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Bonus problems — week 3

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper function, i.e., we assume that the set

$$\mathrm{dom} f = \{x \in \mathbb{R}^n \mid f(x) < +\infty\}$$

is nonempty. Recall that the function $f^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined as

$$f^*(s) = \sup_{x \in \mathbb{R}^n} \Big(s^\top x - f(x) \Big)$$

for each $s \in \mathbb{R}^n$ is called the *conjugate function* of f. Moreover, the *epigraph* of f is defined as the set

$$\mathrm{epi} f = \Big\{ (x, r) \in \mathbb{R}^{n+1} \mid f(x) \le r \Big\} .$$

Suppose that $Z \subseteq \mathbb{R}^m$ is nonempty. We define the *support function* $\sigma_Z : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ of the set Z as

$$\sigma_Z(z) = \sup_{y \in Z} z^\top y$$

for each $z \in \mathbb{R}^m$. Show that

$$\sigma_{\mathrm{epi} f}(s, -1) = f^*(s)$$

for each $s \in \mathbb{R}^n$.

Proof. We are tasked to show that $\sigma_{\mathrm{epi} f}(s, -1) = f^*(s)$, where $f^*(s)$ is the conjugate function of f, and $\mathrm{epi} f$ is the epigraph of f. Recall that the support function $\sigma_Z(z)$ for a set $Z \subseteq \mathbb{R}^m$ is defined as

$$\sigma_Z(z) = \sup_{y \in Z} z^\top y.$$

In our case, $Z = \mathrm{epi} f \subseteq \mathbb{R}^{n+1}$, so the support function for $\mathrm{epi} f$ evaluated at $(s, -1)$ is given by

$$\sigma_{\mathrm{epi} f}(s, -1) = \sup_{(x,r) \in \mathrm{epi} f} (s, -1)^\top (x, r).$$

Expanding the dot product, we get

$$\sigma_{\mathrm{epi} f}(s, -1) = \sup_{(x,r) \in \mathrm{epi} f} \Big(s^\top x - r \Big) .$$

By the definition of the epigraph, $(x, r) \in \mathrm{epi} f$ implies that $f(x) \le r$. Thus, we can rewrite the supremum as

$$\sigma_{\mathrm{epi} f}(s, -1) = \sup_{x \in \mathbb{R}^n,\, r \ge f(x)} \Big(s^\top x - r \Big) .$$

Since r appears with a negative sign, the supremum is maximized when r is as small as possible. Therefore, the maximum is achieved by setting $r = f(x)$. Substituting $r = f(x)$ gives

$$\sigma_{\text{epi}f}(s, -1) = \sup_{x \in \mathbb{R}^n} \left(s^\top x - f(x) \right).$$

Finally, this expression is precisely the definition of the conjugate function $f^*(s)$, i.e.,

$$f^*(s) = \sup_{x \in \mathbb{R}^n} \left(s^\top x - f(x) \right).$$

Thus, we conclude that

$$\sigma_{\text{epi}f}(s, -1) = f^*(s),$$

which completes the proof.

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