Bonus problems — week 4

Let $\gamma > 0$, $g : \mathbb{R} \to \mathbb{R}$ such that

$$g(x) = |x|$$

for each $x \in \mathbb{R}$, and $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$f(x) = \|x\|_1$$

for each $x \in \mathbb{R}^n$.

1. Compute the subdifferential ∂g.
2. Find $\mathrm{prox}_{\gamma g}$ (your answer is not allowed to contain any maximization or minimization).
   Hint: Use Fermat's rule.
3. Find $\mathrm{prox}_{\gamma f}$ (your answer is not allowed to contain any maximization or minimization).

Solution. **1.** The subdifferential of a convex function $g : \mathbb{R}^n \to \mathbb{R}$ at a point x is defined as:

$$\partial g(x) = \{s \in \mathbb{R}^n : g(y) \geq g(x) + s^\top (y - x)\ \forall y \in \mathbb{R}^n\}.$$

For $g(x) = |x|$, we have different behavior depending on whether x is positive, negative, or zero.

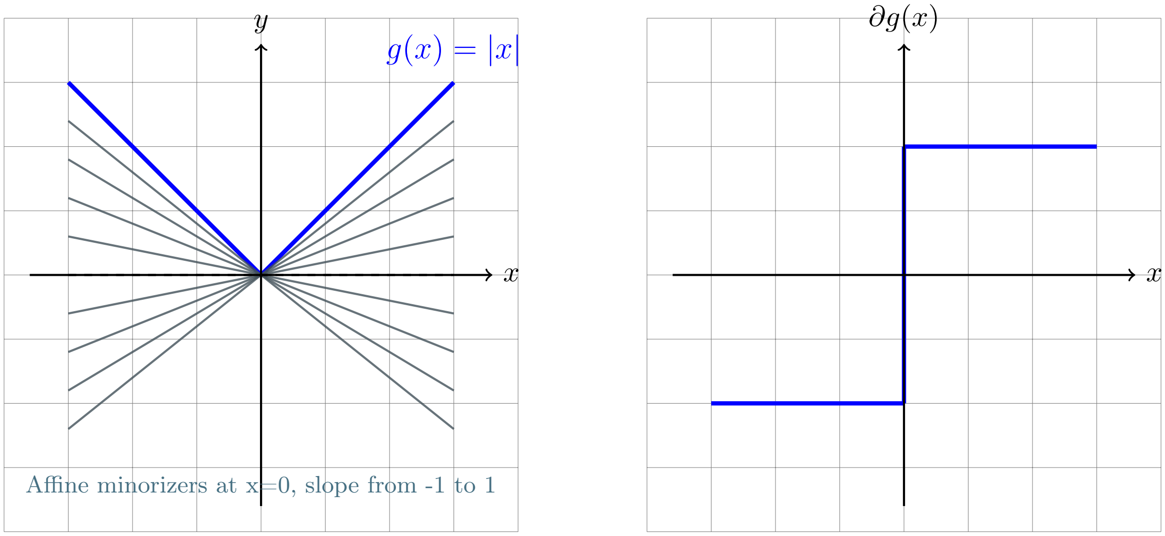
1. If $x > 0$: In this case, $g(x) = x$, so g is differentiable, and $\frac{d}{dx}x = 1$. Therefore, $\partial g(x) = \{1\}$ for $x > 0$.
2. If $x < 0$: In this case, $g(x) = -x$, so g is differentiable, and $\frac{d}{dx}(-x) = -1$. Therefore, $\partial g(x) = \{-1\}$ for $x < 0$.
3. If $x = 0$: At $x = 0$, $g(x) = |x|$ is not differentiable, but by the definition of the subdifferential, we have:

   $$\partial g(0) = \{s \in \mathbb{R} : |y| \geq sy \text{ for all } y \in \mathbb{R}\}.$$

   This holds for all $s \in [-1, 1]$. Thus, $\partial g(0) = [-1, 1]$, as can be seen in figure 1.

In summary, the subdifferential $\partial g(x)$ is given by:

$$\partial g(x) = \begin{cases} \{1\}, & \text{if } x > 0, \\ \{-1\}, & \text{if } x < 0, \\ [-1,\, 1], & \text{if } x = 0. \end{cases}$$



(a) Affine minorizers at $x = 0$.

(b) Illustration of the subdifferential of $g(x) = |x|$.

Figure 1: The subdifferential of g

2. The proximal operator is defined as:

$$\text{prox}_{\gamma g}(z) = \arg\min_x \left(g(x) + \frac{1}{2\gamma}\|x - z\|_2^2 \right).$$

For $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = |x|$ this becomes

$$\text{prox}_{\gamma g}(z) = \arg\min_x \left(|x| + \frac{1}{2\gamma}(x - z)^2 \right).$$

Recall that Fermat's rule gives that $x = \text{prox}_{\gamma f}(z)$ if and only if $0 \in \partial g(x) + \gamma^{-1}(x - z)$. Using this for the three separate cases we have:

- For $x < 0$, we have $\partial f(x) = \{-1\}$. Therefore, we get that $0 = -1 + \gamma^{-1}(x - z)$ or $x = \gamma + z$. Note that $z < -\gamma$ implies the condition $x < 0$.
- For $x > 0$, we have $\partial f(x) = \{1\}$. Therefore, we get that $0 = 1 + \gamma^{-1}(x - z)$ or $x = z - \gamma$. Note that $z > \gamma$ implies the condition $x > 0$.
- For $x = 0$, we have $\partial f(x) = [-1, 1]$. Therefore, we get that $0 \in [-1, 1] - \gamma^{-1}z$ or $z \in [-\gamma, \gamma]$.

Thus, we have

$$\text{prox}_{\gamma g}(z) = \begin{cases} z + \gamma, & \text{if } z < -\gamma, \\ 0, & \text{if } |z| \leq \gamma, \\ z - \gamma, & \text{if } z > \gamma. \end{cases}$$

or more compact:

$$\text{prox}_{\gamma g}(z) = \text{sign}(z) \cdot \max\left(|z| - \gamma,\, 0\right).$$

In the problem formulation it is stated that "your answer is not allowed to contain any maximization or minimization". I assume this concerns $\arg\min_x$ and $\arg\max_x$, but not the $\max(x, y)$ function. If it does concern the $\max(x, y)$ function then refer to

$$\text{prox}_{\gamma g}(z) = \begin{cases} z + \gamma, & \text{if } z < -\gamma, \\ 0, & \text{if } |z| \leq \gamma, \\ z - \gamma, & \text{if } z > \gamma. \end{cases}$$

3. Since $f(x) = \|x\|_1 = \sum_{i=1}^{n} |x_i| = \sum_{i=1}^{n} g(x_i)$, this function is separable. In exercise 4.2 we saw that this implies

$$\text{prox}_{\gamma f}(z) = \begin{bmatrix} \text{prox}_{\gamma g}(z_1) \\ \vdots \\ \text{prox}_{\gamma g}(z_n) \end{bmatrix} = \begin{bmatrix} \text{sign}(z_1) \cdot \max\left(|z_1| - \gamma,\, 0\right) \\ \vdots \\ \text{sign}(z_n) \cdot \max\left(|z_n| - \gamma,\, 0\right) \end{bmatrix}$$

for $z = (z_1, \ldots, z_n)$, since f is closed, proper and convex.

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