MEGR 7090/8090: Advanced Optimal Control

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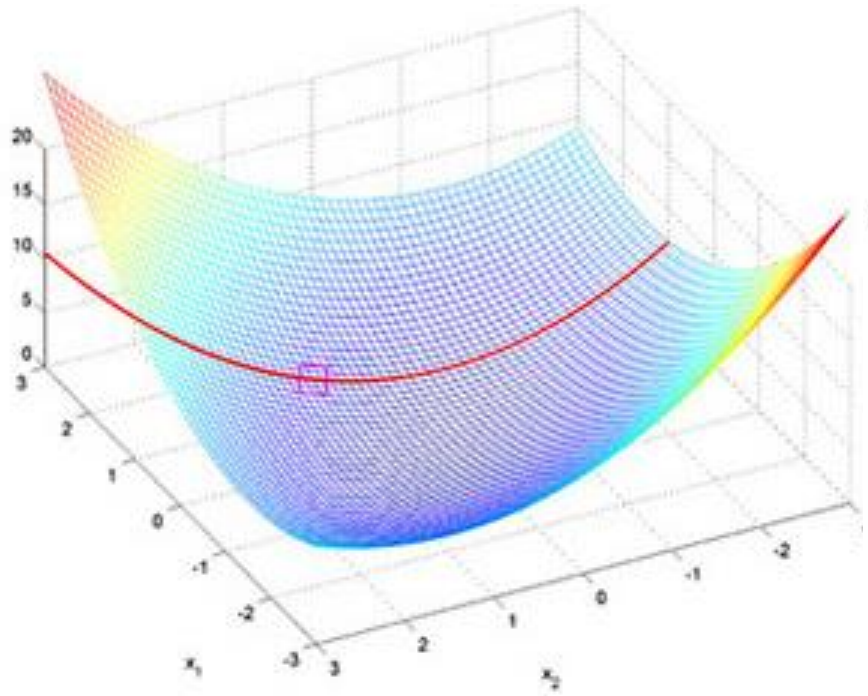


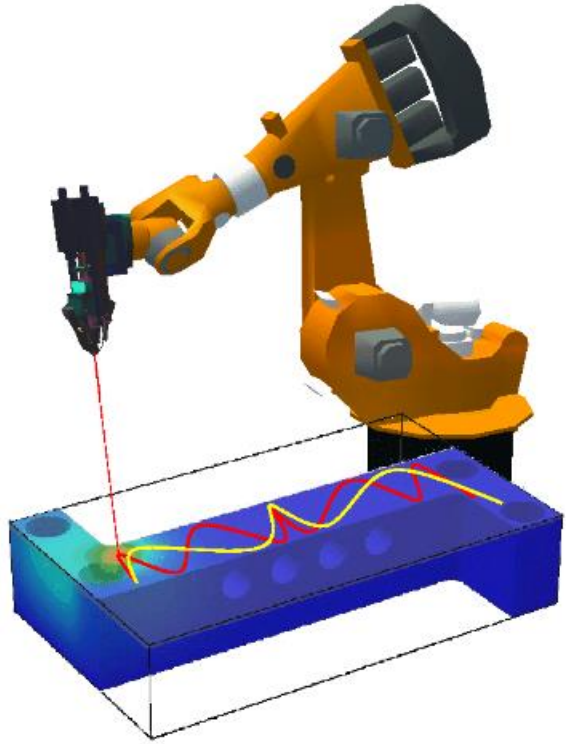
$$V_n(\mathbf{x}_n) = \min_{\{\mathbf{u}_n, \mathbf{u}_{n+1}, \cdots, \mathbf{u}_{N-1}\}} \left[\frac{1}{2} \sum_{k=n}^{N-1} \left(\mathbf{x}_k^T \mathbf{Q}_k \mathbf{x}_k + \mathbf{u}_k^T \mathbf{R} \mathbf{u}_k \right) + \frac{1}{2} \mathbf{x}_N^T \mathbf{Q}_N \mathbf{x}_N \right]$$

$$V_n(\mathbf{x}_n) = \min_{\{\mathbf{u}_n, \mathbf{u}_{n+1}, \cdots, \mathbf{u}_{N-1}\}} \left[\frac{1}{2} \sum_{k=n}^{N-1} \left(\mathbf{x}_k^T \mathbf{Q}_k \mathbf{x}_k + \mathbf{u}_k^T \mathbf{R} \mathbf{u}_k \right) + \frac{1}{2} \mathbf{x}_N^T \mathbf{Q}_N \mathbf{x}_N \right]$$

$$= \min_{\mathbf{u}_n} \left[\frac{1}{2} \left(\mathbf{x}_n^T \mathbf{Q}_n \mathbf{x}_n + \mathbf{u}_n^T \mathbf{R} \mathbf{u}_n \right) + \underbrace{\min_{\{\mathbf{u}_{n+1}, \cdots, \mathbf{u}_{N-1}\}} \left[\frac{1}{2} \sum_{k=n+1}^{N-1} \left(\mathbf{x}_k^T \mathbf{Q}_k \mathbf{x}_k + \mathbf{u}_k^T \mathbf{R} \mathbf{u}_k \right) + \frac{1}{2} \mathbf{x}_N^T \mathbf{Q}_N \mathbf{x}_N \right]}_{V_{n+1}(\mathbf{x}_{n+1})} \right]$$

$$= \min_{\mathbf{u}_n} \left[\frac{1}{2} \left(\mathbf{x}_n^T \mathbf{Q}_n \mathbf{x}_n + \mathbf{u}_n^T \mathbf{R} \mathbf{u}_n \right) + V_{n+1}(\mathbf{x}_{n+1}) \right]$$

$$V_n(\mathbf{x}_n) = \min_{\mathbf{u}_n} \left[\frac{1}{2} \left(\mathbf{x}_n^T \mathbf{Q}_n \mathbf{x}_n + \mathbf{u}_n^T \mathbf{R} \mathbf{u}_n \right) + V_{n+1}(\mathbf{x}_{n+1}) \right]$$



Lecture 18

October 24, 2017

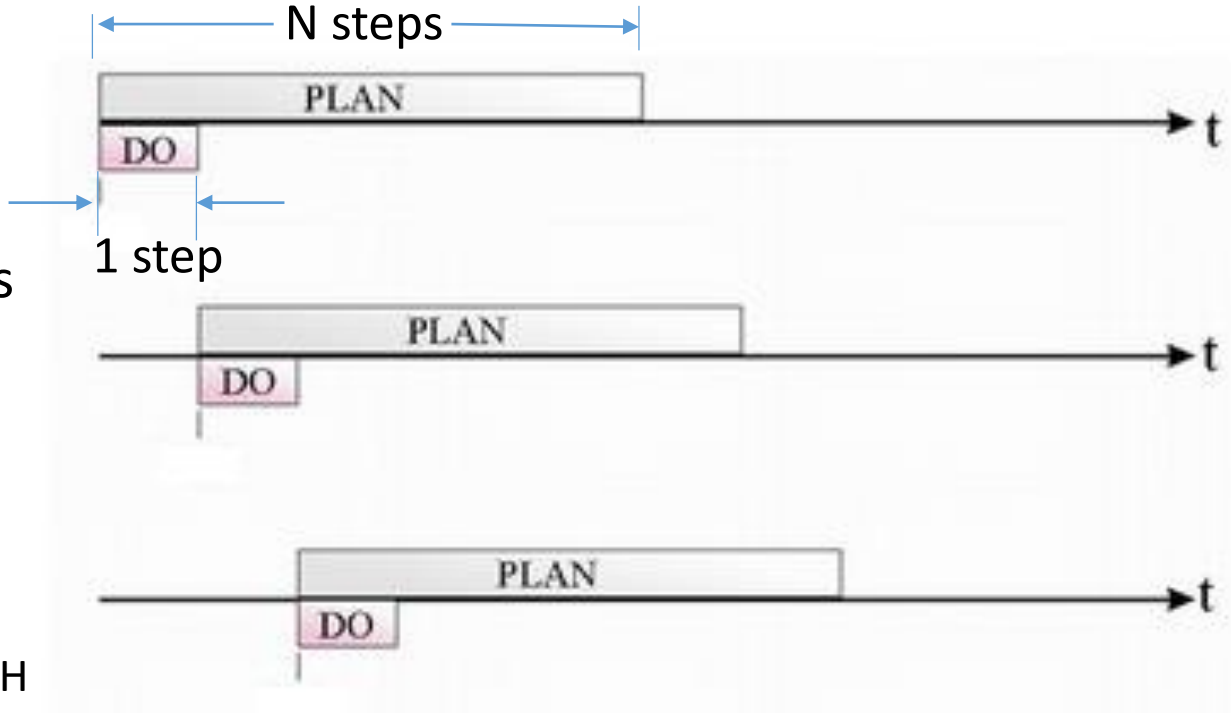
Main Idea of MPC – Reminder

- At time 0, compute the control trajectory that minimizes $J\big(\mathbf{u}, \mathbf{x}(0)\big) = \sum_{i=0}^{N-1} g\big(\mathbf{x}(i), u(i)\big) + h(\mathbf{x}(N))$, subject to constraints. Implement $u^*(0)$.

- At the next time step (1), compute the control signal that minimizes $J\big(\mathbf{u}, \mathbf{x}(1)\big) = \sum_{i=1}^{N} g\big(\mathbf{x}(i), u(i)\big) + h(\mathbf{x}(N+1))$, subject to constraints

- Repeat every time step

- This process is known as ***model predictive control (MPC)***, also known as ***receding horizon control***



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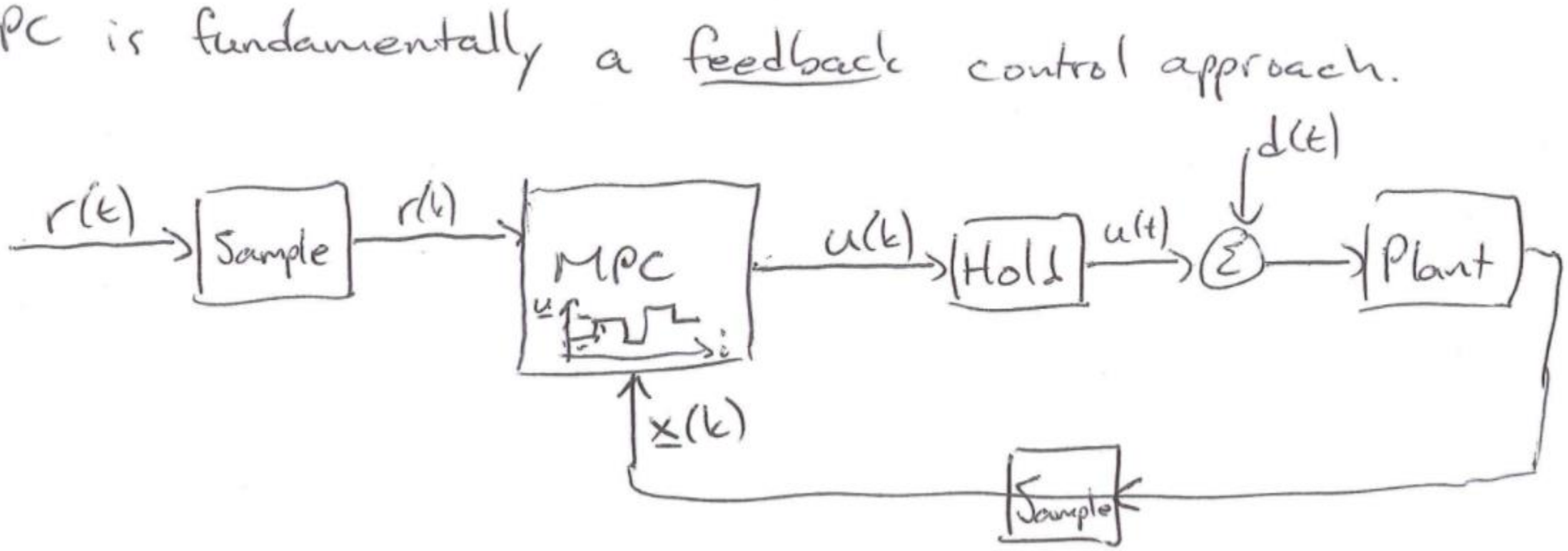
Main Idea of MPC – Reminder



MPC Notation – Reminder

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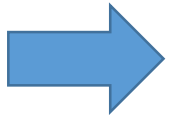
Note: At every time step, MPC optimizes a **sequence** of control signals. Those sequences overlap. (Example: If $N = 10$), then a value for $u(9)$ is computed at steps 0, 1,…, 9. To differentiate between these different values, we use what Dr. V calls "slash notation":

- $\mathbf{u}(k) = [u(k|k) \quad \dots \quad u(k+N-1|k)]^T$ = candidate control sequence at step k
- $\mathbf{x}_{seq}(k) = [\mathbf{x}(k|k) \quad \dots \quad \mathbf{x}(k+N|k)]^T$ = predicted state sequence at step k
- In general, the notation is interpreted as (time step at which the variable is evaluated|time step at which the optimization is performed)

General mathematical formulation for MPC:

$$\mathbf{u}^*(k) = \arg\min \left[\sum_{i=k}^{N-1} g\big(\mathbf{x}(i|k), u(i|k)\big) + h(\mathbf{x}(k+N|k)) \right]$$



$$u(k) = u^*(k|k)$$

(Implement the first step of the optimized sequence)

subject to: $\mathbf{x}(i|k) \in X, i = k \dots k+N$, $u(i|k) \in U, i = k \dots k+N-1$ and: $\mathbf{x}(i+1|k) = f\big(\mathbf{x}(i|k), u(i|k)\big)$

Linear MPC – Reminder

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Constrained **discrete-time LQR problem – also known as** ***linear MPC*****:**

$$\mathbf{u}^*(k) = \arg\min\left[\sum_{i=k}^{N-1}(\mathbf{x}^T(i|k)Q\mathbf{x}(i|k) + Ru^2(i|k)) + \mathbf{x}^T(k+N|k)S\mathbf{x}(k+N|k))\right] \Rightarrow u(k) = u^*(k|k)$$

Subject to: $M_1\mathbf{x}(i|k) - \mathbf{b}_1 \leq 0, i = k \ldots k+N$

$M_2u(i|k) - \mathbf{b}_2 \leq 0, i = k \ldots k+N-1$

and $\mathbf{x}(i+1|k) = A\mathbf{x}(i|k) + Bu(i|k)$

Solution – ***Quadratic programming (QP):***

- Refer to lecture 9 notes for proof that the above problem is convex and therefore QP can be used reliably

Simulink's built-in MATLAB toolbox can be used to perform linear MPC (constrained discrete-time LQR)

Benefit and Limitations to Linear MPC

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Major benefit: Convex constraints on the state and control signal can be enforced

Limitations:

- System dynamics ***must be linear*** (other than constraints) or linearized
- Stage cost ***must be quadratic*** in the states and control signal

Today, we will discuss nonlinear MPC, which removes the above limitations at the expense of computational expensiveness.

General Nonlinear MPC Setup

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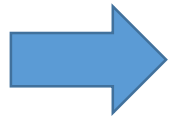
General optimization problem (note the free-form objective function, constraints, and model):

$$\mathbf{u}^*(k) = \arg\min\left[\sum_{i=k}^{N-1} g(\mathbf{x}(i|k), u(i|k)) + h(\mathbf{x}(k+N|k))\right]$$



$$u(k) = u^*(k|k)$$

Subject to:

$$\mathbf{x}(i|k) \in X, i = k \ldots k+N$$
$$u(i|k) \in U, i = k \ldots k+N-1$$

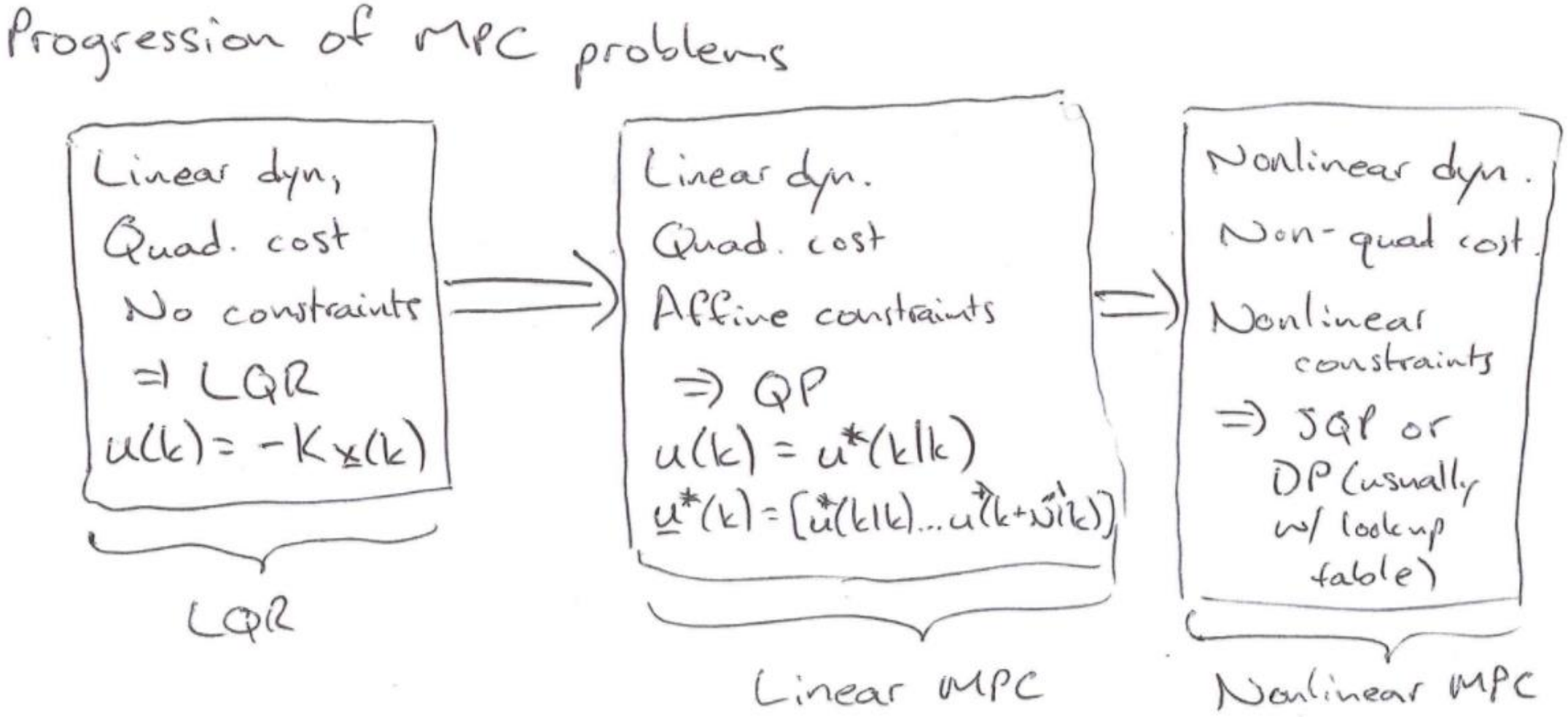
and

$$\mathbf{x}(i+1|k) = f(\mathbf{x}(i|k), u(i|k))$$

Solution techniques:

- Sequential quadratic programming (SQP) – refer to lecture 12-13 notes for a refresher
- Dynamic programming (DP) – usually, the actual DP calculations must be done offline, and a lookup table (generated by DP) can then be used to determine the required control signal

General Nonlinear MPC Setup



Dynamic Programming for MPC (Building a Lookup Table)

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Recall: With **backward recursion**, DP allows you to ***specify a constraint on the terminal state*** $(\mathbf{x}(N))$ and ***determine the optimal control sequence for all possible initial states*** $(\mathbf{x}(0))$

Process for using DP to build a lookup table for MPC:

- Perform DP using backward recursion (gives you the result indicated above) – refer back to lectures 14-16 for a reminder on how to do this)
- For each gridded state value, store $u^*(0)$, noting that with MPC, we will take $u(k) = u^*(0)$
- Use a 1D lookup table (available in Simulink) to determine $u(k)$ for interpolated points

Dynamic Programming for MPC (Building a Lookup Table)

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Possible states:

$\underline{x}_1$ $\quad \underline{u}^*_{OP}(x(0) = \underline{x}_1) = [\ \ldots\ldots\ ]^T$

$\underline{x}_2$ $\quad \underline{u}^*_{OP}(\underline{x}(0) = \underline{x}_2) = [\ \ldots\ldots\ ]^T$

$\underline{x}_3$

$\underline{x}_4$ $\quad$ etc.

$\underline{x}_5$

Dynamic Programming for MPC (Building a Lookup Table)

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For MPC, can take $\underline{u}^*(k) = \underline{u}^*_{interp}(\underline{x}(k))$

interpolation of $\underline{u}^*$ among nearest tabulated values of $\underline{u}^*_{DP}$

$u(k) = \underline{u}^*(k|k)$

Sequential Quadratic Programming (SQP) for MPC

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Recall:

- At each iteration, SQP first derives a quadratic approximation of the Lagrangian (setting up the "QP subproblem")
- SQP then moves in the direction specified by the QP subproblem solution
- fmincon can be used to perform SQP

Challenge: fmincon is not supported by Simulink!!

Workarounds:

- Option 1 – Choose another SQP solver
- Option 2 – Write SQP yourself
- Option 3 – Generate a ***level 2 s-function*** – We will focus on this one

Level 2 s-functions – Main Ideas and References

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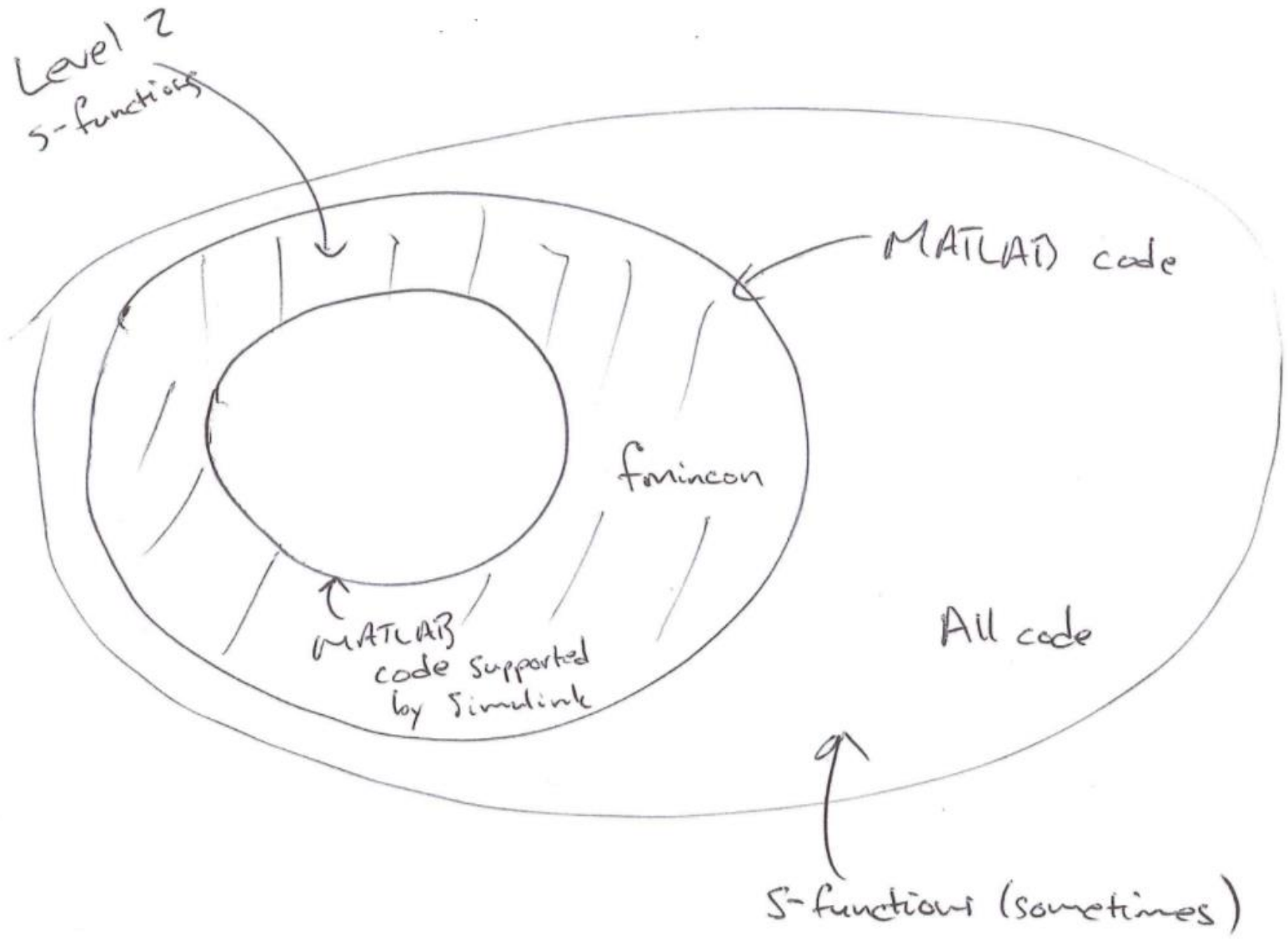
Main ideas:

- Embedded MATLAB functions only support a ***subset*** of MATLAB functionality (for code generation reasons)
- s-functions provide a way for you to take code written in a non-native language or non-supported MATLAB code and run it in Simulink
- Level 2 s-functions are for code that was written in MATLAB but not supported by Simulink blocks (fmincon is an example of this)
- We will work through a level 2 s-function example…a good video tutorial to level 2 s-functions can be found at:

https://www.youtube.com/watch?v=X-qVign6BLg

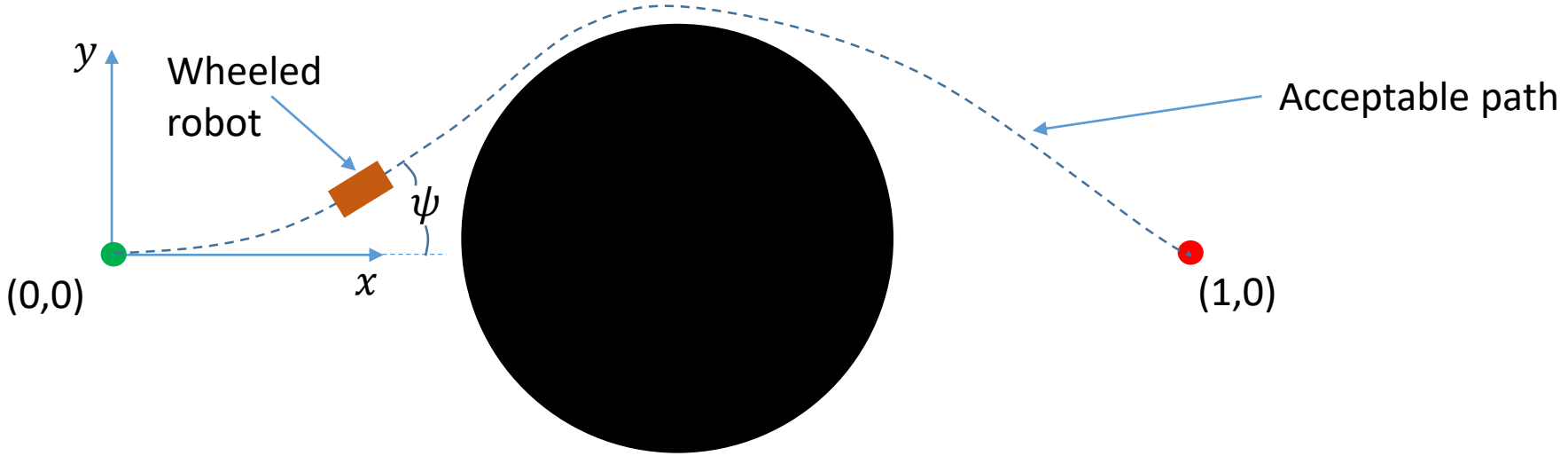
https://www.youtube.com/watch?v=ZnvRoGA23uU

Level 2 s-functions – Main Ideas and References



Nonlinear MPC Example

Scenario: A wheeled mobile robot must travel from point A (0,0) to point B (1,0) while avoiding a circular obstacle of radius $R = 0.25$



Dynamic model:

$$\dot{x} = v\cos\psi$$
$$\dot{y} = v\sin\psi$$
$$\dot{\psi} = u$$

where:

$$v = v_0((x-1)^2 + y^2)$$

The robot slows down automatically as it nears the finish line.

$$-5 \le u(t) \le 5, \forall t$$

Nonlinear MPC Example – Continued

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Tasks:

- Given the scenario described on the previous slide, determine an acceptable stage cost ($g(\mathbf{x}(i|k), \mathbf{u}(i|k))$), terminal cost ($h(\mathbf{x}(k+N|k), \mathbf{u}(k+N|k))$), and constraint sets ($U$ and X)
- Implement a Simulink model of the robot and MPC-based controller, using fmincon and a level 2 s-function

Sample code available on Canvas

Nonlinear MPC Example

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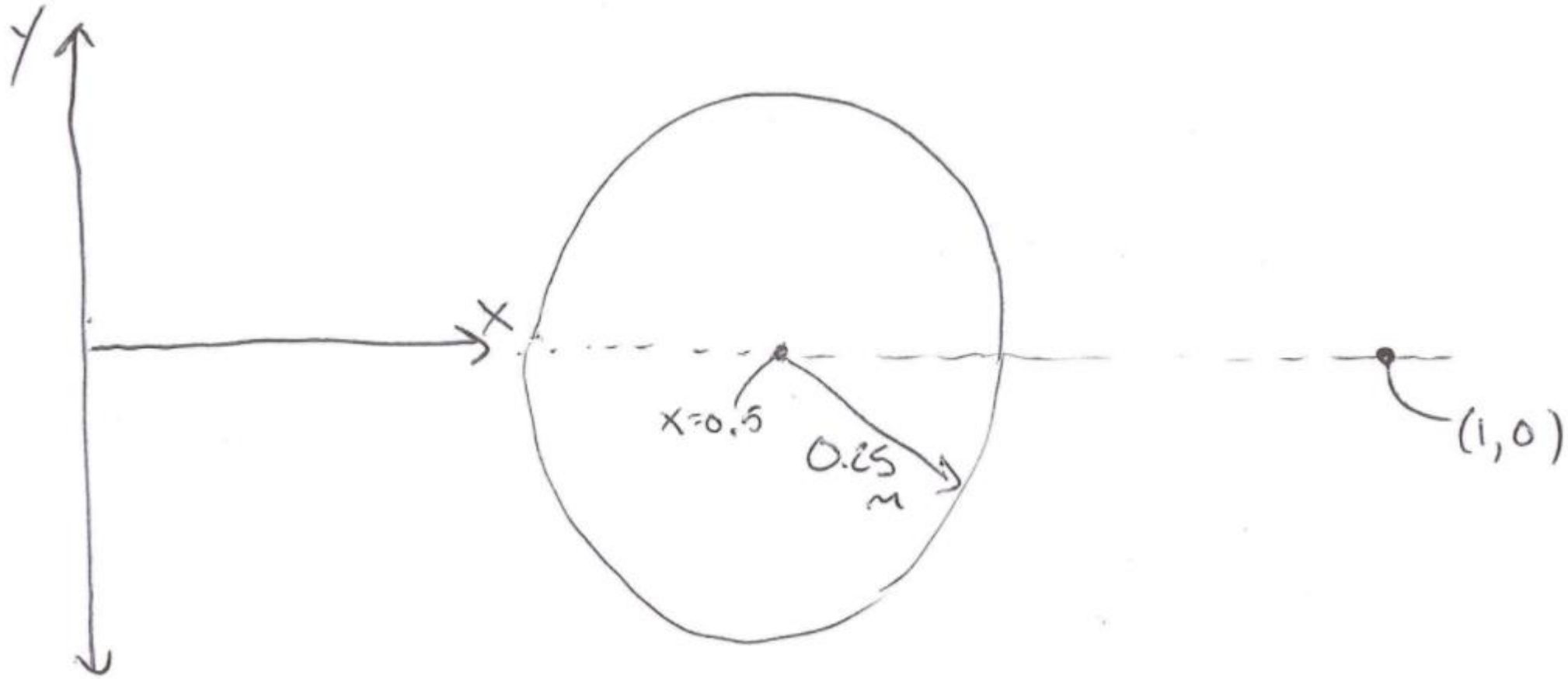

Nonlinear MPC Example

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$$\dot{x} = v\cos\psi$$
$$\dot{y} = v\sin\psi$$
$$\dot{\psi} = u$$

$$v = v_0\left[(x-1)^2 + y^2\right]$$

$$-5 \le u(t) \le 5 \quad \forall\, t$$

$$\Downarrow$$

$$x(k+1) = x(k) + v(k)\cos(\psi(k))\,\Delta t$$
$$y(k+1) = y(k) + v(k)\sin(\psi(k))\,\Delta t$$
$$\psi(k+1) = \psi(k) + u(k)\,\Delta t$$

Nonlinear MPC Example

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$$J_{MPC}(\underline{u}(k); \underline{x}(k)) = \sum_{i=k}^{k+N-1} (y(i|k))^2$$

Constraints: $\underline{x}(i|k) \in X$, $i = k \ldots k+N-1$ (note: $\underline{x} = [x \; y]^T$)

$$X = \{\underline{x} : (x^{(i)} - 0.5)^2 + y_{(i)}^2 \geq 0.25^2\} \quad i = 0 \ldots N-1$$

note: Can take $X = \{\underline{x} : (x^{(i)} - 0.5)^2 + y_{(i)}^2 \geq 0.25^2 + B_i\}$ $i = 0 \ldots N-1$

B_i: buffer. (recognizes that the model is imperfect)

$$U = \{\underline{u} : -5 \leq u(i) \leq 5, \; i = 0 \ldots N-1\}$$

Nonlinear MPC Setup with Disturbances

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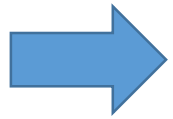
General optimization problem (note the free-form objective function, constraints, and model):

$$\mathbf{u}^*(k) = \arg\min\left[\sum_{i=k}^{N-1} g(\mathbf{x}(i|k), u(i|k)) + h(\mathbf{x}(k+N|k))\right]$$



$$u(k) = u^*(k|k)$$

Subject to:

$$\mathbf{x}(i|k) \in X, i = k \ldots k+N$$
$$u(i|k) \in U, i = k \ldots k+N-1$$

and

$$\mathbf{x}(i+1|k) = f(\mathbf{x}(i|k), u(i|k), d(i|k))$$

Assessment of candidate solution techniques:

- Offline dynamic programming with online lookup table: The DP solution will be ***different for every disturbance profile*** $(d(i|k), i = k \ldots k+N-1))$…this makes the DP-based lookup table ***cumbersome to generate***
- SQP, on the other hand, can take into account the new value of the disturbance at each time step

Nonlinear MPC Setup with Disturbances

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MPC w/ disturbances:

Include disturbance? (d)	MPC	Plant model
d measured?	Yes	Yes
d unmeasured?	No	Yes

Preview of Upcoming Lectures

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October 26 – Stability and robustness of MPC

After October 26 – Optimal control for continuous-time systems