ROYAL INSTITUTE
OF TECHNOLOGY

EP3260: Machine Learning Over Networks

Lecture 2: Centralized Convex ML

(part 1)

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https://sites.google.com/view/mlons/home

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Learning outcomes

- Basic definitions of smooth and convex functions
- Important properties of smooth and convex functions
- Main (deterministic) iterative algorithms for convex problems
- Connections among them
- Pros and cons of them
- Convergence analysis

Outline

1. Student groups
2. Basic definitions and properties
3. Iterative solution approaches
4. Supplements

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Student groups

Any question?

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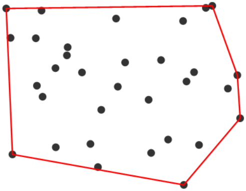
Basic definitions

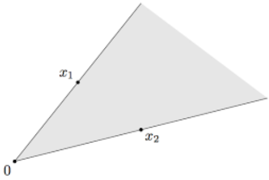
Convex combination of points $\mathcal{X} = \{\boldsymbol{x}_i\}_{i\in[n]}$ is $\sum_{i\in[n]} \theta_i \boldsymbol{x}_i$ where $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_n]$ form a probability simplex; $\boldsymbol{\theta} \geq 0, \boldsymbol{\theta}^T \mathbf{1} = 1$

Conic combination of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ is $\theta_1 \boldsymbol{x}_1 + \theta_2 \boldsymbol{x}_2$ where $\theta_i \geq 0$

Convex hull of $\mathcal{X}$: set of all convex combinations of points in $\mathcal{X}$

Convex cone of $\mathcal{X}$: set of all conic combinations of points in $\mathcal{X}$





Convexity: basic definitions

Convex set $\mathcal{X}$**:** $\forall \boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{X}$ and $\theta \in [0,1]$, $\theta \boldsymbol{x}_1 + (1-\theta)\boldsymbol{x}_2 \in \mathcal{X}$.

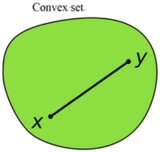
Euclidean/norm ball with radius r centered at $\boldsymbol{x}_c$:

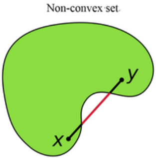
$$\{\boldsymbol{x} \mid \|\boldsymbol{x} - \boldsymbol{x}_c\|_2 \le r\} = \{\boldsymbol{x}_c + r\boldsymbol{u} \mid \|\boldsymbol{u}\|_2 \le 1\}$$

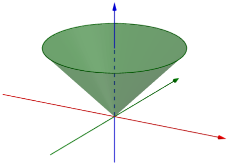
Norm cone: $\{(\boldsymbol{x}, r) \mid \|\boldsymbol{x}\| \le r\}$

Ellipsoid centered at $\boldsymbol{x}_c$: for symmetric positive definite $\boldsymbol{P}$ and square nonsingular $\boldsymbol{A}$:

$$\{(\boldsymbol{x} - \boldsymbol{x}_c)^T \boldsymbol{P}^{-1} (\boldsymbol{x} - \boldsymbol{x}_c) \le 1\} \equiv \{\boldsymbol{x}_c + \boldsymbol{A}\boldsymbol{u} \mid \|\boldsymbol{u}\|_2 \le 1\}$$







Convexity: basic definitions

Convex function $f : \mathcal{X} \to \mathbb{R}$**:** if $\mathcal{X}$ is convex and $\forall \boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{X}$ and $\theta \in [0, 1]$:

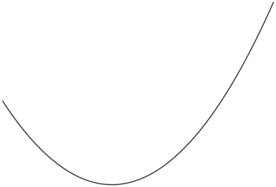
$$f\left(\theta \boldsymbol{x}_1 + (1-\theta)\boldsymbol{x}_2\right) \leq \theta f(\boldsymbol{x}_1) + (1-\theta) f(\boldsymbol{x}_2) \tag{1}$$

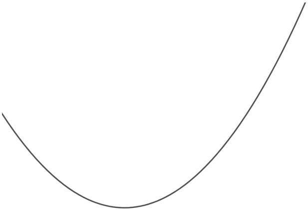
Assuming differentiability of f, (1) is equivalent to

$$f(\boldsymbol{x}_2) \geq f(\boldsymbol{x}_1) + \nabla f(\boldsymbol{x}_1)^T (\boldsymbol{x}_2 - \boldsymbol{x}_1) \tag{2}$$

Local information (gradient) determines a global lower bound

For twice differentiable f, (1) $\leftrightarrow \nabla^2 f(\boldsymbol{x}) \geq 0, \forall \boldsymbol{x} \in \mathcal{X}$: PSD Hessian, non-negative curvature everywhere.





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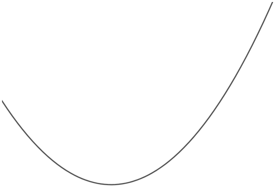

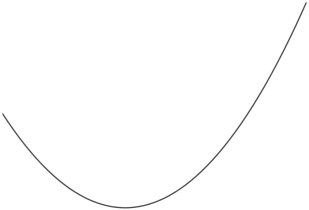

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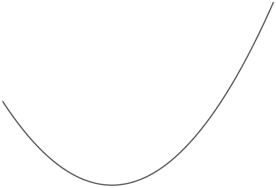
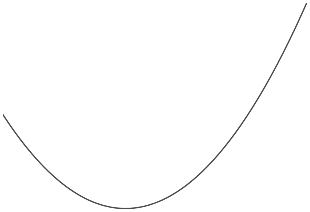




Convexity: some examples

- $\|\boldsymbol{x}\|_p$ for any $p \geq 1$
- Quadratic function $f(\boldsymbol{x}) = \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x} + \boldsymbol{b}^T \boldsymbol{x} + \boldsymbol{c}$ for symmetric matrix $\boldsymbol{A}$

$$\nabla^2 f(\boldsymbol{x}) = 2\boldsymbol{A} \quad \text{convex iff} \quad \boldsymbol{A} \geq 0$$

- $f(\boldsymbol{x}) = \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|_2^2$ is convex for any $\boldsymbol{A}$ (observe $\nabla^2 f(\boldsymbol{x}) = 2\boldsymbol{A}^T\boldsymbol{A}$)
- $\|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|_p$ for any $p \geq 1$
- $\max_{i \in [n]} \|\boldsymbol{A}_i^T \boldsymbol{x} - \boldsymbol{b}_i\|_p$ for any $p \geq 1$
- $\lambda_{\max} = \sup_{\|\boldsymbol{y}\|_2 = 1} \boldsymbol{y}^T \boldsymbol{X} \boldsymbol{y}$ for any symmetric $\boldsymbol{X}$
- $\mathrm{proj}(\boldsymbol{x}, \mathcal{C}) := \inf_{y \in \mathcal{C}} \|\boldsymbol{y} - \boldsymbol{x}\|$ for convex $\mathcal{C}$
- Check Boyd and Vandenberghe (2003) for more examples

  Nonnegative weighted sum, composition with affine function, pointwise maximum and supremum, composition, minimization, and perspective

Convexity: more definitions

- $f : \mathcal{X} \to \mathbb{R}$ is quasi-convex if $\mathcal{X}$ is convex and sub-level sets

$$\{\boldsymbol{x} \in \mathcal{X} \mid f(\boldsymbol{x}) \leq \alpha\}$$

  are convex for all α.

  Equivalently, if for all $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{X}$ and $\theta \in [0, 1]$,

$$f(\theta \boldsymbol{x}_1 + (1-\theta)\boldsymbol{x}_2) \leq \max\{f(\boldsymbol{x}_1), f(\boldsymbol{x}_2)\}$$

- A positive function is log-concave if $\forall \boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{X}$ and $\theta \in [0, 1]$

$$\log f(\theta \boldsymbol{x}_1 + (1-\theta)\boldsymbol{x}_2) \geq \theta \log f(\boldsymbol{x}_1) + (1-\theta) \log f(\boldsymbol{x}_2)$$

  Most of probability densities are log-concave

  For convex $\mathcal{S}$ and random variable $\boldsymbol{y}$ with log-concave PDF,
  $f(\boldsymbol{x}) = \Pr(\boldsymbol{x} + \boldsymbol{y} \in \mathcal{S})$ is log-concave and $\{\boldsymbol{x} \mid f(\boldsymbol{x}) \geq t\}$ is convex

- Majorization ($\boldsymbol{a} \succ \boldsymbol{b}$) and Schur convexity ($\boldsymbol{a} \succ \boldsymbol{b}$ implies $f(\boldsymbol{a}) \geq f(\boldsymbol{b})$)

Standard forms

$$\begin{aligned} \text{minimize} \quad & f_0(\boldsymbol{x}) \\ \text{s.t.} \quad & f_i(\boldsymbol{x}) \leq 0, i \in [m] \\ & h_i(\boldsymbol{x}) = 0, i \in [p] \end{aligned} \tag{3}$$

Linear program: affine objective over a (open/closed) polyhedron

Quadratic program: convex quadratic objective over a polyhedron

Quadratically constrained quadratic program

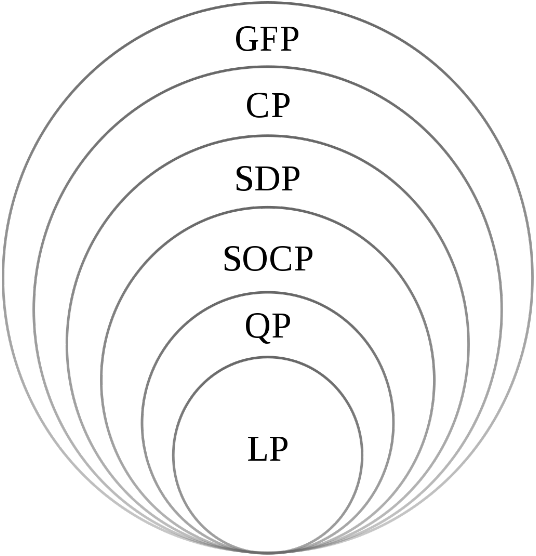
Second-order cone program:

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \|\boldsymbol{A}_i \boldsymbol{x} + \boldsymbol{b}_i\|_2 \leq \boldsymbol{d}_i^T \boldsymbol{x} + \boldsymbol{e}_i, i \in [m], \quad \boldsymbol{F}\boldsymbol{x} = \boldsymbol{g} \end{aligned}$$

Semidefinite program for symmetric $\boldsymbol{A}$ and $\boldsymbol{B}$:

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t} \quad & \boldsymbol{A} + \sum_{i \in [n]} \boldsymbol{x}_i \boldsymbol{B}_i \leq 0, \quad \boldsymbol{D}\boldsymbol{x} = \boldsymbol{e} \end{aligned}$$

Standard forms



Duality

Consider (3) with objective f_0, inequality and equality constraints f_i and h_i, and optimal solution $(x^\star, f^\star = f_0(\boldsymbol{x}^\star))$

Lagrange dual function: $g : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$

$$g(\lambda, \nu) = \inf_{x \in \mathcal{X}} \; L(x, \lambda, \nu) := f_0(\boldsymbol{x}) + \sum_{i \in [m]} \lambda_i f_i(\boldsymbol{x}) + \sum_{i \in [p]} \nu_i h_i(\boldsymbol{x})$$

if $\lambda \geq 0$, then $g(\lambda, \nu) \leq f^\star$.

Proof: given non-negative λ_i, $h_i(\boldsymbol{x}) = 0$, and $\lambda_i f_i(\boldsymbol{x}) \leq 0$ for any feasible point $\boldsymbol{x}$. Therefore,

$$f^\star = f_0(\boldsymbol{x}^\star) \geq L(\boldsymbol{x}^\star, \lambda, \nu) \geq \inf_{x \in \mathcal{X}} L(\boldsymbol{x}, \lambda, \nu) = g(\lambda, \nu)$$

Lagrange dual problem: with solution $d^\star (\leq f^\star, =$ with strong duality)

$$\begin{aligned} \text{maximize} \quad & g(\lambda, \nu) \\ \text{s.t.} \quad & \lambda \geq 0 \end{aligned}$$

Duality

Primal: minimize $\boldsymbol{c}^T\boldsymbol{x}$

s.t. $\boldsymbol{x} \geq 0$

$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$

Dual: maximize $-b^T\nu$

s.t. $\boldsymbol{A}^T\nu + \boldsymbol{c} = 0$

In a network with one unit of communication per constraint, dual is more communication-efficient for tall $\boldsymbol{A}$

- Check Boyd and Vandenberghe (2003) for more details

  Weak and strong duality

  Constraint qualifications

  Slater's constraint qualification

  Complementary slackness

  Karush-Kuhn-Tucker (KKT) conditions

Strong convexity

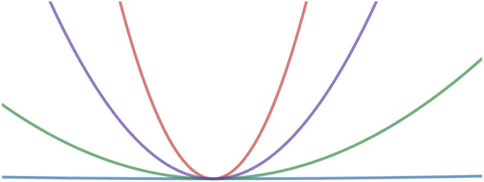
Differentiable function f is μ-strongly convex iff $\forall \boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{X}, \mu > 0$

$$f(\boldsymbol{x}_2) \geq f(\boldsymbol{x}_1) + \nabla f(\boldsymbol{x}_1)^T (\boldsymbol{x}_2 - \boldsymbol{x}_1) + \frac{\mu}{2} \|\boldsymbol{x}_2 - \boldsymbol{x}_1\|_2^2$$

Gradient can be replaced by sub-gradient for non-smooth functions

Main intuition: linear lower bound with convexity, quadratic lower bound with strong convexity

Global definitions not local ($\forall \boldsymbol{x} \in \mathcal{X}$)



Strong convexity

$$f(\boldsymbol{x}_2) \geq f(\boldsymbol{x}_1) + \nabla f(\boldsymbol{x}_1)^T(\boldsymbol{x}_2 - \boldsymbol{x}_1) + \frac{\mu}{2}\|\boldsymbol{x}_2 - \boldsymbol{x}_1\|_2^2 \tag{4}$$

- (4) is equivalent to a minimum positive curvature $\nabla^2 f(\boldsymbol{x}) \geq \mu \boldsymbol{I}_d, \forall \boldsymbol{x} \in \mathcal{X}$
- (4) is equivalent to $\left(\nabla f(\boldsymbol{x}_2) - \nabla f(\boldsymbol{x}_1)\right)^T (\boldsymbol{x}_2 - \boldsymbol{x}_1) \geq \mu\|\boldsymbol{x}_2 - \boldsymbol{x}_1\|_2^2$
- (4) implies
  (a) $f(\boldsymbol{x}) - f^\star \leq \dfrac{1}{2\mu}\|\nabla f(\boldsymbol{x})\|_2^2, \forall \boldsymbol{x}$
  (b) $\|\boldsymbol{x}_2 - \boldsymbol{x}_1\|_2 \leq \dfrac{1}{\mu}\|\nabla f(\boldsymbol{x}_2) - \nabla f(\boldsymbol{x}_1)\|_2, \forall \boldsymbol{x}_1, \boldsymbol{x}_2$
  (c) $\left(\nabla f(\boldsymbol{x}_2) - \nabla f(\boldsymbol{x}_1)\right)^T (\boldsymbol{x}_2 - \boldsymbol{x}_1) \leq \dfrac{1}{\mu}\|\nabla f(\boldsymbol{x}_2) - \nabla f(\boldsymbol{x}_1)\|_2^2, \forall \boldsymbol{x}_1, \boldsymbol{x}_2$
  (d) $f(\boldsymbol{x}) + r(\boldsymbol{x})$ is strongly convex for any convex f and strongly convex r

HW1(a): prove all the statements of this slide

Smoothness

A function $f : \mathbb{R}^d \to \mathbb{R}$ is L-smooth iff it is differentiable and its gradient is L-Lipschitz-continuous (usually w.r.t. norm-2):

$$\forall \boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^d, \|\nabla f(\boldsymbol{x}_2) - \nabla f(\boldsymbol{x}_1)\|_2 \leq L\|\boldsymbol{x}_2 - \boldsymbol{x}_1\|_2 \tag{5}$$

Recall strong convexity result: $\|\nabla f(\boldsymbol{x}_2) - \nabla f(\boldsymbol{x}_1)\|_2 \geq \mu\|\boldsymbol{x}_2 - \boldsymbol{x}_1\|_2$

For twice differentiable f, (5) $\leftrightarrow \nabla^2 f(\boldsymbol{x}) \leq L\boldsymbol{I}_d$

Smoothness: $f(\boldsymbol{x}_2) - f(\boldsymbol{x}_1)$ can be over-estimated by a quadratic function

- (5) implies for all $\boldsymbol{x}_1, \boldsymbol{x}_2$ (**HW1(b):** prove them. Assume convexity if needed)

  (a) $f(\boldsymbol{x}_2) \leq f(\boldsymbol{x}_1) + \nabla f(\boldsymbol{x}_1)^T(\boldsymbol{x}_2 - \boldsymbol{x}_1) + \dfrac{L}{2}\|\boldsymbol{x}_2 - \boldsymbol{x}_1\|_2^2$

  (b) $f(\boldsymbol{x}_2) \geq f(\boldsymbol{x}_1) + \nabla f(\boldsymbol{x}_1)^T(\boldsymbol{x}_2 - \boldsymbol{x}_1) + \dfrac{1}{2L}\|\nabla f(\boldsymbol{x}_2) - \nabla f(\boldsymbol{x}_1)\|_2^2$

  (c) $\left(\nabla f(\boldsymbol{x}_2) - \nabla f(\boldsymbol{x}_1)\right)^T(\boldsymbol{x}_2 - \boldsymbol{x}_1) \geq \dfrac{1}{L}\|\nabla f(\boldsymbol{x}_2) - \nabla f(\boldsymbol{x}_1)\|_2^2$

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Gradient descent

- Problem: $\underset{\boldsymbol{x}}{\text{minimize}}\ f(\boldsymbol{x})$ for some differentiable $f : \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$

Gradient descent (GD), also called batch GD, full GD, ...

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \alpha_k \nabla f(\boldsymbol{x}_k) \tag{6}$$

for some sequence of non-negative step sizes $(\alpha_k)_{k\in\mathbb{N}}$.

Theorem 1: Convergence of GD with constant step size

Convex and L-smooth f with $\alpha \le 1/L$ satisfies $f(\boldsymbol{x}_k) - f^\star \le \dfrac{\|\boldsymbol{x}_0 - \boldsymbol{x}^\star\|_2^2}{2k\alpha}$.

μ**-strongly convex and L-smooth** f with $\alpha \le 2/(\mu + L)$ satisfies $f(\boldsymbol{x}_k) - f^\star \le e^{-ck} L \|\boldsymbol{x}_0 - \boldsymbol{x}^\star\|_2^2/2$ for $c = 2\alpha\mu L/(\mu + L)$. With $\alpha = 2/(\mu + L)$, we have $\|\boldsymbol{x}_k - \boldsymbol{x}^\star\|_2^2 \le \left(1 - \frac{2}{1+L/\mu}\right)^{2k} \|\boldsymbol{x}_0 - \boldsymbol{x}^\star\|_2^2$.

Smooth convex: $\mathcal{O}(1/\epsilon)$ iterations for ϵ-optimality

Smooth strongly-convex: $\mathcal{O}(\log(1/\epsilon))$ iterations for ϵ-optimality

Proofs

Foods for thought

1. Define the conjugate function as

$$f^*(\boldsymbol{y}) = \sup_{y \in \mathbb{R}^d} \boldsymbol{y}^T\boldsymbol{x} - f(\boldsymbol{x}).$$

   Observe that f^* is convex even when f is not (why?). When f is μ-strongly convex and L-smooth, f^* is $\frac{1}{L}$-strongly convex and $\frac{1}{\mu}$-smooth (why?).

2. Define projection operator for convex set $\mathcal{X}$ as

$$\text{proj}(\boldsymbol{x}, \mathcal{X}) := \text{argmin}_{y \in \mathcal{X}} \|\boldsymbol{y} - \boldsymbol{x}\|.$$

   Observe $\|\text{proj}(\boldsymbol{y}, \mathcal{X}) - \boldsymbol{x}\|^2 \le \|\boldsymbol{y} - \boldsymbol{x}\|^2$ for any $\boldsymbol{x} \in \mathcal{X}$ and any $\boldsymbol{y}$. Modify (6) to solve $\min_{\boldsymbol{x} \in \mathcal{X}} f(\boldsymbol{x})$. What is the convergence of this "projected GD" algorithm?

3. Define the set of subgradients of $f : \mathcal{X} \to \mathbb{R}$ at $\boldsymbol{x} \in \mathcal{X}$ as

$$\partial f(\boldsymbol{x}) := \{\boldsymbol{s} \mid f(\boldsymbol{x}) - f(\boldsymbol{y}) \le \boldsymbol{s}^T(\boldsymbol{x} - \boldsymbol{y}) \ \forall \boldsymbol{y} \in \mathcal{X}\}.$$

   Modify (6) to solve $\min_{\boldsymbol{x} \in \mathcal{X}} f(\boldsymbol{x})$ for non-smooth (non-differentiable) functions that are Lipschitz ($|f(\boldsymbol{x}) - f(\boldsymbol{y})| \le \beta\|\boldsymbol{x} - \boldsymbol{y}\|_2$). What is the convergence of this "projected sub-GD" for Lipschitz functions? What is the optimal step size?

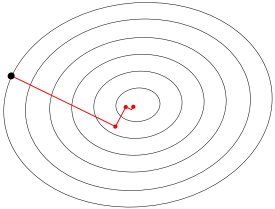
4. Check `https://ee227c.github.io/code/lecture4.html`

GD for smooth and strongly convex functions

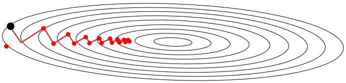
Linear convergence rate $1 - \dfrac{2}{1 + L/\mu}$

GD may need many iterations to converge

Preconditioning to change the space geometry, to make sub-levels similar in all coordinates



small L/μ



large L/μ

Descent methods

Descent methods: $$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \alpha_k \boldsymbol{d}(\boldsymbol{x}_k) \;\text{ s.t. } f(\boldsymbol{x}_{k+1}) < f(\boldsymbol{x}_k) \tag{7}$$

for some sequence of non-negative step sizes $(\alpha_k)_{k\in\mathbb{N}}$ and decent direction $\boldsymbol{d}$

$f(\boldsymbol{x}_{k+1}) < f(\boldsymbol{x}_k)$ implies $-\nabla f(\boldsymbol{x}_k)^T \boldsymbol{d}(\boldsymbol{x}_k) > 0$, same half-space as the negative gradient

Steepest descent: $\boldsymbol{d}(\boldsymbol{x}) = \operatorname{argmin}\{\nabla f(\boldsymbol{x})^T \nu \mid \|\nu\| = 1\}$ in some norm $\|\cdot\|$

* Note that we need to unnormalize the descent direction using the dual-norm

Maximizes the first order prediction of decrease (for small ν):
$f(\boldsymbol{x}+\nu) - f(\boldsymbol{x}) \approx \nabla f(\boldsymbol{x})^T \nu$

Define $\|\boldsymbol{x}\|_{\boldsymbol{P}} = (\boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x})^{1/2}$ for positive definite $\boldsymbol{P}$ (this is called Mahalanobis distance): $\boldsymbol{d}(\boldsymbol{x}) = -\boldsymbol{P}^{-1} \nabla f(\boldsymbol{x})$

Reduces to GD on Euclidian norm $(\boldsymbol{P} = \boldsymbol{I}, \boldsymbol{d}(\boldsymbol{x}) = -\nabla f(\boldsymbol{x}_k)^T)$

Good norm: should be consistent with the geometry of sublevel sets

Same theoretical convergence as of GD, much better in practice

Newton methods

Around optimal point:

$$f(\boldsymbol{x}) \approx f(\boldsymbol{x}^\star) + \cancelto{0}{\nabla f(\boldsymbol{x}^\star)^T(\boldsymbol{x} - \boldsymbol{x}^\star)} + \frac{1}{2}(\boldsymbol{x} - \boldsymbol{x}^\star)^T \nabla^2 f(\boldsymbol{x}^\star)(\boldsymbol{x} - \boldsymbol{x}^\star)$$

Sublevel sets are like ellipsoids (determined by Hessian) near the minimum

Recall $\|\boldsymbol{x}\|_{\boldsymbol{P}} = (\boldsymbol{x}^T \boldsymbol{P} \boldsymbol{x})^{1/2}$ and its descent direction $\boldsymbol{d} = \boldsymbol{P}^{-1} \nabla f(\boldsymbol{x})$

How about steepest descent on the norm induced by Hessian $\nabla^2 f(\boldsymbol{x}^\star)$?

Oops! we do not know $\boldsymbol{x}^\star$

Newton method:

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \alpha_k \nabla^2 f(\boldsymbol{x}_k)^{-1} \nabla f(\boldsymbol{x}_k). \tag{8}$$

Newton methods

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \alpha_k \nabla^2 f(\boldsymbol{x}_k)^{-1} \nabla f(\boldsymbol{x}_k)$$

Theorem 2: Quadratic convergence of Newton's method

Assume f is a twice continuously differentiable and set $\alpha_k = 1$. If $\|\boldsymbol{x}_k - \boldsymbol{x}^\star\|$ is small enough, there exist a positive constant c such that $\|\boldsymbol{x}_{k+1} - \boldsymbol{x}^\star\| \leq c(\|\boldsymbol{x}_k - \boldsymbol{x}^\star\|_2)^2$.

constant $+ \mathcal{O}(\log\log(1/\epsilon))$ iterations for ϵ-optimality

Expensive iterations due to $\nabla^2 f(\boldsymbol{x}_k)$

- **Affine invariance:** apply coordinate change for non-singular matrix $\boldsymbol{A}$. Newton's method have same iterations for $\min_{\boldsymbol{x}} f(\boldsymbol{x})$ and $\min_{\boldsymbol{y}} f(\boldsymbol{A}\boldsymbol{y})$ (namely $\boldsymbol{x}_k = \boldsymbol{A}\boldsymbol{y}_k$), whereas GD has $\nabla f(\boldsymbol{x}) = \boldsymbol{A}^T \nabla(\boldsymbol{A}\boldsymbol{y})$.

  If we change coordinate/metric, GD iterates change

  Finding a good coordinate for GD is usually very hard in high-dimension!

Proximal methods

Smoothness implies $f(\boldsymbol{x}_{k+1}) \leq f(\boldsymbol{x}_k) + \nabla f(\boldsymbol{x}_k)(\boldsymbol{x} - \boldsymbol{x}_k) + \frac{L}{2}\|\boldsymbol{x} - \boldsymbol{x}_k\|_2^2, \forall \boldsymbol{x}$

GD iterations to minimize differentiable f is like *successive quadratic upper-bound minimization*:

$$\boldsymbol{x}_{k+1} = \underset{\boldsymbol{x} \in \mathbb{R}^d}{\operatorname{argmin}}\ f(\boldsymbol{x}_k) + \nabla f(\boldsymbol{x}_k)^T(\boldsymbol{x} - \boldsymbol{x}_k) + \frac{L}{2}\|\boldsymbol{x} - \boldsymbol{x}_k\|_2^2$$

New objective: minimize $f(\boldsymbol{x}) = g(\boldsymbol{x}) + h(\boldsymbol{x})$ for convex differentiable g and convex (possibly) non-differentiable h

Define **proximal mapping** as

$$\operatorname{prox}_{\alpha h}(\boldsymbol{x}) = \operatorname{argmin}_{\boldsymbol{u}}\ h(\boldsymbol{u}) + \frac{1}{2\alpha}\|\boldsymbol{u} - \boldsymbol{x}\|_2^2$$

Proximal method:

$$\boldsymbol{x}_k = \operatorname{prox}_{\alpha_k h}\left(\boldsymbol{x}_{k-1} - \alpha_k \nabla g(\boldsymbol{x}_k)\right)$$

Proximal methods

Smoothness implies $f(\boldsymbol{x}_{k+1}) \leq f(\boldsymbol{x}_k) + \nabla f(\boldsymbol{x}_k)(\boldsymbol{x} - \boldsymbol{x}_k) + \frac{L}{2}\|\boldsymbol{x} - \boldsymbol{x}_k\|_2^2, \forall \boldsymbol{x}$

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Proximal methods

- Observe from the definition of the proximal method

$$
\begin{aligned}
\boldsymbol{x}_k &= \operatorname{argmin}_{\boldsymbol{x}} \ h(\boldsymbol{x}) + \frac{1}{2\alpha_k}\|\boldsymbol{x} - \boldsymbol{x}_{k-1} + \alpha_k \nabla g(\boldsymbol{x}_k)\|_2^2 \\
&= \operatorname{argmin}_{\boldsymbol{x}} \ h(\boldsymbol{x}) + g(\boldsymbol{x}_k) + \nabla g(\boldsymbol{x}_k)^T(\boldsymbol{x} - \boldsymbol{x}_k) + \frac{1}{2\alpha_k}\|\boldsymbol{x} - \boldsymbol{x}_k\|_2^2
\end{aligned}
$$

GD $\leftrightarrow$ proximal method with $h(\boldsymbol{x}) = 0$ and $\alpha = 1/L$

Projected GD $\leftrightarrow$ proximal method with $h(\boldsymbol{x}) = \begin{cases} 0, & \text{if } \ \boldsymbol{x} \in \mathcal{X} \\ \infty, & \text{otherwise.} \end{cases}$

Soft thresholding for ℓ_1 regularization $\leftrightarrow$ proximal method with $h(\boldsymbol{x}) = \lambda\|\boldsymbol{x}\|_1$.

$$
[\operatorname{prox}_h(\boldsymbol{x})]_i = \begin{cases} x_i - \lambda, & \text{if } \ x_i \geq \lambda \\ 0, & \text{if } \ -\lambda \leq x_i \leq \lambda \\ x_i + \lambda, & \text{if } x_i \leq -\lambda. \end{cases}
$$

Resource allocation

HW1(c): Consider

$$\begin{aligned} \text{minimize} \quad & \frac{1}{N} \sum\nolimits_{i \in [N]} f_i(x_i) \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}. \end{aligned}$$

for $\boldsymbol{A} \in \mathbb{R}^{p \times N}$ and $\boldsymbol{x} = [x_1, \ldots, x_N]^T$.

(a) Assume strong-convexity and smoothness on f. How would you solve this problem when $N = 1000$?

(b) What if $N = 10^9$?

(c) Can we use Newton's method for $N = 10^9$? Try efficient method for computing $\nabla^2 f(\boldsymbol{x}_k)$ for $p = 1$ and $b = 1$ (probability simplex constraint). Extend it to $1 \leq p \ll N$.

(d) Now, add twice differentiable $r(\boldsymbol{x})$ to the objective and solve (a)-(c).

Some references

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Outline

Proof sketch for convex and L-smooth f

By convexity of f, $f(\boldsymbol{x}_i) \leq f(\boldsymbol{x}^\star) + \langle \nabla f(\boldsymbol{x}_i), \boldsymbol{x}_i - \boldsymbol{x}^\star \rangle$ $\qquad (*)$

Use smoothness property $f(\boldsymbol{x}_{i+1}) \leq f(\boldsymbol{x}_i) + \nabla f(\boldsymbol{x}_i)^T(\boldsymbol{x}_{i+1} - \boldsymbol{x}_i) + \dfrac{L}{2}\|\boldsymbol{x}_{i+1} - \boldsymbol{x}_i\|_2^2$ and $\alpha \leq 1/L$ to conclude $f(\boldsymbol{x}_{i+1}) \leq f(\boldsymbol{x}_i) - \frac{\alpha}{2}\|\nabla f(\boldsymbol{x}_i)\|_2^2$ $\qquad (**)$

Substitute $(*)$ into $(**)$ and note that from (6): $\nabla f(\boldsymbol{x}_i) = \dfrac{\boldsymbol{x}_i - \boldsymbol{x}_{i+1}}{\alpha}$

Observe $f(\boldsymbol{x}_{i+1}) \leq f^\star + \dfrac{1}{2\alpha}\left(\|\boldsymbol{x}_i - x^\star\|_2^2 - \|\boldsymbol{x}_{i+1} - x^\star\|_2^2\right)$

Summing over iterations: $\dfrac{1}{k}\sum_{i\in[k]}\left(f(\boldsymbol{x}_i) \leq f^\star\right) \leq \dfrac{1}{2\alpha k}\|\boldsymbol{x}_0 - x^\star\|_2^2$

Conclude from the non-increasing property of GD iterates:

$$f(\boldsymbol{x}_k) - f^\star \leq \frac{1}{k}\sum_{i\in[k]} f(\boldsymbol{x}_i) - f^\star \leq \frac{1}{2\alpha k}\|\boldsymbol{x}_0 - x^\star\|_2^2.$$

Return

Proof sketch for strongly-convex and L-smooth f

From smoothness and vanishing gradient of the optimal point, conclude
$f(\boldsymbol{x}_i) - f(\boldsymbol{x}^\star) \le \frac{L}{2}\|\boldsymbol{x}_k - \boldsymbol{x}^\star\|_2^2 \qquad (*)$

Use the coercivity of the gradient (**HW1(d):** prove it)

$$(\nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y}))^T(\boldsymbol{x} - \boldsymbol{y}) \ge \frac{\mu L}{\mu + L}\|\boldsymbol{x} - \boldsymbol{y}\|_2^2 + \frac{1}{\mu + L}\|\nabla f(\boldsymbol{x}) - \nabla f(\boldsymbol{y})\|_2^2$$

Use $\alpha < 2/(L+\mu)$ to obtain $\boldsymbol{x}_i - \boldsymbol{x}^\star\|^2 \le \left(1 - 2t\alpha\frac{\mu L}{\mu + L}\right)\|\boldsymbol{x}_i - \boldsymbol{x}^\star\|_2^2$

Iterate over i and use $(*)$ to obtain

$$f(\boldsymbol{x}_k) - f(\boldsymbol{x}^\star) \le \frac{L}{2}\prod_{i\in[k]}\left(1 - 2t\alpha\frac{\mu L}{\mu + L}\right)\|\boldsymbol{x}_0 - \boldsymbol{x}^\star\|_2^2$$

Use $e^{-x} \ge 1 - x$ to conclude $f(\boldsymbol{x}_k) - f(\boldsymbol{x}^\star) \le \dfrac{L}{2}e^{-ck}\|\boldsymbol{x}_0 - \boldsymbol{x}^\star\|_2^2$ for $c = \frac{2\alpha\mu L}{\mu+L}$

Return