Solutions to Problems in Theoretical Physics

Abdul Ali Khan

This is a collection of my solutions to problems from various subfields of Theoretical Physics. Please e-mail corrections or comments to Ali.Khan.Physics@gmail.com

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1 Quantum Field Theory

Q1 A string of length a, mass per unit length σ and under tension T is fixed at each end. The Lagrangian governing the time evolution of the transverse displacement $y(x,t)$ is

$$L = \int_0^a dx \left[\frac{\sigma}{2} \left(\frac{\partial y}{\partial t} \right)^2 - \frac{T}{2} \left(\frac{\partial y}{\partial x} \right)^2 \right] \tag{1}$$

where x identifies position along the string from one end point. By expressing the displacement as a sine series Fourier expansion in the form

$$y(x,t) = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a} \right) q_n(t) \tag{2}$$

$$\tag{3}$$

(a)
Show that the Lagrangian becomes

$$L = \sum_{n=1}^{\infty} \left[\frac{\sigma}{2} \dot{q}_n^2 - \frac{T}{2} \left(\frac{n\pi}{a} \right)^2 q_n^2 \right] \tag{4}$$

Differentiating (2):

$$y = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a} \right) q_n \tag{5}$$

$$\frac{\partial y}{\partial x} = \pi \sqrt{\frac{2}{a^3}} \sum_{n=1}^{\infty} n \cos\left(\frac{n\pi x}{a} \right) q_n \tag{6}$$

$$\frac{\partial y}{\partial t} = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a} \right) \dot{q}_n \tag{7}$$

Substituting into (1):

$$\begin{aligned}
L &= \int_0^a dx \left[\frac{\sigma}{a} \left(\sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a} \right) \dot{q}_n \right)^2 - \frac{T\pi^2}{a^3} \left(\sum_{n=1}^{\infty} n \cos\left(\frac{n\pi x}{a} \right) q_n \right)^2 \right] && (8) \\
&= \int_0^a dx \left[\frac{\sigma}{a} \sum_{n=1}^{\infty} \sin^2\left(\frac{n\pi x}{a} \right) \dot{q}_n^2 - \frac{T\pi^2}{a^3} \sum_{n=1}^{\infty} n^2 \cos^2\left(\frac{n\pi x}{a} \right) q_n^2 \right] && (9) \\
&= \sum_{n=1}^{\infty} \left[\frac{\sigma}{a} \dot{q}_n^2 \int_0^a dx \sin^2\left(\frac{n\pi x}{a} \right) - \frac{T\pi^2}{a^3} n^2 q_n^2 \int_0^a dx \cos^2\left(\frac{n\pi x}{a} \right) \right] && (10) \\
&= \sum_{n=1}^{\infty} \left[\frac{\sigma}{2} \dot{q}_n^2 - \frac{T}{2} \left(\frac{n\pi}{a} \right)^2 q_n^2 \right] && (11)
\end{aligned}$$

Where on line (9), we used that the orthoganlity of $\sin(n\pi x)$ and $\sin(m\pi x)$ under integration, when $n \neq m$. On line (10), we used that $\int_0^a dx \cos^2\left(\frac{n\pi x}{a}\right) = \frac{a}{2}$.

(b) Derive the equations of motion. Hence show that the string is equivalent to an infinite set of decoupled harmonic oscialltors with frequencies

$$\omega_n = \sqrt{\frac{T}{\sigma}} \left(\frac{n\pi}{a} \right) \tag{12}$$

$$\frac{\partial L}{\partial q_n} = \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_n} \tag{13}$$

$$\frac{\partial L}{\partial q_n} = \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{q}_n} \tag{14}$$

$$-T\left(\frac{n\pi}{a}\right)^2 q_n = \sigma \ddot{q}_n \tag{15}$$

$$\ddot{q}_n = -\frac{T}{\sigma}\left(\frac{n\pi}{a}\right)^2 q_n \tag{16}$$

where $n \in \mathbb{N}$ Therefore, this is an infinite set of decoupled harmonic oscillators with frequencies

$$\omega_n = \sqrt{\frac{T}{\sigma}}\left(\frac{n\pi}{a}\right) \tag{17}$$

Q2 Show directly that if $\phi(x)$ satisfies the Klein-Gordon equation, then $\phi\left(\Lambda^{-1}x\right)$ also satisfies this equation for any Lorentz transformation Λ.

$$\phi(x) \to \phi(\Lambda^{-1}(x)) = \phi(y) \tag{18}$$

$$\partial_\mu \partial^\mu \phi(x) \to \partial_\mu \partial^\mu \phi(y) \tag{19}$$

$$= (\Lambda^{-1})_\mu^{\ \alpha}(\Lambda^{-1})^\mu_{\ \beta}\partial_\alpha \partial^\beta \phi(y) \tag{20}$$

$$= \partial_\alpha \partial^\beta \phi(y) \tag{21}$$

$$\leftarrow (\Lambda^{-1})_\mu^{\ \alpha}(\Lambda^{-1})^\mu_{\ \beta} = g^{\alpha}_{;\ \beta} \tag{22}$$

Q3 The motion of a complex field $\psi(x)$ is governed by the Lagrangian

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi - m^2 \psi^* \psi - \frac{\lambda}{2}\left(\psi^* \psi\right)^2 \tag{23}$$

(a) Write down the Euler-Lagrange fields equations for this system. Verify that the Lagrangian is invariant under the infinitesimal transformation

$$\delta\psi = i\alpha\psi \quad , \quad \delta\psi^* = i\alpha\psi^* \tag{24}$$

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi - m^2 \psi^* \psi - \frac{\lambda}{2}\left(\psi^* \psi\right)^2 \tag{25}$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\right) - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \tag{26}$$

$$\partial^\mu \partial_\mu \psi^* - m^2 \psi^* - \lambda \psi^* |\psi|^2 = 0 \qquad (1) \tag{27}$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*)}\right) - \frac{\partial \mathcal{L}}{\partial \psi^*} = 0 \tag{28}$$

$$\partial^\mu \partial_\mu \psi - m^2 \psi - \lambda \psi |\psi|^2 = 0 \qquad (2) \tag{29}$$

Infinitesimal transformation of Lagrangian:

$$\psi \to \psi + \delta\psi = \psi + i\alpha\psi \tag{30}$$
$$\psi^* \to \psi^* + \delta\psi^* = \psi^* - i\alpha\psi^* \tag{31}$$

$$\mathcal{L} \to \partial_\mu \left(\psi^* - i\alpha\psi^*\right) \partial^\mu \left(\psi + i\alpha\psi\right) - m^2 \left(\psi^* - i\alpha\psi^*\right)\left(\psi + i\alpha\psi\right) - \frac{\lambda}{2}\left(\left(\psi^* - i\alpha\psi^*\right)\left(\psi + i\alpha\psi\right)\right)^2 \tag{32}$$
$$= \left(\partial_\mu\psi^* - i\alpha\partial_\mu\psi^*\right)\left(\partial^\mu\psi + i\alpha\partial^\mu\psi\right) - m^2\left(\psi^*\psi + \alpha^2\psi^*\psi\right) - \frac{\lambda}{2}\left(\psi^*\psi + \alpha^2\psi^*\psi\right)^2 \tag{33}$$
$$= \partial_\mu\psi^*\partial^\mu\psi + \alpha^2\partial_\mu\psi^*\partial^\mu\psi - m^2(\psi^*\psi + \alpha^2\psi^*\psi) - \frac{\lambda}{2}\left(\psi^*\psi\right)^2(1+\alpha^2)^2 \tag{34}$$
$$= \mathcal{L} + \alpha^2\partial_\mu\psi^*\partial^\mu\psi - m^2\alpha^2\psi^*\psi - \lambda\alpha^2\left(\psi^*\psi\right)^2 + -\frac{\lambda}{2}\alpha^4\left(\psi^*\psi\right)^2 \tag{35}$$

$\therefore \delta\mathcal{L} = 0$ to first order in α.

(b) Derive the Noether current associated with this transformation and verify explicitly that it is conserved using the field equations satisfied by ψ.

Noether Current:

$$j^\mu = \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right)\delta\psi + \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^*)}\right)\delta\psi^* \tag{36}$$
$$= i\alpha(\psi\partial^\mu\psi^* - \psi^*\partial^\mu\psi) \tag{37}$$

Consevation of Noether Current:

$$\partial_\mu j^\mu = \psi\partial_\mu\partial^\mu\psi^* - \psi^*\partial_\mu\partial^\mu\psi + \partial_\mu\psi\partial^\mu\psi^* - \partial_\mu\psi^*\partial^\mu\psi \tag{38}$$
$$= \psi\partial_\mu\partial^\mu\psi^* - \psi^*\partial_\mu\partial^\mu\psi \tag{39}$$

$$\leftarrow \partial^\mu\partial_\mu\psi^* = m^2\psi^* - \lambda\psi^*|\psi|^2 \quad (1) \tag{42}$$

$$= m^2|\psi|^2 - \lambda|\psi|^4 - (m^2|\psi|^2 - \lambda|\psi|^4) \tag{40}$$
$$= 0 \tag{41}$$

$$\leftarrow \partial^\mu\partial_\mu\psi = m^2\psi - \lambda\psi|\psi|^2 \quad (2) \tag{43}$$

Q4 The Lagrangian density for a triplet of real fields $\phi_a (a = 1, 2, 3)$ is defined as

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a - \tfrac{1}{2}m^2\phi_a\phi_a \tag{44}$$

(a) Verify that $\mathcal{L}$ is invariant under the infinitesimal $SO(3)$ rotation by θ

$$\phi_a \to \phi_a + \theta\epsilon_{abc}n_b\phi_c \tag{45}$$

where n_a is a unit vector.

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a - \tfrac{1}{2}m^2\phi_a\phi_a \tag{46}$$
$$\mathcal{L} \to \tfrac{1}{2}\partial_\mu\left(\phi_a + \theta\epsilon_{abc}n_b\phi_c\right)\partial^\mu\left(\phi_a + \theta\epsilon_{ade}n_d\phi_e\right) - \tfrac{1}{2}m^2\left(\phi_a + \theta\epsilon_{abc}n_b\phi_c\right)\left(\phi_a + \theta\epsilon_{ade}n_d\phi_e\right) \tag{47}$$
$$= \tfrac{1}{2}\left(\partial_\mu\phi_a + \theta\epsilon_{abc}n_b\partial_\mu\phi_c\right)\left(\partial^\mu\phi_a + \theta\epsilon_{ade}n_d\partial^\mu\phi_e\right) - \tfrac{1}{2}m^2\left(\phi_a + \theta\epsilon_{abc}n_b\phi_c\right)\left(\phi_a + \theta\epsilon_{ade}n_d\phi_e\right) \tag{48}$$
$$= \mathcal{L} + \theta\epsilon_{abc}n_b\partial_\mu\phi_c\partial^\mu\phi_a - \theta\epsilon_{abc}n_b\phi_a\phi_c + O(\theta^2) \tag{49}$$

Where, in the line (49), we multiply out and combine terms by relabelling $d \to b$ and $e \to c$. We observe that the terms linear in θ are the product of a symmetric term and the anti-symmetric term ϵ_{abc}. Therefore, the Lagrangian is invariant under an infinitesimal $SO(3)$ transformation.

(b) Compute the Noether current j^μ.

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\delta\phi}{\delta\theta} \tag{50}$$

$$= \epsilon_{abc} n_b \phi_c \partial^\mu \phi_a \tag{51}$$

$$j^0 = \epsilon_{abc} n_b \dot{\phi}_a \phi_c \tag{52}$$

$$\int d^3x \; j^0 = \int d^3x \; \epsilon_{abc} n_b \phi_c \dot{\phi}_a \tag{53}$$

$$Q_b n_b = \int d^3x \; \epsilon_{abc} n_b \phi_c \dot{\phi}_a \tag{54}$$

(c) Deduce that the three quantities

$$Q_a = \int d^3x \epsilon_{abc} \dot{\phi}_b \phi_c \tag{55}$$

are all conserved and verify this directly using the field equations satisfied by ϕ_a.

$$\dot{Q}_b = \int d^3x \; \epsilon_{abc} \left(\ddot{\phi}_a \phi_c + \dot{\phi}_a \dot{\phi}_c \right) \tag{56}$$

$$= \int d^3x \; \epsilon_{abc} \ddot{\phi}_a \phi_c \tag{57}$$

$$= \int d^3x \; \epsilon_{abc} \phi_c \left(\nabla^2 \phi_a - m^2 \phi_a \right) \tag{58}$$

$$= \int d^3x \; \epsilon_{abc} \phi_c \nabla^2 \phi_a \tag{59}$$

$$= \int d^3x \; \epsilon_{abc} \nabla \phi_c \nabla \phi_a \tag{60}$$

$$= 0 \tag{61}$$

Field equations:

(62)

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0 \tag{63}$$

$$\partial_\mu \partial^\mu \phi_a + m^2 \phi_a = 0 \tag{64}$$

$$\leftarrow \frac{\partial^2 \phi_a}{\partial t^2} - \nabla^2 \phi_a + m^2 \phi_a = 0 \tag{65}$$

$\leftarrow$ Integration by parts with

(66)

$$[\phi_c \nabla \phi_a]_{-\infty}^{+\infty} = 0 \tag{67}$$

Q5 A string has classical Hamiltonian given by

$$H = \sum_{n=1}^{\infty} \left(\tfrac{1}{2} p_n^2 + \tfrac{1}{2} \omega_n^2 q_n^2 \right) \tag{68}$$

where ω_n is the frequency of the nth mode. Compare this Hamiltonian to the Lagrangian in the previous question. The mass per unit length σ, has now been set to unity so as to make various formulae somewhat simpler. After quantization, q_n and p_n become operators satisfying

$$[q_n, q_m] = [p_n, p_m] = 0 \quad \text{and} \quad [q_n, p_m] = i\delta_{nm} \tag{69}$$

(a) Introduce creation and annihilation operators a_n and $a_n^\dagger$,

$$a_n = \sqrt{\frac{\omega_n}{2}} q_n + \frac{i}{\sqrt{2\omega_n}} p_n \quad \text{and} \quad a_n^\dagger = \sqrt{\frac{\omega_n}{2}} q_n - \frac{i}{\sqrt{2\omega_n}} p_n \tag{70}$$

Show that they satisfy the commutation relations

$$[a_n, a_m] = [a_n^\dagger, a_m^\dagger] = 0 \quad \text{and} \quad [a_n, a_m^\dagger] = \delta_{nm} \tag{71}$$

$$\begin{aligned}
[a_n, a_m] &= \left[\sqrt{\frac{\omega_n}{2}} q_n + \frac{i}{\sqrt{2\omega_n}} p_n, \sqrt{\frac{\omega_m}{2}} q_m + \frac{i}{\sqrt{2\omega_m}} p_m\right] && (72)\\
&= \frac{1}{2}\sqrt{\omega_n\omega_m}[q_n, q_m] - \frac{1}{2\sqrt{\omega_n\omega_m}}[p_n, p_m] - i\sqrt{\frac{\omega_m}{\omega_n}}[q_m, p_n] + i\sqrt{\frac{\omega_m}{\omega_n}}[q_n, p_m] && (73)\\
&= -i\sqrt{\frac{\omega_m}{\omega_n}}\delta_{mn} + i\sqrt{\frac{\omega_m}{\omega_n}}\delta_{mn} && (74)\\
&= 0 && (75)
\end{aligned}$$

Q6 Consider the infinitesimal form of the Lorentz transformation derived in question (b): $x^\mu \to x^\mu + \omega^\mu{}_\nu x^\nu$.

(a) Show that a scalar field transforms as

$$\phi(x) \to \phi'(x) = \phi(x) - \omega^\mu{}_\nu x^\nu \partial_\mu \phi(x) \tag{76}$$

$$\begin{aligned}
\phi(x) \to \phi(x') &= \phi((\Lambda^{-1})^\mu{}_\nu x^\nu) && (77)\\
&= \phi((\delta^\mu_\nu - \omega^\mu{}_\nu)x^\nu) && (78)\\
&= \phi(x^\mu - \omega^\mu{}_\nu x^\nu) && (79)\\
&= \phi(x^\mu) - \omega^\mu{}_\nu x^\nu \partial_\mu \phi(x^\mu) && (80)
\end{aligned}$$

(b) Show that the variation of the Lagrangian density is a total derivative

$$\delta\mathcal{L} = -\partial_\mu \left(\omega^\mu{}_\nu x^\nu \mathcal{L}\right) \tag{81}$$

As $\mathcal{L}$ is a Lortentz scalar, it must transform in the same way as ϕ, which we derived in question (a)

$$\begin{aligned}
\delta\mathcal{L} &= -\omega^\mu{}_\nu x^\nu \partial_\mu \mathcal{L} && (82)\\
&= -\omega^\mu{}_\nu x^\nu \partial_\mu \mathcal{L} - \omega^\mu{}_\mu \mathcal{L} && (83)\\
&= -\partial_\mu \left(\omega^\mu{}_\nu x^\nu \mathcal{L}\right) && (84)
\end{aligned}$$

In line 83 we added the trace of of $\omega^\mu{}_\nu$, which we can do because it's antisymmetric and its trace is therefore zero.

(c) Using Noether's theorem deduce the existence of the conserved current

$$j^\mu = -\omega^\rho{}_\nu \left[T^\mu{}_\rho x^\nu\right] \tag{85}$$

The Noether current is given by

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi - F^\mu \tag{86}$$

$$\tag{87}$$

nnn The function F^μ is defined by $\delta\mathcal{L} = \partial_\mu F^\mu$, where $\delta\mathcal{L}$ is the change in the Lagrangian due to an infinitesimal Lorentz transform. However, in question (b) we showed that this can also be expressed as $-\partial_\mu\left(\omega^\mu{}_\nu x^\nu \mathcal{L}\right)$. We therefore have that $F^\mu = \omega^\mu{}_\nu x^\nu \mathcal{L}$ and the Noether current can be expressed as:

$$\begin{aligned} j^\mu &= -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\omega^\rho{}_\nu x^\nu \partial_\rho \phi + \omega^\mu{}_\nu x^\nu \mathcal{L} && (88)\\ &= -\omega^\rho{}_\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} x^\nu \partial_\rho\phi - \delta^\mu_\rho x^\nu \mathcal{L}\right] && (89)\\ &= -\omega^\rho{}_\nu \left[T^\mu{}_\rho x^\nu\right] && (90) \end{aligned}$$

Where $T^\mu{}_\rho$ is the energy-momentum tensor.

(d) The three conserved charges arising from spatial rotational invariance define the *total angular momentum* of the field. Show that these charges are given by

$$Q_i = \epsilon_{ijk} \int d^3x \left(x^j T^{0k} - x^k T^{0j}\right) \tag{91}$$

The charge associated with j^μ is given by

$$\begin{aligned} Q \equiv \int d^3x\, j^0 &= -\omega^\rho{}_\nu \int d^3x \left[T^0{}_\rho x^\nu\right] && (92)\\ &= -\omega^\rho{}_\nu \int d^3x \left(\tfrac{1}{2}\left(T^0{}_\rho x^\nu - T^0{}_\nu x^\rho\right) + \tfrac{1}{2}\left(T^0{}_\rho x^\nu + T^0{}_\nu x^\rho\right)\right) && (93)\\ &= -\frac{\omega^\rho{}_\nu}{2} \int d^3x \left(T^0{}_\rho x^\nu - T^0{}_\nu x^\rho\right) && (94) \end{aligned}$$

On line (93), we split the energy-momentum tensor into antisymmetric and symmetric parts, and on line (94) we use the fact that the product of a symmetric the antisymmetric tensor $\omega^\mu{}_\nu$ is zero to drop the symmetric part.

For pure spatial rotations, we have that $\omega^i{}_j \neq 0$ and $\omega^0{}_i = \omega^i{}_0 = 0$, for $i,j,k = 1,2,3$. Therefore, (94) can be expressed as

$$Q = \frac{\omega_{jk}}{2} \int d^3x \left(T^{0k} x^j - T^{0j} x^k\right) \tag{95}$$

An infinitesimal rotation through angle θ about the vector $v^i = (v_1, v_2, v_3)$ is represented by the matrix

$$\begin{pmatrix} 1 & -v_3\theta & v_2\theta \\ v_3\theta & 1 & -v_1\theta \\ -v_2\theta & v_1\theta & 1 \end{pmatrix} \tag{96}$$

In question (b), we showed that the infinitesimal Lorentz transformations can be expressed as $\Lambda^\mu{}_\nu = \omega^\mu{}_\nu + \delta^\mu_\nu$. Therefore, we have that

$$\omega^\mu{}_\nu = \begin{pmatrix} 0 & -v_3\theta & v_2\theta \\ v_3\theta & 0 & -v_1\theta \\ -v_2\theta & v_1\theta & 0 \end{pmatrix} = v^i\epsilon_{ijk} \tag{97}$$

Each component of v^i is associated with a component of the rotational angular momentum, so we use this to express the charge Q in terms of its components Q_i

$$Q = Q_i v^i \tag{98}$$

Substituting (97 and (98) into (94):

$$Q_i v^i = \frac{v^i\epsilon_{ijk}}{2}\int d^3x\left(T^{0k}x^j - T^{0j}x^k\right) \tag{99}$$

$$Q_i = \frac{\epsilon_{ijk}}{2}\int d^3x\left(T^{0k}x^j - T^{0j}x^k\right) \tag{100}$$

(e) Derive the conserved charges arising from invariance under Lorentz boosts. Show that they imply

$$\frac{d}{dt}\int d^3x_i\left(x^iT^{00}\right) = \text{constant} \tag{101}$$

and interpret this equation

On line (94), we showed

$$Q = -\frac{\omega^\rho{}_\nu}{2}\int d^3x\left(T^0{}_\rho x^\nu - T^0{}_\nu x^\rho\right) \tag{102}$$

For Lorentz boosts, we have that $\omega^i{}_0 \neq 0$, $\omega^0{}_i \neq 0$ and $\omega^i{}_j = 0$ for $i,j,k = 1,2,3$. Therefore, line (102) can be expressed as

$$Q_i = \frac{\omega_{0k}}{2}\int d^3x\left(T^{0k}x^0 - T^{00}x^k\right) \tag{103}$$

Next, we use the conservation of charge

$$\frac{\partial Q_i}{\partial x^0} = 0 = \frac{1}{2}\omega_{0k}\int d^3x\,\frac{\partial}{\partial t}\left(T^{0k}x^0 - T^{00}x^k\right) \tag{104}$$

$$0 = \int d^3x\,\frac{\partial}{\partial x^0}\left(T^{0k}x^0 - T^{00}x^k\right) \tag{105}$$

$$0 = \int d^3x\left(T^{0k} + x^0\frac{\partial}{\partial t}T^{0k} - \frac{\partial}{\partial x^0}\left(T^{00}x^k\right)\right) \tag{106}$$

$$0 = \int d^3x\left(T^{0k} - \frac{\partial}{\partial x^0}\left(T^{00}x^k\right)\right) \tag{107}$$

$$\int d^3x\,\frac{\partial}{\partial x^0}\left(T^{00}x^k\right) = \text{const} \tag{108}$$

On line (107), we used the fact that the linear momentum P^i is given by T^{0i} to drop the derivative of this term, and then on line (108) we used the fact that the total momentum $\int d^3x\ P^i$ is constant.

Furthermore, the energy is given by T^{00}, so (108) can be interpreted to mean that the centre of energy of the field has a constant velocity.

Q7 A Lorentz transformation $x^\mu \to x'^\mu = \Lambda^\sigma{}_\mu x^\nu$ is such that it preserves the Minkowski metirc, $g_{\mu\nu}$, meaning $g_{\mu\nu}x^\mu x^\nu = g_{\mu\nu}x'^\mu x'^\nu$ for all x.

(a) Show that this implies that

$$g_{\mu\nu} = g_{\sigma\tau}\Lambda^{\sigma}{}_{\mu}\Lambda^{\tau}{}_{\nu} \tag{109}$$

$$\begin{aligned}
g_{\mu\nu}x^{\mu}x^{\nu} &= g_{\mu\nu}x'^{\mu}x'^{\nu} && (110)\\
g_{\mu\nu}x^{\mu}x^{\nu} &= g_{\mu\nu}\Lambda^{\mu}{}_{\sigma}\Lambda^{\nu}{}_{\tau}x^{\sigma}x^{\tau} \qquad \leftarrow \text{ Lorentz transformation: } x'^{\mu} = \Lambda^{\mu}{}_{\sigma}x^{\sigma} && (111)\\
g_{\mu\nu}x^{\mu}x^{\nu} &= g_{\sigma\tau}\Lambda^{\sigma}{}_{\mu}\Lambda^{\tau}{}_{\nu}x^{\mu}x^{\nu} \qquad \leftarrow \mu \leftrightarrow \sigma \ , \ \nu \leftrightarrow \tau && (112)\\
\therefore\ g_{\mu\nu} &= g_{\sigma\tau}\Lambda^{\sigma}{}_{\mu}\Lambda^{\tau}{}_{\nu} && (113)
\end{aligned}$$

(b) Use this to show that an infinitesimal transformation of the form

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}{}_{\nu} \tag{114}$$

is a Lorentz transformation when $\omega^{\mu\nu} = -\omega^{\nu\mu}$. How many parameters does such an anti-symmetric 4×4 matrix have?

$$\begin{aligned}
g_{\mu\nu} &= g_{\sigma\tau}\Lambda^{\sigma}{}_{\mu}\Lambda^{\tau}{}_{\nu} && (115)\\
g_{\mu\nu} &= g_{\sigma\tau}\left(\delta^{\sigma}_{\mu} + \omega^{\sigma}{}_{\mu}\right)\left(\delta^{\tau}_{\nu} + \omega^{\tau}{}_{\nu}\right) && (116)\\
g_{\mu\nu} &= g_{\sigma\tau}\delta^{\sigma}_{\mu}\delta^{\tau}_{\nu} + g_{\sigma\tau}\delta^{\sigma}_{\mu}\omega^{\tau}{}_{\nu} + g_{\sigma\tau}\omega^{\sigma}{}_{\mu}\delta^{\tau}_{\nu} + \mathcal{O}(\omega^2) && (117)\\
g_{\mu\nu} &= g_{\mu\nu} + g_{\sigma\tau}\delta^{\sigma}_{\mu}\omega^{\tau}{}_{\nu} + g_{\sigma\tau}\omega^{\sigma}{}_{\mu}\delta^{\tau}_{\nu} && (118)\\
0 &= g_{\sigma\tau}\delta^{\sigma}_{\mu}\omega^{\tau}{}_{\nu} + g_{\sigma\tau}\omega^{\sigma}{}_{\mu}\delta^{\tau}_{\nu} && (119)\\
0 &= g_{\mu\tau}\omega^{\tau}{}_{\nu} + g_{\sigma\nu}\omega^{\sigma}{}_{\mu} && (120)\\
0 &= \omega_{\mu\nu} + \omega_{\nu\mu} && (121)\\
\omega_{\mu\nu} &= -\omega_{\nu\mu} && (122)
\end{aligned}$$

$\implies \omega_{\mu\nu}$ is a 4×4 anti-symmetric matrix and therefore has 6 parameters. These correspond to the 6 continuous symmetries of the Lorentz group: 3 boosts and 3 rotations.

(c) Write down the matrix form of $\Lambda^{\mu}{}_{\nu}$ for a rotation through an infinitesimal angle θ about the x_1 axis. Do the same thing for a boost along the x_1 axis by an infinitesimal velocity v.

Rotation through an infinitesimal angle θ about x^3 axis:

$$\Lambda^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & \cos\theta & -\sin\theta & 0\\ 0 & \sin\theta & \cos\theta & 0\\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\theta\to 0} \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & -\theta & 0\\ 0 & \theta & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{123}$$

(d) Do the same thing for a boost along the x_1 axis by an infinitesimal velocity v.

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[\gamma \to 1]{v \to 0} \begin{pmatrix} 1 & -v & 0 & 0 \\ -v & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{124}$$

Q8 Even though we have set $\hbar = c = 1$, we can still do dimensional analysis because we still have one unit left over, which we choose as mass (or length^{-1})

(a) Consider the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\partial_{\mu}\partial^{\mu} - \sum_{n \geq 2} a_n \phi^n \tag{125}$$

Given that the action is dimensionless when $\hbar = 1$ and working in d space-time dimensions (one time and $d-1$ space), calculate the following

(i) The mass dimension of the scalar field ϕ

(ii) The dimension of the coefficients a_n.

(i) The action S must be unit-less. One way to see this is to consider the term $e^{iS/}$, which is used in path-integral calculations.
(a)

$$[S] = 0$$

$$\left[\int \mathcal{L}\ d^d x\right] = 0 \qquad \leftarrow \text{By definition in d-dimensions}$$

$$\left[\int \left((\partial_{\mu}\phi)^2 - \Sigma_{(n \leq 2)}\ a_n \phi^n\right) d^d x\right] = 0$$

$$\left[\int (\partial_{\mu}\phi)^2\ d^d x\right] = 0 \qquad \leftarrow \text{Both terms must have equal dimension so looking at first term only}$$

$$[(\partial_{\mu}\phi)^2] + [d^d x] = 0 \qquad \leftarrow \text{Square brackets indicate the mass dimension of the term, so these terms can be manipulated like logarithms.}$$

$$2[(\partial_{\mu}\phi)] + [d^d x] = 0$$

$$2[\partial_{\mu}] + 2[\phi] + [d^d x] = 0$$

$$2 + 2[\phi] - d = 0 \qquad \leftarrow \text{Dimension of } \partial_{\mu} \text{ and } d^d x \text{ is } 1/L \text{ and } L^d \text{ respectively, and } [M] = [L]^{-1}. \text{ Therefore } [\partial_{\mu}] = 1 \text{ and } [d^d x] = -d$$

$$[\phi] = \frac{d}{2} - 1$$

(ii)

$$\left[\int \left((\partial_{\mu}\phi)^2 - \Sigma_{(n \leq 2)}\ a_n \phi^n\right) d^d x\right] = 0 \qquad \leftarrow \text{This time analyzing the 2nd term.}$$

$$[a_n] + [\phi^n] + [d^d x] = 0$$

$$[a_n] + n[\phi] + [d^d x] = 0$$

$$[a_n] + n\left(\frac{d}{2} - 1\right) - d = 0$$

$$[a_n] = d - n\left(\frac{d}{2} - 1\right)$$

In four space-time dimensions

$$[a_n] = 4 - n$$

(b) Consider the Lagrangian for a real vector field $A^\mu(x)$

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi \tag{126}$$

Given that $[\bar{\psi} = [\psi]$, and that $[\gamma^\mu] = 0$, what is the mass dimension of ψ in d space-time dimensions?

$$\begin{aligned}
[\mathcal{L}] &= \left[\bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi\right] \\
d &= \left[\bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi\right] \\
d &= \left[\left(i\gamma^\mu \partial_\mu - m\right)\psi^2\right] \\
d &= [m] + \left[\psi^2\right] && \leftarrow \text{Analyzing 2nd term} \\
d &= d + 2\,[\psi] \\
[\psi] &= 0
\end{aligned}$$

Q9 Consider the Lagrangian for a real vector field $A^\mu(x)$

$$\mathcal{L} = -\tfrac{1}{2}\left(\partial_\alpha A_\beta\right)\left(\partial^\alpha A^\beta\right) + \tfrac{1}{2}\left(\partial_\alpha A^\alpha\right)^2 + \tfrac{1}{2}\mu^2 A_\alpha A^\alpha \tag{127}$$

(a) Show that this leads to the field equations

$$\left[g_{\alpha\beta}\left(\partial^2 + \mu^2\right) - \partial_\alpha \partial_\beta\right] A^\beta = 0 \tag{128}$$

$$\begin{aligned}
\mathcal{L} &= -\tfrac{1}{2}(\partial_\alpha A_\beta)(\partial^\alpha A^\beta) + \tfrac{1}{2}(\partial_\alpha A^\alpha)^2 + \frac{\mu^2}{2} A_\alpha A^\alpha \\
&= -\tfrac{1}{2} g^{\alpha\mu} g^{\beta\nu}(\partial_\alpha A_\beta)(\partial_\mu A_\nu) + \tfrac{1}{2} g^{\alpha\beta} g^{\sigma\tau}(\partial_\alpha A_\beta)(\partial_\sigma A_\tau) + \frac{\mu^2}{2} g^{\alpha\beta} A_\alpha A_\beta
\end{aligned}$$

Re-expressed to make all indices different

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial A_\gamma} &= \frac{\mu^2}{2} g^{\alpha\beta}\left(\frac{\partial A_\alpha}{\partial A_\gamma} A_\beta + A_\alpha \frac{\partial A_\beta}{\partial A_\gamma}\right) \\
&= \frac{\mu^2}{2} g^{\alpha\beta}\left(\delta^\gamma_\alpha A_\beta + A_\alpha \delta^\gamma_\beta\right) \\
&= \frac{\mu^2}{2} g^{\alpha\beta}\left(\delta^\gamma_\alpha A_\beta + A_\alpha \delta^\gamma_\beta\right) \\
&= \frac{\mu^2}{2}\left(\delta^\gamma_\alpha A^\alpha + A^\beta \delta^\gamma_\beta\right) \\
&= \frac{\mu^2}{2}\left(A^\gamma + A^\gamma\right) \\
&= \mu^2 A^\gamma
\end{aligned}$$

$$
\begin{aligned}
&\partial_\rho \left(\frac{\partial \mathcal{L}}{\partial(\partial_\rho A_\gamma)} \right) \\
=&\tfrac{1}{2}\partial_\rho \left(-g^{\alpha\mu} g^{\beta\nu} (\partial_\alpha A_\beta) \frac{\partial(\partial_\mu A_\nu)}{\partial(\partial_\rho A_\gamma)} - g^{\alpha\mu} g^{\beta\nu} \frac{\partial(\partial_\alpha A_\beta)}{\partial(\partial_\rho A_\gamma)} (\partial_\mu A_\nu) + g^{\alpha\beta} g^{\sigma\tau} \frac{\partial(\partial_\alpha A_\beta)}{\partial(\partial_\rho A_\gamma)} (\partial_\sigma A_\tau) + g^{\alpha\beta} g^{\sigma\tau} (\partial_\alpha A_\beta) \frac{\partial(\partial_\sigma A_\tau)}{\partial(\partial_\rho A_\gamma)} \right) \\
=&\tfrac{1}{2}\partial_\rho \left(-g^{\alpha\mu} g^{\beta\nu} (\partial_\alpha A_\beta) \delta^\rho_\mu \delta^\gamma_\nu - g^{\alpha\mu} g^{\beta\nu} \delta^\rho_\alpha \delta^\gamma_\beta (\partial_\mu A_\nu) + g^{\alpha\beta} g^{\sigma\tau} \delta^\rho_\alpha \delta^\gamma_\beta (\partial_\sigma A_\tau) + g^{\alpha\beta} g^{\sigma\tau} (\partial_\alpha A_\beta) \delta^\rho_\sigma \delta^\gamma_\tau \right) \\
=&\tfrac{1}{2}\partial_\rho \left(-(\partial^\mu A^\nu) \delta^\rho_\mu \delta^\gamma_\nu - \delta^\rho_\alpha \delta^\gamma_\beta (\partial^\alpha A^\beta) + g^{\rho\gamma} g^{\sigma\tau} (\partial_\sigma A_\tau) + g^{\alpha\beta} g^{\rho\gamma} (\partial_\alpha A_\beta) \right) \\
=&\tfrac{1}{2}\partial_\rho \left(-(\partial^\rho A^\gamma) - (\partial^\rho A^\gamma) + g^{\rho\gamma} g^{\sigma\tau} (\partial_\sigma A_\tau) + g^{\alpha\beta} g^{\rho\gamma} (\partial_\alpha A_\beta) \right) \\
=&-\partial^2 A^\gamma + \tfrac{1}{2} g^{\rho\gamma} \partial_\rho \left(g^{\sigma\tau} (\partial_\sigma A_\tau) + g^{\alpha\beta} (\partial_\alpha A_\beta) \right) \\
=&-\partial^2 A^\gamma + \tfrac{1}{2} \partial^\gamma \left(g^{\sigma\tau} (\partial_\sigma A_\tau) + g^{\alpha\beta} (\partial_\alpha A_\beta) \right) \\
=&-\partial^2 A^\gamma + \tfrac{1}{2} \partial^\gamma \left(\partial_\lambda A^\lambda + \partial_\lambda A^\lambda \right) \\
=&-\partial^2 A^\gamma + \partial^\gamma \partial_\lambda A^\lambda
\end{aligned}
$$

Substituting these into Euler-Lagrange equation

$$
\begin{aligned}
&0 = \frac{\partial \mathcal{L}}{\partial A_\gamma} - \partial_\rho \left(\frac{\partial \mathcal{L}}{\partial(\partial_\rho A_\gamma)} \right) \\
&0 = \mu^2 A^\gamma + \partial^2 A^\gamma - \partial^\gamma \partial_\lambda A^\lambda \\
&0 = g_{\gamma\gamma} \left(\mu^2 A^\gamma + \partial^2 A^\gamma - \partial^\gamma \partial_\lambda A^\lambda \right) \\
&0 = \mu^2 A_\gamma + \partial^2 A_\gamma - \partial_\gamma \partial_\lambda A^\lambda \\
&0 = g_{\gamma\lambda} \mu^2 A^\lambda + g_{\gamma\lambda} \partial^2 A^\lambda - \partial_\gamma \partial_\lambda A^\lambda \\
&0 = \left(g_{\gamma\lambda} \left(\mu^2 + \partial^2 \right) - \partial_\gamma \partial_\lambda \right) A^\lambda \\
&0 = \left(g_{\alpha\beta} \left(\mu^2 + \partial^2 \right) - \partial_\alpha \partial_\beta \right) A^\beta \qquad \Longleftarrow \ \gamma \to \alpha \ , \ \lambda \to \beta
\end{aligned}
$$

(b) Introducing the field-strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{129}$$

show that the Lagrangian for $\mu = 0$ can be written as (integrate by parts freely)

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{130}$$

and that the field equations can be written as

$$\partial_\mu F^{\mu\nu} = 0 \tag{131}$$

$$
\begin{aligned}
-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} &= -\tfrac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) \\
&= -\tfrac{1}{4}\left(\partial^\mu A^\nu \partial_\mu A_\nu - \partial^\mu A^\nu \partial_\nu A_\mu - \partial^\nu A^\mu \partial_\mu A_\nu + \partial^\nu A^\mu \partial_\nu A_\mu\right) \\
&= -\tfrac{1}{2}\partial^\mu A^\nu \partial_\mu A_\nu + \tfrac{1}{2}\partial^\mu A^\nu \partial_\nu A_\mu
\end{aligned}
$$

$\Longleftarrow \mu \leftrightarrow \nu$ in last 2 terms

Integration by parts on 2nd term and disposing of surface term:

$$
\begin{aligned}
&\int \partial^\mu A^\nu \partial_\nu A_\mu \\
&= \int -A^\nu \partial^\mu \partial_\nu A_\mu + [A^\nu \partial_\nu A_\mu]_{\mathbb{R}^4} \\
&= -\int -A^\nu \partial^\mu \partial_\nu A_\mu \\
&= -\int -\partial_\nu A^\nu \partial^\mu A_\mu \\
&= -\int (\partial_\rho A^\rho)^2
\end{aligned}
$$

$$
= -\tfrac{1}{2}\partial^\mu A^\nu \partial_\mu A_\nu + \tfrac{1}{2}\left(\partial_\rho A^\rho\right)^2
$$

Re-expressing the Field Equations in terms of $F^{\mu\nu}$

$$
\begin{aligned}
\partial_\mu F^{\mu\nu} &= \partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) \\
&= \partial^2 A^\nu - \partial_\mu \partial^\nu A^\mu \\
&= \partial^2 A_\nu - \partial_\mu \partial_\nu A^\mu \\
&= g_{\mu\nu}\partial^2 A^\mu - \partial_\mu \partial_\nu A^\mu \\
&= g_{\mu\nu}\partial^2 A^\mu - \partial_\nu \partial_\mu A^\mu \\
&= \left(g_{\mu\nu}\partial^2 - \partial_\nu \partial_\mu\right) A^\mu
\end{aligned}
$$

Therefore $\partial_\mu F^{\mu\nu} = 0$ is equivalent to the field equation above.

Q10 Consider a Lagrangian $\mathcal{L}$ depending not only on ϕ and $\partial_\mu \phi$ but also on the second derivates of the fields

$$
\mathcal{L} = \mathcal{L}\left(\phi, \partial_\mu \phi, \partial_\mu \partial_\nu \phi\right) \tag{132}
$$

(a) For the case that the variations $\delta\phi$ vanish at the endpoints and that $\delta\left(\partial_{\mu_1}\cdots\partial_{\mu_N}\phi\right) = \partial_{\mu_1}\cdots\partial_{\mu_N}\left(\delta\phi\right)$ holds, derive the Euler-Lagrange equations for such a theory.

Expanding $\mathcal{L}$ in terms of small changes in ϕ, $\partial_\mu \phi$ and $\partial_\mu \partial_\nu \phi$

$$
\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \phi)}\delta(\partial_\mu \partial_\nu \phi) \tag{133}
$$

As with the usual derivation of the Euler-Lagrange equation, we observe

$$
\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi) = -\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\delta\phi + \cancel{\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi\right)} \tag{134}
$$

Where the final term is a total derivative, and will therefore disappear in the action integral. Substituting this into (133)

$$
\delta\mathcal{L} = \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\right)\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \phi)}\delta(\partial_\mu \partial_\nu \phi) \tag{135}
$$

We would like to do something similar for the 3rd term.

Consider the term $\partial_\mu\partial_\nu\left(\dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)}\delta\phi\right)$, which can be expressed as:

$$\partial_\mu\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)}\delta\phi\right)=\partial_\mu\left(\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)}\right)\delta\phi+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)}\delta(\partial_\nu\phi)\right)$$

However, the term $\partial_\mu\partial_\nu\left(\dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)}\delta\phi\right)$ is a total derivative and therefore will vanish in the action integral. Therefore, we have

$$0=\partial_\mu\left(\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)}\right)\delta\phi+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)}\delta(\partial_\nu\phi)\right)\tag{136}$$

$$0=\partial_\mu\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)}\right)\delta\phi+2\underbrace{\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)}\right)\delta(\partial_\nu\phi)}+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)}\delta(\partial_\mu\partial_\nu\phi)\tag{137}$$

We can replace the braced term through the following observation:

$$\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)}\right)\delta(\partial_\mu\phi)=-\partial_\mu\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)}\right)\delta\phi+\cancel{\partial_\mu\left(\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)}\right)\delta\phi\right)}$$

Where we disposed of the total derivative term. Substituting this result into (137)

$$\begin{aligned}0&=-\partial_\mu\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)}\right)\delta\phi+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)}\delta(\partial_\mu\partial_\nu\phi)\\
\partial_\mu\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)}\right)\delta\phi&=\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)}\delta(\partial_\mu\partial_\nu\phi)\end{aligned}$$

We now substitute this into (135)

$$\delta\mathcal{L}=\left(\frac{\partial\mathcal{L}}{\partial\phi}-\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)+\partial_\mu\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)}\right)\right)\delta\phi\tag{138}$$

As usual, we use the principle of least action to set $\delta\mathcal{L}=0$, thus obtaining the Euler-Lagrange equation with double derivatives of the field

$$0=\left(\frac{\partial\mathcal{L}}{\partial\phi}-\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)+\partial_\mu\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)}\right)\right)\delta\phi\tag{139}$$

(b) Apply your result to find the equation of motion for ϕ following from the Lagrangian

$$\mathcal{L}=(\partial_t\phi)(\partial_x\phi)+\frac{\alpha}{3}(\partial_x\phi)^3-\frac{\rho}{2}(\partial_\nu\partial_\mu\phi)^2\tag{140}$$

Calculating each term for the Euler-Lagrange equation individually:

$$\begin{aligned}\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)&=\partial_\mu\left((\partial_t\phi)\frac{\partial(\partial_x\phi)}{\partial(\partial_\mu\phi)}+\frac{\partial(\partial_t\phi)}{\partial(\partial_\mu\phi)}(\partial_x\phi)+\alpha(\partial_x\phi)^2\frac{\partial(\partial_x\phi)}{\partial(\partial_\mu\phi)}\right)\\
&=\partial_\mu\left((\partial_t\phi)\delta_x^\mu+\delta_t^\mu(\partial_x\phi)+\alpha(\partial_x\phi)^2\delta_x^\mu\right)\\
&=\partial_x(\partial_t\phi)+\partial_t(\partial_x\phi)+\alpha\partial_x(\partial_x\phi)^2\\
&=\partial_x(\partial_t\phi)+\partial_t(\partial_x\phi)+\alpha\partial_x(\partial_x\phi)^2\\
&=2\partial_t\partial_x\phi+\alpha\partial_x(\partial_x\phi)^2\\
&=2\partial_t\partial_x\phi+2\alpha\partial_x\phi\partial_x^2\phi\end{aligned}$$

$$\begin{aligned}\partial_\mu\partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)}\right)&=-\rho\partial_\mu\partial_\nu\partial_\nu\partial_\mu\phi\\
&=-\rho\partial^4\phi\end{aligned}$$

We also have that $\frac{\partial \mathcal{L}}{\partial \phi} = 0$, so we substitute these into Euler-Lagrange equation derived in (a)

$$\begin{aligned}
\therefore \quad \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) + \partial_\mu \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \phi)} \right) &= 0 \\
-2\partial_t \partial_x \phi - 2\alpha \partial_x \phi \partial_x^2 \phi - \rho \partial^4 \phi &= 0
\end{aligned}$$

Q11 Consider the Klein-Gordon Lagrangian

$$\mathcal{L} = \tfrac{1}{2} \left(\partial_\mu \phi\right)^2 - \tfrac{1}{2} m^2 \phi^2 \tag{141}$$

(a) Write down the energy-momentum tensor $T^{\mu\nu}$ and show explicitly that $\partial_\mu T^{\mu\nu} = 0$

$$\begin{aligned}
T^\mu{}_\nu &\equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \qquad\qquad \mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2} m^2 \phi^2 \\
T^{\mu\nu} &= \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu} \mathcal{L} \\
T^{\mu\nu} &= \partial^\mu \phi \partial^\nu \phi - \tfrac{1}{2} g^{\mu\nu} (\partial_\rho \phi)(\partial^\rho \phi) + \tfrac{1}{2} g^{\mu\nu} m^2 \phi^2
\end{aligned}$$

$$\begin{aligned}
\partial_\mu T^{\mu\nu} &= \left(\partial_\mu \partial^\mu \phi\right)\left(\partial^\nu \phi\right) + \left(\partial^\mu \phi\right)\left(\partial_\mu \partial^\nu \phi\right) \\
&\quad - \tfrac{1}{2} \partial^\nu \left((\partial_\rho \phi)(\partial^\rho \phi)\right) + \tfrac{1}{2} m^2 \partial_\nu \left(\phi^2\right) \\
&= \left(\partial_\mu \partial^\mu \phi\right)\left(\partial^\nu \phi\right) + \left(\partial^\mu \phi\right)\left(\partial_\mu \partial^\nu \phi\right) \\
&\quad - \tfrac{1}{2} (\partial_\rho \phi)(\partial^\nu \partial^\rho \phi) - \tfrac{1}{2} (\partial^\nu \partial_\rho \phi)(\partial^\rho \phi) + m^2 \phi \left(\partial^\nu \phi\right) \\
&= \left(\partial_\mu \partial^\mu \phi\right)\left(\partial^\nu \phi\right) + \left(\partial^\mu \phi\right)\left(\partial_\mu \partial^\nu \phi\right) \qquad \Longleftarrow \ \tfrac{1}{2}(\partial_\nu \partial_\rho \phi)(\partial^\rho \phi) = \tfrac{1}{2}(\partial_\nu \partial^\rho \phi)(\partial_\rho \phi) \\
&\quad - (\partial^\rho \phi)(\partial^\nu \partial_\rho \phi) + m^2 \phi \left(\partial^\nu \phi\right) \\
&= \left(\partial_\mu \partial^\mu \phi\right)\left(\partial^\nu \phi\right) + m^2 \phi \left(\partial_\nu \phi\right) \qquad \Longleftarrow \ (\partial^\rho \phi)(\partial^\nu \partial_\rho \phi) = (\partial^\rho \phi)(\partial_\rho \partial^\nu \phi) = (\partial^\mu \phi)(\partial_\mu \partial^\nu \phi) \\
&= \left(\partial_\mu \partial^\mu \phi + m^2 \phi\right)\left(\partial_\nu \phi\right) \qquad\qquad \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0 \\
&= 0 \qquad\qquad \Longleftarrow \ \partial_\mu \partial^\mu \phi + m^2 \phi = 0
\end{aligned}$$

(b) Give the expressions for the conserved energy E and momentum P^i

---

$$\begin{aligned}
E \equiv \int d^3x \; T^{00} &= \int d^3x \left(\dot{\phi}^2 - \tfrac{1}{2} (\partial_\rho \phi)(\partial^\rho \phi) + \tfrac{1}{2} m^2 \phi^2 \right) \\
&= \int d^3x \left(\dot{\phi}^2 - \tfrac{1}{2} \left(\dot{\phi}^2 - \nabla^2 \phi \right) + \tfrac{1}{2} m^2 \phi^2 \right) \\
&= \int d^3x \left(\tfrac{1}{2} \dot{\phi}^2 + \tfrac{1}{2} \nabla^2 \phi + \tfrac{1}{2} m^2 \phi^2 \right)
\end{aligned}$$

$$P^i \equiv \int d^3x \; T^{0i} = \int d^3x \; \dot{\phi} \; \partial^i \phi$$

Q12 Consider a theory involving a doublet of complex scalar fields

$$\Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \tag{142}$$

appearing in the Lagrangian

$$\mathcal{L} = \partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 \Phi^\dagger \Phi = \sum_{a=1}^{2} \partial_\mu \varphi_a^* \partial^\mu \varphi_a - m^2 \varphi_a^* \varphi_a \tag{143}$$

(a) For a U(1) transformation $\Phi \to e^{i\alpha}\Phi$:

(i) Show that the Lagrangian is invariant under this transformation

(ii) Give the conserved current associated with the transformation

---

(i)

$$\begin{aligned}
\Phi &\to e^{i\alpha}\Phi \\
\Phi^\dagger &\to e^{-i\alpha}\Phi^\dagger \\
\mathcal{L} &\to \mathcal{L}' = \partial_\mu (e^{-i\alpha}\Phi^\dagger) \partial^\mu (e^{i\alpha}\Phi) - m^2 (e^{-i\alpha}\Phi^\dagger)(e^{i\alpha}\Phi) \\
&= \partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 \Phi^\dagger \Phi = \mathcal{L}
\end{aligned}$$

(ii)

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \frac{\delta \Phi}{\delta \alpha} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^\dagger)} \frac{\delta \Phi^\dagger}{\delta \alpha}$$

$$\begin{aligned}
\Phi \to & e^{i\alpha}\Phi \\
&= \Phi + i\alpha\Phi + \mathcal{O}(\alpha^2) \\
\therefore \ \frac{\delta \Phi}{\alpha} &= i\Phi \\
\frac{\delta \Phi^\dagger}{\alpha} &= -i\Phi^\dagger
\end{aligned}$$

$$= i\left(\Phi \partial^\mu \Phi^\dagger - \Phi^\dagger \partial^\mu \Phi\right)$$

(b) For the 3 SU(2) transformations $\Phi \to e^{i\alpha_j \sigma^j/2}\Phi$, where σ^j are Pauli matrices (note j is fixed for each transformation and that one can work to first order in infinitesimal α_j):

(i) Show that the Lagrangian is invariant under this transformation

(ii) Give the conserved current associated with the transformation

---

(i)

$$\begin{aligned}
\Phi &\to e^{(i\alpha_j \sigma^j/2)}\Phi \\
\Phi^\dagger &\to e^{(-i\alpha_j \sigma^j/2)}\Phi^\dagger
\end{aligned}$$

Note: $\sigma^\dagger = \sigma$

$$\begin{aligned}
\mathcal{L} \to \mathcal{L}' &= \partial_\mu (e^{(-i\alpha_j \sigma^j/2)}\Phi^\dagger) \partial^\mu (e^{(i\alpha_j \sigma^j/2)}\Phi) - m^2 (e^{(-i\alpha_j \sigma^j/2)}\Phi^\dagger)(e^{(i\alpha_j \sigma^j/2)}\Phi) \\
&= \partial_\mu \Phi^\dagger \partial^\mu \Phi - m^2 \Phi^\dagger \Phi = \mathcal{L}
\end{aligned}$$

Transformation of field:

$$\Phi \to e^{(i\alpha_j\sigma^j/2)}\Phi$$
$$= \Phi + \tfrac{i}{2}\alpha_j\sigma^j\Phi + \mathcal{O}(\alpha^2)$$
$$= \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} + \tfrac{i}{2}\alpha_j\sigma^j\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$$

$$=^{(j=x)} \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} + \tfrac{i}{2}\alpha_j\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$$
$$=^{(j=x)} \begin{pmatrix}\varphi_1 + \frac{i}{2}\alpha_x\varphi_2\\ \varphi_2 + \frac{i}{2}\alpha_x\varphi_1\end{pmatrix}$$
$$\Longrightarrow \frac{\delta\mathbf{\Phi}}{\alpha} =^{(j=x)} \frac{1}{\alpha_x}\begin{pmatrix}\delta\varphi_1\\ \delta\varphi_2\end{pmatrix} = \begin{pmatrix}\frac{i}{2}\varphi_2\\ \frac{i}{2}\varphi_1\end{pmatrix}$$

$$=^{(j=y)} \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} + \tfrac{i}{2}\alpha_y\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$$
$$=^{(j=y)} \begin{pmatrix}\varphi_1 + \frac{1}{2}\alpha_y\varphi_2\\ \varphi_2 - \frac{1}{2}\alpha_y\varphi_1\end{pmatrix}$$
$$\Longrightarrow \frac{\delta\mathbf{\Phi}}{\alpha} =^{(j=y)} \frac{1}{\alpha_y}\begin{pmatrix}\delta\varphi_1\\ \delta\varphi_2\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\varphi_2\\ -\frac{i}{2}\varphi_1\end{pmatrix}$$

$$=^{(j=z)} \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} + \tfrac{i}{2}\alpha_z\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$$
$$=^{(j=z)} \begin{pmatrix}\varphi_1 + \frac{i}{2}\alpha_z\varphi_1\\ \varphi_2 - \frac{i}{2}\alpha_z\varphi_2\end{pmatrix}$$
$$\Longrightarrow \frac{\delta\mathbf{\Phi}}{\alpha} =^{(j=z)} \frac{1}{\alpha_z}\begin{pmatrix}\delta\varphi_1\\ \delta\varphi_2\end{pmatrix} = \begin{pmatrix}\frac{i}{2}\varphi_1\\ -\frac{i}{2}\varphi_2\end{pmatrix}$$

Transformation of conjugate field:

$$\Phi^\dagger \to e^{(-i\alpha_j\sigma^j/2)}\Phi^\dagger$$
$$= \Phi^\dagger - \tfrac{i}{2}\alpha_j\sigma^j\Phi^\dagger + \mathcal{O}(\alpha^2)$$
$$= \begin{pmatrix}\varphi_1^*\\ \varphi_2^*\end{pmatrix} - \tfrac{i}{2}\alpha_j\sigma^j\begin{pmatrix}\varphi_1^*\\ \varphi_2^*\end{pmatrix}$$

$$=^{(j=x)} \begin{pmatrix}\varphi_1^*\\ \varphi_2^*\end{pmatrix} - \tfrac{i}{2}\alpha_j\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\varphi_1^*\\ \varphi_2^*\end{pmatrix}$$
$$=^{(j=x)} \begin{pmatrix}\varphi_1^* - \frac{i}{2}\alpha_x\varphi_2^*\\ \varphi_2^* - \frac{i}{2}\alpha_x\varphi_1^*\end{pmatrix}$$
$$\Longrightarrow \frac{\delta\mathbf{\Phi}^\dagger}{\alpha} =^{j=x} \frac{1}{\alpha_x}\begin{pmatrix}\delta\varphi_1^*\\ \delta\varphi_2^*\end{pmatrix} = \begin{pmatrix}-\frac{i}{2}\varphi_2^*\\ -\frac{i}{2}\varphi_1^*\end{pmatrix}$$

$$=^{(j=y)} \begin{pmatrix}\varphi_1^*\\ \varphi_2^*\end{pmatrix} - \tfrac{i}{2}\alpha_y\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\varphi_1^*\\ \varphi_2^*\end{pmatrix}$$
$$=^{(j=y)} \begin{pmatrix}\varphi_1^* - \frac{1}{2}\alpha_y\varphi_2^*\\ \varphi_2^* + \frac{1}{2}\alpha_y\varphi_1^*\end{pmatrix}$$
$$\Longrightarrow \frac{\delta\mathbf{\Phi}^\dagger}{\alpha} =^{(j=y)} \frac{1}{\alpha_y}\begin{pmatrix}\delta\varphi_1^*\\ \delta\varphi_2^*\end{pmatrix} = \begin{pmatrix}-\frac{1}{2}\varphi_2^*\\ \frac{i}{2}\varphi_1^*\end{pmatrix}$$

$$=^{(j=z)} \begin{pmatrix}\varphi_1^*\\ \varphi_2^*\end{pmatrix} - \tfrac{i}{2}\alpha_z\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\varphi_1^*\\ \varphi_2^*\end{pmatrix}$$
$$=^{(j=z)} \begin{pmatrix}\varphi_1^* - \frac{i}{2}\alpha_z\varphi_1^*\\ \varphi_2^* + \frac{i}{2}\alpha_z\varphi_2^*\end{pmatrix}$$
$$\Longrightarrow \frac{\delta\mathbf{\Phi}^\dagger}{\alpha} =^{(j=z)} \frac{1}{\alpha_z}\begin{pmatrix}\delta\varphi_1^*\\ \delta\varphi_2^*\end{pmatrix} = \begin{pmatrix}-\frac{i}{2}\varphi_1^*\\ \frac{i}{2}\varphi_2^*\end{pmatrix}$$

(ii)

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_1)}\frac{\delta\varphi_1}{\alpha} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_1^*)}\frac{\delta\varphi_1^*}{\alpha} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_2)}\frac{\delta\varphi_2}{\alpha} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_2^*)}\frac{\delta\varphi_2^*}{\alpha}$$

$$=^{(j=x)} \tfrac{i}{2}\varphi_2\partial^\mu\varphi_1^* - \tfrac{i}{2}\varphi_2^*\partial^\mu\varphi_1 + \tfrac{i}{2}\varphi_1\partial^\mu\varphi_2^* - \tfrac{i}{2}\varphi_2^*\partial^\mu\varphi_2$$

$$=^{(j=y)} \tfrac{1}{2}\varphi_2\partial^\mu\varphi_1^* - \tfrac{1}{2}\varphi_2^*\partial^\mu\varphi_1 - \tfrac{i}{2}\varphi_1\partial^\mu\varphi_2^* + \tfrac{i}{2}\varphi_1^*\partial^\mu\varphi_2$$

$$=^{(j=z)} \tfrac{i}{2}\varphi_1\partial^\mu\varphi_1^* - \tfrac{i}{2}\varphi_1^*\partial^\mu\varphi_1 - \tfrac{i}{2}\varphi_2\partial^\mu\varphi_2^* + \tfrac{i}{2}\varphi_2^*\partial^\mu\varphi_2$$

Q13 In this problem we deal with the Hamiltonian for a free, scalar field ϕ, appearing in the Klein-Gordon Hamiltonian (after dropping the infinite zero-point energy)

$$H = \int \frac{d^3\mathbf{x}}{2}\left(\pi^2 + \nabla\phi\cdot\nabla\phi + m^2\phi^2\right) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\omega_{\mathbf{p}}a_{\mathbf{p}}^\dagger a_{\mathbf{p}} \tag{144}$$

(a) Using the Fourier expansions of ϕ and π, show that

(i)

$$[\phi(\mathbf{x}), \phi(\mathbf{y})] = 0 \tag{145}$$

(ii)

$$[\pi(\mathbf{x}), \pi(\mathbf{y})] = 0 \tag{146}$$

are satisfied. To do so use the commutation relations $\left[a_{\mathbf{p}}, a_{\mathbf{q}}^{\dagger}\right] = (2\pi)^3\,\delta^{(3)}\left(\mathbf{p}-\mathbf{q}\right)$ and $[a_{\mathbf{p}}, a_{\mathbf{q}}] = \left[a_{\mathbf{p}}^{\dagger}, a_{\mathbf{q}}^{\dagger}\right] = 0$

(i) The Fourier expansion of ϕ is

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a_{\mathbf{p}}e^{ipx} + a_{\mathbf{p}}^{\dagger}e^{-ipx}\right)$$

Substituting this into the commutator for the fields:

$$[\phi(\mathbf{x}), \phi(\mathbf{y})] = \frac{1}{2\sqrt{\omega_p\omega_q}}\int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\left[\left(a_{\mathbf{p}}e^{ipx} + a_{\mathbf{p}}^{\dagger}e^{-ipx}\right), \left(a_{\mathbf{q}}e^{iqy} + a_{\mathbf{q}}^{\dagger}e^{-iqy}\right)\right]$$

We begin by reexpressing the integrand

$$\begin{aligned}
&\left[\left(a_{\mathbf{p}}e^{ipx} + a_{\mathbf{p}}^{\dagger}e^{-ipx}\right), \left(a_{\mathbf{q}}e^{iqy} + a_{\mathbf{q}}^{\dagger}e^{-iqy}\right)\right] \\
&= \left(a_{\mathbf{p}}e^{ipx} + a_{\mathbf{p}}^{\dagger}e^{-ipx}\right)\left(a_{\mathbf{q}}e^{iqy} + a_{\mathbf{q}}^{\dagger}e^{-iqy}\right) - \left(a_{\mathbf{q}}e^{iqy} + a_{\mathbf{q}}^{\dagger}e^{-iqy}\right)\left(a_{\mathbf{p}}e^{ipx} + a_{\mathbf{p}}^{\dagger}e^{-ipx}\right) \\
&= a_{\mathbf{p}}a_{\mathbf{q}}e^{ipx}e^{i\mathbf{q}\cdot\mathbf{y}} + a_{\mathbf{p}}a_{\mathbf{q}}^{\dagger}e^{ipx}e^{-iqy} + a_{\mathbf{p}}^{\dagger}a_{\mathbf{q}}e^{-ipx}e^{iqy} + a_{\mathbf{p}}^{\dagger}a_{\mathbf{q}}^{\dagger}e^{-ipx}e^{-iqy} \\
&\quad - \left(a_{\mathbf{q}}a_{\mathbf{p}}e^{ipx}e^{i\mathbf{q}\cdot\mathbf{y}} + a_{\mathbf{q}}^{\dagger}a_{\mathbf{p}}e^{ipx}e^{-iqy} + a_{\mathbf{q}}a_{\mathbf{p}}^{\dagger}e^{-ipx}e^{iqy} + a_{\mathbf{q}}^{\dagger}a_{\mathbf{p}}^{\dagger}e^{-ipx}e^{-iqy}\right) \\
&= [a_{\mathbf{p}}, a_{\mathbf{q}}]e^{ipx}e^{i\mathbf{q}\cdot\mathbf{y}} + [a_{\mathbf{p}}, a_{\mathbf{q}}^{\dagger}]e^{ipx}e^{-iqy} + [a_{\mathbf{p}}^{\dagger}, a_{\mathbf{q}}]e^{-ipx}e^{iqy} + [a_{\mathbf{p}}^{\dagger}, a_{\mathbf{q}}^{\dagger}]e^{-ipx}e^{-iqy} \\
&= 0 + (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})e^{ipx}e^{-iqy} - (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})e^{-ipx}e^{iqy} + 0 \left\{ \begin{array}{l} [a_{\mathbf{p}}, a_{\mathbf{q}}^{\dagger}] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\ [a_{\mathbf{p}}, a_{\mathbf{q}}] = [a_{\mathbf{p}}^{\dagger}, a_{\mathbf{q}}^{\dagger}] = 0 \end{array}\right.
\end{aligned}$$

$$\begin{aligned}
\therefore [\phi(\mathbf{x}), \phi(\mathbf{y})] &= \int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\frac{1}{2\sqrt{\omega_p\omega_q}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\left(e^{ipx}e^{-iqy} - e^{-ipx}e^{iqy}\right) \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\left(e^{ip(x-y)} - e^{-ip(x-y)}\right) \\
&= 0
\end{aligned}$$

Because integrand is symmetric under exchange $\mathbf{p} \to -\mathbf{p}$

(ii) The Fourier expansion of π is

$$\pi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_p}{2}}\left(a_{\mathbf{p}}e^{ipx} - a_{\mathbf{p}}^{\dagger}e^{-ipx}\right)$$

Substituting this into the commutator for the fields:

$$[\pi(\mathbf{x}), \pi(\mathbf{y})] = -\sqrt{\frac{\omega_p\omega_q}{4}}\int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\left[\left(a_{\mathbf{p}}e^{ipx} - a_{\mathbf{p}}^{\dagger}e^{-ipx}\right), \left(a_{\mathbf{q}}e^{iqy} - a_{\mathbf{q}}^{\dagger}e^{-iqy}\right)\right]$$

We repexress the integrand

$$\begin{aligned}
&\left[\left(a_{\mathbf{p}}e^{ipx} - a_{\mathbf{p}}\dagger e^{-ipx}\right), \left(a_{\mathbf{q}}e^{iqy} - a_{\mathbf{q}}\dagger e^{-iqy}\right)\right] \\
&= \left(a_{\mathbf{p}}e^{ipx} - a_{\mathbf{p}}\dagger e^{-ipx}\right)\left(a_{\mathbf{q}}e^{iqy} - a_{\mathbf{q}}\dagger e^{-iqy}\right) - \left(a_{\mathbf{q}}e^{iqy} - a_{\mathbf{q}}\dagger e^{-iqy}\right)\left(a_{\mathbf{p}}e^{ipx} - a_{\mathbf{p}}\dagger e^{-ipx}\right) \\
&= [a_{\mathbf{p}}, a_{\mathbf{q}}]e^{ipx}e^{i\mathbf{q}\cdot\mathbf{y}} - [a_{\mathbf{p}}, a_{\mathbf{q}}\dagger]e^{ipx}e^{-iqy} - [a_{\mathbf{p}}\dagger, a_{\mathbf{q}}]e^{-ipx}e^{iqy} + [a_{\mathbf{p}}\dagger, a_{\mathbf{q}}\dagger]e^{-ipx}e^{-iqy} \\
&= 0 - (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})e^{ipx}e^{-iqy} + (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})e^{-ipx}e^{iqy} + 0
\end{aligned}$$

$$
\begin{aligned}
\therefore\ [\pi(\mathbf{x}),\pi(\mathbf{y})] &= \int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\sqrt{\frac{\omega_p\omega_q}{4}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\left(e^{ipx}e^{-iqy}+e^{-ipx}e^{iqy}\right)\\
&= \int \frac{d^3p}{(2\pi)^3}\frac{\omega_p}{2}\left(e^{ip(x-y)}+e^{-ip(x-y)}\right)\\
&= 0
\end{aligned}
$$

Once again, evoking the symmetry $\mathbf{p}\to-\mathbf{p}$

Q14 Prove the commutation relations

(i) $[H,a_{\mathbf{p}}]=-\omega_{\mathbf{p}}a_{\mathbf{p}}$

(ii) $[H,a\dagger_{\mathbf{p}}]=\omega_{\mathbf{p}}a\dagger_{\mathbf{p}}$

(i)

$$
\begin{aligned}
[H,a_{\mathbf{p}}] &= \int \frac{d^3q}{(2\pi)^3}\omega_q\,[a_{\mathbf{q}}\dagger\, a_{\mathbf{q}},a_{\mathbf{p}}]\\
&= \int \frac{d^3q}{(2\pi)^3}\omega_q\,(a_{\mathbf{q}}\dagger\,[a_{\mathbf{q}},a_{\mathbf{p}}]-[a_{\mathbf{p}},a_{\mathbf{q}}\dagger]a_{\mathbf{q}})\\
&= -\int \frac{d^3q}{(2\pi)^3}\omega_q[a_{\mathbf{p}},a_{\mathbf{q}}\dagger]a_{\mathbf{q}}\\
&= -\int \frac{d^3q}{(2\pi)^3}\omega_q(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})a_{\mathbf{q}}\\
&= -\omega_p a_{\mathbf{p}}
\end{aligned}
$$

(ii)

$$
\begin{aligned}
[H,a_{\mathbf{p}}\dagger] &= \int \frac{d^3q}{(2\pi)^3}\omega_q\,[a_{\mathbf{q}}\dagger\, a_{\mathbf{q}},a_{\mathbf{p}}\dagger]\\
&= \int \frac{d^3q}{(2\pi)^3}\omega_q\,(a_{\mathbf{q}}\dagger\,[a_{\mathbf{q}},a_{\mathbf{p}}\dagger]-[a_{\mathbf{p}}\dagger,a_{\mathbf{q}}\dagger]a_{\mathbf{q}})\\
&= \int \frac{d^3q}{(2\pi)^3}\omega_q a_{\mathbf{q}}\dagger\,[a_{\mathbf{q}},a_{\mathbf{p}}\dagger]\\
&= \int \frac{d^3q}{(2\pi)^3}\omega_q a_{\mathbf{q}}\dagger\,(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\\
&= \omega_p a_{\mathbf{p}}\dagger
\end{aligned}
$$

(a) Consider the number operator

$$
N=\int \frac{d^3\mathbf{p}}{(2\pi)^3}a\,\dagger_{\mathbf{p}}\,a_{\mathbf{p}} \tag{147}
$$

(i) Show that N commutes with the Klein-Gordon Hamiltonian

(ii) Show that $N\,|\mathbf{p}_1,\cdots,\mathbf{p}_n\rangle=n\,|\mathbf{p}_1,\cdots,\mathbf{p}_n\rangle$, where $|\mathbf{p}_1,\cdots,\mathbf{p}_n\rangle=a\,\dagger_{\mathbf{p}_1}\cdots a\,\dagger_{\mathbf{p}_n}\,|0\rangle$, denotes an n-particle state

---

(i)

$$
\begin{aligned}
[H, N] &= \int \frac{d^3\mathbf{q}}{(2\pi)^3} \left[H, a_{\mathbf{p}} \dagger a_{\mathbf{p}}\right] \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left([H, a_{\mathbf{p}}\dagger]\, a_{\mathbf{p}} - a_{\mathbf{p}} \dagger [a_{\mathbf{p}}, H] \right) \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left(\omega_p a_{\mathbf{p}} \dagger a_{\mathbf{p}} - \omega_p a_{\mathbf{p}} \dagger a_{\mathbf{p}} \right) \\
&= 0
\end{aligned}
$$

(ii)

$$
\begin{aligned}
& N \left|\mathbf{p}_1, ..., \mathbf{p}_n\right\rangle \\
&= N a_{\mathbf{p}_1} \dagger a_{\mathbf{p}_2} \dagger ... a_{\mathbf{p}_n} \dagger |0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3} a \dagger_{\mathbf{p}}\, a_{\mathbf{p}} a_{\mathbf{p}_1} \dagger a_{\mathbf{p}_2} \dagger ... a_{\mathbf{p}_n} \dagger |0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3} a \dagger_{\mathbf{p}} \left(a_{\mathbf{p}_1} \dagger a_{\mathbf{p}} + (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}_1) \right) a_{\mathbf{p}_2} \dagger ... a_{\mathbf{p}_n} \dagger |0\rangle \qquad \begin{cases} [a_{\mathbf{p}}, a_{\mathbf{p}_i}\dagger] = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}_i) \\ a_{\mathbf{p}} a_{\mathbf{p}_i}\dagger = a_{\mathbf{p}_i} \dagger a_{\mathbf{p}} + (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}_i) \end{cases} \\
&= \int \frac{d^3p}{(2\pi)^3} a \dagger_{\mathbf{p}}\, a_{\mathbf{p}_1} \dagger a_{\mathbf{p}} a_{\mathbf{p}_2} \dagger ... a_{\mathbf{p}_n} \dagger |0\rangle + \int \frac{d^3p}{(2\pi)^3} a \dagger_{\mathbf{p}}\, (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{p}_1) a_{\mathbf{p}_2} \dagger ... a_{\mathbf{p}_n} \dagger |0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3} a \dagger_{\mathbf{p}}\, a_{\mathbf{p}_1} \dagger a_{\mathbf{p}} a_{\mathbf{p}_2} \dagger ... a_{\mathbf{p}_n} \dagger |0\rangle + a \dagger_{\mathbf{p}_1} a_{\mathbf{p}_2} \dagger ... a_{\mathbf{p}_n} \dagger |0\rangle \\
\\
&= \int \frac{d^3p}{(2\pi)^3} a \dagger_{\mathbf{p}}\, a_{\mathbf{p}_1} \dagger ... a_{\mathbf{p}_n} \dagger a_{\mathbf{p}} |0\rangle + n\ a \dagger_{\mathbf{p}_1} a_{\mathbf{p}_2} \dagger ... a_{\mathbf{p}_n} \dagger |0\rangle \qquad \Longleftarrow \text{Repeating the previous 3 steps for } i = 2, ..., n \\
&= n\ a \dagger_{\mathbf{p}_1} a_{\mathbf{p}_2} \dagger ... a_{\mathbf{p}_n} \dagger |0\rangle \qquad \Longleftarrow\ a_{\mathbf{p}} |0\rangle = 0 \\
&= n \left|\mathbf{p}_1, ..., \mathbf{p}_n\right\rangle
\end{aligned}
$$

(b) Calculate $J_i \left|\mathbf{p} = 0\right\rangle$ with J_i defined as

$$
J_i = \epsilon_{ijk} \int d^3x\ x^j T^{0k} \tag{148}
$$

where the $0i$ componenet of the stress-energy tensor is

$$
T^{0i} = -\pi \partial_i \phi \tag{149}
$$

$$
\begin{aligned}
J_i\left|\mathbf{p}=0\right\rangle =&\epsilon_{ijk}\int d^3x\ x^j T^{0k}\left|\mathbf{p}=0\right\rangle \\
=&-\epsilon_{ijk}\int d^3x\ x^j\pi\partial_k\phi\ \left|\mathbf{p}=0\right\rangle \\
=&-\epsilon_{ijk}\int d^3x\ x^j\left((-i)\int\frac{d^3p}{(2\pi)^3}\sqrt{\frac{\omega_p}{2}}\left(a_{\mathbf{p}}e^{ipx}-a_{\mathbf{p}}\dagger e^{-ipx}\right)\right) \\
&\times\partial_k\left(\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a_{\mathbf{p}}e^{ipx}+a_{\mathbf{p}}\dagger e^{-ipx}\right)\right)\left|\mathbf{p}=0\right\rangle \\
=&\frac{i}{2}\epsilon_{ijk}\int d^3x\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3p}{(2\pi)^3}x^j\left(a_{\mathbf{p}}e^{ipx}-a_{\mathbf{p}}\dagger e^{-ipx}\right) \\
&\times\partial_k\left(a_{\mathbf{p}}e^{ipx}+a_{\mathbf{p}}\dagger e^{-ipx}\right)\left|\mathbf{p}=0\right\rangle \\
=&\frac{i}{2}\epsilon_{ijk}\int d^3x\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3p}{(2\pi)^3}x^j\left(a_{\mathbf{p}}e^{ipx}-a_{\mathbf{p}}\dagger e^{-ipx}\right) \\
&\times ip_k\left(a_{\mathbf{p}}e^{ipx}-a_{\mathbf{p}}\dagger e^{-ipx}\right)\left|\mathbf{p}=0\right\rangle \\
=&-\frac{1}{2}\ \epsilon_{ijk}\int d^3x\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3p}{(2\pi)^3}p_k x^j\left(a_{\mathbf{p}}e^{ipx}-a_{\mathbf{p}}\dagger e^{-ipx}\right) \\
&\times\left(a_{\mathbf{p}}e^{ipx}-a_{\mathbf{p}}\dagger e^{-ipx}\right)\left|\mathbf{p}=0\right\rangle \\
=&-\frac{1}{2}\ \epsilon_{ijk}\int d^3x\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3p}{(2\pi)^3}p_k x^j \\
&\times\left(\left(a_{\mathbf{p}}\right)^2 e^{2ipx}+\left(a_{\mathbf{p}}\dagger\right)^2 e^{-2ipx}-a_{\mathbf{p}}\dagger a_{\mathbf{p}}-a_{\mathbf{p}}a_{\mathbf{p}}\dagger\right)\left|\mathbf{p}=0\right\rangle
\end{aligned}
$$

Q15 The Lagrangian for a complex scalar field obeying the Klein-Gordon equation is

$$\mathcal{L}=\partial_\mu\phi^*\partial^\mu\phi-m^2\phi^*\phi \tag{150}$$

(a) Using the Noether procedure, show that the invariance of the action under the $U(1)$ transformation $\phi(x)\to e^{i\alpha}\phi(x)$ implies the conserved current

$$j^\mu=i\left[\left(\partial^\mu\phi^*\right)\phi-\phi^*\left(\partial^\mu\phi\right)\right] \tag{151}$$

For infinitesimal α, we have that $e^{i\alpha}\phi\to 1+i\alpha\phi$. The infinitesimal change in $\phi(x)$ (independently of the parameter α) is therefore $\delta\phi=i\phi$. Similarly, we have that $\delta\phi^*=-i\phi^*$

$$
\begin{aligned}
j^\mu&=\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\delta\phi^* \\
&=i\left[\left(\partial^\mu\phi^*\right)\phi-\phi^*\left(\partial^\mu\phi\right)\right]
\end{aligned}
$$

(b)

(i) Find the conjugate momenta to $\phi(x)$ and $\phi^*(x)$

(ii) Write down the canonical commutation relations (use the Heisenberg picture)
(iii) Show that the Hamiltonian is

$$H = \int d^3\mathbf{x} \left(\pi \dagger \pi + \nabla\phi \dagger \cdot \nabla\phi + m^2 \phi \dagger \phi \right) \tag{152}$$

(i)

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^*$$
$$\pi\dagger = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*} = \dot{\phi}$$

(ii)

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y})$$

$$[\phi(\mathbf{x}), \pi \dagger (\mathbf{y})] = [\phi \dagger (\mathbf{x}), \phi(\mathbf{y})] = [\pi \dagger (\mathbf{x}), \pi(\mathbf{y})] = 0$$

(iii) Performing a Legendre transform on the operator form of the Lagrangian (150)

$$\begin{aligned}
H &= \int d^3\mathbf{x} \left(\pi\dot{\phi} + \pi \dagger \dot{\phi} \dagger - \mathcal{L} \right) \\
&= \int d^3\mathbf{x} \left(\dot{\phi} \dagger \dot{\phi} + \dot{\phi}\dot{\phi} \dagger - \dot{\phi} \dagger \dot{\phi} + \nabla\phi \dagger \cdot \nabla\phi + m^2 \phi \dagger \phi \right) \\
&= \int d^3\mathbf{x} \left(\dot{\phi}\dot{\phi} \dagger + \nabla\phi \dagger \cdot \nabla\phi + m^2 \phi \dagger \phi \right) \\
&= \int d^3\mathbf{x} \left(\pi \dagger \pi + \nabla\phi \dagger \cdot \nabla\phi + m^2 \phi \dagger \phi \right)
\end{aligned}$$

(c)
(i) Derive the Heisenberg equation of motion for $\phi(x)$
(ii) Derive the Heisenberg equation of motion for $\pi(x)$
(iii) Show that (i) and (ii) imply the Klein-Gordon equation

(i)

$$\begin{aligned}
\dot{\phi} &= i\,[H(\mathbf{y}), \phi(\mathbf{x})] \\
&= i \int d^3\mathbf{y} \left[\left(\pi \dagger (\mathbf{y})\pi(\mathbf{y}) + \nabla\phi \dagger (\mathbf{y}) \cdot \nabla\phi(\mathbf{y}) + m^2 \phi \dagger (\mathbf{y})\phi(\mathbf{y}) \right), \phi(\mathbf{x}) \right] \\
&= i \int d^3\mathbf{y} \left([\pi \dagger (\mathbf{y})\pi(\mathbf{y}), \phi(\mathbf{x})] + \cancel{[\nabla\phi \dagger (\mathbf{y}) \cdot \nabla\phi(\mathbf{y}), \phi(\mathbf{x})]} + m^2 \cancel{[\phi \dagger (\mathbf{y})\phi(\mathbf{y}), \phi(\mathbf{x})]} \right) \\
&= i \int d^3\mathbf{y} \left(\pi \dagger (\mathbf{y}) [\pi(\mathbf{y}), \phi(\mathbf{x})] + \cancel{[\pi \dagger (\mathbf{y}), \phi(\mathbf{x})]}\pi(\mathbf{y}) \right) \\
&= i \int d^3\mathbf{y}\ \pi \dagger (\mathbf{y})(-i)\delta^{(3)}(\mathbf{x} - \mathbf{y}) \\
&= \pi \dagger (\mathbf{x})
\end{aligned}$$

(ii)

$$
\begin{aligned}
\dot{\pi} &= i\left[H(\mathbf{y}), \pi(\mathbf{x})\right] &(153)\\
&= i\int d^3\mathbf{y}\left[\left(\pi\dagger(\mathbf{y})\pi(\mathbf{y}) + \nabla_{\mathbf{y}}\phi\dagger(\mathbf{y})\cdot\nabla_{\mathbf{y}}\phi(\mathbf{y}) + m^2\phi\dagger(\mathbf{y})\phi(\mathbf{y})\right), \pi(\mathbf{x})\right] &(154)\\
&= i\int d^3\mathbf{y}\Big(\cancel{[\pi\dagger(\mathbf{y})\pi(\mathbf{y}), \pi(\mathbf{x})]} + \left[\nabla_{\mathbf{y}}\phi\dagger(\mathbf{y})\cdot\nabla_{\mathbf{y}}\phi(\mathbf{y}), \pi(\mathbf{x})\right] &(155)\\
&\qquad + m^2\left[\phi\dagger(\mathbf{y})\phi(\mathbf{y}), \pi(\mathbf{x})\right]\Big) &(156)\\
&= i\int d^3\mathbf{y}\Big(\nabla_{\mathbf{y}}\phi\dagger(\mathbf{y})\cdot\left[\nabla_{\mathbf{y}}\phi(\mathbf{y}), \pi(\mathbf{x})\right] + \left[\nabla_{\mathbf{y}}\phi\dagger(\mathbf{y}), \pi(\mathbf{x})\right]\cdot\nabla_{\mathbf{y}}\phi(\mathbf{y}) &(157)\\
&\qquad + m^2\phi\dagger(\mathbf{y})\left[\phi(\mathbf{y}), \pi(\mathbf{x})\right] + m^2\left[\phi\dagger(\mathbf{y}), \pi(\mathbf{x})\right]\phi(\mathbf{y})\Big) &(158)\\
&= i\int d^3\mathbf{y}\Big(\nabla_{\mathbf{y}}\phi\dagger(\mathbf{y})\cdot\nabla_{\mathbf{y}}\left[\phi(\mathbf{y}), \pi(\mathbf{x})\right] + \nabla_{\mathbf{y}}\cancel{\left[\phi\dagger(\mathbf{y}), \pi(\mathbf{x})\right]}\cdot\nabla_{\mathbf{y}}\phi(\mathbf{y}) &(159)\\
&\qquad + m^2\phi\dagger(\mathbf{y})\left[\phi(\mathbf{y}), \pi(\mathbf{x})\right] + m^2\cancel{\left[\phi\dagger(\mathbf{y}), \pi(\mathbf{x})\right]}\phi(\mathbf{y})\Big) &(160)\\
&= i\int d^3\mathbf{y}\left(\nabla_{\mathbf{y}}\phi\dagger(\mathbf{y})\cdot\nabla_{\mathbf{y}}\left(i\delta^{(3)}(\mathbf{x}-\mathbf{y})\right) + m^2\phi\dagger(\mathbf{y})\left(i\delta^{(3)}(\mathbf{x}-\mathbf{y})\right)\right) &(161)\\
&= -\int d^3\mathbf{y}\left(\nabla_{\mathbf{y}}\phi\dagger(\mathbf{y})\cdot\nabla_{\mathbf{y}}\left(\delta^{(3)}(\mathbf{x}-\mathbf{y})\right) + m^2\phi\dagger(\mathbf{y})\left(\delta^{(3)}(\mathbf{x}-\mathbf{y})\right)\right) &(162)\\
&= -\int d^3\mathbf{y}\left(-\nabla^2_{\mathbf{y}}\phi\dagger(\mathbf{y})\left(\delta^{(3)}(\mathbf{x}-\mathbf{y})\right) + m^2\phi\dagger(\mathbf{y})\left(\delta^{(3)}(\mathbf{x}-\mathbf{y})\right)\right) &(163)\\
&= \nabla^2_{\mathbf{y}}\phi\dagger(\mathbf{x}) - m^2\phi\dagger(\mathbf{x}) &(164)
\end{aligned}
$$

Where we used integration by parts to obtain line (163).

(iii) We substitute the result of (i) ($\dot{\phi} = \dot{\pi}$) into the resulty of (ii) ($\dot{\pi} = \nabla^2\phi\dagger(\mathbf{x}) - m^2\phi\dagger(\mathbf{x})$) to obtain

$$
\begin{aligned}
\frac{\partial^2\phi}{\partial t^2} &= \nabla^2\phi\dagger(\mathbf{x}) - m^2\phi\dagger(\mathbf{x}) &(165)\\
0 &= -\frac{\partial^2\phi\dagger}{\partial t^2} + \nabla^2\phi\dagger - m^2\phi\dagger &(166)\\
0 &= \partial_\mu\partial^\mu\phi\dagger - m^2\phi\dagger &(167)
\end{aligned}
$$

This is the equation of motion obtained from the Klein-Gordon equation. The corresponding equations of motion for Klein-Gordon equation for ϕ can be obtained through the equations of motion for $\dot{\pi}\dagger = i[H, \pi\dagger]$ and $\dot{\phi}\dagger = i[H, \phi\dagger]$.

(d) Diagonalize the Hamiltonian by introducing creation and annihilation operators by introducing creation and anniilation operators, and show that the theory contains two kinds of particles, both having mass m. To do this, note that for a complex field the Fourier decomposition of $\phi(x)$ contains two types of ladder operators which you can call $b_{\mathbf{p}}$ and $c_{\mathbf{p}}\dagger$, and note that the mode expansion of π must be consistent with the equation of motion derived in (c). Explain why there are two types of ladder operators, not one.

The Fourier decomposition of ϕ and π in terms of $b_{\mathbf{p}}$ and $c\dagger_{\mathbf{p}}$

$$\phi \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(b_{\mathbf{p}} e^{-ipx} + c_{\mathbf{p}} \dagger e^{ipx} \right)$$

$$\phi\dagger \equiv \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(b_{\mathbf{p}} \dagger e^{ipx} + c_{\mathbf{p}} e^{-ipx} \right)$$

$$\pi \equiv -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_p}{2}} \left(b_{\mathbf{p}} e^{-ipx} - c_{\mathbf{p}} \dagger e^{ipx} \right)$$

$$\pi\dagger \equiv i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_p}{2}} \left(b_{\mathbf{p}} \dagger e^{ipx} - c_{\mathbf{p}} e^{-ipx} \right)$$

Re-expressing the Hamiltonian in terms of these fields, considering each term separately:

$$\int d^3\mathbf{x}\ \pi \dagger \pi = \left(i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_p}{2}} \left(b_{\mathbf{p}} \dagger e^{ipx} - c_{\mathbf{p}} e^{-ipx} \right) \right) \left(-i \int \frac{d^3q}{(2\pi)^3} \sqrt{\frac{\omega_q}{2}} \left(b_{\mathbf{q}} e^{-iqx} - c_{\mathbf{q}} \dagger e^{iqx} \right) \right)$$

$$= \sqrt{\frac{\omega_p}{2}} \sqrt{\frac{\omega_q}{2}} \int d^3\mathbf{x} \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^3} \left(b_{\mathbf{p}} \dagger b_{\mathbf{q}} e^{i(p-q)x} + c_{\mathbf{p}} c_{\mathbf{q}}^\dagger e^{-i(p-q)x} - b_{\mathbf{p}}^\dagger c_{\mathbf{q}}^\dagger e^{i(p+q)x} - c_{\mathbf{p}} b_{\mathbf{q}} e^{-i(p+q)x} \right)$$

$$\begin{cases} \int d^3\mathbf{x} e^{\pm i(p-q)x} = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \\ \int d^3\mathbf{x} e^{\pm i(p+q)x} = (2\pi)^3 \delta^{(3)}(\mathbf{p} + \mathbf{q}) \end{cases}$$

$$= \sqrt{\frac{\omega_p}{2}} \sqrt{\frac{\omega_q}{2}} \int \frac{d^3p}{(2\pi)^3} \int d^3q \left(b_{\mathbf{p}}^\dagger b_{\mathbf{q}} \delta^{(3)}(\mathbf{p} - \mathbf{q}) + c_{\mathbf{p}} c_{\mathbf{q}}^\dagger \delta^{(3)}(\mathbf{p} - \mathbf{q}) - b_{\mathbf{p}}^\dagger c_{\mathbf{q}}^\dagger \delta^{(3)}(\mathbf{p} + \mathbf{q}) - c_{\mathbf{p}} b_{\mathbf{q}} \delta^{(3)}(\mathbf{p} + \mathbf{q}) \right)$$

$$\begin{aligned} = & \sqrt{\frac{\omega_p}{2}} \sqrt{\frac{\omega_q}{2}} \int \frac{d^3p}{(2\pi)^3} \int d^3q \delta^{(3)}(\mathbf{p} - \mathbf{q}) \left(b_{\mathbf{p}}^\dagger b_{\mathbf{q}} + c_{\mathbf{p}} c_{\mathbf{q}}^\dagger \right) \\ & - \sqrt{\frac{\omega_p}{2}} \sqrt{\frac{\omega_q}{2}} \int \frac{d^3p}{(2\pi)^3} \int d^3q \delta^{(3)}(\mathbf{p} + \mathbf{q}) \left(b_{\mathbf{p}}^\dagger c_{\mathbf{q}}^\dagger + c_{\mathbf{p}} b_{\mathbf{q}} \right) \end{aligned}$$

$$\begin{aligned} = & \frac{\omega_p}{2} \int \frac{d^3p}{(2\pi)^3} \left(b_{\mathbf{p}}^\dagger b_{\mathbf{p}} + c_{\mathbf{p}} c_{\mathbf{p}}^\dagger \right) \\ & - \frac{\omega_p}{2} \int \frac{d^3p}{(2\pi)^3} \left(b_{\mathbf{p}}^\dagger c_{(-\mathbf{p})}^\dagger + c_{\mathbf{p}} b_{(-\mathbf{p})} \right) \end{aligned}$$

$$= \frac{\omega_p}{2} \int \frac{d^3p}{(2\pi)^3} \left(b_{\mathbf{p}}^\dagger b_{\mathbf{p}} + c_{\mathbf{p}} c_{\mathbf{p}}^\dagger - b_{\mathbf{p}}^\dagger c_{(-\mathbf{p})}^\dagger - c_{\mathbf{p}} b_{(-\mathbf{p})} \right)$$

$$\int d^3\mathbf{x}\ \nabla\phi^\dagger \cdot \nabla\phi$$

$$=\frac{1}{\sqrt{2\omega_p}}\frac{1}{\sqrt{2\omega_q}}\int d^3\mathbf{x}\int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\nabla\left(b^\dagger_{\mathbf{p}}e^{ipx}+c_{\mathbf{q}}e^{-ipx}\right)\cdot\nabla\left(b_{\mathbf{q}}e^{-i\mathbf{q}\cdot\mathbf{x}}+c^\dagger_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{x}}\right)$$

$$=-\frac{1}{\sqrt{2\omega_p}}\frac{1}{\sqrt{2\omega_q}}\int d^3\mathbf{x}\int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\mathbf{p}\left(b^\dagger_{\mathbf{p}}e^{ipx}-c_{\mathbf{q}}e^{-ipx}\right)\cdot\mathbf{q}\left(c^\dagger_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{x}}-b_{\mathbf{q}}e^{-i\mathbf{q}\cdot\mathbf{x}}\right)$$

$$=-\frac{1}{\sqrt{2\omega_p}}\frac{1}{\sqrt{2\omega_q}}\int d^3\mathbf{x}\int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\mathbf{p}\cdot\mathbf{q}\left(b^\dagger_{\mathbf{p}}c^\dagger_{\mathbf{q}}e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}+c_{\mathbf{q}}b_{\mathbf{q}}e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}-b^\dagger_{\mathbf{p}}b_{\mathbf{q}}e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}-c_{\mathbf{q}}c^\dagger_{\mathbf{q}}e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\right)$$

$$=-\frac{1}{\sqrt{2\omega_p}}\frac{1}{\sqrt{2\omega_q}}\int \frac{d^3p}{(2\pi)^3}\int d^3q\ \mathbf{p}\cdot\mathbf{q}\left(\delta^{(3)}(\mathbf{p}+\mathbf{q})\left(b^\dagger_{\mathbf{p}}c^\dagger_{\mathbf{q}}+c_{\mathbf{q}}b_{\mathbf{q}}\right)-\delta^{(3)}(\mathbf{p}-\mathbf{q})\left(b^\dagger_{\mathbf{p}}b_{\mathbf{q}}+c_{\mathbf{q}}c^\dagger_{\mathbf{q}}\right)\right)$$

$$=\frac{1}{\sqrt{2\omega_p}}\frac{1}{\sqrt{2\omega_q}}\int \frac{d^3p}{(2\pi)^3}\int d^3q\ \mathbf{p}\cdot\mathbf{q}\delta^{(3)}(\mathbf{p}-\mathbf{q})\left(b^\dagger_{\mathbf{p}}b_{\mathbf{q}}+c_{\mathbf{p}}c^\dagger_{\mathbf{q}}\right)$$
$$-\frac{1}{\sqrt{2\omega_p}}\frac{1}{\sqrt{2\omega_q}}\int \frac{d^3p}{(2\pi)^3}\int d^3q\ \mathbf{p}\cdot\mathbf{q}\delta^{(3)}(\mathbf{p}+\mathbf{q})\left(b^\dagger_{\mathbf{p}}c^\dagger_{\mathbf{q}}+c_{\mathbf{p}}b_{\mathbf{q}}\right)$$

$$=\frac{1}{2\omega_p}\int \frac{d^3p}{(2\pi)^3}\ p^2\left(b^\dagger_{\mathbf{p}}b_{\mathbf{p}}+c_{\mathbf{p}}c^\dagger_{\mathbf{p}}\right)$$
$$-\frac{1}{2\omega_p}\int \frac{d^3p}{(2\pi)^3}\ (-p^2)\left(b^\dagger_{\mathbf{p}}c^\dagger_{(-\mathbf{p})}+c_{\mathbf{p}}b_{(-\mathbf{p})}\right)$$

$$=\frac{p^2}{2\omega_p}\int \frac{d^3p}{(2\pi^2)}\left(b^\dagger_{\mathbf{p}}b_{\mathbf{p}}+c_{\mathbf{p}}c^\dagger_{\mathbf{p}}+b^\dagger_{\mathbf{p}}c^\dagger_{(-\mathbf{p})}+c_{\mathbf{p}}b_{(-\mathbf{p})}\right)$$

$$m^2\int d^3\mathbf{x}\ \phi^\dagger\phi$$

$$=m^2\frac{1}{\sqrt{2\omega_p}}\frac{1}{\sqrt{2\omega_q}}\int d^3\mathbf{x}\int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\left(b^\dagger_{\mathbf{p}}e^{ipx}+c_{\mathbf{p}}e^{-ipx}\right)\left(b_{\mathbf{q}}e^{-i\mathbf{q}\cdot\mathbf{x}}+c^\dagger_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{x}}\right)$$

$$=m^2\frac{1}{\sqrt{2\omega_p}}\frac{1}{\sqrt{2\omega_q}}\int d^3\mathbf{x}\int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\left(b^\dagger_{\mathbf{p}}c^\dagger_{\mathbf{q}}e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}+c_{\mathbf{p}}b_{\mathbf{q}}e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}+b^\dagger_{\mathbf{p}}b_{\mathbf{q}}e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}+c_{\mathbf{p}}c^\dagger_{\mathbf{q}}e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\right)$$

$$=m^2\frac{1}{\sqrt{2\omega_p}}\frac{1}{\sqrt{2\omega_q}}\int \frac{d^3p}{(2\pi)^3}\int d^3q\left(\delta^{(3)}(\mathbf{p}+\mathbf{q})\left(b^\dagger_{\mathbf{p}}c^\dagger_{\mathbf{q}}+c_{\mathbf{p}}b_{\mathbf{q}}\right)+\delta^{(3)}(\mathbf{p}-\mathbf{q})\left(b^\dagger_{\mathbf{p}}b_{\mathbf{q}}+c_{\mathbf{p}}c^\dagger_{\mathbf{q}}\right)\right)$$

$$=m^2\frac{1}{\sqrt{2\omega_p}}\frac{1}{\sqrt{2\omega_q}}\int \frac{d^3p}{(2\pi)^3}\int d^3q\delta^{(3)}(\mathbf{p}+\mathbf{q})\left(b^\dagger_{\mathbf{p}}c^\dagger_{\mathbf{q}}+c_{\mathbf{p}}b_{\mathbf{q}}\right)$$
$$+m^2\frac{1}{\sqrt{2\omega_p}}\frac{1}{\sqrt{2\omega_q}}\int \frac{d^3p}{(2\pi)^3}\int d^3q\delta^{(3)}(\mathbf{p}-\mathbf{q})\left(b^\dagger_{\mathbf{p}}b_{\mathbf{q}}+c_{\mathbf{p}}c^\dagger_{\mathbf{q}}\right)$$

$$=\frac{m^2}{2\omega_p}\int \frac{d^3p}{(2\pi)^3}\left(b^\dagger_{\mathbf{p}}c^\dagger_{\mathbf{p}}+c_{\mathbf{p}}b_{\mathbf{p}}\right)$$
$$+\frac{m^2}{2\omega_p}\int \frac{d^3p}{(2\pi)^3}\left(b^\dagger_{\mathbf{p}}b_{(-\mathbf{p})}+c_{\mathbf{p}}c^\dagger_{(-\mathbf{p})}\right)$$

$$=\frac{m^2}{2\omega_p}\int \frac{d^3p}{(2\pi)^3}\left(b^\dagger_{\mathbf{p}}c^\dagger_{\mathbf{q}}+c_{\mathbf{p}}b_{\mathbf{q}}+b^\dagger_{\mathbf{p}}b_{(-\mathbf{p})}+c_{\mathbf{p}}c^\dagger_{(-\mathbf{p})}\right)$$

$$
\begin{aligned}
\therefore H =& \int d^3\mathbf{x} \left(\pi^\dagger \pi + \nabla\phi^\dagger \cdot \nabla\phi + m^2\phi^\dagger\phi\right) \\
=& \frac{\omega_p}{2}\int \frac{d^3p}{(2\pi)^3}\left(b^\dagger_{\mathbf{p}} b_{\mathbf{p}} + c_{\mathbf{p}} c^\dagger_{\mathbf{p}} - b^\dagger_{\mathbf{p}} c^\dagger_{(-\mathbf{p})} - c_{\mathbf{p}} b_{(-\mathbf{p})}\right) \\
&+ \frac{p^2}{2\omega_p}\int \frac{d^3p}{(2\pi^2)}\left(b^\dagger_{\mathbf{p}} b_{\mathbf{p}} + c_{\mathbf{p}} c^\dagger_{\mathbf{p}} + b^\dagger_{\mathbf{p}} c^\dagger_{(-\mathbf{p})} + c_{\mathbf{p}} b_{(-\mathbf{p})}\right) \\
&+ \frac{m^2}{2\omega_p}\int \frac{d^3p}{(2\pi)^3}\left(b^\dagger_{\mathbf{p}} c^\dagger_{\mathbf{q}} + c_{\mathbf{p}} b_{\mathbf{q}} + b^\dagger_{\mathbf{p}} b_{(-\mathbf{p})} + c_{\mathbf{p}} c^\dagger_{(-\mathbf{p})}\right)
\end{aligned}
$$

$$
\begin{aligned}
=& \frac{\omega_p}{2}\int \frac{d^3p}{(2\pi)^3}\left(b^\dagger_{\mathbf{p}} b_{\mathbf{p}} + c_{\mathbf{p}} c^\dagger_{\mathbf{p}} - b^\dagger_{\mathbf{p}} c^\dagger_{p} - c_{\mathbf{p}} b_{p}\right) \\
&+ \frac{\left(p^2+m^2\right)}{2\omega_p}\int \frac{d^3p}{(2\pi^2)}\left(b^\dagger_{\mathbf{p}} b_{\mathbf{p}} + c_{\mathbf{p}} c^\dagger_{\mathbf{p}} + b^\dagger_{\mathbf{p}} b_{(-\mathbf{p})} + c_{\mathbf{p}} c^\dagger_{(-\mathbf{p})}\right)
\end{aligned}
$$

$$
\begin{aligned}
=& \frac{\omega_p}{2}\int \frac{d^3p}{(2\pi)^3}\left(b^\dagger_{\mathbf{p}} b_{\mathbf{p}} + c_{\mathbf{p}} c^\dagger_{\mathbf{p}} - b^\dagger_{\mathbf{p}} c^\dagger_{p} - c_{\mathbf{p}} b_{p}\right) \\
&+ \frac{\omega_p}{2}\int \frac{d^3p}{(2\pi^2)}\left(b^\dagger_{\mathbf{p}} b_{\mathbf{p}} + c_{\mathbf{p}} c^\dagger_{\mathbf{p}} + b^\dagger_{\mathbf{p}} b_{(-\mathbf{p})} + c_{\mathbf{p}} c^\dagger_{(-\mathbf{p})}\right) && \leftarrow \omega_p \equiv \sqrt{p^2+m^2}
\end{aligned}
$$

$$
\begin{aligned}
=& \omega_p\int \frac{d^3p}{(2\pi)^3}\left(b^\dagger_{\mathbf{p}} b_{\mathbf{p}} + c_{\mathbf{p}} c^\dagger_{\mathbf{p}}\right) \\
&&& \leftarrow \text{Normal ordering} \\
=& \omega_p\int \frac{d^3p}{(2\pi)^3}\left(b^\dagger_{\mathbf{p}} b_{\mathbf{p}} + c^\dagger_{\mathbf{p}} c_{\mathbf{p}}\right)
\end{aligned}
$$

$$
\therefore\ H = \omega_p\int \frac{d^3p}{(2\pi)^3}\left(b^\dagger_{\mathbf{p}} b_{\mathbf{p}} + c^\dagger_{\mathbf{p}} c_{\mathbf{p}}\right)
$$

We can check that this is diagonalized by acting on it with $|k,k'\rangle = b^\dagger_{\mathbf{k}} c^\dagger_{\mathbf{k}} |0,0\rangle$

$$
\begin{aligned}
H|k,k'\rangle =& \omega_p\int \frac{d^3p}{(2\pi)^3}\left(b^\dagger_{\mathbf{p}} b_{\mathbf{p}} + c^\dagger_{\mathbf{p}} c_{\mathbf{p}}\right) b^\dagger_{\mathbf{k}} c^\dagger_{\mathbf{k}'} |0,0\rangle \\
=& \omega_p\int \frac{d^3p}{(2\pi)^3}\left(b^\dagger_{\mathbf{p}} b_{\mathbf{p}} b^\dagger_{\mathbf{k}} c^\dagger_{\mathbf{k}'} + c^\dagger_{\mathbf{p}} c_{\mathbf{p}} b^\dagger_{\mathbf{k}} c^\dagger_{\mathbf{k}'}\right)|0,0\rangle \\
=& \omega_p\int \frac{d^3p}{(2\pi)^3}\left(c^\dagger_{\mathbf{k}'} b^\dagger_{\mathbf{p}} b_{\mathbf{p}} b^\dagger_{\mathbf{k}} + b^\dagger_{\mathbf{k}} c^\dagger_{\mathbf{p}} c_{\mathbf{p}} c^\dagger_{\mathbf{k}'}\right)|0,0\rangle \\
=& \omega_p\int \frac{d^3p}{(2\pi)^3}\left(c^\dagger_{\mathbf{k}'} b^\dagger_{\mathbf{p}}\left(b^\dagger_{\mathbf{k}} b_{\mathbf{p}} + (2\pi)^3\delta(\mathbf{p}-\mathbf{k})\right) + b^\dagger_{\mathbf{k}} c^\dagger_{\mathbf{p}}\left(c^\dagger_{\mathbf{k}'} c_{\mathbf{p}} + (2\pi)^3\delta(\mathbf{p}-\mathbf{k}')\right)\right)|0,0\rangle && \left\{\begin{array}{c} b_{\mathbf{p}}|0,0\rangle \\ = c_{\mathbf{p}}|0,0\rangle \\ = 0 \end{array}\right. \\
=& \omega_p\int \frac{d^3p}{(2\pi)^3}\left(c^\dagger_{\mathbf{k}'} b^\dagger_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{k}) + b^\dagger_{\mathbf{k}} c^\dagger_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{k}')\right)|0,0\rangle \\
=& \omega_p\int d^3p\left(c^\dagger_{\mathbf{k}'} b^\dagger_{\mathbf{p}}\delta^{(3)}(\mathbf{p}-\mathbf{k}) + b^\dagger_{\mathbf{k}} c^\dagger_{\mathbf{p}}\delta^{(3)}(\mathbf{p}-\mathbf{k}')\right)|0,0\rangle \\
=& \int d^3p\ \omega_p c^\dagger_{\mathbf{k}'} b^\dagger_{\mathbf{p}}\delta^{(3)}(\mathbf{p}-\mathbf{k}) + \int d^3p\ \omega_p b^\dagger_{\mathbf{k}} c^\dagger_{\mathbf{p}}\delta^{(3)}(\mathbf{p}-\mathbf{k}')|0,0\rangle \\
=& \left(\omega_k c^\dagger_{\mathbf{k}} b^\dagger_{\mathbf{k}} + \omega_{\mathbf{k}'} b^\dagger_{\mathbf{k}} c^\dagger_{\mathbf{k}'}\right)|0,0\rangle \\
H|k,k'\rangle =& \left(\omega_k + \omega_{k'}\right)|k,k'\rangle
\end{aligned}
$$

As expected for a diagonalized Hamiltonian.

(e)

(i) Show that (up to an arbitrary rescaling), the conserved charge associated with the current j^μ in part (a) can be expressed as

$$Q = \int d^3\mathbf{x} \; \tfrac{1}{2} i \left(\phi^\dagger \pi^\dagger - \pi \phi \right) \tag{168}$$

(ii) Rewrite this in terms of creation and annihilation operators. Interpret the physical significance of your result.

(i) In (a), we found that the Noether current is $j^\mu = i \left[\left(\partial^\mu \phi^* \right) \phi - \left(\partial^\mu \phi \right) \phi^* \right]$. The corresponding conserved charge is therefore

$$Q \equiv \int d^3\mathbf{x} \; j^0 = \int d^3\mathbf{x} \; i \left(\left[\partial^0 \phi^\dagger \right) \phi - \left(\partial^0 \phi \right) \phi^\dagger \right] \tag{169}$$

$$= \int d^3\mathbf{x} \; i \left(\dot{\phi}^\dagger \phi - \dot{\phi} \phi^\dagger \right) \tag{170}$$

$$= \int d^3\mathbf{x} \; i \left(\pi \phi - \phi^\dagger \pi^\dagger \right) \tag{171}$$

To obtain (171), we used that $\pi^\dagger = \dot{\phi}$ (which we derived in (c)(i)) and that $\pi = \dot{\phi}^\dagger$ (which can be shown by evaluating the Heisenberg equation of motion $\dot{\phi}^\dagger = i \left[H, \phi^\dagger \right]$, similar to our calculations of (c)(i) and (c)(ii)).

(ii)

$$
\begin{aligned}
Q &= \int d^3\mathbf{x}\; i\left(\dot{\phi}^\dagger\phi - \dot{\phi}\phi^\dagger\right)\\
&= i\int d^3\mathbf{x}\Bigg(\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(b_{\mathbf{p}}e^{-ipx} + c^\dagger_{\mathbf{p}}e^{ipx}\right)\frac{\partial}{\partial t}\left(\int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_q}}\left(b^\dagger_{\mathbf{q}}e^{iqx} + c_{\mathbf{q}}e^{-iqx}\right)\right)\\
&\qquad - \int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_q}}\left(b^\dagger_{\mathbf{q}}e^{iqx} + c_{\mathbf{q}}e^{-iqx}\right)\frac{\partial}{\partial t}\left(\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(b_{\mathbf{p}}e^{-ipx} + c^\dagger_{\mathbf{p}}e^{ipx}\right)\right)\Bigg)\\
&= \frac{i}{2}\int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\int d^3\mathbf{x}\Bigg(\frac{1}{\sqrt{\omega_p\omega_q}}\left(b_{\mathbf{p}}e^{-ipx} + c^\dagger_{\mathbf{p}}e^{ipx}\right)\frac{\partial}{\partial t}\left(\left(b^\dagger_{\mathbf{q}}e^{iqx} + c_{\mathbf{q}}e^{-iqx}\right)\right)\\
&\qquad - \frac{1}{\sqrt{\omega_p\omega_q}}\left(b^\dagger_{\mathbf{q}}e^{iqx} + c_{\mathbf{q}}e^{-iqx}\right)\frac{\partial}{\partial t}\left(b_{\mathbf{p}}e^{-ipx} + c^\dagger_{\mathbf{p}}e^{ipx}\right)\Bigg)\\
&= \frac{i}{2}\int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\int d^3\mathbf{x}\Bigg(\frac{1}{\sqrt{\omega_p\omega_q}}\left(b_{\mathbf{p}}e^{-ipx} + c^\dagger_{\mathbf{p}}e^{ipx}\right)i\omega_q\left(\left(b^\dagger_{\mathbf{q}}e^{iqx} - c_{\mathbf{q}}e^{-iqx}\right)\right)\\
&\qquad - \frac{1}{\sqrt{\omega_p\omega_q}}\left(b^\dagger_{\mathbf{q}}e^{iqx} + c_{\mathbf{q}}e^{-iqx}\right)i\omega_p\left(-b_{\mathbf{p}}e^{-ipx} + c^\dagger_{\mathbf{p}}e^{ipx}\right)\Bigg)\\
&= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\int d^3\mathbf{x}\Bigg(\frac{\omega_q}{\sqrt{\omega_p\omega_q}}\left(b^\dagger_{\mathbf{p}}e^{ipx} + c_{\mathbf{p}}e^{-ipx}\right)\left(-b_{\mathbf{q}}e^{-iqx} + c^\dagger_{\mathbf{q}}e^{iqx}\right)\\
&\qquad - \frac{\omega_p}{\sqrt{\omega_p\omega_q}}\left(b_{\mathbf{q}}e^{-iqx} + c^\dagger_{\mathbf{q}}e^{iqx}\right)\left(b^\dagger_{\mathbf{p}}e^{ipx} - c_{\mathbf{p}}e^{-ipx}\right)\Bigg)\\
&= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}d^3q\Bigg(\frac{\omega_q}{\sqrt{\omega_p\omega_q}}\delta^{(3)}(\mathbf{p}-\mathbf{q})(c_pc^\dagger_q - b^\dagger_pb_q) + \delta^{(3)}(\mathbf{p}+\mathbf{q})(b^\dagger_pc^\dagger_q - c_pb_q)\\
&\qquad - \frac{\omega_p}{\sqrt{\omega_p\omega_q}}\left(\delta^{(3)}(\mathbf{p}-\mathbf{q})(b_qb^\dagger_p - c^\dagger_qc_p) + \delta^{(3)}(\mathbf{p}+\mathbf{q})(c^\dagger_qb^\dagger_p - b_qc_p)\right)\Bigg)\\
&= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\left(c_pc^\dagger_p - b^\dagger_pb_p + b^\dagger_pc^\dagger_{-p} - c_pb_{-p} - (b_pb^\dagger_p - c^\dagger_pc_p) - (c^\dagger_{-p}b^\dagger_p - b_{-p}c_p)\right)\\
&= \int \frac{d^3p}{(2\pi)^3}\left(c_pc^\dagger_p - b^\dagger_pb_p\right)\\
&= \int \frac{d^3p}{(2\pi)^3}\left(c^\dagger_pc_p - b^\dagger_pb_p\right)\\
&= \int \frac{d^3p}{(2\pi)^3}\left(N_c - N_b\right)
\end{aligned}
$$

Where N is the number operator for each particle.
The physical interpretation of this result is that particles b and c correspond to the field and the conjugate field respectively, and have equal and opposite charge.

(f)

(i) Calculate the two propagation amplitudes $D_b(x-y) = \langle 0|\,\phi(x)\phi^\dagger(y)\,|0\rangle$ and $D_c(x-y) = \langle 0|\,\phi^\dagger(x)\phi(y)\,|0\rangle$ and interpret their meaning.

(ii) Express the commutator $\left[\phi(x), \phi^\dagger(y)\right]$ in terms of these functions. and explain how to turn the result into a statement about causality.

(i)

$$
\begin{aligned}
D_b\left(x-y\right) &= \langle 0|\, \phi(x)\phi^\dagger(y)\, |0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{4\omega_p\omega_{p'}}}\langle 0|\, b_{\mathbf{p}}b^\dagger_{\mathbf{p}'}\, |0\rangle\, e^{-ipx+ip'y} \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{4\omega_p\omega_{p'}}}\langle 0|\left[b_{\mathbf{p}}, b^\dagger_{\mathbf{p}'}\right]|0\rangle\, e^{-ipx+ip'y} \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{4\omega_p\omega_{p'}}}\langle 0|\,(2\pi)^3\,\delta^{(4)}\left(\mathbf{p}-\mathbf{p}'\right)|0\rangle\, e^{-ipx+ip'y} \\
&= \int \frac{d^3p}{(2\pi)^3}d^3p'\frac{1}{\sqrt{4\omega_p\omega_{p'}}}\delta^{(4)}\left(\mathbf{p}-\mathbf{p}'\right)e^{-ipx+ip'y} \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}e^{-ip(x-y)}
\end{aligned}
$$

This is the amplitude for the particle to propagate from x to y.

$$
\begin{aligned}
D_c\left(x-y\right) &= \langle 0|\, \phi^\dagger(x)\phi(y)\, |0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{4\omega_p\omega_{p'}}}\langle 0|\, c_{\mathbf{p}}c^\dagger_{\mathbf{p}'}\, |0\rangle\, e^{-ipx+ip'y} \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{4\omega_p\omega_{p'}}}\langle 0|\left[c_{\mathbf{p}}, c^\dagger_{\mathbf{p}'}\right]|0\rangle\, e^{-ipx+ip'y} \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{4\omega_p\omega_{p'}}}\langle 0|\,(2\pi)^3\,\delta^{(4)}\left(\mathbf{p}-\mathbf{p}'\right)|0\rangle\, e^{-ipx+ip'y} \\
&= \int \frac{d^3p}{(2\pi)^3}d^3p'\frac{1}{\sqrt{4\omega_p\omega_{p'}}}\delta^{(4)}\left(\mathbf{p}-\mathbf{p}'\right)e^{-ipx+ip'y} \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}e^{-ip(x-y)}
\end{aligned}
$$

This is the amplitude for the anti-particle to propagate from x to y.

(ii)

$$
\begin{aligned}
\langle 0|\left[\phi(x), \phi^\dagger(y)\right]|0\rangle &= \langle 0|\, \phi(x)\phi^\dagger(y)\, |0\rangle - \langle 0|\, \phi^\dagger(y)\phi(x)\, |0\rangle \\
&= D_b(x-y) - D_c(y-x) \\
&\sim e^{-ip(x-y)} - e^{-ip(y-x)} \\
&= 0
\end{aligned}
$$

Where the last step follows because it can be shown that, when $x-y$ is spacelike, $D_b(x-y) = D_c(y-x) \sim e^{\mu|x-y|}$.

The fact that D_b and D_c are non-zero for spacelike $x-y$ may appear to violate causality. However, in the above calculation we show that the amplitude for a particle to traverse a spacelike separation $x-y$ is exactly cancelled by a the ampltiude fo an anti-particle to propagate from $y-x$. Therefore, causality is preserved.

Q16 An infinitesimal Lorentz transformation on a four-vector V^α can be written as

$$
V^a \to V'^a = \Lambda^\alpha{}_\beta V^\beta = \left(\delta^\alpha_\beta - \frac{i}{2}\omega_{\mu\nu}\left(\mathcal{J}^{\mu\nu}\right)^\alpha{}_\beta\right)V^\beta \tag{172}
$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$ are small parameters (such as angles) parameterizing the transformation, and

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} = i\left(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\mu}_{\beta}\delta^{\nu}_{\alpha}\right) \tag{173}$$

are six 'generators' (antisymmetric in $\mu \leftrightarrow \nu$) of the Lorentz group acting on the vector representation.

(a) Use the metric tensor to raise the index α and show that

$$(\mathcal{J}^{\mu\nu})^{\alpha}{}_{\beta} = i\left(g^{\alpha\mu}\delta^{\nu}_{\beta} - g^{\alpha\nu}\delta^{\mu}_{\beta}\right) \tag{174}$$

$$\begin{aligned}
(\mathcal{J}^{\mu\nu})_{\alpha'\beta} =& i\left(\delta^{\mu}{}_{\alpha'}\delta^{\nu}{}_{\beta} - \delta^{\mu}{}_{\beta}\delta^{\nu}{}_{\alpha'}\right)\\
g^{\alpha\alpha'}(\mathcal{J}^{\mu\nu})_{\alpha'\beta} =& i\left(g^{\alpha\alpha'}\delta^{\mu}{}_{\alpha'}\delta^{\nu}{}_{\beta} - g^{\alpha\alpha'}\delta^{\mu}{}_{\beta}\delta^{\nu}{}_{\alpha'}\right)\\
(\mathcal{J}^{\mu\nu})^{\alpha}{}_{\beta} =& i\left(g^{\alpha\mu}\delta^{\nu}{}_{\beta} - g^{\alpha\nu}\delta^{\mu}{}_{\beta}\right)
\end{aligned}$$

(b) Consider the Lorentz transformation where $\omega_{12} = -\omega_{21} = \theta$, with all other $\omega_{\mu\nu}$ set to zero, and show that (a) implies the matrix equation

$$V \to \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & -\theta & 0\\ 0 & \theta & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} V \tag{175}$$

Interpret the result.

$$V^{\alpha} \to V'^{\alpha} = \Lambda^{\alpha}{}_{\beta}V^{\beta} = \left(\delta^{\alpha}{}_{\beta} - \tfrac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})^{\alpha}{}_{\beta}\right)V^{\beta}$$

$$\begin{aligned}
\Lambda^{\alpha}{}_{\beta} =& \delta^{\alpha}{}_{\beta} - \tfrac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})^{\alpha}{}_{\beta}\\
=& \delta^{\alpha}{}_{\beta} + \tfrac{1}{2}\omega_{\mu\nu}\left(g^{\alpha\mu}\delta^{\nu}{}_{\beta} - g^{\alpha\nu}\delta^{\mu}{}_{\beta}\right) && \leftarrow (\mathcal{J}^{\mu\nu})^{\alpha}{}_{\beta} = i\left(g^{\alpha\mu}\delta^{\nu}{}_{\beta} - g^{\alpha\nu}\delta^{\mu}{}_{\beta}\right)\\
=& \delta^{\rho}{}_{\rho} + \tfrac{1}{2}\omega_{\mu\nu}\left(g^{\rho\mu}\delta^{\nu}{}_{\rho} - g^{\rho\nu}\delta^{\mu}{}_{\rho}\right) && \leftarrow \left(\text{Considering the diagonal by setting } \alpha \text{ and } \beta \text{ to } \rho\right)\\
=& \delta^{\rho}{}_{\rho} + \tfrac{1}{2}\omega_{\mu\nu}\left(g^{\nu\mu} - g^{\mu\nu}\right) && \leftarrow g^{\mu\nu} = g^{\nu\mu}\\
=& \delta^{\rho}{}_{\rho} && \text{(Therefore, diagonal elements of } \Lambda \text{ are 1.)}
\end{aligned}$$

Now considering off-diagonal elements such that $\alpha \neq \beta$

$$\begin{aligned}
\Lambda^{\alpha}{}_{\beta} =& \underbrace{\delta^{\alpha}{}_{\beta}}_{0} + \tfrac{1}{2}\omega_{\mu\nu}\left(g^{\alpha\mu}\delta^{\nu}{}_{\beta} - g^{\alpha\nu}\delta^{\mu}{}_{\beta}\right)\\
=& \tfrac{1}{2}\omega_{\mu\nu}\left(g^{\alpha\mu}\delta^{\nu}{}_{\beta} - g^{\alpha\nu}\delta^{\mu}{}_{\beta}\right)\\
=& \tfrac{1}{2}\omega_{12}\left(g^{\alpha1}\delta^{2}{}_{\beta} - g^{\alpha2}\delta^{1}{}_{\beta}\right) + \tfrac{1}{2}\omega_{21}\left(g^{\alpha2}\delta^{1}{}_{\beta} - g^{\alpha1}\delta^{2}{}_{\beta}\right) && \leftarrow \text{Summing explicitly}\\
=& \tfrac{1}{2}\theta\left(g^{\alpha1}\delta^{2}{}_{\beta} - g^{\alpha2}\delta^{1}{}_{\beta}\right) - \tfrac{1}{2}\theta\left(g^{\alpha2}\delta^{1}{}_{\beta} - g^{\alpha1}\delta^{2}{}_{\beta}\right)\\
=& \theta\left(g^{\alpha1}\delta^{2}{}_{\beta} - g^{\alpha2}\delta^{1}{}_{\beta}\right)\\
\Lambda^{1}{}_{2} =& \theta g^{11} = -\theta\\
\Lambda^{2}{}_{1} =& -\theta g^{22} = \theta
\end{aligned}$$

All other off-diagonal elements are zero.

$$\therefore V \to \Lambda^{\alpha}{}_{\beta}V^{\beta} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & -\theta & 0\\ 0 & \theta & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} V$$

(c) Consider the equation

$$S^{-1}(\Lambda)\gamma^\alpha S(\Lambda) = \Lambda^\alpha{}_\beta \gamma^\beta \tag{176}$$

where

$$S(\Lambda) = \exp\left(-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right) \tag{177}$$

is the matrix representing Lorentz transformations on the Dirac field, and $\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]$. Verify this equation by expanding to first order in $\omega_{\mu\nu}$ and using (174). As a first step, show that

$$\left[\gamma^\alpha, \sigma^{\mu\nu}\right] = 2\left(\mathcal{J}^{\mu\nu}\right)^\alpha{}_\beta \gamma^\beta \tag{178}$$

$$
\begin{aligned}
\left[\gamma^\alpha, \sigma^{\mu\nu}\right] =& \tfrac{i}{2}\left[\gamma^\alpha, \left[\gamma^\mu, \gamma^\nu\right]\right] && \leftarrow \sigma^{\mu\nu} = \tfrac{i}{2}\left[\gamma^\mu, \gamma^\nu\right] \\
=& i\left[\gamma^\alpha, \left(\gamma^\mu\gamma^\nu - g^{\mu\nu}\right)\right] && \leftarrow \left[\gamma^\mu, \gamma^\alpha\right] = 2\gamma^\mu\gamma^\alpha - 2g^{\mu\alpha} \\
=& i\left[\gamma^\alpha, \gamma^\mu\gamma^\nu\right] \\
=& i\left[\gamma^\alpha, \gamma^\mu\gamma^\nu\right] \\
=& i\left(\gamma^\mu[\gamma^\alpha, \gamma^\nu] + [\gamma^\alpha, \gamma^\mu]\gamma^\nu\right) \\
=& 2i\left(\gamma^\mu(\gamma^\alpha\gamma^\nu - g^{\alpha\nu}) + (\gamma^\alpha\gamma^\mu - g^{\alpha\mu})\gamma^\nu\right) && \{\gamma^\mu, \gamma^\alpha\} = 2g^{\mu\alpha} \\
=& 2i\left(\underbrace{\gamma^\mu\gamma^\alpha}\gamma^\nu - g^{\alpha\nu}\gamma^\mu + \gamma^\alpha\gamma^\mu\gamma^\nu - g^{\alpha\mu}\gamma^\nu\right) && \leftarrow \gamma^\mu\gamma^\alpha = 2g^{\mu\alpha} - \gamma^\alpha\gamma^\mu \\
=& 2i\left(\left(2g^{\mu\alpha} - \gamma^\alpha\gamma^\mu\right)\gamma^\nu - g^{\alpha\nu}\gamma^\mu + \gamma^\alpha\gamma^\mu\gamma^\nu - g^{\alpha\mu}\gamma^\nu\right) \\
=& 2i\left(2g^{\mu\alpha}\gamma^\nu - \cancel{\gamma^\alpha\gamma^\mu\gamma^\nu} - g^{\alpha\nu}\gamma^\mu + \cancel{\gamma^\alpha\gamma^\mu\gamma^\nu} - g^{\alpha\mu}\gamma^\nu\right) \\
=& 2i\left(2g^{\alpha\mu}\gamma^\nu - g^{\alpha\nu}\gamma^\mu - g^{\alpha\mu}\gamma^\nu\right) && \leftarrow g^{\mu\alpha} = g^{\alpha\mu} \\
=& 2i\left(g^{\alpha\mu}\gamma^\nu - g^{\alpha\nu}\gamma^\mu\right) \\
=& 2i\left(g^{\alpha\mu}\delta^\nu{}_\beta\gamma^\beta - g^{\alpha\nu}\delta^\mu{}_\beta\gamma^\beta\right) \\
=& 2i\left(g^{\alpha\mu}\delta^\nu{}_\beta - g^{\alpha\nu}\delta^\mu{}_\beta\right)\gamma^\beta \\
=& 2\left(\mathcal{J}^{\mu\nu}\right)^\alpha{}_\beta\gamma^\beta
\end{aligned}
$$

$$
\begin{aligned}
S^{-1}(\Lambda)\gamma^\alpha S(\Lambda) =& \left(1 + \tfrac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right)\gamma^\alpha\left(1 - \tfrac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right) + \mathcal{O}(\omega^2) && S\left(\Lambda\right) = \exp\left(-\tfrac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right) \\
=& \left(1 + \tfrac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right)\left(\gamma^\alpha - \gamma^\alpha\tfrac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right) && S^{-1}\left(\Lambda\right) = \exp\left(\tfrac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right) \\
=& \gamma^\alpha + \tfrac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\gamma^\alpha - \gamma^\alpha\tfrac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu} + \mathcal{O}(\omega^2) \\
=& \gamma^\alpha + \tfrac{i}{4}\omega_{\mu\nu}[\sigma^{\mu\nu}, \gamma^\alpha] \\
=& \gamma^\alpha - \tfrac{i}{2}\omega_{\mu\nu}\left(\mathcal{J}^{\mu\nu}\right)^\alpha{}_\beta\gamma^\beta && \Lambda^\alpha{}_\beta\gamma^\beta = \left(\delta^\alpha{}_\beta - \tfrac{i}{2}\omega_{\mu\nu}\left(\mathcal{J}^{\mu\nu}\right)^\alpha{}_\beta\right)\gamma^\beta \\
=& \Lambda^\alpha{}_\beta\gamma^\beta \qquad \Longleftarrow && = \gamma^\alpha - \tfrac{i}{2}\omega_{\mu\nu}\left(\mathcal{J}^{\mu\nu}\right)^\alpha{}_\beta\gamma^\beta
\end{aligned}
$$

(d) Parity is the transformation $x^\mu \to x'^\mu = \left(x^0, -\mathbf{x}\right)$, which can be thought of as mapping the world onto its mirror image.

Show that if $\psi(x)$ is a solution to the Dirac equation, then so is $\psi'(x') = \gamma^0\psi(x)$ in the parity transformed world. In other words, start with $\left(i\gamma^\mu\partial_\mu - m\right)\psi(x) = 0$, and manipulate it into the form

$$\left(i\gamma^\mu\partial'_\mu - m\right)\psi'\left(x'\right) = 0 \tag{179}$$

$$
\begin{aligned}
0 &= (i\gamma^\mu \partial_\mu - m)\,\psi(x) \\
&= \left(i\gamma^0\partial_0 - i\gamma^i\partial_i - m\right)\psi(x) \\
&= \left(i\gamma^0\gamma^0\partial_0 - i\gamma^0\gamma^i\partial_i - m\gamma^0\right)\psi(x) && \leftarrow \text{Multiplying on left by } \gamma^0 \\
&= \left(i\gamma^0\gamma^0\partial_0 + i\gamma^i\gamma^0\partial_i - m\gamma^0\right)\psi(x) && \leftarrow \gamma^0\gamma^i = -\gamma^i\gamma^0 \\
&= \left(i\gamma^0\partial_0 + i\gamma^i\partial_i - m\right)\gamma^0\psi(x) && \leftarrow \text{Factorizing } \gamma^0 \text{ on the right} \\
&= \left(i\gamma^0\partial_0' - i\gamma^i\partial_i' - m\right)\gamma^0\psi(x) && \leftarrow \partial_i = -\partial_i' \;\; \partial_0 = \partial_0' \\
&= \left(i\gamma^\mu\partial_\mu' - m\right)\gamma^0\psi(x) \\
&= \left(i\gamma^\mu\partial_\mu' - m\right)\psi'(x')
\end{aligned}
$$

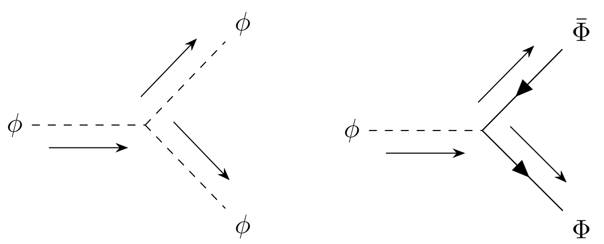
Q17 Consider the following Lagrangian involving a complex scalar field Φ and a real scalar field ϕ:

$$\mathcal{L} = \partial_\mu\Phi^*\partial^\mu\Phi - M^2\Phi^*\Phi + \tfrac{1}{2}\left(\partial_\mu\phi\right)^2 - \tfrac{1}{2}m^2\phi^2 - \frac{g}{6}\phi^3 - g\Phi^*\Phi\phi \tag{180}$$

The quantized theory contains a particle Φ and an antiparticle $\bar{\Phi}$, associated with the complex scalar field, and in addition a particle ϕ associated with the real scalar field. Interactions between particles are mediated through the terms in the Lagrangian proportional to g.

(a) Draw the two interaction vertices in the quantized theory and write down the momentum-space Feynman rules associated with them

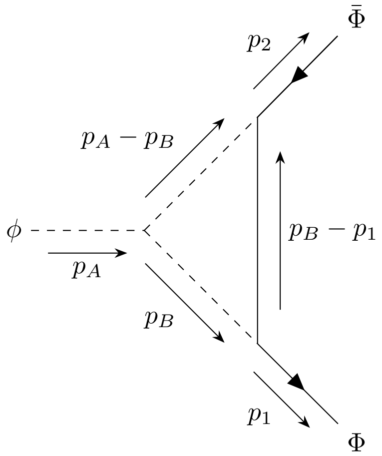
---



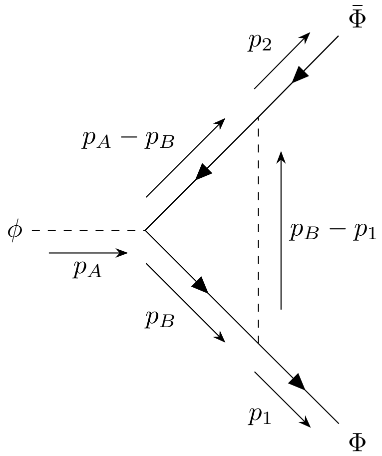
1. Factor of $-ig$ associated with vertex
2. Term $\dfrac{i}{p^2 - m^2 + i\epsilon}$ associated with propagators
3. External legs are 1
4. Momentum conservation at vertex

(b) Draw and evaluate all fully connected, amputated Feynman diagrams contributing to the decay amplitude $\phi(p_A) \to \Phi(p_1)\,\bar{\Phi}(p_2)$ to order g^3 (write down any loop integrals but don't evaluate them).

Introducing the unknown momentum p_B

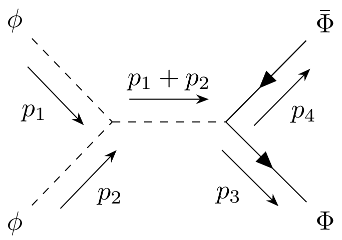


$$= \int \frac{d^4 p_B}{(2\pi)^4} \frac{i}{(p_B^2 - m^2 + i\epsilon)} \frac{i}{\left((p_A - p_B)^2 - m^2 + i\epsilon\right)} \frac{i}{\left((p_B - p_1)^2 - M^2 + i\epsilon\right)}$$
$$\times (-ig)^3 \times (2\pi)^4 \delta^{(4)}(p_A - p_1 - p_2)$$

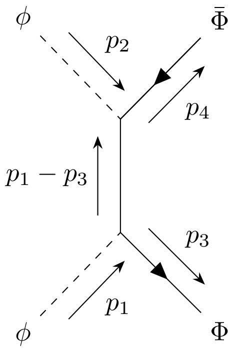


$$= \int \frac{d^4 p_B}{(2\pi)^4} \frac{i}{(p_B^2 - M^2 + i\epsilon)} \frac{i}{\left((p_A - p_B)^2 - M^2 + i\epsilon\right)} \frac{i}{\left((p_B - p_1)^2 - m^2 + i\epsilon\right)}$$
$$\times (-ig)^3 \times (2\pi)^4 \delta^{(4)}(p_A - p_1 - p_2)$$

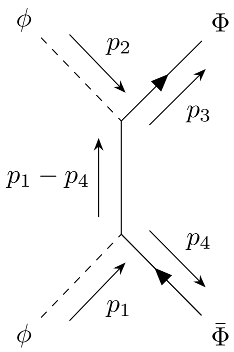
(c) Draw and evaluate all Feynman diagrams contributing to the scattering process $\phi(p_1)\phi(p_2) \to \Phi(p_3)\bar{\Phi}(p_4)$ at lowest non-trivial order in g.



$$= \frac{1}{(p_1 + p_2)^2 - m^2 + i\epsilon} \times (-ig)^2 \times (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4)$$



$$= \frac{1}{\left(p_1 - p_3\right)^2 - m^2 + i\epsilon} \times \left(-ig\right)^2 \times \left(2\pi\right)^4 \delta^{(4)}\left(p_1 + p_2 - p_3 - p_4\right)$$



$$= \frac{1}{\left(p_1 - p_4\right)^2 - m^2 + i\epsilon} \times \left(-ig\right)^2 \times \left(2\pi\right)^4 \delta^{(4)}\left(p_1 + p_2 - p_3 - p_4\right)$$

---

2 Path Integral QFT

Q1 Compute the generating functional

$$Z[J] = \mathcal{N} \int \mathcal{D}q \exp\left(\frac{i}{\hbar}\int_{-\infty}^{\infty} dt \left(m\frac{\dot{q}^2}{2} - m\omega^2\frac{q^2}{2} - \lambda\frac{q^4}{4!}\right) + \int_{-\infty}^{\infty} dt\ J(t)q(t)\right) \tag{181}$$

in perturbation theory about the harmonic oscillator to order λ using $Z[0] = 1$ to fix $\mathcal{N}$. Use the result to obtain the two-point Green function $G(t_1, t_2)$ to order λ.

$$\begin{aligned}
Z[J] =& \mathcal{N} \int \mathcal{D}q \exp\left(\frac{i}{\hbar}\int_{-\infty}^{\infty} dt \left(m\frac{\dot{q}^2}{2} - m\omega^2\frac{q^2}{2} - \lambda\frac{q^4}{4!}\right) + \int_{-\infty}^{\infty} dt J(t)q(t)\right)\\
=& \mathcal{N}\left\{1 - \frac{i}{\hbar}\int_{-\infty}^{\infty} dt\frac{\lambda}{4!}\left(\frac{\delta}{\delta J(t)}\right)^4\right\}\int \mathcal{D}q \exp\left(\frac{i}{\hbar}\int_{-\infty}^{\infty} dt\ m\left(\frac{\dot{q}^2}{2} + \omega^2\frac{q^2}{2}\right) + \int_{-\infty}^{\infty} dt\ J(t)q(t)\right)\\
=& \mathcal{N}\left\{1 - \frac{i}{\hbar}\int_{-\infty}^{\infty} dt\frac{\lambda}{4!}\left(\frac{\delta}{\delta J(t)}\right)^4\right\}\exp\left(\frac{1}{2}\int_{-\infty}^{\infty} dt_1 dt_2 J(t_1)G_0(t_1,t_2)J(t_2)\right)
\end{aligned}$$

The path integral in the second line is that of the harmonic oscillator so we replaced it with the harmonic oscillator solution. Now, we calculate:

$$\left(\frac{\delta}{\delta J(t)}\right)^4 \exp\left(\frac{1}{2}\int_{-\infty}^{\infty} dt_1 dt_2 J(t_1)G_0(t_1,t_2)J(t_2)\right)$$

$$= \left(\frac{\delta}{\delta J(t)}\right)^3\left[\left(\int_{-\infty}^{\infty} dt_3 G_0(t,t_3)J(t_3)\right)\exp\left(\frac{1}{2}\int_{-\infty}^{\infty} dt_1 dt_2 J(t_1)G_0(t_1,t_2)J(t_2)\right)\right]$$

$$= \left(\frac{\delta}{\delta J(t)}\right)^2\left[\left(G_0(t,t) + \left(\int_{-\infty}^{\infty} dt_3 G_0(t,t_3)J(t_3)\right)^2\right)\exp\left(\frac{1}{2}\int_{-\infty}^{\infty} dt_1 dt_2 J(t_1)G_0(t_1,t_2)J(t_2)\right)\right]$$

$$\begin{aligned}
= \left(\frac{\delta}{\delta J(t)}\right)&\left[\left(G_0(t,t) + \left(\int_{-\infty}^{\infty} dt_3 G_0(t,t_3)J(t_3)\right)^2\right)\left(\int_{-\infty}^{\infty} dt_3 G_0(t,t_3)J(t_3)\right)\exp\left(\frac{1}{2}\int_{-\infty}^{\infty} dt_1 dt_2 J(t_1)G_0(t_1,t_2)J(t_2)\right)\right.\\
&\left. + \left(2G_0(t,t_3)\left(\int_{-\infty}^{\infty} dt_3 G_0(t,t)J(t_3)\right)\right)\exp\left(\frac{1}{2}\int_{-\infty}^{\infty} dt_1 dt_2 J(t_1)G_0(t_1,t_2)J(t_2)\right)\right]
\end{aligned}$$

$$= \left(\frac{\delta}{\delta J(t)}\right)\left[\left\{3G_0(t,t) + \left(\int_{-\infty}^{\infty} dt_3 G_0(t,t_3)J(t_3)\right)^2\right\}\left(\int_{-\infty}^{\infty} dt_3 G_0(t,t_3)J(t_3)\right)\exp\left(\frac{1}{2}\int_{-\infty}^{\infty} dt_1 dt_2 J(t_1)G_0(t_1,t_2)J(t_2)\right)\right]$$

$$= \left(\frac{\delta}{\delta J(t)}\right)\left[\left\{3G_0(t,t)\int_{-\infty}^{\infty} dt_3 G_0(t,t_3)J(t_3) + \left(\int_{-\infty}^{\infty} dt_3 G_0(t,t_3)J(t_3)\right)^3\right\}\exp\left(\frac{1}{2}\int_{-\infty}^{\infty} dt_1 dt_2 J(t_1)G_0(t_1,t_2)J(t_2)\right)\right]$$

$$\begin{aligned}
=& \left\{3G_0(t,t)\left(\int_{-\infty}^{\infty} dt_3 G_0(t,t_3)J(t_3)\right)^2 + \left(\int_{-\infty}^{\infty} dt_3 G_0(t,t_3)J(t_3)\right)^4\right\}\exp\left(\frac{1}{2}\int_{-\infty}^{\infty} dt_1 dt_2 J(t_1)G_0(t_1,t_2)J(t_2)\right)\\
&+ \left\{3\left(G_0(t,t)\right)^2 + 3G_0(t,t)\left(\int_{-\infty}^{\infty} dt_3 G_0(t,t_3)J(t_3)\right)^2\right\}\exp\left(\frac{1}{2}\int_{-\infty}^{\infty} dt_1 dt_2 J(t_1)G_0(t_1,t_2)J(t_2)\right)\\
=& \left\{3\left(G_0(t,t)\right)^2 + 6G_0(t,t)\left(\int_{-\infty}^{\infty} dt_3 G_0(t,t_3)J(t_3)\right)^2 + \left(\int_{-\infty}^{\infty} dt_3 G_0(t,t_3)J(t_3)\right)^4\right\}\\
&\times \exp\left(\frac{1}{2}\int_{-\infty}^{\infty} dt_1 dt_2 J(t_1)G_0(t_1,t_2)J(t_2)\right)
\end{aligned}$$

Substituting this back into the expression for $Z[J]$:

$$Z[J] = \mathcal{N}\left\{\overbrace{1 - \frac{i\lambda}{4!\hbar}\int_{-\infty}^{\infty} dt\left[3\left(G_0(t,t)\right)^2 + 6G_0(t,t)\left(\int_{-\infty}^{\infty} dt_3 G_0(t,t_3)J(t_3)\right)^2 + \left(\int_{-\infty}^{\infty} dt_3 G_0(t,t_3)J(t_3)\right)^4\right]}^{\text{Define this as } V}\right\}$$
$$\times \exp\left(\underbrace{\frac{1}{2}\int_{-\infty}^{\infty} dt_1 dt_2 J(t_1)G_0(t_1,t_2)J(t_2)}_{\text{Define this as } W}\right)$$
$$Z[J] = \mathcal{N} V e^W$$

The 2-point Green function can be calculated using:

$$\begin{aligned}
G(t_1,t_2) &= \left.\frac{\delta^2 Z[J]}{\delta J(t_1)\delta J(t_2)}\right|_{J=0} \\
&= \mathcal{N}\left.\frac{\delta^2}{\delta J(t_1)\delta J(t_2)}\left(Ve^W\right)\right|_{J=0} \\
&= \mathcal{N}\frac{\delta}{\delta J(t_2)}\left.\left[\left(\frac{\delta V}{\delta J(t_1)} + V\frac{\delta W}{\delta J(t_1)}\right)e^W\right]\right|_{J=0} \\
&= \mathcal{N}\left.\left[\left(\frac{\delta^2 V}{\delta J(t_1)\delta J(t_2)} + \frac{\delta V}{\delta J(t_2)}\frac{\delta W}{\delta J(t_1)} + V\frac{\delta^2 W}{\delta J(t_1)\delta J(t_2)}\right)e^W + \left(\frac{\delta V}{\delta J(t_1)} + V\frac{\delta W}{\delta J(t_1)}\right)\frac{\delta W}{\delta J(t_2)}e^W\right]\right|_{J=0} \\
&= \mathcal{N}\left.\left[\frac{\delta^2 V}{\delta J(t_1)\delta J(t_2)} + \frac{\delta V}{\delta J(t_2)}\frac{\delta W}{\delta J(t_1)} + V\frac{\delta^2 W}{\delta J(t_1)\delta J(t_2)} + \left(\frac{\delta V}{\delta J(t_1)} + V\frac{\delta W}{\delta J(t_1)}\right)\frac{\delta W}{\delta J(t_2)}\right]e^W\right|_{J=0}
\end{aligned}$$

Calculating these components separately, ($i = 1, 2$):

$$\begin{aligned}
V &= 1 - \frac{i\lambda}{4!\hbar}\int_{-\infty}^{\infty} dt\left[3\left(G_0(t,t)\right)^2 + 6G_0(t,t)\left(\int_{-\infty}^{\infty} dt_3 G_0(t,t_3)J(t_3)\right)^2 + \left(\int_{-\infty}^{\infty} dt_3 G_0(t,t_3)J(t_3)\right)^4\right] \\
\frac{\delta V}{\delta J(t_i)} &= -\frac{i\lambda}{4!\hbar}\int_{-\infty}^{\infty} dt\left[12G_0(t,t)\left(\int_{-\infty}^{\infty} dt_3 G_0(t,t_3)J(t_3)\right)G_0(t,t_i) + 4\left(\int_{-\infty}^{\infty} dt_3 G_0(t,t_3)J(t_3)\right)^3 G_0(t,t_i)\right] \\
\frac{\delta^2 V}{\delta J(t_1)\delta J(t_2)} &= -\frac{i\lambda}{4!\hbar}\int_{-\infty}^{\infty} dt\left[12G_0(t,t)G_0(t,t_2)G_0(t,t_1) + 12\left(\int_{-\infty}^{\infty} dt_3 G_0(t,t_3)J(t_3)\right)^2 G_0(t,t_2)G_0(t,t_1)\right]
\end{aligned}$$

$$\begin{aligned}
W &= \frac{1}{2}\int_{-\infty}^{\infty} dt_1 dt_2 J(t_1)G_0(t_1,t_2)J(t_2) \\
\frac{\delta W}{\delta J(t_i)} &= \int_{-\infty}^{\infty} dt_3 G_0(t_i,t_3)J(t_3) \\
\frac{\delta^2 W}{\delta J(t_1)\delta J(t_2)} &= G_0(t_1,t_2)
\end{aligned}$$

It can be seen that

$$\left.\frac{\delta V}{\delta J(t_1)}\right|_{J=0} = \left.\frac{\delta V}{\delta J(t_2)}\right|_{J=0} = \left.\frac{\delta W}{\delta J(t_1)}\right|_{J=0} = \left.\frac{\delta W}{\delta J(t_2)}\right|_{J=0} = 0,\ \left.e^W\right|_{J=0} = 1$$

Returning to our expression for the 2-point Green function:

$$
\begin{aligned}
&G(t_1,t_2)\\
&=\mathcal{N}\left[\frac{\delta^2 V}{\delta J(t_1)\delta J(t_2)}+\frac{\delta V}{\delta J(t_2)}\frac{\delta W}{\delta J(t_1)}+V\frac{\delta^2 W}{\delta J(t_1)\delta J(t_2)}+\left(\frac{\delta V}{\delta J(t_1)}+V\frac{\delta W}{\delta J(t_1)}\right)\frac{\delta W}{\delta J(t_2)}\right]e^W\bigg|_{J=0}\\
&=\mathcal{N}\left[\frac{\delta^2 V}{\delta J(t_1)\delta J(t_2)}+V\frac{\delta^2 W}{\delta J(t_1)\delta J(t_2)}\right]\bigg|_{J=0}\\
&=\mathcal{N}\left(-\frac{i\lambda}{4!\hbar}\int_{-\infty}^{\infty}dt\,(12G_0(t,t)G_0(t,t_1)G_0(t,t_2))+\left(1-\frac{i\lambda}{4!\hbar}\int_{-\infty}^{\infty}dt\;3G_0(t,t)^2\right)G_0(t_1,t_2)\right)\\
&=\mathcal{N}\left(G_0(t_1,t_2)-\frac{i\lambda}{2\hbar}\int_{-\infty}^{\infty}dtG_0(t,t)G_0(t,t_1)G_0(t,t_2)-\frac{i\lambda}{8\hbar}\int_{-\infty}^{\infty}dt\;G_0(t,t)^2G_0(t_1,t_2)\right)
\end{aligned}
$$

$$
\text{Calculating Normalization constant:}\begin{cases} Z[0]=1\\ \mathcal{N}\left\{1-\frac{i\lambda}{8\hbar}\int_{-\infty}^{\infty}dt\,(G_0(t,t))^2\right\}=1\\ \mathcal{N}=\left\{1-\frac{i\lambda}{8\hbar}\int_{-\infty}^{\infty}dt\,(G_0(t,t))^2\right\}^{-1}\end{cases}
$$

$$
\begin{aligned}
&=\left(1-\frac{i\lambda}{8\hbar}\int_{-\infty}^{\infty}dt\,(G_0(t,t))^2\right)^{-1}\left(G_0(t_1,t_2)-\frac{i\lambda}{2\hbar}\int_{-\infty}^{\infty}dt\;G_0(t,t)G_0(t,t_1)G_0(t,t_2)-\frac{i\lambda}{8\hbar}\int_{-\infty}^{\infty}dt\;G_0(t,t)^2G_0(t_1,t_2)\right)\\
&\approx\left(1+\frac{i\lambda}{8\hbar}\int_{-\infty}^{\infty}dt\,(G_0(t,t))^2\right)\left(G_0(t_1,t_2)-\frac{i\lambda}{2\hbar}\int_{-\infty}^{\infty}dt\;G_0(t,t)G_0(t,t_1)G_0(t,t_2)-\frac{i\lambda}{8\hbar}\int_{-\infty}^{\infty}dt\;G_0(t,t)^2G_0(t_1,t_2)\right)\\
&=G_0(t_1,t_2)-\frac{i\lambda}{2\hbar}\int_{-\infty}^{\infty}dt\;G_0(t,t)G_0(t,t_1)G_0(t,t_2)-\cancel{\frac{i\lambda}{8\hbar}\int_{-\infty}^{\infty}dt\;G_0(t,t)^2G_0(t_1,t_2)}+\cancel{\frac{i\lambda}{8\hbar}\int_{-\infty}^{\infty}dt\;(G_0(t,t))^2G_0(t_1,t_2)}\\
&=G_0(t_1,t_2)-\frac{i\lambda}{2\hbar}\int_{-\infty}^{\infty}dt\,(G_0(t,t)G_0(t,t_2)G_0(t,t_1))
\end{aligned}
$$

---

3 Advanced Quantum Field Theory

Q1

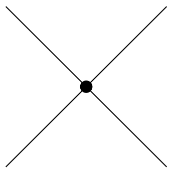
(a) Write down the momentum space Feynman rules for the theory with the action

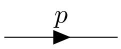
$$S[\phi] = \int d^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \right) \tag{182}$$

Where ϕ is a real scalar field.

(b) Use these rules to write down the connected part of the scattering amplitude $i\mathcal{M}$ to order λ for two incoming particles with momenta p_1^μ and p_2^μ and two outgoing particles with momenta $p_{1'}^\mu$ and $p_{2'}^\mu$.

(i)

1. Term associated with a vertex: 

2. Term associated with a propogator: 

$= \dfrac{i}{p^2 - m^2 + i\epsilon}$

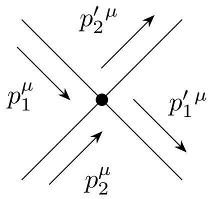
3. Term associated with external lines: 1

4. Momentum conservation at vertex

5. Integrate over undetermined momentum with integration measure $\int \dfrac{d^4k}{(2\pi)^4}$

6. Divide by the symmetry factor

(ii)



$$i\mathcal{M} = -i\lambda$$

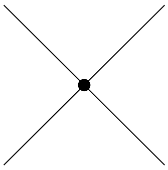
Q2 Consider the one-dimensional integral

$$Z[\lambda] = \int dx \ \exp\left(-\frac{1}{2}x^2 - \frac{\lambda}{4!}x^4 \right) \equiv Z[0]\exp\left(W[\lambda]\right) \tag{183}$$

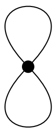
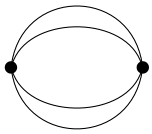
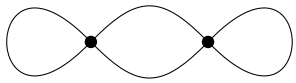
which can be thought of as the partition function of an interacting quantum field theory in $0+0$ dimensions.

(a) Consider performaing a perturbative expansion of this partition function in powers of λ. This expansion can be organized in terms of Feynman diagrams. Write down the Feynman rules for this theory.

Propagator: $= 1$

Interaction vertex:  $= -i\lambda$

(b) Draw the Feynman diagrams contributing up to order to λ^2 in $W[\lambda]$. Evaluate the diagrams using the Feynamn rules found in (a).

 $+$  $+$  $+\mathcal{O}\left(\lambda^3\right)$

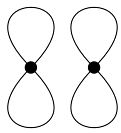
$$W[\lambda] = \quad -\frac{\lambda}{8} \quad + \quad \frac{\lambda^2}{48} \quad + \quad \frac{\lambda^2}{16} \quad + \mathcal{O}\left(\lambda^3\right)$$

(c) Using (b), write down an expression for the ratio $Z[\lambda]/Z[0]$ up to order λ^2. Check that this expression gives 0.988306 when $\lambda = 0.1$.

$$\frac{Z[\lambda]}{Z[0]} = e^{W[\lambda]} = 1 + W[\lambda] + \tfrac{1}{2}W[\lambda]^2 + \mathcal{O}\left(\lambda^3\right) \tag{184}$$

$$= 1 - \frac{\lambda}{8} + \frac{\lambda^2}{48} + \frac{\lambda^2}{16} + \tfrac{1}{2}W[\lambda]^2 + \mathcal{O}\left(\lambda^3\right) \tag{185}$$

Squaring our expression for $W[\lambda]$ derived in (b)

$W[\lambda]^2 =$  $+\mathcal{O}\left(\lambda^3\right) = \left(-\frac{\lambda}{8}\right)^2 + \mathcal{O}\left(\lambda^3\right) = \frac{\lambda^2}{64} + \mathcal{O}\left(\lambda^3\right)$

Substituting this into (185)

$$\begin{aligned}\frac{Z[\lambda]}{Z[0]} &= 1 - \frac{\lambda}{8} + \frac{\lambda^2}{48} + \frac{\lambda^2}{16} + \frac{\lambda^2}{128} + \mathcal{O}\left(\lambda^3\right)\\ \frac{Z[0.1]}{Z[0]} &\approx 1 - \frac{0.1}{8} + \frac{0.1^2}{48} + \frac{0.1^2}{16} + \frac{0.1^2}{128}\\ &\approx 0.988411\end{aligned}$$

Error of approximately 0.01% when compared with the given value of 0.988306.

Q3

In this problem we will study a theory of N self-interacting scalar fields $\phi^i, i \in 1, \cdots, N$ with an $SO(N)$ symmetry. Consider the action

$$Z[\lambda] = \int dx\ \exp\left(-\frac{1}{2}x^2 - \frac{\lambda}{4!}x^3\right) \equiv Z[0]\exp(W[\lambda]) \tag{186}$$

where ${\phi^i}^2 = \phi^i \phi^i$ (i.e. the magnitude-squared of the N-componenet vector ϕ^i), and where

$$V\left(\phi^i\right) = \frac{m^2}{2}\left(\phi^i\right)^2 + \frac{\lambda}{4}\left(\left(\phi^i\right)^2\right)^2 \tag{187}$$

(a) This theory has an $SO(N)$ symmetry whose action on the fields is

$$\delta\phi^I = \epsilon^{ij}\phi^j \tag{188}$$

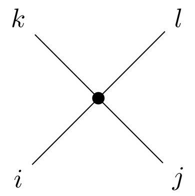
where ϵ^{ij} is an antisymmetric matrix. Write down (in terms of ϵ^{ij}) the Noether current associated to the symmetry (188). As a function of N, how many independent Noether currents does this theory have?

Expression for the Noether currents

$$\left(\frac{\partial\mathcal{L}}{\partial(\partial\phi^i)}\right)\delta\phi^i = \left(\partial\phi^i\right)\epsilon^{ij}\phi^j$$

ϕ^i has N components and ϵ is an $N \times N$ antisymmetric matrix. Such a matrix has $\frac{1}{2}N(N-1)$ independent components, therefore there are as many Noether currents.

(b) First, consider the theory with $m^2 > 0$. Write down the Feynman rules for this theory. In particular, show that there is one type of vertex, which is



$$= -2i\lambda\left(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}\right)$$

This follows from the form of the interaction term in the Lagrangian: $-\frac{\lambda}{4}\left(\left(\phi^i\right)^2\right)$
If there are 4 of the same field, $\left(\phi^1\phi^1\right)\left(\phi^1\phi^1\right)$, there are 4! ways of arranging these fields on the vertex. Multiplying this with the coefficient $-\frac{\lambda}{4}$ and the usual factor of i gives $-6i\lambda$, in agreement wih the given expression when $i = j = k = l$.

If the interaction was $\left(\phi^1\phi^1\right)\left(\phi^2\phi^2\right)$, there is a factor of 2 for exchanging the two ϕ^1's, another factor of 2 for exchanging the two ϕ^2's and another factor of 2 for exchanging the ϕ^1's with ϕ^2's. This gives a vertex of $-\frac{\lambda}{4} \times 8 \times i = -2i\lambda$, which is also in agreement with the given expression if any 2 pairs of indices are equal.

This exhausts the possible interactions and, as demonstrated, the correct vertex is given by the expression: $-2i\lambda\left(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right)$.
Also, the usual Feynman rules of conservation of momentum, external legs = 1 and integrating over undetermined momenta apply to this interaction.

(c) Now consider the theory with $m^2 < 0$, i.e. we write $m^2 \equiv -\mu^2$, where $\mu^2 > 0$. Now the point in the potential V with $\phi^i = 0$ is a local maximum rather than a minimum, and this the equilibrium value of ϕ^i will now have a nonzero value that minimizes the potential. By an $SO(N)$ rotation we can choose this nonzero value to point in the $i = N$ direction: thus we single out the $i = N$ component of ϕ^i and write the theory in terms of new fields σ and π^i:

$$\phi^N(x) = v + \sigma(x) \tag{189}$$

$$\phi^i(x) = \pi^i(x) \qquad i \in i \cdots N-1 \tag{190}$$

where v is a constant that is picked to minimize V. What is the value of v? What are the masses of the π^i and σ fields?

$$\begin{aligned} V =& \frac{\mu^2}{2}\left(\phi^i\right)^2 - \frac{\lambda}{4}\left(\left(\phi^i\right)^2\right)^2 \\ \frac{\partial V}{\partial((\phi^i)^2)} = 0 =& \frac{\mu^2}{2} - \frac{\lambda}{2}\left(\phi^i_0\right)^2 \\ \left(\phi^i_0\right)^2 =& \frac{\mu^2}{\lambda} \\ \left(\phi^i_0\right) =& \boxed{v = \frac{\mu}{\sqrt{\lambda}}} \end{aligned}$$

The Lagrangian transforms as follows under the shift $\left(\phi^i\right) \rightarrow \left(\pi^k \quad v+\sigma(x)\right)$,k=1,$\cdots$, $N-1$

$$\begin{aligned} \mathcal{L} =& \tfrac{1}{2}\left(\partial\phi^i\right)^2 + \frac{\mu^2}{2}\left(\phi^i\right)^2 - \frac{\lambda}{4}\left(\left(\phi^i\right)^2\right)^2 \\ \rightarrow& \frac{1}{2}\begin{pmatrix}\partial\pi^k \\ \partial\sigma\end{pmatrix}\cdot\left(\partial\pi^k \quad \partial\sigma\right) + \frac{\mu^2}{2}\begin{pmatrix}\pi^k \\ v+\sigma\end{pmatrix}\cdot\left(\pi^k \quad v+\sigma\right) - \frac{\lambda}{4}\left(\begin{pmatrix}\pi^k \\ v+\sigma\end{pmatrix}\cdot\left(\pi^k \quad v+\sigma\right)\right)^2 \\ =& \tfrac{1}{2}\left(\partial\pi^k\right)^2 + \tfrac{1}{2}\left(\partial\sigma^k\right)^2 + \frac{\mu^2}{2}\left(\left(\pi^k\right)^2 + (v+\sigma)^2\right) - \frac{\lambda}{4}\left(\left(\pi^k\right)^2 + (v+\sigma)^2\right)^2 \\ =& \tfrac{1}{2}\left(\partial\pi^k\right)^2 + \tfrac{1}{2}\left(\partial\sigma^k\right)^2 + \frac{\mu^2}{2}\left(\left(\pi^k\right)^2 + v^2 + \sigma^2 + 2v\sigma\right) - \frac{\lambda}{4}\left(\left(\pi^k\right)^4 + (v+\sigma)^4 + 2\left(\pi^k\right)^2(v+\sigma)^2\right) \\ =& \tfrac{1}{2}\left(\partial\pi^k\right)^2 + \tfrac{1}{2}\left(\partial\sigma^k\right)^2 + \frac{\mu^2}{2}\left(\left(\pi^k\right)^2 + v^2 + \sigma^2 + 2v\sigma\right) \\ & - \frac{\lambda}{4}\left(\left(\pi^k\right)^4 + v^4 + 4v^3\sigma + 6v^2\sigma^2 + 4v\sigma^3 + \sigma^4 + 2\left(\pi^k\right)^2\left(v^2+\sigma^2+2v\sigma\right)\right) \\ =& \tfrac{1}{2}\left(\partial\pi^k\right)^2 + \tfrac{1}{2}\left(\partial\sigma^k\right)^2 + \frac{\mu^2}{2}\left(\pi^k\right)^2 + \frac{\mu^2 v^2}{2} + \frac{\mu^2}{2}\sigma^2 + \mu^2 v\sigma \\ & - \frac{\lambda}{4}\left(\pi^k\right)^4 - \frac{\lambda v^4}{4} - \lambda v^3\sigma - \frac{3\lambda v^2}{2}\sigma^2 - \lambda v\sigma^3 - \frac{\lambda}{4}\sigma^4 - \frac{\lambda v^2}{2}\left(\pi^k\right)^2 - \frac{\lambda}{2}\left(\pi^k\right)^2\sigma^2 - \lambda v\left(\pi^k\right)^2\sigma \\ =& \tfrac{1}{2}\left(\partial\pi^k\right)^2 + \tfrac{1}{2}\left(\partial\sigma^k\right)^2 + \cancel{\frac{\mu^2}{2}\left(\pi^k\right)^2} + \frac{\mu^4}{2\lambda} + \frac{\mu^2}{2}\sigma^2 + \cancel{\frac{\mu^3}{\sqrt{\lambda}}\sigma} \\ & - \frac{\lambda}{4}\left(\pi^k\right)^4 - \frac{\mu^4}{4\lambda} - \cancel{\frac{\mu^3}{\sqrt{\lambda}}\sigma} - \frac{3}{2}\mu^2\sigma^2 - \sqrt{\lambda}\mu\sigma^3 - \frac{\lambda}{4}\sigma^4 - \cancel{\frac{\mu^2}{2}\left(\pi^k\right)^2} - \frac{\lambda}{2}\left(\pi^k\right)^2\sigma^2 - \mu\sqrt{\lambda}\left(\pi^k\right)^2\sigma \\ =& \tfrac{1}{2}\left(\partial\pi^k\right)^2 + \tfrac{1}{2}\left(\partial\sigma^k\right)^2 + \frac{\mu^4}{4\lambda} - \underbrace{\mu^2\sigma^2} - \frac{\lambda}{4}\left(\pi^k\right)^4 - \sqrt{\lambda}\mu\sigma^3 - \frac{\lambda}{4}\sigma^4 - \frac{\lambda}{2}\left(\pi^k\right)^2\sigma^2 - \mu\sqrt{\lambda}\left(\pi^k\right)^2\sigma \end{aligned}$$

Comparing this $-\frac{1}{2}m_\sigma^2\sigma^2$ gives the mass of the σ field:

$$\boxed{m_\sigma = \sqrt{2}\mu}$$

The lack of a term proportional to $\left(\pi^k\right)^2 \implies$ that there are $N-1$ massless π fields.

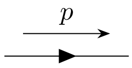
$$\boxed{m_\pi = 0}$$

(d)

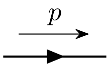
(i) Work out the self interactions of the σ and π^i fields and write down the Feynman rules (i.e. the new propagators and vertices) in terms of these fields.

(ii) Calculate (to order λ) the connected part of the matrix element $i\mathcal{M}$ for the scattering of $\pi^1\pi^1 \to \pi^2\pi^2$, where the incoming particles have momenta p_1, k_1 and the outgoing particles have momenta p_2, k_2. Express all your answers in terms of the parameters in the Lagrangian μ and λ.

(i) Propagators

σ 

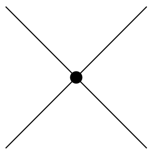
$= \dfrac{i}{p^2 - 2\mu^2}$

π^i 

π^j $= \dfrac{i}{p^2 - 2\mu^2}$

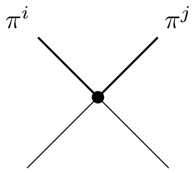
The amplitude associated with a particular vertex is given by the coefficient of the corresponding term in the Lagrangian, multiplied by i, multiplied by the symmetry factor associated with its Feynman diagram.

σ^4 :



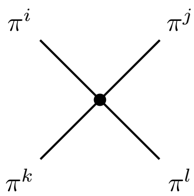
$$= -\frac{\lambda}{3} \times i \times 4! = -6i\lambda$$

$\sigma^2\left(\pi^i\right)^2$:



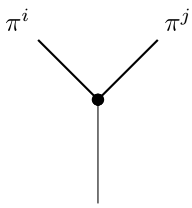
$$= -\frac{\lambda}{2} \times i \times 4 \times \delta^{ij} = -2i\lambda\delta^{ij}$$

From (b):



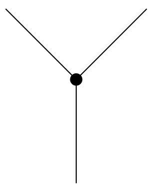
$$= -2i\lambda\left(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}\right)$$

$\sigma\left(\pi^i\right)^2$



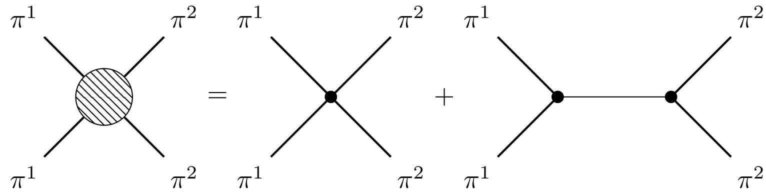
$$= -\lambda v \times i \times 2 \times \delta^{ij} = -2i\sqrt{\lambda}\mu\delta^{ij}$$

σ^3 :



$$= -\lambda v \times i \times 3! = -6i\sqrt{\lambda}\mu$$

(ii) Defining $p_1 + k_1 = p_2 + k_2 \equiv p$:

$i\mathcal{M} =$ 

$+\mathcal{O}\left(\lambda^3\right)$

$$\begin{aligned}
&= -2i\lambda + \left(-2i\sqrt{\lambda}\mu\right)\frac{i}{p^2 - 2\mu^2}\left(-2i\sqrt{\lambda}\mu\right) \\
&= -2i\lambda - \frac{4i\lambda\mu^2}{p^2 - 2\mu^2} \\
&= -2i\lambda\left(1 + \frac{2\mu^2}{p^2 - 2\mu^2}\right) \\
&= -2i\lambda\left(\frac{p^2}{p^2 - 2\mu^2}\right) \\
&= \frac{2i\lambda p^2}{2\mu^2 - p^2}
\end{aligned}$$

---

4 Lie Algebra

Q1

(a)

(i) Decompose the tensor product of $SU(3)$ reps $3 \otimes 3 \otimes 3$ into irreducibles representations using the Littlewood-Richardson rules

(ii) Check that the dimensions of the irreducible representations add up correctly

(iii) How can you write this decompisition using the notation (p, q) introduced to label the irreducible representations of $SU(3)$.

(i)

$$\begin{ytableau} \; \end{ytableau} \otimes \begin{ytableau} \; \end{ytableau} \otimes \begin{ytableau} \; \end{ytableau} = \left(\begin{ytableau} \; \end{ytableau} \otimes \begin{ytableau} a \end{ytableau} \right) \otimes \begin{ytableau} \; \end{ytableau} \tag{191}$$

$$= \left(\begin{ytableau} \; & a \end{ytableau} \oplus \begin{ytableau} \; \\ a \end{ytableau} \right) \otimes \begin{ytableau} \; \end{ytableau} \tag{192}$$

$$= \left(\begin{ytableau} \; & \; \end{ytableau} \oplus \begin{ytableau} \; \\ \; \end{ytableau} \right) \otimes \begin{ytableau} a \end{ytableau} \tag{193}$$

$$= \begin{ytableau} \; & \; & a \end{ytableau} \oplus \begin{ytableau} \; & \; \\ a \end{ytableau} \oplus \begin{ytableau} \; & a \\ \; \end{ytableau} \oplus \begin{ytableau} \; \\ \; \\ a \end{ytableau} \tag{194}$$

$$= \begin{ytableau} \; & \; & a \end{ytableau} \oplus \begin{ytableau} \; & \; \\ a \end{ytableau} \oplus \begin{ytableau} \; & a \\ \; \end{ytableau} \oplus \mathbb{1} \tag{195}$$

(ii)

$$\dim \left(\begin{ytableau} \; & \; & a \end{ytableau} \oplus \begin{ytableau} \; & \; \\ a \end{ytableau} \oplus \begin{ytableau} \; & a \\ \; \end{ytableau} \oplus \mathbb{1} \right)$$

$$= \dim \left(\begin{ytableau} \; & \; & \; \end{ytableau} \right) + 2 \dim \left(\begin{ytableau} \; & \; \\ \; \end{ytableau} \right) + \dim (\mathbb{1})$$

$$= 10 + 2 \times 8 + 1 = 27$$

$$\left(\begin{array}{l} \dim \left(\begin{ytableau} \; & \; & \; \end{ytableau} \right) = \dfrac{\begin{ytableau} 3 & 4 & 5 \end{ytableau}}{\begin{ytableau} 3 & 2 & 1 \end{ytableau}} = 10 \\[2em] \dim \left(\begin{ytableau} \; & \; \\ \; \end{ytableau} \right) = \dfrac{\begin{ytableau} 3 & 4 \\ 2 \end{ytableau}}{\begin{ytableau} 3 & 1 \\ 1 \end{ytableau}} = 8 \end{array} \right)$$

Which is what we expect as $\dim (3 \otimes 3 \otimes 3) = 3 \times 3 \times 3 = 27$

(iii) The irreducible representations can be labelled as follows

$$\begin{ytableau} \; & \; & \; \end{ytableau} : (3, 0) \tag{196}$$

$$\begin{ytableau} \; & \; \\ \; \end{ytableau} : (1, 1) \tag{197}$$

$$\mathbb{1} : (0, 0) \tag{198}$$

(b)

(i) Find the tensor product of $\mathbf{6} \otimes \mathbf{6}$ in $SU(4)$ using the Littlewood-Richardson rules

(ii) Check that the dimensions of the irreducible representations add up correctly

---

(i)

$$\ydiagram{1,1} \otimes \ydiagram{1,1} = \ydiagram{1,1} \otimes \ytableaushort{a,b} \tag{199}$$

$$= \left(\ytableaushort{{}a,{}} \oplus \ytableaushort{{},{},a} \right) \otimes \ytableaushort{b} \tag{200}$$

$$= \cancel{\ytableaushort{{}ab,{}}} \oplus \ytableaushort{{}a,{}b} \oplus \ytableaushort{{}a,{},b} \oplus \cancel{\ytableaushort{{}b,{},a}} \oplus \ytableaushort{{},{},a,b} \tag{201}$$

$$= \ytableaushort{{}a,{}b} \oplus \ytableaushort{{}a,{},b} \oplus \ytableaushort{{},{},a,b} \tag{202}$$

$$= \ytableaushort{{}a,{}b} \oplus \ytableaushort{{}a,{},b} \oplus \mathbb{1} \tag{203}$$

(ii)

$$\begin{aligned}
&\dim\left(\ytableaushort{{}a,{}b} \oplus \ytableaushort{{}a,{},b} \oplus \mathbb{1} \right) \\
&= \dim\left(\ydiagram{2,2}\right) + \dim\left(\ydiagram{2,1,1}\right) + \dim\left(\mathbb{1}\right) \\
&= 20 + 15 + 1 \\
&= 36
\end{aligned}$$

$$\left(\begin{aligned}
&\dim\left(\ydiagram{2,2}\right) = \frac{\ytableaushort{45,34}}{\ytableaushort{32,21}} = 20 \\
&\dim\left(\ydiagram{2,1,1}\right) = \frac{\ytableaushort{45,3,2}}{\ytableaushort{41,2,1}} = 15
\end{aligned} \right)$$

Which is what we expect as $\dim(6 \otimes 6) = 6 \times 6 = 36$

(c) The Lie algebras $SU(4)$ and $SO(6)$ are isomorphic. The **6** representation of $SU(4)$ is the fundamental representation of $SO(6)$ with representation space ϕ^i where $i \in 1, \cdots, 6$.

(i) Perform the decomposition of $\mathbf{6} \otimes \mathbf{6}$ above from the point of view of $SO(6)$ explicitly using the indices: $\phi^i \psi^j$.

(ii) Explain how the dimensions of the irreducible representations agree with the decomposition in question (b).

(i)

$$\varphi^i\psi^j = \underbrace{\frac{\varphi^i\pi^j+\varphi^j\psi^i}{2}-\frac{\delta^{ij}\varphi^k\psi^k}{N}}_{\substack{\text{Symmetric matrix has } \frac{N(N+1)}{2} \text{ constraints, but}\\ \text{it's also traceless so 1 trace element is fixed}\\ \implies \text{1 less constraint}}} + \underbrace{\frac{1}{N}\left(\delta^{ij}\varphi^k\psi^k\right)}_{\text{Trace trivial, 1 constraint}} + \underbrace{\frac{1}{2}\left(\varphi^i\psi^j-\varphi^j\psi^i\right)}_{\text{Anti-summetric matrix, } \frac{N(N-1)}{2} \text{ constraints.}} \quad (204),\ (205)$$

$$\therefore \dim\left(\varphi^i\psi^j\right) = \frac{N(N+1)}{2}-1 \quad + \quad 1 \quad + \quad \frac{N(N-1)}{2}$$

$$= 20 \quad + \quad 1 \quad + \quad 15 \quad = 36$$

$$\ydiagram{2,2} \qquad \mathbb{1} \qquad \ydiagram{2,1,1} \tag{206}$$

(ii) Line (206) shows how each of the three parts of this decomposition corresponds to the Young-Tableux calculated in question (b). This is because the **6** representation of SU(4) is the fundamental representation of SO(6):

$$\ydiagram{1,1} \to \phi^i$$

$$\ydiagram{1,1} \otimes \ydiagram{1,1} \to \phi^i\psi^j$$

So we expect the same irreducible representations.

(d) Mesons are particles composed of one quark and one anti-quark. The quark transforms as the **3** of a suitable $SU(3)$ symmetry group while the anti-quark transforms as the $\bar{\mathbf{3}}$ of the same $SU(3)$ group. Using Young Tableaux express the mesons as a sum of irreducible representations add up correctly.

Finding the tensor product of $\bar{\mathbf{3}} \otimes \mathbf{3}$ in SU(3)

$$\begin{aligned}\bar{\mathbf{3}} \otimes \mathbf{3} &= \ydiagram{1,1} \otimes \ydiagram{1} \\ &= \ydiagram{2,1} \oplus \ydiagram{1,1,1} \\ &= \ydiagram{2,1} \oplus \mathbb{1}\end{aligned}$$

We already showed in question (b) that $\dim\left(\ydiagram{2,1}\right) = 8$, so the dimension is

$$\dim\left(\ydiagram{2,1} \oplus \mathbb{1}\right) = \dim\left(\ydiagram{2,1}\right) + \dim(1) = 8+1 = 9$$

The 8 dimensional irreducible representation corresponds to an octet of degenerate energy eigenstates with the same mass: the meson octet. The 1 dimensional irreducible representation is a singlet and corresponds to a different energy eigenstate.

(e) Let ϕ^i, $i \in \{1,2,3\}$, transform in the fundamental representation of $SU(3)$ and consider the subgroup H of $SU(3)$ defined by the matrices of the form

$$H = \left\{ \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} \text{with } U \in SU(2) \right\} \subset SU(3) \tag{207}$$

(i) Check that H is indeed a subgroup of $SU(3)$

(ii) Decompose ϕ^i into irreducible representations of H

(i) H trivially satisfies the definition of SU(3), $H = H^\dagger$ and $|H| = 1$

Conditions for H being a subgroup of SU(3):

Identity element: $\begin{pmatrix} \mathbb{1} & 0 \\ 0 & 1 \end{pmatrix}$

Closure: $H_1 H_2 = \begin{pmatrix} \mathbf{U}_1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{U}_2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{U}_1\mathbf{U}_2 & 0 \\ 0 & 1 \end{pmatrix}$

$\mathbf{U}_1\mathbf{U}_2 \in SU(N) \therefore H_1 H_2 \in H$

Inverse: $H^\dagger H = \begin{pmatrix} \mathbf{U}^\dagger & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{U} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{U}^\dagger\mathbf{U} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 1 \end{pmatrix}$

$\therefore H^{-1} = H^\dagger \in H$

Associativity: Elements are matrices and matrix multiplication is known to be associative

(ii)

$$\begin{pmatrix} \mathbf{U} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi^{(1,2)} \\ \phi^3 \end{pmatrix} = \begin{pmatrix} \mathbf{U}\phi^{(1,2)} \\ \phi^3 \end{pmatrix} \qquad \text{where } \phi^{(1,2)} = \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix}$$

---

5 Standard Model

Q1 Consider a model in which the Standard Model is extended by the addition of a real scalar $SU(2)_L$ Higgs triplet with zero weak hypercharge, in addition to the usual complex scalar $SU(2)_L$ Higgs doublet with weak hypercharge $Y_W = 1$.

The kinetic term for the Higgs fields is

$$\mathcal{L}^{\text{kinetic}}_{\text{Higgs}} = \left(D^\mu \Phi\right)^\dagger D_\mu \Phi + \tfrac{1}{2}\left(D^\mu H\right)^\dagger D_\mu H \tag{208}$$

where Φ is the $SU(2)_L$ doublet field and H the $SU(2)_L$ triplet field. The covariant derivative is

$$D^\mu = \partial^\mu - \tfrac{1}{2} i g_1 B^\mu - i g_2 t^i W^{i\mu} \tag{209}$$

where B^μ is the $U(1)_Y$ gauge boson, g_1 is the hypercharge coupling, g_2 is the weak coupling, $W^{i\mu}$ are the $SU(2)_L$ gauge bosons and t^i the generators of $SU(2)_L$ in the appropriate representation.
The $W^{3\mu}$ and B^μ field mix

$$B^\mu = -\sin\theta_W Z^\mu + \cos\theta_W A^\mu \tag{210}$$

$$W^{3\mu} = \cos\theta_W Z^\mu + \sin\theta_W A^\mu \tag{211}$$

to give the Z^0 boson, Z^μ, and photon, A^μ, where θ_W is the weak mixing angle.
After spontaneous symmetry breaking the Higgs doublets can be expanded around the minimum of the potential giving

$$\Phi = \begin{pmatrix} \phi^+ \\ \frac{1}{\sqrt{2}}\left(b + h^0 + i\phi^0\right) \end{pmatrix} \qquad H = \begin{pmatrix} \eta^+ \\ \frac{1}{2} v \tan\beta + \eta^0 \\ \eta^- \end{pmatrix} \tag{212}$$

where ϕ^+ and ϕ^0 are the Goldstone bosons eaten by the $W^\pm$ and Z^0 bosons, v is the doublet vacuum expectation value, β describes the mixing between the doublet and triplet vacuum expectation values, and h^0 and $\eta^{0,+,-}$ are scalar fields.

(a) Calculate the following:

(i) The masses of the $W^\pm$ and Z bosons

(ii) The relationship between g_1 and g_2

(iii) The Feynman rules for the couplings of the scalar h^0 and η^0 bosons to $W^= W^-$ and $Z^0 Z^0$.

(i)

$$\begin{aligned}\mathcal{L}^{\text{kinetic}}_{\text{Higgs}} &= \left(D^\mu \Phi\right)^\dagger D_\mu \Phi + \frac{1}{2}\left(D^\mu H\right)^\dagger D^\mu H \\ &= \left|D^\mu \Phi\right|^2 + \frac{1}{2}\left|D^\mu H\right|^2\end{aligned}$$

Calculating the doublet term first:

$$
\begin{aligned}
&\left|D^\mu \Phi\right|^2 \\
&= \left|\left(\partial^\mu - i g_2 t^i W^{i\mu} - i\frac{g_1}{2}B^\mu\right)\begin{pmatrix}\phi^+ \\ \frac{1}{\sqrt{2}}\left(v + h^0 + i\phi^0\right)\end{pmatrix}\right|^2 \\
&\ni \left|\left(-i g_2 t^i W^{i\mu} - i\frac{g_1}{2}B^\mu\right)\begin{pmatrix}0 \\ \frac{1}{\sqrt{2}}v\end{pmatrix}\right|^2
\end{aligned}
$$

Where we dropped the ∂^μ as it doesn't contribute to the $W^{+\mu}W^-_\mu$ term or the $Z^\mu Z_\mu$ term. We also choose Φ as the vacuum such that $\phi^+ = \phi^0 = 0$.

$$
\begin{aligned}
&= \tfrac{1}{2}v^2\left|\left(g_2 t^i W^{i\mu} + \frac{g_1}{2}B^\mu\right)\begin{pmatrix}0\\1\end{pmatrix}\right|^2 \\
&= \tfrac{1}{2}v^2\left|\left(g_2 t^1 W^{1\mu} + g_2 t^2 W^{2\mu} + g_2 t^3 W^{3\mu} + \frac{g_1}{2}B^\mu\right)\begin{pmatrix}0\\1\end{pmatrix}\right|^2 \\
&= \tfrac{1}{2}v^2\left|\left(\frac{1}{2}\begin{pmatrix}0 & g_2 W^{1\mu}\\ g_2 W^{1\mu} & 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0 & -i g_2 W^{2\mu}\\ i g_2 W^{2\mu} & 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}g_2 W^{3\mu} & 0\\ 0 & -g_2 W^{3\mu}\end{pmatrix} + \begin{pmatrix}\frac{1}{2}g_1 B^\mu & 0\\ 0 & \frac{1}{2}g_1 B^\mu\end{pmatrix}\right)\begin{pmatrix}0\\1\end{pmatrix}\right|^2 \\
&= \tfrac{1}{8}v^2\left|\left(\begin{pmatrix}0 & g_2 W^{1\mu}\\ g_2 W^{1\mu} & 0\end{pmatrix} + \begin{pmatrix}0 & -i g_2 W^{2\mu}\\ i g_2 W^{2\mu} & 0\end{pmatrix} + \begin{pmatrix}g_2 W^{3\mu} & 0\\ 0 & -g_2 W^{3\mu}\end{pmatrix} + \begin{pmatrix}g_1 B^\mu & 0\\ 0 & g_1 B^\mu\end{pmatrix}\right)\begin{pmatrix}0\\1\end{pmatrix}\right|^2 \\
&= \tfrac{1}{8}v^2\left|\begin{pmatrix}g_1 B^\mu + g_2 W^{3\mu} & g_2\left(W^{1\mu} - i W^{2\mu}\right)\\ g_2\left(W^{1\mu} + i W^{2\mu}\right) & g_1 B^\mu - g_2 W^{3\mu}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right|^2
\end{aligned}
$$

$$
= \tfrac{1}{8}v^2\left|\begin{pmatrix}g_1 B^\mu + g_2 W^{3\mu} & W^{+\mu} g_2\sqrt{2}\\ W^{-\mu} g_2 \sqrt{2} & -Z^\mu\sqrt{g_1^2 + g_2^2}\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right|^2 \quad \leftarrow \quad
\begin{aligned}
W^{\pm\mu} &\equiv \frac{W^{1\mu} \mp i W^{2\mu}}{\sqrt{2}} \\
Z^\mu &\equiv \frac{-g_1 B^\mu + g_2 W^{3\mu}}{\sqrt{g_1^2 + g_2^2}} \\
A^\mu &\equiv \frac{g_2 B^\mu + g_1 W^{3\mu}}{\sqrt{g_1^2 + g_2^2}}
\end{aligned}
$$

$$
\begin{aligned}
&= \tfrac{1}{8}v^2\left|\begin{pmatrix}W^{+\mu} g_2\sqrt{2}\\ -Z^\mu\sqrt{g_1^2 + g_2^2}\end{pmatrix}\right|^2 \\
&= \tfrac{1}{4}v^2 g_2^2 W^-_\mu W^{+\mu} + \tfrac{1}{8}v^2\left(g_1^2 + g_2^2\right) Z_\mu Z^\mu
\end{aligned}
$$

Now calculating the triplet term:

$$= \tfrac{1}{2}\left|D^\mu H\right|^2$$

$$= \tfrac{1}{2}\left|\left(\partial^\mu - ig_2 t^i W^{i\mu} - i\frac{g_1}{2}B^\mu\right)\begin{pmatrix}\eta^+\\ \tfrac{1}{2}v\tan\beta+\eta^0\\ \eta^-\end{pmatrix}\right|^2$$

$$= \tfrac{1}{2}\left|-ig_2 t^i W^{i\mu}\begin{pmatrix}0\\ \tfrac{1}{2}v\tan\beta\\ 0\end{pmatrix}\right|^2$$

Where we dropped terms as for the previous calculation. We also drop B^μ as the Higgs triplet field has zero weak hypercharge and is not acted on by B^μ.

$$= \tfrac{1}{8}v^2 g_2^2\tan^2\beta\left|t^i W^{i\mu}\begin{pmatrix}0\\1\\0\end{pmatrix}\right|^2$$

$$= \tfrac{1}{8}v^2 g_2^2\tan^2\beta\left|\left(t^1W^{1\mu}+t^2W^{2\mu}+t^3W^{3\mu}\right)\begin{pmatrix}0\\1\\0\end{pmatrix}\right|^2$$

$$= \tfrac{1}{8}v^2 g_2^2\tan^2\beta\left|\left(\frac{1}{\sqrt{2}}\begin{pmatrix}0 & W^{1\mu} & 0\\ W^{1\mu} & 0 & W^{1\mu}\\ 0 & W^{1\mu} & 0\end{pmatrix}+\frac{1}{\sqrt{2}}\begin{pmatrix}0 & -iW^{2\mu} & 0\\ iW^{2\mu} & 0 & -iW^{2\mu}\\ 0 & iW^{2\mu} & 0\end{pmatrix}+\begin{pmatrix}W^{3\mu} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -W^{3\mu}\end{pmatrix}\right)\begin{pmatrix}0\\1\\0\end{pmatrix}\right|^2$$

$$= \tfrac{1}{8}v^2 g_2^2\tan^2\beta\left|\begin{pmatrix}W^{3\mu} & \dfrac{W^{1\mu}-iW^{2\mu}}{\sqrt{2}} & 0\\ \dfrac{W^{1\mu}+iW^{2\mu}}{\sqrt{2}} & 0 & \dfrac{W^{1\mu}-iW^{2\mu}}{\sqrt{2}}\\ 0 & \dfrac{W^{1\mu}+iW^{2\mu}}{\sqrt{2}} & W^{3\mu}\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\right|^2$$

$$= \tfrac{1}{8}v^2 g_2^2\tan^2\beta\left|\begin{pmatrix}W^{3\mu} & W^{+\mu} & 0\\ W^{-\mu} & 0 & W^{+\mu}\\ 0 & W^{-\mu} & W^{3\mu}\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\right|^2$$

$$= \tfrac{1}{8}v^2 g_2^2\tan^2\beta\left|\begin{pmatrix}W^{+\mu}\\ 0\\ W^{-\mu}\end{pmatrix}\right|^2$$

$$= \tfrac{1}{8}v^2 g_2^2\tan^2\beta\left(W^-_\mu W^{+\mu}+W^+_\mu W^{-\mu}\right)$$

$$= \tfrac{1}{4}v^2 g_2^2\tan^2\beta\; W^-_\mu W^{+\mu}$$

Therefore the Lagrangian is:

$$\begin{aligned}
\mathcal{L}^{\text{kinetic}}_{\text{Higgs}} &= \left|D^\mu\Phi\right|^2+\frac{1}{2}\left|D^\mu H\right|^2\\
&= \tfrac{1}{4}v^2g_2^2W^-_\mu W^{+\mu}+\tfrac{1}{8}v^2\left(g_1^2+g_2^2\right)Z_\mu Z^\mu+\tfrac{1}{4}v^2g_2^2\tan^2\beta\;W^-_\mu W^{+\mu}\\
&= \tfrac{1}{4}v^2g_2^2\left(\tan^2\beta+1\right)W^-_\mu W^{+\mu}+\tfrac{1}{8}v^2\left(g_1^2+g_2^2\right)Z_\mu Z^\mu\\
&= \tfrac{1}{4}v^2g_2^2\sec^2\beta\;W^-_\mu W^{+\mu}+\tfrac{1}{8}v^2\left(g_1^2+g_2^2\right)Z_\mu Z^\mu\\
&= \tfrac{1}{8}v^2g_2^2\sec^2\beta\left|W^+\right|^2+\tfrac{1}{8}v^2g_2^2\sec^2\beta\left|W^-\right|^2+\tfrac{1}{8}v^2\left(g_1^2+g_2^2\right)\left|Z\right|^2\\
&\equiv \tfrac{1}{2}M^2_{W^+}\left|W^+\right|^2+\tfrac{1}{2}M^2_{W^-}\left|W^-\right|^2+\tfrac{1}{2}M_Z^2\left|Z\right|^2\\
&\implies \boxed{\begin{aligned}M_{W^\pm}&=\tfrac{1}{2}vg_2\sec\beta\\ M_Z&=\tfrac{1}{2}v\sqrt{g_1^2+g_2^2}\end{aligned}}
\end{aligned}$$

(ii)

$$
\begin{aligned}
\begin{pmatrix} B^\mu \\ W^{3\mu} \end{pmatrix} &= \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} A^\mu \\ Z^\mu \end{pmatrix} \\
\begin{pmatrix} A^\mu \\ Z^\mu \end{pmatrix} &= \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix}^{-1} \begin{pmatrix} B^\mu \\ W^{3\mu} \end{pmatrix} \\
\begin{pmatrix} A^\mu \\ Z^\mu \end{pmatrix} &= \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} B^\mu \\ W^{3\mu} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
A^\mu &= \cos\theta_W B^\mu + \sin\theta_W W^{3\mu} \\
Z^\mu &= -\sin\theta_W B^\mu + \cos\theta_W W^{3\mu}
\end{aligned}
$$

Compare with:

$$
\begin{aligned}
A^\mu &\equiv \frac{g_2 B^\mu + g_1 W^{3\mu}}{\sqrt{g_1^2 + g_2^2}} \\
Z^\mu &\equiv \frac{-g_1 B^\mu + g_2 W^{3\mu}}{\sqrt{g_1^2 + g_2^2}}
\end{aligned}
$$

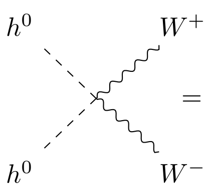
$$
\implies \begin{aligned}
\frac{g_1}{\sqrt{g_1^2 + g_2^2}} &= \sin\theta_W \\
\frac{g_2}{\sqrt{g_1^2 + g_2^2}} &= \cos\theta_W
\end{aligned}
$$

$$
g_1^2 + g_2^2 = 1
$$

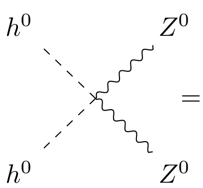
(iii)

$$
\begin{aligned}
\mathcal{L} =& \tfrac{1}{4} v^2 g_2^2 W_\mu^- W^{+\mu} + \tfrac{1}{8} v^2 \left(g_1^2 + g_2^2\right) Z_\mu Z^\mu + \tfrac{1}{4} v^2 g_2^2 \tan^2\beta\ W_\mu^- W^{+\mu} \\
\to& \tfrac{1}{4} \left(v + h^0\right)^2 g_2^2 W_\mu^- W^{+\mu} + \tfrac{1}{8} \left(v + h^0\right)^2 \left(g_1^2 + g_2^2\right) Z_\mu Z^\mu + g_2^2 \left(\tfrac{1}{2} v \tan\beta + \eta^0\right)^2 W_\mu^- W^{+\mu}
\end{aligned}
$$

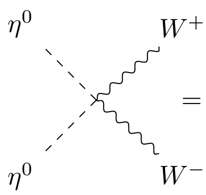
$$
\begin{aligned}
\ni& \tfrac{1}{4} \left(h^0\right)^2 g_2^2 W_\mu^- W^{+\mu} + \tfrac{1}{2} v h^0 g_2^2 W_\mu^- W^{+\mu} + \tfrac{1}{8} \left(h^0\right)^2 \left(g_1^2 + g_2^2\right) Z_\mu Z^\mu + \tfrac{1}{4} v h^0 \left(g_1^2 + g_2^2\right) Z_\mu Z^\mu \\
&+ g_2^2 \left(\eta^0\right)^2 W_\mu^- W^{+\mu} + g_2^2 v \eta^0 \tan\beta W_\mu^- W^{+\mu}
\end{aligned}
$$



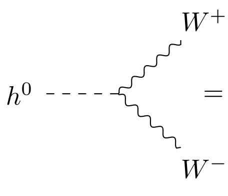
$= \frac{1}{4} i g_2^2$



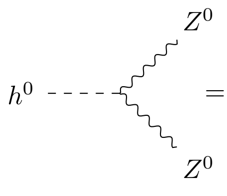
$= \frac{1}{8} i \left(g_1^2 + g_2^2\right)$



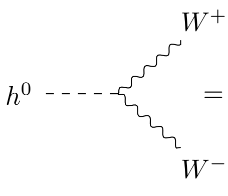
$= i g_2^2$



$= \frac{1}{2} i v g_2^2$



$= \frac{1}{4} i v \left(g_1^2 + g_2^2\right)$



$= i v g_2^2 \tan\beta$

(b) Calculate the ρ parameterx

$$
\rho = \frac{M_W^2}{M_z^2 \cos^2\theta_W} \tag{213}
$$

where M_W and M_Z are the masses of the $W^\pm$ and Z^0 bosons. Comment on the implications of this result for the value of β.

The ρ-parameter can be shown from QED, and is confirmed by experiment to be (within a good approximation) equal to 1.

$$
\begin{aligned}
\rho = 1 &= \frac{M_W^2}{M_Z^2}\frac{1}{\cos^2\theta_W} \\
1 &= \frac{\left(\frac{1}{2}vg_2\sec\beta\right)^2}{\left(\frac{1}{2}v\sqrt{g_1^2+g_2^2}\right)^2}\frac{1}{\cos^2\theta_W} \\
1 &= \frac{(g_2\sec\beta)^2}{(g_1^2+g_2^2)}\frac{1}{\cos^2\theta_W} \\
\cos^2\beta &= \frac{g_2^2}{(g_1^2+g_2^2)}\frac{1}{\cos^2\theta_W} \\
\cos^2\beta &= \cos^2\theta_W\frac{1}{\cos^2\theta_W} \\
y\cos^2\beta &= 1 \\
\beta &= 0
\end{aligned}
$$

This implies that the vacuum expectation value for the triplet is negligible.

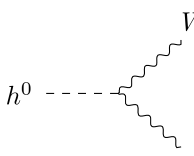
(c) For a two-body decay to two equal mass particles the rate is

$$
\Gamma(a \to b + c) = \frac{1}{16\pi m_a}\sqrt{1-\frac{4m_b^2}{m_a^2}}\,\left|\bar{M}\right|^2 \tag{214}
$$

where m_a is the mass of the decaying particle, m_b is the mass of one of the decay products and $\left|\bar{M}\right|^2$ is the spin-averaged matrix element squared. Compute the decay rates for:

(i) $h^0 \to W^+W^-$

(ii) $\eta^0 \to W^+W^-$



$= ivg_2^2 = ig_2 M_W \cos\beta$

(i)

Calculating the spin-averaged matrix element by summing the amplitude over the polarization vectors, ϵ, of W.

$$
\begin{aligned}
|\mathcal{M}|^2 &= g_2^2 M_W^2 \cos^2\beta \sum \left(\epsilon^\mu \epsilon^{*\nu}\right)\left(\epsilon_\mu \epsilon_\nu^*\right) \\
&= g_2^2 M_W^2 \cos^2\beta \sum \left(\epsilon^\mu \epsilon^{*\nu}\right)\left(\epsilon_\mu \epsilon_\nu^*\right) \\
&= g_2^2 M_W^2 \cos^2\beta \left(-g^{\mu\nu} + \frac{p^\mu_{W^+} p^\nu_{W^+}}{M_W^2}\right)\left(-g_{\mu\nu} + \frac{p_{W^-\mu} p_{W^-\nu}}{M_W^2}\right) \\
&= g_2^2 M_W^2 \cos^2\beta \left(g^{\mu\nu} g_{\mu\nu} - \frac{p^2_{W^+}}{M_W^2} - \frac{p^2_{W^-}}{M_W^2} + \frac{p^\mu_{W^+} p^\nu_{W^+} p_{W^-\mu} p_{W^-\nu}}{M_W^4}\right) \\
&= g_2^2 M_W^2 \cos^2\beta \left(2 + \frac{p^\mu_{W^+} p^\nu_{W^+} p_{W^-\mu} p_{W^-\nu}}{M_W^4}\right) \\
&= g_2^2 M_W^2 \cos^2\beta \left(2 + \frac{\left(M_h^2 - 2M_W^2\right)^2}{4M_W^4}\right) \\
&= g_2^2 M_W^2 \cos^2\beta \left(2 + \left(\frac{M_h^2}{2M_W^2} - 1\right)^2\right) \\
&= g_2^2 M_W^2 \cos^2\beta \left(\frac{M_h^4}{4M_W^4} - \frac{M_h^2}{M_W^2} + 3\right) \\
&= \frac{g_2^2 \cos^2\beta}{4M_W^2}\left(M_h^4 - 4M_h^2 M_W^2 + 12 M_W^4\right)
\end{aligned}
$$

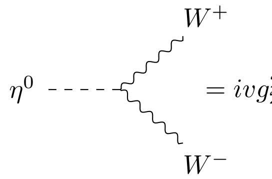
$$
\leftarrow \sum \left(\epsilon^\mu \epsilon^{*\nu}\right) = -\,g_{\mu\nu} + \frac{p_\mu p_\nu}{M_W^2}
$$

$$
\begin{aligned}
p^\mu_{h_0} &= p^\mu_{W^+} + p^\mu_{W^-} \\
p^2_{h_0} &= 2p^\mu_{W^+} p_{W^-\mu} + p^2_{W^+} + p^2_{W^-} \\
M_h^2 &= 2p^\mu_{W^+} p_{W^-\mu} + 2M_W^2 \\
\leftarrow 2p^\mu_{W^+} p_{W^-\mu} &= M_h^2 - 2M_W^2
\end{aligned}
$$

The decay rate is therefore

$$
\Gamma(h_0 \to W^+ + W^-) = \frac{1}{16\pi M_h}\sqrt{1 - \frac{4M_W^2}{M_h^2}}\,|\bar{\mathcal{M}}|^2
$$

$$
\Gamma(h_0 \to W^+ + W^-) = \frac{g_2^2 \cos^2\beta M_h^4}{64\pi M_W^2}\sqrt{1 - \frac{4M_W^2}{M_h^2}}\left(1 - \frac{4M_W^2}{M_h^2} + 12\frac{M_W^4}{M_h^4}\right)
$$



$= ivg_2^2 \tan\beta = 2ig_2 M_W \sin\beta$

(ii)

Calculation of the decay rate is the same as for $h_0 \to W^+ + W^-$, but with $\cos\beta \to \sin\beta$, $M_h \to M_\eta$ and a factor of 2 in $\mathcal{M}$ which results in a factor of 4 in the Decay Rate:

$$
\Gamma(\eta_0 \to W^+ + W^-) = \frac{g_2^2 \sin^2\beta M_\eta^4}{16\pi M_W^2}\sqrt{1 - \frac{4M_W^2}{M_\eta^2}}\left(1 - \frac{4M_W^2}{M_\eta^2} + 12\frac{M_W^4}{M_\eta^4}\right)
$$

6 Renormalization Group

Q1 Scalar QED describes the interaction of a photon with a complex scalar field. In d dimensions it is defined by the action

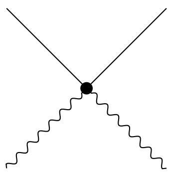
$$\mathcal{L} = \int d^3x \left(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D^\mu\phi)^* D_\mu\phi - m^2\phi^*\phi\right) \tag{215}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu\phi = \partial_\mu\phi + ieA_\mu\phi$. In this problem we will consider the case of a massless scalar field and set $m = 0$.

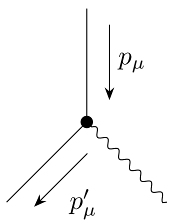
(a) Write down the Feynman rules for the scalar field

$$= \quad \frac{i}{p^2 - m^2 + i\epsilon}$$

$$= \quad -\frac{i}{p^2 + i\epsilon}\left[g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right]$$



$$= \quad 2ie^2 g_{\mu\nu}$$

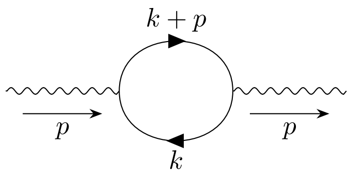


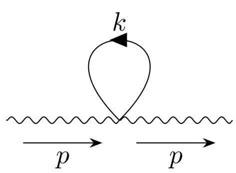
$$= \quad ie\left(p_\mu + p'_\mu\right)$$

(b) Compute the contribution of the scalar field to the β function of the gauge theory and show that the full β function is

$$\beta(g) = \frac{g^3}{48\pi^2} \tag{216}$$

How does the theory behave at low energies? [Useful fact: Let us call the counterterm for the cubic scalar-gauge boson vertex δ_1 and the counterterm for the scalar propagator δ_2. The difference $\delta_1 - \delta_2$ is determined purely by the gauge sector and does not depend on the matter content. You may use this fact without proof.]

$i\Pi^{\mu\nu}(p) =$ 

$+$ 

$$=(ie)^2\int\frac{d^dk}{(2\pi)^d}(2k+p)^\mu\frac{i}{(k+p)^2}(2k+p)^\nu\frac{i}{k^2} + 2ie^2\int\frac{d^dk}{(2\pi)^d}\frac{ig^{\mu\nu}}{k^2} \tag{217}$$

$$=e^2\int\frac{d^dk}{(2\pi)^d}\frac{(2k+p)^\mu(2k+p)^\nu}{k^2(k+p)^2} - 2e^2\int\frac{d^dk}{(2\pi)^d}\frac{g^{\mu\nu}}{k^2} \tag{218}$$

$$=e^2\int\frac{d^dk}{(2\pi)^d}\frac{(2k+p)^\mu(2k+p)^\nu - 2g^{\mu\nu}(k+p)^2}{k^2(k+p)^2} \tag{219}$$

The denominator of this expression can be re-expressed as follows

$$\begin{aligned}
\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2(k+p)^2} &= \frac{\Gamma(2)}{\Gamma(1)^2}\int_0^1 \frac{dx}{((k+p)^2x+k^2(1-x))^2} \\
&= \int_0^1 \frac{dx}{((k^2+p^2+2pk)x+k^2(1-x))^2} \\
&= \int_0^1 \frac{dx}{(p^2x+2pkx+k^2)^2} \\
&= \int_0^1 \frac{dx}{((k+px)^2+p^2x-p^2x^2)^2} \\
&= \int_0^1 \frac{dx}{(l^2-\Delta)^2}
\end{aligned}$$

Where in the last line, we made the substitutions

$$\begin{aligned}
l &\equiv k+px \\
\Delta &\equiv p^2x^2-p^2x
\end{aligned}$$

Substituting this into (219)

$$
\begin{aligned}
i\Pi^{\mu\nu}(p) =& \\
=& e^2\int_0^1 dx \int \frac{d^dl}{(2\pi)^d}\frac{(2(l-px)+p)^\mu(2(l-px)+p)^\nu - 2g^{\mu\nu}(l-px+p)^2}{(l^2-\Delta)^2} \\
=& e^2\int_0^1 dx \int \frac{d^dl}{(2\pi)^d}\frac{(2l^\mu+p^\mu(1-2x))(2l^\nu+p^\nu(1-2x)) - 2g^{\mu\nu}(l+p(1-x))^2}{(l^2-\Delta)^2} \\
=& e^2\int_0^1 dx \int \frac{d^dl}{(2\pi)^d}\frac{4l^\mu l^\nu + p^\mu p^\nu(1-2x)^2 - 2g^{\mu\nu}(l^2+p^2(1-x)^2)}{(l^2-\Delta)^2} \qquad \text{Terms with odd powers of } l \text{ vanish under integration} \\
=& e^2\int_0^1 dx \int \frac{d^dl}{(2\pi)^d}\frac{\frac{4l^2}{d}g^{\mu\nu} + p^\mu p^\nu(1-2x)^2 - 2g^{\mu\nu}(l^2+p^2(1-x)^2)}{(l^2-\Delta)^2} \qquad \leftarrow l^\mu l^\nu = \frac{1}{d}l^2 g^{\mu\nu} \\
=& e^2\int_0^1 dx \int \frac{d^dl}{(2\pi)^d}\frac{\frac{4l^2}{d}g^{\mu\nu} + p^\mu p^\nu(1-2x)^2 - 2g^{\mu\nu}l^2 - 2g^{\mu\nu}p^2(1-x)^2}{(l^2-\Delta)^2} \\
=& e^2\int_0^1 dx \int \frac{d^dl}{(2\pi)^d}\frac{\left(\frac{4}{d}-2\right)g^{\mu\nu}l^2 + p^\mu p^\nu(1-2x)^2 - 2g^{\mu\nu}p^2(1-x)^2}{(l^2-\Delta)^2} \\
=& e^2\int_0^1 dx \left(g^{\mu\nu}\left(\frac{4}{d}-2\right)\int\frac{d^dl}{(2\pi)^d}\frac{l^2}{(l^2-\Delta)^2} + \left(p^\mu p^\nu(1-2x)^2 - 2g^{\mu\nu}p^2(1-x)^2\right)\int\frac{d^dl}{(2\pi)^d}\frac{1}{(l^2-\Delta)^2}\right) \\
=& e^2\int_0^1 dx \left(g^{\mu\nu}\left(\frac{4}{d}-2\right)\int\frac{d^dl}{(2\pi)^d}\left[\frac{1}{l^2-\Delta}+\frac{\Delta}{(l^2-\Delta)^2}\right] + \left(p^\mu p^\nu(1-2x)^2 - 2g^{\mu\nu}p^2(1-x)^2\right)\int\frac{d^dl}{(2\pi)^d}\frac{1}{(l^2-\Delta)^2}\right) \\
=& e^2\int_0^1 dx \left(g^{\mu\nu}\underbrace{\left(\frac{4}{d}-2\right)\int\frac{d^dl}{(2\pi)^d}\frac{1}{l^2-\Delta}}_{\substack{=\left(\frac{4}{d}-2\right)\frac{-i}{(4\pi)^{d/2}}\frac{\Gamma\left(1-\frac{d}{2}\right)}{\Gamma(1)}\Delta^{d/2-1} \\ =-\frac{i}{(4\pi)^{d/2}}\frac{4}{d}\left(1-\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)\Delta^{d/2-1} \\ =-\frac{i}{(4\pi)^{d/2}}\frac{4}{d}\Gamma\left(2-\frac{d}{2}\right)\Delta^{d/2-1}}} + \left(\left(\frac{4}{d}-2\right)g^{\mu\nu}\Delta + p^\mu p^\nu(1-2x)^2 - 2g^{\mu\nu}p^2(1-x)^2\right)\underbrace{\int\frac{d^dl}{(2\pi)^d}\frac{1}{(l^2-\Delta)^2}}_{\substack{=\frac{i}{(4\pi)^{d/2}}\frac{\Gamma\left(2-\frac{d}{2}\right)}{\Gamma(2)}\Delta^{d/2-2} \\ =\frac{i}{(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)\Delta^{d/2-2}}}\right) \\
=& \frac{ie^2}{(4\pi)^{d/2}}\int_0^1 dx\ \Gamma\left(2-\frac{d}{2}\right)\left(-\frac{4}{d}g^{\mu\nu}\Delta^{d/2-1} + \left(\left(\frac{4}{d}-2\right)g^{\mu\nu}\Delta + p^\mu p^\nu(1-2x)^2 - 2g^{\mu\nu}p^2(1-x)^2\right)\Delta^{d/2-2}\right) \\
=& \frac{ie^2}{(4\pi)^{d/2}}\int_0^1 dx\ \Gamma\left(2-\frac{d}{2}\right)\left(-\frac{4}{d}g^{\mu\nu}\Delta^{d/2-1} + \left(\frac{4}{d}-2\right)g^{\mu\nu}\Delta^{d/2-1} + \left(p^\mu p^\nu(1-2x)^2 - 2g^{\mu\nu}p^2(1-x)^2\right)\Delta^{d/2-2}\right) \\
=& \frac{ie^2}{(4\pi)^{d/2}}\int_0^1 dx\ \Gamma\left(2-\frac{d}{2}\right)\left(-2g^{\mu\nu}\Delta^{d/2-1} + \left(p^\mu p^\nu(1-2x)^2 - 2g^{\mu\nu}p^2(1-x)^2\right)\Delta^{d/2-2}\right) \\
=& \frac{ie^2}{(4\pi)^{d/2}}\int_0^1 dx\ \Gamma\left(2-\frac{d}{2}\right)\Delta^{d/2-2}\left(-2g^{\mu\nu}\Delta + p^\mu p^\nu(1-2x)^2 - 2g^{\mu\nu}p^2(1-x)^2\right) \\
=& \frac{ie^2}{(4\pi)^{d/2}}\int_0^1 dx\ \Gamma\left(2-\frac{d}{2}\right)\Delta^{d/2-2}\left(-2g^{\mu\nu}p^2\left(x^2-x\right) + p^\mu p^\nu(1-2x)^2 - 2g^{\mu\nu}p^2(1-2x+x^2)\right) \\
=& \frac{ie^2}{(4\pi)^{d/2}}\int_0^1 dx\ \Gamma\left(2-\frac{d}{2}\right)\Delta^{d/2-2}\left(p^\mu p^\nu(1-2x)^2 - 2g^{\mu\nu}p^2\left(2x^2-3x+1\right)\right) \\
=& \frac{ie^2}{(4\pi)^{d/2}}\int_0^1 dx\ \Gamma\left(2-\frac{d}{2}\right)\Delta^{d/2-2}\left(p^\mu p^\nu(1-2x)^2 - 2g^{\mu\nu}p^2(1-x)(1-2x)\right) \\
=& \frac{ie^2}{(4\pi)^{d/2}}\int_0^1 dx\ \Gamma\left(2-\frac{d}{2}\right)\Delta^{d/2-2}\left(p^\mu p^\nu(1-2x)^2 - g^{\mu\nu}p^2(1-2x)^2 - g^{\mu\nu}p^2\cancel{(1-2x)}\right) \qquad \text{This term is odd under } x\to 1-x \\
=& -\frac{ie^2}{(4\pi)^{d/2}}\left(g^{\mu\nu}p^2 - p^\mu p^\nu\right)\int_0^1 dx(1-2x)^2\Gamma\left(2-\frac{d}{2}\right)\Delta^{d/2-2} \\
\xrightarrow{d=4}& -\frac{ie^2}{(4\pi)^2}\left(g^{\mu\nu}p^2 - p^\mu p^\nu\right)\left(\frac{2}{\epsilon} - \log\Delta - \gamma + \log(4\pi)\right)\int_0^1 dx(1-2x)^2
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{ie^2}{(4\pi)^2}\left(g^{\mu\nu}p^2 - p^\mu p^\nu\right)\left(\frac{2}{\epsilon} - \log\Delta - \gamma + \log(4\pi)\right)\int_0^1 dx\left(1 - 4x + 4x^2\right) \\
&= -\frac{ie^2}{(4\pi)^2}\left(g^{\mu\nu}p^2 - p^\mu p^\nu\right)\left(\frac{2}{\epsilon} - \log\Delta - \gamma + \log(4\pi)\right)\left[x - 2x^2 + \tfrac{4}{3}x^3\right]_0^1 \\
i\Pi^{\mu\nu}(p) &= -\frac{ie^2}{48\pi^2}\left(g^{\mu\nu}p^2 - p^\mu p^\nu\right)\left(\frac{2}{\epsilon} - \log\Delta - \gamma + \log(4\pi)\right) \\
\Pi^{\mu\nu}(p) &= -\frac{e^2}{48\pi^2}\left(g^{\mu\nu}p^2 - p^\mu p^\nu\right)\left(\frac{2}{\epsilon} - \log\Delta - \gamma + \log(4\pi)\right)
\end{aligned}
$$

Therefore, the β function is

$$
\begin{aligned}
\beta(e) =& \mu\frac{\partial}{\partial\mu}\left[e\left(-\delta_1 + \tfrac{1}{2}\left(2\delta_2 + \delta_3\right)\right)\right] \\
=& \mu\frac{\partial}{\partial\mu}\left[e\delta_3\right] \\
=& \frac{\partial}{\partial(\log\mu)}\left[-\frac{1}{2}\frac{e^3}{48\pi^2}\left(\frac{1}{\epsilon} - \log\left(\mu^2\right)\right)\right] \\
=& \frac{e^3}{48\pi^2}
\end{aligned}
$$

$\delta_1 - \delta_2 = 0$ does not depend on matter content so can be taken to be 0, as is the case with no matter

The coupling constant decreases as the energy decreases, therefore the theory is not asymptotically free.

---

7 Amplitudes

Q1 Using Little group scaling and locality, argue that the 3-pt MHV (maximlly helicity violating amplitude) and anti-MHV amplitudes for spin-2 particles (gravitons) are (up to an overall factor)

$$M\left(1^-,2^-,3^+\right)=\left(\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\right)^2 \tag{220}$$

$$M\left(1^+,2^+,3^-\right)=\left(\frac{[12]^3}{[23][31]}\right)^2 \tag{221}$$

The amplitude must be in terms of only angle brackets or only square brackets.
In the case of angle brackets, the most general form of the amplitude is:

$$M\left(1^-,2^-,3^+\right)=\langle 12\rangle^x\langle 23\rangle^y\langle 31\rangle^z$$

We act on this with the helicity operator $h_i=\frac{1}{2}\left(-\lambda_i^\alpha\frac{\partial}{\partial\lambda_i^\alpha}+\tilde{\lambda}_i^{\dot{\alpha}}\frac{\partial}{\partial\tilde{\lambda}_i^{\dot{\alpha}}}\right)$

$$\begin{aligned}
h_1M\left(1^-,2^-,3^+\right)&=-\tfrac{1}{2}(x+z)M=(-2)M & & x=+6\\
h_2M\left(1^-,2^-,3^+\right)&=-\tfrac{1}{2}(x+y)M=(-2)M &\implies & y=-2\\
h_3M\left(1^-,2^-,3^+\right)&=-\tfrac{1}{2}(y+z)M=(+2)M & & z=-2
\end{aligned}$$

$$M\left(1^-,2^-,3^+\right)=\langle 12\rangle^{+6}\langle 23\rangle^{-2}\langle 31\rangle^{-2}$$

$$\boxed{M\left(1^-,2^-,3^+\right)=\left(\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\right)^2}$$

In the case of square brackets, the most general form of the amplitude is:

$$M\left(1^-,2^-,3^+\right)=[12]^x[23]^y[31]^z$$

$$\begin{aligned}
h_1M\left(1^-,2^-,3^+\right)&=+\tfrac{1}{2}(x+z)M=(-2)M & & x=-6\\
h_2M\left(1^-,2^-,3^+\right)&=+\tfrac{1}{2}(x+y)M=(-2)M &\implies & y=+2\\
h_3M\left(1^-,2^-,3^+\right)&=+\tfrac{1}{2}(y+z)M=(+2)M & & z=+2
\end{aligned}$$

$$\begin{aligned}
M\left(1^-,2^-,3^+\right)&=[12]^{-6}[23]^{+2}[31]^{+2}\\
&=\left(\frac{[23][31]}{[12]^3}\right)^2
\end{aligned}$$

However, we see that this expression has negative mass dimension (we can see from $\langle pq\rangle[pq]=2p\cdot q=(p+q)^2$ that both angle and square brackets have mass dimension 1). In a 3 particle vertex, this would require a derivative to an inverse power, which violates locality. Therefore, the expression for $M\left(1^-,2^-,3^+\right)$ is the one in terms of angle brackets.

In the case of $M(1^+, 2^+, 3^-)$:
Angle brackets:

$$M\left(1^+, 2^+, 3^-\right) = \langle 12\rangle^x \langle 23\rangle^y \langle 31\rangle^z$$
$$\begin{aligned} h_1 M(1^+, 2^+, 3^-) &= -\tfrac{1}{2}(x+z)M = (+2)M & & x = -6 \\ h_2 M(1^+, 2^+, 3^-) &= -\tfrac{1}{2}(x+y)M = (+2)M & \implies & y = +2 \\ h_3 M(1^+, 2^+, 3^-) &= -\tfrac{1}{2}(y+z)M = (-2)M & & z = +2 \end{aligned}$$
$$M\left(1^+, 2^+, 3^-\right) = \langle 12\rangle^{-6} \langle 23\rangle^{+2} \langle 31\rangle^{+2}$$
$$M\left(1^+, 2^+, 3^-\right) = \left(\frac{\langle 23\rangle \langle 31\rangle}{\langle 12\rangle^3}\right)^2$$

Which is invalid for the same reason as the expression for $M(1^-, 2^-, 3^+)$ in terms of square brackets.

Square brackets:

$$M\left(1^+, 2^+, 3^-\right) = [12]^x [23]^y [31]^z$$
$$\begin{aligned} h_1 M(1^+, 2^+, 3^-) &= +\tfrac{1}{2}(x+z)M = (+2)M & & x = +6 \\ h_2 M(1^+, 2^+, 3^-) &= +\tfrac{1}{2}(x+y)M = (+2)M & \implies & y = -2 \\ h_3 M(1^+, 2^+, 3^-) &= +\tfrac{1}{2}(y+z)M = (-2)M & & z = -2 \end{aligned}$$
$$M\left(1^+, 2^+, 3^-\right) = [12]^{+6} [23]^{-2} [31]^{-2}$$
$$M\left(1^+, 2^+, 3^-\right) = \left(\frac{[12]^3}{[23][31]}\right)^2$$

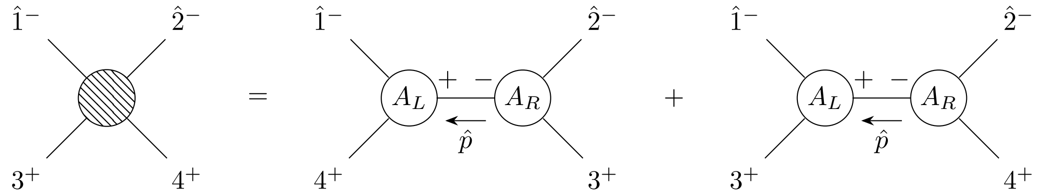
Q2 Compute the tree-level 4-pt gravitational MHV amplitude $M_4(1^-, 2^-, 3^+, 4^+)$ using the following BCFW shift:

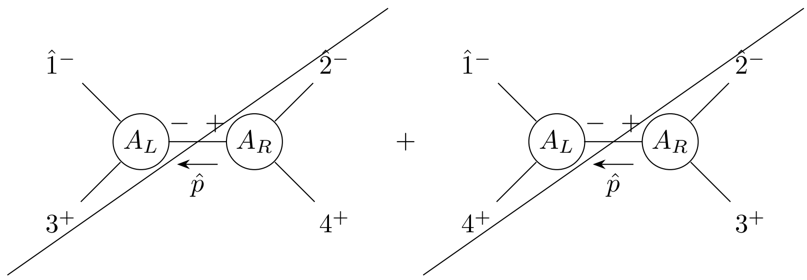
$$\hat{\lambda}_1 = \lambda_1 + z\lambda_2 \tag{222}$$
$$\hat{\tilde{\lambda}}_2 = \tilde{\lambda}_2 + z\tilde{\lambda}_1 \tag{223}$$

You may assume that $\lim_{z\to\infty} \hat{M}_4(z) = 0$. Note that, unlike color-ordered gluon amplitudes, gravity amplitudes are permutatino invariant. Using the above shift, only two diagrams contribute and they are related by exchanging legs 3 and 4.

Using the BCFW theorem, the four-point graviton amplitude (with deformed momentum on 2 external legs) can be split into 2 three-point subamplitudes connected by a deformed propogator. You must then sum this diagram over the possible permutations of external legs. In the case of gravity, any order of external legs is valid but $\hat{1}^-$ and $\hat{2}^-$ must be connected to different subamplitudes so that the deformed momentum can flow through the propogator. Therefore, we obtain 2 diagrams by exchanging 3^+ and 4^+. For each of these 2 diagrams, there are 2 diagrams for the possible helicities on the propogator legs:

$$M_4\left(1^-2^-3^+4^+\right) \quad = \quad M_1 \quad + \quad M_2$$





The latter two diagrams vanish because the subamplitudes $A_R\left(p^+\hat{2}^-3^+\right)$ and $A_R\left(p^+\hat{2}^-4^+\right)$ are anti-MHV amplitudes, and it will not be possible to choose a deformed momentum such that the kinematics are satisfied at these subamplitudes.

Momentum shift:

$$\hat{p}_1(z) = p_1 + zq$$
$$\hat{p}_2(z) = p_2 - zq$$

Deformed momenta in terms of spinor variables:

$$\begin{aligned} \hat{p}_1(z) &= \hat{\lambda}_1\hat{\tilde{\lambda}}_1 = (\lambda_1 + z\lambda_2)\tilde{\lambda}_1 = p_1 + z\lambda_2\tilde{\lambda}_1 \\ \hat{p}_2(z) &= \hat{\lambda}_2\hat{\tilde{\lambda}}_2 = \lambda_2(\tilde{\lambda}_2 - z\tilde{\lambda}_1) = p_2 - z\lambda_2\tilde{\lambda}_1 \end{aligned} \leftarrow \left\{ \begin{aligned} \hat{\lambda}_1 &= \lambda_1 + z\lambda_2 \\ \hat{\tilde{\lambda}}_2 &= \tilde{\lambda}_2 - z\tilde{\lambda}_1 \end{aligned} \right.$$

$$\implies q = \lambda_2\tilde{\lambda}_1$$

Obtaining an expresion for z:

$$\begin{aligned}
0 &= \hat{p}^2 \\
0 &= \left(-(\hat{p}_2 + p_3)\right)^2 \\
0 &= \left(-(p_2 + p_3) - zq\right)^2 \\
0 &= \left(p_2 + p_3\right)^2 + 2zq\left(p_2 + p_3\right) \\
z &= -\frac{\left(p_2 + p_3\right)^2}{2q \cdot (p_2 + p_3)} \\
z &= -\frac{p_2 \cdot p_3}{q \cdot p_3} && \leftarrow q \cdot p_2 = 0 \\
z &= -\frac{\langle 23\rangle\,[32]}{\langle q3\rangle\,[3q]} \\
z &= -\frac{\langle 23\rangle\,[32]}{\langle 23\rangle\,[31]} && \leftarrow q = \lambda_2\tilde{\lambda}_1 \\
z &= -\frac{[32]}{[31]} \\
z &= -\frac{[23]}{[13]}
\end{aligned}$$

Substituting this into the expression for $\hat{p}$:

$$\begin{aligned}
\hat{p} =& -\left(p_2+p_3\right)-zq \\
=& -\left(p_2+p_3\right)+\frac{[23]}{[13]}\lambda_2\tilde{\lambda}_1 \\
=& -\lambda_2\tilde{\lambda}_2-\lambda_3\tilde{\lambda}_3+\frac{[23]}{[13]}\lambda_2\tilde{\lambda}_1 \\
=& \frac{\lambda_2}{[13]}\left(-[13]\tilde{\lambda}_2+[23]\tilde{\lambda}_1\right)-\lambda_3\tilde{\lambda}_3 \\
=& \frac{\lambda_2}{[13]}\left([31]\tilde{\lambda}_2+[23]\tilde{\lambda}_1\right)-\lambda_3\tilde{\lambda}_3 \\
=& \frac{-\lambda_2[12]\tilde{\lambda}_3}{[13]}-\lambda_3\tilde{\lambda}_3 && \leftarrow \text{Schouten Identity: } [12]\tilde{\lambda}_3+[23]\tilde{\lambda}_1+[31]\tilde{\lambda}_2=0
\end{aligned}$$

$$\begin{aligned}
M_1 =& \frac{1}{p^2}A_L\left((-p)^+\hat{1}^-4^+\right)A_R\left(p^-\hat{2}^-3^+\right) \\
=& \frac{1}{(p_1+p_4)^2}\left(\frac{[4(-\hat{p})]^3}{[\hat{p}\hat{1}][\hat{1}4]}\right)^2\left(\frac{\langle\hat{2}\hat{p}\rangle^3}{\langle\hat{p}3\rangle\langle\hat{2}3\rangle}\right)^2 \\
=& \frac{1}{(p_1+p_4)^2}\left(\frac{[4(-\hat{p})]^3}{[\hat{p}1][14]}\right)^2\left(\frac{\langle 2\hat{p}\rangle^3}{\langle\hat{p}3\rangle\langle 23\rangle}\right)^2 && \leftarrow \begin{cases} \left|\hat{1}\right] & = |1] \\ \left|\hat{2}\right\rangle & = |2\rangle \end{cases} \\
=& \frac{1}{(p_1+p_4)^2}\left(\frac{\left([4(-\hat{p})]\langle 2\hat{p}\rangle\right)^3}{\langle\hat{p}3\rangle[\hat{p}1][14]\langle 23\rangle}\right)^2 \\
=& \frac{1}{(p_1+p_4)^2}\left(\frac{\left(\langle 12\rangle[14]\right)^3}{\langle\hat{p}3\rangle[\hat{p}1][14]\langle 23\rangle}\right)^2 && \leftarrow \begin{cases} [4(-\hat{p})]\langle 2\hat{p}\rangle & = -\langle 2\hat{p}\rangle[\hat{p}4] \\ & = -\langle 2|\,\hat{p}\,|4] \\ & = \langle 2|\left(\hat{p}_1+p_4\right)|4] \\ & = -\langle 2|\left(p_1+z\lambda_2\tilde{\lambda}_1+p_4\right)|4] \\ & = -\langle 2|\,p_1\,|4] \\ & = -\langle 21\rangle[14] \\ & = -\langle 21\rangle[14] \\ & = \langle 12\rangle[14] \end{cases} \\
=& \frac{1}{(p_1+p_4)^2}\left(\frac{\langle 12\rangle^3[14]^2}{\langle\hat{p}3\rangle[\hat{p}1]\langle 23\rangle}\right)^2 \\
=& \frac{1}{(p_1+p_4)^2}\left(\frac{\langle 12\rangle^3[14]^2}{\langle 34\rangle[14]\langle 23\rangle}\right)^2 && \leftarrow \begin{cases} \langle\hat{p}3\rangle[\hat{p}1] & = \langle 3\hat{p}\rangle[\hat{p}1] \\ & = -\langle 3|\,\hat{p}\,|1] \\ & = -\langle 3|\left(\hat{p}_1+p_4\right)|1] \\ & = -\langle 3|\,p_4\,|1] \\ & = -\langle 34\rangle[41] \\ & = \langle 34\rangle[14] \end{cases} \\
=& \frac{1}{(p_1+p_4)^2}\frac{\langle 12\rangle^6[14]^2}{\langle 23\rangle^2\langle 34\rangle^2} \\
=& \frac{\langle 12\rangle^6[14]^2}{\langle 14\rangle[14]\langle 32\rangle^2\langle 34\rangle^2} && \leftarrow (p_1+p_4)^2=\langle 14\rangle[14] \\
=& \frac{\langle 12\rangle^6[14]}{\langle 14\rangle\langle 32\rangle^2\langle 34\rangle^2}
\end{aligned}$$

$$
\begin{aligned}
&M_4\left(1^-2^-3^+4^+\right)\\
&= M_1 + M_2\\
&= M_1 + M_1^{(3\leftrightarrow 4)}\\
&= \frac{\langle 12\rangle^6 [14]}{\langle 14\rangle \langle 23\rangle^2 \langle 34\rangle^2} + \frac{\langle 12\rangle^6 [13]}{\langle 13\rangle \langle 24\rangle^2 \langle 34\rangle^2}\\
&= \frac{\langle 12\rangle^6}{\langle 34\rangle^2}\left(\frac{[14]}{\langle 14\rangle \langle 23\rangle^2} + \frac{[13]}{\langle 13\rangle \langle 24\rangle^2}\right)\\
&= \frac{\langle 12\rangle^6}{\langle 34\rangle^2}\left(\frac{[14]\langle 13\rangle \langle 24\rangle^2 + [13]\langle 14\rangle \langle 23\rangle^2}{\langle 14\rangle \langle 23\rangle^2 \langle 13\rangle \langle 24\rangle^2}\right)\\
&= \frac{\langle 12\rangle^6}{\langle 34\rangle^2}\left(\frac{[14]\langle 13\rangle \langle 24\rangle^2 [12] + [13]\langle 14\rangle \langle 23\rangle^2 [12]}{\langle 14\rangle \langle 23\rangle^2 \langle 13\rangle \langle 24\rangle^2 [12]}\right)\\
&= \frac{\langle 12\rangle^6}{\langle 34\rangle^2}\left(\frac{[14]\langle 13\rangle \langle 24\rangle \langle 34\rangle [13] - [13]\langle 14\rangle \langle 23\rangle \langle 34\rangle [14]}{\langle 14\rangle \langle 23\rangle^2 \langle 13\rangle \langle 42\rangle^2 [12]}\right) && \leftarrow \left\{\begin{aligned} [12]\langle 24\rangle &= [13]\langle 34\rangle\\ [12]\langle 23\rangle &= [14]\langle 43\rangle\\ [12]\langle 23\rangle &= -[14]\langle 34\rangle \end{aligned}\right.\\
&= \frac{\langle 12\rangle^6 [14][13]}{\langle 34\rangle}\left(\frac{\langle 13\rangle \langle 24\rangle - \langle 14\rangle \langle 23\rangle}{\langle 14\rangle \langle 23\rangle^2 \langle 13\rangle \langle 24\rangle^2 [12]}\right)\\
&= -\frac{\langle 12\rangle^6 [14][13]}{\langle 34\rangle}\left(\frac{\langle 12\rangle \langle 34\rangle}{\langle 14\rangle \langle 23\rangle^2 \langle 13\rangle \langle 24\rangle^2 [12]}\right) && \leftarrow \left\{\begin{aligned} &\text{Schouten Identity:}\\ &\langle 12\rangle \lambda_3 + \langle 23\rangle \lambda_1 + \langle 31\rangle \lambda_2 &&= 0\\ &\langle 12\rangle \langle 3| + \langle 23\rangle \langle 1| + \langle 31\rangle \langle 2| &&= 0\\ &\langle 12\rangle \langle 34\rangle + \langle 23\rangle \langle 14\rangle + \langle 31\rangle \langle 24\rangle &&= 0\\ &\langle 12\rangle \langle 34\rangle + \langle 23\rangle \langle 14\rangle + \langle 31\rangle \langle 24\rangle &&= 0\\ &\langle 12\rangle \langle 34\rangle + \langle 23\rangle \langle 14\rangle - \langle 13\rangle \langle 24\rangle &&= 0 \end{aligned}\right.\\
&= -\frac{\langle 12\rangle^7 [14][13]}{\langle 14\rangle \langle 23\rangle^2 \langle 13\rangle \langle 24\rangle^2 [12]}\\
&= -\frac{\langle 12\rangle^7 [14][13]}{\langle 14\rangle \langle 23\rangle^2 \langle 13\rangle \langle 24\rangle \langle 34\rangle [13]} && \leftarrow \left\{\begin{aligned} [12]\langle 24\rangle &= \langle 42\rangle [21]\\ &= \langle 43\rangle [31]\\ &= \langle 34\rangle [13] \end{aligned}\right.\\
&= -\frac{\langle 12\rangle^7 [14]}{\langle 14\rangle \langle 23\rangle^2 \langle 13\rangle \langle 24\rangle \langle 34\rangle}\\
&= -\frac{\langle 12\rangle^7 [14]\langle 34\rangle}{\langle 14\rangle \langle 23\rangle^2 \langle 13\rangle \langle 42\rangle \langle 34\rangle^2}\\
&= \frac{\langle 12\rangle^7 \langle 23\rangle [12]}{\langle 14\rangle \langle 23\rangle^2 \langle 13\rangle \langle 24\rangle \langle 34\rangle^2} && \leftarrow \left\{\begin{aligned} \langle 34\rangle [14] &= -\langle 34\rangle [41]\\ &= -\langle 32\rangle [21]\\ &= -\langle 23\rangle [12] \end{aligned}\right.\\
&= \frac{\langle 12\rangle^7 [12]}{\langle 14\rangle \langle 23\rangle \langle 13\rangle \langle 24\rangle \langle 34\rangle^2}
\end{aligned}
$$

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8 Conformal Field Theory

Q1 In a free scalar field theory of two massless scalar fields ϕ_1, ϕ_2, in four dimensional space-time, the OPE can be written as

$$\phi_1(x)\phi_2(0) = \frac{1}{x^2} + :\phi_1(x)\phi_2(0): \tag{224}$$

The possible primary operators appearing in the sum on RHS to this order are

$$\mathcal{O} =: \phi_1\phi_2 : \tag{225}$$
$$\mathcal{O}_\mu =: \phi_1\partial_\mu\phi_2 - \phi_2\partial_\mu\phi_1 : \tag{226}$$
$$\mathcal{O}_{\mu\nu} =: \phi_1\partial^2_{\mu\nu}\phi_2 + \phi_2\partial^2_{\mu\nu}\phi_1 - 2\partial_\mu\phi_1\partial_\nu\phi_2 - 2\partial_\mu\phi_2\partial_\nu\phi_1 + \partial_\rho\phi_1\partial^\rho\phi_2\eta_{\mu\nu} : \tag{227}$$

(a) Taylor expand the second term around $x = 0$ to second order.

$$\begin{aligned}
\phi_1(x)\phi_2(0) &= \frac{1}{x^2} + :\left(\phi_1(0) + x^\mu\partial_\mu\phi_1(0) + \tfrac{1}{2}x^\mu x^\nu\partial_\mu\partial_\nu\phi_1(0) + \mathcal{O}(x^3)\right)\phi_2(0): \\
&= \frac{1}{x^2} + :\left(\phi_1 + x^\mu\partial_\mu\phi_1 + \tfrac{1}{2}x^\mu x^\nu\partial_\mu\partial_\nu\phi_1\right)\phi_2: && (228) \\
&= \frac{1}{x^2} + :\boxed{\phi_1\phi_2} + x^\mu\boxed{\phi_2\partial_\mu\phi_1} + \tfrac{1}{2}x^\mu x^\nu\boxed{\phi_2\partial_\mu\partial_\nu\phi_1}: && (229)
\end{aligned}$$

From line (228), we're implicitly denoting $\phi_1(0)$ as ϕ_1.

(b) Express $\boxed{\phi_2\partial_\mu\phi_1}$ purely in terms of primaries and descendants.

Differentiating (225)

$$\begin{aligned}
\partial_\mu\mathcal{O} &= \partial_\mu(\phi_1\phi_2) \\
\partial_\mu\mathcal{O} &= \phi_1\partial_\mu\phi_2 + \phi_2\partial_\mu\phi_1 \\
\tfrac{1}{2}\partial_\mu\mathcal{O} &= \tfrac{1}{2}\phi_1\partial_\mu\phi_2 + \tfrac{1}{2}\phi_2\partial_\mu\phi_1 \\
\tfrac{1}{2}\partial_\mu\mathcal{O} &= \tfrac{1}{2}\phi_1\partial_\mu\phi_2 - \tfrac{1}{2}\phi_2\partial_\mu\phi_1 + \phi_2\partial_\mu\phi_1 \\
\tfrac{1}{2}\partial_\mu\mathcal{O} &= \tfrac{1}{2}\mathcal{O}_\mu + \phi_2\partial_\mu\phi_1 \\
\phi_2\partial_\mu\phi_1 &= \tfrac{1}{2}(\partial_\mu\mathcal{O} - \mathcal{O}_\mu) && (230)
\end{aligned}$$

(c) Express $\boxed{\phi_2\partial_\mu\partial_\nu\phi_1}$ purely in terms of primaries and descendants.

We twice differentiate (225).

$$\begin{aligned}
\partial_\mu\partial_\nu\mathcal{O} &= \partial_\mu\partial_\nu(\phi_1\phi_2) && (231) \\
\partial_\mu\partial_\nu\mathcal{O} &= \partial_\mu(\phi_1\partial_\nu\phi_2 + \phi_2\partial_\nu\phi_1) && (232) \\
\partial_\mu\partial_\nu\mathcal{O} &= \phi_1\partial_\mu\partial_\nu\phi_2 + \phi_2\partial_\mu\partial_\nu\phi_1 + \partial_\mu\phi_1\partial_\nu\phi_2 + \partial_\nu\phi_1\partial_\mu\phi_2 && (233) \\
2\partial_\mu\partial_\nu\mathcal{O} &= 2\phi_1\partial_\mu\partial_\nu\phi_2 + 2\phi_2\partial_\mu\partial_\nu\phi_1 + 2\partial_\mu\phi_1\partial_\nu\phi_2 + 2\partial_\nu\phi_1\partial_\mu\phi_2 && (234) \\
2\partial_\mu\partial_\nu\mathcal{O} + \mathcal{O}_{\mu\nu} &= 3\phi_1\partial_\mu\partial_\nu\phi_2 + 3\phi_2\partial_\mu\partial_\nu\phi_1 + \partial_\rho\phi_1\partial^\rho\phi_2\eta_{\mu\nu} && (235)
\end{aligned}$$

We can make progress by differentiating (226)

$$\begin{aligned}
\partial_\mu \mathcal{O}_\nu =& \partial_\mu \left(\phi_1 \partial_\nu \phi_2 - \phi_2 \partial_\nu \phi_1\right) \\
\partial_\mu \mathcal{O}_\nu =& \phi_1 \partial_\mu \partial_\nu \phi_2 - \phi_2 \partial_\mu \partial_\nu \phi_1 + \partial_\mu \phi_1 \partial_\nu \phi_2 - \partial_\nu \phi_1 \partial_\mu \phi_2 \\
\partial_\mu \mathcal{O}_\nu + \partial_\nu \mathcal{O}_\mu =& \phi_1 \partial_\mu \partial_\nu \phi_2 - \phi_2 \partial_\mu \partial_\nu \phi_1 + \partial_\mu \phi_1 \partial_\nu \phi_2 - \partial_\nu \phi_1 \partial_\mu \phi_2 + \left(\phi_1 \partial_\mu \partial_\nu \phi_2 - \phi_2 \partial_\mu \partial_\nu \phi_1 - \partial_\mu \phi_1 \partial_\nu \phi_2 + \partial_\nu \phi_1 \partial_\mu \phi_2\right) \\
\partial_\mu \mathcal{O}_\nu + \partial_\nu \mathcal{O}_\mu =& 2\phi_1 \partial_\mu \partial_\nu \phi_2 - 2\phi_2 \partial_\mu \partial_\nu \phi_1 \\
\phi_1 \partial_\mu \partial_\nu \phi_2 =& \phi_2 \partial_\mu \partial_\nu \phi_1 + \tfrac{1}{2}\left(\partial_\mu \mathcal{O}_\nu + \partial_\nu \mathcal{O}_\mu\right) \qquad (236)
\end{aligned}$$

We now return to (235) and substitute in (236).

$$\begin{aligned}
2\partial_\mu \partial_\nu \mathcal{O} + \mathcal{O}_{\mu\nu} =& 3\left(\phi_2 \partial_\mu \partial_\nu \phi_1 + \tfrac{1}{2}\left(\partial_\mu \mathcal{O}_\nu + \partial_\nu \mathcal{O}_\mu\right)\right) + 3\phi_2 \partial_\mu \partial_\nu \phi_1 + \partial_\rho \phi_1 \partial^\rho \phi_2 \eta_{\mu\nu} \qquad (237) \\
2\partial_\mu \partial_\nu \mathcal{O} + \mathcal{O}_{\mu\nu} =& 6\phi_2 \partial_\mu \partial_\nu \phi_1 + 3\partial_\mu \mathcal{O}_\nu + 3\partial_\nu \mathcal{O}_\mu + \partial_\rho \phi_1 \partial^\rho \phi_2 \eta_{\mu\nu} \qquad (238) \\
2\partial_\mu \partial_\nu \mathcal{O} + \mathcal{O}_{\mu\nu} =& 6\phi_2 \partial_\mu \partial_\nu \phi_1 + 3\partial_\mu \mathcal{O}_\nu + 3\partial_\nu \mathcal{O}_\mu + \partial_\rho \phi_1 \partial^\rho \phi_2 \eta_{\mu\nu} \qquad (239)
\end{aligned}$$

We can further eliminate the non-descendant/primary term by twice differentiating (225).

$$\begin{aligned}
\partial^2 \mathcal{O} =& \partial^2 \left(\phi_1 \phi_2\right) \\
\partial^2 \mathcal{O} =& \phi_1 \partial^2 \phi_2 + \phi_2 \partial^2 \phi_1 + 2\partial^\mu \phi_1 \partial_\mu \phi_2 \\
\partial^\mu \phi_1 \partial_\mu \phi_2 =& \tfrac{1}{2}\left(\partial^2 \mathcal{O} - \left(\underbrace{\phi_1 \partial^2 \phi_2 + \phi_2 \partial^2 \phi_1}\right)\right) \qquad (240)
\end{aligned}$$

Now, we prove that the highlighted term in (240) is the trace of (227)

$$\begin{aligned}
\mathcal{O}_{\mu\nu} =& \phi_1 \partial_\mu \partial_\nu \phi_2 + \phi_2 \partial_\mu \partial_\nu \phi_1 - 2\partial_\mu \phi_1 \partial_\nu \phi_2 - 2\partial_\nu \phi_1 \partial_\mu \phi_2 + \partial_\rho \phi_1 \partial^\rho \phi_2 \eta_{\mu\nu} \\
\mathcal{O}^\mu{}_\mu =& \phi_1 \partial^2 \phi_2 + \phi_2 \partial^2 \phi_1 - 2\partial_\mu \phi_1 \partial^\mu \phi_2 - 2\partial^\mu \phi_1 \partial_\mu \phi_2 + \partial_\rho \phi_1 \partial^\rho \phi_2 \eta^\mu_\mu \\
=& \phi_1 \partial^2 \phi_2 + \phi_2 \partial^2 \phi_1 - 4\partial_\mu \phi_1 \partial^\mu \phi_2 + 4\partial_\rho \phi_1 \partial^\rho \phi_2 \\
=& \phi_1 \partial^2 \phi_2 + \phi_2 \partial^2 \phi_1 \qquad (241)
\end{aligned}$$

Therefore, (240) can be written as

$$\partial^\mu \phi_1 \partial_\mu \phi_2 = \tfrac{1}{2}\left(\partial^2 \mathcal{O} - \mathcal{O}^\mu{}_\mu\right) \qquad (242)$$

Substituting into (239)

$$\begin{aligned}
2\partial_\mu \partial_\nu \mathcal{O} + \mathcal{O}_{\mu\nu} =& 6\phi_2 \partial_\mu \partial_\nu \phi_1 + 3\partial_\mu \mathcal{O}_\nu + 3\partial_\nu \mathcal{O}_\mu + \tfrac{1}{2}\left(\partial^2 \mathcal{O} - \mathcal{O}^\rho{}_\rho\right)\eta_{\mu\nu} \qquad (243) \\
\phi_2 \partial_\mu \partial_\nu \phi_1 =& \tfrac{1}{6}\mathcal{O}_{\mu\nu} + \tfrac{1}{3}\partial_\mu \partial_\nu \mathcal{O} + \tfrac{1}{12}\mathcal{O}^\rho{}_\rho \eta_{\mu\nu} - \tfrac{1}{2}\partial_\mu \mathcal{O}_\nu - \tfrac{1}{2}\partial_\nu \mathcal{O}_\mu - \tfrac{1}{12}\partial^2 \mathcal{O} \qquad (244)
\end{aligned}$$

(d) Use the results of (b) and (c) to express the Taylor expansion in (a) purely in terms of primaries and descendants. Thus obtain the terms in the OPE coefficients $C_{\mathcal{O}}(x, \partial_y)$ for the operator $\mathcal{O}(y) =: \phi_1(y)\phi_2(y) :$ up to quadratic order in x.

$$\begin{aligned}
&\phi_1(x)\phi_2(0) = \\
&= \frac{1}{x^2} + : \phi_1 \phi_2 + x^\mu \phi_2 \partial_\mu \phi_1 + \tfrac{1}{2} x^\mu x^\nu \phi_2 \partial_\mu \partial_\nu \phi_1 : \\
&= \frac{1}{x^2} + : \mathcal{O} + \tfrac{1}{2} x^\mu \left(\partial_\mu \mathcal{O} - \mathcal{O}_\mu\right) + \tfrac{1}{2} x^\mu x^\nu \left(\tfrac{1}{6}\mathcal{O}_{\mu\nu} + \tfrac{1}{3}\partial_\mu \partial_\nu \mathcal{O} + \tfrac{1}{12}\mathcal{O}^\rho{}_\rho \eta_{\mu\nu} - \tfrac{1}{2}\partial_\mu \mathcal{O}_\nu - \tfrac{1}{2}\partial_\nu \mathcal{O}_\mu - \tfrac{1}{12}\partial^2 \mathcal{O}\right) : \\
&= \frac{1}{x^2} + : \left(1 + \frac{1}{2} x^\mu \partial_\mu + \frac{1}{6} x^\mu x^\nu \partial_\mu \partial_\nu - \frac{1}{24} x^2 \partial^2\right) \mathcal{O} - \tfrac{1}{2} x^\mu \mathcal{O}_\mu + \tfrac{1}{12} x^\mu x^\nu \mathcal{O}_{\mu\nu} + \tfrac{1}{24} x^2 \mathcal{O}^\rho{}_\rho - \tfrac{1}{4} x^\mu x^\nu \partial_\mu \mathcal{O}_\nu - \tfrac{1}{4} x^\mu x^\nu \partial_\nu \mathcal{O}_\mu :
\end{aligned}$$

The co-efficients of $\mathcal{O}$ give $C_{\mathcal{O}}(x, \partial_y)$

$$C_{\mathcal{O}}(x, \partial_y) = 1 + \frac{1}{2} x^\mu \partial_\mu + \frac{1}{6} x^\mu x^\nu \partial_\mu \partial_\nu - \frac{1}{24} x^2 \partial^2 + \cdots \qquad (245)$$

Q2

(a) By matching the 3-point function to the OPE, show that, for $\Delta_1 = \Delta_2 = \delta$ and $\Delta_\Phi = \Delta$, we have that

$$C_\Phi\left(x, \partial_y\right) = \frac{1}{|x|^{2\delta - \Delta}}\left[1 + \tfrac{1}{2}x^\mu \partial_\mu +^\mu x^\nu \partial_{\mu\nu} +^2 \partial^2 + \cdots\right] \tag{246}$$

where

$$\alpha = \frac{\Delta + 2}{8(\Delta+1)} \qquad\qquad \beta = -\frac{\Delta}{16\left(\Delta - \frac{D}{2} + 1\right)(\Delta+1)} \tag{247}$$

$$\begin{aligned}
\langle \phi_1(x)\phi_2(0)\Phi(z)\rangle =& \sum_{\mathcal{O}} C_{\mathcal{O}}(x,\partial_y)\left\langle \mathcal{O}(y)|_{y=0}\ \Phi(z)\right\rangle \\
=& C_\Phi(x,\partial_y)\ \left\langle \Phi(y)\Phi(z)\right\rangle|_{y=0} \\
=& C_\Phi(x,\partial_y)\ \left.\frac{1}{|y-z|^{2\Delta}}\right|_{y=0} \\
=& \left(A + Bx^\mu\partial_\mu + \alpha x^\mu x^\nu \partial_\mu\partial_\nu + \beta x^2\partial^2 + \mathcal{O}\left(x^3 y\right)\right)\left.|y-z|^{-2\Delta}\right|_{y=0}
\end{aligned} \tag{248}$$

$$(5) = (6)$$

$$\frac{|z|^{-2\Delta}}{|x|^{2\delta-\Delta}}\left(1 + \Delta\frac{\mathbf{x}\cdot\mathbf{z}}{|z|^2} - \frac{\Delta}{2}\frac{|x|^2}{|z|^2} + \Delta\left(\Delta+2\right)\left(\frac{\mathbf{x}\cdot\mathbf{z}}{|z|^2}\right)^2\right) = \left(A + Bx^\mu\partial_\mu + \alpha x^\mu x^\nu\partial_\mu\partial_\nu + \beta x^2\partial^2\right)\left.|y-z|^{-2\Delta}\right|_{y=0}$$

Comparing first terms on each side:

$$\begin{aligned}
\frac{|z|^{-2\Delta}}{|x|^{2\delta-\Delta}} =& A\left.|y-z|^{-2\Delta}\right|_{y=0} \\
\frac{|z|^{-2\Delta}}{|x|^{2\delta-\Delta}} =& A|z|^{-2\Delta}
\end{aligned}$$

$$\boxed{A = \frac{1}{|x|^{2\delta-\Delta}}}$$

Second terms:

$$\begin{aligned}
\frac{|z|^{-2\Delta}}{|x|^{2\delta-\Delta}}\left(\Delta\frac{\mathbf{x}\cdot\mathbf{z}}{|z|^2}\right) =& Bx^\mu\partial_\mu\ \left.|y-z|^{-2\Delta}\right|_{y=0} \\
\frac{|z|^{-2\Delta-2}}{|x|^{2\delta-\Delta}}\Delta\mathbf{x}\cdot\mathbf{z} =& Bx^\mu\partial_\mu\ \left.\left[\left((y-z)^2\right)^{-\Delta}\right]\right|_{y=0} \\
\frac{|z|^{-2\Delta-2}}{|x|^{2\delta-\Delta}}\Delta\mathbf{x}\cdot\mathbf{z} =& Bx^\mu\ \left.\left[-2(y-z)_\mu\Delta\left((y-z)^2\right)^{-\Delta-1}\right]\right|_{y=0} \\
\frac{|z|^{-2\Delta-2}}{|x|^{2\delta-\Delta}}\mathbf{x}\cdot\mathbf{z} =& -2Bx^\mu\ \left.\left[(y-z)_\mu\left((y-z)^2\right)^{-\Delta-1}\right]\right|_{y=0} \\
\frac{|z|^{-2\Delta-2}}{|x|^{2\delta-\Delta}}\mathbf{x}\cdot\mathbf{z} =& 2Bx^\mu z_\mu\left(z^2\right)^{-\Delta-1} \\
\frac{|z|^{-2\Delta-2}}{|x|^{2\delta-\Delta}}\mathbf{x}\cdot\mathbf{z} =& 2B\mathbf{x}\cdot\mathbf{z}|z|^{-2\Delta-2} \\
\frac{1}{|x|^{2\delta-\Delta}} =& 2B
\end{aligned}$$

$$\boxed{B = \frac{1}{2}\frac{1}{|x|^{2\delta-\Delta}}}$$

Calculating the term in α

$$
\begin{aligned}
\alpha x^\mu x^\nu \partial_\mu \partial_\nu \left|y-z\right|^{-2\Delta}\Big|_{y=0} =& \alpha x^\mu x^\nu \partial_\mu \partial_\nu \left[(y-z)^2\right]^{-\Delta}\Big|_{y=0} \\
=& \alpha x^\mu x^\nu \partial_\mu \left[-2\Delta (y-z)_\nu \left((y-z)^2\right)^{-\Delta-1}\right]\Big|_{y=0} \\
=& -2\Delta \alpha x^\mu x^\nu \partial_\mu \left[(y-z)_\nu \left((y-z)^2\right)^{-\Delta-1}\right]\Big|_{y=0} \\
=& -2\alpha\Delta x^\mu x^\nu \left[\delta_{\mu\nu}\left((y-z)^2\right)^{-\Delta-1} - 2(\Delta+1)(y-z)_\mu (y-z)_\nu \left((y-z)^2\right)^{-\Delta-2}\right]\Big|_{y=0} \\
=& -2\alpha\Delta x^\mu x^\nu \left[\delta_{\mu\nu}|y-z|^{-2\Delta-2} - 2(\Delta+1)(y-z)_\mu (y-z)_\nu |y-z|^{-2\Delta-4}\right]\Big|_{y=0} \\
=& -2\alpha\Delta x^\mu x^\nu \left[\delta_{\mu\nu}|z|^{-2\Delta-2} - 2(\Delta+1) z_\mu z_\nu |z|^{-2\Delta-4}\right] \\
=& -2\alpha\Delta x^\mu x^\nu |z|^{-2\Delta-2} \left[\delta_{\mu\nu} - 2(\Delta+1) z_\mu z_\nu |z|^{-2}\right] \\
=& -2\alpha\Delta |z|^{-2\Delta-2} \left[x^2 - 2(\Delta+1)\left(\mathbf{x}\cdot\mathbf{z}\right)^2 |z|^{-2}\right] \\
=& |z|^{-2\Delta}\left(-2\alpha\Delta\frac{|x|^2}{|z|^2} + 4\alpha\Delta(\Delta+1)\left(\frac{\mathbf{x}\cdot\mathbf{z}}{|z|^2}\right)^2\right)
\end{aligned}
\tag{249}
$$

Calculating the term in β

$$
\begin{aligned}
\beta x^2 \partial^2 \left|y-z\right|^{-2\Delta}\Big|_{y=0} =& \beta x^2 \partial^\mu \partial_\mu \left((y-z)^2\right)^{-\Delta}\Big|_{y=0} \\
=& -\beta\Delta x^2 \partial^\mu \left[2(y-z)_\mu \left((y-z)^2\right)^{-\Delta-1}\right]\Big|_{y=0} \\
=& -2\beta\Delta x^2 \partial^\mu \left[(y-z)_\mu \left((y-z)^2\right)^{-\Delta-1}\right]\Big|_{y=0} \\
=& -2\beta\Delta x^2 \partial^\mu \left[\delta^\mu_\mu \left((y-z)^2\right)^{-\Delta-1} - 2(\Delta+1)(y-z)^\mu (y-z)_\mu \left((y-z)^2\right)^{-\Delta-2}\right]\Big|_{y=0} \\
=& -2\beta\Delta x^2 \left[D\left((y-z)^2\right)^{-\Delta-1} - 2(\Delta+1)(y-z)^2 \left((y-z)^2\right)^{-\Delta-2}\right]\Big|_{y=0} \\
=& -2\beta\Delta x^2 \left[D\left((y-z)^2\right)^{-\Delta-1} - 2(\Delta+1) \left((y-z)^2\right)^{-\Delta-1}\right]\Big|_{y=0} \\
=& -2\beta\Delta x^2 \left(D - 2(\Delta+1)\right)\left((y-z)^2\right)^{-\Delta-1}\Big|_{y=0} \\
=& -2\beta\Delta x^2 \left(D - 2(\Delta+1)\right)|y-z|^{-2\Delta-2}\Big|_{y=0} \\
=& -2\beta\Delta x^2 \left(D - 2(\Delta+1)\right)|z|^{-2\Delta-2} \\
=& -2\beta\Delta x^2 \left(D - 2(\Delta+1)\right)|z|^{-2\Delta-2} \\
=& |z|^{-2\Delta}\left(-2\beta\Delta\left(D-2(\Delta+1)\right)\frac{|x|^2}{|z|^2}\right)
\end{aligned}
\tag{250}
$$

Now equating the 3rd and 4th terms in (5) with the 3rd and 4th terms in (6), and inserting (7) and (8) :

$$
\begin{aligned}
\frac{|z|^{-2\Delta}}{|x|^{2\delta-\Delta}}\left(-\frac{\Delta}{2}\frac{|x|^2}{|z|^2} + \frac{\Delta\left(\Delta+2\right)}{2}\left(\frac{\mathbf{x}\cdot\mathbf{z}}{|z|^2}\right)^2\right) &= \left(\alpha x^\mu x^\nu \partial_\mu\partial_\nu + \beta x^2\partial^2\right)\left|y-z\right|^{-2\Delta}\Big|_{y=0} \\
\frac{|z|^{-2\Delta}}{|x|^{2\delta-\Delta}}\left(-\frac{\Delta}{2}\frac{|x|^2}{|z|^2} + \frac{\Delta\left(\Delta+2\right)}{2}\left(\frac{\mathbf{x}\cdot\mathbf{z}}{|z|^2}\right)^2\right) &= |z|^{-2\Delta}\left(-2\alpha\Delta\frac{|x|^2}{|z|^2} + 4\alpha\Delta(\Delta+1)\left(\frac{\mathbf{x}\cdot\mathbf{z}}{|z|^2}\right)^2 - 2\beta\Delta\left(D-2(\Delta+1)\right)\frac{|x|^2}{|z|^2}\right) \\
\frac{1}{|x|^{2\delta-\Delta}}\left(-\frac{1}{2}\frac{|x|^2}{|z|^2} + \frac{\left(\Delta+2\right)}{2}\left(\frac{\mathbf{x}\cdot\mathbf{z}}{|z|^2}\right)^2\right) &= -2\alpha\frac{|x|^2}{|z|^2} + 4\alpha(\Delta+1)\left(\frac{\mathbf{x}\cdot\mathbf{z}}{|z|^2}\right)^2 - 2\beta\left(D-2(\Delta+1)\right)\frac{|x|^2}{|z|^2} \\
\frac{1}{|x|^{2\delta-\Delta}}\left(-\frac{1}{2}\frac{|x|^2}{|z|^2} + \frac{\left(\Delta+2\right)}{2}\left(\frac{\mathbf{x}\cdot\mathbf{z}}{|z|^2}\right)^2\right) &= \left(-2\alpha + -2\beta\left(D-2(\Delta+1)\right)\right)\frac{|x|^2}{|z|^2} + 4\alpha(\Delta+1)\left(\frac{\mathbf{x}\cdot\mathbf{z}}{|z|^2}\right)^2
\end{aligned}
$$

Comparing the coefficients of $\left(\frac{\mathbf{x}\cdot\mathbf{z}}{|z|^2}\right)^2$

$$\begin{aligned}
\frac{1}{|x|^{2\delta-\Delta}}\frac{(\Delta+2)}{2}\left(\frac{\mathbf{x}\cdot\mathbf{z}}{|z|^2}\right)^2 =&4\alpha(\Delta+1)\left(\frac{\mathbf{x}\cdot\mathbf{z}}{|z|^2}\right)^2\\
\frac{1}{|x|^{2\delta-\Delta}}\frac{\Delta+2}{2} =&4\alpha(\Delta+1)\\
&\boxed{\alpha=\frac{1}{x^{2\delta-\Delta}}\frac{\Delta+2}{8(\Delta+1)}}
\end{aligned}$$

Comparing the coefficients of $\frac{|x|^2}{|z|^2}$

$$\begin{aligned}
-\frac{1}{2}\frac{1}{|x|^{2\delta-\Delta}} =&-2\alpha-2\beta\left(D-2(\Delta+1)\right)\\
-\frac{1}{2}\frac{1}{|x|^{2\delta-\Delta}} =&-2\left(\frac{1}{x^{2\delta-\Delta}}\frac{\Delta+2}{8(\Delta+1)}\right)-2\beta\left(D-2(\Delta+1)\right)\\
-\frac{1}{2}\frac{1}{|x|^{2\delta-\Delta}} =&-\frac{1}{x^{2\delta-\Delta}}\frac{\Delta+2}{4(\Delta+1)}-2\beta\left(D-2(\Delta+1)\right)\\
\frac{1}{x^{2\delta-\Delta}}\left(\frac{(\Delta+2)}{4(\Delta+1)}-\frac{1}{2}\right) =&-2\beta\left(D-2(\Delta+1)\right)\\
\frac{1}{x^{2\delta-\Delta}}\left(\frac{\Delta+2-2(\Delta+1)}{4(\Delta+1)}\right) =&-2\beta\left(D-2(\Delta+1)\right)\\
\frac{1}{x^{2\delta-\Delta}}\left(\frac{-\Delta}{4(\Delta+1)}\right) =&-2\beta\left(D-2(\Delta+1)\right)\\
\beta =&-\frac{1}{x^{2\delta-\Delta}}\left(\frac{-\Delta}{8\left(D-2(\Delta+1)\right)(\Delta+1)}\right)\\
&\boxed{\beta=-\frac{1}{x^{2\delta-\Delta}}\left(\frac{\Delta}{16\left(\Delta+1-\frac{D}{2}\right)(\Delta+1)}\right)}
\end{aligned}$$

Now returning to the expression for $C_\Phi(x,\partial_y)$

$$C_\Phi(x,\partial_y)=\left(A+Bx^\mu\partial_\mu+\alpha x^\mu x^\nu\partial_\mu\partial_\nu+\beta x^2\partial^2+\cdots\right)|y-z|^{-2\Delta}\Big|_{y=0} \tag{251}$$

$$=\frac{1}{x^{2\delta-\Delta}}\left(1+\frac{1}{2}x^\mu\partial_\mu+\frac{\Delta+2}{8(\Delta+1)}x^\mu x^\nu\partial_\mu\partial_\nu-\frac{\Delta}{16\left(\Delta+1-\frac{D}{2}\right)(\Delta+1)}x^2\partial^2+\cdots\right) \tag{252}$$

(b) Check that this is consistent with the results of question 1.

The result (252) corresponds to (245) if we set $\Delta=2$ (because this is the weight of ϕ^2). Also, we were given in question 1 that we are in four dimensional spacetime, so $D=4$.

$$C_\Phi(x,\partial_y)=\frac{1}{x^{2\delta-\Delta}}\left(1+\frac{1}{2}x^\mu\partial_\mu+\frac{1}{6}x^\mu x^\nu\partial_\mu\partial_\nu-\frac{1}{24}x^2\partial^2+\cdots\right)$$

---

9 General Relativity

Q1 Prove the following

(a) if P^μ is time-like and $P^\mu S_\mu = 0$ then S^μ is space-like

---

Because P^μ is timelike, we choose a frame in which $P^\mu = \left(P^0, \mathbf{0}\right)$.

$$P^\mu S_\mu = S^0 = 0 \implies S^\mu = (0, \mathbf{S})$$

Therefore, we have that $S^\mu S_\mu > 0$, and S^μ is spacelike.

(b) if P^μ and Q^μ are time-like and $P^\mu Q_\mu < 0$ then either both are future-pointing or both are past-pointing

---

Choose a frame in which $P^\mu = \left(P^0, \mathbf{0}\right)$ and $Q^\mu = \left(Q^0, \mathbf{0}\right)$

$$\begin{aligned} P^\mu Q_\mu &< 0 \\ -P^0 Q^0 &< 0 \\ P^0 Q^0 &> 0 \end{aligned}$$

Therefore, we have that either $P^0,\ Q^0 > 0$ or $P^0,\ Q^0 < 0$.

(c) if P^μ and Q^μ are null and $P^\mu Q_\mu = 0$ then P^μ and Q^μ are proportional to each other

---

Using the fact that P^μ and Q^μ are null

$$-\left(P^0\right)^2 + |\mathbf{P}|^2 = 0 \tag{253}$$
$$\left(P^0\right)^2 = |\mathbf{P}|^2 \tag{254}$$

So we will also have $\left(Q^0\right)^2 = |\mathbf{Q}|^2$. If we denote the angle between $\mathbf{P}$ and $\mathbf{Q}$ as θ, we can express the second condition as

$$0 = P^\mu Q_\mu \tag{255}$$
$$0 = -P^0 Q^0 + |\mathbf{P}|\,|\mathbf{Q}| \cos\theta \tag{256}$$
$$0 = \left(P^0\right)^2 \left(Q^0\right)^2 + |\mathbf{P}|^2 |\mathbf{Q}|^2 \cos^2\theta \tag{257}$$
$$0 = \left(P^0\right)^2 \left(Q^0\right)^2 \left(1 + \cos^2\theta\right) \tag{258}$$

Assuming that P^0 and Q^0 are non-zero, we have that $\theta = 0$ or $\theta = \pi$. In the case that $\theta = 0$, so we can express $\mathbf{P}$ as $c\mathbf{Q}$ where $c \in \mathbb{R}_{>0}$. Returning to (254)

$$P^0 = \pm |\mathbf{P}| \tag{259}$$
$$P^0 = \pm c\,|\mathbf{Q}| \tag{260}$$
$$P^0 = cQ^0 \qquad \begin{cases} \text{Using line (256)} \\ P^0 Q^0 = |\mathbf{P}|\,|\mathbf{Q}| \\ \dfrac{P^0}{|\mathbf{P}|} = \dfrac{|\mathbf{Q}|}{Q^0} \\ \text{Therefore, if } P^0 = |\mathbf{P}| \text{ then } Q^0 = |\mathbf{Q}| \\ \text{ and if } P^0 = -|\mathbf{P}| \text{ then } Q^0 = -|\mathbf{Q}| \end{cases} \tag{261}$$

Therefore $P^\mu = \begin{pmatrix} P^0 \\ \mathbf{P} \end{pmatrix} = c \begin{pmatrix} Q^0 \\ \mathbf{Q} \end{pmatrix}$

Similarly, in the case that $\theta = \pi$, so we can express $\mathbf{P}$ as $-c\mathbf{Q}$ where $c \in \mathbb{R}_{>0}$. Returning to (254)

$$P^0 = \pm |\mathbf{P}| \tag{262}$$

$$P^0 = \mp c |\mathbf{Q}| \tag{263}$$

$$P^0 = cQ^0 \qquad \begin{cases} \text{Using line (256)} \\ P^0 Q^0 = -|\mathbf{P}||\mathbf{Q}| \\ \dfrac{P^0}{|\mathbf{P}|} = -\dfrac{|\mathbf{Q}|}{Q^0} \\ \text{Therefore, if } P^0 = |\mathbf{P}| \text{ then } Q^0 = -|\mathbf{Q}| \\ \text{ and if } P^0 = -|\mathbf{P}| \text{ then } Q^0 = |\mathbf{Q}| \end{cases} \tag{264}$$

Therefore we have that $P^\mu = \begin{pmatrix} P^0 \\ \mathbf{P} \end{pmatrix} = c \begin{pmatrix} Q^0 \\ \mathbf{Q} \end{pmatrix}$ in this case too.

(d) if P^μ is null and $P^\mu R_\mu = 0$ then either R^μ is space-like or R^μ is proportional to P^μ

$$P^\mu R_\mu = 0 \tag{265}$$

$$-P^0 R^0 + |\mathbf{P}||\mathbf{R}| \cos\theta = 0 \tag{266}$$

$$P^0 R^0 = |\mathbf{P}||\mathbf{R}| \cos\theta \tag{267}$$

$$\left(P^0\right)^2 \left(R^0\right)^2 - |\mathbf{P}|^2 |\mathbf{R}|^2 \cos^2\theta = 0 \tag{268}$$

$$\left(P^0\right)^2 \left(\left(R^0\right)^2 - |\mathbf{R}|^2 \cos^2\theta\right) = 0 \tag{269}$$

$$\left(R^0\right)^2 = |\mathbf{R}|^2 \cos^2\theta \tag{270}$$

$$|R^0| \leq |\mathbf{R}| \tag{271}$$

On line (269), we used that P^μ is null so that $|\mathbf{P}|^2 = \left(P^0\right)^2$. From line (271), we have that either $|R^0| < |\mathbf{R}|$ (which means that R^μ is space-like) or $|R^0| = |\mathbf{R}|$ (which means that R^μ is null). In the latter case, we have the same situation as question (c), so we deduce that P^μ is proportional to R^μ.

Q2 Show that if k^μ is a Killing vector and $M_{\mu\nu} = \nabla_{[\mu} k_{\nu]}$ then $\nabla_\mu M^{\mu\nu} = 0$ provided the background geometry satisfies Einstein's equations in the vacuum.

First, we prove that $\nabla_\mu \nabla_\nu k_\mu = 0$:

Because k is a Killing vector:

$$0 = \nabla_\nu k_\lambda + \nabla_\lambda k_\nu \tag{272}$$

$$(\nabla_\mu \rightarrow) \qquad 0 = \nabla_\mu \nabla_\nu k_\lambda + \nabla_\mu \nabla_\lambda k_\nu \tag{273}$$

... and expressing this in different indices:

$$0 = \nabla_\mu k_\nu + \nabla_\nu k_\mu \tag{274}$$

$$(\nabla_\lambda \rightarrow) \qquad 0 = \nabla_\lambda \nabla_\mu k_\nu + \nabla_\lambda \nabla_\nu k_\mu \tag{275}$$

$$
\begin{aligned}
0 &= (2) - (4) \\
0 &= \nabla_\mu \nabla_\nu k_\lambda + \nabla_\mu \nabla_\lambda k_\nu - (\nabla_\lambda \nabla_\mu k_\nu + \nabla_\lambda \nabla_\nu k_\mu) \\
&= \nabla_\mu \nabla_\nu k_\lambda + [\nabla_\mu, \nabla_\lambda] k_\nu - \nabla_\lambda \nabla_\nu k_\mu \\
&= \nabla_\mu \nabla_\nu k_\lambda + [\nabla_\mu, \nabla_\lambda] k_\nu - [\nabla_\lambda, \nabla_\nu] k_\mu - \nabla_\nu \nabla_\lambda k_\mu && \text{Killing condition again :} \\
&= \nabla_\mu \nabla_\nu k_\lambda + [\nabla_\mu, \nabla_\lambda] k_\nu - [\nabla_\lambda, \nabla_\nu] k_\mu + \nabla_\nu \nabla_\mu k_\lambda && \leftarrow \nabla_\lambda k_\mu = -\nabla_\mu k_\lambda \\
&= 2\nabla_\mu \nabla_\nu k_\lambda + [\nabla_\mu, \nabla_\lambda] k_\nu - [\nabla_\lambda, \nabla_\nu] k_\mu + [\nabla_\nu, \nabla_\mu] k_\lambda \\
&= 2\nabla_\mu \nabla_\nu k_\lambda + [\nabla_\mu, \nabla_\lambda] k_\nu + [\nabla_\nu, \nabla_\lambda] k_\mu + [\nabla_\nu, \nabla_\mu] k_\lambda \\
&= 2\nabla_\mu \nabla_\nu k_\lambda - \left(R^\alpha{}_{\nu\mu\lambda} + R^\alpha{}_{\mu\nu\lambda} + R^\alpha{}_{\lambda\nu\mu}\right) k_\alpha && \leftarrow R^\alpha{}_{\mu\nu\lambda} k_\alpha \equiv -[\nabla_\mu, \nabla_\nu] k_\lambda \\
& && R^\alpha{}_{\nu\mu\lambda} + R^\alpha{}_{\lambda\nu\mu} + R^\alpha{}_{\mu\lambda\nu} = 0 \\
&= 2\nabla_\mu \nabla_\nu k_\lambda - \left(R^\alpha{}_{\mu\nu\lambda} - R^\alpha{}_{\mu\lambda\nu}\right) k_\alpha && \leftarrow R^\alpha{}_{\nu\mu\lambda} + R^\alpha{}_{\lambda\nu\mu} = -R^\alpha{}_{\mu\lambda\nu} \\
& && \leftarrow R^\alpha{}_{\mu\nu\lambda} = -R^\alpha{}_{\mu\lambda\nu} \\
0 &= 2\nabla_\mu \nabla_\nu k_\lambda + 2R^\alpha{}_{\mu\nu\lambda} k_\alpha \\
0 &= \nabla_\mu \nabla_\nu k_\lambda + R^\alpha{}_{\mu\nu\lambda} k_\alpha \\
&= \nabla_\mu \nabla_\nu k_\lambda - R^\mu{}_{\alpha\nu\lambda} k_\alpha && \leftarrow R^\alpha{}_{\mu\nu\lambda} k_\alpha = -R^\mu{}_{\alpha\nu\lambda} \\
&= \nabla_\mu \nabla_\nu k_\mu - R^\mu{}_{\alpha\nu\mu} k_\alpha && \leftarrow \{\lambda \to \mu\} \\
&= \nabla_\mu \nabla_\nu k_\mu - R_{\alpha\nu} k_\alpha
\end{aligned}
$$

$R_{\alpha\nu}$ is the Ricci tensor, which is zero in vacuum

$$\therefore \nabla_\mu \nabla_\nu k_\mu = 0 \qquad (5)$$

We are given that:

$$
\begin{aligned}
M_{\mu\nu} &= \nabla_{[\mu} k_{\nu]} \\
M_{\mu\nu} &= \tfrac{1}{2}(\nabla_\mu k_\nu - \nabla_\nu k_\mu) \\
\nabla^\mu M_{\mu\nu} &= \tfrac{1}{2}(\nabla^\mu \nabla_\mu k_\nu - \nabla^\mu \nabla_\nu k_\mu) && \text{From Killing condition:} \\
\nabla^\mu M_{\mu\nu} &= -\nabla^\mu \nabla_\nu k_\mu && \leftarrow \nabla_\mu k_\nu = -\nabla_\nu k_\mu \\
& && \Longleftarrow (5)
\end{aligned}
$$

$$\boxed{\nabla^\mu M_{\mu\nu} = 0}$$

Q3 Consider the covariant actions of a scalar field ϕ and Maxwell 1-form field A_μ given by

$$S_{\text{scalar}} = \int d^4x \sqrt{-g} \left(-\tfrac{1}{2}\nabla_\mu \phi \nabla^\mu \phi - V(\phi)\right) \qquad (276)$$

$$S_{\text{Maxwell}} = \int d^4x \sqrt{-g} \left(-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}\right) \qquad (277)$$

with V any real function and $F_{\mu\nu}$ defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad (278)$$

(a) Derive the equations of motion for ϕ.

First, we need to derive the Euler-Lagrange equations for Lagrangian of this form. The action S is given by the integral of the Lagrangian over 4-volume V.

$$S = \int_V d^4x\sqrt{-g}\,(\mathcal{L}(\Phi, \nabla_\mu\Phi)) \tag{279}$$

$$\delta S = \int_V d^4x\sqrt{-g}\left(\frac{\partial\mathcal{L}}{\partial\Phi}\delta\Phi + \frac{\partial\mathcal{L}}{\partial(\nabla_\mu\Phi)}\delta(\nabla_\mu\Phi)\right) \tag{280}$$

$$= \int_V d^4x\sqrt{-g}\left(\frac{\partial\mathcal{L}}{\partial\Phi}\delta\Phi + \frac{\partial\mathcal{L}}{\partial(\nabla_\mu\Phi)}\nabla_\mu(\delta\Phi)\right) \tag{281}$$

$$= \int_V d^4x\sqrt{-g}\left(\frac{\partial\mathcal{L}}{\partial\Phi}\delta\Phi + \nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\Phi)}(\delta\Phi)\right) - \nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\Phi)}\right)\delta\Phi\right) \tag{282}$$

$$= \int_V d^4x\sqrt{-g}\left(\left(\frac{\partial\mathcal{L}}{\partial\Phi} - \nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\Phi)}\right)\right)\delta\Phi + \nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\Phi)}(\delta\Phi)\right)\right) \tag{283}$$

$$= \int_V d^4x\sqrt{-g}\left(\frac{\partial\mathcal{L}}{\partial\Phi} - \nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\Phi)}\right)\right)\delta\Phi + \int_V d^4x\sqrt{-g}\;\nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\Phi)}(\delta\Phi)\right) \tag{284}$$

Stokes' theorem allows us to express an integral of a field over a volume as the integral of the curl of the field over the surface. As the surface is at infinity, we can assume that this is zero.

$$\delta S = \int_V d^4x\sqrt{-g}\left(\frac{\partial\mathcal{L}}{\partial\Phi} - \nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\Phi)}\right)\right)\delta\Phi \tag{285}$$

From the principle of least action, we have that $\delta S = 0$. Using this, we arrive at the Euler-Lagrange equation.

$$\frac{\partial\mathcal{L}}{\partial\Phi} - \nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\Phi)}\right) = 0 \tag{286}$$

From (276), we have that the Lagrangian for S_{scalar} is

$$\mathcal{L}_{\text{scalar}} = -\tfrac{1}{2}\nabla_\mu\varphi\nabla^\mu\varphi - V(\varphi) \tag{287}$$

Substituting this into (286), we obtain the Euler-Lagrange equation for the scalar field

$$\frac{\partial V(\varphi)}{\partial\varphi} - \nabla_\mu\nabla^\mu\varphi = 0$$

(b) Derive the equations of motion for A_μ

We apply the Euler-Lagrange equation, derived on line (**??**)

$$\frac{\partial\mathcal{L}_{\text{Maxwell}}}{\partial A_\mu} - \nabla_\nu\left(\frac{\partial\mathcal{L}_{\text{Maxwell}}}{\partial(\nabla_\nu A_\mu)}\right) = 0 \tag{288}$$

Where $\mathcal{L}_{\text{Maxwell}}$ is defined by (292).

$$\mathcal{L}_{\text{Maxwell}} = -\tfrac{1}{4}\sqrt{-g}\;F_{\mu\nu}F^{\mu\nu} \tag{289}$$

We would like to express $\mathcal{L}_{\text{Maxwell}}$ in terms of A_μ. Therefore, we begin by defining $F_{\mu\nu}$ in terms of the covariant derivative using its definition $\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\alpha_{\mu\nu}A_\alpha$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{290}$$

$$= \nabla_\mu A_\nu + \Gamma^\alpha_{\mu\nu}A_\alpha - \left(\nabla_\nu A_\mu + \Gamma^\alpha_{\nu\mu}A_\alpha\right) \tag{291}$$

$$= \nabla_\mu A_\nu - \nabla_\nu A_\mu \tag{292}$$

Where we've obtained line (292) using the fact that the connection is torsion-free (which implies that $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}$). We now express the Lagrangian (277) in terms of A^μ

$$\mathcal{L}_{\text{Maxwell}} = -\tfrac{1}{4}\sqrt{-g}\,(\nabla_\mu A_\nu - \nabla_\nu A_\mu)\,(\nabla^\mu A^\nu - \nabla^\nu A^\mu) \tag{293}$$

From this we have the trivial result

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = 0 \tag{294}$$

The other calculation that we'll need is

$$\frac{\partial F_{\alpha\beta}}{\partial(\nabla_\sigma A_\tau)} = \frac{\partial(\nabla_\alpha A_\beta)}{\partial(\nabla_\sigma A_\tau)} - \frac{\partial(\nabla_\beta A_\alpha)}{\partial(\nabla_\sigma A_\tau)} \tag{295}$$

$$= \delta^\sigma_\alpha \delta^\tau_\beta - \delta^\sigma_\beta \delta^\tau_\alpha \tag{296}$$

Now, we may calculate the Euler-Lagrange equation (288

$$\begin{aligned}
0 =& \frac{\partial \mathcal{L}}{\partial A_\kappa} - \nabla_\lambda \left(\frac{\partial \mathcal{L}}{\partial(\nabla_\lambda A_\kappa)} \right) \\
=& \nabla_\lambda \left(\frac{\partial \mathcal{L}}{\partial(\nabla_\lambda A_\kappa)} \right) \\
=& -\tfrac{1}{4} g^{\gamma\alpha} g^{\delta\beta} \nabla_\lambda \left(\frac{\partial F_{\alpha\beta}}{\partial(\nabla_\lambda A_\kappa)} F_{\gamma\delta} + F_{\alpha\beta} \frac{\partial F_{\gamma\delta}}{\partial(\nabla_\lambda A_\kappa)} \right) \\
=& -\tfrac{1}{4} g^{\gamma\alpha} g^{\delta\beta} \nabla_\lambda \left(\left(\delta^\lambda_\alpha \delta^\kappa_\beta - \delta^\lambda_\beta \delta^\kappa_\alpha \right) F_{\gamma\delta} + F_{\alpha\beta} \left(\delta^\lambda_\gamma \delta^\kappa_\delta - \delta^\lambda_\delta \delta^\kappa_\gamma \right) \right) \\
=& -\tfrac{1}{4} g^{\gamma\alpha} g^{\delta\beta} \nabla_\lambda \left(\left(\delta^\lambda_\alpha \delta^\kappa_\beta - \delta^\lambda_\beta \delta^\kappa_\alpha \right) F_{\gamma\delta} + F_{\alpha\beta} \left(\delta^\lambda_\gamma \delta^\kappa_\delta - \delta^\lambda_\delta \delta^\kappa_\gamma \right) \right) \\
0 =& -\tfrac{1}{4} \nabla_\lambda \left\{ \left(g^{\gamma\lambda} g^{\delta\kappa} - g^{\gamma\kappa} g^{\delta\lambda} \right) F_{\gamma\delta} + \left(g^{\lambda\alpha} g^{\kappa\beta} - g^{\kappa\alpha} g^{\lambda\beta} \right) F_{\alpha\beta} \right\} \\
=& -\tfrac{1}{4} \nabla_\lambda \left\{ \left(g^{\gamma\lambda} g^{\delta\kappa} - g^{\gamma\kappa} g^{\delta\lambda} \right) \left(\nabla_\gamma A_\delta - \nabla_\delta A_\gamma \right) \right. \\
& \left. + \left(\nabla_\alpha A_\beta - \nabla_\beta A_\alpha \right) \left(g^{\lambda\alpha} g^{\kappa\beta} - g^{\kappa\alpha} g^{\lambda\beta} \right) \right. && \textit{program@epstopdf} \\
=& \tfrac{1}{4} \nabla_\lambda \left\{ g^{\gamma\lambda} g^{\delta\kappa} g^{\gamma\kappa} g^{\delta\lambda} \nabla_\delta A_\gamma \nabla_\delta A_\gamma \right. \\
& \left. + \nabla_\alpha A_\beta \nabla_\beta A_\alpha g^{\lambda\alpha} g^{\kappa\beta} g^{\kappa\alpha} g^{\lambda\beta} \right. && \textit{program@epstopdf} \\
=& \tfrac{1}{4} \nabla_\lambda \left\{ \nabla^\kappa A^\lambda + \nabla^\kappa A^\lambda + \nabla^\lambda A^\kappa + \nabla^\lambda A^\kappa + \nabla^\lambda A^\kappa + \nabla^\kappa A^\lambda + \nabla^\kappa A^\lambda + \nabla^\lambda A^\kappa \right\} \\
=& \nabla_\lambda \nabla^\lambda A^\kappa + \nabla_\lambda \nabla^\kappa A_\lambda \\
=& \nabla^2 A^\kappa + \nabla_\lambda \nabla^\kappa A_\lambda \\
=& \nabla^2 A^\kappa + \nabla_\lambda \nabla^\kappa A_\lambda
\end{aligned}$$

There is an alternative route from line (**??**):

$$\begin{aligned}
0 =& -\tfrac{1}{4} \nabla_\lambda \left\{ \left(g^{\gamma\lambda} g^{\delta\kappa} - g^{\gamma\kappa} g^{\delta\lambda} \right) F_{\gamma\delta} + \left(g^{\lambda\alpha} g^{\kappa\beta} - g^{\kappa\alpha} g^{\lambda\beta} \right) F_{\alpha\beta} \right\} \\
=& -\tfrac{1}{4} \nabla_\lambda \left\{ F^{\lambda\kappa} - F^{\kappa\lambda} + F^{\lambda\kappa} - F^{\kappa\lambda} \right\} \\
=& -\tfrac{1}{2} \nabla_\lambda \left\{ F^{\lambda\kappa} - F^{\kappa\lambda} \right\} \\
=& -\nabla_\lambda F^{\lambda\kappa} \\
=& \nabla_\lambda F^{\lambda\kappa}
\end{aligned}$$

Where we obtained line (297) using the antisymmetry of $F_{\mu\nu}$. Substitute in line (292) to see that this is equivalent to (**??**).

(c) Derive an expression for the stress tensor $T^{\text{scalar}}_{\mu\nu}$ by considering the variation of these actions with respect to the background metric.

First we will need to prove $\delta(\sqrt{-g}) = -\frac{1}{2}\sqrt{-g}\; g_{\mu\nu}\delta g^{\mu\nu}$

$$\delta(\sqrt{-g}) = -\frac{1}{2}\frac{1}{\sqrt{-g}}\delta g = \frac{1}{2}\frac{g}{\sqrt{-g}} g_{\mu\nu}\delta g^{\mu\nu} = -\frac{1}{2}\sqrt{-g}\; g_{\mu\nu}\delta g^{\mu\nu} \tag{297}$$

Where in the second step, we used that $\delta g_{\mu\nu} = -g_{\mu\alpha}g_{\nu\beta}\delta^{\alpha\beta}$.
Now, we vary the action

$$S_{\text{scalar}} = \int \sqrt{-g}\left(-\tfrac{1}{2}g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - V(\varphi)\right) \tag{298}$$

$$\delta S_{\text{scalar}} = \int \sqrt{-g}\left(-\tfrac{1}{2}\delta g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi\right) + \int \delta\left(\sqrt{-g}\right)\left(-\tfrac{1}{2}g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - V(\varphi)\right) \tag{299}$$

$$= \int \sqrt{-g}\left(-\tfrac{1}{2}\delta g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi\right) - \tfrac{1}{2}\int \sqrt{-g}\; g_{\mu\nu}\delta g^{\mu\nu}\left(-\tfrac{1}{2}g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - V(\varphi)\right) \tag{300}$$

$$= -\tfrac{1}{2}\int \sqrt{-g}\;\delta g_{\mu\nu}\left\{\nabla_\mu\varphi\nabla_\nu\varphi - \left(\tfrac{1}{2}g_{\mu\nu}g^{\rho\sigma}\nabla_\rho\varphi\nabla_\sigma\varphi + V(\varphi)\right)\right\} \tag{301}$$

By definition of the energy-momentum tensor:

$$T^{\text{scalar}}_{\mu\nu} \equiv -2\frac{1}{\sqrt{-g}}\frac{\delta S^{\text{scalar}}}{\delta g^{\mu\nu}} \tag{302}$$

$$T^{\text{scalar}}_{\mu\nu} = \nabla_\mu\varphi\nabla_\nu\varphi - \tfrac{1}{2}g_{\mu\nu}g^{\rho\sigma}\nabla_\rho\varphi\nabla_\sigma\varphi - g_{\mu\nu}V(\varphi) \tag{303}$$

(d) Derive the stress tensor $T^{\text{Maxwell}}_{\mu\nu}$.

$$S_{\text{Maxwell}} = -\int \tfrac{1}{4}\sqrt{-g}\; g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}$$

$$\delta S_{\text{Maxwell}} = -\int \tfrac{1}{4}\delta(\sqrt{-g})g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} - \int \tfrac{1}{4}\sqrt{-g}\;\delta g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} - \int \tfrac{1}{4}\sqrt{-g}\; g^{\mu\alpha}\delta g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}$$

$$= \int \tfrac{1}{8}\sqrt{-g}\; g_{\mu\nu}\delta g^{\mu\nu}g^{\rho\alpha}g^{\sigma\beta}F_{\rho\sigma}F_{\alpha\beta} - \int \tfrac{1}{4}\sqrt{-g}\;\delta g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} - \int \tfrac{1}{4}\sqrt{-g}\; g^{\mu\alpha}\delta g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}$$

$$= \int \tfrac{1}{8}\sqrt{-g}\; g_{\mu\nu}\delta g^{\mu\nu}g^{\rho\alpha}g^{\sigma\beta}F_{\rho\sigma}F_{\alpha\beta} - \int \tfrac{1}{4}\sqrt{-g}\;\delta g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} - \underbrace{\int \tfrac{1}{4}\sqrt{-g}\; g^{\nu\beta}\delta g^{\mu\alpha}F_{\nu\mu}F_{\beta\alpha}}_{\{\mu\leftrightarrow\nu,\ \alpha\leftrightarrow\beta\}}$$

$$= \int \tfrac{1}{8}\sqrt{-g}\; g_{\mu\nu}\delta g^{\mu\nu}g^{\rho\alpha}g^{\sigma\beta}F_{\rho\sigma}F_{\alpha\beta} - \int \tfrac{1}{4}\sqrt{-g}\;\delta g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} - \underbrace{\int \tfrac{1}{4}\sqrt{-g}\; g^{\nu\beta}\delta g^{\mu\alpha}F_{\mu\nu}F_{\alpha\beta}}_{\{F_{\nu\mu} = -F_{\mu\nu},\ F_{\beta\alpha} = -F_{\alpha\beta}\}}$$

$$= \int \tfrac{1}{8}\sqrt{-g}\; g_{\mu\nu}\delta g^{\mu\nu}g^{\rho\alpha}g^{\sigma\beta}F_{\rho\sigma}F_{\alpha\beta} - \tfrac{1}{2}\int \sqrt{-g}\;\delta g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}$$

$$= \int \tfrac{1}{8}\sqrt{-g}\; g_{\mu\nu}\delta g^{\mu\nu}g^{\rho\alpha}g^{\sigma\beta}F_{\rho\sigma}F_{\alpha\beta} - \tfrac{1}{2}\int \sqrt{-g}\;\delta g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}$$

$$= \int \tfrac{1}{8}\sqrt{-g}\; g_{\mu\nu}\delta g^{\mu\nu}g^{\rho\alpha}g^{\sigma\beta}F_{\rho\sigma}F_{\alpha\beta} - \underbrace{\tfrac{1}{2}\int \sqrt{-g}\;\delta g^{\mu\nu}g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}}_{\{\mu\leftrightarrow\alpha\}}$$

$$= \tfrac{1}{2}\int \sqrt{-g}\;\delta g^{\mu\nu}\left(\tfrac{1}{4}g_{\mu\nu}g^{\rho\alpha}g^{\sigma\beta}F_{\rho\sigma}F_{\alpha\beta} - g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta}\right)$$

Therefore, the stress tensor is

$$T^{\text{Maxwell}}_{\mu\nu} = -2\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}$$

$$T^{\text{Maxwell}}_{\mu\nu} = g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} - \tfrac{1}{4}g_{\mu\nu}g^{\rho\alpha}g^{\sigma\beta}F_{\rho\sigma}F_{\alpha\beta}$$

(e) Check explicitly that $\nabla_\mu T^{\mu\nu}_{\text{scalar}} = 0$ using the equations you derived in question 2.

$$\begin{aligned}
T^{\mu\nu}_{\text{scalar}} =& \nabla^\mu\varphi\nabla^\nu\varphi - \tfrac{1}{2}g^{\mu\nu}g^{\rho\sigma}\nabla_\rho\varphi\nabla_\sigma\varphi - g^{\mu\nu}V(\varphi) \\
\nabla_\mu T^{\mu\nu}_{\text{scalar}} =& \nabla_\mu\left(\nabla^\mu\varphi\nabla^\nu\varphi - \tfrac{1}{2}g^{\mu\nu}g^{\rho\sigma}\nabla_\rho\varphi\nabla_\sigma\varphi - g^{\mu\nu}V(\varphi)\right) \\
=& \nabla_\mu\nabla^\mu\varphi\nabla^\nu\varphi + \nabla^\mu\varphi\nabla_\mu\nabla^\nu\varphi - g^{\mu\nu}g^{\rho\sigma}\nabla_\rho\varphi\nabla_\mu\nabla_\sigma\varphi - g^{\mu\nu}\nabla_\mu V(\varphi) \\
=& \nabla_\mu\nabla^\mu\varphi\nabla^\nu\varphi + \nabla^\mu\varphi\nabla_\mu\nabla^\nu\varphi - \nabla_\rho\varphi\nabla^\nu\nabla^\rho\varphi - \nabla^\nu V(\varphi) \\
=& \nabla_\mu\nabla^\mu\varphi\nabla^\nu\varphi + \nabla^\mu\varphi\nabla_\mu\nabla^\nu\varphi - \nabla_\rho\varphi\nabla^\rho\nabla^\nu\varphi - \nabla^\nu V(\varphi) \\
& \{\nabla^\nu\nabla^\rho\varphi = \partial^\nu\partial^\rho\varphi = \partial^\rho\partial^\nu\varphi = \nabla^\rho\nabla^\nu\varphi\}\uparrow \\
=& \nabla_\mu\nabla^\mu\varphi\nabla^\nu\varphi + \cancel{\nabla^\mu\varphi\nabla_\mu\nabla^\nu\varphi} - \cancel{\nabla_\rho\varphi\nabla^\rho\nabla^\nu\varphi} - \nabla^\nu V(\varphi) \\
=& \nabla_\mu\nabla^\mu\varphi\nabla^\nu\varphi - \nabla^\nu V(\varphi) \\
=& \nabla_\mu\nabla^\mu\varphi\nabla^\nu\varphi - \frac{\partial V(\varphi)}{\partial\varphi}\nabla^\nu\varphi \\
=& \left(\nabla_\mu\nabla^\mu\varphi - \frac{\partial V(\varphi)}{\partial\varphi}\right)\nabla^\nu\varphi \\
=& 0
\end{aligned}$$

$$\begin{aligned}
\nabla_\mu\varphi =& \partial_\mu\varphi \\
& \because \varphi \text{ is a scalar}
\end{aligned}$$

$$\begin{aligned}
\nabla^\nu V(\varphi) = \partial^\nu V(\varphi) =& \frac{\partial V(\varphi)}{\partial\varphi}\partial^\nu\varphi \\
\Longleftarrow \quad =& \frac{\partial V(\varphi)}{\partial\varphi}\nabla^\nu\varphi
\end{aligned}$$

$\Longleftarrow$ Equations of motion:

$$0 = \frac{\partial V(\varphi)}{\partial\varphi} - \nabla_\mu\nabla^\mu\varphi$$

$$\therefore \boxed{\nabla_\mu T^{\mu\nu}_{\text{scalar}} = 0}$$

(f) Check explicitly that $\nabla_\mu T^{\mu\nu}_{\text{Maxwell}} = 0$.

First proving a result we'll need, $\nabla_\mu F_{\nu\beta} + \nabla_\beta F_{\mu\nu} + \nabla_\nu F_{\beta\mu} = 0$

$$\begin{aligned}
\nabla_\mu F_{\nu\beta} + \nabla_\beta F_{\mu\nu} + \nabla_\nu F_{\beta\mu} =& \partial_\mu F_{\nu\beta} - \Gamma^\alpha_{\mu\nu}F_{\alpha\beta} - \overbrace{\Gamma^\alpha_{\mu\beta}F_{\nu\alpha}} \\
& + \partial_\beta F_{\mu\nu} - \Gamma^\alpha_{\beta\mu}F_{\alpha\nu} - \Gamma^\alpha_{\beta\nu}F_{\mu\alpha} \\
& + \partial_\nu F_{\beta\mu} - \Gamma^\alpha_{\nu\beta}F_{\alpha\mu} - \underbrace{\Gamma^\alpha_{\nu\mu}F_{\beta\alpha}} \\
& \{F_{\sigma\tau} = -F_{\tau\sigma} \text{ and } \Gamma^\alpha_{\kappa\lambda} = \Gamma^\alpha_{\lambda\kappa}\} \\
=& \partial_\mu F_{\nu\beta} - \Gamma^\alpha_{\mu\nu}F_{\alpha\beta} + \Gamma^\alpha_{\beta\mu}F_{\alpha\nu} \\
& + \partial_\beta F_{\mu\nu} - \Gamma^\alpha_{\beta\mu}F_{\alpha\nu} + \Gamma^\alpha_{\nu\beta}F_{\alpha\mu} \\
& + \partial_\nu F_{\beta\mu} - \Gamma^\alpha_{\nu\beta}F_{\alpha\mu} + \Gamma^\alpha_{\mu\nu}F_{\alpha\beta} \\
=& \partial_\mu F_{\nu\beta} + \partial_\beta F_{\mu\nu} + \partial_\nu F_{\beta\mu} \\
=& \partial_\mu\left(\partial_\nu A_\beta - \partial_\beta A_\nu\right) \\
& + \partial_\beta\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \\
& + \partial_\nu\left(\partial_\beta A_\mu - \partial_\mu A_\beta\right) \\
=& 0
\end{aligned}$$

Using (7):

$$\begin{aligned}
\leftarrow F_{\sigma\tau} =& \nabla_\sigma A_\tau - \nabla_\tau A_\sigma \\
=& \partial_\sigma A_\tau - \partial_\tau A_\sigma
\end{aligned}$$

$$\therefore \nabla_\mu F_{\nu\beta} + \nabla_\beta F_{\mu\nu} + \nabla_\nu F_{\beta\mu} = 0 \qquad (9)$$

$$
\begin{aligned}
T^{\mu\nu}_{Maxwell} =& g_{\alpha\beta} F^{\mu\alpha} F^{\nu\beta} - \tfrac{1}{4} g^{\mu\nu} g^{\rho\alpha} g^{\sigma\beta} F_{\rho\sigma} F_{\alpha\beta} \\
=& g_{\alpha\beta} F^{\mu\alpha} F^{\nu\beta} - \tfrac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}
\end{aligned}
$$

$$
\begin{aligned}
\nabla_\mu T^{\mu\nu}_{Maxwell} =& g_{\alpha\beta} \left(\nabla_\mu F^{\mu\alpha}\right) F^{\nu\beta} + g_{\alpha\beta} F^{\mu\alpha} \nabla_\mu F^{\nu\beta} - \tfrac{1}{2} F_{\alpha\beta} \nabla^\nu F^{\alpha\beta} && \text{Equation of motion} \implies \nabla_\mu F^{\mu\alpha} = 0 \\
=& g_{\alpha\beta} \cancel{\left(\nabla_\mu F^{\mu\alpha}\right)} F^{\nu\beta} + g_{\alpha\beta} F^{\mu\alpha} \nabla_\mu F^{\nu\beta} - \tfrac{1}{2} F_{\alpha\beta} \nabla^\nu F^{\alpha\beta} \\
=& g_{\alpha\beta} F^{\mu\alpha} \nabla_\mu F^{\nu\beta} - \tfrac{1}{2} F_{\alpha\beta} \nabla^\nu F^{\alpha\beta} \\
=& F_{\mu\beta} \nabla^\mu F^{\nu\beta} - \tfrac{1}{2} F_{\alpha\beta} \nabla^\nu F^{\alpha\beta} \\
=& F_{\mu\beta} \nabla^\mu F^{\nu\beta} - \tfrac{1}{2} F_{\mu\beta} \nabla^\nu F^{\mu\beta} \\
=& F_{\mu\beta} \left(\nabla^\mu F^{\nu\beta} - \tfrac{1}{2} \nabla^\nu F^{\mu\beta}\right) && (*) \quad 0 = \nabla^\mu F^{\nu\beta} + \nabla^\beta F^{\mu\nu} + \nabla^\nu F^{\beta\mu} \\
=& F_{\mu\beta} \left(-\nabla^\beta F^{\mu\nu} - \nabla^\nu F^{\beta\mu} - \tfrac{1}{2} \nabla^\nu F^{\mu\beta}\right) && \leftarrow \nabla^\mu F^{\nu\beta} = -\nabla^\beta F^{\mu\nu} - \nabla^\nu F^{\beta\mu} \\
=& F_{\mu\beta} \left(-\nabla^\beta F^{\mu\nu} + \nabla^\nu F^{\mu\beta} - \tfrac{1}{2} \nabla^\nu F^{\mu\beta}\right) && \leftarrow F^{\beta\mu} = -F^{\mu\beta} \\
=& F_{\mu\beta} \left(-\nabla^\beta F^{\mu\nu} + \tfrac{1}{2} \nabla^\nu F^{\mu\beta}\right) \\
=& - F_{\mu\beta} \nabla^\beta F^{\mu\nu} + \tfrac{1}{2} F_{\mu\beta} \nabla^\nu F^{\mu\beta} \\
=& F_{\mu\beta} \nabla^\beta F^{\nu\mu} + \tfrac{1}{2} F_{\mu\beta} \nabla^\nu F^{\mu\beta} && \leftarrow F^{\mu\nu} = -F^{\nu\mu} \\
=& F_{\beta\mu} \nabla^\mu F^{\nu\beta} + \tfrac{1}{2} F_{\mu\beta} \nabla^\nu F^{\mu\beta} && \leftarrow \{\mu \leftrightarrow \beta\} \text{ in 1st term} \\
=& - F_{\mu\beta} \nabla^\mu F^{\nu\beta} + \tfrac{1}{2} F_{\mu\beta} \nabla^\nu F^{\mu\beta} && \leftarrow F^{\beta\mu} = -F^{\mu\beta} \\
=& - F_{\mu\beta} \left(\nabla^\mu F^{\nu\beta} - \tfrac{1}{2} \nabla^\nu F^{\mu\beta}\right) && (*)
\end{aligned}
$$

Comparing the 2 (*) terms, it can be seen that $\nabla_\mu T^{\mu\nu}_{Maxwell} = -\nabla_\mu T^{\mu\nu}_{Maxwell}$. Therefore:

$$\boxed{\nabla_\mu T^{\mu\nu}_{Maxwell} = 0}$$

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10 Cosmology

Q1

(a) Write down the general form of the Friedmann-Robertson-Walker metric in four-dimensional spacetime and give its interpretation. What is the expression of this metric in an open universe. Explain your answer

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$$ds^2 = dt^2 - R^2\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega\right) \qquad \text{In an open universe, } k = -1$$

$$\begin{aligned} ds^2 =& dt^2 - a(t)^2\left[d\chi^2 + f_k(\chi)\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \\ =& dt^2 - a(t)^2\left[d\chi^2 + \frac{\sinh\left(\sqrt{-K}\right)\chi}{\sqrt{-K}}\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \qquad K < 0 \end{aligned}$$

An open universe implies locally hyperbolic spatial sections.

(b) How does a physical length evolve with the scale factor? Give the physical explanation for this evolution and deduce the evolution and deduce the evolution of the particle number density with the scale factor.

Distances, $l(t)$, evolve with respect to the scale factor, a, according to:

$$\frac{dl(t)}{dt} = \frac{\dot{a}}{a}l(t) = H(t)l(t)$$

Where $H(t) = \dot{a}/a$ is the Hubble rate. This evolution is due to the expansion of the universe and is also related to the receding velocity of galaxies, v, according to $H(t)l(t) = v$.
We expect particle number density to be proportional to a^{-3}.

(c) Compute the age of the universe as a function of the Hubble parameter H_0 starting from the definition of the Hubble rate H and assuming that the universe is flat with a cosmological parameter $\Omega_{m,0} = 1$.

$$\begin{aligned} H(z) =& -\frac{\frac{dz}{dt}}{1+z} \\ dt =& -\int\frac{dz}{(1+z)H(z)} \\ t =& -\int\frac{dz}{(1+z)H(z)} \\ t =& -\int\frac{dz}{(1+z)H_0\sqrt{\Omega_{m,0}(1+z)^3}} \\ t =& -\int\frac{dz}{(1+z)H_0\sqrt{(1+z)^3}} \\ t =& \frac{2}{3H_0} \end{aligned}$$

$$\begin{aligned} H(z) =& H_0E(z) \\ =& H_0\sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_\Lambda} \\ \leftarrow =& H_0\sqrt{\Omega_{m,0}(1+z)^3} \text{ In matter-dominated universe} \\ \leftarrow \Omega_{m,0} \approx& 1 \end{aligned}$$

Q2

(a) Show that to first order in slow roll the comoving Hubble radius is

$$\mathcal{H}^{-1} = (\epsilon - 1)\tau \tag{304}$$

where we have chosen our time coordinate so that the Hubble radius vanishes at late times.

$$\begin{aligned}
0 =& H' - H^2 + \tfrac{1}{2}\kappa^2\phi'^2 \\
0 =& \frac{H'}{H^2} - 1 + \frac{\kappa^2\phi'^2}{2H^2} \\
0 =& \frac{H'}{H^2} - 1 + \epsilon \\
0 =& -\left(H\right)^{-1} - 1 + \epsilon \\
\left(H^{-1}\right)' =& \epsilon - 1 \\
\int_0^\tau dt \left(H^{-1}\right)' =& \int_0^\tau dt \left(\epsilon - 1\right) \\
\left(H\right)^{-1} =& \left(\epsilon - 1\right)\tau
\end{aligned}$$

(b) Defining $z = a\kappa\varphi'/\mathcal{H} = a\sqrt{2\epsilon}$, show that

$$\frac{z''}{z} = \frac{\alpha^2 - \frac{1}{4}}{\tau^2} \tag{305}$$

where to first order in slow roll

$$\alpha = \frac{3}{2} + \frac{1}{2}\eta + \epsilon \tag{306}$$

You can assume that η' is of second order in slow roll. Note that $\frac{z''}{z} = \left(\frac{z'}{z}\right)' + \left(\frac{z'}{z}\right)^2$

$$\begin{aligned}
\frac{z''}{z} =& \left(\frac{z'}{z}\right)' + \left(\frac{z'}{z}\right)^2 \\
\frac{z''}{z} =& H'\left(1 + \tfrac{1}{2}\eta\right) + H^2\left(1 + \tfrac{1}{2}\eta\right)^2 \quad \leftarrow \\
\frac{z''}{z} \approx& H'\left(1 + \tfrac{1}{2}\eta\right) + H^2\left(1 + \eta\right) \\
\frac{z''}{z} =& H^2\left(\frac{H'}{H^2}\left(1 + \tfrac{1}{2}\eta\right) + 1 + \eta\right) \\
\frac{z''}{z} =& H^2\left(\left(1 - \epsilon\right)\left(1 + \tfrac{1}{2}\eta\right) + 1 + \eta\right) \\
\frac{z''}{z} =& H^2\left(2 + \tfrac{3}{2}\eta - \epsilon\right) \\
\frac{z''}{z} =& \frac{2 + \frac{3}{2}\eta - \epsilon}{(\epsilon - 1)^2\tau^2} \\
\frac{z''}{z} \approx& \frac{\left(2 + \frac{3}{2}\eta - \epsilon\right)\left(1 + 2\epsilon\right)}{\tau^2} \\
\frac{z''}{z} \approx& \frac{\left(2 + \frac{3}{2}\eta + 3\epsilon\right)}{\tau^2} \\
\frac{z''}{z} \approx& \frac{\left(\frac{3}{2} + \frac{1}{2}\eta + \epsilon\right)^2 - \frac{1}{4}}{\tau^2} \\
\frac{z''}{z} =& \frac{\alpha^2 - \frac{1}{4}}{\tau^2}
\end{aligned}$$

$$\boxed{\begin{aligned}
z =& a\sqrt{2\epsilon} \\
z' =& a'\sqrt{2\epsilon} + \tfrac{1}{2}a\epsilon'\sqrt{\frac{2}{\epsilon}} \\
z' =& a'\sqrt{2\epsilon} + \tfrac{1}{2}a\eta H\epsilon\sqrt{\frac{2}{\epsilon}} \\
z' =& a'\sqrt{2\epsilon} + \tfrac{1}{2}a\eta H\sqrt{2\epsilon} \\
z' =& aH\sqrt{2\epsilon} + \tfrac{1}{2}aH\eta\sqrt{2\epsilon} \\
z' =& Hz\left(1 + \tfrac{1}{2}\eta\right) \\
\frac{z'}{z} =& H\left(1 + \tfrac{1}{2}\eta\right)
\end{aligned}}$$

Q3 Expanding the inflationary action to quadratic order in the perturbations, one finds the mode function $\nu_q(\tau)$ obeys the equation of motion.

$$0 = \nu_q'' + \left(q^2 - \frac{z''}{z}\right)\nu_q \tag{307}$$

Show that upon making the substitutions $x = -q\tau$ and $\nu_q(\tau) = \sqrt{x}y(x)$, this reduces to Bessel's equation

$$0 = x^2 y_{,xx} + xy_{,x} + \left(x^2 - \alpha^2\right)y \tag{308}$$

$$\begin{aligned}
\nu_q &= \sqrt{x}y \\
\nu_q' &= -q\frac{\partial \nu_q}{\partial x} \\
\nu_q' &= -q\left(\sqrt{x}y_{,x} + \frac{y}{2\sqrt{x}}\right) \\
\nu_q'' &= q^2\frac{\partial}{\partial x}\left(\sqrt{x}y_{,x} + \frac{y}{2\sqrt{x}}\right) \\
\Longleftarrow \quad \nu_q'' &= q^2\left(\sqrt{x}y_{,xx} + \frac{y_{,x}}{\sqrt{x}} - \frac{y}{4\sqrt{x^3}}\right)
\end{aligned}$$

$$\begin{aligned}
0 &= \nu_q'' + \left(q^2 - \frac{z''}{z}\right)\nu_q \\
0 &= \nu_q'' + \left(q^2 - \frac{\alpha^2 - \frac{1}{4}}{\tau^2}\right)\nu_q \\
0 &= q^2\left(\sqrt{x}y_{,xx} + \frac{y_{,x}}{\sqrt{x}} - \frac{y}{4\sqrt{x^3}}\right) + \left(q^2 - \frac{\alpha^2 - \frac{1}{4}}{\left(\frac{-x}{q}\right)^2}\right)\sqrt{x}y \\
0 &= \sqrt{x}y_{,xx} + \frac{y_{,x}}{\sqrt{x}} - \frac{y}{4\sqrt{x^3}} + \left(1 - \frac{\alpha^2 - \frac{1}{4}}{x^2}\right)\sqrt{x}y \\
0 &= x^2 y_{,xx} + xy_{,x} - \tfrac{1}{4}y + \left(1 - \frac{\alpha^2 - \frac{1}{4}}{x^2}\right)x^2 y \\
0 &= x^2 y_{,xx} + xy_{,x} + \left(x^2 - \alpha^2\right)y
\end{aligned}$$

Q4 Bessel's equation can be solved in terms of Hankel functions,

$$y(x) = A_q^{(1)}H_\alpha^{(1)}(x) + A_q^{(2)}H_\alpha^{(2)}(x) \tag{309}$$

which are combinations of the usual Bessel functions ($H_\alpha^{(1)}(x) = J_\alpha(x) + iY_\alpha(X)$ and $H_\alpha^{(2)}(x) = H_\alpha^{(1)*}(x)$), while $A_q^{(1)}$ and $A_q^{(2)}$ are constants of integration.
In the limit $x \to \infty$, we can approximate

$$H_\alpha^{(1)}(x) \approx \sqrt{\frac{2}{\pi x}}\exp\left[i\left(x - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right)\right] \tag{310}$$

while in the limit $x \to 0$, we find instead

$$H_\alpha^{(1)}(x) \approx \frac{i}{\pi}\Gamma(\alpha)\left(\frac{x}{2}\right)^{-\alpha} \tag{311}$$

At early times, the effects of spacetime curvature are negligible and we wish to impose the Bunch-Davies vacuum (setting $\hbar = 1$):

$$\nu_q(\tau) \to \frac{1}{\sqrt{2q}}e^{-iq\tau} \text{ as } \tau \to -\infty \tag{312}$$

. Show that this requires

$$A_q^{(2)} = 0, \; \left|A_q^{(1)}\right| = \sqrt{\frac{\pi}{4q}} \tag{313}$$

$$
\begin{aligned}
\nu_q =& \sqrt{x} y \\
\nu_q =& \sqrt{x} \left(A_q^{(1)} H_\alpha^{(1)} + A_q^{(2)} H_\alpha^{(2)} \right) \\
\xrightarrow[\tau \to -\infty]{x \to \infty} \frac{1}{\sqrt{2q}} e^{-iq\tau} =& \sqrt{x} \left(A_q^{(1)} \sqrt{\frac{2}{\pi x}} \exp\left[i \left(x - \frac{\alpha\pi}{2} - \frac{\pi}{4} \right) \right] + A_q^{(2)} \sqrt{\frac{2}{\pi x}} \exp\left[-i \left(x - \frac{\alpha\pi}{2} - \frac{\pi}{4} \right) \right] \right) \\
\frac{1}{\sqrt{2q}} e^{ix} =& A_q^{(1)} \sqrt{\frac{2}{\pi}} \exp\left[i \left(x - \frac{\alpha\pi}{2} - \frac{\pi}{4} \right) \right] + A_q^{(2)} \sqrt{\frac{2}{\pi}} \exp\left[-i \left(x - \frac{\alpha\pi}{2} - \frac{\pi}{4} \right) \right] \\
\frac{1}{\sqrt{2q}} =& A_q^{(1)} \sqrt{\frac{2}{\pi}} \exp\left[i \left(- \frac{\alpha\pi}{2} - \frac{\pi}{4} \right) \right] + A_q^{(2)} \sqrt{\frac{2}{\pi}} \exp\left[-i \left(2x - \frac{\alpha\pi}{2} - \frac{\pi}{4} \right) \right] \\
\frac{1}{\sqrt{2q}} =& A_q^{(1)} \sqrt{\frac{2}{\pi}} \exp\left[i \left(- \frac{\alpha\pi}{2} - \frac{\pi}{4} \right) \right] + A_q^{(2)} \sqrt{\frac{2}{\pi}} \exp\left[-i \left(2x - \frac{\alpha\pi}{2} - \frac{\pi}{4} \right) \right]
\end{aligned}
$$

Comparing coefficients of $e^{-i2x} \implies A_q^{(2)} = 0$
Comparing constant terms:

$$
\begin{aligned}
\frac{1}{\sqrt{2q}} =& A_q^{(1)} \sqrt{\frac{2}{\pi}} \exp\left[i \left(- \frac{\alpha\pi}{2} - \frac{\pi}{4} \right) \right] \\
\frac{1}{\sqrt{2q}} =& \left| A_q^{(1)} \right| \sqrt{\frac{2}{\pi}} \left| \exp\left[i \left(- \frac{\alpha\pi}{2} - \frac{\pi}{4} \right) \right] \right| \\
\frac{1}{\sqrt{2q}} =& \left| A_q^{(1)} \right| \sqrt{\frac{2}{\pi}} \\
\left| A_q^{(1)} \right| =& \sqrt{\frac{\pi}{4q}}
\end{aligned}
$$

Q5 The quantised curvature perturbations are described by the operator

$$
\hat{\zeta}(\tau, \mathbf{q}) = \zeta_q(\tau) \hat{a}(\mathbf{q}) + \zeta_q^* \hat{a}^\dagger(-\mathbf{q}) \tag{314}
$$

where $\zeta_q(\tau) = \kappa \nu_q(\tau)(\tau)$ and the creation and annihilation operators obey the usual commutation relations

(a) Show that the 2-point function can be expressed as

$$
\langle 0 | \hat{\zeta}(\tau, \mathbf{q}) \hat{\zeta}(\tau, \mathbf{q}') | 0 \rangle = |\zeta_q(\tau)|^2 (2\pi)^3 \delta(\mathbf{q} + \mathbf{q}') \tag{315}
$$

$$
\begin{aligned}
\langle 0 | \zeta(\tau, \mathbf{q}) \zeta(\tau, \mathbf{q}') | 0 \rangle &= \langle 0 | \left(\zeta_q(\tau) a(\mathbf{q}) + \zeta_q^*(\tau) a^\dagger(-\mathbf{q}) \right) \left(\zeta_{q'}(\tau) a(\mathbf{q}') + \zeta_{q'}^*(\tau) a^\dagger(-\mathbf{q}') \right) | 0 \rangle \\
&= \langle 0 | \zeta_q(\tau) \zeta_{q'}^*(\tau) a(\mathbf{q}) a^\dagger(-\mathbf{q}') | 0 \rangle \\
&= \langle 0 | \zeta_q(\tau) \zeta_{q'}^*(\tau) (2\pi)^3 \delta(\mathbf{q} + \mathbf{q}') | 0 \rangle \\
&= |\zeta_q(\tau)|^2 (2\pi)^3 \delta(\mathbf{q} + \mathbf{q}')
\end{aligned}
$$

(b) In position space, show that this corresponds to

$$
\langle 0 | \hat{\zeta}(\tau, \mathbf{x}) \hat{\zeta}(\tau, \mathbf{x}') | 0 \rangle = \int \frac{d^3\mathbf{q}}{(2\pi)^3} |\zeta_q(\tau)|^2 e^{i\mathbf{q} \cdot (\mathbf{x} - \mathbf{x}')} \tag{316}
$$

$$
\begin{aligned}
\langle 0|\, \zeta(\tau,\mathbf{x})\zeta(\tau,\mathbf{x}')\, |0\rangle &= |\zeta_q(\tau)|^2\, (2\pi)^3\delta(\mathbf{q}+\mathbf{q}') \\
&= \int \frac{d^3\mathbf{q}'}{(2\pi)^3} e^{i\mathbf{q}'\cdot\mathbf{x}'}\, |\zeta_q(\tau)|^2\, (2\pi)^3\delta(\mathbf{q}+\mathbf{q}') \\
&= e^{-i\mathbf{q}\cdot\mathbf{x}'}\, |\zeta_q(\tau)|^2 \\
&= \int \frac{d^3\mathbf{q}}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}} e^{-i\mathbf{q}\cdot\mathbf{x}'}\, |\zeta_q(\tau)|^2 \\
&= \int \frac{d^3\mathbf{q}}{(2\pi)^3} e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}\, |\zeta_q(\tau)|^2
\end{aligned}
$$

(c) Show that the power spectrum, defined by

$$
\langle 0|\, \hat{\zeta}(\tau,\mathbf{x}')^2\, |0\rangle = \int d(\ln q)\Delta_S^2(q) \tag{317}
$$

is then given by

$$
\Delta_S^2(q) = \frac{q^3}{2\pi^2}\, |\zeta_q(\tau)|^2 \tag{318}
$$

$$
\begin{aligned}
\int (d\ln q)\Delta_S^2(q) &= \langle 0|\, \zeta(\tau,\mathbf{x})^2\, |0\rangle \\
\int (d\ln q)\Delta_S^2(q) &= \int \frac{d^3\mathbf{q}}{(2\pi)^3}\, |\zeta_q(\tau)|^2 \\
\int (d\ln q)\Delta_S^2(q) &= \int \frac{4\pi q^2 dq}{(2\pi)^3}\, |\zeta_q(\tau)|^2 \\
\int (d\ln q)\Delta_S^2(q) &= \int (d\ln q)\frac{q^3}{2\pi^2}\, |\zeta_q(\tau)|^2 \\
\implies \Delta_S^2(q) &= \frac{q^3}{2\pi^2}\, |\zeta_q(\tau)|^2
\end{aligned}
$$

Q6

(a) Show that to first order in slow roll

$$
\frac{z'}{z} = \frac{\frac{1}{2}-\alpha}{\tau} \tag{319}
$$

and hence

$$
z(\tau) = z(\tau_*)\left(\frac{\tau}{\tau_*}\right)^{1/2-\alpha} \tag{320}
$$

where τ_* is some reference time.

$$
\begin{aligned}
\frac{z'}{z} =& H\left(1+\tfrac{1}{2}\eta\right) && \text{From 2(b)}\\
\frac{z'}{z} =& \frac{1+\frac{1}{2}\eta}{(\epsilon-1)\tau}\\
\frac{z'}{z} =& -\frac{\left(1+\frac{1}{2}\eta\right)(\epsilon+1)}{\tau}\\
\frac{z'}{z} =& -\frac{1+\frac{1}{2}\eta+\epsilon}{\tau}\\
\frac{z'}{z} =& -\frac{\alpha-\frac{1}{2}}{\tau}\\
\frac{z'}{z} =& \frac{\frac{1}{2}-\alpha}{\tau}\\
\frac{\frac{\partial z}{\partial \tau}}{z} =& \frac{\frac{1}{2}-\alpha}{\tau}\\
\frac{1}{z}\frac{\partial z}{\partial \tau} =& \frac{\frac{1}{2}-\alpha}{\tau}\\
\int_{z(\tau_*)}^{z(\tau)} \frac{\partial z}{z} =& \left(\frac{1}{2}-\alpha\right)\int_{\tau_*}^{\tau}\frac{\partial \tau}{\tau}\\
\ln\left(\frac{z(\tau)}{z(\tau_*)}\right) =& \left(\frac{1}{2}-\alpha\right)\ln\left(\frac{\tau}{\tau_*}\right)\\
z(\tau) =& z(\tau_*)\left(\frac{\tau}{\tau_*}\right)^{\frac{1}{2}-\alpha}
\end{aligned}
$$

(b) Choosing τ_* as the moment of horizon crossing, $q = \mathcal{H}(\tau_*)$, show that

$$\tau_* = -\frac{(1+\epsilon_*)}{q} \tag{321}$$

where $\epsilon_* = \epsilon(\tau_*)$.

$$
\begin{aligned}
H^{-1} =& (\epsilon-1)\tau\\
yq^{-1} =& (\epsilon_*-1)\tau_* && c \Longleftarrow q = H(\tau_*)\\
\tau_* =& \frac{1}{q(\epsilon_*-1)}\\
\tau_* \approx& -\frac{1+\epsilon_*}{q}\\
x_* =& 1+\epsilon_*
\end{aligned}
$$

(c) Show that

$$z(\tau_*) = \frac{q\sqrt{2\epsilon_*}}{H_*} \tag{322}$$

where H_* is the *proper* Hubble rate at horizon crossing.

$$
\begin{aligned}
z &= a\sqrt{2\epsilon} \\
z_* &= a_*\sqrt{2\epsilon_*} \\
z_* &= \frac{H_*}{H_*}\sqrt{2\epsilon_*} \\
z_* &= \frac{q\sqrt{2\epsilon_*}}{H_*}
\end{aligned}
$$

(d) Show that at late times, to learing order in slow roll

$$
|\zeta_q| \to \frac{2^\alpha \Gamma(\alpha)\kappa H_*}{\sqrt{8\pi\epsilon_* q^3}} \tag{323}
$$

$$
\begin{aligned}
\zeta_q =& \frac{\kappa \nu_q}{z} \\
\zeta_q =& \frac{\kappa\sqrt{x}y}{z} \\
\zeta_q =& \frac{\kappa\sqrt{x}}{z}\left(A_q^{(1)}H_\alpha^{(1)} + A_q^{(2)}H_\alpha^{(2)}\right) \\
|\zeta_q| =& \frac{\kappa\sqrt{x}}{z}\sqrt{\frac{\pi}{4q}}\left|H_\alpha^{(1)}\right| \\
\xrightarrow{x\to 0}\ |\zeta_q| =& \frac{\kappa\sqrt{x}}{z}\sqrt{\frac{\pi}{4q}}\left(\frac{1}{\pi}\Gamma(\alpha)\left(\frac{x}{2}\right)^{-\alpha}\right) \\
|\zeta_q| =& \frac{2^\alpha\Gamma(\alpha)\kappa}{\sqrt{4\pi q}}\frac{x^{\frac{1}{2}-\alpha}}{z} \\
|\zeta_q| =& \frac{2^\alpha\Gamma(\alpha)\kappa}{\sqrt{4\pi q}}\frac{x^{\frac{1}{2}-\alpha}}{z_*\left(\frac{\tau}{\tau_*}\right)^{\frac{1}{2}-\alpha}} && \Longleftarrow\ z = z_*\frac{\tau}{\tau_*} \\
|\zeta_q| =& \frac{2^\alpha\Gamma(\alpha)\kappa}{\sqrt{4\pi q}}\frac{x_*^{\frac{1}{2}-\alpha}}{z_*} \\
|\zeta_q| =& \frac{2^\alpha\Gamma(\alpha)\kappa H_*}{\sqrt{8\pi\epsilon_* q^3}}x_*^{\frac{1}{2}-\alpha} && \Longleftarrow\ z_* = \frac{q\sqrt{2\epsilon_*}}{H_*} \\
|\zeta_q| =& \frac{2^\alpha\Gamma(\alpha)\kappa H_*}{\sqrt{8\pi\epsilon_* q^3}}(1+\epsilon_*)^{\frac{1}{2}-\alpha} && \Longleftarrow\ x_* = 1+\epsilon_* \\
|\zeta_q| =& \frac{2^\alpha\Gamma(\alpha)\kappa H_*}{\sqrt{8\pi\epsilon_* q^3}}(1+\epsilon_*)^{-1-\frac{1}{2}\eta-\epsilon} \\
|\zeta_q| \approx& \frac{2^\alpha\Gamma(\alpha)\kappa H_*}{\sqrt{8\pi\epsilon_* q^3}}\left(1+\left(-1-\frac{1}{2}\eta-\epsilon\right)\epsilon_*\right) \\
|\zeta_q| \approx& \frac{2^\alpha\Gamma(\alpha)\kappa H_*}{\sqrt{8\pi\epsilon_* q^3}}
\end{aligned}
$$

(e) Finally, approximating $2^\alpha\Gamma(\alpha) \approx 2^{3/2}\Gamma(3/2) = \sqrt{2\pi}$, show that the late-time scalar power spectrum is

$$
\Delta_S^2(Q) = \frac{\kappa^2 H_*^2}{8\pi^2\epsilon_*} \tag{324}
$$

$$
\begin{aligned}
\Delta_S^2(q) =& \frac{q^3}{2\pi^2} \left|\zeta_q(\tau)\right|^2 \\
=& \frac{q^3}{2\pi^2} \left| \frac{2^\alpha \Gamma(\alpha) \kappa H_*}{\sqrt{8\pi\epsilon_* q^3}} \right|^2 \\
=& \frac{q^3}{2\pi^2} \left| \frac{\sqrt{2\pi} \kappa H_*}{\sqrt{8\pi\epsilon_* q^3}} \right|^2 \\
=& \frac{q^3}{2\pi^2} \left| \frac{\kappa H_*}{\sqrt{4\epsilon_* q^3}} \right|^2 \\
=& \frac{\kappa^2 H_*^2}{8\pi^2 \epsilon_*}
\end{aligned}
$$

11 String Theory

Q1 Consider the relativistic point particle action

$$I = \int_{t_A}^{t_B} dt \left(\dot{X}^m P_m - \tfrac{1}{2} e \left(P^2 + m^2 \right) \right) \tag{325}$$

(a) Verify that it is invariant under the Diff_1 gauge transformations

$$\delta_\xi X = \xi \dot{X} \tag{326}$$
$$\delta_\xi P = \xi \dot{P} \tag{327}$$
$$\delta_\xi e = \tfrac{d}{dt}(e\xi) \tag{328}$$

where $\xi(t)$ is an arbitrary infinitesimal function satisfying $\xi(t_A) = \xi(t_B) = 0$.

$$\delta_\xi I = \int_{t_A}^{t_B} dt \left(\tfrac{d}{dt}\left(\delta_\xi X^m\right) P_m + \dot{X}^m \delta_\xi P_m - \tfrac{1}{2}\left(P^2 + m^2\right) \delta_\xi e - e P^m \delta_\xi P_m \right) \tag{329}$$
$$= \int_{t_A}^{t_B} dt \left(\boxed{\tfrac{d}{dt}\left(\xi \dot{X}^m\right) P_m} + \xi \dot{X}^m \dot{P}_m - \tfrac{1}{2} \boxed{\left(P^2 + m^2\right) \tfrac{d}{dt}(e\xi)} - \xi e P^m \dot{P}_m \right) \tag{330}$$

Performing integration by parts on the boxed terms. The boundary terms vanish because $\xi(t_A) = \xi(t_B) = 0$.

$$= \int_{t_A}^{t_B} dt \left(-\xi \dot{X}^m \dot{P}_m + \xi \dot{X}^m \dot{P}_m - \xi e P^m \dot{P}_m - \xi e P^m \dot{P}_m \right) \tag{331}$$
$$= 0 \tag{332}$$

(b) Now verfiy that it is also invariant under the "canonical" gauge transformation

$$\delta_\alpha X = \alpha P \tag{333}$$
$$\delta_\alpha P = 0 \tag{334}$$
$$\delta_\alpha e = \dot{\alpha} \tag{335}$$

where $\alpha(t)$ is an arbitrary infinitesimal function satisfying $\alpha(t_A) = \alpha(t_B) = 0$.

$$\delta_\alpha I = \int_{t_A}^{t_B} dt \left(\tfrac{d}{dt}\left(\delta_\alpha X^m\right) P_m + \tfrac{1}{2} \delta_\alpha e \left(P^2 + m^2\right) \right) \tag{336}$$
$$= \int_{t_A}^{t_B} dt \left(\tfrac{d}{dt}\left(\alpha P^m\right) P_m + \tfrac{1}{2} \delta_\alpha e \left(P^2 + m^2\right) \right) \tag{337}$$
$$\tag{338}$$

---

p

(c) Show that any action functional $I[\psi, \phi]$ is invariant under the infinitesimal gauge transformation

$$\delta_f \psi = f \frac{\delta I}{\delta \phi} \tag{339}$$
$$\delta_f \phi = -f \frac{\delta I}{\delta \psi} \tag{340}$$

for an *arbitrary* function f. Such "trivial" gauge invariances have no physical effect. Use this observation to explain why the "Diff_1" and "canonical" gauge invariances of the point particle are physically equivalent.

Q2 In a canonical quantization of the worldsheet scalar field X^μ, the field and its canonically conjugate momentum $P^\mu = \partial_\tau X^\mu$ are required to satisfy the commutation relations $[X^\mu(\tau,\sigma), P^\nu(\tau,\sigma')] = i\eta^{\mu\nu}\delta(\sigma-\sigma')$. By using the closed string mode expansion for $X^\mu = X^\mu_R + X^\mu_L$.

$$X^\mu_R = \frac{x^\mu}{2} + \frac{p^\mu\sigma^-}{2\pi} + \frac{i}{2\sqrt{\pi}}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}e^{-2in\sigma^-} \tag{341}$$

$$X^\mu_L = \frac{x^\mu}{2} + \frac{p^\mu\sigma^+}{2\pi} + \frac{i}{2\sqrt{\pi}}\sum_{n\neq 0}\frac{\tilde{\alpha}^\mu_n}{n}e^{-2in\sigma^+} \tag{342}$$

where $\sigma^\pm = \tau \pm \sigma$, derive the commutation relation for the creation/annihilation operators

$$[x^\mu, p^\nu] = i\eta^{\mu\nu} \tag{343}$$

$$[\alpha^\mu_m, \alpha^\nu_n] = m\delta_{n+m,0}\eta^{\mu\nu} \tag{344}$$

$$[\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\delta_{n+m,0}\eta^{\mu\nu} \tag{345}$$

$$[\alpha^\mu_m, \tilde{\alpha}^\nu_n] = 0 \tag{346}$$

X^μ is given by

$$X^\mu = X^\mu_L + X^\mu_R = x^\mu + \frac{p^\mu\tau}{\pi} + \frac{i}{2\sqrt{\pi}}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}e^{-2in\sigma^-} + \frac{i}{2\sqrt{\pi}}\sum_{n\neq 0}\frac{\tilde{\alpha}^\mu_n}{n}e^{-2in\sigma^+} \tag{347}$$

We also calculate P^μ

$$P^\mu(\sigma,\tau) = \partial_\tau X^\mu = \frac{p^\mu}{\pi} + \frac{i}{2\sqrt{\pi}}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}(-2in)e^{-2in\sigma^-} + \frac{i}{2\sqrt{\pi}}\sum_{n\neq 0}\frac{\tilde{\alpha}^\mu_n}{n}(-2in)e^{-2in\sigma^+} \tag{348}$$

$$= \frac{p^\mu}{\pi} + \frac{1}{\sqrt{\pi}}\sum_{n\neq 0}\alpha^\mu_n e^{-2in\sigma^-} + \frac{1}{\sqrt{\pi}}\sum_{n\neq 0}\tilde{\alpha}^\mu_n e^{-2in\sigma^+} \tag{349}$$

Now we use these to calculate the commutator

$$
\begin{aligned}
&[X^\mu(\tau,\sigma), P^\nu(\tau,\sigma')] \\
&= \left[x^\mu + \frac{p^\mu \tau}{\pi} + \frac{i}{2\sqrt{\pi}} \sum_{n\neq 0} \frac{1}{n} \left(\alpha_n^\mu e^{-2in\sigma^-} + \tilde{\alpha}_n^\mu e^{-2in\sigma^+} \right), \frac{p^\nu}{\pi} + \frac{1}{\sqrt{\pi}} \sum_{m\neq 0} \left(\alpha_m^\nu e^{-2im\sigma'^-} + \tilde{\alpha}_m^\nu e^{-2im\sigma'^+} \right) \right] \\
&= \frac{1}{\pi}[x^\mu, p^\nu] + \left[\frac{i}{2\sqrt{\pi}} \sum_{n\neq 0} \frac{1}{n} \left(\alpha_n^\mu e^{-2in\sigma^-} + \tilde{\alpha}_n^\mu e^{-2in\sigma^+} \right), \frac{1}{\sqrt{\pi}} \sum_{m\neq 0} \left(\alpha_m^\nu e^{-2im\sigma'^-} + \tilde{\alpha}_m^\nu e^{-2im\sigma'^+} \right) \right] \\
&= \frac{1}{\pi}[x^\mu, p^\nu] + \frac{i}{2\pi} \sum_{m,n\neq 0} \frac{1}{n} \left[\left(\alpha_n^\mu e^{-2in\sigma^-} + \tilde{\alpha}_n^\mu e^{-2in\sigma^+} \right), \left(\alpha_m^\nu e^{-2im\sigma'^-} + \tilde{\alpha}_m^\nu e^{-2im\sigma'^+} \right) \right] \\
&= \frac{1}{\pi}[x^\mu, p^\nu] + \frac{i}{2\pi} \sum_{m,n\neq 0} \frac{1}{n} \left(\begin{array}{l} \left[\alpha_n^\mu e^{-2in\sigma^-}, \alpha_m^\nu e^{-2im\sigma'^-} \right] + \left[\alpha_n^\mu e^{-2in\sigma^-}, \tilde{\alpha}_m^\nu e^{-2im\sigma'^+} \right] \\ + \left[\tilde{\alpha}_n^\mu e^{-2in\sigma^+}, \alpha_m^\nu e^{-2im\sigma'^-} \right] + \left[\tilde{\alpha}_n^\mu e^{-2in\sigma^+}, \tilde{\alpha}_m^\nu e^{-2im\sigma'^+} \right] \end{array} \right) \\
&= \frac{1}{\pi}[x^\mu, p^\nu] + \frac{i}{2\pi} \sum_{m,n\neq 0} \frac{1}{n} \left(\begin{array}{l} e^{-2i(n\sigma^- + m\sigma'^-)} [\alpha_n^\mu, \alpha_m^\nu] + e^{-2i(n\sigma^- + m\sigma'^+)} [\alpha_n^\mu, \tilde{\alpha}_m^\nu] \\ + e^{-2i(n\sigma^+ + m\sigma'^-)} [\tilde{\alpha}_n^\mu, \alpha_m^\nu] + e^{-2i(n\sigma^+ + m\sigma'^+)} [\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu] \end{array} \right) \\
&= \frac{1}{\pi}[x^\mu, p^\nu] + \frac{i}{2\pi} \sum_{m,n\neq 0} \frac{1}{n} \left(\begin{array}{l} e^{-2i(n(\tau-\sigma) + m(\tau-\sigma'))} [\alpha_n^\mu, \alpha_m^\nu] + e^{-2i(n(\tau-\sigma) + m(\tau+\sigma'))} [\alpha_n^\mu, \tilde{\alpha}_m^\nu] \\ + e^{-2i(n(\tau+\sigma) + m(\tau-\sigma'))} [\tilde{\alpha}_n^\mu, \alpha_m^\nu] + e^{-2i(n(\tau+\sigma) + m(\tau+\sigma'))} [\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu] \end{array} \right) \\
&= \frac{1}{\pi}[x^\mu, p^\nu] + \frac{i}{2\pi} \sum_{m,n\neq 0} \frac{e^{-2i(n+m)\tau}}{n} \left(\begin{array}{l} e^{-2i(-n\sigma - m\sigma')} [\alpha_n^\mu, \alpha_m^\nu] + e^{-2i(-n\sigma + m\sigma')} [\alpha_n^\mu, \tilde{\alpha}_m^\nu] \\ + e^{-2i(n\sigma - m\sigma')} [\tilde{\alpha}_n^\mu, \alpha_m^\nu] + e^{-2i(n\sigma + m\sigma')} [\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu] \end{array} \right) \\
&= \frac{1}{\pi}[x^\mu, p^\nu] + \frac{i}{2\pi} \sum_{m,n\neq 0} \frac{e^{-2i(n+m)\tau}}{n} \left(\begin{array}{l} e^{2i(n\sigma + m\sigma')} [\alpha_n^\mu, \alpha_m^\nu] + e^{2i(n\sigma - m\sigma')} [\alpha_n^\mu, \tilde{\alpha}_m^\nu] \\ + e^{-2i(n\sigma - m\sigma')} [\tilde{\alpha}_n^\mu, \alpha_m^\nu] + e^{-2i(n\sigma + m\sigma')} [\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu] \end{array} \right)
\end{aligned}
$$

Equating this to the expression of $[X^\mu(\tau,\sigma), P^\nu(\tau,\sigma')]$ given in the question

$$
\begin{aligned}
[X^\mu(\tau,\sigma), P^\nu(\tau,\sigma')] &= i\eta^{\mu\nu}\delta(\sigma-\sigma') = \frac{i\eta^{\mu\nu}}{\pi} \sum_N e^{-2iN(\sigma-\sigma')} \\
\frac{[x^\mu, p^\nu]}{\pi} + \sum_{m,n\neq 0} \frac{ie^{-2i(n+m)\tau}}{2\pi n} \left(\begin{array}{l} e^{2i(n\sigma + m\sigma')} [\alpha_n^\mu, \alpha_m^\nu] + e^{2i(n\sigma - m\sigma')} [\alpha_n^\mu, \tilde{\alpha}_m^\nu] \\ + e^{-2i(n\sigma - m\sigma')} [\tilde{\alpha}_n^\mu, \alpha_m^\nu] + e^{-2i(n\sigma + m\sigma')} [\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu] \end{array} \right) &= \frac{i\eta^{\mu\nu}}{\pi} \sum_N e^{-2iN(\sigma-\sigma')} \\
[x^\mu, p^\nu] + \sum_{m,n\neq 0} \frac{ie^{-2i(n+m)\tau}}{2n} \left(\begin{array}{l} e^{2i(n\sigma + m\sigma')} [\alpha_n^\mu, \alpha_m^\nu] + e^{2i(n\sigma - m\sigma')} [\alpha_n^\mu, \tilde{\alpha}_m^\nu] \\ + e^{-2i(n\sigma - m\sigma')} [\tilde{\alpha}_n^\mu, \alpha_m^\nu] + e^{-2i(n\sigma + m\sigma')} [\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu] \end{array} \right) &= i\eta^{\mu\nu} \sum_N e^{-2iN(\sigma-\sigma')} \\
[x^\mu, p^\nu] + \frac{i}{2} \sum_{m,n\neq 0} \frac{1}{n} \left(\begin{array}{l} e^{2i(n\sigma - n\sigma')} [\alpha_n^\mu, \alpha_m^\nu] + e^{2i(n\sigma + n\sigma')} [\alpha_n^\mu, \tilde{\alpha}_m^\nu] \\ + e^{-2i(n\sigma + n\sigma')} [\tilde{\alpha}_n^\mu, \alpha_m^\nu] + e^{-2i(n\sigma - n\sigma')} [\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu] \end{array} \right) &= i\delta_{n+m,0}\eta^{\mu\nu} \sum_N e^{-2iN(\sigma-\sigma')} \\
[x^\mu, p^\nu] + \frac{i}{2} \sum_{m,n\neq 0} \frac{1}{n} \left(\begin{array}{l} e^{2in(\sigma - \sigma')} [\alpha_n^\mu, \alpha_m^\nu] + e^{2in(\sigma + \sigma')} [\alpha_n^\mu, \tilde{\alpha}_m^\nu] \\ + e^{-2in(\sigma + \sigma')} [\tilde{\alpha}_n^\mu, \alpha_m^\nu] + e^{-2in(\sigma - \sigma')} [\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu] \end{array} \right) &= i\delta_{n+m,0}\eta^{\mu\nu} \sum_N e^{-2iN(\sigma-\sigma')}
\end{aligned}
$$

The term for $N = 0$ corresponds to the first term on the LHS: $\boxed{[x^\mu, p^\nu] = i\eta^{\mu\nu}}$. So the remaining terms are:

$$
\begin{aligned}
\frac{i}{2} \sum_{m,n\neq 0} \frac{1}{n} \left(\begin{array}{l} e^{2in(\sigma - \sigma')} [\alpha_n^\mu, \alpha_m^\nu] + e^{2in(\sigma + \sigma')} [\alpha_n^\mu, \tilde{\alpha}_m^\nu] \\ + e^{-2in(\sigma + \sigma')} [\tilde{\alpha}_n^\mu, \alpha_m^\nu] + e^{-2in(\sigma - \sigma')} [\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu] \end{array} \right) &= i\delta_{n+m,0}\eta^{\mu\nu} \sum_{N\neq 0} e^{-2iN(\sigma-\sigma')} \\
\frac{1}{2} \sum_{m,n\neq 0} \frac{1}{n} \left(\begin{array}{l} e^{2in(\sigma - \sigma')} [\alpha_n^\mu, \alpha_m^\nu] + e^{2in(\sigma + \sigma')} [\alpha_n^\mu, \tilde{\alpha}_m^\nu] \\ + e^{-2in(\sigma + \sigma')} [\tilde{\alpha}_n^\mu, \alpha_m^\nu] + e^{-2in(\sigma - \sigma')} [\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu] \end{array} \right) &= \delta_{n+m,0}\eta^{\mu\nu} \sum_{N\neq 0} e^{-2iN(\sigma-\sigma')} \\
\frac{1}{2} \sum_{m,n\neq 0} \frac{1}{n} \left(e^{2in(\sigma - \sigma')} [\alpha_n^\mu, \alpha_m^\nu] + e^{-2in(\sigma - \sigma')} [\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu] \right) &= \delta_{n+m,0}\eta^{\mu\nu} \sum_{N\neq 0} e^{-2iN(\sigma-\sigma')} \qquad (350)
\end{aligned}
$$

In the final step, we observe that $e^{2in(\sigma+\sigma')}$ and $e^{-2in(\sigma+\sigma')}$ do not match the terms on the RHS for any allowed value of n or N. Therefore these terms must vanish, which implies $\boxed{[\alpha_n^\mu, \tilde{\alpha}_m^\nu] = [\tilde{\alpha}_n^\mu, \alpha_m^\nu] = 0}$. Now we show that $[\alpha_n^\mu, \alpha_m^\nu] = [\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu]$, starting from the canonical equal-time commutation relation for conjugate momenta:

$$
\begin{aligned}
0 &= [P^\mu(\sigma,\tau), P^\nu(\sigma',\tau)] \\
0 &= \left[\frac{p^\mu}{\pi} + \frac{1}{\sqrt{\pi}}\sum_{n\neq 0}\alpha_n^\mu e^{-2in\sigma^-} + \frac{1}{\sqrt{\pi}}\sum_{n\neq 0}\tilde{\alpha}_n^\mu e^{-2in\sigma^+}, \frac{p^\nu}{\pi} + \frac{1}{\sqrt{\pi}}\sum_{m\neq 0}\alpha_m^\nu e^{-2im\sigma'^-} + \frac{1}{\sqrt{\pi}}\sum_{m\neq 0}\tilde{\alpha}_m^\nu e^{-2im\sigma'^+}\right] \\
0 &= \sum_{m,n\neq 0}\left(e^{-2i(n\sigma^- + m\sigma'^-)}\,[\alpha_n^\mu, \alpha_m^\nu] + e^{-2i(n\sigma^+ + m\sigma'^+)}\,[\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu]\right) \\
0 &= \sum_{m,n\neq 0} e^{-2i(n+m)\tau}\left(e^{-2i(-n\sigma - m\sigma')}\,[\alpha_n^\mu, \alpha_m^\nu] + e^{-2i(n\sigma + m\sigma')}\,[\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu]\right) \\
0 &= \sum_{m,n\neq 0}\left(e^{-2i(-n\sigma + n\sigma')}\,[\alpha_n^\mu, \alpha_m^\nu] + e^{-2i(n\sigma - n\sigma')}\,[\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu]\right) \qquad \leftarrow n = -m \\
0 &= \sum_{m,n\neq 0}\left(e^{2in(\sigma-\sigma')}\,[\alpha_n^\mu, \alpha_m^\nu] + e^{-2in(\sigma-\sigma')}\,[\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu]\right)
\end{aligned}
$$

Switching m to $-n$ in second exponent.

$$
0 = \sum_{m,n\neq 0}\left(e^{2in(\sigma-\sigma')}\,[\alpha_n^\mu, \alpha_m^\nu] + e^{2im(\sigma-\sigma')}\,[\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu]\right)
$$

Symmetry in n, m and symmetry in μ, ν allows us to relabel the second term $n \leftrightarrow m$ and $\mu \leftrightarrow \nu$

$$
\begin{aligned}
0 &= \sum_{m,n\neq 0}\left(e^{2in(\sigma-\sigma')}\,[\alpha_n^\mu, \alpha_m^\nu] + e^{2in(\sigma-\sigma')}\,[\tilde{\alpha}_m^\nu, \tilde{\alpha}_n^\mu]\right) \\
0 &= [\alpha_n^\mu, \alpha_m^\nu] + [\tilde{\alpha}_m^\nu, \tilde{\alpha}_n^\mu] \\
[\alpha_n^\mu, \alpha_m^\nu] &= -\,[\tilde{\alpha}_m^\nu, \tilde{\alpha}_n^\mu] \\
[\alpha_n^\mu, \alpha_m^\nu] &= [\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu]
\end{aligned}
$$

Returning to the line (350)

$$
\frac{1}{2}\sum_{n\neq 0}\frac{1}{n}\left(e^{2in(\sigma-\sigma')}\,[\alpha_n^\mu, \alpha_m^\nu] + e^{-2in(\sigma-\sigma')}\,[\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu]\right) = \delta_{n+m,0}\eta^{\mu\nu}\sum_{N\neq 0}e^{-2iN(\sigma-\sigma')}
$$

As the sum is over positive and negative integers, we can take $n \to -n$ in the co-efficient of $[\alpha_n^\mu, \alpha_m^\nu]$

$$
\frac{1}{2}\sum_{n\neq 0}\frac{1}{n}\left(e^{-2in(\sigma-\sigma')}\,[\alpha_n^\mu, \alpha_m^\nu] + e^{-2in(\sigma-\sigma')}\,[\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu]\right) = \delta_{n+m,0}\eta^{\mu\nu}\sum_{N\neq 0}e^{-2iN(\sigma-\sigma')}
$$

Next we use that $[\alpha_n^\mu, \alpha_m^\nu] = [\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu]$

$$
\frac{1}{2}\sum_{n\neq 0}\frac{1}{n}\left(e^{-2in(\sigma-\sigma')}\,[\alpha_n^\mu, \alpha_m^\nu] + e^{-2in(\sigma-\sigma')}\,[\alpha_n^\mu, \alpha_m^\nu]\right) = \delta_{n+m,0}\eta^{\mu\nu}\sum_{N\neq 0}e^{-2iN(\sigma-\sigma')} \tag{351}
$$

$$
\sum_{n\neq 0}\frac{1}{n}e^{-2in(\sigma-\sigma')}\,[\alpha_n^\mu, \alpha_m^\nu] = \delta_{n+m,0}\eta^{\mu\nu}\sum_{N\neq 0}e^{-2iN(\sigma-\sigma')} \tag{352}
$$

$$
\frac{1}{n}\,[\alpha_n^\mu, \alpha_m^\nu] = \delta_{n+m,0}\eta^{\mu\nu} \tag{353}
$$

$$
\frac{1}{(-m)}\,[\alpha_n^\mu, \alpha_m^\nu] = \delta_{n+m,0}\eta^{\mu\nu} \tag{354}
$$

$$
\frac{1}{m}\,[\alpha_m^\nu, \alpha_n^\mu] = \delta_{n+m,0}\eta^{\mu\nu} \tag{355}
$$

$$
[\alpha_m^\nu, \alpha_n^\mu] = m\delta_{n+m,0}\eta^{\mu\nu} \tag{356}
$$

Using symmetry in μ, ν to relabel $\mu \leftrightarrow \nu$ on LHS, we bring the expression into the desired form.

$$[\alpha^\mu_m, \alpha^\nu_n] = m\delta_{n+m,0}\eta^{\mu\nu}$$

And it follows from $[\alpha^\mu_n, \alpha^\nu_m] = [\tilde{\alpha}^\mu_n, \tilde{\alpha}^\nu_m]$ that

$$[\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\delta_{n+m,0}\eta^{\mu\nu}$$

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12 Classical and Quantum Solitons

Q1 Consider a field theory with Lagrangian $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - (1 - 2\cos(2\phi))$.

(a) What are the corresponding kinetic and potential energy densities, and what are the vacua in this theory?

$$
\begin{aligned}
\mathcal{L} &= \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - (1 - \cos 2\phi) \\
&= \tfrac{1}{2}\phi_t^2 - \tfrac{1}{2}\phi_x^2 - (1 - \cos 2\phi) \\
&\implies \text{Vacua: } \phi = n\pi,\ n \in Z
\end{aligned}
$$

Therefore, the kinetic energy density is $\frac{1}{2}\phi_t^2$, the potential energy density is $\frac{1}{2}\phi_x^2 + (1 - \cos 2\phi)$ and the vacua are at $\phi = n\pi$ where $n \in Z$.

(b)

(i) Find a lower (Bogomolnyi) bound for the energy of a single kink in this theory

(ii) Find a one-parameter family of solutions to the equation of motion which saturate this bound.

(i)

$$
\begin{aligned}
E &\geq \left| \int_{\phi_-}^{\phi_+} \sqrt{2U(\phi)}d\phi \right| \\
&= \int_0^{N\pi} \left|\sqrt{2(1 - \cos(2\phi))}\right| d\phi \\
&= \int_0^{N\pi} \left|\sqrt{4\sin^2\phi}\right| d\phi \\
&= 2\int_0^{N\pi} |\sin\phi|\, d\phi \\
&= 2|N| \left[-\cos\phi\right]_0^\pi \\
&= 4|N|
\end{aligned}
$$

(ii)

$$
\begin{aligned}
\phi' &= \pm\sqrt{2U(\phi)} \\
\frac{\partial\phi}{\partial x} &= \pm 2\sin\phi \\
\int \frac{d\phi}{\sin\phi} &= \pm 2\int dx
\end{aligned}
$$

We will use the result $\dfrac{f}{\sin f} = \log\tan\left(\dfrac{f}{2}\right)$

$$
\begin{aligned}
\log\tan\left(\frac{\phi}{2}\right) &= \pm 2\,(x + a) \\
\tan\left(\frac{\phi}{2}\right) &= e^{\pm 2(x+a)} \\
\phi &= 2\arctan e^{\pm 2(x+a)}
\end{aligned}
$$

(c) Compute the asymptotic force between a pair of these kinks, separated by a distance R, for $R \gg 1$

To calculate the asymptotic force between a pair of kinks, consider a kink at $-R/2$ and a kink at $+R/2$, such that $-R \ll 0 \ll R/2$. We require a field configuration which reduces to a single kink in the limit that $x \to \infty$ and $x \to -\infty$, which can be formed by the superposition:

$$\phi_{kk}(x) = 2\arctan e^{2\left(x+\frac{R}{2}\right)} + 2\arctan e^{2\left(x-\frac{R}{2}\right)} - 2\pi$$

Force in the interval in the interval $[a, b]$:

$$F = \frac{d}{dt}\int_a^b dx\ \rho(x)$$

$$= -\frac{d}{dt}\int_a^b dx\ \phi_x\phi_t \qquad \leftarrow \text{Momentum density: } \rho(x) = -\phi_x\phi_t$$

$$= -\int_a^b dx\,(\phi_{xt}\phi_t + \phi_x\phi_{tt})$$

$$\left\{\begin{array}{l} \text{Euler-Lagrange equation:} \\ 0 = \partial_\mu\left(\dfrac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\right) - \dfrac{\partial\mathcal{L}}{\partial\phi} \\ 0 = \partial_\mu\partial^\mu\phi + \dfrac{\partial U}{\partial\phi} \\ 0 = \phi_{tt} - \phi_{xx} + \dfrac{\partial U}{\partial\phi} \\ \phi_{tt} = \phi_{xx} - \dfrac{\partial U}{\partial\phi} \end{array}\right.$$

$$= -\int_a^b dx\left(\phi_{xt}\phi_t + \phi_x\left(\phi_{xx} - \frac{\partial U}{\partial\phi}\right)\right)$$

$$= -\int_a^b dx\left(\phi_{xt}\phi_t + \phi_x\phi_{xx} - \phi_x\frac{\partial U}{\partial\phi}\right)$$

$$= -\int_a^b dx\left(\phi_{xt}\phi_t + \phi_x\phi_{xx} - \frac{\partial U}{\partial x}\right)$$

$$= -\int_a^b dx\left(\frac{1}{2}\frac{\partial\left(\phi_t^2\right)}{\partial x} + \frac{1}{2}\frac{\partial\left(\phi_x^2\right)}{\partial x} - \frac{\partial U}{\partial x}\right)$$

$$= -\left[\tfrac{1}{2}\phi_t^2 + \tfrac{1}{2}\phi_x^2 - U(\phi)\right]_a^b$$

$$= -\left[\tfrac{1}{2}\phi_t^2 + \tfrac{1}{2}\phi_x^2 + \cos(2\phi) - 1\right]_{-\infty}^0 \qquad \left\{\begin{array}{l}\text{To find force on kink at } -R/2\text{, choose the} \\ \text{interval such that } [a,b] = [-\infty, 0]\end{array}\right.$$

$$\left\{\begin{array}{l} \phi_t(0) = 0 \\ \phi(-\infty) = \phi_t(-\infty) = \phi_x(-\infty) = 0 \end{array}\right.$$

$$= 1 - \tfrac{1}{2}\left(\phi_x(0)\right)^2 - \cos(2\phi(0))$$

$$= 1 - \tfrac{1}{2}\left(\phi_x(0)\right)^2 - \cos\left(4\arctan e^R + 4\arctan e^{-R} - 4\pi\right)$$

$$\left\{\begin{array}{lll} \text{For large R:} & & \\ \arctan\left(e^R\right) & \to \frac{\pi}{2} - \arctan\left(e^{-R}\right) \\ \arctan(e^{-R}) & \to e^{-R} \end{array}\right.$$

$$= 1 - \tfrac{1}{2}\left(\phi_x(0)\right)^2 - \cos(2\pi)$$

$$= -\tfrac{1}{2}\left(\phi_x(0)\right)^2$$

Now we calculate $\phi_x(0)$

$$\frac{\partial \phi_{kk}(x)}{\partial x} = \frac{\partial\left(2\arctan e^{2\left(x+\frac{R}{2}\right)} + 2\arctan e^{2\left(x-\frac{R}{2}\right)}\right)}{\partial x} \tag{357}$$

$$= \frac{4e^{2\left(x+\frac{R}{2}\right)}}{1+e^{4\left(x+\frac{R}{2}\right)}} + \frac{4e^{2\left(x-\frac{R}{2}\right)}}{1+e^{4\left(x-\frac{R}{2}\right)}} \tag{358}$$

$$\frac{\partial \phi_{kk}(0)}{\partial x} = \frac{4e^{R}}{1+e^{2R}} + \frac{4e^{-R}}{1+e^{-2R}} \tag{359}$$

$$= \frac{8\left(e^{R}+e^{-R}\right)}{\left(1+e^{2R}\right)\left(1+e^{-2R}\right)} \tag{360}$$

$$= \frac{8e^{-R}\left(e^{2R}+1\right)}{\left(1+e^{2R}\right)\left(1+e^{-2R}\right)} \tag{361}$$

$$= \frac{8e^{-R}}{1+e^{-2R}} \tag{362}$$

Substituting this into (357)

$$\begin{aligned} F =& -\frac{1}{2}\left(\frac{8e^{-R}}{1+e^{-2R}}\right)^2 \\ =& \frac{-32e^{-2R}}{\left(1+e^{-2R}\right)^2} \\ \xrightarrow{R\to\infty}& -32e^{-2R} \end{aligned}$$

(d) Let Y be a manifold without boundary, with selected base point $\mathbf{y}_0$, and consider $\pi_1(Y)$, the set of homotopy equivalence classes of based maps $\psi : S^1 \to Y$. Assume the S^1 is parametrised by $\theta \in [0, 2\pi]$ with $\psi(0) = \psi(2\pi) = \mathbf{y}_0$.
You may use the same symbol for a homotopy equivalence class and for a particular representative of that class. Given a loop $\psi \in \pi_1(Y)$, let ψ' be the same loop 'done backwards', so that $\psi'(\theta) = \psi(2\pi - \theta)$.

By constructing a suitable homotopy $\tilde{\psi}(\theta, \tau) : S^1 \times [0, 1] \to Y$, show that the composition of ψ and ψ' is homotopically equivalent to the trivial loop ψ_e, defined by $\psi_e(\theta) = \mathbf{y}_0\ \forall\ \theta \in [0, 2\pi]$, and hence that ψ' is indeed the inverse of ψ, as claimed in lectures.

A representation for $\psi \circ \psi'$ is:

$$\theta \mapsto \begin{cases} \psi(2\theta) & 0 \le \theta < \pi \\ \psi'(2(\theta - \pi)) & \pi \le \theta \le 2\pi \end{cases}$$

Construct a homotopy $\tilde{\psi}(\theta, \tau) : S^1 \times [0, 1] \to Y$ by inserting the trivial loop between ψ and ψ':

$$\tilde{\psi}(\theta, \tau) = \begin{cases} \psi(2\theta) & 0 \le \theta \le \pi\tau \\ \psi(\tau) & \pi\tau < \theta < \pi(2-\tau) \\ \psi'(2(\theta - \pi)) & \pi(2-\tau) \le \theta \le 2\pi \end{cases}$$

This homotopes to $\psi_e(\theta) = \mathbf{y}_0\ \forall\ \theta \in [0, 2\pi]$. Therefore, ψ' is the inverse of ψ.
