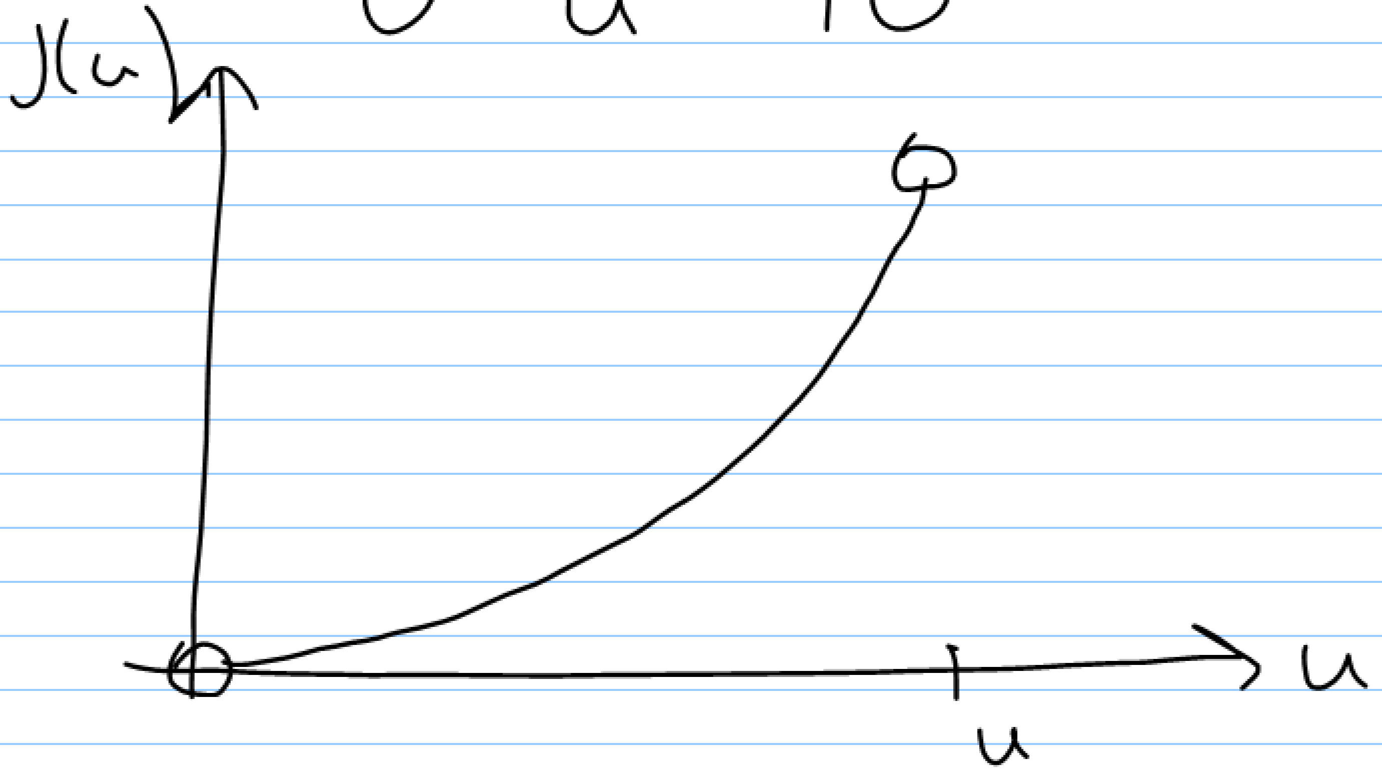
Subject to the constraint: $0 < u < 10$

**Why is this optimization problem ill-posed, and what can be done to make it well-posed?**

# Example Ill-Posed Static Optimization

Minimize $J(u) = u^2$ subj. to

$$0 < u < 10$$



# Example Ill-Posed Static Optimization

$u^* = 0.01$ ... nope ... $u = 0.005$ results in lower $J(u)$

$u^* = 0.001$ ... nope ... $u = 0.0005$ results in lower $J(u)$

Suppose $u^* = \varepsilon$. $u = \varepsilon/2$ results in lower cost.

$\therefore u^*$ is not optimal! (for any $\varepsilon$)

# Another Example of an Ill-Posed Optimal Control Problem

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Suppose that the motion of a 1-dof system is modeled by:

$$\dot{x} = v$$
$$\dot{v} = u$$

where $x \in \mathbb{R}$ is the position, $v \in \mathbb{R}$ is the velocity, and $u \in \mathbb{R}$ is the acceleration input.

Suppose that we want to minimize the amount of time required for the system to move from a position $x(0) = 0$ to a position $x(t_f) = 1$, through the following optimization:

Minimize $J(u(t)) = \int_0^{t_f} dt$

Subject to $x(t_f) = 1$

# Another Example of an Ill-Posed Optimal Control Problem

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Ill-posed ex. 2: $u^*(t) = \infty, \quad 0 \le t \le t_f$

Fix: $|u(t)| \le u_{max}, \quad 0 \le t \le t_f$

# Example Poorly Posed Optimal Control Problem

Suppose that a satellite's rotational position ($\theta$) and velocity ($\omega$) are modeled by:

$$\dot{\theta} = \omega$$

$$\dot{\omega} = \frac{1}{J} u$$

$u$ = control torque

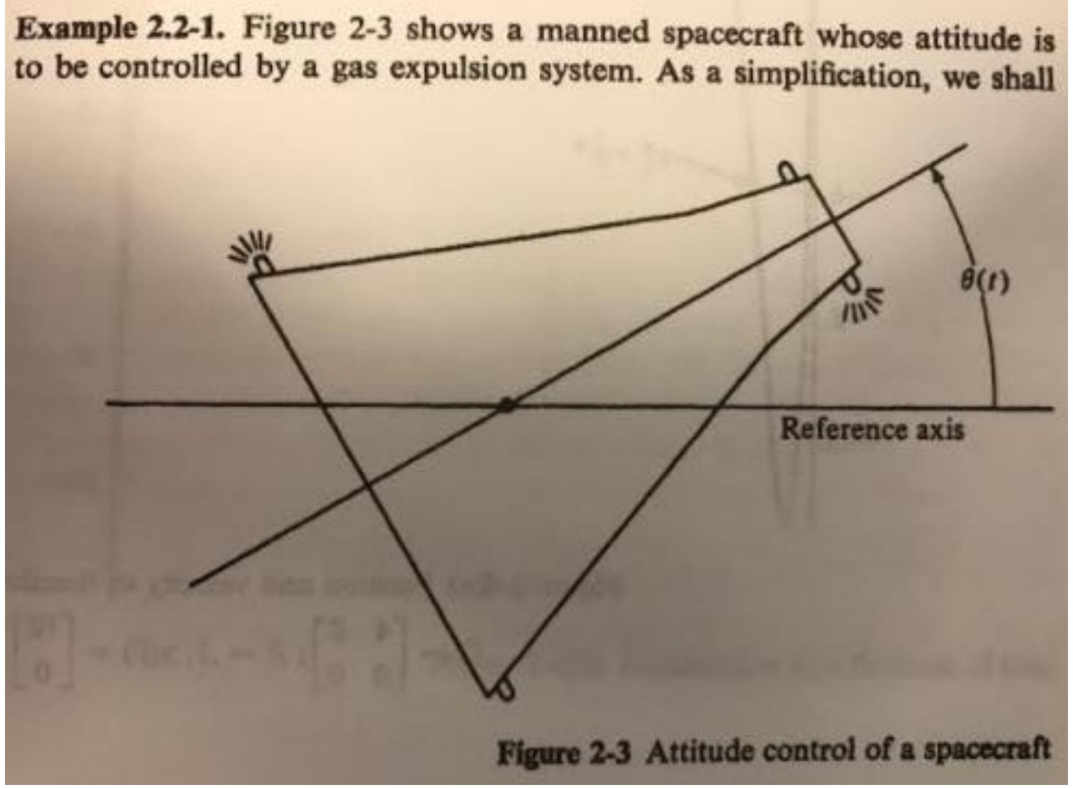
**Example 2.2-1.** Figure 2-3 shows a manned spacecraft whose attitude is to be controlled by a gas expulsion system. As a simplification, we shall

**Figure 2-3** Attitude control of a spacecraft

Suppose the control objective is to minimize the control effort through the following cost function:

$$J(u(t); \theta(0), \omega(0)) = \int_0^{t_f} u(t)^2 dt \qquad \text{subject to:} \qquad |u(t)| \leq u_{max}, \forall t \geq 0$$

# Example Poorly Posed Optimal Control Problem

Poorly posed ex: $J(u(t), \underline{x}(0)) = \int_0^{t_f} u(t)^2 \, dt$

subj to $|u(t)| \leq u_{max} \ \forall t$

+ model

$u^*(t) = 0, \quad 0 \leq t \leq t_f$

# Side Note

Minimizer of the following functionals is the same:

1) $J(u(t); \underline{x}(0)) = \int_0^{t_f} u(t)^2 \, dt$

2) $J(u(t); \underline{x}(0)) = \int_0^{t_f} R\, u(t)^2 \, dt \quad (R \in \mathbb{R})$

3) $J(u(t); \underline{x}(0)) = C + \int_0^{t_f} R\, u(t)^2 \, dt$

# Preview of next lecture (and beyond)

## Finite-dimensional optimal control problems in discrete time:

- Converting continuous time systems to discrete time models
- Equivalence between finite-dimensional design optimization and discrete time optimal control problems
- Convexity