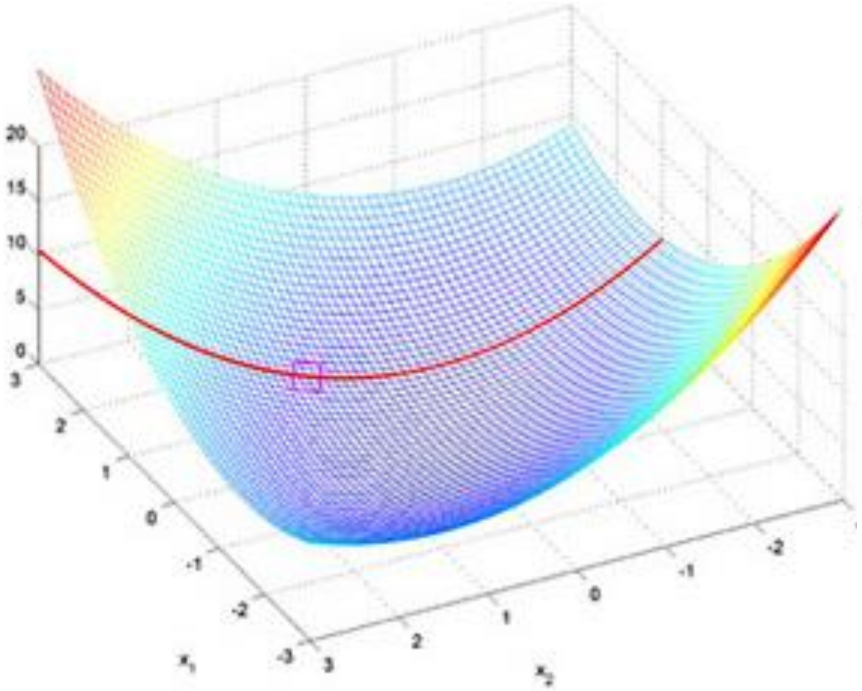
MEGR 7090/8090: Advanced Optimal Control

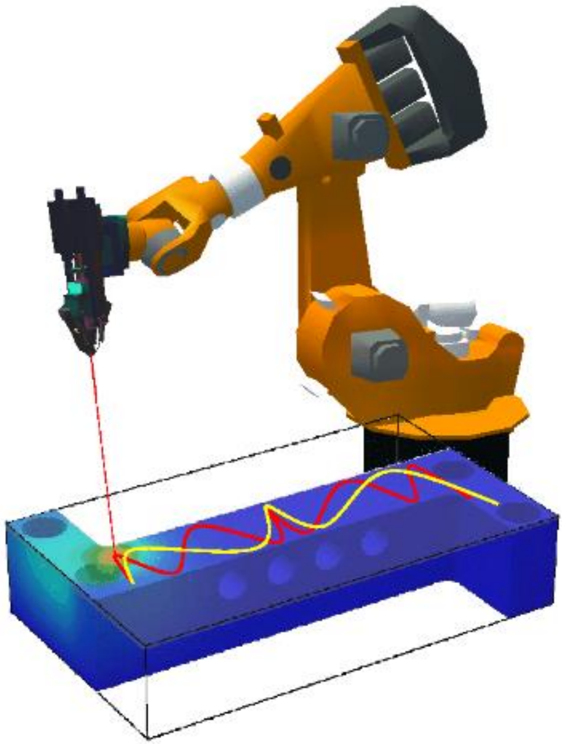
UNC CHARLOTTE



$$\boxed{V_n(\mathbf{x}_n) = \min_{\{\mathbf{u}_n,\mathbf{u}_{n+1},\cdots,\mathbf{u}_{N-1}\}} \left[\frac{1}{2}\sum_{k=n}^{N-1}\left(\mathbf{x}_k^T\mathbf{Q}_k\mathbf{x}_k + \mathbf{u}_k^T\mathbf{R}\mathbf{u}_k\right) + \frac{1}{2}\mathbf{x}_N^T\mathbf{Q}_N\mathbf{x}_N\right]}$$

$$\begin{aligned}
V_n(\mathbf{x}_n) &= \min_{\{\mathbf{u}_n,\mathbf{u}_{n+1},\cdots,\mathbf{u}_{N-1}\}} \left[\frac{1}{2}\sum_{k=n}^{N-1}\left(\mathbf{x}_k^T\mathbf{Q}_k\mathbf{x}_k + \mathbf{u}_k^T\mathbf{R}\mathbf{u}_k\right) + \frac{1}{2}\mathbf{x}_N^T\mathbf{Q}_N\mathbf{x}_N\right] \\
&= \min_{\mathbf{u}_n}\left[\frac{1}{2}\left(\mathbf{x}_n^T\mathbf{Q}_n\mathbf{x}_n + \mathbf{u}_n^T\mathbf{R}\mathbf{u}_n\right) + \underbrace{\min_{\{\mathbf{u}_{n+1},\cdots,\mathbf{u}_{N-1}\}}\left[\frac{1}{2}\sum_{k=n+1}^{N-1}\left(\mathbf{x}_k^T\mathbf{Q}_k\mathbf{x}_k + \mathbf{u}_k^T\mathbf{R}\mathbf{u}_k\right) + \frac{1}{2}\mathbf{x}_N^T\mathbf{Q}_N\mathbf{x}_N\right]}_{V_{n+1}(\mathbf{x}_{n+1})}\right] \\
&= \min_{\mathbf{u}_n}\left[\frac{1}{2}\left(\mathbf{x}_n^T\mathbf{Q}_n\mathbf{x}_n + \mathbf{u}_n^T\mathbf{R}\mathbf{u}_n\right) + V_{n+1}(\mathbf{x}_{n+1})\right]
\end{aligned}$$

$$\boxed{V_n(\mathbf{x}_n) = \min_{\mathbf{u}_n}\left[\frac{1}{2}\left(\mathbf{x}_n^T\mathbf{Q}_n\mathbf{x}_n + \mathbf{u}_n^T\mathbf{R}\mathbf{u}_n\right) + V_{n+1}(\mathbf{x}_{n+1})\right]}$$



Lecture 15

October 12, 2017

Bellman's Principle of Optimality – Reminder

Conditions: Bellman's principle applies to discrete-time systems (also can be used for continuous-time systems) of the form:

$$\mathbf{x}(k+1) = f(\mathbf{x}(k), u(k))$$

(Successor state ($\mathbf{x}(k+1)$) depends on the originating state ($\mathbf{x}(k)$) and the control signal at stage k)

Notation: Given an objective function structure $J\big(u, \mathbf{x}(0)\big) = \sum_{i=0}^{N-1} g(\mathbf{x}(i), u(i))$:

- $J^*_{\mathbf{x}_i(j)\rightarrow\mathbf{x}_f(N)}$ = Optimal *cost to go* from intermediate state x_i at step j to final state $\mathbf{x}_f$ at step N
- $J^*_{\mathbf{x}_o(j-M)\rightarrow\mathbf{x}_i(j)}$ = Optimal *cost to arrive* at intermediate state $\mathbf{x}_i$ at step j from originating state $\mathbf{x}_o$ at step j-M (often the optimal cost to arrive is the ***only*** cost to arrive, since there is only one way to get from $\mathbf{x}_o$ to $\mathbf{x}_i$ in one step
- $J^*_{\mathbf{x}_o(j-M)\rightarrow\mathbf{x}_i(j)\rightarrow\mathbf{x}_f(N)}$ = Optimal cost to go from originating state $\mathbf{x}_o$ at step j-1 to final state $\mathbf{x}_f$ at step N, through intermediate state $\mathbf{x}_i$ at step j

Bellman's principle of optimality: $J^*_{\mathbf{x}_o(j-M)\rightarrow\mathbf{x}_i(j)\rightarrow\mathbf{x}_f(N)} = J^*_{\mathbf{x}_o(j-M)\rightarrow\mathbf{x}_i(j)} + J^*_{\mathbf{x}_i(j)\rightarrow\mathbf{x}_f(N)}$

Bellman's Principle of Optimality – Reminder

If $\underline{x}$ is a valid state (given knowledge of $\underline{x}(k)$ & $u(i), i \geq k$, we can predict $\underline{x}(k+i), i \geq 1$ w/o knowing $\underline{x}(k-j), j \geq 1$ or $u(k-j), j \geq 1$)

Corollary: $$J^*_{\underline{x}_0(j-M) \rightarrow \underline{x}_i(j) \rightarrow x_f(j+N)} = J^*_{\underline{x}_0(j-M) \rightarrow x_i(j)} + J^*_{\underline{x}_i(j) \rightarrow \underline{x}_f(j+N)}$$

Dynamic Programming – Backward Recursion – Reminder

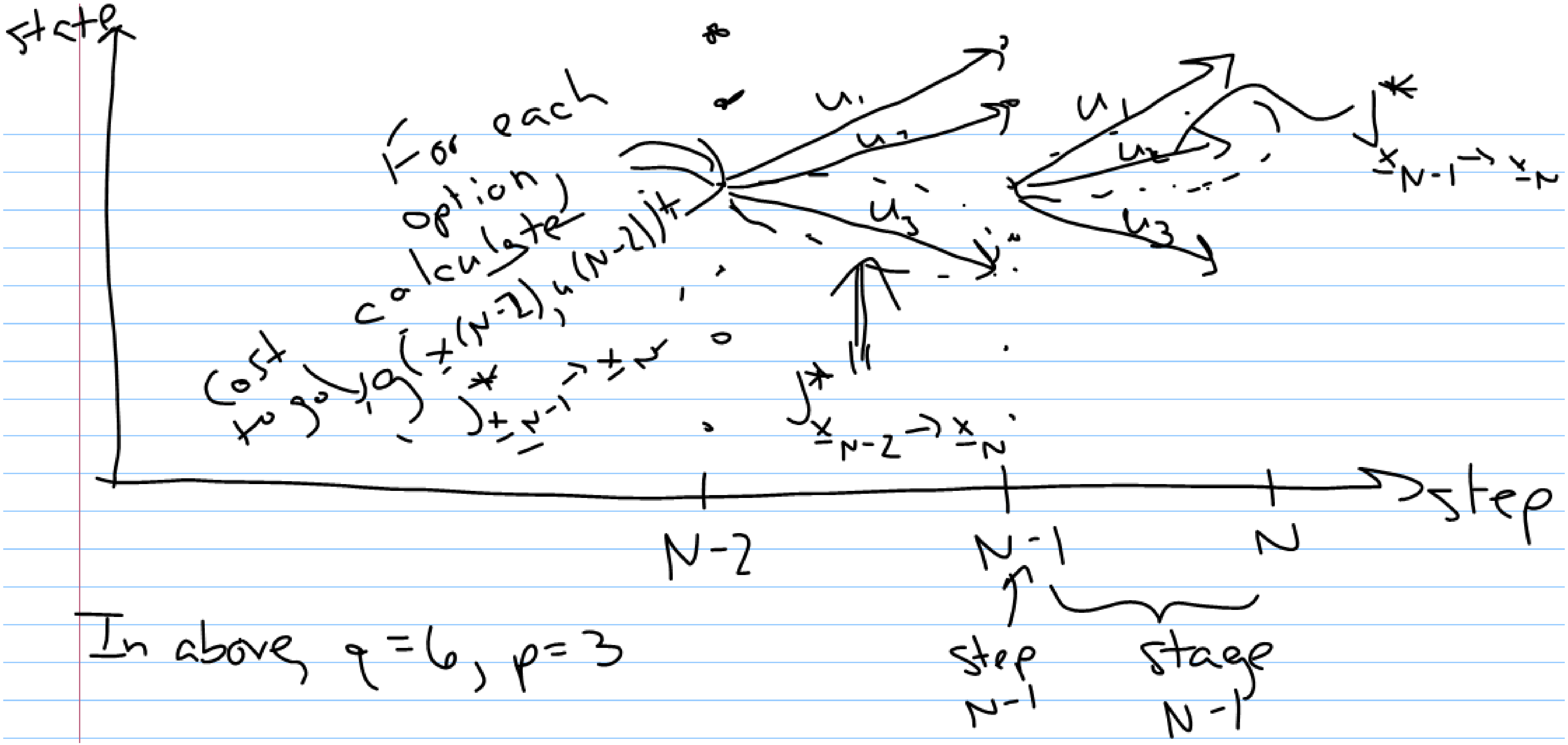
Getting set up:

- Quantize control variables into p discrete values
- Quantize state variables into q discrete values

Solution algorithm:

- Start at step N-1. For each of the q allowable state variables, calculate the stage cost $(g(\mathbf{x}(N-1), u(N-1))$ for each of the p allowable control variables that lead to constraint satisfaction. Control variables that do not satisfy constraints are termed ***inadmissible***. Record the optimal control signals and corresponding stage costs for each originating state.
- Move to step N-2. For each of the q allowable state variables, calculate the stage cost $(g(\mathbf{x}(N-1), u(N-1))$ and associated intermediate state, $\mathbf{x}_i(N-1)$, for each of the p allowable control variables that lead to constraint satisfaction. To determine which control variable is optimal, compute the total cost to go as $J^*_{\mathbf{x}_o(N-2)\to\mathbf{x}_i(N-1)\to\mathbf{x}_f(N)} = J^*_{\mathbf{x}_o(N-2)\to\mathbf{x}_i(N-1)} + J^*_{\mathbf{x}_i(N-1)\to\mathbf{x}_f(N)}$
- Move to step N-3 and repeat the process (total cost to go is now $J^*_{\mathbf{x}_o(N-3)\to\mathbf{x}_i(N-2)\to\mathbf{x}_f(N)} = J^*_{\mathbf{x}_o(N-3)\to\mathbf{x}_i(N-2)} + J^*_{\mathbf{x}_i(N-2)\to\mathbf{x}_f(N)}$). Keep stepping backward in time until step 0.

Dynamic Programming – Backward Recursion – Reminder



In above, $q=6$, $p=3$

Dynamic Programming – Backward Recursion – Reminder

At every stage, we create a chart:

$N-j$	u_1	u_2	u_3	
$\underline{x}_1$	$\underline{x}_{N-j+1}$			
$\underline{x}_2$				
$\underline{x}_3$				
$\underline{x}_4$				
$\underline{x}_5$				
$\underline{x}_6$				

$g(\underline{x}(N-j), u(N-j)) + J^*_{\underline{x}_{N-j+1} \to \underline{x}_N}$

Dynamic Programming – Forward Recursion – Reminder

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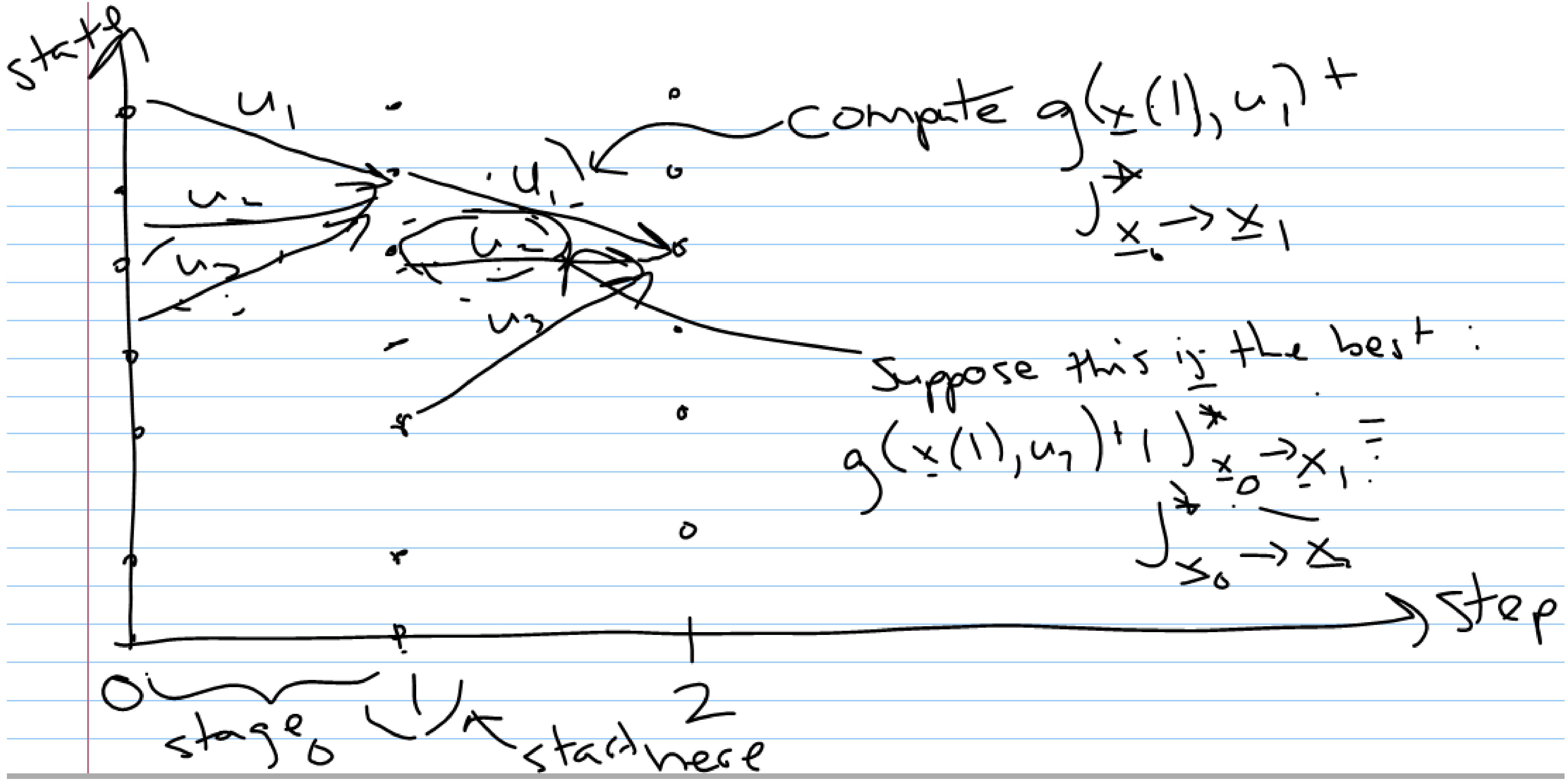
Getting set up:

- Quantize control variables into p discrete values
- Quantize state variables into q discrete values
- Usually the initial state, $\mathbf{x}(0)$, is specified (you don't get to choose your initial state)

Solution algorithm:

- Start at step 1. For each of the q allowable values of $\mathbf{x}(1)$, back-compute the value of $\mathbf{x}(0)$ and stage cost that results from each of the p allowable control values. For each admissible control value (whose associated $\mathbf{x}(0)$ satisfies initial condition requirements), determine the optimal stage cost– this cost, denoted by $J^*_{\mathbf{x}_0(0)\to\mathbf{x}_t(1)}$, is the ***optimal cost to arrive*** at state $\mathbf{x}_t(1)$.
- Move to step 2. For each of the q allowable values of $\mathbf{x}(2)$, back-compute $\mathbf{x}(1)$ and the associated stage cost for each of the p allowable control variables that lead to constraint satisfaction. To determine which control sequence is optimal, compute the total cost to arrive as $J^*_{\mathbf{x}_o(0)\to\mathbf{x}_i(1)\to\mathbf{x}_f(2)} = J^*_{\mathbf{x}_o(0)\to\mathbf{x}_i(1)} + J^*_{\mathbf{x}_i(1)\to\mathbf{x}_f(2)}$
- Move to step 2 and repeat the process. Keep stepping forward in time until step N.

Dynamic Programming – Forward Recursion – Reminder



Problems with Recursion Algorithms

Reminder: State values are quantized as $\mathbf{x}_i \in \{\bar{\mathbf{x}}_1, \dots, \bar{\mathbf{x}}_q\}, i = 0 \dots N$, and control values are quantized as $u_i \in \{\bar{u}_1, \dots, \bar{u}_p\}, i = 0 \dots N-1$

Problem with backward recursion: What happens when the application of some control input u_i at some initial state $\mathbf{x}_i$ results in a successor state $\mathbf{x}_{i+1}$ that is not contained in $\{\bar{\mathbf{x}}_1, \dots, \bar{\mathbf{x}}_q\}$? How do we compute the cost to go?

Problem with forward recursion: What happens when we back-compute $\mathbf{x}_i$ from $\mathbf{x}_{i+1}$ and u_i, only to find that $\mathbf{x}_i \notin \{\bar{\mathbf{x}}_1, \dots, \bar{\mathbf{x}}_q\}$? How do we compute the cost to arrive?

The solution to both of the above problems involves ***interpolation***, which is ***essential to most applications of dynamic programming*** (except for some very simple examples)

Problems with Recursion Algorithms

$x_a, x_b, x_i \in \mathbb{R}$

1

x_a, J_a^*

u_i

x_i, J_i^*?

x_b, J_b^*

i

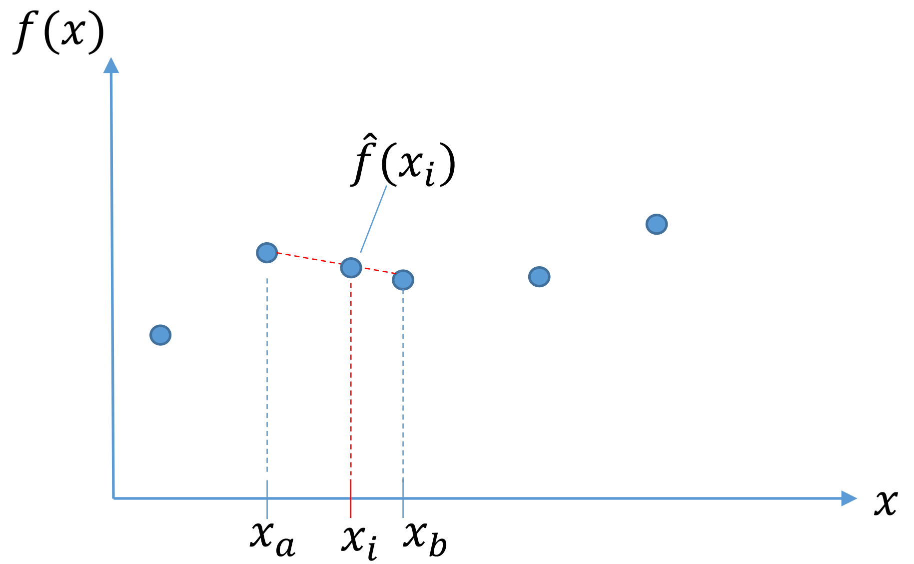
Linear Interpolation – Scalar State

Generic situation: We need to estimate the value of some function $f(x_i)$, but exact values of f are only known at neighboring grid points x_a and x_b, where $x_a < x_i, x_b > x_i$

- In the ***specific*** case of backward recursion, $f(x_i) = J^*_{i\to N}(x_i)$, i.e., the function value to be estimated is the ***cost to go***
- In the ***specific*** case of forward recursion, $f(x_i) = J^*_{0\to i}(x_i)$, i.e., the function value to be estimated is the ***cost to arrive***

An estimate of $f(x_i)$ is obtained through 1D linear interpolation:

$$\hat{f}(x_i) = \frac{(x_b - x_i)f(x_a) + (x_i - x_a)f(x_b)}{x_b - x_a}$$



$\hat{f}(x_i)$

Vector of x values for which $f(x)$ is tabulated

Vector of corresponding $f(x)$ values

x_i

MATLAB syntax: `fi = interp1(x_vec,f_vec,xi);`

Linear Interpolation – Scalar State

$$J_i^* = w J_a^* + (1-w) J_b^*$$

$$= \frac{(x_i - x_b)}{x_a - x_b} J_a^* + \frac{(x_a - x_i)}{x_a - x_b} J_b^*$$

DP-specific syntax:

$J^* = \text{interp1}(\text{admissible states, corresponding } J^*, x_i)$

Backward: successor

Fwd: predecessor

2D "Bilinear" Interpolation – Two State Variables

Situation: We need to estimate the value of some function $f(\mathbf{x}_i)$, but exact values of f are only known at neighboring 2D grid points $[x_{1a} \quad x_{2a}]^T$, $[x_{1b} \quad x_{2a}]^T$, $[x_{1a} \quad x_{2b}]^T$, and $[x_{1b} \quad x_{2b}]^T$, where $x_{1a} < x_{1i}, x_{1b} > x_i, x_{2a} < x_{1i}, x_{2b} > x_i$

An estimate of $f(\mathbf{x}_i)$ is obtained through 2D linear interpolation:

Perform two 1D interpolations in the x_1 direction

$$f_{prelim}([x_{1i} \quad x_{2a}]^T) = \frac{(x_{1b} - x_{1i})(f([x_{1a} \quad x_{2a}]^T)) + (x_{1i} - x_{1a})(f([x_{1b} \quad x_{2a}]^T))}{x_{1b} - x_{1a}}$$

$$f_{prelim}([x_{1i} \quad x_{2b}]^T) = \frac{(x_{1b} - x_{1i})(f([x_{1a} \quad x_{2b}]^T)) + (x_{1i} - x_{1a})(f([x_{1b} \quad x_{2b}]^T))}{x_{1b} - x_{1a}}$$

Perform a 1D interpolation in the x_1 direction:

$$\hat{f}([x_{1i} \quad x_{2i}]^T) = \frac{(x_{2b} - x_{2i})(f_{prelim}([x_{1i} \quad x_{2a}]^T)) + (x_{1i} - x_{1a})(f([x_{1i} \quad x_{2b}]^T))}{x_{2b} - x_{2a}}$$

2D "Bilinear" Interpolation – Two State Variables

Examples of vector state variables:

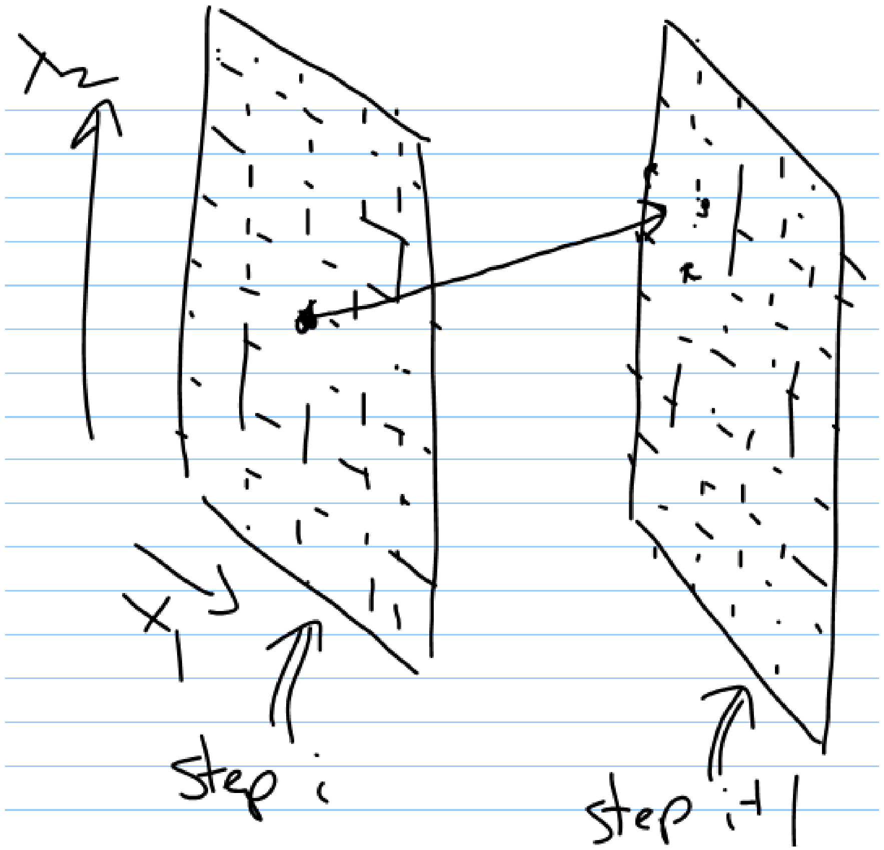
HW 3, P2: Mass-spring-damper

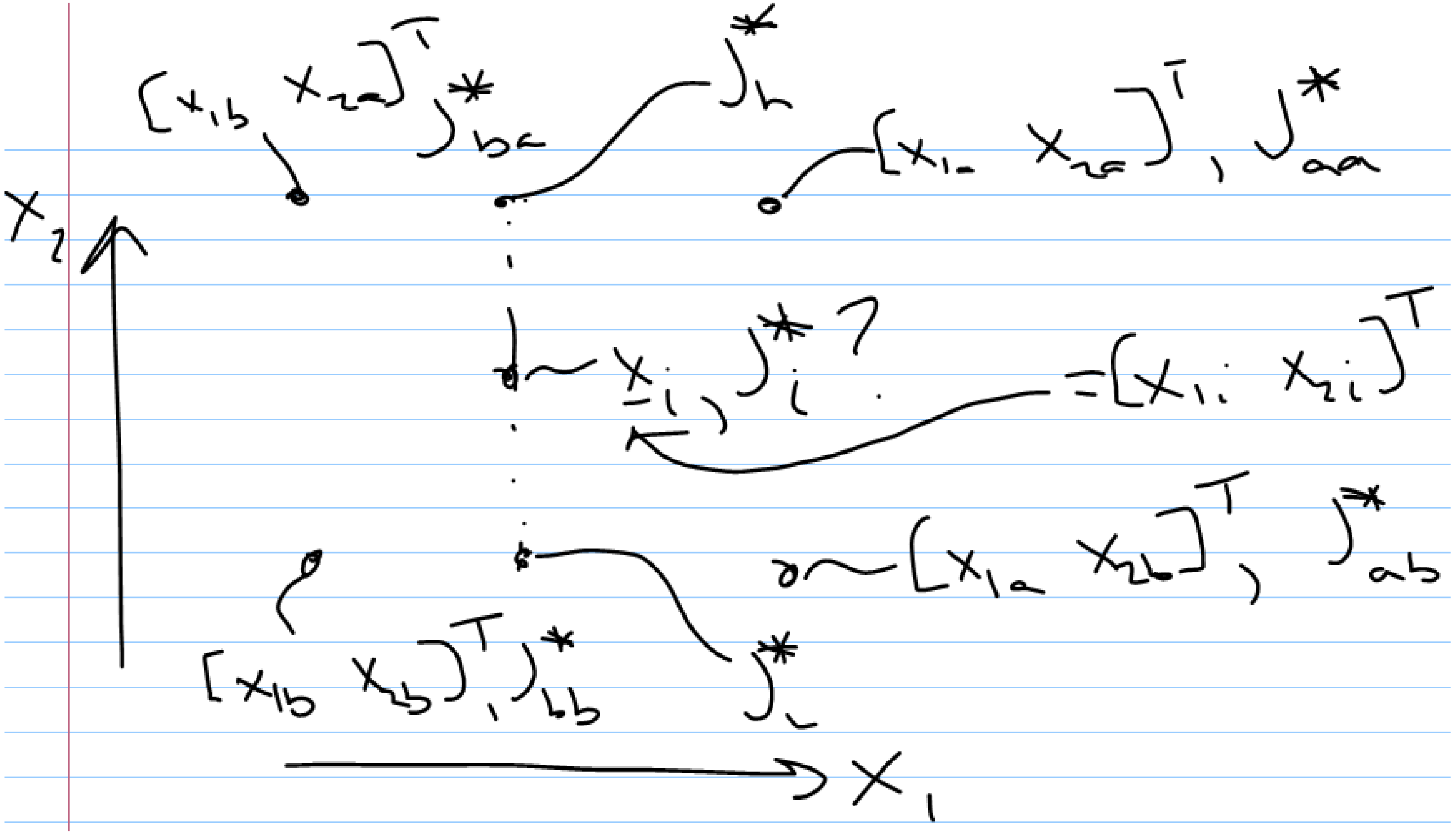
$$\underline{x} = [y \quad \dot{y}]^T \quad \ldots \quad \underline{x} \in \mathbb{R}^2$$

HW 3, P3: Vehicle

$$\underline{x} = [p \quad v]^T \quad \ldots \quad \underline{x} \in \mathbb{R}^2$$

2D "Bilinear" Interpolation – Two State Variables





2D "Bilinear" Interpolation – Two State Variables

$$J^*_h = w_h J^*_{ba} + (1 - w_h) J^*_{aa} = \frac{x_{1a} - x_{1i}}{x_{1a} - x_{1b}} J^*_{ba} + \frac{x_{1b} - x_{1i}}{x_{1a} - x_{1b}} J^*_{aa}$$

$$J^*_L = \frac{x_{1a} - x_{1i}}{x_{1a} - x_{1b}} J^*_{bb} + \frac{x_{1b} - x_{1i}}{x_{1a} - x_{1b}} J^*_{ab}$$

$$J^*_i = \frac{x_{2i} - x_{2b}}{x_{2a} - x_{2b}} J^*_h + \frac{x_{2a} - x_{2i}}{x_{2a} - x_{2b}} J^*_L$$

2D "Bilinear" Interpolation – Two State Variables

$$\underline{u}^*_h = w_h \underline{u}^*_{ba} + (1-w_h)\underline{u}^*_{aa}$$

$$\underline{u}^*_\ell = w_h \underline{u}^*_{bb} + (1-w_h)\underline{u}^*_{ab}$$

$$\Rightarrow \underline{u}^*_i = \frac{x_{2i}-x_{2b}}{x_{2a}-x_{2b}}\underline{u}^*_h + \frac{x_{2a}-x_{2i}}{x_{2a}-x_{2b}}\underline{u}^*_\ell$$

Key point: The interpolation weights on the control sequences are the same as the interpolation weights on the optimal costs to go.

n-D Interpolation – 2+ State Variables

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MATLAB syntax for 2D interpolation:

Vector of x_1 values for which $f(\mathbf{x})$ is tabulated

Matrix of corresponding $f(\mathbf{x})$ values

$\hat{f}(\mathbf{x}_i)$

```
fi = interp2(x1_vec,x2_vec,f_mat,x1i,x2i);
```

Vector of x_2 values for which $f(\mathbf{x})$ is tabulated

x_{1i}

x_{2i}

Fortunately, MATLAB syntax exists for nD linear interpolation

$\hat{f}(\mathbf{x}_i)$

n-dimensional matrix of corresponding $f(\mathbf{x})$ values

Syntax: fi = interpn(x1_vec,…,xn_vec,f_mat,x1i,…xni)

Vectors of $x_1 \ldots x_n$ values for which $f(\mathbf{x})$ is tabulated (note: you need to fill in the "…" in MATLAB

$x_{1i} \ldots x_{ni}$ (note: you need to fill in the "…" in MATLAB

Dynamic Programming – Backward Recursion Example with Interpolation

System dynamics: $x(k+1) = 1.5x(k) + 0.5u(k)$

Optimization problem: Minimize $J(\mathbf{u}, x(0)) = \sum_{i=0}^{N-1} 5x(i+1)^2 + u(i)^2$

Subject to: $-1 \leq x(i) \leq 1, i = 1 \dots N$

where $x(0) = 1$ and $u_i \in \{-3, -2, -1, 0, 1, 2, 3\}, i = 0 \dots N-1$

- Perform dynamic programming, using interpolation, with state variables quantized as $x_i \in \{-1, -0.5, 0, 0.5, 1\}, i = 1 \dots N$ … take $N = 5$ …sample code on Canvas
- Note: This was the same example that gave us problems earlier

Dynamic Programming – Forward Recursion Example

System dynamics: $x(k+1) = 1.5x(k) + 0.5u(k)$

Optimization problem: Minimize $J(\mathbf{u}, x(0)) = \sum_{i=0}^{N-1} 5x(i+1)^2 + u(i)^2$

Subject to: $-1 \leq x(i) \leq 1, i = 1 \dots N$

where $x(0) = 1$ and $u_i \in \{-3, -2, -1, 0, 1, 2, 3\}, i = 0 \dots N-1$

- Determine the optimal control sequence, $\mathbf{u}^*$, using dynamic programming, with state variables quantized as $x_i \in \{-1, -0.5, 0, 0.5, 1\}, i = 1 \dots N$, for $N = 5$ and $N = 10$ …Sample code available on Canvas

- How many cost function evaluations are required? How does this compare with that of an exhaustive search, and how does the number depend on N?

Side Note on Forward Recursion

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Instead of back-computing the predecessor state (then needing to interpolate when that state does not belong to the quantized set), why not compute the required control signal for a specified predecessor state?

$$\underbrace{\underline{x}(k+1)}_{\checkmark} = A\underset{?}{\underline{x}(k)} + \underbrace{Bu(k)}_{\checkmark}$$

$$\underbrace{\underline{x}(k)}_{\checkmark} = A\underset{?}{\underline{x}(k-1)} + \underbrace{Bu(k-1)}_{\checkmark}$$

$$A\underline{x}(k-1) = Bu(k-1) - \underline{x}(k)$$

$$\Rightarrow \underline{x}(k-1) = A^{-1}\left(Bu(k-1) - \underline{x}(k)\right)$$

Key point: Predecessor state can be computed reasonably easily.

Side Note on Forward Recursion

Instead of back-computing the predecessor state (then needing to interpolate when that state does not belong to the quantized set), why not compute the required control signal for a specified predecessor state?

$$\underbrace{\underline{x}(k)}_{\checkmark} = \underbrace{A\underline{x}(k-1)}_{\checkmark} + \underbrace{Bu(k-1)}_{?}$$

$$\Rightarrow Bu(k-1) = \underline{x}(k) - A\underline{x}(k-1)$$

$$\Rightarrow u(k-1) = B^{-1}(\underline{x}(k) - A\underline{x}(k-1))$$

not square.

Key point: Required control input cannot be computed in general, since B is non-square (and is generally narrow, so a solution will typically not exist)

Dynamic Programming – Insights

Key question: When our recursive algorithm is complete and the dust has settled, what have we learned?

With ***backward recursion***, we have learned the optimal control sequence for all candidate (gridded) ***initial*** states (values of $\mathbf{x}(0)$)

- …And we can specify a constraint set on terminal states (or any intermediate state)

With ***forward recursion***, we have learned the optimal control sequence for all candidate (gridded) ***terminal*** states (values of $\mathbf{x}(N)$)

- …And we can specify a constraint set of initial states (or any intermediate state)

Dynamic Programming – Insights

Key question: Just how ***global*** is dynamic programming? After all, we know that the faster convex optimization techniques (LP, QP, SQP) are ***just local***, and an exhaustive search is ***global but slow***

Answer: The dynamic programming solution is ***globally optimal up to the grid resolution*** (sometimes referred to as the “mesh”)

Interpretation: There may be control sequences, $\mathbf{u}$, for which $J(\mathbf{u}) < J(\mathbf{u}_{DP}^*)$, but these sequences include elements that are not in the quantized set of control signals (i.e., $\exists u_i$ such that $u_i \in \{\bar{u}_1, \ldots, \bar{u}_p\}$)

Dynamic Programming – Computational Complexity

Key question: How many objective function evaluations are required by dynamic programming?

Answer: At every stage (there are N of these), for every candidate state variable (there are q of these), each candidate control variable must be evaluated (there are p of these). So the number of objective function evaluations (E) is:

$$E = Npq$$

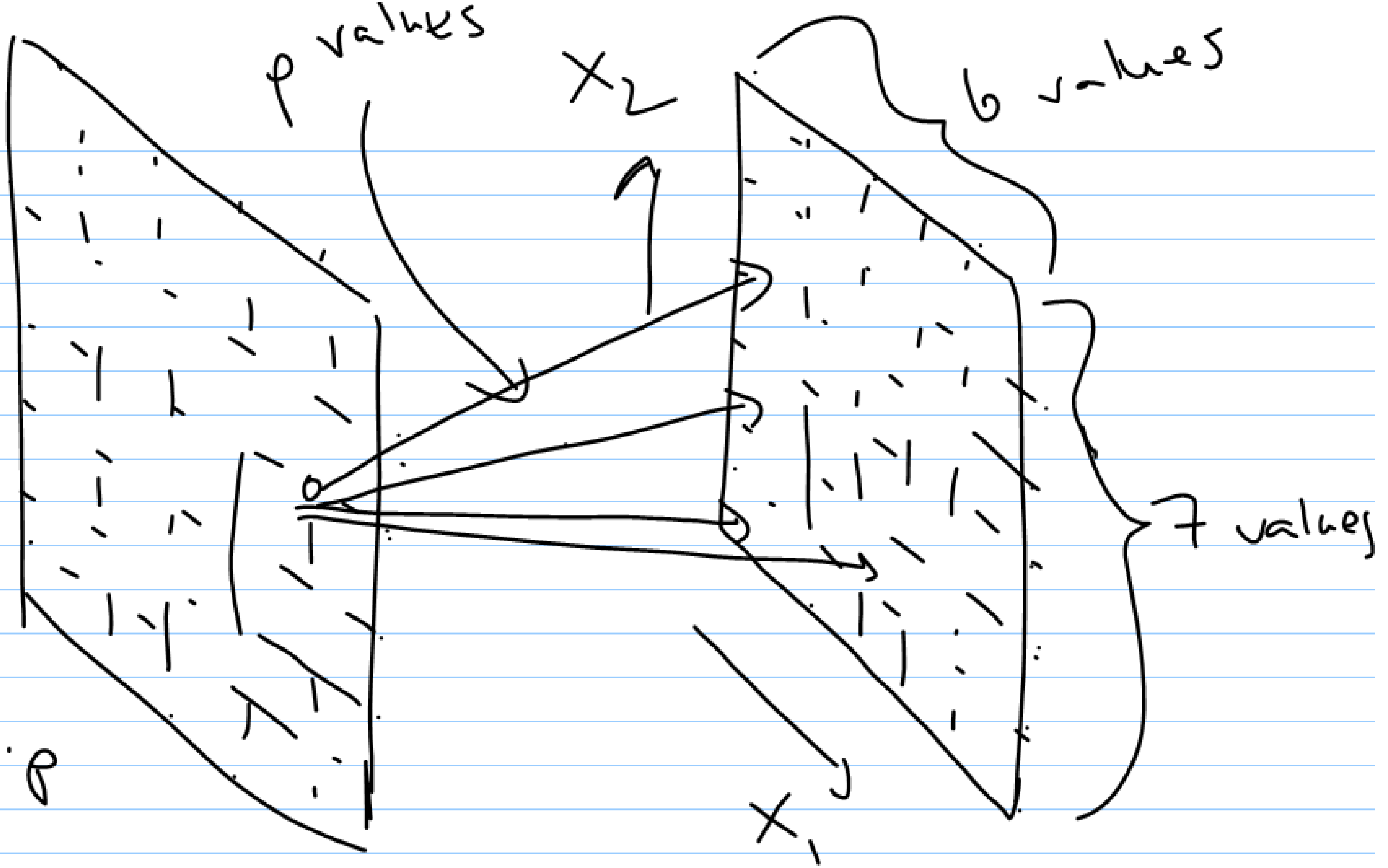
Interpretation:

- Computational complexity is **linear** in the horizon length (N), number of candidate state variables (q), and number of candidate control variables (p) – Nice!
- …but suppose there are n states, each of which is quantized into $\bar{q}$ values. Then $q = \bar{q}^n$, and $E = Np\bar{q}^n$ …this is known as the ***curse of dimensionality!***

Dynamic Programming – Computational Complexity

Total cost fn. evals. per stage = $6 \cdot 7 \cdot p$

$\Rightarrow$ Total cost fn evals. = $6 \cdot 7 \cdot N \cdot p$



Dynamic Programming – Computational Complexity

In general, total cost fun. evals: $N_p \cdot \prod_{i=1}^{\dim} q_i$,

"curse of dimensionality"

In our ex.,
$q_1 = 6,\ q_2 = 7$

Preview of Upcoming Lectures

More dynamic programming:

- More sophisticated examples
- Using Bellman's principle of optimality to derive the control law for the discrete-time linear quadratic regulator