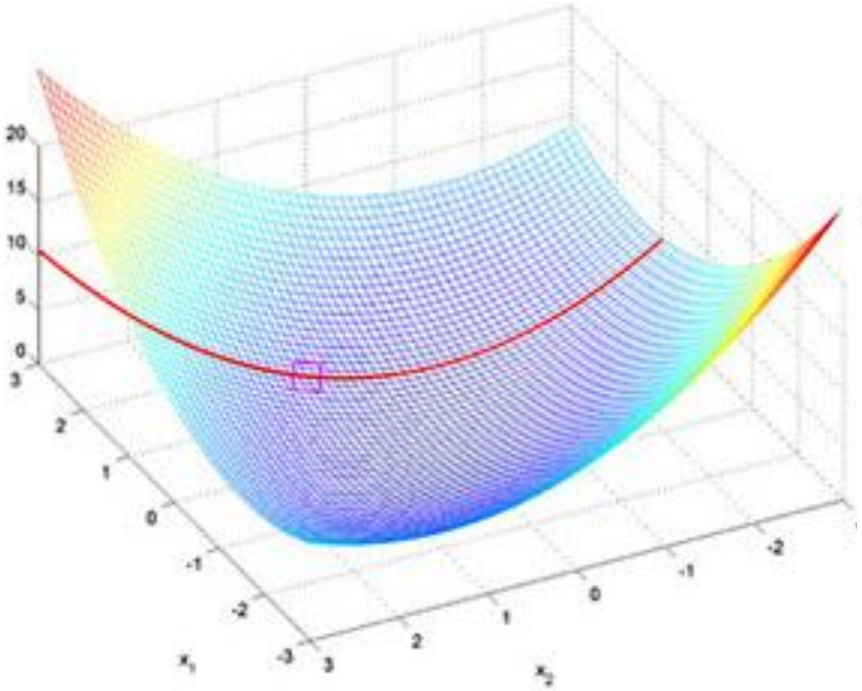
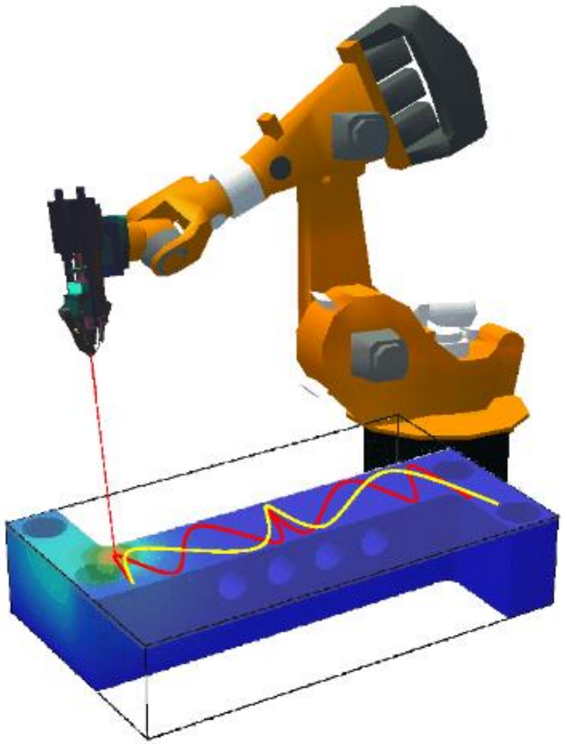
MEGR 3090/7090/8090: Advanced Optimal Control



$$\boxed{V_n(\mathbf{x}_n) = \min_{\{\mathbf{u}_n,\mathbf{u}_{n+1},\cdots,\mathbf{u}_{N-1}\}} \left[\frac{1}{2}\sum_{k=n}^{N-1}\left(\mathbf{x}_k^T\mathbf{Q}_k\mathbf{x}_k + \mathbf{u}_k^T\mathbf{R}\mathbf{u}_k\right) + \frac{1}{2}\mathbf{x}_N^T\mathbf{Q}_N\mathbf{x}_N \right]}$$

$$\begin{aligned}
V_n(\mathbf{x}_n) &= \min_{\{\mathbf{u}_n,\mathbf{u}_{n+1},\cdots,\mathbf{u}_{N-1}\}} \left[\frac{1}{2}\sum_{k=n}^{N-1}\left(\mathbf{x}_k^T\mathbf{Q}_k\mathbf{x}_k + \mathbf{u}_k^T\mathbf{R}\mathbf{u}_k\right) + \frac{1}{2}\mathbf{x}_N^T\mathbf{Q}_N\mathbf{x}_N \right] \\
&= \min_{\mathbf{u}_n}\left[\frac{1}{2}\left(\mathbf{x}_n^T\mathbf{Q}_n\mathbf{x}_n + \mathbf{u}_n^T\mathbf{R}\mathbf{u}_n\right) + \underbrace{\min_{\{\mathbf{u}_{n+1},\cdots,\mathbf{u}_{N-1}\}} \left[\frac{1}{2}\sum_{k=n+1}^{N-1}\left(\mathbf{x}_k^T\mathbf{Q}_k\mathbf{x}_k + \mathbf{u}_k^T\mathbf{R}\mathbf{u}_k\right) + \frac{1}{2}\mathbf{x}_N^T\mathbf{Q}_N\mathbf{x}_N \right]}_{V_{n+1}(\mathbf{x}_{n+1})} \right] \\
&= \min_{\mathbf{u}_n}\left[\frac{1}{2}\left(\mathbf{x}_n^T\mathbf{Q}_n\mathbf{x}_n + \mathbf{u}_n^T\mathbf{R}\mathbf{u}_n\right) + V_{n+1}(\mathbf{x}_{n+1}) \right]
\end{aligned}$$

$$\boxed{V_n(\mathbf{x}_n) = \min_{\mathbf{u}_n}\left[\frac{1}{2}\left(\mathbf{x}_n^T\mathbf{Q}_n\mathbf{x}_n + \mathbf{u}_n^T\mathbf{R}\mathbf{u}_n\right) + V_{n+1}(\mathbf{x}_{n+1}) \right]}$$



Lecture 9

September 19, 2017

Constrained Optimization with <u>*Equality*</u> Constraints - Reminder

Optimization problem:

Minimize $J(\mathbf{u})$

Subject to: $h(\mathbf{u}) = \mathbf{0}$

Note that $h(\mathbf{u}) \in \mathbb{R}^m$, where m is the number of equality constraints

Key point: The gradient of $J(\mathbf{u})$ **must be a linear combination** of the gradients of $h(\mathbf{u})$ **at the optimum point** $(\mathbf{u}^*)$

$$\Longrightarrow \nabla J(\mathbf{u}^*) = -\boldsymbol{\lambda}^T \nabla h(\mathbf{u}^*) \text{ for some } \boldsymbol{\lambda} \in \mathbb{R}^m$$
$$\Longrightarrow \nabla J(\mathbf{u}^*) + \boldsymbol{\lambda}^T \nabla h(\mathbf{u}^*) = 0$$

- $\nabla h(\mathbf{u}^*)$ is now an $m \times p$ matrix, where m is the number of constraints and p is the number of the design variables
- $\boldsymbol{\lambda}$ is now an m-element vector (one Lagrange multiplier for each constraint)

Constrained Optimization with *Inequality* Constraints - Reminder

Optimization problem: Minimize $J(\mathbf{u})$

Subject to: $g(\mathbf{u}) \leq \mathbf{0}$

Optimality requirements: $\nabla J(\mathbf{u}^*) + \boldsymbol{\mu}^T \nabla g(\mathbf{u}^*) = 0$

Same condition as for equality constraints… The "patch" below addresses inactive constraints

$$\boldsymbol{\mu}^T g(\mathbf{u}^*) = 0$$
$$\Longrightarrow \mu_i g(u_i^*) = 0, \forall i$$

If a constraint is inactive, then $\mu_i = 0$, thereby "turning off" the consideration of the constraint in the above equation. This is called the *complementary slackness constraint.*

$$\mu_i \geq 0, \forall i$$

Constrained Optimization with ***Mixed*** Constraints – The Karush-Kuhn-Tucker (KKT) Conditions

Optimization problem: Minimize $J(\mathbf{u})$

Subject to: $g(\mathbf{u}) \leq \mathbf{0}$

$h(\mathbf{u}) = \mathbf{0}$

Optimality requirements: $\nabla J(\mathbf{u}^*) + \boldsymbol{\mu}^T \nabla g(\mathbf{u}^*) + \boldsymbol{\lambda}^T \nabla h(\mathbf{u}^*) = 0$

Combines additional terms from inequality and equality constraints

Complementary slackness: $\mu_i g(u_i^*) = 0, \forall i$

Feasibility: $\mu_i \geq 0, \forall i$

Constraint satisfaction: $g(\mathbf{u}^*) \leq 0$

$h(\mathbf{u}^*) = \mathbf{0}$

Note: If $\mathbf{u}^*$ is a minimizer, then the KKT conditions will be satisfied (i.e., they are necessary). In general, however, satisfaction of the KKT conditions does not guarantee that u^* is a unique global minimizer (or even a minimizer)!

KKT Conditions - Assessment

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Notation: p = number of decision variables, q = number of inequality constraints, r = number of equality constraints

Optimality requirements – equalities that must be satisfied:

$$\nabla J(\mathbf{u}^*) + \boldsymbol{\mu}^T \nabla g(\mathbf{u}^*) + \boldsymbol{\lambda}^T \nabla h(\mathbf{u}^*) = \mathbf{0} \longleftarrow p \text{ equations}$$

$$\mu_i g(u_i^*) = 0, \forall i \longleftarrow q \text{ equations}$$

$$h(\mathbf{u}^*) = \mathbf{0} \longleftarrow r \text{ equations}$$

$p + q + r$ equations,
$p + q + r$ unknowns

Optimality requirements – inequalities that must be satisfied:

$$g(\mathbf{u}^*) \leq \mathbf{0} \longleftarrow q \text{ inequalities}$$

$$\mu_i \geq 0, \forall i \longleftarrow q \text{ more inequalities}$$

$2q$ inequalities,
$p + q$ unknowns (same as earlier unknowns)

KKT Conditions - Sufficiency

In general, the KKT conditions are merely **necessary** for a minimizer but do not guarantee that a minimizer exists

There are a couple of *<u>very important</u>* exceptions…

Exception #1:

Define $L(\mathbf{u}, \boldsymbol{\mu}, \boldsymbol{\lambda}) \triangleq J(\mathbf{u}) + \boldsymbol{\mu} g(\mathbf{u}) + \boldsymbol{\lambda} h(\mathbf{u})$…this is the **generalized Lagrangian** (accounting for inequality constraints)

Suppose that the hessian of the generalized Lagrangian is positive definite at $\mathbf{u}^*$ (i.e., $H(L(\mathbf{u}^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) \succ 0)$, where $\mathbf{u}^*$ satisfies the KKT conditions. Then $\mathbf{u}^*$ is a **local minimizer** (still no guarantees about globality)

Notes About the Lagrangian

Condition for optimality:

$$\nabla J(\underline{u}^*) + \underline{\mu}^T \nabla g(\underline{u}^*) + \underline{\lambda}^T h(\underline{u}^*) = 0$$

From the notes: $L(\underline{u}, \underline{\mu}, \underline{\lambda}) = J(\underline{u}) + \underline{\mu}^T g(\underline{u}) + \underline{\lambda}^T h(\underline{u})$

Minimizing $J(\underline{u})$ subj. to constraints $(g(\underline{u}) \leq 0,\ h(\underline{u}) = 0)$ is equivalent to minimizing $L(\underline{u}, \underline{\mu}, \underline{\lambda})$ subject to the same constraints.

KKT Conditions - Sufficiency

In general, the KKT conditions are merely **necessary** for a minimizer but do not guarantee that a minimizer exists

There are a couple of *very important* exceptions…

Exception #2:

Suppose that $J(\mathbf{u})$ is a (globally) convex ***function***, $\{\mathbf{u}: g(\mathbf{u}) \leq 0\}$ and $\{\mathbf{u}: h(\mathbf{u}) = 0\}$ are both convex **sets**, and the KKT conditions are satisfied at $\mathbf{u}^*$. Then $\mathbf{u}^*$ is a **unique global minimizer!!**

- This is why convexity is such a big deal in optimization and optimal control
- In order to make use of the above exception, we need to understand the notion of a ***convex set***
- From time to time, you might hear people talk about convex optimization ***problems*** and wonder "how the hell is a "problem" defined mathematically?" An ***optimization problem*** is convex if both the objective function ($J(\mathbf{u})$) and the constraint sets ($\{\mathrm{u}: g(\mathbf{u}) \leq 0\}$ and $\{\mathbf{u}: h(\mathbf{u}) = 0\}$) are convex

Convex (Constraint) Sets - Details

First off, a **set** is a **collection of points in some space.**

…Thus, the constraints ($g(\mathbf{u}) \leq 0$ and $h(\mathbf{u}) = 0$) are **not sets**…however, constraint **sets** be written (as they were on the previous slides) as:

- $G = \{\mathbf{u}: g(\mathbf{u}) \leq 0\}$ – In words: "The set of all $\mathbf{u}$ for which $g(\mathbf{u}) \leq 0$"
- $H = \{\mathbf{u}: h(\mathbf{u}) = 0\}$ – In words: "The set of all $\mathbf{u}$ for which $h(\mathbf{u}) = 0$"

Thus, the optimization problem could be rewritten as: Minimize $J(\mathbf{u})$

Subject to: $g(\mathbf{u}) \in G$
$h(\mathbf{u}) \in H$

For the ***problem*** to be convex, $J(\mathbf{u})$, G, and H must be convex
…We know how to evaluate convexity of $J(\mathbf{u})$…now we will consider how to evaluate convexity of G and H

Convex (Constraint) Sets - Details

A set (in $\mathbb{R}^n$) is a collection of points (in $\mathbb{R}^n$).

Inequality constraint: $g(\underline{u}) \leq 0$ ← not a set

The corresponding constraint set can be denoted as

$G \triangleq \{\underline{u} : g(\underline{u}) \leq 0\}$ ← a set

Equality constraint: $h(\underline{u}) = 0$ ← not a set

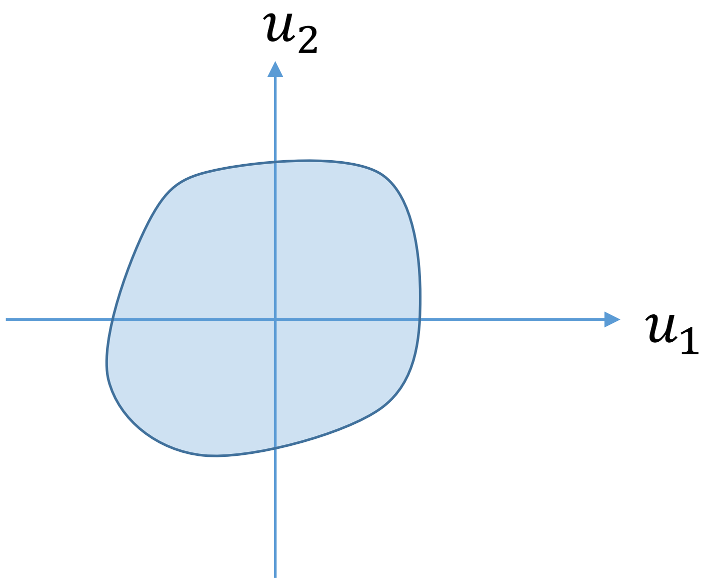
Corresponding set: $H \triangleq \{\underline{u} : h(\underline{u}) = 0\}$ ← a set

Convex (Constraint) Sets - Details

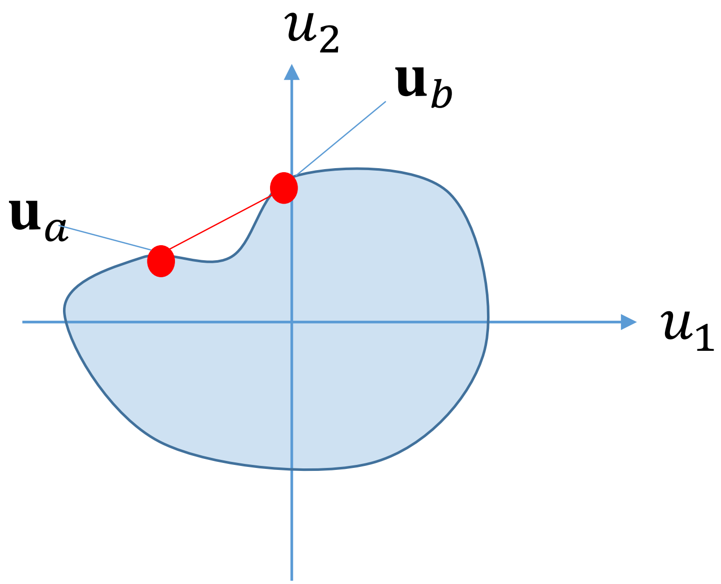
Definition: A set G is convex if and only if $\alpha\mathbf{u}_a + (1-\alpha)\mathbf{u}_b \in G, \forall \mathbf{u}_a, \mathbf{u}_b \in G$, as long as $0 \leq \alpha \leq 1$

Interpretation: For any two (arbitrary) points in G, the line segment connecting those points is contained entirely in G

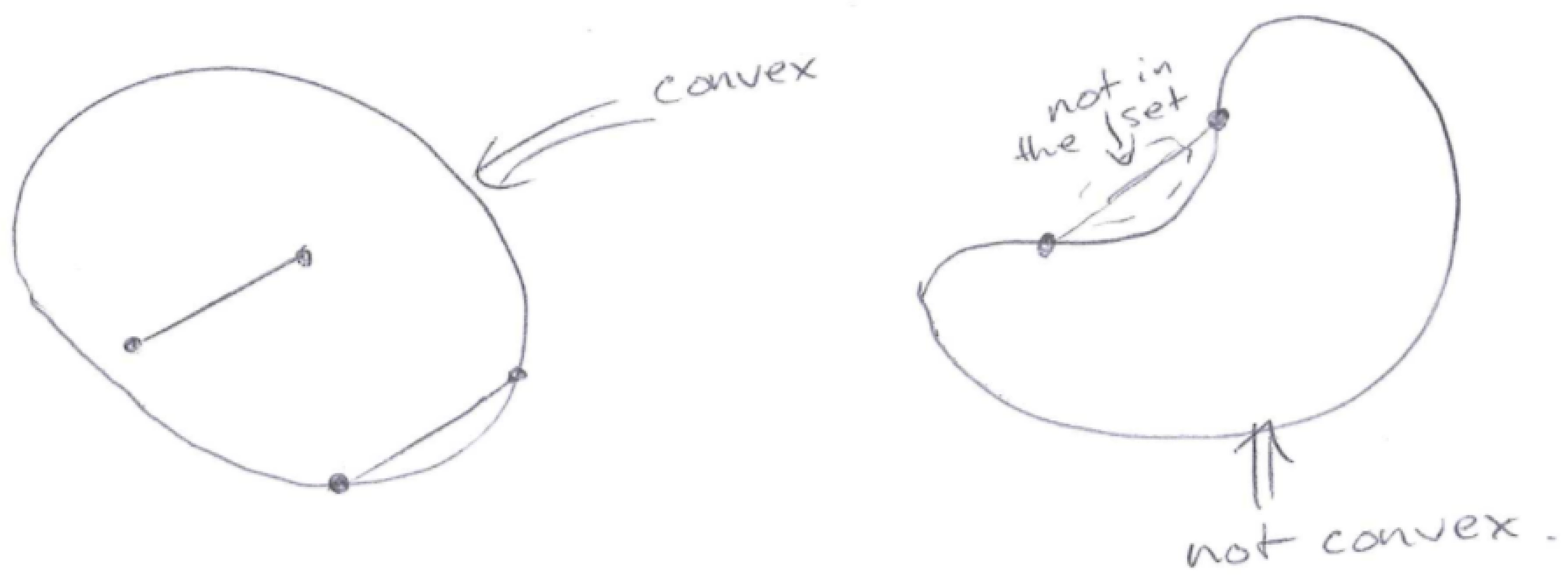
Convex set (example):



Non-convex set (example):



Convex (Constraint) Sets - Details



A set G is convex if $u_1 \in G,\ u_2 \in G \Rightarrow \underline{u}_3 = \alpha \underline{u}_1 + (1-\alpha)\underline{u}_2 \in G$, for all $\alpha \in [0 \quad 1]$.

Convex Sets – Important Properties

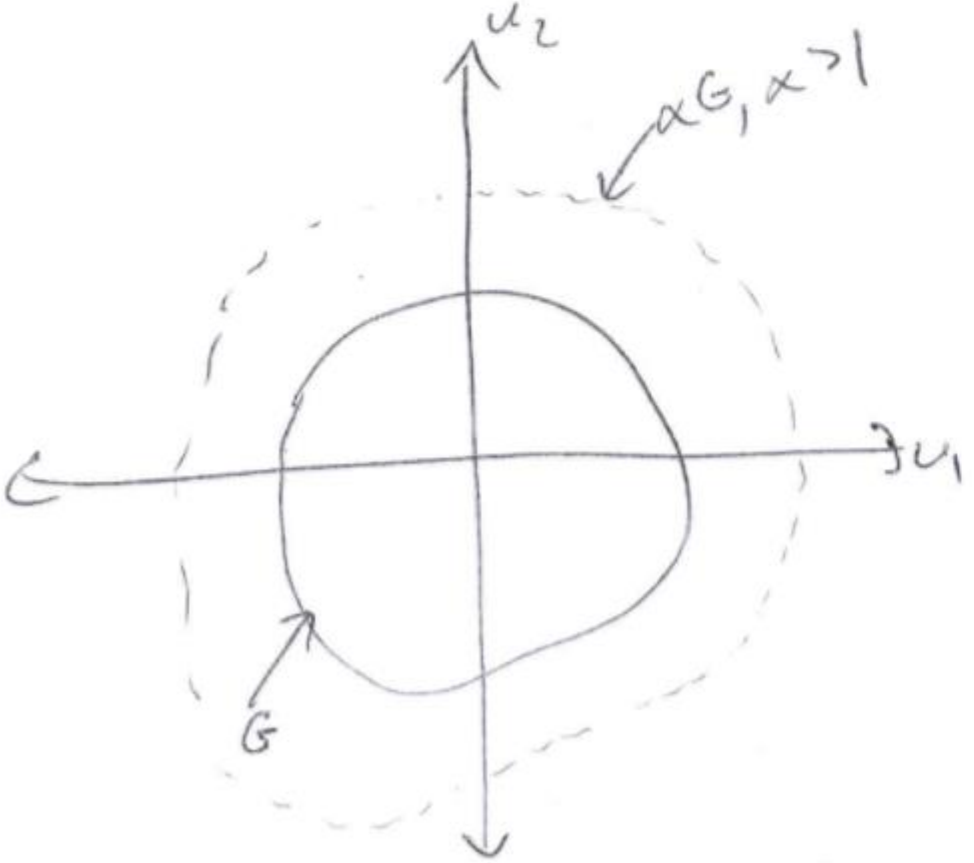
Scaling: If G is a convex set, then αG is also a convex set, for any $\alpha \in \mathbb{R}, \alpha \geq 0$

Additivity: If G_1 and G_2 are convex sets, then $G_1 + G_2$ is also convex, where $G_1 + G_2 \triangleq \{\mathbf{y}: \mathbf{y} = \mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 \in G_1, \mathbf{u}_2 \in G_2\}$

Intersection (*<u>very important</u>*): If G_1 and G_2 are convex, then $G_1 \cap G_2$ is convex

- This is especially important since there are usually multiple constraints involved with an optimization problem…the overall constraint set is the intersection of the individual constraint sets.

Convex Sets – Important Properties



$$\alpha G = \{ \underline{u} : \frac{1}{\alpha} \underline{u} \in G \}$$

Convex Optimization Problem - Example

Optimization problem: Minimize: $J(\mathbf{u}) = k^T \mathbf{u}$

Subject to:
$$A_1 \mathbf{u} - \mathbf{b}_1 \leq \mathbf{0}$$
$$A_2 \mathbf{u} - \mathbf{b}_2 = \mathbf{0}$$

Prove that the optimization problem above is convex.

Fun fact: This optimization problem is known as a ***linear program.***

Convex Optimization Problem - Example

Ex: Minimize $J(\underline{u}) = \underline{k}^T \underline{u}$, $\underline{u} \in \mathbb{R}^p$

Subject to: $A_1 \underline{u} - \underline{b}_1 \leq 0$

$A_2 \underline{u} - b_2 = 0$

$$\left.\begin{array}{l} \nabla J = \underline{k}^T \\ H = O_{p \times p} \end{array}\right\} \quad \begin{array}{l} \underline{u} = \alpha \underline{u}_1 + (1-\alpha)\underline{u}_2 \\ \left.\begin{array}{l} J(\alpha \underline{u}_1 + (1-\alpha)\underline{u}_2) = \underline{k}^T[\alpha \underline{u}_1 + (1-\alpha)\underline{u}_2] \\ \alpha J(\underline{u}_1) + (1-\alpha) J(\underline{u}_2) = \alpha \underline{k}^T \underline{u}_1 + (1-\alpha) k^T \underline{u}_2 \end{array}\right\} = \end{array}$$

$\Rightarrow J(\underline{u})$ is convex

Convex Optimization Problem - Example

$$g(\underline{u}) = A_1 \underline{u} - \underline{b}_1$$

$$g(\alpha \underline{u}_1 + (1-\alpha)\underline{u}_2) = A_1(\alpha \underline{u}_1 + (1-\alpha)\underline{u}_2) - \underline{b}_1$$

$$= \underbrace{\alpha(A\underline{u}_1 - \underline{b}_1)}_{\leq 0} + \underbrace{(1-\alpha)(A\underline{u}_2 - \underline{b}_1)}_{\leq 0} \in G$$

$$\leq 0$$

Convex Optimization Problem - Example

Optimization problem: Minimize: $J(\mathbf{u}) = \mathbf{u}^T Q\mathbf{u} + \mathbf{r}^T\mathbf{u} + s$

Subject to:
$$A_1\mathbf{u} - \mathbf{b}_1 \leq \mathbf{0}$$
$$A_2\mathbf{u} - \mathbf{b}_2 = \mathbf{0}$$

Prove that the optimization problem above is convex for positive definite Q.

Fun fact: This optimization problem is known as a ***quadratic program***.

Convex Optimization Problem - Example

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Ex: Minimize $J(\underline{u}) = \underline{u}^T Q \underline{u} + \underline{c}^T \underline{u} + s$

Subj. to: $A_1 \underline{u} - b_1 \leq 0$

$A_2 \underline{u} - b_2 \leq 0$

Same as before

We already verified constraints are convex

$\nabla J = 2\underline{u}^T Q + \underline{c}^T$

$H = 2Q > 0$

Objective function is convex

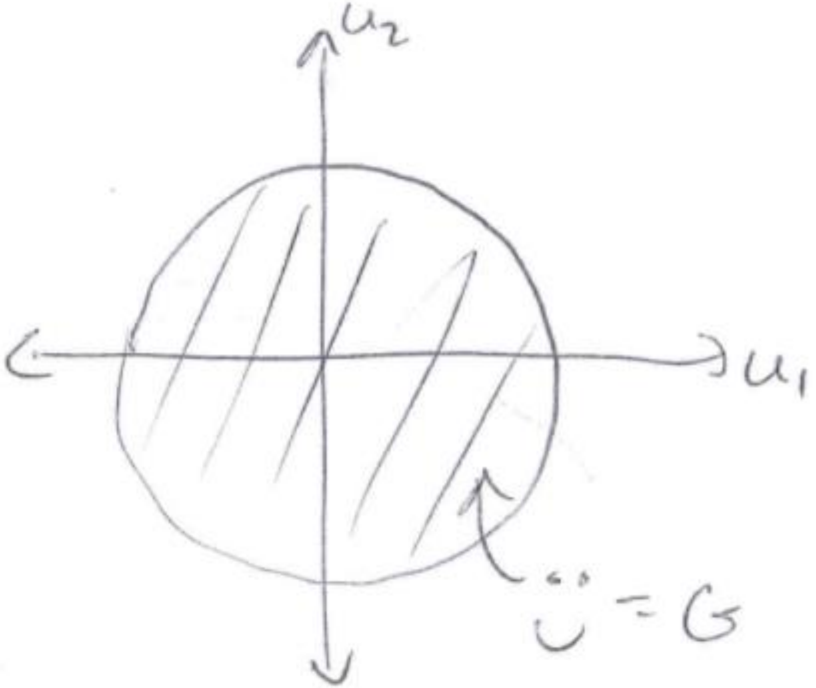
Question From Class

A question was asked: Does a constraint need to be linear for its corresponding constraint set to be convex?

Answer: NO! See the counter-example below (which is one of many)

$$G \triangleq \{\underline{u} : \|\underline{u}\| \leq u_{max}\}$$

$$\|\underline{u}\| = (u_1^2 + u_2^2)^{1/2}$$



Convex Optimal Control Problem – Example – 5 Bonus Points on Exam 1

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Optimal control problem:

Minimize: $$J(\mathbf{u}, \mathbf{x}_0) = \sum_{i=0}^{N-1} \left(\mathbf{x}(i+1)^T Q \mathbf{x}(i+1) + R u(i)^2\right)$$

Subject to: $$M_1 \mathbf{u}(i) - \mathbf{b}_1 \leq \mathbf{0}, i = 0 \ldots N-1$$
$$M_2 \mathbf{x}(i) - \mathbf{b}_2 \leq \mathbf{0}, i = 1 \ldots N$$
$$\mathbf{x}(i+1) = A\mathbf{x}(i) + Bu(i)$$

Prove that the optimization problem above is convex, assuming that Q is positive definite and $R > 0$.

Fun fact: This optimization problem is known as a constrained ***linear quadratic regulator***.

Preview of Upcoming Lectures

Next lecture – Using the KKT conditions to derive general solutions to "classic" convex optimization problems:

- Linear program
- Quadratic program
- Linear quadratic regulator

Dealing with more general (possibly non-convex) optimization problems:

- Constraint softening
- Sequential quadratic programming