ROYAL INSTITUTE
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EP3260: Machine Learning Over Networks

Lecture 7: Communication Efficiency

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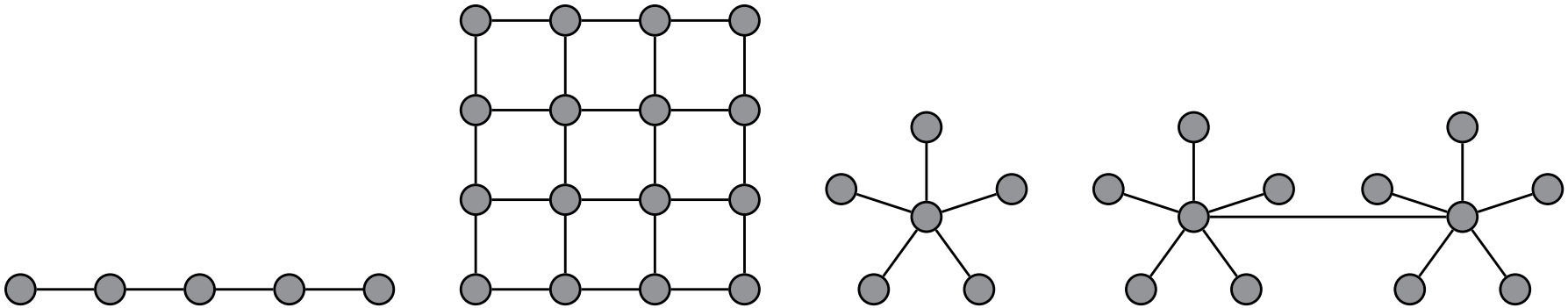
Learning outcomes

- What is the computation-communication tradeoff in a general approach to primal-dual optimizations in ML?
- How quantization affects Gradient Descent Algorithm in ML?
- How quantization affects Stochasitc Gradient Descent Algorithm in ML?

Outline

1. Computation-communication tradeoff in a general approach
2. Quantized Distributed Gradient Descent
3. Parallel Quantized Stochastic Gradient Descent

Recap of previous two lectures



- ML over Master-Workers networks
  - Duality methods (Lec 5)
  - Alternating Direction Methods of Multipliers (ADMM) (Lec 6)
- ML over general networks
  - Duality methods with consensus (Lec 5)
  - ADMM with consensus (Lec 6)

Outline

1. Computation-communication tradeoff in a general approach
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A general framework for primal-dual methods

- **Definition** (L-Lipschitz Continuity). A function $h : \mathbb{R}^m \to \mathbb{R}$ is L-Lipschitz Continuos if $\forall\ \boldsymbol{u}$ and $\boldsymbol{v} \in \mathbb{R}^m$, we have $|h(\boldsymbol{u}) - h(\boldsymbol{v})| \leq L\|\boldsymbol{u} - \boldsymbol{v}\|$

- **Definition** (L-Bounded Support). A function $h : \mathbb{R}^m \to \mathbb{R} \cup +\infty$ has L bounded support if its effective domain is bounded by L
  $h(\boldsymbol{u}) < +\infty \implies \|\boldsymbol{u}\| \leq L$

- **Definition** ($\frac{1}{\mu}$-Smoothness). A function $h : \mathbb{R}^m \to \mathbb{R}$ is $\frac{1}{\mu}$ smooth if it is differentiable and its derivative is $\frac{1}{\mu}$-Lipschitz continuous

$$h(\boldsymbol{u}) \leq h(\boldsymbol{v}) + \nabla h(\boldsymbol{v})^T(\boldsymbol{u} - \boldsymbol{v}) + \frac{1}{2\mu}\|\boldsymbol{u} - \boldsymbol{v}\|^2 \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^m$$

- **Definition** (μ-Strong Convexity). A function $h : \mathbb{R}^m \to \mathbb{R}$ is μ strongly convex for $\mu \geq 0$ if

$$h(\boldsymbol{u}) \geq h(\boldsymbol{v}) + \boldsymbol{s}^T(\boldsymbol{u} - \boldsymbol{v}) + \frac{\mu}{2}\|\boldsymbol{u} - \boldsymbol{v}\|^2 \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^m$$

  for any $\boldsymbol{s} \in \partial h(\boldsymbol{v})$, where $\partial h(\boldsymbol{v})$ denotes the subgradient of h at $\boldsymbol{v}$

A general framework for primal-dual methods

- We now study a general framework to ML problems having the form

$$\min_{\boldsymbol{u}\in\mathbb{R}^n} \ell(\boldsymbol{u}) + r(\boldsymbol{u}) \qquad (I)$$

for convex functions $\ell(\boldsymbol{u}) = \sum_i \ell_i(\boldsymbol{u})$ (the loss function) and $r(\boldsymbol{u})$ (the regularizer function, e.g. $\lambda\|\boldsymbol{u}\|_p$).

- This formulation includes ML problems such as Support Vector Machines, Linear and Logistic Regression, Lasso or Sparse Logistic Regression

- This general framework maps the ML problem (I) into one of the two following problems

$$\min_{\boldsymbol{\alpha}\in\mathbb{R}^n} O_A(\boldsymbol{\alpha}) = f(A\boldsymbol{\alpha}) + g(\boldsymbol{\alpha}) \qquad (A)$$

$$\min_{\boldsymbol{w}\in\mathbb{R}^n} O_B(\boldsymbol{w}) = f^*(\boldsymbol{w}) + g^*(-A^T\boldsymbol{w}) \qquad (B)$$

where $\boldsymbol{\alpha} \in \mathbb{R}^n$ and $\boldsymbol{w} \in \mathbb{R}^m$, $A = [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n] \in \mathbb{R}^{m\times n}$ is a data matrix with column vectors $\boldsymbol{x}_i \in \mathbb{R}^m$ $\forall i$, and f^* and g^* are the convex conjugates of f and g respectively. (A) is called primal. (B) dual.

A general framework for primal-dual methods

- Optimization Problem (A) and (B) are equivalent according to the Fenchel-Rockafellar duality
- Given $\boldsymbol{\alpha}$ from (A), we achieve $\boldsymbol{w}$ of (B) as $\boldsymbol{w} = \boldsymbol{w}(\boldsymbol{\alpha}) := \nabla f(A\boldsymbol{\alpha})$
- (A) and (B) give the duality gap $G(\boldsymbol{\alpha}) := O_A(\boldsymbol{\alpha}) - [-O_B(\boldsymbol{w}(\boldsymbol{\alpha}))]$
- Recall that the duality gap is always non negative and is zero if the pair $(\boldsymbol{\alpha}^*, \boldsymbol{w}^*)$ is optimal. It gives an upper bound on the unknown primal or dual optimization error (certificate of the suboptimality) since

$$O_A(\boldsymbol{\alpha}) \geq O_A(\boldsymbol{\alpha}^*) \geq -O_B(\boldsymbol{w}^*) \geq -O_B(\boldsymbol{w}(\boldsymbol{\alpha}))$$

- Assumption: Problem (A) is with f $\frac{1}{\tau}$-smooth and the function g are separable $g(\boldsymbol{\alpha}) = \sum_i g_i(\boldsymbol{\alpha})$, with $g_i(\boldsymbol{\alpha})$ having L-bounded support.
- Given the equivalence between (A) and (B), this gives that in problem (B) f^* is τ-strongly convex and the function $g^*(-A^T\boldsymbol{w}) = \sum_i g_i^*(-\boldsymbol{x}_i^T\boldsymbol{w})$ is separable with each g_i^* being L-Lipschitz

Common Losses and Regularizers

(i) Losses

Loss	Obj	f / g^*
Least Squares	(A)	$f = \frac{1}{2}\|A\boldsymbol{\alpha} - \mathbf{b}\|_2^2$
	(B)	$g^* = \frac{1}{2}\|A^\top \mathbf{w} - \mathbf{b}\|_2^2$
Logistic Reg.	(A)	$f = \frac{1}{m}\sum_j \log(1+\exp(b_j \mathbf{x}_j^\top \boldsymbol{\alpha}))$
	(B)	$g^* = \frac{1}{n}\sum_i \log(1+\exp(b_i \mathbf{x}_i^\top \mathbf{w}))$
SVM	(B)	$g^* = \frac{1}{n}\sum_i \max(0, 1 - y_i \mathbf{x}_i^\top \mathbf{w})$
Absolute Dev.	(B)	$g^* = \frac{1}{n}\sum_i \|\mathbf{x}_i^\top \mathbf{w} - y_i\|$

(ii) Regularizers

Regularizer	Obj	g / f^*
Elastic Net	(A)	$g = \lambda(\eta\|\boldsymbol{\alpha}\|_1 + \frac{1-\eta}{2}\|\boldsymbol{\alpha}\|_2^2)$
	(B)	$f^* = \lambda(\eta\|\mathbf{w}\|_1 + \frac{1-\eta}{2}\|\mathbf{w}\|_2^2)$
L_2	(A)	$g = \frac{\lambda}{2}\|\boldsymbol{\alpha}\|_2^2$
	(B)	$f^* = \frac{\lambda}{2}\|\mathbf{w}\|_2^2$
L_1	(A)	$g = \lambda\|\boldsymbol{\alpha}\|_1$
Group Lasso	(A)	$g = \lambda\sum_p \|\boldsymbol{\alpha}_{\mathcal{I}_p}\|_2$, $\mathcal{I}_p \subseteq [n]$

Assumptions

- Our main interest is now to apply (A) or (B) for deriving a distributed solution to the initial ML problem (I).

- The data set A is distributed over K machines according to a partition $\{\mathcal{P}_k\}_{k=1}^K$ of the columns of $A \in \mathbb{R}^{m\times n}$. The size of the partition on the machine k is $n_k = |\mathcal{P}_k|$

- For machine $k \in \{1, \ldots, K\}$ and vector $\boldsymbol{\alpha} \in \mathbb{R}^n$, let $\boldsymbol{\alpha}_{[k]} \in \mathbb{R}^n$ a vector with elements $(\boldsymbol{\alpha}_{[k]})_i := \alpha_i$ if $i \in \mathcal{P}_k$ and $(\boldsymbol{\alpha}_{[k]})_i := 0$ otherwise

- Analogously, let $A_{[k]}$ be a matrix with columns corresponding to those of A according to the partition, and zeros elsewehere

- The function g in (A) can be easily distributed, since $g(\boldsymbol{\alpha}) = \sum_i g_i(\boldsymbol{\alpha})$

- However, the function $f(A\boldsymbol{\alpha})$ is not in general separable

- The main idea of the general framework for primal-dual methods is a separable approximation of the function $O_A(\boldsymbol{\alpha})$. See next

Approximation of $O_A(\boldsymbol{\alpha})$

- Let $\boldsymbol{v} := A\boldsymbol{\alpha} \in \mathbb{R}^m$ and let $\boldsymbol{\alpha}_{[k]}^{(t+1)} := \boldsymbol{\alpha}_{[k]}^{(t)} + \gamma \Delta\boldsymbol{\alpha}_{[k]}$, where $\Delta\boldsymbol{\alpha}_{[k]}$ denotes a certain change of variables $\boldsymbol{\alpha}_i$ for $i \in \mathcal{P}_k$ and $(\Delta\boldsymbol{\alpha}_{[k]})_i := 0 \ \forall i \ni \mathcal{P}_k$
- Then, $O_A(\boldsymbol{\alpha})$ can be exactly decomposed as follows

$$\sum_{i\in[n]} g_i(\alpha_i^{(t)} + \Delta\alpha_i) + f(\boldsymbol{v}^{(t)}) + \nabla f(\boldsymbol{v}^{(t)})^T A\Delta\boldsymbol{\alpha} +$$

$$\frac{\sigma'}{2\tau}\Delta\boldsymbol{\alpha}^T \begin{bmatrix} A_{[1]}^T A_{[1]} & & 0 \\ & \ddots & \\ 0 & & A_{[K]}^T A_{[K]} \end{bmatrix} \Delta\boldsymbol{\alpha} = \sum_{k=1}^{K} G_k^{\sigma'}(\Delta\boldsymbol{\alpha}_k; \boldsymbol{v}^{(t)}, \boldsymbol{\alpha}_{[k]})$$

$$G_k^{\sigma'}(\Delta\boldsymbol{\alpha}_k; \boldsymbol{v}^{(t)}, \boldsymbol{\alpha}_{[k]}) := \frac{1}{K} f(\boldsymbol{v}^{(t)}) + \boldsymbol{w}^T A_{[k]}\Delta\boldsymbol{\alpha}_{[k]} + \frac{\sigma'}{2\tau}\left\|A_{[k]}\Delta\boldsymbol{\alpha}_{[k]}\right\|^2 + \sum_{i\in\mathcal{P}_k} g_i(\alpha_i^{(t)} + \Delta\boldsymbol{\alpha}_{[k]_i})$$

Approximation of $O_A(\boldsymbol{\alpha})$

- The function $G_k^{\sigma'}(\Delta\boldsymbol{\alpha}_k; \boldsymbol{v}^{(t)}, \boldsymbol{\alpha}_{[k]}^{(t)})$ is completely local at processor k except the coupling variable $\boldsymbol{v}^{(t)} = A\boldsymbol{\alpha}^{(t)}$ which is global
- The decomposition of $O_A(\boldsymbol{\alpha})$ suggests that we can iteratively solve local problems and exchange $\boldsymbol{\alpha}_k$ to reconstruct $\boldsymbol{v}^{(t)}$

$$\min_{\Delta\boldsymbol{\alpha}_k \in \mathbb{R}^n} G_k^{\sigma'}(\Delta\boldsymbol{\alpha}_k; \boldsymbol{v}^{(t)}, \boldsymbol{\alpha}_{[k]}^{(t)})$$

- Each processor can do the local minimisation and just exchange to the others the variables $\boldsymbol{\alpha}_k$ at each iteration t
- Note that the minimization is done independently from other processors k and thus the resulting $G_k^{\sigma'}(\Delta\boldsymbol{\alpha}_k; \boldsymbol{v}^{(t)}, \boldsymbol{\alpha}_{[k]}^{(t)})$ will not give the exact term to perfectly reconstruct $O_A(\boldsymbol{\alpha})$. However, this is enough to approximately compute the optimal solution with approximation Θ

Algorithm 1: Generalized primal-dual algorithm

Algorithm 1: Generalized primal-dual algorithm

Input Data matrix A distributed column-wise according to the partition $\{\mathcal{P}_k\}_{k=1}^K$, aggregation parameter $\gamma \in (0, 1]$, and σ'.

Starting point $\boldsymbol{\alpha}^{(0)} := 0 \in \mathbb{R}^n$, $\boldsymbol{v}^{(0)} := 0 \in \mathbb{R}^m$

for $t = 0, 1, \ldots$ **do**

  for $k = 1, 2, \ldots, K$ in parallel in each processor **do**

    Compute a Θ approximate solution to

$$\min_{\Delta\boldsymbol{\alpha}_k \in \mathbb{R}^n} G_k^{\sigma'}(\Delta\boldsymbol{\alpha}_k; \boldsymbol{v}^{(t)}, \boldsymbol{\alpha}_{[k]}^{(t)})$$

    $\boldsymbol{\alpha}_{[k]}^{(t+1)} := \boldsymbol{\alpha}_{[k]}^{(t)} + \gamma\Delta\boldsymbol{\alpha}_{[k]}$

    $\Delta\boldsymbol{v}_k := A_{[k]}\Delta\boldsymbol{\alpha}_{[k]}$. Transmit to the other processors $\Delta\boldsymbol{v}_k$

  end for

  Compute $\boldsymbol{v}^{(t+1)} = \boldsymbol{v}^{(t)} + \gamma \sum_{k=1}^K \Delta\boldsymbol{v}_k$

end for

Application to primal and dual

Algorithm 2: Primal mapping

Map Problem (I) into (A)

Distribute dataset A by columns (here typically features) according to the partition $\{\mathcal{P}_k\}_{k=1}^K$

Run Algorothm 1 with appropriate choice of parameter γ and subproblem parameter σ'

Algorithm 3: Dual mapping

Map Problem (I) into (B)

Distribute dataset A by columns (here typically training points) according to the partition $\{\mathcal{P}_k\}_{k=1}^K$

Run Algorothm 1 with appropriate choice of parameter γ and subproblem parameter σ'

Algorithm 1 for convex g_i and L-Lipschiz g_i^*

- **Theorem 1**: Consider Algorithm 1 with $\gamma := 1$, and let Θ be the quality of the local solver at processor k. Let g_i have L bounded support, and let f be $\frac{1}{\tau}$-mooth. Let T be such that

$$T \geq T_0 + \max\left(\left\lceil \frac{1}{1-\Theta} \right\rceil, \frac{4L^2}{\tau\varepsilon_G(1-\Theta)}\right)$$

$$T_0 \geq t_0 + \left\lceil \frac{2}{1-\Theta}\left(\frac{8L^2}{\tau\varepsilon_G} - 1\right)\right\rceil$$

$$t_0 \geq \max\left(0, \left\lceil \frac{1}{1-\Theta}\log\left(\frac{\tau n(O_A(\boldsymbol{\alpha}^{(0)})) - O_A(\boldsymbol{\alpha}^*))}{2L^2K}\right)\right\rceil\right)$$

Then

$$\mathbb{E}[O_A(\bar{\boldsymbol{\alpha}}) - (-O_B(\boldsymbol{w}(\bar{\boldsymbol{\alpha}})))] \leq \varepsilon_G \qquad \bar{\boldsymbol{\alpha}} = \frac{1}{T-T_0}\sum_{t=T_0+1}^{T-1}\boldsymbol{\alpha}^{(t)}$$

Algorithm 1 for strong. convex g_i and smooth g_i^*

- **Theorem 2**: Consider Algorithm 1 with $\gamma := 1$, and let Θ be the quality of the local solver. Let g_i be μ strongly convex $\forall i$ and let f be $\frac{1}{\tau}$-mooth. Let T be such that

$$T \geq \frac{1}{1-\Theta} \frac{\mu\tau + 1}{\mu\tau} \log \frac{1}{\varepsilon_{O_A}}$$

Then $\mathbb{E}[O_A(\boldsymbol{\alpha}^{(T)}) - O_A(\boldsymbol{\alpha}^*)] \leq \varepsilon_{O_A}$
Moreover, if

$$T \geq \frac{1}{1-\Theta} \frac{\mu\tau + 1}{\mu\tau} \log \left(\frac{1}{1-\Theta} \frac{\mu\tau + 1}{\mu\tau} \frac{1}{\varepsilon_{O_A}} \right)$$

then the expected duality gap

$$\mathbb{E}[O_A(\boldsymbol{\alpha}^{(T)}) - (-O_B(\boldsymbol{w}(\boldsymbol{\alpha}^T))] \leq \varepsilon_G$$

Criteria for Running Algorithms 2 vs. 3

	Smooth ℓ	Non-smooth and separable ℓ
Strongly convex r	Alg. 2 or 3	Alg. 3
Non-strongly convex and separable r	Alg. 2	-

	Smooth ℓ	Non-smooth and separable ℓ
Strongly convex r	Theorem 3	Theorem 2
Non-strongly convex and separable r	Theorem 2	-

Comparison with ADMM

- We can apply consensus ADMM to (B) (or (A)):

$$\min_{\boldsymbol{w}_1,\ldots,\boldsymbol{w}_k,\boldsymbol{w}} \sum_{k=1}^{K} \sum_{i\in\mathcal{P}_k} g^*(-\boldsymbol{x}_i^T \boldsymbol{w}_k) + f^*(\boldsymbol{w}) \qquad \text{s.t.} \quad \boldsymbol{w}_k = \boldsymbol{w} \quad \forall k$$

- We solve the problem by the augmented Lagrangian

$$\boldsymbol{w}_k^{(t+1)} = \arg\min_{\boldsymbol{w}_k} \sum_{i\in\mathcal{P}_k} g^*(-\boldsymbol{x}_i^T \boldsymbol{w}_k) + \rho \boldsymbol{u}_k^{(t)^T} (\boldsymbol{w}_k - \boldsymbol{w}^{(t)}) + \frac{\rho}{2} \|\boldsymbol{w}_k - \boldsymbol{w}^{(t)}\|^2$$

$$\boldsymbol{w}^{(t+1)} = \arg\min_{\boldsymbol{w}} f^*(\boldsymbol{w}) + \frac{\rho K}{2} \|\boldsymbol{w} - (\bar{\boldsymbol{w}}_k^{(t+1)} - \bar{\boldsymbol{u}}_k^{(t)})\|^2$$

$$\boldsymbol{u}_k^{(t+1)} = u_k^{(t)} + \boldsymbol{w}_k^{(t+1)} - \boldsymbol{w}^{(t+1)}$$

- ADMM has the drawback of the proximal updating

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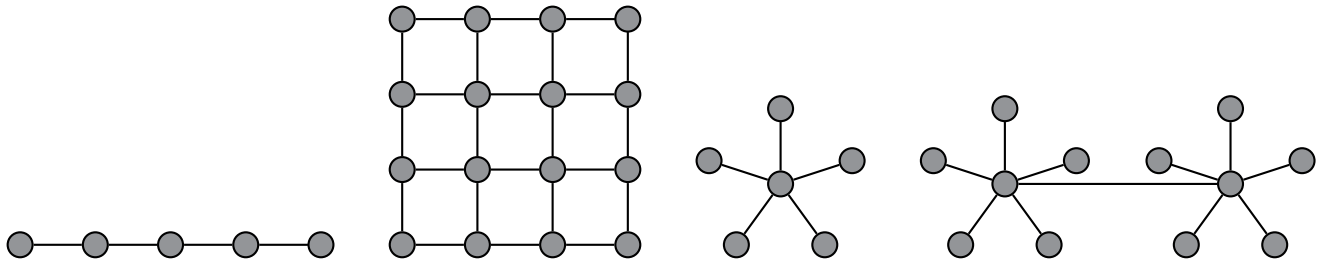
Problem formulation



- Set of n nodes $\mathcal{V} = (1, \ldots, n)$, a set of edges $\mathcal{E} = \mathcal{V} \times \mathcal{V}$. The nodes communicate over a connected an undirected graph $\mathcal{G} = (\mathcal{G}, \mathcal{E})$
- $\mathcal{N}_i$ is the set of neighbours that node i communicates with
- Each node i has a strongly convex and smooth function $f_i(\boldsymbol{w}) : \mathbb{R}^p \to \mathbb{R}$
- All the nodes wish to solve the ML optimization problem
  $\underset{\boldsymbol{w} \in \mathbb{R}^p}{\text{minimize}}\ f(\boldsymbol{w}) = \underset{\boldsymbol{w} \in \mathbb{R}^p}{\text{minimize}}\ \frac{1}{|\mathcal{N}_i|} \sum_{i \in \mathcal{N}_i} f_i(\boldsymbol{w})$
- Clearly, $f(\boldsymbol{w})$ is strongly convex and smooth and there is a unique minimizer $\boldsymbol{w}^*$

Problem formulation

- A node has only access to its local function and it can communicate only with the neighbours $\mathcal{N}_i$
- As we have seen in the previous lectures, we can equivalently rewrite the ML optimization problem by the consensus method as

$$\begin{aligned} \underset{\boldsymbol{w}\in\mathbb{R}^p}{\text{minimize}} \quad & \frac{1}{|\mathcal{N}_i|}\sum_{i\in\mathcal{N}_i} f_i(\boldsymbol{w}) \\ \text{s.t.} \quad & \boldsymbol{w}_i = \boldsymbol{w}_j \quad \forall i,j \in \mathcal{N}_i \end{aligned}$$

- We could solve the problem by the methods of the previous lectures with local iterates
- However, the nodes cannot exchange the decision variables $\boldsymbol{w}_{i,t}$, but a quantized version $\boldsymbol{z}_{i,t} = Q(\boldsymbol{w}_{i,t})$, where $Q(\cdot)$ is a quantizer function
- The quantization can substantially reduce the amount of information to exchange, which is very important, e.g., in IoT applications

Quantized Distributed Gradient Descent (QDGD)

Algorithm 4: QDGD

Node i requires Weights $\{a_{i,j}\}_{j=1}^{n}$

Set $\boldsymbol{w}_{i,0} = 0$ and compute $\boldsymbol{z}_{i,0} = Q(\boldsymbol{w}_{i,0})$

for $t = 0, 1, \ldots, T-1$ **do**

Transmit $\boldsymbol{z}_{i,t} = Q(\boldsymbol{w}_{i,t})$ to $\mathcal{N}_i$ and receive $\boldsymbol{z}_{j,t}$

Compute the local decision variable as

$$\boldsymbol{w}_{i,t+1} = (1 - \varepsilon + \varepsilon a_{i,i})\boldsymbol{w}_{i,t} + \varepsilon \sum_{j \in \mathcal{N}_i} a_{i,j} \boldsymbol{z}_{j,t} - \alpha\varepsilon \nabla f_i(\boldsymbol{w}_{i,t})$$

end for

Return $\boldsymbol{w}_{i,T}$

- ε and α are positive step sizes to be appropriately chosen
- There are no particular restrictions on the type of quantizer (see later)

QDGD Convergence analysis

- **Assumption 1**: $\forall\ \boldsymbol{w} \in \mathbb{R}^p$, $\boldsymbol{y} \in \mathbb{R}^p$, f_i is differentiable and smooth with parameter L

$$\|\nabla f_i(\boldsymbol{w}) - \nabla f_i(\boldsymbol{y})\| \leq L\|\boldsymbol{w} - \boldsymbol{y}\| \qquad \forall i$$

- **Assumption 2**: $\forall\ \boldsymbol{w} \in \mathbb{R}^p$, $\boldsymbol{y} \in \mathbb{R}^p$, f_i is strongly convex with parameter μ

$$(\nabla f_i(\boldsymbol{w}) - \nabla f_i(\boldsymbol{y}))^T(\boldsymbol{w} - \boldsymbol{y}) \geq \mu\|\boldsymbol{w} - \boldsymbol{y}\|^2 \qquad \forall i$$

- **Assumption 3**: The quantizer is unbiased and has a bounded variance:

$$\mathbb{E}[Q(\boldsymbol{w})|\boldsymbol{w}] = \boldsymbol{w} \qquad \mathbb{E}\left[\|Q(\boldsymbol{w}) - \boldsymbol{w}\|^2|\boldsymbol{w}\right] \leq \sigma^2$$

- **Assumption 4**: The matrix $\boldsymbol{A} = [a_{i,j}] \in \mathbb{R}^{n,n}$ is symmetric and doubly stochastic:

$$\boldsymbol{A} = \boldsymbol{A}^T \qquad \boldsymbol{A}\mathbf{1} = \mathbf{1} \qquad \boldsymbol{A}^T\mathbf{1} = \mathbf{1}$$

QDGD Convergence analysis

- **Theorem 4**: Consider the QDGD Algorithm. Suppose Assumptions 1 $\sim$ 4 hold. Let δ be an arbitrary scalar in $(0, 1/2)$ and let $\varepsilon = c_1/T^{3\delta/2}$ and $\alpha = c_2/T^{\delta/2}$, where c_1 and c_2 are arbitrary positive constants independent of T. The, for each node i

$$\mathbb{E}\left[\|\boldsymbol{w}_{i,T} - \boldsymbol{w}^*\|^2\right] \leq \mathcal{O}\left(\left(\frac{4nc_2^2D^2(3+2L/\mu)^2}{(1-\beta)^2} + \frac{2c_1n\sigma^2\|\boldsymbol{A}-\boldsymbol{A}_D\|}{\mu c_2}\right)\frac{1}{T^\delta}\right)$$

  where

$$D^2 = 2L\sum_{i=1}^{n}(f_i(0) - f_i^*), \qquad f_i^* = \min_{\boldsymbol{w}\in\mathbb{R}^p} f_i(\boldsymbol{w})$$

- The theorem shows that QDGD provides an approximation solution with vanishing deviation from the optimal solution, despite the quantization noise that does not vanish with the iterations

- The convergence rate is sublinear

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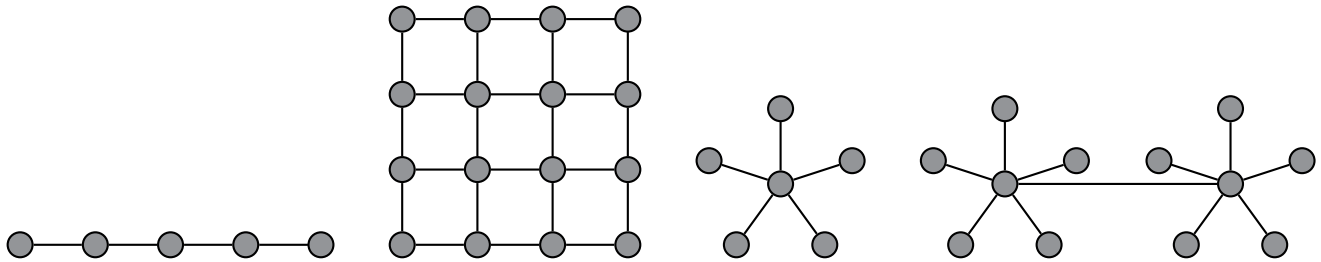
Stochastic Gradient Descent (SGD)



- Set of n nodes $\mathcal{V} = (1, \ldots, n)$, a set of edges $\mathcal{E} = \mathcal{V} \times \mathcal{V}$. The nodes communicate over a connected an undirected graph $\mathcal{G} = (\mathcal{G}, \mathcal{E})$
- Let $\mathcal{W}$ be a known convex set. There is a global function $f(\boldsymbol{w}) : \mathcal{W} \to \mathbb{R}$ which is unknown to the nodes
- Each node i has access to its measurement of the stochastic gradient of $f(\boldsymbol{w})$
- All the nodes wish to solve the ML optimization problem $\underset{\boldsymbol{w} \in \mathbb{R}^p}{\text{minimize}} \; f(\boldsymbol{w})$

SGD

- **Definition 1**: Given the function $f(\boldsymbol{w}) : \mathcal{W} \to \mathbb{R}$, a stochastic gradient of f is a random function $\tilde{g}(\boldsymbol{w})$ so that $\mathbb{E}\left[\tilde{g}(\boldsymbol{w})\right] = \nabla f(\boldsymbol{w})$
- **Definition 2**: The stochastic gradient has second oder moment at most B if $\mathbb{E}\left[\|\tilde{g}(\boldsymbol{w})\|^2\right] \leq B$ for $\boldsymbol{w} \in \mathcal{W}$
- **Definition 3**: The stochastic gradient has variance at most σ^2 if $\mathbb{E}\left[\|\tilde{g}(\boldsymbol{w}) - \nabla f(\boldsymbol{w})\|^2\right] \leq \sigma^2$ for $\boldsymbol{w} \in \mathcal{W}$.

SGD

- A standard instance of the Stochastic Gradient Descent (SGD) is

$$\boldsymbol{w}_{t+1} = \boldsymbol{w}_t - \eta_t \tilde{g}(\boldsymbol{w}_t)$$

  where η_t is variable step size

- **Theorem 5**: Let $\mathcal{W} \subseteq \mathbb{R}^n$ be convex and let the function $f(\boldsymbol{w}) : \mathcal{W} \to \mathbb{R}$ be unknown, convex, and L-smooth. Let $\boldsymbol{w}_0 \in \mathcal{W}$ be given and let $R^2 = \sup_{\boldsymbol{w} \in \mathcal{W}} \|\boldsymbol{w} - \boldsymbol{w}_0\|^2$. Let $T \geq 0$ be fixed. Given repeated and independent access to stocastic gradients with variance bound σ^2, the SGD with constant step size $\eta_t = 1/(L + 1/\gamma)$ where $\gamma = R/\sigma\sqrt{2/T}$ achieves

$$\mathbb{E}\left[f\left(\frac{1}{T}\sum_{t=0}^{T} \boldsymbol{w}_t\right)\right] - \min_{\boldsymbol{w} \in \mathcal{W}} f(\boldsymbol{w}) \leq R\sqrt{\frac{2\sigma^2}{T}} + \frac{LR^2}{T}$$

Parallel SGD

- If we have K processors each making an independent measurement of the stochastic gradient $\tilde{g}^i(\boldsymbol{w})$, and each processor i communicates to each other such measurement at every time step t, a parallel SGD is

$$\boldsymbol{w}_{t+1} = \boldsymbol{w}_t - \frac{\eta_t}{K} \sum_{i=1}^{K} \tilde{g}^i(\boldsymbol{w}_t)$$

- **Corolary 1**: Let $\mathcal{W}$, $f(\boldsymbol{w})$, $\boldsymbol{w}_0$ and R as in the previous theorem. Fix $\varepsilon \geq 0$. Suppose to run parallel SGD on K processors each with access to independent stochastic gradients with second moment bound B, with step size $\eta_t = 1/(L + \sqrt{K}/\gamma)$ with γ as in the previous theorem. If $T = \mathcal{O}\left(R^2 \max\left(\frac{2B}{K\varepsilon^2}, \frac{L}{\varepsilon}\right)\right)$ then

$$\mathbb{E}\left[f\left(\frac{1}{T}\sum_{t=0}^{T} \boldsymbol{w}_t\right)\right] - \min_{\boldsymbol{w}\in\mathcal{W}} f(\boldsymbol{w}) \leq \varepsilon$$

Parallel Quantized SGD

Algorithm 5: PQSGD

for $t = 0, 1, \ldots, T-1$ **do**

Let $\tilde{g}^i(\boldsymbol{w}_t)$ be an independent stochastic gradient

Broadcast $\boldsymbol{z}_{i,t} = Q(\tilde{g}^i(\boldsymbol{w}_t))$ to all nodes and receive $\boldsymbol{z}_{j,t}$

Compute the local estimate of the global decision variable as

$$\boldsymbol{w}_{i,t+1} = \boldsymbol{w}_{i,t} - \frac{\eta_t}{K} \sum_{i=1}^{K} \boldsymbol{z}_{i,t}$$

end for

Return $\boldsymbol{w}_{i,T}$

- Where $Q(\cdot)$ is a quantizer (see below)
- Does the algorithm converge? Not in general...

Quantization

- Let $\boldsymbol{v} \in \mathbb{R}^n$ with $\boldsymbol{v} \neq 0$, and let $s \geq 1$. The "low precision quantizer" is

$$Q_s(\boldsymbol{v}) = [Q_s(v_i) = \|\boldsymbol{v}\|_2 \operatorname{sgn}(v_i)\xi_i(\boldsymbol{v}, s)]$$

  where $\xi_i(\boldsymbol{v}, s)$ are independent random variables with outcome

$$\xi_i(\boldsymbol{v}, s) = \begin{cases} \ell/s & \text{with probability } 1 - p\left(\frac{|v_i|}{\|\boldsymbol{v}\|_2}, s\right) \\ (\ell+1)/s & \text{otherwise} \end{cases}$$

  with $p(a, s) = as - \ell$ for any $a \in [0, 1]$, and the integer $0 \leq \ell < s$ to be chosen such that $|w_i|/\|\boldsymbol{w}\| \in [\ell/s, (l+1)/s]$

- ℓ is the quantization index, and s is the upper bound of the quantization levels

- Example: if $s = 1$, the quantization levels are $0, 1, -1$

Quantization

- Motivation: $\xi_i(\boldsymbol{v}, s)$ has minimal variance over distributions with support $\{0, 1/s, \ldots, 1\}$
- **Lemma**: For any vector $\boldsymbol{v} \in \mathbb{R}^n$, 1) $\mathbb{E}[Q_s(\boldsymbol{v})] = \boldsymbol{v}$ (unbiasedness) 2) $\mathbb{E}[\|Q_s(\boldsymbol{v}) - \boldsymbol{v}\|_2^2 \leq \min(n/s^2, \sqrt{n}/s)\|\boldsymbol{v}\|_2^2$ (variance bund), and 3) $\mathbb{E}[\|Q_s(\boldsymbol{v})\|_0] \leq s(s + \sqrt{n})$
- **Theorem**: Let $f : \mathbb{R}^n \to \mathbb{R}$ be fixed, and let $\boldsymbol{w} \in \mathbb{R}^n$ be arbitrary. Fix $s \geq 2$ quantization levels. If $\tilde{g}(\boldsymbol{w})$ is a stochastic gradient for f at $\boldsymbol{w}$ with second order moment B, then $Q_s(\tilde{g}(\boldsymbol{w}))$ is a stochastic gradient for f at $\boldsymbol{w}$ with variance bound $\min(n/s^2, \sqrt{n}/s)B$. There is an encoding scheme so that in expectation, the number of bits to communicate $Q_s(\tilde{g}(\boldsymbol{w}))$ is upper bounded by

$$\left(3 + \left(\frac{3}{2} + o(1)\right) \log\left(\frac{2(s^2 + n)}{s(s + \sqrt{n})}\right)\right) s(s + \sqrt{n}) + 32$$

Convergence of Parallel QSGD

- **Theorem 6** (Smooth Convex Parallel QSGD). Let $\mathcal{W}$, $f(\boldsymbol{w})$, $\boldsymbol{w}_0$, R and γ as in the main SGD convergence theorem. Let $\varepsilon > 0$. Suppose to run the Parallel QSGD algorithm on K processors accessing independent stochastic gradients with second moment bound B, with step size $\eta_t = 1/(L + \sqrt{K}/\gamma)$ with $\sigma = B'$ with $B' = \min(\frac{n}{s^2}, \frac{\sqrt{n}}{s})B$. If $T = \mathcal{O}\left(R^2 \max\left(\frac{2B'}{K\varepsilon^2}, \frac{L}{\varepsilon}\right)\right)$ then

$$\mathbb{E}\left[f\left(\frac{1}{T}\sum_{t=0}^{T}\boldsymbol{w}_t\right)\right] - \min_{\boldsymbol{w}\in\mathcal{W}} f(\boldsymbol{w}) \leq \varepsilon$$

Moreover, the Parallel QSGD requires a number of bits given by the previous theorem per communication round. If $s = \sqrt{n}$, the number of bits is reduced to $2.8n + 32$.

Convergence of Parallel QSGD

- **Theorem** (Smooth non Convex Parallel QSGD). Let $\mathcal{W}$, $\boldsymbol{w}_0$, R and γ as in the main SGD convergence theorem. Let $f(\boldsymbol{w})$ be an L-smooth possibly non-convex function, and let $\boldsymbol{w}_1$ be an arbitrary initial point. Let $T > 0$ be fixed, and $s > 0$.

  Then there is a random stopping time R supported on $\{1, \ldots, N\}$ so that the Parallel QSGD with quantization level s constant stepsizes $\eta = \mathcal{O}(1/L)$ and access to stochastic gradients of f with second moment bound B satisfies

$$\frac{1}{L}\mathbb{E}\left[\|\nabla f(\boldsymbol{w})\|_2^2\right] \leq \mathcal{O}\left(\frac{\sqrt{L(f(\boldsymbol{w}_1) - f^*}}{N} + \frac{B\min(n/s^2, \sqrt{n}/s)}{L}\right)$$

  Moreover, the number of bits to communicate for each gradient transmission is the same as in the previous theorem

CA 5: Communication efficiency

Split "MNIST" dataset to 10 random disjoint subsets, each for one worker, and consider SVM classifier in the form of $\min_{\boldsymbol{w}} \frac{1}{N} \sum_{i \in [N]} f_i(\boldsymbol{w})$ with $N = 10$.

a) Run decentralized GD (from Lecture 5) with 10 workers. Characterize the convergence against the total number of signaling exchanges among all nodes, denoted by T.

b) Consider a two-star topology with communication graph (1,2,3,4)-5-6-(7,8,9,10) and run decentralized subgradient method (from Lecture 5) and ADMM over network (from Lecture 6). Characterize the convergence against T. Tune hyperparameters to improve the convergence rate.

c) Propose an approach to reduce T with a marginal impact on the convergence. Do not limit your imaginations and feel free to propose any change or any solution. While being nonsense in some applications, your solution may actually make very sense in some other applications. Discuss pros and cons of your solution and possibly provide numerical evidence that it reduces T.

d) An alternative approach to improve communication-efficiency is to compress the information message to be exchanged (usually gradients – either in primal or dual forms). Consider two compression/quantization methods for a vector: (Q1) keep only K values of a vector and set the rest to zero and (Q2) represent every element with fewer bits (e.g., 4 bits instead of 32 bits).

e) Repeat parts a-b using Q1 and Q2. Can you integrate Q1/Q2 to your solution in part d? Discuss.

f) How do you make SVRG and SAG communication efficient for large-scale ML?

Some references

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