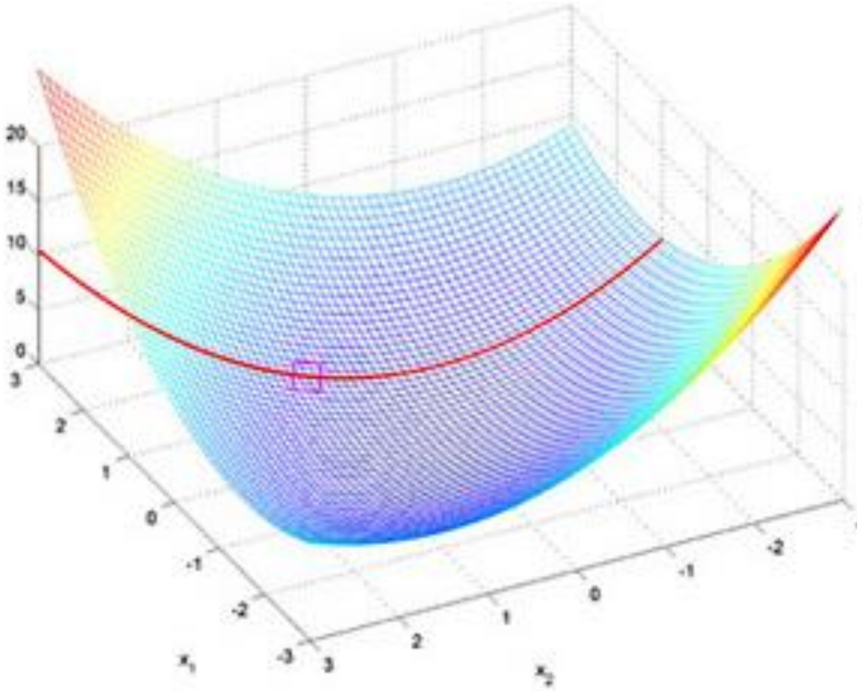
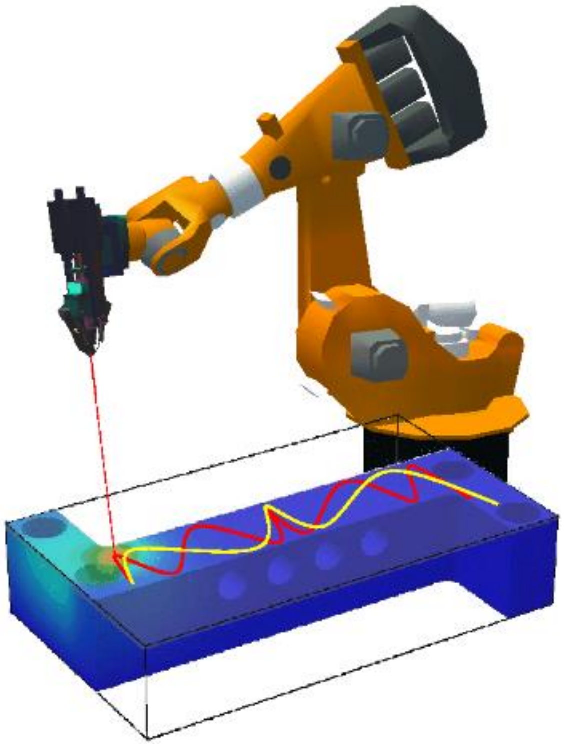
MEGR 3090/7090/8090: Advanced Optimal Control



$$\boxed{V_n(\mathbf{x}_n) = \min_{\{\mathbf{u}_n,\mathbf{u}_{n+1},\cdots,\mathbf{u}_{N-1}\}} \left[\frac{1}{2}\sum_{k=n}^{N-1}\left(\mathbf{x}_k^T\mathbf{Q}_k\mathbf{x}_k + \mathbf{u}_k^T\mathbf{R}\mathbf{u}_k\right) + \frac{1}{2}\mathbf{x}_N^T\mathbf{Q}_N\mathbf{x}_N\right]}$$

$$\begin{aligned}
V_n(\mathbf{x}_n) &= \min_{\{\mathbf{u}_n,\mathbf{u}_{n+1},\cdots,\mathbf{u}_{N-1}\}} \left[\frac{1}{2}\sum_{k=n}^{N-1}\left(\mathbf{x}_k^T\mathbf{Q}_k\mathbf{x}_k + \mathbf{u}_k^T\mathbf{R}\mathbf{u}_k\right) + \frac{1}{2}\mathbf{x}_N^T\mathbf{Q}_N\mathbf{x}_N\right] \\
&= \min_{\mathbf{u}_n}\left[\frac{1}{2}\left(\mathbf{x}_n^T\mathbf{Q}_n\mathbf{x}_n + \mathbf{u}_n^T\mathbf{R}\mathbf{u}_n\right) + \underbrace{\min_{\{\mathbf{u}_{n+1},\cdots,\mathbf{u}_{N-1}\}} \left[\frac{1}{2}\sum_{k=n+1}^{N-1}\left(\mathbf{x}_k^T\mathbf{Q}_k\mathbf{x}_k + \mathbf{u}_k^T\mathbf{R}\mathbf{u}_k\right) + \frac{1}{2}\mathbf{x}_N^T\mathbf{Q}_N\mathbf{x}_N\right]}_{V_{n+1}(\mathbf{x}_{n+1})}\right] \\
&= \min_{\mathbf{u}_n}\left[\frac{1}{2}\left(\mathbf{x}_n^T\mathbf{Q}_n\mathbf{x}_n + \mathbf{u}_n^T\mathbf{R}\mathbf{u}_n\right) + V_{n+1}(\mathbf{x}_{n+1})\right]
\end{aligned}$$

$$\boxed{V_n(\mathbf{x}_n) = \min_{\mathbf{u}_n}\left[\frac{1}{2}\left(\mathbf{x}_n^T\mathbf{Q}_n\mathbf{x}_n + \mathbf{u}_n^T\mathbf{R}\mathbf{u}_n\right) + V_{n+1}(\mathbf{x}_{n+1})\right]}$$



Lecture 8

September 14, 2017

General ***Constrained*** Optimization Framework

Mathematical framework: $$\mathbf{u}^* = \arg\min_{\mathbf{u}} J(\mathbf{u})$$

Subject to: $\boxed{g(\mathbf{u}) \leq 0}$ ← **Inequality constraints**

$\boxed{h(\mathbf{u}) = 0}$ ← **Equality constraints**

Note – General constraints can be written in the form above…

Examples:

- $u_1 + u_2^2 \leq 7$ **can be written as** $u_1 + u_2^2 - 7 \leq 0$ (here, $g(\mathbf{u}) = u_1 + u_2^2 - 7$)
- $u_1 + u_2^2 = 7$ **can be written as** $u_1 + u_2^2 - 7 = 0$ (here, $h(\mathbf{u}) = u_1 + u_2^2 - 7$)

General ***Constrained*** Optimization Framework

3 ingredients to optimal control problems:

1) Objective function (i.e., cost function)
2) Dynamic model

Unconstrained optimal control involves these

3) Constraints

1 ingredient to unconstrained design optimization:

1) Objective function

To use unconstrained design optimization tools to solve an unconstrained optimal control problem, use $\underbrace{\underline{x}(k+1) = f(\underline{x}(k), u(k))}_{\text{dyn. model}}$

to write $J(\underline{u}; \underline{x}(0))$ entirely in terms of $\underline{u}$ & $\underline{x}(0)$

(i.e., eliminate $\underline{x}(i)$, $i = 1 \ldots N$), then throw away dyn. model.

General ***Constrained*** Optimization Framework

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2 ingredients to constrained design optimization:

1) Obj. fn.
2) Constraints

Minimize $J(\underline{u})$

Subject to $\underline{u} \in U$ ← a set

To use constrained design optimization tools to solve a constrained optimal control problem, we must:

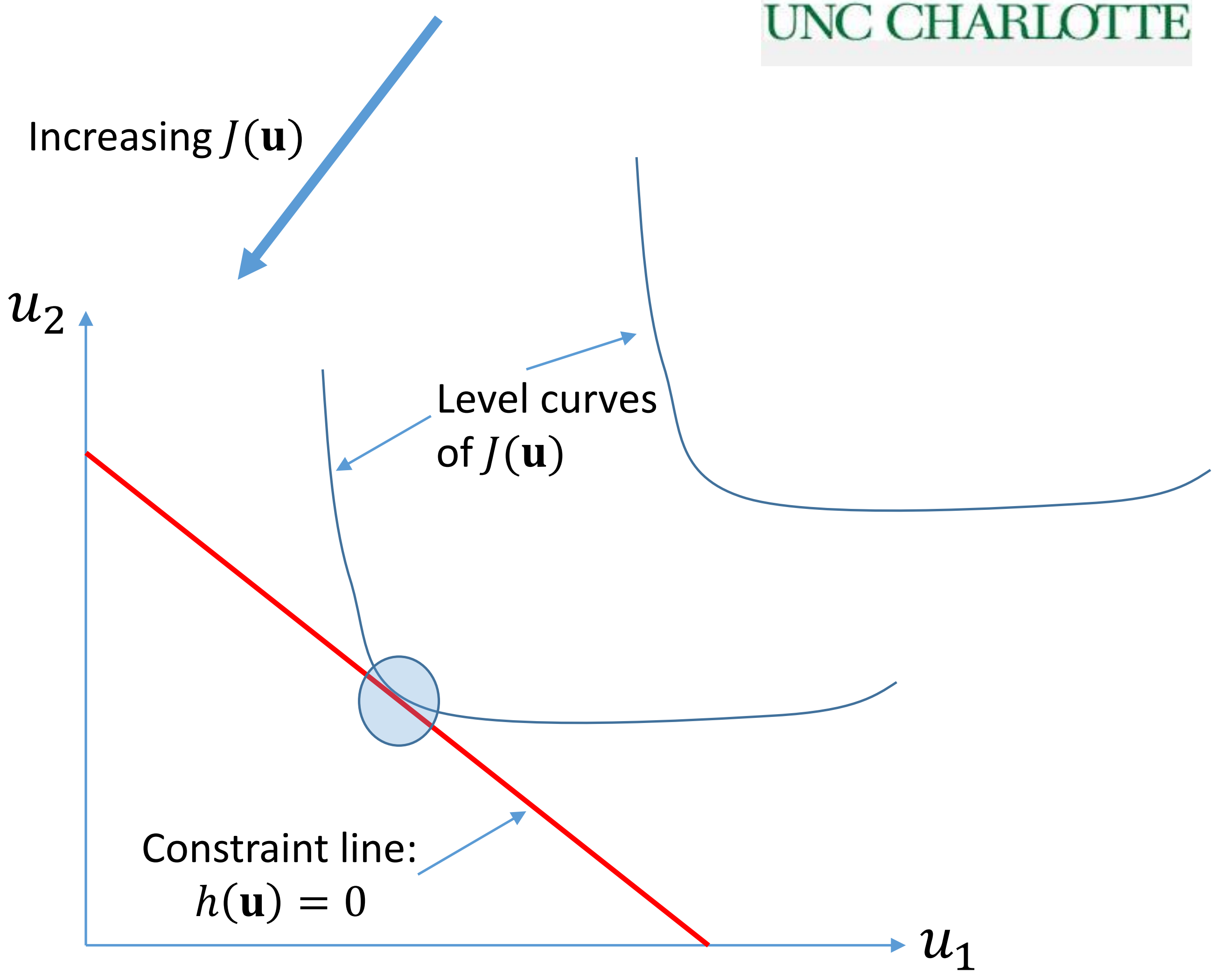
1) Use $\underline{x}(k+1) = f(\underline{x}(k), u(k))$ to write $J(\underline{u}; \underline{x}(0))$ entirely in terms of $\underline{u}$ & $\underline{x}(0)$.
2) Learn how to solve the constrained design optimization problem.

Constrained Optimization – Single *Equality* Constraint

Optimization problem:

Minimize $J(\mathbf{u})$

Subject to: $h(\mathbf{u}) = 0$



Key point: The gradient of $J(\mathbf{u})$ and the gradient of $h(\mathbf{u})$ **must point in the same direction at the optimum point $(\mathbf{u}^*)$**

$$\Longrightarrow \nabla J(\mathbf{u}^*) = -\lambda \nabla h(\mathbf{u}^*) \text{ for some } \lambda \in \mathbb{R}$$
$$\Longrightarrow \nabla J(\mathbf{u}^*) + \lambda \nabla h(\mathbf{u}^*) = 0$$

Constrained Optimization – Single *Equality* Constraint

If two vectors $\underline{x}$ and $\underline{y}$ lie in the same direction, then there exists $\lambda \in \mathbb{R}$ s.t. $\underline{x} = -\lambda \underline{y}$

$$\Rightarrow \underline{x} + \lambda \underline{y} = \underline{0}$$

Goal: Minimize $J(\underline{u})$
Subject to $\underbrace{h(\underline{u})}_{\in \mathbb{R} \Rightarrow 1 \text{ equation}} = 0$ } 1 constraint

$$\underbrace{\nabla J(\underline{u}^*)}_{\in \mathbb{R}^{1\times N}} + \lambda \underbrace{\nabla h(\underline{u}^*)}_{\in \mathbb{R}^{1\times N}} = 0 \Rightarrow N \text{ equations}$$

} N+1 equations.

Unknowns: $\underbrace{\underline{u}^*}_{\in \mathbb{R}^N}$, $\underbrace{\lambda}_{\in \mathbb{R}}$ } N+1 unknowns

Note – Lagrange Multipliers and the Lagrangian

Optimization problem: Minimize $J(\mathbf{u})$

Subject to: $h(\mathbf{u}) = 0$

Key point: The gradient of $J(\mathbf{u})$ and the gradient of $h(\mathbf{u})$ **must point in the same direction at the optimum point** $(\mathbf{u}^*)$ - λ is called the Lagrange multiplier

$$\Rightarrow \nabla J(\mathbf{u}^*) = -\lambda \nabla h(\mathbf{u}^*) \text{ for some } \lambda \in \mathbb{R}$$
$$\Rightarrow \nabla J(\mathbf{u}^*) + \lambda \nabla h(\mathbf{u}^*) = 0$$

Corollary: The **constrained** optimization problem above can be cast as the following **unconstrained** optimization problem:

Minimize $L(\mathbf{u}, \lambda) = J(\mathbf{u}) + \lambda h(\mathbf{u})$…$L(\mathbf{u}, \lambda)$ is sometimes called the Lagrangian

Optimization with a Single Equality Constraint - Example

Optimization problem: Minimize $J(\mathbf{u}) = u_1^2 + u_2^2$

Subject to: $2u_1 + u_2 = 4$

Key questions:

- What is $h(\mathbf{u})$ in this case?
- What is the geometric interpretation of this optimization problem (draw the constraint line and level curves)?
- What numerical value is obtained for λ? What is the interpretation of that numerical value?

Optimization with a Single Equality Constraint - Example

Example: Minimize $J(\underline{u}) = u_1^2 + u_2^2$

Subject to: $2u_1 + u_2 = 4$

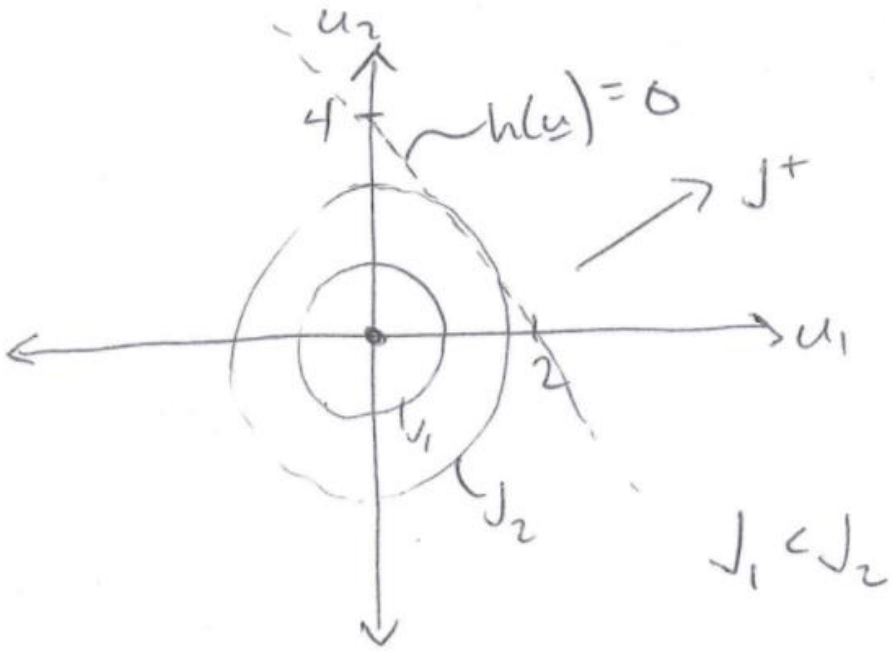
Constraint: $\underbrace{2u_1 + u_2 - 4}_{h(\underline{u})} = 0$

i) $h(\underline{u})$?

ii) Geometric interpretation of problem

iii) $\underline{u}^*$

iv) Significance of λ (bone-in ribeye?)



Optimization with a Single Equality Constraint - Example

$$\nabla J(\underline{u}^*) + \lambda \nabla h(\underline{u}^*) = 0$$

$$[2u_1^* \quad 2u_2^*] + \lambda [2 \quad 1] = 0$$

$$\Rightarrow \left.\begin{aligned} 2u_1^* + 2\lambda &= 0 \\ 2u_2^* + \lambda &= 0 \\ 2u_1^* + u_2^* &= 4 \end{aligned}\right\} \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} u_1^* \\ u_2^* \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

$$\boxed{\underline{u}^* = [1.6 \quad 0.8]^*} \qquad \lambda = -1.6$$

$\Uparrow$

λ indicates the sensitivity of $J(\underline{u}^*)$ to variations in $\underbrace{h(\underline{u})}_{\text{constraint}}$.

Constrained Optimization – Single *Inequality* Constraint

Optimization problem:

Minimize $J(\mathbf{u})$

Subject to: $g(\mathbf{u}) \leq 0$

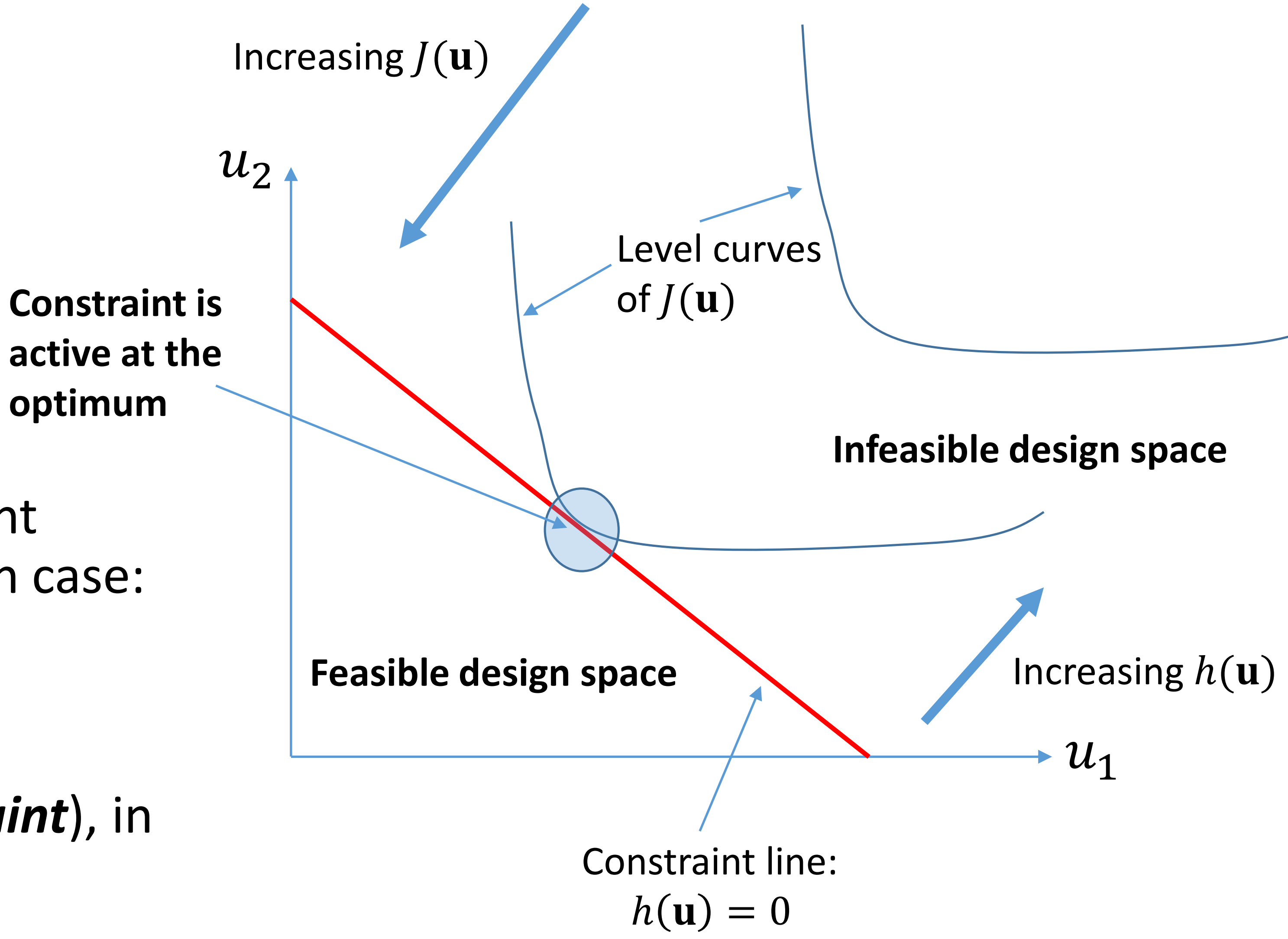
Two possibilities:

- The optimum ($\mathbf{u}^*$) lies on the constraint surface (an ***active constraint***), in which case:

$$\Longrightarrow \nabla J(\mathbf{u}^*) + \lambda \nabla g(\mathbf{u}^*) = 0$$

- The optimum (u^*) does not lie on the constraint surface (an ***inactive constraint***), in which case:

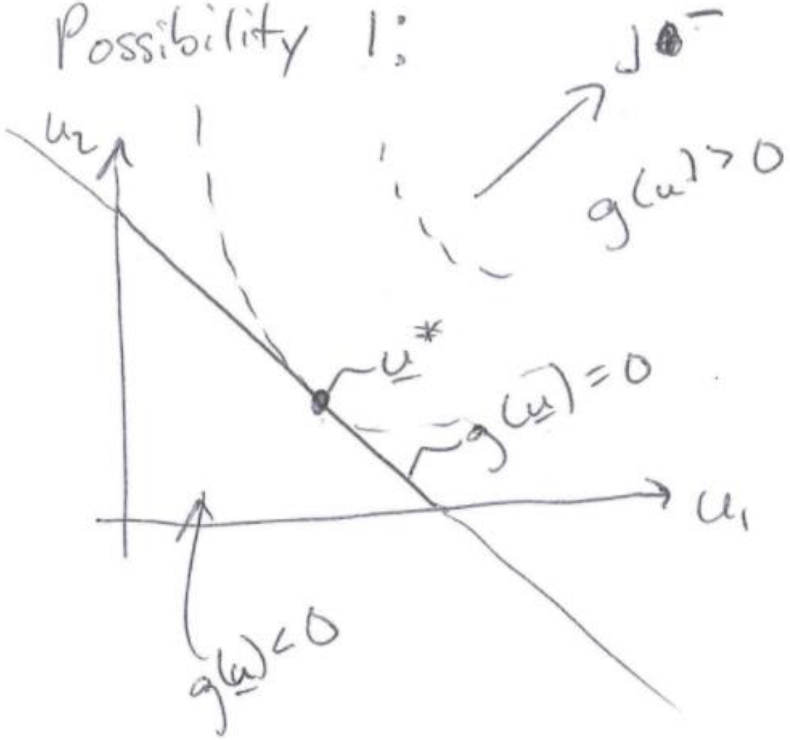
$$\Longrightarrow \nabla J(\mathbf{u}^*) = 0$$



Constrained Optimization – Single *<u>Inequality</u>* Constraint

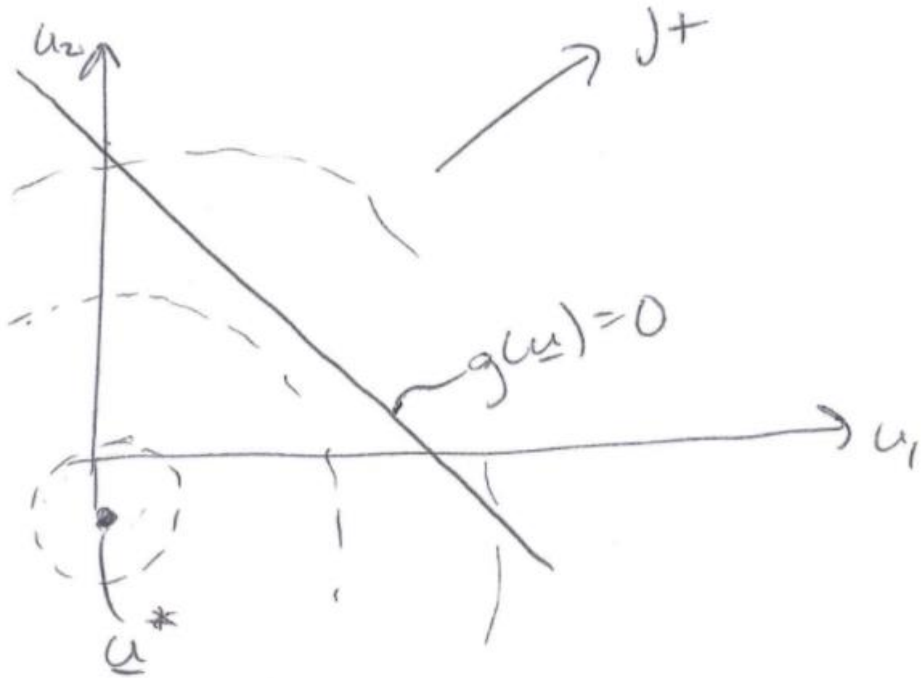
Goal: Minimize $J(\underline{u})$ subj. to $g(\underline{u}) \le 0$

Possibility 1:



Active constraint

Possibility 2:



Inactive constraint.

Constrained Optimization – Single *Inequality* Constraint

Optimization problem: Minimize $J(\mathbf{u})$

Subject to: $g(\mathbf{u}) \leq 0$

Two possibilities (reminder):

- The optimum ($\mathbf{u}^*$) lies on the constraint surface (an ***active constraint***), in which case:

$$\nabla J(\mathbf{u}^*) + \lambda \nabla g(\mathbf{u}^*) = 0$$

- The optimum ($\mathbf{u}^*$) does not lie on the constraint surface (an ***inactive constraint***), in which case: $\nabla J(\mathbf{u}^*) = 0$

How to determine which possibility yields the true optimum:

- Try the inactive constraint approach first. If it yields a solution that **satisfies the inequality constraint**, you're done
- If the inactive constraint approach results in $\mathbf{u}^*$ that violates $g(\mathbf{u}^*) \leq 0$, then solve the active constraint optimization problem (an equality problem)

Constrained Optimization – Single *Inequality* Constraint

Inequality constraint mathematical solution for a single constraint:

$$\nabla J(\underline{u}^*) + \mu \nabla g(\underline{u}^*) = 0 \Longleftarrow N \text{ eq's.}$$

$$\left.\mu\, g(\underline{u}^*) = 0\right\} \text{ Either } \mu = 0 \leftarrow \text{inactive}$$

$$\text{or } g(\underline{u}^*) = 0 \leftarrow \text{active}$$

$$g(\underline{u}^*) \leq 0$$

complementary slackness. ← 1 eq.

Constrained Optimization – Generalization to Multiple *Equality* Constraints

Optimization problem:

Minimize $J(\mathbf{u})$

Subject to: $h(\mathbf{u}) = \mathbf{0}$

Note that $h(\mathbf{u}) \in \mathbb{R}^m$, where m is the number of equality constraints

Key point: The gradient of $J(\mathbf{u})$ **must be a linear combination** of the gradients of $h(\mathbf{u})$ **at the optimum point** ($\mathbf{u}^*$)

$$\Rightarrow \nabla J(\mathbf{u}^*) = -\boldsymbol{\lambda}^T \nabla h(\mathbf{u}^*) \text{ for some } \boldsymbol{\lambda} \in \mathbb{R}^m$$

$$\Rightarrow \nabla J(\mathbf{u}^*) + \boldsymbol{\lambda}^T \nabla h(\mathbf{u}^*) = 0$$

- $\nabla h(\mathbf{u}^*)$ is now an $m \times p$ matrix, where m is the number of constraints and p is the number of the design variables
- $\boldsymbol{\lambda}$ is now an m-element vector (one Lagrange multiplier for each constraint)

Constrained Optimization – Generalization to Multiple *Equality* Constraints

$$\nabla J(\underline{u}^*) = -\underline{\lambda}^T \nabla h(\underline{u}^*)$$

$$\Rightarrow \underbrace{\nabla J(\underline{u}^*)}_{\in \mathbb{R}^{1\times N}} + \underbrace{\underline{\lambda}}_{\in \mathbb{R}^{M\times 1}}{}^T \underbrace{\nabla \underline{h}(\underline{u}^*)}_{\in \mathbb{R}^{M\times N}} = 0$$

Note: p was used in place of N in the notes.

Generalization to Multiple Equality Constraints - Example

Optimization problem:

Minimize $J(\mathbf{u}) = u_1^2 + u_2^2 + u_3^2$

Subject to:
$$5u_1 + 4u_2 + u_3 = 20$$
$$u_1 + u_2 - u_3 = 0$$

Key steps:

- Write both $\nabla J(\mathbf{u})$ and $\nabla h(\mathbf{u})$ in vector/matrix form
- Set up a system of equations (note – what are the unknowns?)
- Solve the system of equations

Generalization to Multiple Equality Constraints - Example

Minimize $J(\underline{u}) = u_1^2 + u_2^2 + u_3^2$

Subj. to: $5u_1 + 4u_2 + u_3 = 20$ (4)

$u_1 + u_2 - u_3 = 0$ (5)

$$h(\underline{u}^*) = \begin{bmatrix} 5u_1 + 4u_2 + u_3 - 20 \\ u_1 + u_2 - u_3 \end{bmatrix}$$

$$\nabla h(\underline{u}^*) = \begin{bmatrix} 5 & 4 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\nabla J(\underline{u}^*) = [2u_1^* \quad 2u_2^* \quad 2u_3^*]$$

Condition: $\nabla J(\underline{u}^*) + [\lambda_1 \quad \lambda_2] \begin{bmatrix} 5 & 4 & 1 \\ 1 & 1 & -1 \end{bmatrix} = 0$

Generalization to Multiple Equality Constraints - Example

$$\Rightarrow [2u_1^* \quad 2u_2^* \quad 2u_3^*] + [5\lambda_1 + \lambda_2 \quad 4\lambda_1 + \lambda_2 \quad \lambda_1 - \lambda_2] = 0$$

$$\Rightarrow 2u_1^* + 5\lambda_1 + \lambda_2 = 0 \quad (1)$$

$$2u_2^* + 4\lambda_1 + \lambda_2 = 0 \quad (2)$$

$$2u_3^* + \lambda_1 - \lambda_2 = 0 \quad (3)$$

$$\begin{bmatrix} 2 & 0 & 0 & 5 & 1 \\ 0 & 2 & 0 & 4 & 1 \\ 0 & 0 & 2 & 1 & -1 \\ 5 & 4 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1^* \\ u_2^* \\ u_3^* \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 20 \\ 0 \end{bmatrix}$$

$$\underline{u}^* = [2.26 \quad 1.29 \quad 3.55]^T$$

Constrained Optimization – Generalization to Multiple *Inequality* Constraints

Optimization problem: Minimize $J(\mathbf{u})$

Subject to: $g(\mathbf{u}) \leq \mathbf{0}$

Optimality requirements: $\nabla J(\mathbf{u}^*) + \boldsymbol{\mu}^T \nabla g(\mathbf{u}^*) = 0$

Same condition as for equality constraints… The "patch" below addresses inactive constraints

$$\boldsymbol{\mu}^T g(\mathbf{u}^*) = 0$$

If a constraint is inactive, then $\mu_i = 0$, thereby "turning off" the consideration of the constraint in the above equation. This is called the *complementary slackness constraint.*

$$\mu_i \geq 0, \forall i$$

Generalization to Multiple Inequality Constraints - Example

Optimization problem: Minimize $J(\mathbf{u}) = 8u_1^2 - 8u_1u_2 + 3u_2^2$

Subject to:
$$u_1 - 4u_2 \leq -3$$
$$u_1 - 2u_2 \geq 0$$

Lingering Issues and Preview of Next Lecture

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Combined equality and inequality constraints:

- Relatively straightforward, given today's results
- Results in the famous Karush-Kuhn-Tucker (KKT) conditions

Necessity vs. sufficiency:

- Mathematical criteria we derived so far were ***necessary*** but ***not sufficient*** to guarantee $\mathbf{u}^*$ was indeed a ***unique minimizer***
- The criteria discussed today could also be satisfied if $\mathbf{u}^*$ were a maximizer or if other local optima existed
- ***Convexity of the constraint set*** turns out to be a ***sufficient condition*** for today's conditions to guarantee a ***unique minimizer***…we will discuss convex sets in the next lecture