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Bonus problems — week 2

Consider the logistic loss function $f : \mathbb{R} \to \mathbb{R}$ defined as:

$$f(x) = \ln(1 + e^{-x}).$$

Show that f is strictly convex, but not strongly convex.

Proof. We remind ourselves of the second-order condition for strict/strong convexity as seen in class. For $f : \mathbb{R}^n \to \mathbb{R}$ twice differentiable, the conditions for strict and strong convexity are:

- f is strictly convex if
$$\nabla^2 f(x) \succ 0 \quad \text{for all } x \in \mathbb{R}^n$$
(i.e., the Hessian is positive definite).
- f is σ-strongly convex if and only if
$$\nabla^2 f(x) \succeq \sigma I \quad \text{for all } x \in \mathbb{R}^n,$$
where $\sigma > 0$ and I is the identity matrix.

In the context of a function from $\mathbb{R}$ to $\mathbb{R}$, which is a univariate case, the second-order condition for convexity involves examining the second derivative rather than the positive definiteness of the Hessian matrix. Specifically:

- A function $f : \mathbb{R} \to \mathbb{R}$ is *strictly convex* if its second derivative $f''(x) > 0$ for all $x \in \mathbb{R}$.
- A function $f : \mathbb{R} \to \mathbb{R}$ is *σ-strongly convex* if there exists some $\sigma > 0$ such that $f''(x) \geq \sigma$ for all $x \in \mathbb{R}$.

The first and second derivatives of f are computed as follows:

$$f'(x) = -\frac{e^{-x}}{1 + e^{-x}},$$

$$f''(x) = \frac{e^{-x}}{(1 + e^{-x})^2}.$$

It is simple to deduce that $f''(x) > 0$ for all x, by observing that the numerator and denominator are always positive. By the second-order condition, f is strictly convex.

The strong convexity condition would require a $\sigma > 0$ such that:

$$f''(x) \geq \sigma \quad \text{for all } x.$$

However, $f''(x)$ can be made arbitrarily small, by taking large x. In other words $\lim_{x\to\infty} f''(x) = 0$, showing that no σ such that $f''(x) \geq \sigma > 0$ can exist. Thus, f is not strongly convex. ■