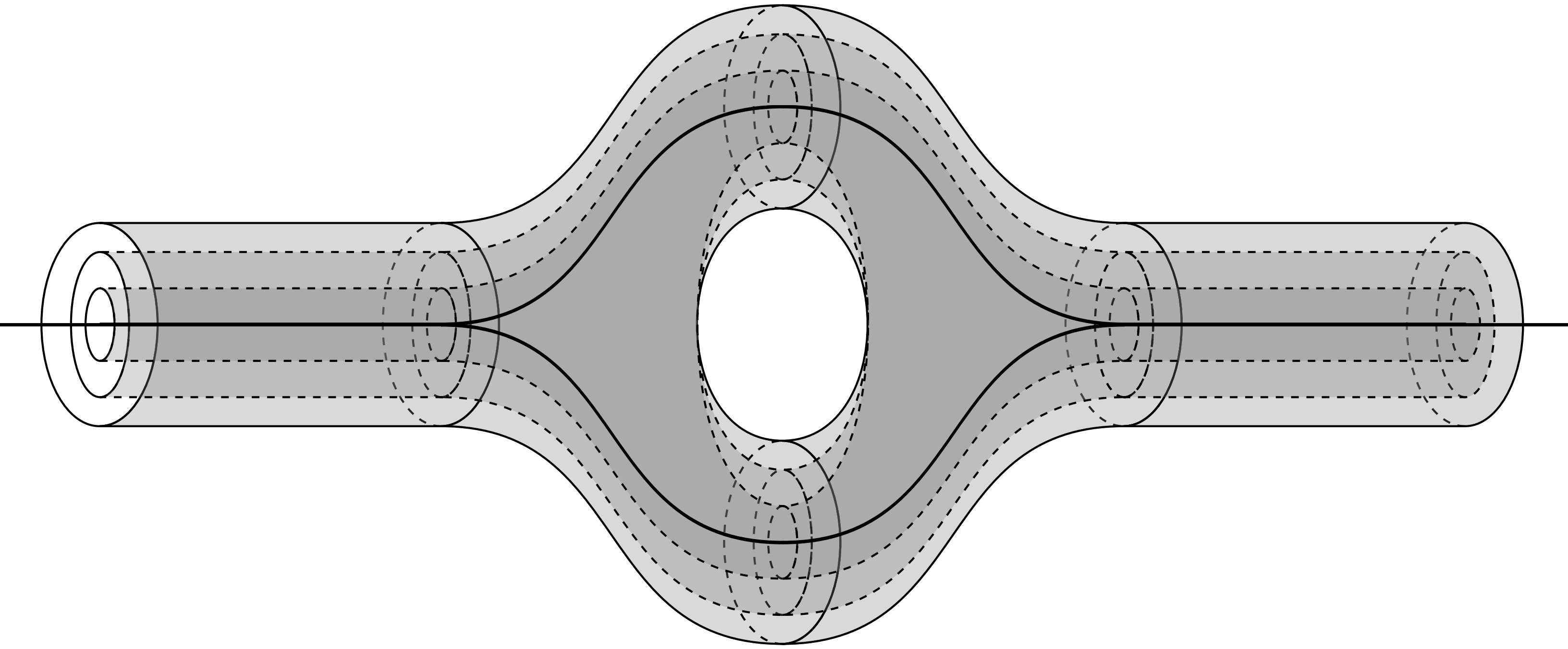
The Worldline Formulation of Quantum Field Theory



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Contents

1 Introduction

1.1 String Theory Correspondence

The beauty and power of string theory is that the vibrational modes of the string are interpreted as different *particles*. These vibrational modes give rise to both fermions and bosons, giving us a unified description of matter and radiation. We find that one of the bosons that emerge from this theory is the *graviton* and this makes string theory a promising framework for the unification of quantum field theory and gravity.

Whether or not string theory is an accurate description of nature, we require that it reduces to theories we *already* know to be accurate. Indeed, if we reduce the strings to particles, we find that string theory reduces to a theory governing the behavior of particles: Quantum Field Theory.

The form of QFT that we obtain is the subject of this thesis and it's known as the *Worldline Formulation of Quantum Field Theory*. It is conceptually and computationally distinct from the usual canonical quantization approach of QFT.

Consider a string as depicted in Figure 1 that exists in 2D space and allow the string to move throughout space. The string has a particular tension, you can imagine that if the tension goes to infinity, the string constricts to a point. Thus, the Worldline Formalism is equivalent to the *infinite tension limit* of String Theory.

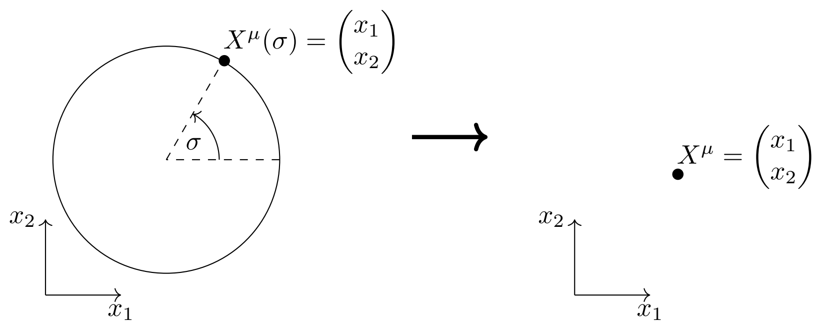


Figure 1: $\sigma \in [0, 2\pi)$ is the coordinate on the string and $X^\mu(\sigma)$ maps a point on the string to its location in space. As this is a closed string, we have $X^\mu(\sigma) = X^\mu(\sigma + 2\pi)$. In going to the infinite tension limit, we lose the dependence on σ.

We plot this 2D space against a time-axis and what we obtain is a cylindrical surface which describes the motion of the string throughout spacetime, as shown in Figure 2. This cylindrical surface is the *worldsheet* of the string.

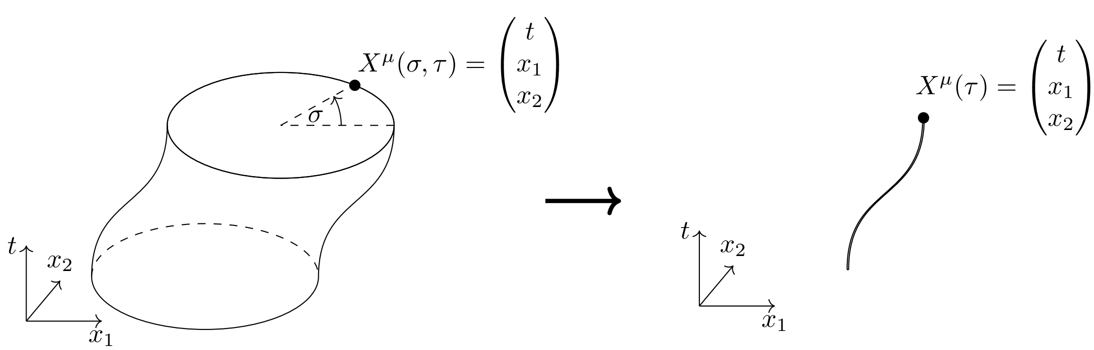


Figure 2: The spacetime diagram of the string in Figure (1) (and its infinite tension limit) illustrates the embedding of the worldsheet and worldline in Minkowski spacetime.

As shown, the worldline is parametrized by τ, which uniquely labels every point *on* the worldline. The position variable $X^\mu(\tau)$ maps the worldline co-ordinate τ to its co-ordinate in Minkowski spacetime. More generally, a theory in which a Riemann manifold is mapped to a target space is known as a σ-model.

If two strings meet in spacetime, they may merge as shown in Figure 1.1 and this is how we model interactions between strings. You can also see how a 1-loop diagram is the infinite tension limit of a string which splits in two and then recombines. Constricting the worldsheet to a vertex introduces the divergences that we are familiar with in Quantum Field Theory.

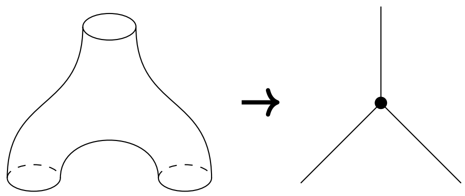


Figure 3: On the left, we have two strings merging into one string and the corresponding Feynman diagram: a 3-point vertex. On the right, we have a a string splitting and recombining, which corresponds to a 1-loop diagram

The *genus* of a worldsheet is the number of holes it has. For example, the one-loop worldsheet in Figure has genus 1. All worldsheets with a particular value of the genus are topologically equivalent to each other, meaning they can be smoothly deformed into each other. For example, all the following genus 2 worldsheets are topologically equivalent to each other:

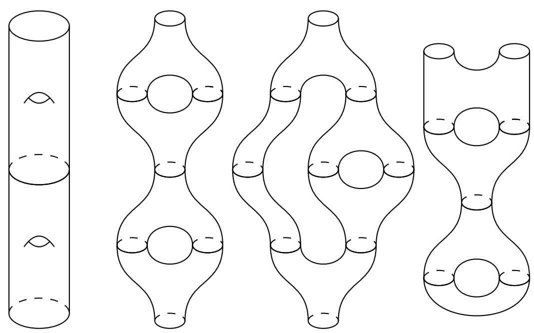


Figure 4: A worldsheet can be continuously deformed into any worldsheet that has the same genus

When considering worldsheets of genus 2 or higher, we find that there are multiple ways of taking the infinite tension limit. Therefore, a string scattering diagram of genus greater than 2 results in multiple Feynman diagrams. In taking the infinite tension limit, we must also decide what interactions are allowed as this will determine what Feynman diagrams we obtain, as illustrated in Figure 5

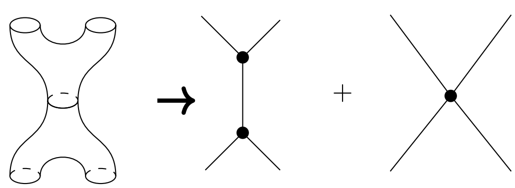


Figure 5: This string scattering diagram can reduce to a 4-point vertex or two 3-point vertices or both depending on what type of vertices we allow

If we allow only 3-point vertices, a genus 2 worldsheet reduces to the following four Feynman diagrams.

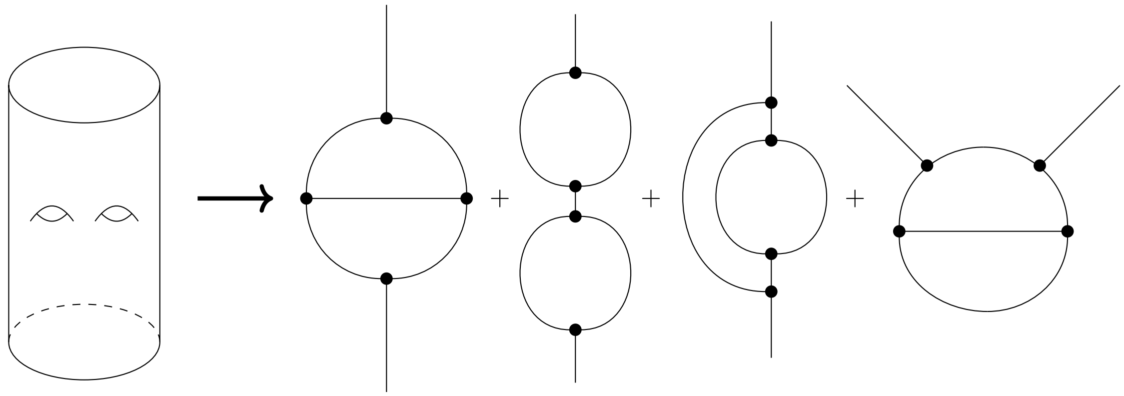


Figure 6: Infinite tension limit of a genus 2 worldsheet if we allow 3-point vertices. The genus 2 worldsheets in Figure 4 were chosen to demonstrate the reduction to these Feynman diagrams.

This correspondence between string theory and the worldline formulation is why it's often described as the *string-inspired* formulation of quantum field theory.

1.2 Physics on the Worldline

String theory fulfills the basic requirement that it must correspond to quantum field theory. However, the worldline formulation (WL) is a theory of particles and it can be derived independently from, but analogously to, string theory (ST).

Both ST and WL are formulated in terms of the Riemannian manifold generated by the fundamental object as it propagates through spacetime. For ST this is the worldsheet, for WL it's the world*line*. The embedding of the Riemannian manifold in Minkowski spacetime is given by a *sigma model*, which maps the co-ordinates of the Riemannian manifold to positions in spacetime. This embedding is described by the position variable $X^\mu(\tau)$, which is a Lorentz 4-vector in Minkowski spacetime but can be viewed as *a set of 4 scalar fields* on the Riemannian manifold. Other fields are also introduced on the manifold in order to make the theory consistent, and the dynamics of these fields result in a *field theory* on the Riemannian manifold. On the worldsheet, we obtain a 2-dimensional conformal field theory but on the worldline we obtain a 1-dimensional quantum mechanical field theory.

One of the fields we introduce to the worldline is a *worldline metric* and we will see how we can use this to show that the action is equivalent to that of a *1-dimensional general relativity*. Unlike ST, there is of course no graviton in QFT and this is because there is no operator-state correspondence, so that fluctuation of the metric don't result in flucututations of an external state.

You will have learnt quantum mechanics as the theory that governs the quantum behaviour of non-relativistic particles. However, on the worldline exists a *relativistic* quantum mechanical field theory which can be cast into a manifestly Lorentz invariant format. The duality between physics on the Riemannian manifold and in Minkowski spacetime is a powerful feature of WL and, in the coming chapters, we will see how amplitudes in the worldline formalism are calculated as correlation functions of the quantum mechanical field theory that lives on the worldline.

Like ST, WL is a *first-quantized* theory where the particle is treated quantum mechanically but the background field is treated classically. Our approach will be analogous to atomic physics, where the electron is treated quantum mechanically but is coupled to an external electromagnetic field, which is treated classically. The interactions with this field modify the Hamiltonian of the electron, and thereby the Hilbert space of quantum mechanical states that the electron occupies. In Section 4, we will model interactions by treating a particle propagating around a loop as the quantum mechanical object, acted on by vertex operators which correspond to external states. This is equivalent to the string theory approach, where a string propagating in a loop generates a genus-1 worldsheet and external states are inserted on the surface of the worldsheet.

We will deduce the dynamics of the worldline field theory by formulating a Lagrangian in terms of the position variable $X^\mu(\tau)$. While you will have studied the Lagrangians of non-relativistic particles, such as the free particle $\frac{1}{2}\dot{\mathbf{x}}$, and the Lagrangian for relativistic quantum fields, such as the Klein-Gordon equation $\partial^\mu\phi\partial_\mu\phi + m^2\phi^2$, it is peculiar to use Lagrangians in terms of relativistic particle paths $X^\mu(\tau)$. We do this by establishing the sigma model, thereby introducing a redundancy to our theory in the form of the worldline parameter τ. This redundancy can be seen as a *local gauge invariance* because the theory is invariant under local transformations $\tau \to \eta(\tau)$.

Using the Lagrangian, we compute amplitudes using *path integral formulation*, summing e^{iS} for all possible worldlines and metrics that satisfy the boundary conditions, where S is the action. Unlike ST, WL if valid off-shell and we will see how this allows us to construct complex diagrams from simples ones.

The path integral reflects the symmetries of the theory and, as a result, we obtain expressions for the amplitude that are gauge-invariant. This has important consequences for the efficiency of calculations. In the canonically quantized (CQ) formalism, we perturbatively calculate amplitudes by summing Feynman diagrams to arbitrary order with respect to the coupling constants. However, individual Feynman diagrams are not generally gauge-invariant. Amplitudes in the worldline formalism combine these diagrams in such a way that they are gauge-invariant and can therefore reduce the number of terms involved.
The increase in efficiency is most pronounced for QCD, where we must Feynman diagrams of many orders in the coupling constant to obtain the desired accuracy. WL is especially advantageous for one-loop amplitudes because, unlike the CQ approach, there are no loop momenta and this greatly reduces the number of kinematic variables. There is no need to calculate momentum integrals or to evaluate Dirac traces explicitly, and the calculations are greatly simplified.

Indeed, the paper that launched the popularity of the worldline method was an application of the infinite tension limit to calculate one-loop gluon scattering, as we will see in the next Section.

1.3 History of the Worldline Formulation

The possibility of integrating over relativistic particle paths was first noted by Feynman in Appendix A of the 1950 paper [14]. He didn't place much emphasis on this and, while there were some further studies in the literature [18][19]) [16], it wasn't considered a useful alternative to CQ for calculating amplitudes.

This changed in 1991 when Bern and Kosower derived the one-loop scattering amplitude of $2 \to 2$ gluons [7] using the infinite tension limit of string theory. Calculations of this type have since become known as Bern-Kosower rules and the authors conducted a wider systematic study of the infinite tension limit.

Serious interest in the subject only began after 1988 when several authors used the infinite tension limit of a genus one open-string worldsheet coupled to a background gauge field in order to calculate the one-loop β-function for Yang-Mills theory [27][28][22]. This work explained the well-known fact that this β-function vanishes in $D = 26$, the critical dimension of the bosonic string when calculated in dimensional regularization. This quantum field theoretic result suggests the use of string theory, and this may explain why the detailed formulation of the worldline formulation emerged from string theory.

Prior to this, the infinite tension limit was not used for computation but there were notable studies [32][29][38][33] and it is discussed in Polyakov's book [31]. In fact, the fact that string theory should correspond to QFT in the infinite tension limit was clear since the very inception of string theory.

It was Strassler who derived the Bern-Kosower rules from a pure worldline approach in 1992 [37]. This is the approach we shall use but first we need to formulate a Lagrangian for our relativistic particle.

2 Lagrangian Formulation of a Relativistic Particle

2.1 The Non-Relativistic Paricle

In non-relativistic mechanics, the spatial position of a particle is given as a function of time $\mathbf{x}[t]$.

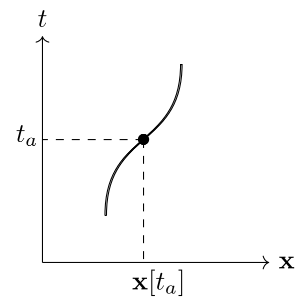


Figure 7: The spacetime diagram of the particle, illustrating the trajectory given by $\mathbf{x}[t]$

The Lagrangian for this particle is given by the classical expression for the kinetic energy of a particle $\frac{1}{2}m\dot{\mathbf{x}}^2$, and the action S is found by integrating this over time:

$$S(\mathbf{x}[t]) = \int dt \, \frac{m}{2} \left(\frac{d\mathbf{x}[t]}{dt} \right)^2 \tag{1}$$

As we wish to describe a relativistic particle, we would like to modify this action to make it *Poincaré invariant*. The *proper time* elapsed along the worldline, ds, is a Poincaré invariant quantity and is given by the spacetime interval:

$$-ds^2 = -dt^2 + d\mathbf{x}[t]^2 \tag{2}$$

$$ds = dt\sqrt{1 - \left(\frac{d\mathbf{x}[t]}{dt} \right)^2} \tag{3}$$

Based on this, we construct a Lagrangian that is proportional to ds, and therefore Poincaré invariant. We shall refer to the resulting action as the *Poincaré Action*

$$S(\mathbf{x}[t]) = \int dt \left(-m\sqrt{1 - \left(\frac{d\mathbf{x}[t]}{dt} \right)^2} \right) \tag{4}$$

Box 1. *We verify that it is a valid description of a relativistic particle by deriving the conjugate momentum* $\mathbf{p}$ *and energy* E *from the Poincaré Lagrangian:*

- $\mathbf{p} = \dfrac{d\mathcal{L}}{d\left(\frac{d\mathbf{x}[t]}{dt}\right)} = \dfrac{m\left(\frac{d\mathbf{x}[t]}{dt}\right)}{\sqrt{1 - \left(\frac{d\mathbf{x}[t]}{dt}\right)^2}}$ - *This is the correct special relativistic expression for momentum.*
- $E = \mathbf{p} \cdot \left(\frac{d\mathbf{x}[t]}{dt}\right) - \mathcal{L} = \sqrt{\mathbf{p}^2 + m^2}$ - *This is the mass-shell condition.*

The Poincaré Action describes a relativistic particle but it treats space and time on a different footing. The spatial co-ordinates are *dynamical degrees of freedom* whereas time is a mere *label* that we integrate the Lagrangian over, and which parametrizes the trajectory $\mathbf{x}(t)$. In order to derive quantum field theoretic results, we seek a formulation that is *manifestly Lorentz invariant.*

2.2 The Sigma-Model

In the canonical quantization approach, manifest Lorentz invariance is achieved by treating the spatial co-ordinates as labels, thus placing them on the same footing as time. As a result, we have a form of the action which integrates over all spacetime co-ordinates $S = \int d^4x\ \mathcal{L}$.

In the worldline formulation, we will take the opposite approach and treat time as a dynamical variable, thus placing it on the same footing as the spatial co-ordinates. These dynamical variables are parametrized by τ, which is the co-ordinate *along* the worldline. The worldline is then seen as a 1-dimensional Riemann manifold and the *position variable* X^μ defines a 1-to-1 mapping between points on the Riemann manifold and points in the target space, which is 4-dimensional Minowski spacetime.

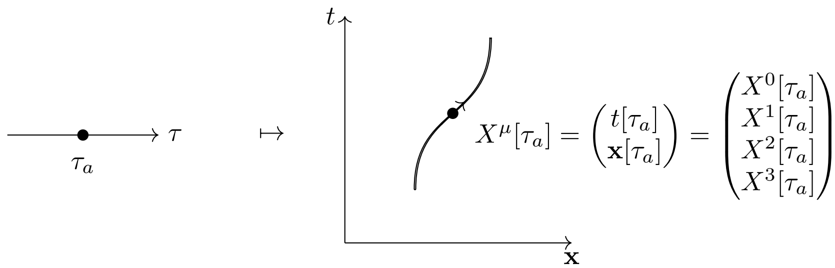


Figure 8: The sigma model of the worldline formulation.

As the purpose of the worldline parameter τ is to uniquely label every spacetime point on the worldline and does not carry any physical meaning by itself, we should be able to exchange it with any monotonic function of the worldline parameter $f(\tau)$ and that'll be an equally good parameter.

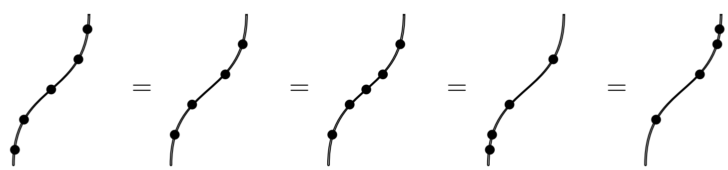


Figure 9: Reparametrizations of the worldline co-ordinate τ.

The invariance of the theory under $\tau \to f(\tau)$ is known as *reparametrization invariance* and (because the transformation is a function of local co-ordinates) it is a *local symmetry.* This is frequently to as a *local gauge symmetry* in the literaure and, while this is techinically not correct, we will adopt this terminology as it simplifies the language somewhat.

We deduce how the position variable transforms under an infinitesimal transformation of the worldline parameter. Consider an infinitesimal shift $\eta\,(\tau)$ in the worldline parameter which brings us from $X^\mu\,(\tau_a)$ to $X^\mu\,(\tau_b = \tau_a + \eta\,(\tau_a))$. The infinitesimal in the position variable induced by the worldline parameter shift $\eta(\tau)$ is denoted as $\delta_\eta X^\mu(\tau)$:

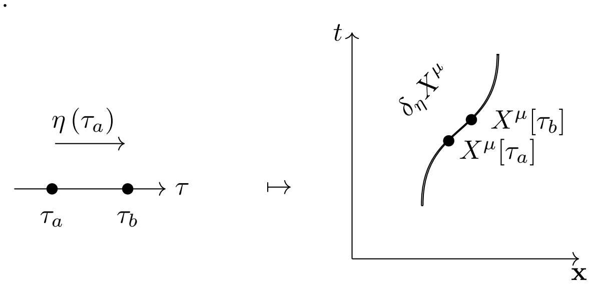


Figure 10: An infinitesimal transformation of the worldline parameter.

We see that the position variable $X^\mu(\tau)$ transforms as:

$$\delta_\eta X^\mu = \eta \frac{dX^\mu}{d\tau} \tag{5}$$

Thus the position variable, which transforms as a Lorentz 4-vector in Minkowski spacetime, transforms as a set of 4 scalar feilds on the worldline. We wish to reformulate the Poincaré action in terms of X^μ, which we can accomplish using the property of reparametrization invariance.

2.3 The Geometric Action

We wish to re-formulate the Poincaré action in a manifestly Lorentz invariant form.

$$S(\mathbf{x}[t]) = \int dt \left(-m\sqrt{1 - \left(\frac{d\mathbf{x}[t]}{dt}\right)^2} \right) \tag{6}$$

$$\to S(\mathbf{x}[\tau]) = \int d\tau \left(-m\sqrt{1 - \left(\frac{d\mathbf{x}[\tau]}{d\tau}\right)^2} \right) \tag{7}$$

The system is no longer parametrized by t and this role is now played by τ.

$$\to S(\mathbf{x}[\tau]) = \int d\tau \left(-m\sqrt{\underbrace{\left(\frac{d\tau}{d\tau}\right)^2} - \left(\frac{d\mathbf{x}[\tau]}{d\tau}\right)^2} \right) \tag{8}$$

This may seem trivial but the point is that the object being differentiated here is a particular gauge choice of $t[\tau]$. The reparametrization invariance discussed in Section 1.1 allows us to set the worldline parameter equal to the temporal component of X^μ, such that $\tau = t[\tau]$. (9)

$$\to S(\mathbf{x}[\tau]) = \int d\tau \left(-m\sqrt{\left(\frac{dt[\tau]}{d\tau}\right)^2 - \left(\frac{d\mathbf{x}[\tau]}{d\tau}\right)^2} \right)$$

Expressing this term more generally as $t[\tau]$, we see that the spatial and temporal parts of $X^\mu(\tau)$ are now on the same footing.

$$\to S\left(X^\mu[\tau]\right) = \int d\tau \left(-m\sqrt{-\eta_{\mu\nu}\left(\frac{dX^\mu[\tau]}{d\tau}\right)\left(\frac{dX^\nu[\tau]}{d\tau}\right)} \right)$$

This is a manifestly Lorentz invariant form of the action and is in terms of the position variable $X^\mu(\tau)$.

We will refer to this as the *Geometric action.* If we express the term $\sqrt{-\dot{X}^2}$ heuristically as $d_\tau|X|$, we can identify this as a length element on the worldline and $\int d\tau\sqrt{-\dot{X}^2}$ as the total length of the worldline in spacetime. The action is thus proportional to the worldline length and this is in accordance with the principle of least action, so that minimizing the action corresponds to minimizing the length of the particle's trajectory.

We also note that under the reparametrization $\tau \to \tilde{\tau}(\tau)$, the terms in S transforms as:

$$\frac{dX^\mu[\tau]}{d\tau} = \frac{d\tilde{\tau}}{d\tau}\frac{dX^\mu[\tau]}{d\tilde{\tau}} \tag{10}$$

$$d\tau = \left|\frac{d\tau}{d\tilde{\tau}}\right| d\tilde{\tau} \tag{11}$$

Substituting this into the Geometric action, we find that terms from (10) cancel with terms from (11) to give:

$$S\left(X^\mu[\tau]\right) \to S\left(X^\mu[\tilde{\tau}]\right) = \int d\tilde{\tau} \left(-m\sqrt{-\eta_{\mu\nu}\left(\frac{dX^\mu[\tilde{\tau}]}{d\tilde{\tau}}\right)\left(\frac{dX^\nu[\tilde{\tau}]}{d\tilde{\tau}}\right)} \right) \tag{12}$$

Therefore, this form of the action is reparametrization invariant. We will now let the parameter $[\tau]$ be implicit, define $A^2 \equiv \eta_{\mu\nu}A^\mu A^\nu$, $A \cdot B \equiv \eta_{\mu\nu}A^\mu B^\nu$ and $\dot{A} \equiv \dfrac{dA}{d\tau}$ so that we can express the Geometric action simply as:

$$S\left(X^{\mu}\right)=\int d\tau\left(-m\sqrt{-\dot{X}^{2}}\right) \tag{13}$$

It may seem problematic that we have introduced a new dynamical variable X^0 because it is unphysical to change the number of degrees of freedom in a system. However, we have seen that this is not a true dynamical degree of freedom through a particular gauge choice, as explained on line (9). Therefore, our system still has $D-1$ degrees of freedom.

2.4 The Classical Equations of Motion

We now use the Geometric Lagrangian $-m\sqrt{-\dot{X}^2}$ to find the equations of motion for X^{μ}:

$$\frac{d}{d\tau}\left(\frac{d\mathcal{L}}{d\left(\dot{X}_{\mu}\right)}\right)=\frac{d\mathcal{L}}{dX_{\mu}} \tag{14}$$

$$\frac{d}{d\tau}\left(\frac{m\dot{X}^{\mu}}{\sqrt{-\dot{X}^{2}}}\right)=0 \tag{15}$$

This can be rendered in a more familiar form if we now consider the conjugate momentum:

$$P^{\mu}=\frac{\partial\mathcal{L}}{\partial\dot{X}_{\mu}} \tag{16}$$

$$P^{\mu}=\frac{m\dot{X}^{\mu}}{\sqrt{-\dot{X}^{2}}} \tag{17}$$

Box 2. *Squaring (17) gives the Lorentz invariant form of the mass-shell condition:*

$$P^{2}+m^{2}=0 \tag{18}$$

Thus, the Geometric action is a valid description for a relativistic particle. Notice that only 3 of these momenta are independent and the 4th one is determined by this mass shell condition. This re-inforces the fact that we have only 3 degrees of freedom.

Substituting the conjugate momentum (17) into the equation of motion (15) gives the expected expression for a free particle:

$$\frac{dP^{\mu}}{d\tau}=0 \tag{19}$$

We wish to find the equation of motion for X^{μ} as a function of τ. We can greatly simplify the process by making the following gauge choice, which will be important throughout our discussion of the worldline formalism:

Box 3. *Consider the spacetime interval along the worldline:*

$$-ds^{2}=-d\tau^{2}+d\mathbf{x}^{2} \tag{20}$$

$$\left(\frac{ds}{d\tau}\right)^{2}=1-\left(\frac{d\mathbf{x}}{d\tau}\right)^{2} \tag{21}$$

We will use the proper time gauge, where we set $d\tau=ds$, so that equation (21) gives:

$$\frac{d\mathbf{x}}{ds}=0 \tag{22}$$

This allows us to make the simplification:

$$\dot{X}^2 = -\left(\frac{ds}{d\tau}\right)^2 + \left(\frac{d\mathbf{x}}{d\tau}\right)^2 \tag{23}$$

$$\to -\left(\frac{ds}{ds}\right)^2 + \left(\frac{d\mathbf{x}}{ds}\right)^2 \tag{24}$$

$$= -1 \tag{25}$$

This is in accordance with the special relativistic definition of 4-velocity:

$$\frac{dX^\mu}{ds} = U^\mu = \frac{1}{\sqrt{1 - \left(\frac{d\mathbf{x}}{dt}\right)^2}} \begin{pmatrix} 1 \\ \frac{d\mathbf{x}}{dt} \end{pmatrix} \tag{26}$$

$$U^2 = -1 \tag{27}$$

In the proper time gauge, the conjugate momentum (17) is:

$$P^\mu = -m\dot{X}^\mu \tag{28}$$

$$\dot{X}^\mu = -\frac{P^\mu}{m} \tag{29}$$

It's trivial to integrate this (as we showed that P^μ is independent of τ in (19)) to obtain the *trajectory* of the particle:

$$X^\mu = Q^\mu - \tau\left(\frac{P^\mu}{m}\right) \tag{30}$$

Q^μ is a constant of integration, and it's the position of the particle when $\tau = 0$. As expected, this is the trajectory of a paricle moving in a straight line.

We saw in Section 2.2 that the position variable transforms as a set of scalar fields on the worldline. Because the trajectory of the free particle is linear in τ, we see that the transformation holds for am arbitrary shift in worldline parameter: $\tau \to \tau + \rho(\tau)$, where $\rho(\tau)$ is not necessarily infinitesimal:

$$X^\mu(\tau) = Q^\mu - \tau\left(\frac{P^\mu}{m}\right) \tag{31}$$

$$\to X^\mu(\tau + \rho) = Q^\mu - \tau\left(\frac{P^\mu}{m}\right) - \rho\left(\frac{P^\mu}{m}\right) \qquad \rho \text{ is a function of } \tau \text{ but we will let this be implicit} \tag{32}$$

$$\delta_\rho X^\mu = -\rho\left(\frac{P^\mu}{m}\right) \tag{33}$$

$$\delta_\rho X^\mu = \rho\left(\frac{dX^\mu}{d\tau}\right) \tag{34}$$

We have found the expected equations of motion and transformation laws in terms of the position variable $X^\mu(\tau)$ and momentum $P^\mu(\tau)$. We wish to treat these as quantum mechanical operators and find the evolution of the states in Hilbert space using the *Hamiltonian*. We found this in terms of 3-vectors for the Poincaré action in Box 1:

$$H = \sqrt{\mathbf{p}^2 + m^2} \tag{35}$$

In first quantization, we treat $\mathbf{p}$ as the operator $i\dfrac{\partial}{\partial x}$:

$$H = \sqrt{-\frac{\partial^2}{\partial x^2} + m^2} \tag{36}$$

However, having the square root of a differential operator results in a Schröedinger equation that is non-local in spacetime. We solve this in the next section by reformulating the action once more such that it does not have a square root.

2.5 The Brink-di Vecchia-Howe action

Consider the following Lagrangian, which we will call the *Quadratic Lagrangian*:

$$\mathcal{L} = \frac{1}{2} m\dot{X}^2 \tag{37}$$

The conjugate momentum and equations of motion for this Lagrangian are:

$$P^\mu = \frac{d\mathcal{L}}{d\dot{X}} \tag{38}$$

$$= m\dot{X}^\mu \tag{39}$$

$$\frac{d}{d\tau}\left(\frac{d\mathcal{L}}{d\left(\dot{X}_\mu\right)}\right) = \frac{d\mathcal{L}}{dX_\mu} \tag{40}$$

$$\ddot{X} = 0 \tag{41}$$

These are the same as those derived from Geometric action but the action is quadratic in $\dot{X}^\mu$, making it trivial to quantize.

We start from the Quadratic Lagrangian and impose special relativistic behaviour on the particle using the *Lagrange multiplier procedure*. You can read about this in-depth here [21], and we will outline the procedure now:

Box 4. *We begin by forming an extended Lagrangian in terms of the Hamiltonian H, a constraint equation $C = 0$ and a Lagrange multiplier $e[\tau]$:*

$$\mathcal{L}[X^\mu] \to \mathcal{L}[X^\mu, e] = P \cdot \dot{X} - H - eC \tag{42}$$

$$= P \cdot \dot{X} - \frac{P^2}{2m} - eC \tag{43}$$

The Hamiltonian is obtained trivially from (37) and we express it in phase space.

$$= P \cdot \dot{X} - \frac{P^2}{2m} - e\left(P^2 - m^2\right) \tag{44}$$

We impose the mass-shell condition as the constraint $C = P^2 - m^2 = 0$.

$$= P \cdot \dot{X} - \frac{P^2}{2m} - \frac{e}{2m}\left(P^2 - m^2\right) \tag{45}$$

Rescaling the Lagrange multiplier $e \to \frac{e}{2m}$.

$$= P \cdot \dot{X} - \frac{P^2}{2m}(1 + e) - \frac{me}{2} \tag{46}$$

$$= \frac{m\dot{X}^2}{2(1+e)} - \frac{me}{2} \tag{47}$$

$\left\{ \begin{array}{l} \textit{Using (42) to find the Euler-Lagrange equation for } P^\mu\textit{:} \\ P^\mu = \dfrac{m\dot{X}}{1+e} \end{array} \right.$

$$= \frac{m\dot{X}^2}{2e} - \frac{me}{2} + \cancel{\frac{m}{2}} \tag{48}$$

Redefining $e \to e - 1$ and disposing of the constant term which does not affect the equations of motion.

$$= \frac{\dot{X}^2}{2e} - \frac{m^2 e}{2} \tag{49}$$

Now rescaling $e \to me$

We now interpret $e(\tau)$ as a field on the worldline and the action that we have derived is known as the Brink-di Vecchia-Howe action:

$$S\left[X^\mu, e\right] = \int d\tau \frac{1}{2}\left(\frac{\dot{X}^2}{e} - m^2 e\right) \tag{50}$$

This was first formulated by the authors after whom it's named in 1976 [9]. It's easy to see that it has the same equations of motion for X^μ as we had for the quadratic action (and therefore the Geometric action). The conjugate momentum now takes the form:

$$P^\mu = \frac{\dot{X}^\mu}{e} \tag{51}$$

We use this to express the Brink-di Vecchia-Howe action in terms of momentum:

$$S = \int d\tau \frac{1}{2}\left(\frac{\dot{X}^2}{e} - m^2 e\right) = \int d\tau \; \frac{e}{2}\left(P^2 - m^2\right) \tag{52}$$

The Hamiltonian of the relativistic particle is found through a Legendre transformation of the Lagrangian:

$$H = P \cdot \dot{X} - \mathcal{L} = \frac{\dot{X}^2}{e} - \frac{1}{2}\left(\frac{\dot{X}^2}{e} - m^2 e\right) = \frac{1}{2}\left(\frac{\dot{X}^2}{e} + m^2 e\right) = \frac{e}{2}\left(P^2 + m^2\right) \tag{53}$$

We expect this to vanish if the mass-shell condition is satisfied. We had imposed this in the Lagrange multiplier process in Box 4, and this can be seen explicitly by substituting the conjugate momentum (51) into the equation of motion for e:

$$\frac{d}{d\tau}\left(\frac{d\mathcal{L}}{d\dot{e}}\right) = \frac{d\mathcal{L}}{de} \tag{54}$$

$$\frac{\dot{X}^2}{e^2} + m^2 = 0 \tag{55}$$

$$P^2 + m^2 = 0 \tag{56}$$

Line (55) proves that we haven't introduced a new degree of freedom to our system, as e is completely determined by this equation. Solving (55) for e, we find:

$$e = \pm \frac{\sqrt{-\dot{X}^2}}{m} \tag{57}$$

If we substitute the positive solution of this into the Brink-di Vecchia-Howe action, we restore the Geometric action. Thus, we dispose of the negative solution through the requirement:

$$e > 0 \tag{58}$$

$$e = \frac{\sqrt{-\dot{X}^2}}{m} \tag{59}$$

We have reformulated the action in a way that is entirely analogous to string theory, where we make the transition from the Nambu-Goto action to the Polyakov action in order to remove the square root. As an unintended consuquence, we have also created an action which is valid for *massless particles*, unlike the previous forms of S that vanish for $m \to 0$.

We were able to do this by introducing the field e, and we will now examine the properties of this field and verify that the Brink-di Vecchia-Howe action is reparametrization invariant.

2.6 The Einbein Field $e(\tau)$

The action for 1-dimensional general relativity is given by the following:

$$S = -\frac{1}{2}\int d\tau\sqrt{-g_{\tau\tau}}\left(g^{\tau\tau}\partial_\tau X^\mu\partial_\tau X^\nu\eta_{\mu\nu} + m^2\right) \tag{60}$$

$$S = -\frac{1}{2}\int d\tau\sqrt{-g_{\tau\tau}}\left(g^{\tau\tau}\dot{X}^2 + m^2\right) \tag{61}$$

$g_{\tau\tau}$ is the induced metric on the worldline. This will be familiar to readers who have studied string theory, where the Polyakov action is expressed in terms of an induced metric on the worldsheet. However, in 1-dimension this is a scalar quantity so the inverse metric can be expressed straightforwardly as: $g^{\tau\tau} = 1/g_{\tau\tau}$.

$$S\left[X^\mu, \sqrt{-g_{\tau\tau}}\right] = \frac{1}{2}\int d\tau\left(\frac{\dot{X}^2}{\sqrt{-g_{\tau\tau}}} - m^2\sqrt{-g_{\tau\tau}}\right) \tag{62}$$

This is the Brink-di Vecchia-Howe action with $e = \sqrt{-g}$. Therefore, the Brink-di Vecchia-Howe action can be interpreted as a 1-dimensional general relativity with $-e^2$ acting as the worldline metric. While we will restrict ourselves to flat spacetime, we could change the Minkowski metric in (60) so that $\eta_{\mu\nu} \to g_{\mu\nu}$ so that the action describes the propagation of a relativistic particle in *curved* spacetime.

e can be heuristically regarded as the square root of the metric and such objects are important tools in general relativity, where they are a locally defined set of linearly independent vector fields that are used as a basis to represent tensors. You can find more information information about them in an introductory text on general relativity, such as Sean Carroll's book [10]. In 4-dimensions, this object is known as the *tetrad* or *vier*bein, as *vier* is German for 4. In 1-dimension, it is known as the *ein*bein and this is how we will refer to $e(\tau)$ from now on.

The metric is also used to find the Lorentz invariant interval on the manifold. We know that the invariant interval between a given pair of spacetime points is found by contracting the 4-vector separating them, dX^μ, with itself through the Minkowski metric $g^{\mu\nu}$:

$$ds^2 = g_{\mu\nu}dX^\mu dX^\nu \tag{63}$$

In the same way, the invariant interval between a given pair of worldline points is found by contracting the distance between them, $d\tau$, with itself through the worldline metric g:

$$ds^2 = -g\,(d\tau)^2 \tag{64}$$

$$ds = e\ d\tau \tag{65}$$

The invariance of this expression means that the einbein transforms in the following way under reparametrizations:

$$e(\tau)\ d\tau = e\left(\tau'\right)d\tau' \tag{66}$$

This allows us to gauge-fix the einbein as a constant.

We can also derive the transformation of e under the reparametrization $\tau \to \tau + \rho$, so that we can verify the reparametrization invariance of the Brink-di Vecchia-Howe action:

$$\delta_\rho e = e(\tau+\rho) - e(\tau) \tag{67}$$

$$= \frac{1}{m}\left(\sqrt{-\left(\frac{\partial X^\mu(\tau+\rho)}{\partial\tau}\right)^2} - \sqrt{-\left(\frac{\partial X^\mu(\tau)}{\partial\tau}\right)^2}\right) \tag{68}$$

$$= \frac{1}{m}\left(\sqrt{-\left(\frac{\partial(X^\mu+\delta_\rho X^\mu)}{\partial t}\right)^2} - \sqrt{-\left(\frac{\partial X^\mu}{\partial\tau}\right)^2}\right) \quad \text{Letting the } (\tau) \text{ be implicit} \tag{69}$$

$$= \frac{1}{m}\left(\sqrt{-\left(\frac{\partial\left(X^\mu+\rho\dot{X}^\mu\right)}{\partial t}\right)^2} - \sqrt{-\left(\frac{\partial X^\mu}{\partial\tau}\right)^2}\right) \quad \text{Using the transformation of } X^\mu \text{ from (5)} \tag{70}$$

$$= \frac{1}{m}\left(\sqrt{-\left(\dot{X}^\mu+\dot{\rho}\dot{X}^\mu+\rho\ddot{X}^\mu\right)^2} - \sqrt{-\dot{X}^2}\right) \tag{71}$$

We wish to find the transformation law of e as a function of (e,ρ) (where ρ is the parameter shift) as we had done for X^μ in (5)). However, we cannot eliminate X^μ from (71) so we shall consider infinitesimal reparametrizations. We will let $\rho(\tau)$ become an infinitesimal parameter $\eta(\tau)$, which evolves slowly in time such that $\dot{\eta}$ can also be taken as infinitesimal:

$$\delta_\eta e = \frac{1}{m}\left(\sqrt{-\left(\dot{X}^\mu+\dot{\eta}\dot{X}^\mu+\eta\ddot{X}^\mu\right)^2} - \sqrt{-\dot{X}^2}\right) \tag{72}$$

$$= \frac{1}{m}\left(\sqrt{-\left(\dot{X}^2+2\dot{\eta}\dot{X}^2+2\eta X^\mu\ddot{X}_\mu\right)} - \sqrt{-\dot{X}^2}\right) \tag{73}$$

$$= \frac{1}{m}\left(\sqrt{-\dot{X}^2\left(1+2\dot{\eta}+2\frac{\eta X^\mu\ddot{X}_\mu}{\dot{X}^2}\right)} - \sqrt{-\dot{X}^2}\right) \tag{74}$$

$$= \frac{1}{m}\left(\sqrt{-\dot{X}^2}\left(1+\dot{\eta}+\frac{\eta X^\mu\ddot{X}_\mu}{\dot{X}^2}\right) - \sqrt{-\dot{X}^2}\right) \quad \text{Disposing of terms order } \mathcal{O}\left(\eta^2\right) \text{ and higher.} \tag{75}$$

$$= \frac{1}{m}\left(\sqrt{-\dot{X}^2}\left(1+\dot{\eta}+\frac{\eta X^\mu\ddot{X}_\mu}{\dot{X}^2}\right) - \sqrt{-\dot{X}^2}\right) \tag{76}$$

$$= \dot{\eta}\frac{\sqrt{-\dot{X}^2}}{m} - \eta\frac{\dot{X}^\mu\dot{X}_\mu}{m\sqrt{-\dot{X}^2}} \tag{77}$$

$$= \dot{\eta}\frac{\sqrt{-\dot{X}^2}}{m} + \eta\frac{\partial}{\partial\tau}\left(\frac{\sqrt{-\dot{X}^2}}{m}\right) \tag{78}$$

$$= \dot{\eta}e + \eta\dot{e} \tag{79}$$

$$= \frac{\partial}{\partial\tau}\left(\eta e\right) \tag{80}$$

This shows that e transforms as a *field density* on the worldline. We may use this to prove that Brink-di Vecchia-Howe Lagrangian is invariant under infinitesimal reparametrizations of the worldline and the proof is in Appendix A.1.

We have studied the classical behaviour of the relativistic particle and, in doing so, formulated an action that allows us to *quantize* the particle. We proceed to study the quantum behaviour of the particle in terms of a quantum mechanical field theory that lives on the worldline.

3 Free, Relativistic Particle

In this Section, we will apply the results of the previous section to derive the properties of the quantum mechanical field theory on the worldline and use this to calculate amplitudes.

First, we Wick rotate the worldline parameter $i\tau \to \tau$ and the temporal component of the Minkowski 4-vector $iX^0 \to X^0$: We perform a Wick rotation and all calculations henceforth will be in Eucliean space, except where explicitly stated otherwise:

$$t \quad \to \quad -it \tag{81}$$

$$X^2 \equiv \eta_{\mu\nu} X^\mu X^\nu \quad \to \quad X^2 \equiv \delta_{\mu\nu} X^\mu X^\nu \tag{82}$$

$$\mathcal{L}(X^\mu) = \frac{m}{2}\left(\frac{dX^\mu}{d\tau}\right)^2 - V \quad \to \quad \mathcal{L}(X^\mu) = -\frac{m}{2}\left(\frac{dX^\mu}{d\tau}\right)^2 - V \tag{83}$$

$$S(X^\mu) = \int d\tau \left(\frac{m}{2}\left(\frac{dX^\mu}{dt}\right)^2 - V\right) \quad \to \quad S(X^\mu) = i\int d\tau \left(\frac{m}{2}\left(\frac{dX^\mu}{d\tau}\right)^2 + V\right) \tag{84}$$

The Wick rotated action will be defined as follows:

$$S(X^\mu) \equiv \int d\tau \left(\frac{m}{2}\left(\frac{dX^\mu}{dt}\right)^2 + V\right) \tag{85}$$

3.1 First Quantization on the Worldline

In non-relativistic quantum mechanics, we perform first-quantization by identifying position $\mathbf{x}$ and momentum $\mathbf{p}$ with the operators that act on the states $|\psi\rangle$ that span the Hilbert space, which can be expressed as states of well-defined position $|\mathbf{x}\rangle$ as well as states of well-defined momentum $|\mathbf{p}\rangle$.

$$\mathbf{p} \to \mathbb{p}\,|\mathbf{p}\rangle = i\frac{d}{d\mathbf{x}}\,|\mathbf{p}\rangle \tag{86}$$

$$= \mathbf{p}\,|\mathbf{p}\rangle \tag{87}$$

$$\mathbf{x} \to \mathbb{x}\,|\mathbf{x}\rangle = \mathbf{x}\,|\mathbf{x}\rangle \tag{88}$$

The obey the usual commutation relation:

$$\left[x^i, p_j\right] = i\delta^i_j \tag{89}$$

The Hamiltonian for a free, non-relativistic particle is treated as an operator using the momentum operator (87):

$$H = \frac{\mathbf{p}^2}{2m} \to \mathbb{H} = -\frac{1}{2m}\nabla^2 \tag{90}$$

The Schröedinger equation then gives the evolution of these states with respect to time in terms of the Hamiltonian operator $\mathbb{H}$:

$$i\frac{d\,|\psi(t)\rangle}{dt} = \mathbb{H}\,|\psi(t)\rangle \tag{91}$$

$$|\psi(t)\rangle = e^{-i\mathbb{H}t}\,|\psi(0)\rangle \tag{92}$$

Therefore, the amplitude for $|\mathbf{x}_a\rangle$ to evolve to $|\mathbf{x}_b\rangle$ in a time T is given by:

$$\left\langle \psi_a \left| e^{-i\mathbb{H}T} \right| \psi_a \right\rangle \tag{93}$$

Upon transitioning from the non-relativistic picture to the sigma model in Section 2.2, we describe the dynamics of the system through the position 4-vector X^μ. Correspondingly, we will quantize the system by identifying the Lorentz invariant position and momentum as quantum mechanical operators that act on states on the Hilbert space $|\Psi\rangle$:

$$P^\mu \to \mathbb{P}^\mu |P\rangle = i\frac{\partial}{\partial X_\mu} |P\rangle \tag{94}$$
$$= P^\mu |P\rangle \tag{95}$$

$$X^\mu \to \mathbb{X}^\mu |X\rangle = X^\mu |X\rangle \tag{96}$$

The obey the usual commutation relation:

$$[\mathbb{X}^\mu, \mathbb{P}_\nu] = i\delta^\mu_\nu \tag{97}$$

The states $|\Psi\rangle$ can be expressed in terms of the wavefunctions that are a function of τ and the eigenvalues and eigenvectors of $\mathbb{X}$

$$|\Psi\rangle = \int d^4X^\mu \;\; \Psi\left(X^\mu, \tau\right)|X\rangle \tag{98}$$

We derived the Hamiltonian in Section 2.5 on line (53) and saw that it vanishes due to it's equality to the mass-shell condition. Therefore, the corresponding Hamiltonian operator acts on the states of the Hilbert space such that:

$$H \to \mathbb{H}|\Psi\rangle = \left(\mathbb{P}^2 + m^2\right)|\Psi\rangle = 0 \tag{99}$$
$$= \left(-\Box + m^2\right)|\Psi\rangle = 0 \tag{100}$$

This equation enforces the mass-shell condition on the states spanning the Hilbert space. The Hamiltonian is also used for the Schröedinger equation of our theory. In shifting to the sigma model in Section 2.2, we promoted time to a dynamical variable and instead parametrize the system with the worldline parameter τ. Thus, the Schröedinger equation (which we express in Euclideanized form) is such that it gives the evolution of states *along the worldline* rather than with respect to time.

$$\frac{d|\Psi(\tau)\rangle}{d\tau} = -\mathbb{H}|\Psi(\tau)\rangle \tag{101}$$
$$|\Psi(\tau)\rangle = e^{-\mathbb{H}\tau}|\Psi(0)\rangle \tag{102}$$
$$|\Psi(\tau)\rangle = |\Psi(0)\rangle \tag{103}$$

This show that the states in the Hilbert space are independent of the worldline parameter. This is what we expect because, as we elaborated in Section 2.2, τ itself has no physical meaning so the wavefunction ought not depend on it. The states of Hilbert space are now represented as:

$$|\Psi\rangle = \int d^4X \;\; \Psi\left(X\right)|X\rangle \tag{104}$$

Furthermore, we have seen in Section 13 that the worldline can be parametrized such that τ is the proper time elapsed for the relativistic particle. Therefore, the amplitude for the particle to propagate from $|X_a\rangle$ to $|X_b\rangle$ in the proper time T is given by:

$$\left\langle X_a \left| e^{-T\mathbb{H}} \right| X_a \right\rangle \tag{105}$$

Among the states in Hilbert space, those that satisfy the mass-shell constraint (100) are the *physical* states $|\phi\rangle = \int d^4X^\mu \; \phi\left(X^\mu\right)|X\rangle$:

$$\left(\mathbb{P}^2 + m^2\right)|\phi\rangle = 0 \tag{106}$$
$$\left(\mathbb{P}^2 + m^2\right)|X\rangle\langle X|\phi\rangle = 0 \tag{107}$$
$$\left(-\Box + m^2\right)\phi\left(X^\mu\right) = 0 \tag{108}$$

We have obtained the Klein-Gordon equation, which is familiar from canonically quantized QFT as the equation of motion for free, massive scalar *fields*.

In analogy to ordinary non-relativistic first quantization, we have performed a first quantization of the relativistic, scalar particle whose dynamics are described by the Brink-di Vechia-Howe action. What we obtain is a relativistic, quantum mechanical field theory that lives on the worldline.

It's significant that we have been able to derive a quantum field theoretic result from this first-quantized approach. In canonically quantized QFT, the Klein-Gordon equation is used to derive the *propagator*, which is the amplitude for the free, scalar field to propagate in spacetime,

In the next section, we see how the propogator can be derived from the worldline perspective, i.e. as the correlation function of the relativistic quantum mechanical field theory that exists on the worldline.

3.2 The Free, Relativistic Propagator

In the usual approach, we find the *Green's function* of the Klein-Gordon equation. This is the inverse of the Klein-Gordon equation and, in Euclidean space, is defined by:

$$\left(\mathbb{P}^2 + m^2\right) \Delta\left(X_a, X_b\right) = \delta\left(X_b - X_a\right) \tag{109}$$

Then, we have the amplitude for a field to propagate from X_a^μ to X_b^μ according to $\langle \phi(X_a^\mu)\phi(X_b^\nu)\rangle = \eta^{\mu\nu}\Delta(X_b, X_a)$.

For the Klein-Gordon equation, this is referred to as the *free propagator* and the *free Green's function.*

In momentum space, the propagator represented as: $\overset{P}{\longrightarrow} = \left\langle P \left| \dfrac{1}{\mathbb{P}^2 + m^2} \right| P \right\rangle = \dfrac{1}{P^2 + m^2}$

In position space, the propagator represented as: $\overset{X_a}{\bullet} \text{- - - - - -} \overset{X_b}{\bullet} = \left\langle X_b \left| \dfrac{1}{\mathbb{P}^2 + m^2} \right| X_a \right\rangle = \displaystyle\int \frac{d^D p}{(2\pi)^D} \frac{e^{iP\cdot(X_b - X_a)}}{P^2 + m^2}$

The position space representation is obtained through a Fourier transform of the momentum space representation in D-dimensions. We now manipulate this expression using the *Schwinger Proper Time parametrization.*

Box 5. *The Schwinger Proper Time parametrization uses the following mathematical identity to re-express a denominator as an exponential:*

$$\frac{1}{A} = \int_0^\infty dT\ e^{-TA} \tag{110}$$

This is a trivial manipulation, but it will become clear that T is the proper time for the particular to propagate from X_a to X_b. The utility of introducing this parameter to solve Quantum Field Theoretic problems was noted by Feynman [14] and Schwinger [34] in 1950.

Expressing the position space representation of the propagator in the Schwinger proper time parametrization implies the worldline approach:

$$\Delta\left(X_a, X_b\right) = \left\langle X_b \left| \frac{1}{\mathbb{P}^2 + m^2} \right| X_a \right\rangle = \int_0^\infty dT \left\langle X_b \left| e^{-T\left(\mathbb{P}^2 + m^2\right)} \right| X_a \right\rangle = \int_0^\infty dT\ e^{-m^2 T} \left\langle X_b \left| e^{-T\mathbb{P}^2} \right| X_a \right\rangle \tag{111}$$

But we saw on line (105) of Section 3.1 that $\langle X_b| e^{-T\left(\mathbb{P}^2\right)} |X_a\rangle$ is the amplitude for a relativistic particle with Hamiltonian $\mathbb{P}^2$ to propagate from X_a to X_a in proper time T. We see that this is then integrated over T to give the manifestly Lorentz invariant result.

This elucidates the statement made in Section 1.2, that the worldline formalism calculates the transition amplitude for a relativistic particle as a correlation function of a 1-dimensional, relativistic, quantum mechanical field theory that lives on the worldline.

In non-relativistic quantum mechanics, we may calculate amplitudes of the form $\langle \mathbf{x}_b| e^{iT\mathbb{H}} |\mathbf{x}_a\rangle$ in the *path integral formalism.* In the context of the sigma-model, the same technique can be used to calculate relativistic amplitudes of the form $\langle X_b| e^{-T\left(\mathbb{P}^2 + m^2\right)} |X_a\rangle$. This is the basis of the worldline formalism.

This is the subject of the next section but first, we proceed to find an explicit expression for the free propagator in position space so that we may verify the result obtained from the worldline approach.

For simplicity, let's take the boundary conditions for the free propagator from $\{X_a^\mu, X_b^\mu\}$ to $\{0^\mu, X^\mu\}$:

$$\Delta = \int \frac{d^D P}{(2\pi)^D} \frac{e^{iP\cdot X}}{P^2 + m^2} \tag{112}$$

Now express this using Schwinger proper time parametrization:

$$\Delta = \int_0^\infty dT \int \frac{d^D P}{(2\pi)^D} e^{iP\cdot X} e^{-T\left(P^2+m^2\right)} \tag{113}$$

$$= \int_0^\infty dT\; e^{-Tm^2} \int \frac{d^D P}{(2\pi)^D} \exp\left(iP\cdot X - TP^2\right) \tag{114}$$

We now complete the square on the terms inside the exponential so that the integral is in a Gaussian form:

$$\Delta = \int_0^\infty dT\; e^{-Tm^2} \int \frac{d^D P}{(2\pi)^D} \exp\left(-T\left(P^\mu - \frac{iX^\mu}{2T}\right)^2 - \frac{X^2}{4T}\right) \tag{115}$$

$$= \int_0^\infty dT\; e^{-Tm^2} e^{-X^2/4T} \int \frac{d^D P}{(2\pi)^D} \exp\left(-T\left(P^\mu - \frac{iX^\mu}{T}\right)^2\right) \tag{116}$$

As Gaussian integrals are translation invariant, we can take $P^\mu \to P^\mu + \dfrac{iX^\mu}{T}$:

$$\Delta = \int_0^\infty dT\; e^{-Tm^2} e^{-X^2/4T} \int \frac{d^D P}{(2\pi)^D} \exp\left(-TP^2\right) \tag{117}$$

We insert the result of this standard Gaussian integral to obtain the expression of the free propagator in position space::

$$\Delta = \int_0^\infty dT\; e^{-Tm^2} \underbrace{\frac{e^{-X^2/4T}}{(4\pi T)^{D/2}}} \tag{118}$$

In the next section, we see how the indicated part of this expression can be obtained in the path integral formalism.

3.3 The Path Integral Formalism

The path integral formulation of quantum mechanics was proposed by Richard Feynman in his 1948 paper 'Space-Time Approach to Non-Relativistic Quantum Mechanics [15]. According to this approach, we can calculate the amplitude for a particle to propagate from $\mathbf{x}_a$ to $\mathbf{x}_b$ in a time T by considering all the trajectories $\mathbf{x}(t)$ that connect these 2 points:

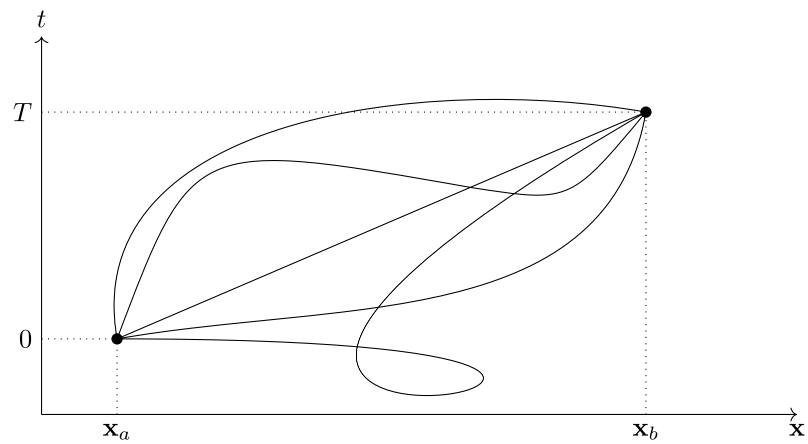


Figure 11: The classical path for the free particle to propagate from $\mathbf{x}_a$ to $\mathbf{x}_b$ is the straight line joining the points. However, in the path integral formalism, we consider all paths satisfying the boundary conditions $\{\mathbf{x}(0) = \mathbf{x}_a, \mathbf{x}(T) = \mathbf{x}_b\}$, including seemingly non-sensical paths.

Each path is given by a trajectory $\mathbf{x}(t)$ and each trajectory gives an action $S[\mathbf{x}(t)]$. The amplitude for the particle to propagate from $\mathbf{x}_a$ to $\mathbf{x}_b$ is found by summing $\exp(-S[\mathbf{x}(t)])$ for all paths $\{\mathbf{x}(t)\}$ satisfying the boundary conditions $\{\mathbf{x}(0) = \mathbf{x}_a, \mathbf{x}(T) = \mathbf{x}_b\}$.

$$\left\langle \mathbf{x}_b \left| e^{iT\mathbb{H}} \right| \mathbf{x}_a \right\rangle = \sum_{\{\mathbf{x}(t)\}} e^{-S[\mathbf{x}(t)]} \tag{119}$$

In the infinitesimal limit, this is represented as a functional integral of $\exp(-S[\mathbf{x}(t)])$ with respect to trajectories $\mathbf{x}(t)$. The boundary conditions $\{\mathbf{x}(0) = \mathbf{x}_a, \mathbf{x}(T) = \mathbf{x}_b\}$ give the functional integration limits.

$$\left\langle \mathbf{x}_b \left| e^{-T\mathbb{H}} \right| \mathbf{x}_a \right\rangle = \int_{\mathbf{x}(0)=\mathbf{x}_a}^{\mathbf{x}(T)=\mathbf{x}_b} \mathcal{D}\mathbf{x}(t)\ e^{-S[\mathbf{x}(t)]} = \int_{\mathbf{x}(0)=\mathbf{x}_a}^{\mathbf{x}(T)=\mathbf{x}_b} \mathcal{D}\mathbf{x}(t)\ e^{-\int d\tau \mathcal{L}[\mathbf{x}(t)]} \tag{120}$$

This is readily applicable to a variety of systems whose Lagrangians are known, such as the free particle or the harmonic oscillator. However, as we saw in Section 3.2, we wish to calculate amplitudes of the form $\langle X_b| e^{-T\mathbb{P}^2} |X_a\rangle$ in the worldline formalism. We transition to the sigma model to express the path integral *on the worldline*, and we also introduce the notation for the amplitude as K, whose argument is the boundary conditions of the path integral:

$$\left\langle X_b \left| e^{-T\mathbb{H}} \right| X_a \right\rangle = K(X_a, 0; X_b, T) = \int_{X(0)=X_a}^{X(T)=X_b} \mathcal{D}X(\tau)\ e^{-S[X(\tau)]} \tag{121}$$

Rather than the non-relativistic picture of Figure 11, we now calculate the amplitude for a particle to propagate from X_a to X_b in a proper time T by considering all *worldlines* that connect these spacetime points:

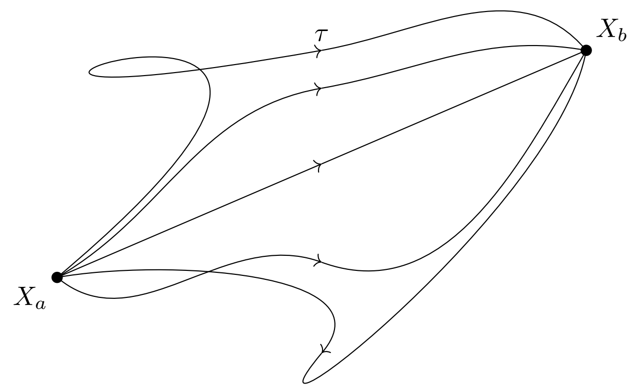


Figure 12: An example of worldlines $X(\tau)$ that satisfy the boundary conditions $\{X(0) = X_a, X(T) = X_b\}$. The arrows are to emphasize that, unlike the non-relativistic case, the variable that parametrizes the system is the co-ordinate *on* the paths.

As we saw on line (111) in Section 3.2, the expression for the free, relativistic propagator contains a term which takes the form of a quantum mechanical transition amplitude, which we now express as a path integral:

$$\left\langle X_b \left| e^{-T\mathbb{P}^2} \right| X_a \right\rangle = \int_{X(0)=X_a}^{X(T)=X_b} \mathcal{D}X(\tau)\; e^{-S[X(\tau)]} \tag{122}$$

In order to deduce the action $S[X(\tau)]$ which corresponds to the Hamiltonian $\mathbb{H} = \mathbb{P}^2$, we consider the Hamiltonian for a free, scalar, non-relativistic particle (which we saw on line (90) in Section 3.1) and the corresponding Lagrangian:

$$\mathbb{H} = \frac{\mathbb{p}^2}{2m} = -\frac{1}{2m}\nabla^2 \qquad\Longrightarrow\qquad \mathcal{L}[\mathbf{x}(t)] = \frac{m}{2}\frac{d\mathbf{x}(t)}{dt}^2 \tag{123}$$

We transition to the sigma model through the formal replacements:

$$t \to \tau \tag{124}$$

$$\mathbf{x}(t) \to X^\mu(\tau) \tag{125}$$

$$\mathbb{p} \to \mathbb{P} \tag{126}$$

$$m \to \frac{1}{2} \tag{127}$$

In doing so, we find the Lagrangian:

$$\mathbb{H} = \mathbb{P}^2 = -\Box \qquad\Longrightarrow\qquad \mathcal{L}[X(\tau)] = \frac{1}{4}\left(\frac{dX(\tau)}{d\tau}\right)^2 \tag{128}$$

Therefore we deduce that the free, relativistic propagator can be expressed in terms of the path integral:

$$\left\langle X_b \left| e^{-T\mathbb{P}^2} \right| X_a \right\rangle = \int_{X(0)=X_a}^{X(T)=X_b} \mathcal{D}X(\tau) \exp\left(-\int_0^T d\tau \frac{1}{4}\left(\frac{dX(\tau)}{d\tau}\right)^2\right) \tag{129}$$

$$\tag{130}$$

This appears to be the propagator for a fictitious particle of Hamiltonian $\mathbb{P}^2$, propagating in D spatial dimensions and zero temporal dimension. However, we will gain insight on why the path integral takes this form in the Section 3.5. First, we should explicitly calculate this path integral and this is the subject of the next chapter.

3.4 Propagator of the Fictitious Particle

To evaluate the path integral (129), we first simplify the calculation by taking the boundary conditions from $\{X_a^\mu, X_b^\mu\}$ to $\{0^\mu, X^\mu\}$:

$$K(X,T) = \int\limits_{X(0)=0}^{X(T)=X} \mathcal{D}X(\tau) \exp\left(-\int_0^T d\tau \frac{1}{4}\left(\frac{dX(\tau)}{d\tau}\right)^2\right) \tag{131}$$

We do this by expressing the path $X(\tau)$ as the sum of the *classical path* $X_C(\tau)$ and the quantum fluctuations around that path $Q(\tau)$.

$$X(\tau) = X_C(\tau) + Q(\tau) \tag{132}$$

The classical path is the solution of the Euler-Lagrange equation for the Lagrangian (128) subject to the appropriate boundary conditions. This gives a straight line connecting 0 and X:

$$X_C(\tau) = \frac{\tau}{T} X \tag{133}$$

Substituting this into (132) and evaluating it at the boundaries implies that the boundary condition for $Q(\tau)$ is that it vanishes at the boundaries:

$$Q(0) = Q(\tau) = 0 \tag{134}$$

All this information is summarized by the diagram:

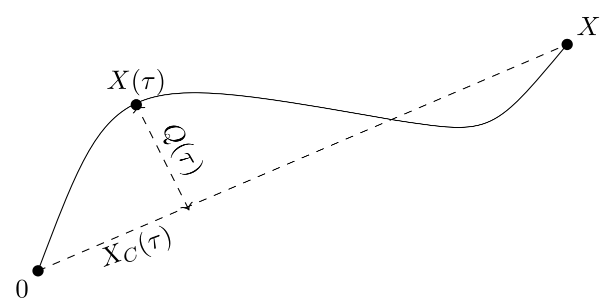


Figure 13: $X(\tau)$ is an example of the infinitely many paths connecting 0 and X, each of which can be expressed as the sum of the classical path $X_C(\tau)$ and quantum fluctuations around that path $Q(\tau)$.

We now express the action in terms of these new co-ordinates:

$$S\left[X(\tau)\right] = \frac{1}{4}\int_0^T d\tau\ \frac{dX(\tau)}{d\tau}^2 \tag{135}$$

$$= \frac{1}{4}\int_0^T d\tau \left(\frac{dX_C(\tau)}{d\tau} + \frac{dQ(\tau)}{d\tau}\right)^2 \tag{136}$$

$$= \frac{1}{4}\int_0^T d\tau \left(\frac{X}{T} + \frac{dQ(\tau)}{d\tau}\right)^2 \tag{137}$$

$$= \frac{1}{4}\int_0^T d\tau \left(\left(\frac{X}{T}\right)^2 + \left(\frac{dQ(\tau)}{d\tau}\right)^2\right) + \frac{X}{2T}\int_0^T d\tau\ \frac{dQ(\tau)}{d\tau} \tag{138}$$

$$= \frac{X^2}{4T} + \frac{1}{4}\int_0^T d\tau \left(\frac{dQ(\tau)}{d\tau}\right)^2 \tag{139}$$

We substitute this into the path integral (131) and the functional integral over $X(\tau)$ becomes a functional integral over $Q(\tau)$:

$$K(X,T) = \int\limits_{X(0)=0}^{X(T)=X} \mathcal{D}Q(\tau)\exp\left(-\left(\frac{X^2}{4T}+\frac{1}{4}\int_0^T d\tau\left(\frac{dQ(\tau)}{d\tau}\right)^2\right)\right) \tag{140}$$

$$= e^{-X^2/4T}\int\limits_{Q(0)=Q(T)=0} \mathcal{D}Q(\tau)\exp\left(-\frac{1}{4}\int_0^T d\tau\ \left(\frac{dQ(\tau)}{d\tau}\right)^2\right) \tag{141}$$

The classical solution has factored out and the path integral has reduced to the same form that we started with (131), but in terms of the quantum fluctuation variable $Q(\tau)$ and with Dirichlet boundary conditions. We shall simply denote the result of this path integral as $\mathcal{N}(T)$:

$$K(X,T) = \left\langle X^\mu \left| e^{-T\mathbb{P}^2} \right| 0^\mu \right\rangle = \langle X^\mu, T|0^\mu, 0\rangle = K(X^\mu, T; 0^\mu, 0) = \mathcal{N}(T)e^{-X^2/4T} \tag{142}$$

Where we have expressed the result as the inner product of the eigenstates that specify the boundary conditions in terms of spacetime points and the proper time elapsed $|X^\mu, T\rangle$. We see that inserting a complete set of these eigenstates $|X'^\mu, \tau\rangle$ is equivalent to inserting an intermediary boundary that reduces our initial boundary conditions $\{0^m u, 0, X^\mu, T\}$ into 2 boundary conditions: $\{0^\mu, 0, X'^\mu, \tau\}$ and $\{X'^\mu, \tau, X^\mu, T\}$:

$$\langle X^\mu, T|0^\mu, 0\rangle = \int dX'\ \langle X^\mu, T|X'^\mu, t\rangle\langle X'^\mu, t|0^\mu, 0\rangle \tag{143}$$

$$K(X^\mu, T; 0^\mu, 0) = \int dX'\ K\left(X^\mu, T; X'^\mu, \tau\right) K(X'^\mu, \tau; 0^\mu, 0) \tag{144}$$

$$\mathcal{N}(T)\exp\left(-\frac{X^2}{4T}\right) = \int dX'\ \mathcal{N}(T-\tau)\exp\left(-\frac{(X-X')^2}{4(T-\tau)}\right)\mathcal{N}(\tau)\exp\left(-\frac{X'^2}{4\tau}\right) \tag{145}$$

$$\frac{\mathcal{N}(T)}{\mathcal{N}(T-\tau)\mathcal{N}(\tau)} = \int dX'\exp\left(-\frac{1}{4}\left(\frac{(X-X')^2}{(T-\tau)}+\frac{X'^2}{\tau}-\frac{X^2}{T}\right)\right) \tag{146}$$

This is a Gaussian integral and we can perform a linear shift of the co-ordinates $X \to 0$.

$$\frac{\mathcal{N}(T)}{\mathcal{N}(T-\tau)\mathcal{N}(\tau)} = \int dX'\exp\left(-\frac{1}{4}\left(\frac{X'^2}{(T-\tau)}+\frac{X'^2}{\tau}\right)\right) \tag{147}$$

$$= \int dX'\exp\left(-\frac{1}{4}\left(\frac{X'^2 T}{(T-\tau)\tau}\right)\right) \tag{148}$$

$$= \sqrt{\frac{4\pi(T-\tau)\tau}{T}}^{D} \tag{149}$$

Where, in the final line, we inserted standard result of the Gaussian integral in D-dimensions. We write the result in a suggestive form:

$$\frac{\mathcal{N}(T)}{\mathcal{N}(T-\tau)\mathcal{N}(\tau)} = \left(\frac{\sqrt{\frac{1}{4\pi T}}}{\sqrt{\frac{1}{4\pi(T-\tau)}}\sqrt{\frac{1}{4\pi\tau}}}\right)^D \tag{150}$$

Comparing the left and right hand sides of this equation, we deduce the form of $\mathcal{N}(T)$:

$$\mathcal{N}(T) = \sqrt{\frac{1}{4\pi T}}^{D} \tag{151}$$

Inserting this into line (142):

$$K(X,T) = \frac{e^{-X^2/4T}}{(4\pi T)^{D/2}} \tag{152}$$

We have successfully showed that the part of the free propagator indicated on line (118) at the end of Section 3.2 can be obtained as the path integral of a fictitious particle with Lagrangian $\dot{X}^2/4$ propagating in D-dimensions:

$$K(X,T) = \int\limits_{X(0)=0}^{X(T)=X} \mathcal{D}X(\tau) \exp\left(-\int_0^T d\tau \frac{1}{4}\left(\frac{dX(\tau)}{d\tau}\right)^2\right) = \frac{e^{-X^2/4T}}{(4\pi T)^{D/2}} \tag{153}$$

We begin the following section by considering the *full* expression for the free propagator could be obtained as a path integral.

3.5 Worldline Representation of Free Propagator

We derived the position space representation of the free propagator on line (118) at the end of Section 3.2 with boundary conditions $\{0, X\}$, which we now express with boundary conditions $\{X_a, X_b\}$

$$\Delta(X_a, X_b) = \int_0^\infty dT\ e^{-m^2T} \left\langle X_b \left| e^{-T\mathbb{P}^2} \right| X_a \right\rangle \tag{154}$$

$$= \int_0^\infty dT\ e^{-m^2T}\ \frac{e^{-(X_b-X_a)^2/4T}}{(4\pi T)^{D/2}} \tag{155}$$

Using line (153) at the end of Section 3.3 to insert the path integral:

$$\Delta(X_a, X_b) = \int_0^\infty dT\ e^{-m^2T} \int_{X(0)=X_a}^{X(T)=X_b} \mathcal{D}X \exp\left(-\frac{1}{4}\int_0^T d\tau \left(\frac{dX}{d\tau}\right)^2\right) \tag{156}$$

We rescale the worldline co-ordinates $\tau \to T\tau$ so that it's range is $\{0, 1\}$:

$$\Delta(X_a, X_b) = \int_0^\infty dT\ e^{-m^2T} \int_{X(0)=X_a}^{X(T)=X_b} \mathcal{D}X \exp\left(-\frac{1}{4}\int_0^T d(T\tau) \left(\frac{1}{T}\frac{dX(\tau)}{d\tau}\right)^2\right) \tag{157}$$

$$= \int_0^\infty dT\ e^{-m^2T} \int_{X(0)=X_a}^{X(1)=X_b} \mathcal{D}X(\tau) \exp\left(-\frac{1}{4}\int_0^1 d\tau \frac{\dot{X}^2}{T}\right) \tag{158}$$

$$= \int_0^\infty dT \int_{X(0)=X_a}^{X(1)=X_b} \mathcal{D}X(\tau) \exp\left(-\int_0^1 d\tau \left(\frac{\dot{X}^2}{4T} + m^2T\right)\right) \tag{159}$$

We have manipulated the path integral such that we are now calculating the amplitude for a particle with Lagrangian:

$$\mathcal{L}[X(\tau)] = \frac{\dot{X}^2}{4T} + m^2T \tag{160}$$

Compare this with the Euclideanized version of the Brink-di Vecchia-Howe Lagrangian (which describes the dynamics of a relativistic particle, as we saw in Section 2):

$$\mathcal{L}[X(\tau), e(\tau)] = \frac{1}{2}\left(\frac{\dot{X}^2}{e} + m^2e\right) \tag{161}$$

We know from Section 2.5 that there is a freedom to gauge-fix the einbein and we see from the above comparison that:

$$\mathcal{L}[X(\tau)] = \mathcal{L}[X(\tau), e = 2T] \tag{162}$$

This is consistent with line (66) of Section 2.5 where we saw that the quantity $e\ d\tau$ is gauge-invariant, if we define the gauge-fixing according to:

$$\int_0^T d\tau\ e = 2T \tag{163}$$

In fact, the *only* invariant quantity on a 1-dimensional Riemannian manifold is the *length*, which is the *proper time* in the context of worldlines. Therefore, T is the *only* quantity which can be used to gauge-fix the einbein.

This leads us to consider whether the path integral (159) can emerge from the imposition of this gauge-fixing on the path integral for a relativistic particle, which now includes a functional integral over the einbein field.:

$$\Delta\left(X_a, X_b\right) = \int_{X(0)=X_a}^{X(1)=X_b} \mathcal{D}X\mathcal{D}e \exp\left(-\frac{1}{2}\int_0^T d\tau \left(\frac{\dot{X}^2}{e} + m^2 e\right)\right) \tag{164}$$

A gauge-fixed path integral may run over values of the gauge-fixed field that are equivalent to each other through diffeomorphisms. The path integral (165) therefore overcounts the field configurations by a factor of the *volume of this diffeomorphism group*, which we denote as Vol(Diffeos.). The correct form of the path integral is therefore:

$$\Delta\left(X_a, X_b\right) = \int \frac{\mathcal{D}e}{\mathrm{Vol(Diffeos.)}} \int_{X(0)=X_a}^{X(1)=X_b} \mathcal{D}X \exp\left(-\frac{1}{2}\int_0^T d\tau \left(\frac{\dot{X}^2}{e} + m^2 e\right)\right) \tag{165}$$

Readers familiar with string theory may have studied the analogous procedure of gauge-fixing the intrinsic worldsheet metric in the Polyakov action. For the worldline, the diffeomorphism group (whose volume is *infinite*) is the reparametrization $\tau \to \tau + \eta(\tau)$, as we showed in Section 2.6 that the Brink-di Vecchia-Howe action is invariant under such gauge transformations.

For a gauge-fixed value of the einbein, $\hat{e}$, there exists a family of gauge-equivalent einbeins $\{\hat{e}_\alpha\}$ which constitute a 1-dimensional gauge group known as a *gauge orbit*. Any einbein on the gauge orbit can be transformed to $\hat{e}$ through a diffeomorphism:

$$\frac{d\alpha}{d\tau} = e \tag{166}$$

Where the function $\alpha(\tau)$ is a map from this 1-dimensional gauge group to the worldline. We wish to perform the path integral such that we integrate over a single representative value of the einbein for each gauge group. Another way of saying this is that we wish to perform the path integral such that we cut each gauge orbit once.

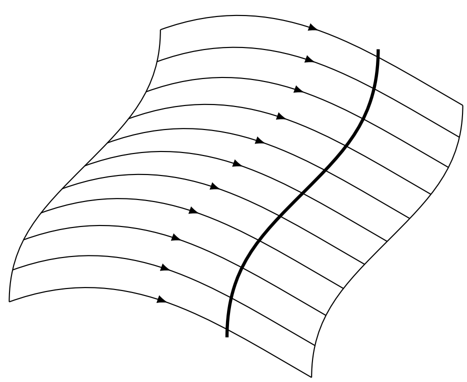


Figure 14: An illustration of the full field space of e, where the lines with arrows are the gauge orbits. The bold line is the *gauge slice*, and is perpendicular to the gauge orbits such that it cuts each orbit exactly once.

We may decompose the functional integral over the full space of einbein fields into an integral over the physically inequivalent values of the einbein along the gauge slice and the physically equivalent einbein fields along the gauge orbits. This is accomplished by observing that the gauge-equivalent classes of e can be labelled by T according to (163):

$$2T = \int_0^T d\tau\ e = \int_0^{\tilde{\tau}(1)} d\tilde{\tau}\ \tilde{e} \tag{167}$$

Therefore an arbitrary einbein e can be expressed as the sum of $2T$ and a diffeomorphism within the gauge group:

$$e(\tau) = 2T + \dot{\alpha}(\tau) \tag{168}$$

The functional integral over the einbein field can then be expressed as an ordinary integral over the proper time (which we know can vary from 0 to ∞) and a functional integral over $\mathcal{D}\alpha$:

$$\int \mathcal{D}e = \Delta_{FP} \int_0^\infty dT \int \mathcal{D}\alpha \tag{169}$$

Δ_{FP} is the Jacobian factor introduced by this change in co-ordinates, and is gauge-invariant (and therefore independent of α) because it's defined as an average over the gauge group. We now use the delta function defined by $\int \mathcal{D}e\ \delta(e - 2T) = 1$, which restricts the integral to the gauge slice:

$$1 = \Delta_{FP} \int_0^\infty dT \int \mathcal{D}\alpha\ \delta(e - 2T) \tag{170}$$

Inserting this into the path integral (165):

$$\Delta(X_a, X_b) = \Delta_{FP} \int_0^\infty dT \int \frac{\mathcal{D}e\mathcal{D}\alpha\ \delta(e-2T)}{\text{Vol(Diffeos.)}} \int_{X(0)=X_a}^{X(1)=X_b} \mathcal{D}X \exp\left(-\frac{1}{2}\int_0^T d\tau \left(\frac{\dot{X}^2}{e} + m^2 e\right)\right) \tag{171}$$

$$= \Delta_{FP} \int_0^\infty dT \int \frac{\mathcal{D}\alpha}{\text{Vol(Diffeos.)}} \int_{X(0)=X_a}^{X(1)=X_b} \mathcal{D}X \exp\left(-\int_0^T d\tau \left(\frac{\dot{X}^2}{4T} + m^2 T\right)\right) \tag{172}$$

The position variable can be transformed under the diffeomorphism $X \to X_\alpha$ so that the path integral is gauge invariant. The functional integral over α is then simply evaluated as the volume of the diffeomorphism group, thus canceling out the Vol(Diffeos.) term. Furthermore, the Jacobian Δ_{FP} is in fact a Faddeev-Popov ghost, which we will regard as a normalization factor that can be fixed as unity by comparison with (159). Finally, we arrive at the path integral representation of the amplitude for a relativistic particle to propagate from X_a^μ to X_b^μ:

$$\Delta(X_a, X_b) = \int_0^\infty dT \int_{X(0)=X_a}^{X(1)=X_b} \mathcal{D}X \exp\left(-\int_0^T d\tau \left(\frac{\dot{X}^2}{4T} + m^2 T\right)\right) \tag{173}$$

As this matches with (159), we conclude that the free, relativistic propagator can be derived from the path integral over relativistic particle paths. The amplitude we have formulated is known as the worldline path integral representation of the free, scalar propagator.

Just as in string theory, the topology of the Riemannian manifold has played an important role and we have derived the amplitude for worldlines of topology $\mathbb{R}^1$. Naturally, we should now consider worldlines of topology S^1. In other words, we would like to extend the above formalism to the case where the worldline is a *loop*.

3.6 One-loop Effective Action

We have seen that the worldline representation of the free propagator was a path integral for the relativistic particle with boundary conditions $\{X(0) = X_a, X(T) = X_b\}$. We extend this method to a particle propagating around a *loop* in spacetime, whose boundary conditions are given by $\{X(0) = X(T) = X_0\}$.

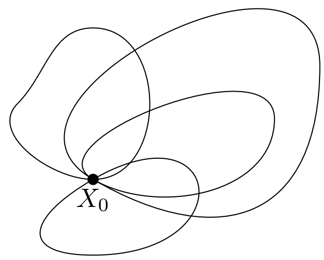


Figure 15: An example of worldlines $X(\tau)$ that satisfy the boundary conditions $\{X(0) = X(T) = X_0\}$.

However, we are not interested in the loops intersecting the arbitrary spacetime point X_0. Rather, we'd like to calculate the amplitude for all loops in spacetime, which can be achieved by integrating the amplitude for loops that satisfy $\{X(0) = X(T) = X_0\}$ with respect to X_0.

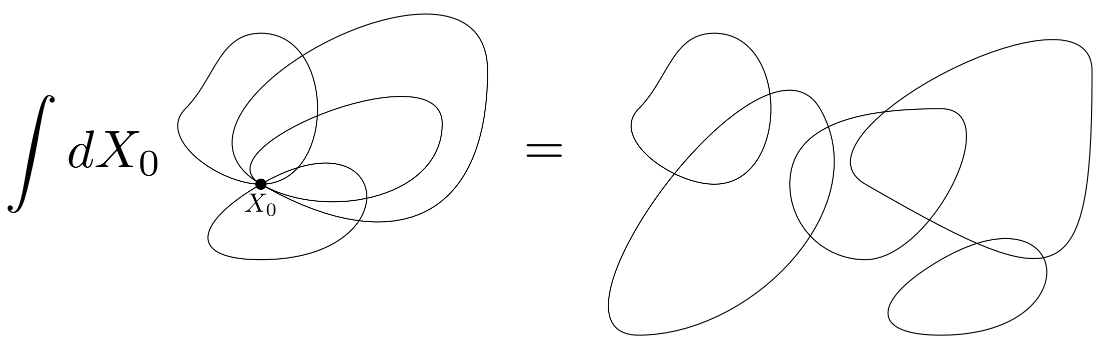


Figure 16: Integrating with respect to the fixed point over spacetime gives all loops that satisfy the periodic boundary condition $X(0) = X(T)$

Extrapolating the results we obtain in Section 2, we obtain this amplitude from the path integral for a relativistic particle with the appropriate boundary conditions and denote it as Γ_{eff} for reasons we will see in Section 3.7:

$$\Gamma_{\text{eff}} = \int \frac{\mathcal{D}e}{\text{Vol(Diffeos.)}} \int dX_0 \int\limits_{X(0)=X(1)=X_0} \mathcal{D}X \exp\left(-\frac{1}{2}\int_0^T d\tau \left(\frac{\dot{X}^2}{e} + m^2 e\right)\right) \tag{174}$$

$$= \int \frac{\mathcal{D}e}{\text{Vol(Diffeos.)}} \int\limits_{X(0)=X(1)} \mathcal{D}X \exp\left(-\frac{1}{2}\int_0^T d\tau \left(\frac{\dot{X}^2}{e} + m^2 e\right)\right) \tag{175}$$

In gauge-fixing the einbein we follow the same procedure outlined in Section 3.5, but now there is an additional diffeomorphism because we require the einbein field to obey the periodic boundary condition $e(0) = e(T)$. The volume of this diffeomorphism is the length of the loop, which is the proper time T. Therefore, the gauge-fixed path integral reads:

$$\Gamma_{\text{eff}} = \int_0^\infty \frac{dT}{T} \oint \mathcal{D}X \exp\left(-\int_0^T d\tau \left(\frac{\dot{X}^2}{4T} + m^2\right)\right) \tag{176}$$

$$= \int_0^\infty \frac{dT}{T} e^{-m^2 T} \oint \mathcal{D}X \exp\left(-\int_0^T d\tau \frac{\dot{X}^2}{4T}\right) \tag{177}$$

$$= \int_0^\infty \frac{dT}{T} e^{-m^2 T} \oint \mathcal{D}X \exp\left(-\frac{1}{4}\int_0^T d\tau\ \dot{X}^2\right) \tag{178}$$

The path integral can be evaluated as $(4\pi T)^{-D/2}$, which we demonstrate up to a constant using the technique of zeta-function regularization in Appendix C. This can also be seen by comparison with the corresponding computation in canonically quantized QFT, which we discuss next.

3.7 Vacuum Expectation Value for Complex Scalar Field

In canonically quantized QFT, the amplitude for a quantum field to propagate between vacuum states is the *vacuum expectation value* of the field. The amplitude for such processes can be found using the Lagrangian for a complex scalar field:

$$\mathcal{L} = \partial^\mu \phi^\dagger \partial_\mu \phi + m^2 \phi^\dagger \phi \tag{179}$$

A path integral with this Lagrangian gives the generating functional Z:

$$Z = \int \mathcal{D}\phi \mathcal{D}\phi^\dagger\ \exp\left(-\int d^4x \left(\partial^\mu \phi^\dagger \partial_\mu \phi + m^2 \phi^\dagger \phi\right)\right) \tag{180}$$

$$= \int \mathcal{D}\phi \mathcal{D}\phi^\dagger\ \exp\left(-\int d^4x \left(\phi^\dagger \left(-\Box + m^2\right) \phi\right)\right) \tag{181}$$

Having integrated by parts in the last step, discretizing the integral brings it into the form of a higher-dimension Gaussian integral that is evaluated in 4.2:

$$Z = \frac{1}{\text{Det}\left(-\Box + m^2\right)} \tag{182}$$

The constants introduced by discretization and evaluation of the Gaussian integrals have been neglected because the amplitude we wish to calculate is defined as the logarithm of $-Z$, and is denoted as the *effective action* Γ_{eff}:

$$\Gamma_{\text{eff}} = -\ln Z \tag{183}$$

$$= \ln \text{Det}\left(-\Box + m^2\right) \tag{184}$$

Now, we use the identity $\ln\det = \text{tr}\ln$:

Box 6. *For an n-dimensional diagonal matrix* $\mathbf{D} = \text{diag}\left(\lambda_i, \cdots, \lambda_n\right)$*, where* λ_i *are the eigenvalues:*

$$\ln\det \mathbf{D} = \ln \prod_{i=1}^n \lambda_i \tag{185}$$

$$= \sum_{i=1}^n \ln \lambda_i \tag{186}$$

$$= \text{tr} \ln \mathbf{D} \tag{187}$$

This not only applies for diagonalized matrices, but for real, symmetric matrices $\mathbf{H}$*. This is because they can be diagonalized via the relation* $\mathbf{H} = \mathbf{R}^T\mathbf{D}\mathbf{R}$ *(where* $\mathbf{R}$ *is an* $O(n)$ *matrix), and we can use the* *cyclicity of trace and the orthogonality of* $\mathbf{R}$ *to show:*

$$\operatorname{tr}\mathbf{H} = \operatorname{tr}\left(\mathbf{R}^T\mathbf{D}\mathbf{R}\right) \tag{188}$$
$$= \operatorname{tr}\left(\mathbf{R}\mathbf{R}^T\mathbf{D}\right) \tag{189}$$
$$= \operatorname{tr}\mathbf{D} \tag{190}$$

The effective action is now:

$$\Gamma_{\text{eff}} = \operatorname{tr}\ln\left(-\Box + m^2\right) \tag{191}$$

For the propagator, we used the Schwinger proper time representation specified in Box 5. For the effective action, this takes a different form:

Box 7. *Consider the following integral:*

$$\ln A = \int_0^\infty \frac{dT}{T} e^{-AT} \tag{192}$$

To see this is valid, differentiate the defining identity for the Schwinger time representation of the propagator:

$$\frac{1}{A} = \int_0^\infty dT\ e^{-AT} \tag{193}$$

Identity 192 can be formally extended for the operator $\mathbb{A}$ *[20][36] and is then known as Frullani's integral.*

Using Box 7, we find the Schwinger proper time representation of the effective action:

$$\Gamma_{\text{eff}} = \int_0^\infty \frac{dT}{T} \operatorname{Tr} e^{-T\left(-\Box+m^2\right)} \tag{194}$$

Inserting a complete set of position eigenstates $\int dX_0 \left|X_0\right\rangle \left\langle X_0\right|$ and using the cyclicity of trace:

$$\Gamma_{\text{eff}} = \int_0^\infty \frac{dT}{T} \int dX_0 \operatorname{Tr}\left(\left|X_0\right\rangle \left\langle X_0\right| e^{-T\left(-\Box+m^2\right)}\right) \tag{195}$$
$$= \int_0^\infty \frac{dT}{T} \int dX_0 \operatorname{Tr}\left\langle X_0 \left| e^{-T\left(-\Box+m^2\right)} \right| X_0\right\rangle \tag{196}$$
$$= \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int dX_0 \operatorname{Tr}\underbrace{\left\langle X_0 \left| e^{T\Box} \right| X_0\right\rangle} \tag{197}$$

As elaborated in Section 3.3, the indicated term can be expressed as a path integral of a particle with Hamiltonian $\Box$ and boundary conditions $X(0) = X(1) = X_0$. We also saw in Section 3.3 that the operator $\Box$ corresponds to the action $\frac{1}{4}\dot{X}^2$:

$$\Gamma_{\text{eff}} = \int_0^\infty \frac{dT}{T} \int dX_0 \int_{X(0)=X(1)=X_0} \mathcal{D}X\ e^{-\frac{1}{4}\int_0^T d\tau \dot{X}^2} \tag{198}$$

Which is the same expression we obtained from the gauge-fixed path integral of the relativistic particle (178).

We can compute the path integral simply by evaluating the functional integral, obtaining an Gaussian integral over the momentum P:

$$\operatorname{Tr}\left\langle X_0 \left| e^{-T\left(-\Box+m^2\right)} \right| X_0\right\rangle = \int \frac{d^D P}{\left(4\pi\right)^2} e^{-TP^2} = \left(4\pi T\right)^{-D/2} \tag{199}$$

4 Scalar Interactions

4.1 Effective Action for Cubic Interaction

Starting from the usual Lagrangian for a field that can interact via a 3-point vertex:

$$S[\phi] = \int d^D x \left((\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{3!} \lambda \phi^3 \right) \tag{200}$$

In the background field method [23], we express the field as the sum of a classical background field ϕ and quantum fluctuations φ:

$$\phi \to \phi + \varphi \tag{201}$$

In expanding the action $S[\phi]$, the next to leading order term is:

$$S_2[\varphi] = \int d^D x \left((\partial_\mu \varphi)^2 + m^2 \varphi^2 + \lambda \varphi^2 \phi \right) \tag{202}$$

$$= \int d^D x \; \varphi \left(-\Box + m^2 + \lambda \phi \right) \varphi \tag{203}$$

Just as in Section 3.7, we use Appendix 4.2 to evaluate the path integral using this action:

$$Z[\varphi] = \int \mathcal{D}\varphi \; e^{-S_2[\varphi]} \tag{204}$$

$$= \frac{1}{\mathrm{Det}\left(-\Box + m^2 + \lambda \phi \right)} \tag{205}$$

We also saw in Section 3.7 that the effective as action is defined as $\Gamma_{\text{eff}} = -\ln Z[\varphi]$ and can be expressed in the Schwinger Proper Time Representation to find the Worldline Path Intergal Representation:

$$\Gamma[\phi] = -\ln Z[\varphi] \tag{206}$$

$$= \ln \mathrm{Det} \left(-\Box + m^2 + \lambda \phi \right) \tag{207}$$

$$= \mathrm{Tr} \ln \left(-\Box + m^2 + \lambda \phi \right) \tag{208}$$

$$= \int_0^\infty \frac{dT}{T} \, \mathrm{Tr} \, e^{-T\left(-\Box + m^2 + \lambda \phi \right)} \tag{209}$$

$$= \int_0^\infty \frac{dT}{T} \int dX_0 \left\langle X_0 \left| e^{-T\left(-\Box + m^2 + \lambda \phi \right)} \right| X_0 \right\rangle \tag{210}$$

$$= \int_0^\infty \frac{dT}{T} \int dX_0 \int_{X(0)=X(T)=X_0} \mathcal{D}X e^{-\int_0^T d\tau \left(\frac{1}{4} \dot{X}^2 + m^2 + \lambda \phi \right)} \tag{211}$$

$$= \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int dX_0 \int_{X(0)=X(T)=X_0} \mathcal{D}X e^{-\frac{1}{4} \int_0^T d\tau \dot{X}^2} \underbrace{\exp\left(-\lambda \int_0^T d\tau \; \phi \right)} \tag{212}$$

We have found the worldline representation of the amplitude for a particle traveling around a loop, interacting with classical background field ϕ. This procedure has introduced the interaction term indicated on the last line. We now expand the field ϕ into the sum of N plane waves:

$$\phi \to \sum_{i=1}^{N} e^{iK_i \cdot X(\tau)} \tag{213}$$

We also insert N delta functions that enforce the intersection of the plane wave with points on the worldline loop $X(\tau_i)$. The interaction term then becomes:

$$\prod_{i=1}^{N} \delta\left(X(\tau) - X(\tau_i) \right) \exp\left(-\lambda \int_0^T d\tau \sum_{i=1}^{N} e^{iK_i \cdot X(\tau)} \right) \tag{214}$$

In order to calculate the effective action illustrated in Figure 17, we consider the term of order λ^N in the Taylor expansion of the exponential.

$$\left(\prod_{i=1}^{N} \delta\left(X\left(\tau\right) - X\left(\tau_i\right)\right)\right) \frac{1}{N!} \left(-\lambda \int_0^T d\tau \sum_{i=1}^{N} e^{iK_i \cdot X(\tau)}\right)^N \tag{215}$$

Of these terms, we consider only those that mix all the external states $e^{iK_i \cdot X}$. In other words, we consider the states proportional to $\prod_i^N e^{iK_i \cdot X}$. There are $N!$ of these, one for every permutation of the external states, so the interaction term becomes:

$$\begin{aligned}
&(-\lambda)^N \left(\prod_{i=1}^{N} \delta\left(X\left(\tau\right) - X\left(\tau_i\right)\right)\right) \left(\int_0^T d\tau \prod_{i=1}^{N} e^{iK_i \cdot X(\tau)}\right) && (216)\\
&= (-\lambda)^N \int_0^T d\tau_1 \cdots \int_0^T d\tau_N \prod_{i=1}^{N} e^{iK_i \cdot X(\tau_i)} && (217)\\
&= (-\lambda)^N \int_0^T d\tau_1 \; e^{iK_1 \cdot X(\tau_1)} \cdots \int_0^T d\tau_N \; e^{iK_N \cdot X(\tau_N)} && (218)\\
&= (-\lambda)^N \, V^\phi\left[K_1\right] \cdots V^\phi\left[K_N\right] && (219)
\end{aligned}$$

In the final step, we have defined the *vertex operator* for the interaction between a scalar particle and scalar field ϕ:

$$V^\phi\left[K\right] = \int_0^T d\tau \; e^{iK \cdot X(\tau)} \tag{220}$$

As explained in Section 1.2, string theory models interactions by inserting vertex operators that correspond to external states onto the surface of the Riemannian manifold. For a genus-1 worldsheet, this Riemannian manifold would be a torus and in our case it's a loop.

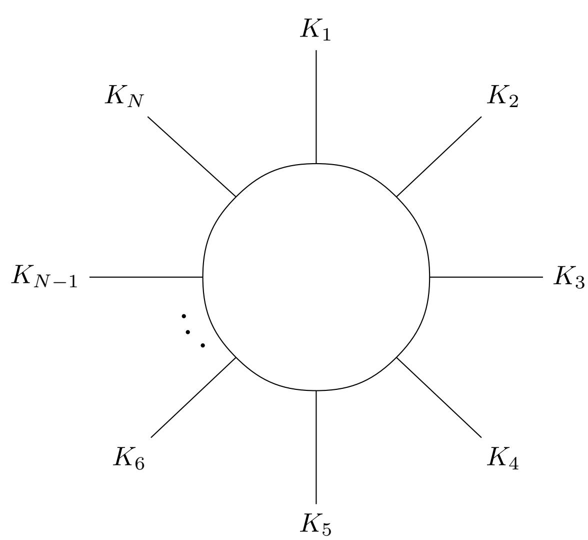


Figure 17: A scalar particle propagating around a loop, interacting with scalar fields at vertices whose worldline co-ordinates are τ_i for the field of momentum K_i. In Section 4.3, we will arrive at a general formula for this

Each vertex operator $V^\phi\left[K_i\right]$ is inserted onto the loop at the worldline co-ordinate τ_i, which is then integrated around the loop. This also encompasses *every permutation* of the external legs because, for example, the integral accounts for $\tau_1 > \tau_2$ and $\tau_2 > \tau_1$.

By expressing the effective action (212) in terms of the vertex operators (220), we see that we can calculate the amplitude for one-loop, N-scalar interaction as a quantum mechanical correlation function of these vertex operators:

$$\Gamma\left[K_1, \cdots, K_N\right] = (-\lambda)^N \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int dX_0 \int_{X(0)=X(T)=X_0} \mathcal{D}X e^{-\frac{1}{4}\int_0^T d\tau \dot{X}^2} \left\langle V^\phi\left[K_1\right] \cdots V^\phi\left[K_{{}_N}\right]\right\rangle \tag{221}$$

We have obtained a gauge-invariant form of the amplitude. This is one of the main advantages of the worldline formalism because, as explained in Section 1.2, individual Feynman diagrams are not generally gauge invariant so this amplitude would have to combine them in such a way that they are.

4.2 Centre-of-Mass Coordinates

In Section 4.1, we found a formula for the one-loop interaction of N-scalar fields:

$$\Gamma\left[K_1, \cdots, K_N\right] = (-\lambda)^N \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int dX_0 \int\limits_{X(0)=X(T)=X_0} \mathcal{D}X e^{-\frac{1}{4}\int_0^T d\tau \dot{X}^2} \prod_{i=1}^N \int_0^T d\tau_i \; e^{iK_i \cdot X} \tag{222}$$

$$= (-\lambda)^N \int_0^\infty \frac{dT}{T} e^{-m^2 T} \prod_{i=1}^N \int_0^T d\tau_i \int dX_0 \int\limits_{X(0)=X(T)=X_0} \mathcal{D}X e^{-\frac{1}{4}\int_0^T d\tau \dot{X}^2} \; e^{i\sum_{i=1}^N K_i \cdot X} \tag{223}$$

$$= (-\lambda)^N \int_0^\infty \frac{dT}{T} e^{-m^2 T} \prod_{i=1}^N \int_0^T d\tau_i \int dX_0 \underbrace{\int\limits_{X(0)=X(T)=X_0} \mathcal{D}X e^{-\frac{1}{4}\int_0^T d\tau \dot{X}^2 + i\sum_{i=1}^N K_i \cdot X}} \tag{224}$$

We have expressed the quantum mechanical correlation function as a path integral. In Section 3.6, we split the position variable $X(\tau)$ into a fixed point X_0 and a loop co-ordinate $Q(\tau)$ that obeys boundary conditions $Q(0) = Q(T) = 0$:

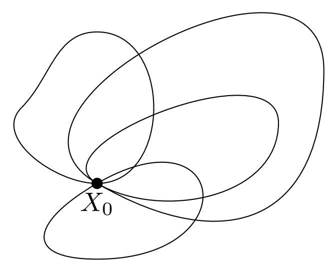


Figure 18: An example of worldlines $X(\tau)$ that satisfy the boundary conditions $\{X(0) = X(T) = X_0\}$.

However, we will split the co-ordinates such that X_0 is the 'centre-of-mass' of the worldline:

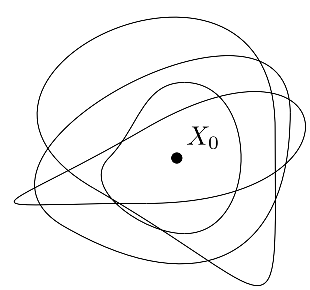


Figure 19: Worldline loops can be expressed as the sum of X_0, which is the average spacetime position of the worldline, and $Q(\tau)$

We can switch to these co-ordinates by defining X_0^μ as the average space-time position of $X^\mu(\tau)$ as shown below, or equivalently by setting the integral of $Q^\mu(\tau)$ around the loop equal to zero:

$$X_0 \equiv \frac{1}{T}\int_0^T d\tau \; X = 0 \qquad \Longleftrightarrow \qquad \int_0^T d\tau \; Q = 0 \tag{225}$$

The position variable can then be expressed as:

$$X^\mu(\tau) = X_0^\mu + Q^\mu(\tau) \tag{226}$$

Noting that $Q(\tau)$ obeys periodic boundary conditions $Q(0) = Q(T)$, we re-express the effective action (224) in terms of these new co-ordinates:

$$\Gamma\left[K_1, \cdots, K_N\right] = (-\lambda)^N \int_0^\infty \frac{dT}{T} e^{-m^2 T} \prod_{i=1}^N \int_0^T d\tau_i \int dX_0 \int^{X(0)=X(T)=X_0} \mathcal{D}X \; e^{-\frac{1}{4}\int_0^T d\tau \dot{X}^2 + i\sum_{i=1}^N K_i \cdot X} \tag{227}$$

$$= (-\lambda)^N \int_0^\infty \frac{dT}{T} e^{-m^2 T} \prod_{i=1}^N \int_0^T d\tau_i \int dX_0 \int^{Q(0)=Q(T)} \mathcal{D}Q \; e^{-\frac{1}{4}\int_0^T d\tau \dot{Q}^2 + i\sum_{i=1}^N K_i \cdot (Q + X_0)} \tag{228}$$

$$= (-\lambda)^N \int_0^\infty \frac{dT}{T} e^{-m^2 T} \prod_{i=1}^N \int_0^T d\tau_i \underbrace{\int dX_0 \; e^{i\sum_{i=1}^N K_i \cdot X_0}} \int^{Q(0)=Q(T)} \mathcal{D}Q \; e^{-\frac{1}{4}\int_0^T d\tau \dot{Q}^2 + i\sum_{i=1}^N K_i \cdot Q} \tag{229}$$

The Fourier transform of this term gives a delta-function which enforces the *conservation of momentum*:

$$\int dX_0 \; e^{i\sum_{i=1}^N K_i \cdot X_0} = (2\pi)^D \, \delta\left(\sum_{i=1}^N K_i\right) \tag{230}$$

Now we have expressed the path integral in this new co-ordinate system:

$$\Gamma\left[K_1, \cdots, K_N\right] = (-\lambda)^N (2\pi)^D \, \delta\left(\sum_{i=1}^N K_i\right) \int_0^\infty \frac{dT}{T} e^{-m^2 T} \prod_{i=1}^N \int_0^T d\tau_i \int^{Q(0)=Q(T)} \mathcal{D}Q \; e^{-\frac{1}{4}\int_0^T d\tau \dot{Q}^2 + i\sum_{i=1}^N K_i \cdot Q} \tag{231}$$

Next, we will evaluate this path integral and obtain the general formula for the N-scalar, one-loop amplitude.

4.3 Bern-Kosower Master Formula for N-Scalars

In Section 4.2, we brought the quantum mechanical correlation function into the form of the path integral:

$$\int_{Q(0)=Q(T)} \mathcal{D}Q \exp\left(\int_0^T d\tau \ -\frac{\dot{Q}^2}{4} + i\sum_{i=1}^{N} K_i \cdot Q\right) \tag{232}$$

We define the source term $J^\mu(\tau)$:

$$J^\mu(\tau) = i\sum_{i=1}^{N} \delta(\tau - \tau_i) K_i^\mu \tag{233}$$

This allows us to express the correlation function as:

$$\int_{Q(0)=Q(T)} \mathcal{D}Q \exp \int_0^T d\tau \left(-\frac{\dot{Q}^2}{4} + J \cdot Q\right) \tag{234}$$

In Appendix 4.3, we show how this can be reduced to the free particle path integral and an expression which is a function of the source term and a Green's function:

$$\int \mathcal{D}X \exp \int_0^T d\tau \left(-\frac{\dot{X}^2}{4} + J(\tau) \cdot X\right) = \int \mathcal{D}X e^{-\frac{1}{4}\int_0^T d\tau \dot{X}^2} \exp\left(-\int_0^T d\tau \int_0^T d\tau' J(\tau) \cdot \left(\frac{\partial}{\partial t}\right)^{-2} \cdot J(\tau')\right)$$

We apply this to our path integral, inserting the result of the free particle path integral from Section 3.7:

$$\int_{Q(0)=Q(T)} \mathcal{D}Q \exp \int_0^T d\tau \left(-\frac{\dot{Q}^2}{4} + J(\tau) \cdot Q\right) = \frac{1}{(4\pi T)^{D/2}} \exp\left(-\frac{1}{2}\int_0^T d\tau \int_0^T d\tau' J(\tau) \Delta(\tau, \tau') J(\tau')\right) \tag{235}$$

Where we define the Green's function:

$$\Delta(\tau, \tau') = 2\left\langle \tau \left| \left(\frac{\partial}{\partial \tau}\right)^{-2} \right| \tau' \right\rangle \tag{236}$$

As we know, the Green's function is a one-dimensional propagator and we may attempt to define analogously to the Green's function for the Klein-Gordon operator:

$$\frac{\partial^2 \Delta(\tau, \tau')}{\partial \tau^2} = 2\delta(\tau - \tau') \tag{237}$$

However, this has no solution on the loop, analogously to solving Poission's for a charge in compact space. In the latter case, we find that the potential is infinite but this is cured by adding a background charge density that makes the total space neutral.

In our scenario, we see that the 'charge' is 2 (because it's given by equation (237) when $\tau = \tau'$) and so we need to subtract a charge density of $2/T$ from the worldline. We therefore define a Green's function that satisfies (236).

$$\frac{\partial^2 \Delta(\tau, \tau')}{\partial \tau^2} = 2\delta(\tau - \tau') - \frac{2}{T} \tag{238}$$

This equation has a solution when the periodicity $\tau \to \tau + T$ is imposed. It's easy to see that this is given by:

$$\frac{\partial \Delta(\tau, \tau')}{\partial \tau} = \operatorname{sign}(\tau - \tau') - \frac{2(\tau - \tau')}{T} \tag{239}$$

$$\Delta(\tau, \tau') = |\tau - \tau'| - \frac{(\tau - \tau')^2}{T} \tag{240}$$

We have set the constant of integration to zero because it'd later vanish in the full expression for the one-loop effective action due to momentum conservation. In originally developing this method, Bern and Kosower used the Green's function for a torus in the limit where $\tau - \tau'$ is much larger than the inner-radius of the torus.

Because we imposed a boundary condition on the worldline co-ordinate $\int_0^T d\tau Q(\tau) = 0$ that respects translation invariance on the loop, we have obtained a Green's function which is a function of only $\tau - \tau'$ and therefore also obeys translation invariance.

We now return to quantum mechanical correlation function (235) and re-express it using $\Delta(\tau, \tau')$ and the definition of the source term (233).

$$\frac{1}{(4\pi T)^{D/2}} \exp\left(\frac{1}{2}\sum_{i,j=1}^{N}\int_0^T d\tau \int_0^T d\tau' \; \delta(\tau - \tau_i)\,\delta(\tau' - \tau_j)\, K_i \cdot K_j \Delta(\tau, \tau')\right) \tag{241}$$

$$= \frac{1}{(4\pi T)^{D/2}} \exp\left(\frac{1}{2}\sum_{i,j=1}^{N} K_i \cdot K_j \Delta_{ij}\right) \tag{242}$$

Where we have used the shorthand:

$$\Delta(\tau_i, \tau_j) \equiv \Delta_{ij} \tag{243}$$

Inserting this into the full expression for the N-scalar, one-loop amplitude (231):

$$\Gamma[K_1, \cdots, K_N] = (-\lambda)^N (2\pi)^D \delta\left(\sum_{i=1}^{N} K_i\right) \prod_{i=1}^{N} \int_0^T d\tau_i \int_0^\infty \frac{dT}{T} \frac{e^{-m^2 T}}{(4\pi T)^{D/2}} \; e^{\frac{1}{2}\sum_{i,j=1}^{N} K_i \cdot K_j \Delta_{ij}} \tag{244}$$

Finally, we rescale to worldline co-ordinates u_i, so that the worldline is of unit length, through the relation $\tau_i = T u_i$.

$$\Gamma[K_1, \cdots, K_N] = (-\lambda)^N (2\pi)^D \delta\left(\sum_{i=1}^{N} K_i\right) \prod_{i=1}^{N} \int_0^1 du_i \int_0^\infty \frac{dT}{T^{1-N}} \frac{e^{-m^2 T}}{(4\pi T)^{D/2}} \; e^{\frac{1}{2}T\sum_{i,j=1}^{N} K_i \cdot K_j \Delta_{ij}} \tag{245}$$

We have obtained the *Bern-Kosower Master formula.* The function Δ_{ij} was found on line (238):

$$\Delta_{ij} = |u_i - u_j| - \frac{(u_i - u_j)^2}{T} \tag{246}$$

We proceed to find the Bern-Kosower Master Formula for the N-photon propagator, which we will also use to compute the 3-point amplitude and compare it to the known result from Quantum Field Theory.

4.4 Tree Level Master Formula for N-Scalars

We combine the worldline path integral representation developed in Section 3.5 with the form of the action we derived in the Section 4 to find the amplitude for a particle, that can interact via the 3-pt vertex, to propagate from X_a to X_b:

$$\mathcal{A}\left[X_a, X_b, \phi\right] = \int_0^\infty dT \int_{X(0)=X_a}^{X(T)=X_b} \mathcal{D}X e^{-\int_0^T d\tau\left(\frac{1}{4}\dot{X}^2+m^2+\lambda\phi\right)} \tag{247}$$

$$= \int_0^\infty dT\ e^{-m^2T} \int_{X(0)=X_a}^{X(T)=X_b} \mathcal{D}X e^{-\frac{1}{4}\int_0^T d\tau \dot{X}^2} \exp\left(-\int_0^T d\tau\ \lambda\phi\right) \tag{248}$$

As in Section 4, we expand the background field into N plane waves to find the amplitude for N interactions, each with an external state of momentum K_i:

$$\phi \to \sum_{i=1}^N e^{iK_i\cdot X} \tag{249}$$

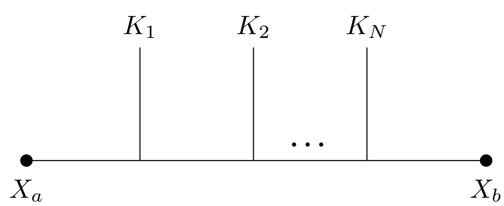


Figure 20: The particle propagates from X_a to X_b, interacting with N scalar fields along the way.

Repeating steps (212) to (224), we obtain the same formula as we did for the effective action but with the appropriate boundary conditions for the propagator:

$$\mathcal{A}\left[X_a, X_b; K_1, \cdots, K_N\right] = (-\lambda)^N \int_0^\infty dT e^{-m^2T} \prod_{i=1}^N \int_0^T d\tau_i \underbrace{\int_{X(0)=X_a}^{X(T)=X_b} \mathcal{D}X e^{-\frac{1}{4}\int\limits_0^T d\tau \dot{X}^2 + i\sum\limits_{i=1}^N K_i\cdot X}} \tag{250}$$

We wish to evaluate the quantum mechanical correlation function:

$$\int_{X(0)=X_a}^{X(T)=X_b} \mathcal{D}X \exp\left(-\frac{1}{4}\int_0^T d\tau \dot{X}^2 + i\sum_{i=1}^N K_i\cdot X\right) \tag{251}$$

As in Section 3.4, we decompose the worldline into a straight line connecting the end-points plus fluctuations around that line that must obey the boundary conditions $Q(0) = Q(T) = 0$.

$$X^\mu(\tau) = X_a^\mu + \frac{\tau}{T}\left(X_b^\mu - X_a^\mu\right) + Q^\mu(\tau) \tag{252}$$

We substitute this into the path integral, recalling that terms in $\dot{Q}$ vanish due to boundary conditions:

$$\int_{Q(0)=Q(T)=0} \mathcal{D}Q \exp\left(-\frac{1}{4}\int_0^T d\tau \left(\dot{Q}^2 + \frac{(X_b - X_a)^2}{T^2}\right) + i\sum_{i=1}^{N} K_i \cdot \left(X_a + \frac{\tau_i}{T}(X_b - X_a) + Q(\tau)\right)\right) \tag{253}$$

$$= \int_{Q(0)=Q(T)=0} \mathcal{D}Q \exp\left(-\frac{1}{4}\int_0^T d\tau\ \dot{Q}^2 - \frac{(X_b - X_a)^2}{4T} + i\sum_{i=1}^{N} K_i \cdot \left(X_a + \frac{\tau_i}{T}(X_b - X_a)\right) + i\sum_{i=1}^{N} K_i \cdot Q\right) \tag{254}$$

$$= \exp\left(i\sum_{i=1}^{N} K_i \cdot \left(X_a + \frac{\tau_i}{T}(X_b - X_a)\right) - \frac{(X_b - X_a)^2}{4T}\right) \underbrace{\int_{Q(0)=Q(T)=0} \mathcal{D}Q \exp\left(-\frac{1}{4}\int_0^T d\tau\ \dot{Q}^2 + i\sum_{i=1}^{N} K_i \cdot Q\right)} \tag{255}$$

We showed in Section 4.3 that the indicated path integral can be decomposed into the free particle path integral and a term involving the Green's function:

$$\int_{Q(0)=Q(T)=0} \mathcal{D}Q e^{-\frac{1}{4}\int_0^T d\tau\ \dot{Q}^2} \exp\left(\sum_{i,j=1}^{N} K_i \cdot K_j \left\langle \tau_i \middle| \left(\frac{\partial}{\partial \tau}\right)^{-2} \middle| \tau_j \right\rangle\right) \tag{256}$$

Using the results of Section 3.4, we insert the result of the quantum fluctuation path integral:

$$\frac{1}{(4\pi T)^{D/2}} \exp\left(\sum_{i,j=1}^{N} K_i \cdot K_j \left\langle \tau_i \middle| \left(\frac{\partial}{\partial \tau}\right)^{-2} \middle| \tau_j \right\rangle\right) \tag{257}$$

$$= \frac{1}{(4\pi T)^{D/2}} \exp\left(\sum_{i,j=1}^{N} K_i \cdot K_j \Delta_{ij}\right) \tag{258}$$

We have defined the Green's function for the propagator:

$$\Delta(\tau_i, \tau_j) = \Delta_{ij} = \left\langle \tau_i \middle| \left(\frac{\partial}{\partial \tau}\right)^{-2} \middle| \tau_j \right\rangle \tag{259}$$

The Green's function for the propagator is defined by:

$$\frac{\partial^2 \Delta_{ij}}{\partial \tau_i^2} = \frac{\partial^2 \Delta_{ij}}{\partial \tau_j^2} = \delta(\tau_i - \tau_j) \tag{260}$$

We require a solution that satisfies this and also the boundary conditions on $Q(\tau)$, so we must have $\Delta(\tau_i, 0) = \Delta(\tau_i, T) = \Delta(0, \tau_j) = \Delta(T, \tau_j) = 0$. It can be seen that these conditions are satisfied by:

$$\Delta_{ij} = \frac{\tau_i \tau_j}{T} + \frac{|\tau_i - \tau_j|}{2} - \frac{\tau_i + \tau_j}{2} \tag{261}$$

The QM correlation function (255) can now be expressed as:

$$\frac{1}{(4\pi T)^{D/2}} \underbrace{\exp\left(i\sum_{i=1}^{N} K_i \cdot \left(X_a + \frac{\tau_i}{T}(X_b - X_a)\right) - \frac{(X_b - X_a)^2}{4T}\right)} \exp\left(\sum_{i,j=1}^{N} K_i \cdot K_j \Delta_{ij}\right) \tag{262}$$

Turning our attention to the indicated term:

$$\exp\left(i\sum_{i=1}^{N} K_i \cdot \left(X_a + \frac{\tau_i}{T}(X_b - X_a)\right) - \frac{(X_b - X_a)^2}{4T}\right) \tag{263}$$

We will switch to momentum space and calculate the amplitude as $\mathcal{A}\left[P_a, P_b; K_1, \cdots, K_N\right]$.

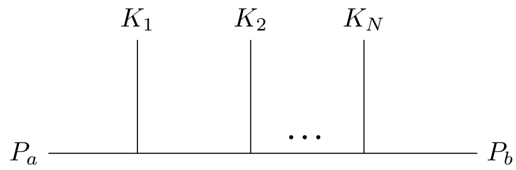


Figure 21: P_a and P_b are the momenta of the particle when it is at X_a and X_b respectively.

We switch to momentum space through Fourier transforms on (263):

$$\int dX_a\ e^{iP_a\cdot X_a} \int dX_b\ e^{iP_b\cdot X_b} \exp\left(i\sum_{i=1}^{N} K_i\cdot\left(X_a + \frac{\tau_i}{T}\left(X_b - X_a\right)\right) - \frac{\left(X_b - X_a\right)^2}{4T}\right) \tag{264}$$

$$= \int dX_a \int dX_b \exp\left(iP_a\cdot X_a + iP_b\cdot X_b + i\sum_{i=1}^{N} K_i\cdot\left(X_a + \frac{\tau_i}{T}\left(X_b - X_a\right)\right) - \frac{\left(X_b - X_a\right)^2}{4T}\right) \tag{265}$$

We redefine the co-ordinates in terms of the 'centre-of-mass' co-ordinate X_+ and the 'distance' co-ordinate X_-:

$$X_- \equiv X_b - X_a \tag{266}$$

$$X_+ \equiv \tfrac{1}{2}\left(X_a + X_b\right) \tag{267}$$

In terms of these co-ordinates, line (265) becomes:

$$\int dX_+ \int dX_-\ \exp\left(iP_a\cdot\left(X_+ - \tfrac{1}{2}X_-\right) + iP_b\cdot\left(X_+ + \tfrac{1}{2}X_-\right) + i\sum_{i=1}^{N} K_i\cdot\left(X_+ - \tfrac{1}{2}X_- + \frac{\tau_i}{T}X_-\right) - \frac{X_-^2}{4T}\right) \tag{268}$$

$$= \int dX_+ \int dX_-\ \exp\left(i\left(P_a + P_b + \sum_{i=1}^{N} K_i\right)\cdot X_+ + \frac{i}{2}\left(P_b - P_a + \sum_{i=1}^{N} K_i\left(\frac{2\tau_i}{T} - 1\right)\right)\cdot X_- - \frac{X_-^2}{4T}\right) \tag{269}$$

$$= \underbrace{\int dX_+ e^{i\left(P_a + P_b + \sum_{i=1}^{N} K_i\right)\cdot X_+}}_{} \int dX_-\ \exp\left(\frac{i}{2}\left(P_b - P_a + \sum_{i=1}^{N} K_i\left(\frac{2\tau_i}{T} - 1\right)\right)\cdot X_- - \frac{X_-^2}{4T}\right) \tag{270}$$

$$\tag{271}$$

The indicated term gives the delta-function which enforces conservation of momentum:

$$\int dX_+ e^{i\left(P_a + P_b + \sum_{i=1}^{N} K_i\right)\cdot X_+} = \delta\left(P_a + P_b + \sum_{i=1}^{N} K_i\right) \tag{272}$$

We have now expressed the QM correlation function (262) as:

$$\frac{\delta\left(P_a + P_b + \sum_{i=1}^{N} K_i\right)}{\left(4\pi T\right)^{D/2}} e^{\sum_{i,j=1}^{N} K_i\cdot K_j \Delta_{ij}} \underbrace{\int dX_-\ \exp\left(\frac{i}{2}\left(P_b - P_a + \sum_{i=1}^{N} K_i\left(\frac{2\tau_i}{T} - 1\right)\right)\cdot X_- - \frac{X_-^2}{4T}\right)}_{} \tag{273}$$

Now we consider the integral over X_-

$$\int dX_-\ \exp\left(\frac{i}{2}\left(P_b - P_a + \frac{2}{T}\sum_{i=1}^{N} K_i\tau_i - \underbrace{\sum_{i=1}^{N} K_i}_{}\right)\cdot X_- - \frac{X_-^2}{4T}\right) \tag{274}$$

Using conservation of momentum to replace the indicated term $\sum_{i=1}^{N} K_i = -P_a - P_b$:

$$\int dX_- \exp\left(i\left(P_b + \frac{1}{T}\sum_{i=1}^{N} K_i\tau_i\right)\cdot X_- - \frac{X_-^2}{4T}\right) \tag{275}$$

Next, we complete the square in the integral:

$$\int dX_- \exp\left(-\frac{1}{4T}\left(X_- - 2iT\left(P_b + \frac{1}{T}\sum_{i=1}^{N} K_i\tau_i\right)\right)^2 - T\left(P_b + \frac{1}{T}\sum_{i=1}^{N} K_i\tau_i\right)^2\right) \tag{276}$$

$$= e^{-T\left(P_b + \sum_{i=1}^{N} K_i\tau_i/T\right)^2} \int dX_- \exp\left(-\frac{1}{4T}\left(X_- - 2iT\left(P_b + \frac{1}{T}\sum_{i=1}^{N} K_i\tau_i\right)\right)^2\right) \tag{277}$$

The integral over X_- has been brought to Gaussian form, so we can replace $X_- \to X_- + 2iT\left(P_b + \frac{1}{T}\sum_{i=1}^{N} K_i\tau_i\right)$ without affecting the integral.

$$e^{-T\left(P_b + \sum_{i=1}^{N} K_i\tau_i/T\right)^2} \int dX_- \exp\left(-\frac{X_-^2}{4T}\right) \tag{278}$$

$$= e^{-T\left(P_b + \sum_{i=1}^{N} K_i\tau_i/T\right)^2} (4\pi T)^{D/2} \tag{279}$$

We have inserted the result of the Gaussian integral, which is equal to the value of the quantum fluctuation path integral and will cancel it when we substitute this result into the QM correlation function (273):

$$\delta\left(P_a + P_b + \sum_{i=1}^{N} K_i\right) \underbrace{\exp\left(-T\left(P_b + \frac{1}{T}\sum_{i=1}^{N} K_i\tau_i\right)^2 + \sum_{i,j=1}^{N} K_i\cdot K_j \Delta_{ij}\right)} \tag{280}$$

Considering this term, we will insert the expression for Δ_{ij} from line (368), as this will allow us to cancel terms:

$$e^{-TP_b^2} \exp\left(-2P_b\cdot\sum_{i=1}^{N} K_i\tau_i - \frac{1}{T}\sum_{i,j=1}^{N} K_i\cdot K_j\tau_i\tau_j + \sum_{i,j=1}^{N} K_i\cdot K_j\Delta_{ij}\right) \tag{281}$$

$$= e^{-TP_b^2} \exp\left(-2P_b\cdot\sum_{i=1}^{N} K_i\tau_i - \frac{1}{T}\sum_{i,j=1}^{N} K_i\cdot K_j\tau_i\tau_j + \sum_{i,j=1}^{N} K_i\cdot K_j\left(\frac{\tau_i\tau_j}{T} + \frac{|\tau_i - \tau_j|}{2} - \frac{\tau_i + \tau_j}{2}\right)\right) \tag{282}$$

$$= e^{-TP_b^2} \exp\left(-2P_b\cdot\sum_{i=1}^{N} K_i\tau_i + \sum_{i,j=1}^{N} K_i\cdot K_j\left(\frac{|\tau_i - \tau_j|}{2} - \frac{\tau_i + \tau_j}{2}\right)\right) \tag{283}$$

$$= e^{-TP_b^2} \exp\left(-2P_b\cdot\sum_{i=1}^{N} K_i\tau_i + \tfrac{1}{2}\sum_{i,j=1}^{N} K_i\cdot K_j|\tau_i - \tau_j| + \boxed{-\frac{1}{2}\sum_{i,j=1}^{N} K_i\cdot K_j\tau_i - \frac{1}{2}\sum_{i,j=1}^{N} K_i\cdot K_j\tau_j}\right) \tag{284}$$

Using symmetry to relabel $i \leftrightarrow j$ in the second term, the boxed terms equal $-\sum_{i,j=1}^{N} K_i\cdot K_j\tau_j$ and then we use momentum conservation to substitute $\sum_{j=1}^{N} K_j = -P_a - P_b$.

$$e^{-TP_b^2} \exp\left(-2P_b \cdot \sum_{i=1}^{N} K_i\tau_i + \tfrac{1}{2}\sum_{i,j=1}^{N} K_i \cdot K_j \left|\tau_i - \tau_j\right| + (P_a + P_b)\sum_{i,j=1}^{N} K_i \cdot K_j\tau_i\right) \tag{285}$$

$$= e^{-TP_b^2} \exp\left((P_b - P_a) \cdot \sum_{i=1}^{N} K_i\tau_i + \tfrac{1}{2}\sum_{i,j=1}^{N} K_i \cdot K_j \left|\tau_i - \tau_j\right|\right) \tag{286}$$

The QM correlation function (280) is then:

$$\delta\left(P_a + P_b + \sum_{i=1}^{N} K_i\right) e^{-TP_b^2} \exp\left((P_b - P_a) \cdot \sum_{i=1}^{N} K_i\tau_i + \sum_{i,j=1}^{N} K_i \cdot K_j \tfrac{1}{2}\left|\tau_i - \tau_j\right|\right) \tag{287}$$

We substitute this into the formula for the N-Scalar Propagator (250):

$$\mathcal{A}\left[P_a, P_b; K_1, \cdots, K_N\right] = (-\lambda)^N \,\delta\left(P_a + P_b + \sum_{i=1}^{N} K_i\right) \int_0^\infty dT\ e^{-T\left(P_b^2 + m^2\right)} \prod_{i=1}^{N} \int_0^T d\tau_i \tag{288}$$

$$\times\, e^{(P_b - P_a)\cdot \sum\limits_{i=1}^{N} K_i\tau_i} \exp\left(\tfrac{1}{2}\sum_{i,j=1}^{N} K_i \cdot K_j \left|\tau_i - \tau_j\right|\right) \tag{289}$$

Rescaling the worldline co-ordinate to the co-ordinate of unit length u_i, through the transformation $\tau_i = Tu_i$:

$$\mathcal{A}\left[P_a, P_b; K_1, \cdots, K_N\right] = (-\lambda)^N \,\delta\left(P_a + P_b + \sum_{i=1}^{N} K_i\right) \int_0^\infty dT\ T^N e^{-T\left(P_b^2 + m^2\right)} \prod_{i=1}^{N} \int_0^1 du_i \tag{290}$$

$$\times\, e^{T(P_b - P_a)\cdot \sum\limits_{i=1}^{N} K_i u_i} \exp\left(\tfrac{1}{2}T\sum_{i,j=1}^{N} K_i \cdot K_j \left|u_i - u_j\right|\right) \tag{291}$$

We have arrived at the Bern-Kosower Master Formula for the N-scalar propagator. We see that, unlike the Bern-Kosower Master Formula for the loop derived in Section 4.3, it does not vanish for $N = 1$.

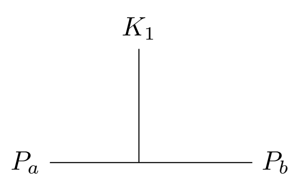


Figure 22: A single interaction of the particle with the scalar field.

We use this formula to calculate the amplitude for this 3-point vertex, redefining $u_1 \equiv u$ and $K_1 \equiv K$

$$\mathcal{A}\left[P_a, P_b; K\right] = \delta\left(P_a + P_b + K\right)(-\lambda) \int_0^\infty dT\ Te^{-T\left(P_b^2 + m^2\right)} \int_0^1 du\ e^{uT(P_b - P_a)\cdot K} \tag{292}$$

The integrals are simple to calculate and we find:

$$\mathcal{A}\left[P_a, P_b; K\right] = \delta\left(P_a + P_b + K\right)(-\lambda)\, \frac{1}{\left(P_a^2 + m^2\right)\left(P_b^2 + m^2\right)} \tag{293}$$

The Bern-Kosower Master Formula gives amplitudes that are not truncated with respect to the external legs of the propagating particle. This is why the propagators of these legs are present in the answer. The truncated answer amplitude is exactly what we require: a vertex term and the conservation of momentum term: $\delta\left(P_a+P_b+K\right)(-\lambda)$.

The above procedure is straightforwardly generalized to any number of interactions and, as we saw, it gives the amplitude *for all permutations of the external legs.* Of course, we have only allowed for a 3-point vertex but the above method can also be applied to 4-point interactions etc.

Another context where we expect 4-point vertices is QED and, in the next section, we will see how the Bern-Kosower Master Formula for Scalar QED generate both 3-point and 4-point vertices, and we will demonstrate the ability of the Master Formula to compactify multiple Feynman diagrams into a single term.

5 Scalar QED Interactions

5.1 Effective Action for Scalar QED

In Section 1.2, we introduced the notion of interactions in the worldline formulation in analogy to atomic physics, where the orbiting electron is treated quantum mechanically and the external electromagnetic field is treated classically. The electron is coupled to the electromagnetic field by modifying it's Hamiltonian and therefore the physical states of Hilbert space.

We derived the Hamiltonian of the Brink-di Vecchia-Howe action on line (53) of Section 2.5:

$$H = \frac{e}{2}\left(P^2 + m^2\right) \tag{294}$$

We treated this as a quantum mechanical operator in Section 3.1 which reduced the full Hilbert space to the physical Hilbert space, as we saw on line (100). Now we couple the relativistic particle, which has charge q, to an external electromagnetic field, whose vector potential is A^μ, through the *gauge transformation*:

$$P^\mu \to \pi^\mu = P^\mu + qA^\mu \tag{295}$$

The Hamiltonian of the particle becomes:

$$H = \frac{e}{2}\left(\pi^2 + m^2\right) \tag{296}$$

Through a Legendre transform of this Hamiltonian, we deduce the form of the Lagrangian in Min kowski space:

$$\mathcal{L}\left[X^\mu, e, A^\mu\right] = P \cdot \dot{X} - \frac{e}{2}H \tag{297}$$

$$= P \cdot \dot{X} - \frac{e}{2}\left(\pi^2 + m^2\right) \tag{298}$$

We eliminate P^μ using (423) and requiring that the gauge-transformed momentum is equal to the conjugate momentum for the Brink-di Vecchia-Howe action (which we derived on line (51)), so that $\pi^\mu = \dot{X}^\mu/e$.

$$\mathcal{L}\left[X^\mu, e, A^\mu\right] = \left(\pi^\mu - qA^\mu\right)\dot{X}_\mu - \frac{e}{2}\left(\pi^2 - m^2\right) \tag{299}$$

$$= \left(\frac{\dot{X}^\mu}{e} - qA^\mu\right)\dot{X}_\mu - \frac{e}{2}\left(\frac{\dot{X}^2}{e^2} - m^2\right) \tag{300}$$

$$= \frac{1}{2}\left(\frac{\dot{X}^2}{e} - m^2 e\right) + qA \cdot \dot{X} \tag{301}$$

We perform a Wick rotation to express the Lagrangian in Euclidean space:

$$\mathcal{L}\left[X^\mu, e, A^\mu\right] = \frac{1}{2}\left(\frac{\dot{X}^2}{e} + m^2 e\right) + iqA \cdot \dot{X} \tag{302}$$

We have derived the effective Lagrangian of a relativistic particle of charge q coupled to an Abelian vector field A^μ. We use this to replace the action in (198), thus finding the effective action for Scalar QED:

$$\Gamma[A] = \int_0^\infty \frac{dT}{T}\, e^{-m^2 T} \int\limits_{X(0)=X(T)=X_0} \mathcal{D}X e^{-\int_0^T d\tau\left(\frac{\dot{X}^2}{4} + iqA\cdot\dot{X}\right)} \tag{303}$$

5.2 Bern-Kosower Master Formula for N-Photons

We repeat the analysis of Section 4.3 to find the one-loop amplitude for N-photons. The difference in the derivation of the Bern-Kosower master formula for cubic interactions and for scalar QED is that now the external states have polarization vectors ε_i:

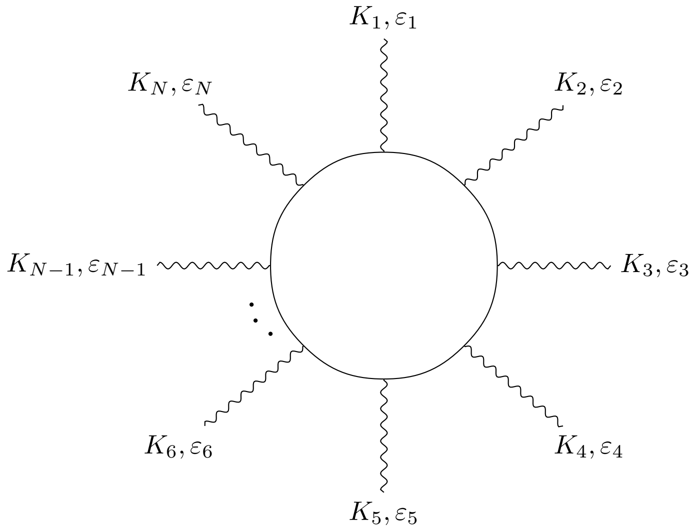


Figure 23: A scalar particle propagating around a loop, interacting with photons at vertices whose worldline co-ordinates are τ_i for the photon of momentum K_i

Consider the effective action derived in Section 5.1.:

$$\Gamma[A] = \int_0^\infty \frac{dT}{T}\, e^{-m^2 T} \int\limits_{X(0)=X(T)=X_0} \mathcal{D}X e^{-\int_0^T d\tau\left(\frac{\dot{X}^2}{4} + iqA\cdot\dot{X}\right)} \tag{304}$$

The interaction term is:

$$\exp\left(-iq\int_0^T d\tau A\cdot\dot{X}\right) \tag{305}$$

As we did for the scalar field on line (213), we expand the electromagnetic potential as the sum of N plane waves of momentum K_i^μ and polarization ε_i^μ.

$$A^\mu = \sum_{i=1}^N \varepsilon_i^\mu e^{iP_i\cdot X} \tag{306}$$

Repeating the analysis that led to the scalar vertex operator (220), we see that the photon vertex operator is:

$$V^A[K,\varepsilon] \equiv \int_0^T d\tau\ \varepsilon\cdot\dot{X}\ e^{iK\cdot X} \tag{307}$$

As with line (221), we express the effective action in terms of a correlation function of the vertex operators:

$$\Gamma[K_1,\varepsilon_1;\cdots;K_N,\varepsilon_N] = \tag{308}$$

$$= (-iq)^N \int_0^\infty \frac{dT}{T} e^{-m^2T} \int dX_0 \int\limits_{X(0)=X(T)=X_0} \mathcal{D}X e^{-\frac{1}{4}\int_0^T d\tau \dot{X}^2} \left\langle V^A[K_1,\varepsilon_1]\cdots V^A[K_N,\varepsilon_N]\right\rangle \tag{309}$$

$$= (-iq)^N \int_0^\infty \frac{dT}{T} e^{-m^2T} \int dX_0 \int\limits_{X(0)=X(T)=X_0} \mathcal{D}X e^{-\frac{1}{4}\int_0^T d\tau \dot{X}^2} \prod_{i=1}^N \int_0^T d\tau_i\ \varepsilon_i\cdot\dot{X}\ e^{iK_i\cdot X} \tag{310}$$

In order to compute the path integral, we exponentiate the $\varepsilon_i \cdot \dot{X}$ and, at the end of the calculation, we will dispose of all terms unless they are linear with respect to *each* polarization vector ε_i.

$$\Gamma\left[K_1, \varepsilon_1; \cdots; K_N, \varepsilon_N\right] = \tag{311}$$

$$= (-iq)^N \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int dX_0 \int_{X(0)=X(T)=X_0} \mathcal{D}X e^{-\frac{1}{4}\int_0^T d\tau \dot{X}^2} \prod_{i=1}^N \int_0^T d\tau_i \;\; e^{iK_i\cdot X + \varepsilon_i \cdot \dot{X}}\Big|_{\text{lin}\,\varepsilon_i} \tag{312}$$

$$= (-iq)^N \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int dX_0 \int_{X(0)=X(T)=X_0} \mathcal{D}X e^{-\frac{1}{4}\int_0^T d\tau \dot{X}^2} \prod_{i=1}^N \int_0^T d\tau_i \;\; e^{iK_i\cdot X + \varepsilon_i \cdot \dot{X}}\Big|_{\text{lin}\,\varepsilon_i} \tag{313}$$

$$= (-iq)^N \int_0^\infty \frac{dT}{T} \; e^{-m^2 T} \int dX_0 \int_{X(0)=X(T)=X_0} \mathcal{D}X e^{-\frac{1}{4}\int_0^T d\tau \dot{X}^2} \prod_{i=1}^N \int_0^T d\tau_i \; e^{\sum\limits_{i=1}^N \left(iK_i\cdot Q + \varepsilon_i\cdot\dot{Q}\right)}\Bigg|_{\text{lin}\,\varepsilon_i} \tag{314}$$

$$= (-iq)^N \int_0^\infty \frac{dT}{T} \; e^{-m^2 T} \prod_{i=1}^N \int_0^T d\tau_i \int dX_0 \int_{X(0)=X(T)=X_0} \mathcal{D}X \; e^{-\frac{1}{4}\int\limits_0^T d\tau \dot{X}^2 + \sum\limits_{i=1}^N \left(iK_i\cdot Q + \varepsilon_i\cdot\dot{Q}\right)}\Bigg|_{\text{lin}\,\varepsilon_i} \tag{315}$$

Applying the results of Section 4.2, we use the centre-of-mass co-ordinates, factor out the zero mode and Fourier transform it exactly as we did to arrive at line (231):

$$\Gamma\left[K_1, \varepsilon_1; \cdots; K_N, \varepsilon_N\right] = \tag{316}$$

$$= (-iq)^N \int_0^\infty \frac{dT}{T} \; e^{-m^2 T} \prod_{i=1}^N \int_0^T d\tau_i \int dX_0 \int_{X(0)=X(T)=X_0} \mathcal{D}X \; e^{-\frac{1}{4}\int\limits_0^T d\tau \dot{Q}^2 + \sum\limits_{i=1}^N \left(iK_i\cdot Q + \varepsilon_i\cdot\dot{Q}\right)}\Bigg|_{\text{lin}\,\varepsilon_i} \tag{317}$$

$$= (-iq)^N (2\pi)^D \delta\left(\sum_{i=1}^N K_i\right) \int_0^\infty \frac{dT}{T} e^{-m^2 T} \prod_{i=1}^N \int_0^T d\tau_i \underbrace{\int_{Q(0)=Q(T)} \mathcal{D}Q e^{-\frac{1}{4}\int\limits_0^T d\tau \dot{Q}^2 + i\sum\limits_{i=1}^N \left(iK_i\cdot Q + \varepsilon_i\cdot\dot{Q}\right)}}\Bigg|_{\text{lin}\,\varepsilon_i} \tag{318}$$

Next, we look at the indicated part of the expression, the path integral over $Q^\mu(\tau)$:

$$\int_{Q(0)=Q(T)} \mathcal{D}Q \exp\left(-\frac{1}{4}\int_0^T d\tau \; \dot{Q}^2 + \sum_{i=1}^N \left(iK_i \cdot Q + \varepsilon_i \cdot \dot{Q}\right)\right) \tag{319}$$

As in Section 4.3, we define a source term:

$$J^\mu(\tau) = \sum_{i=1}^N \delta\left(\tau - \tau_i\right)\left(iK_i^\mu + \varepsilon_i^\mu \frac{\partial}{\partial \tau_i}\right) \tag{320}$$

We apply the results of Appendix 4.3 to this just as we did for the scalar interaction on line (235):

$$\int_{Q(0)=Q(T)} \mathcal{D}Q \exp \int_0^T d\tau \left(-\frac{\dot{Q}^2}{4} + J(\tau)\cdot Q\right) = \frac{1}{(4\pi T)^{D/2}} \exp\left(-\frac{1}{2}\int_0^T d\tau \int_0^T d\tau' J\left(\tau\right) \Delta\left(\tau, \tau'\right) J\left(\tau'\right)\right) \tag{321}$$

We saw on lines (240), (239) and (238) that the inverse of M and its derivatives are given by:

$$\Delta\left(\tau, \tau'\right) = \left|\tau - \tau'\right| - \frac{\left(\tau - \tau'\right)^2}{T} \tag{322}$$

$$\frac{\partial \Delta\left(\tau, \tau'\right)}{\partial \tau} = \operatorname{sign}\left(\tau - \tau'\right) - \frac{2\left(\tau - \tau'\right)}{T} \tag{323}$$

$$\frac{\partial^2 \Delta\left(\tau, \tau'\right)}{\partial \tau^2} = 2\delta\left(\tau - \tau'\right) - \frac{2}{T} \tag{324}$$

We also note that, because this Green's function and its derivatives are functions only of only $\tau - \tau'$, the derivatives are related by:

$$\frac{\partial}{\partial \tau} = -\frac{\partial}{\partial \tau'} \tag{325}$$

We now express the correlation function (321) in terms of $\Delta(\tau, \tau')$, its derivatives and the source term defined on line (320. Also, because this line has J acting on Δ from the right-hand side, we will use an arrow above the differential operator to make this clear.

$$\begin{aligned}
&\frac{1}{(4\pi T)^{D/2}} \exp \sum_{i,j=1}^{N} \left(-\frac{1}{2} \int_0^T d\tau \int_0^T d\tau' \, \delta(\tau - \tau_i)\, \delta(\tau' - \tau_j) \left(iK_i^\mu + \varepsilon_i^\mu \frac{\overrightarrow{\partial}}{\partial \tau_i} \right) \Delta(\tau, \tau') \left(iK_{j\mu} + \varepsilon_{\mu j} \frac{\overleftarrow{\partial}}{\partial \tau_j} \right) \right) \\
&= \frac{1}{(4\pi T)^{D/2}} \exp \sum_{i,j=1}^{N} \frac{1}{2} \left(K_i \cdot K_j \Delta(\tau_i, \tau_j) - iK_j \cdot \varepsilon_i \frac{\partial \Delta(\tau_i, \tau_j)}{\partial \tau_i} - \underbrace{iK_i \cdot \varepsilon_j \frac{\partial \Delta(\tau_j, \tau_i)}{\partial \tau_j}} - \varepsilon_i \cdot \varepsilon_j \frac{\partial^2 \Delta(\tau_i, \tau_j)}{\partial \tau_i \partial \tau_j} \right)
\end{aligned}$$

In the 3rd term, we may exchange i and j but obtain a minus sign due to the anti-symmetry of (323), and we obtain another minus sign in changing the derivative according to (325). Therefore, it is equal to the second term.

$$\frac{1}{(4\pi T)^{D/2}} \exp \left(\sum_{i,j=1}^{N} \left(\tfrac{1}{2} K_i \cdot K_j \Delta(\tau_i, \tau_j) - iK_j \cdot \varepsilon_i \frac{\partial \Delta(\tau_i, \tau_j)}{\partial \tau_i} - \tfrac{1}{2} \varepsilon_i \cdot \varepsilon_j \frac{\partial^2 \Delta(\tau_i, \tau_j)}{\partial \tau_i \partial \tau_j} \right) \right) \tag{326}$$

We introduce the shorthand:

$$\Delta(\tau_i, \tau_j) \equiv \Delta_{ij} \tag{327}$$

We will also use the notation that a dot on the upper-left and upper-right of Δ_{ij} implies a time derivative with respect to its first and second argument respectively:

$$\frac{\partial \Delta_{ij}}{\partial \tau_j} \equiv \Delta^{\bullet}_{ij} \tag{328}$$

$$\frac{\partial \Delta_{ij}}{\partial \tau_i} \equiv {}^{\bullet}\Delta_{ij} \tag{329}$$

The correlation function (326) can now be expressed more compactly:

$$\frac{1}{(4\pi T)^{D/2}} \exp \left(\sum_{i,j=1}^{N} \left(\tfrac{1}{2} K_i \cdot K_j \Delta_{ij} - iK_i \cdot \varepsilon_j {}^{\bullet}\Delta_{ij} - \tfrac{1}{2} \varepsilon_i \cdot \varepsilon_j {}^{\bullet}\Delta^{\bullet}_{ij} \right) \right) \tag{330}$$

$$= \frac{1}{(4\pi T)^{D/2}} \exp \left(\sum_{i,j=1}^{N} \left(\tfrac{1}{2} K_i \cdot K_j \Delta_{ij} + iK_i \cdot \varepsilon_j \Delta^{\bullet}_{ij} + \tfrac{1}{2} \varepsilon_i \cdot \varepsilon_j \Delta^{\bullet\bullet}_{ij} \right) \right) \tag{331}$$

Inserting this into the full N-photon, one-loop amplitude (318), we obtain the Bern-Kosower Master formula for Scalar QED:

$$\Gamma[K_1, \varepsilon_1; \cdots; K_N, \varepsilon_N] = (-iq)^N (2\pi)^D \delta\left(\sum_{i=1}^{N} K_i \right) \int_0^\infty \frac{dT}{T} \frac{e^{-m^2 T}}{(4\pi T)^{D/2}} \prod_{k=1}^{N} \int_0^T d\tau_k \tag{332}$$

$$\times \exp \left(\sum_{i,j=1}^{N} \left(\tfrac{1}{2} K_i \cdot K_j \Delta_{ij} + iK_i \cdot \varepsilon_j \Delta^{\bullet}_{ij} + \tfrac{1}{2} \varepsilon_i \cdot \varepsilon_j \Delta^{\bullet\bullet}_{ij} \right) \right) \Bigg|_{\mathrm{lin}\, \varepsilon_n} \tag{333}$$

Where the function Δ_{ij} and its derivatives were found on lines (324), (323) and (322):

$$\Delta_{ij} = |\tau_i - \tau_j| - \frac{(\tau_i - \tau_j)^2}{T} \tag{334}$$

$$\Delta^{\bullet}_{ij} = \frac{2\,(\tau_i - \tau_j)}{T} - \operatorname{sign}(\tau_i - \tau_j) \tag{335}$$

$$\Delta^{\bullet\bullet}_{ij} = 2\delta\,(\tau_i - \tau_j) - \frac{2}{T} \tag{336}$$

The form of the Bern-Kosower Master Formula is identical to that for interactions with the scalar field , We proceed to calculate the amplitude for $N = 2$.

5.3 Vacuum Polarization of Photon Propagator

For $N = 2$, the Bern-Kosower Master Formula 333 gives the one-loop contribution to the photon propagator:

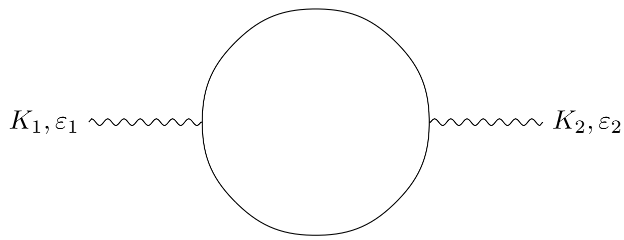


Figure 24: A scalar particle propagating around a loop, interacting with photons at vertices whose worldline co-ordinates are τ_i for the photon of momentum K_i

2-point function:

$$\Gamma\left[K_1, \varepsilon_1; K_2, \varepsilon_2\right] = -\, q^2\,(2\pi)^D\,\delta\,(K_1 + K_2) \int_0^\infty \frac{dT}{T}\,\frac{e^{-m^2 T}}{(4\pi T)^{D/2}} \int_0^T d\tau_1 \int_0^T d\tau_2 \tag{337}$$

$$\times\ \exp\left(\sum_{i,j=1}^{2}\left(\tfrac{1}{2}K_i \cdot K_j \Delta_{ij} + iK_i \cdot \varepsilon_j \Delta^{\bullet}_{ij} + \tfrac{1}{2}\varepsilon_i \cdot \varepsilon_j \Delta^{\bullet\bullet}_{ij}\right)\right)\Bigg|_{\mathrm{lin}\,\varepsilon_1,\varepsilon_2} \tag{338}$$

We have $\Delta_{11} = \Delta_{22} = \Delta^{\bullet}_{11} = \Delta^{\bullet}_{22} = 0$. We also have that the co-efficients of $\Delta^{\bullet\bullet}_{11} = \Delta^{\bullet\bullet}_{22}$ vanish as they are higher than 1st order in a polarization vector. We now write the quantum correlation function with the non-vanishing terms and simplify it using the symmetry of Δ_{ij} and $\Delta^{\bullet\bullet}_{ij}$, and the anti-symmetry of $\Delta^{\bullet}_{ij}$:

$$\exp\left\{\tfrac{1}{2}K_1 \cdot K_2 \Delta_{12} + \tfrac{1}{2}K_2 \cdot K_1 \Delta_{21} + iK_1 \cdot \varepsilon_2 \Delta^{\bullet}_{12} + iK_2 \cdot \varepsilon_1 \Delta^{\bullet}_{21} + \tfrac{1}{2}\varepsilon_1 \cdot \varepsilon_2 \Delta^{\bullet\bullet}_{12} + \tfrac{1}{2}\varepsilon_2 \cdot \varepsilon_1 \Delta^{\bullet\bullet}_{21}\right\}\Big|_{\mathrm{lin}\,\varepsilon_1,\varepsilon_2} \tag{339}$$

$$= \exp\left\{K_1 \cdot K_2 \Delta_{12} + i\,(K_1 \cdot \varepsilon_2 - K_2 \cdot \varepsilon_1)\,\Delta^{\bullet}_{12} + \varepsilon_1 \cdot \varepsilon_2 \Delta^{\bullet\bullet}_{12}\right\}\Big|_{\mathrm{lin}\,\varepsilon_1,\varepsilon_2} \tag{340}$$

$$= \underbrace{\exp\left\{i\,(K_1 \cdot \varepsilon_2 - K_2 \cdot \varepsilon_1)\,\Delta^{\bullet}_{12} + \varepsilon_1 \cdot \varepsilon_2 \Delta^{\bullet\bullet}_{12}\right\}\Big|_{\mathrm{lin}\,\varepsilon_1,\varepsilon_2}}\, e^{K_1 \cdot K_2 \Delta_{12}} \tag{341}$$

The second term in the Taylor expansion of the indicated term contributes a term linear in $\varepsilon_1, \varepsilon_2$:

$$\left(\cancel{i\,(K_1 \cdot \varepsilon_2 - K_2 \cdot \varepsilon_1)\,\Delta^{\bullet}_{12}} + \varepsilon_1 \cdot \varepsilon_2 \Delta^{\bullet\bullet}_{12}\right)\Big|_{\mathrm{lin}\,\varepsilon_1,\varepsilon_2} = \varepsilon_1 \cdot \varepsilon_2 \Delta^{\bullet\bullet}_{12} \tag{342}$$

The third term contributes the following:

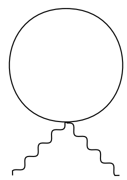
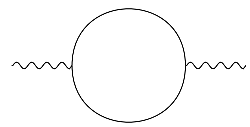
$$\tfrac{1}{2}\left(i\,(K_1 \cdot \varepsilon_2 - K_2 \cdot \varepsilon_1)\,\Delta^{\bullet}_{12} + \cancel{\varepsilon_1 \cdot \varepsilon_2 \Delta^{\bullet\bullet}_{12}}\right)^2\Big|_{\mathrm{lin}\,\varepsilon_1,\varepsilon_2} = -\tfrac{1}{2}\,(K_1 \cdot \varepsilon_2 - K_2 \cdot \varepsilon_1)^2\,(\Delta^{\bullet}_{12})^2\Big|_{\mathrm{lin}\,\varepsilon_1,\varepsilon_2} \tag{343}$$

$$= K_1 \cdot \varepsilon_2\ K_2 \cdot \varepsilon_1\,(\Delta^{\bullet}_{12})^2 \tag{344}$$

There are no terms linear in $\varepsilon_1, \varepsilon_2$ in higher-order expansions, so line (341) is given by:

$$\left(\varepsilon_1 \cdot \varepsilon_2 \Delta^{\bullet\bullet}_{12} + K_1 \cdot \varepsilon_2 \ K_2 \cdot \varepsilon_1 \left(\Delta^{\bullet}_{12}\right)^2\right) e^{K_1 \cdot K_2 \Delta_{12}} \tag{345}$$

$$= \varepsilon_1 \cdot \varepsilon_2 \Delta^{\bullet\bullet}_{12} e^{K_1 \cdot K_2 \Delta_{12}} + K_1 \cdot \varepsilon_2 \ K_2 \cdot \varepsilon_1 \left(\Delta^{\bullet}_{12}\right)^2 e^{K_1 \cdot K_2 \Delta_{12}} \tag{346}$$

=  + 

As we saw on (336), $\Delta^{\bullet\bullet}_{12}$ contains the delta function $\delta(\tau_1 - \tau_2)$, which is where both vertex operators $V[K_1, \varepsilon_1]$ and $V[K_2, \varepsilon_2]$ are inserted at the same point on the loop. This corresponds to the Feynman diagram containing containing the seagull vertex as shown, and the τ integral in the full expression for the amplitude (338) integrates the seagull vertex around the loop. The second term corresponds to the case of two distinct vertices.

The combination of these diagrams is gauge-invariant, as it must be, and this is an example of the compactification of terms in the worldline formulation. As elaborated in Section 1.2, the benefits of this approach are substantial in QCD where hundreds of terms can be compressed into only a couple. Now we switch $\Delta^{\bullet\bullet}_{12} = {}^{\bullet\bullet}\Delta_{12}$ to make the next step more clear:

$$\varepsilon_1 \cdot \varepsilon_2 {}^{\bullet\bullet}\Delta_{12} e^{K_1 \cdot K_2 \Delta_{12}} + K_1 \cdot \varepsilon_2 \ K_2 \cdot \varepsilon_1 \left(\Delta^{\bullet}_{12}\right)^2 e^{K_1 \cdot K_2 \Delta_{12}} \tag{347}$$

In the full 2-point amplitude (338), this expression will be integrated over τ_1 and τ_2, and we will use the integral over τ_1 to perform integration by parts, observing that the surface term vanishes due to the periodicity of $\Delta^{\bullet}_{12}$:

$$-K_1 \cdot K_2 \ \varepsilon_1 \cdot \varepsilon_2 \left(\Delta^{\bullet}_{12}\right)^2 e^{K_1 \cdot K_2 \Delta_{12}} + K_1 \cdot \varepsilon_2 \ K_2 \cdot \varepsilon_1 \left(\Delta^{\bullet}_{12}\right)^2 e^{K_1 \cdot K_2 \Delta_{12}} \tag{348}$$

$$= \left(K_1 \cdot \varepsilon_2 \ K_2 \cdot \varepsilon_1 - K_1 \cdot K_2 \ \varepsilon_1 \cdot \varepsilon_2\right) \left(\Delta^{\bullet}_{12}\right)^2 e^{K_1 \cdot K_2 \Delta_{12}} \tag{349}$$

Inserting this into the 2-point function (338),

$$\Gamma\left[K_1, \varepsilon_1; K_2, \varepsilon_2\right] = -\, q^2 \left(2\pi\right)^D \delta\left(K_1 + K_2\right) \left(K_1 \cdot \varepsilon_2 \ K_2 \cdot \varepsilon_1 - K_1 \cdot K_2 \ \varepsilon_1 \cdot \varepsilon_2\right) \tag{350}$$

$$\times \int_0^\infty \frac{dT}{T} \frac{e^{-m^2 T}}{\left(4\pi T\right)^{D/2}} \int_0^T d\tau_1 \int_0^T d\tau_2 \left(\Delta^{\bullet}_{12}\right)^2 e^{K_1 \cdot K_2 \Delta_{12}} \tag{351}$$

The delta function enforces conservation of momentum $K_1 = -K_2 \equiv K$.

$$\Gamma\left[K_1, \varepsilon_1; K_2, \varepsilon_2\right] = -\, q^2 \left(2\pi\right)^D \left(K^2 \ \varepsilon_1 \cdot \varepsilon_2 - K \cdot \varepsilon_2 \ K \cdot \varepsilon_1\right) \tag{352}$$

$$\times \int_0^\infty \frac{dT}{T} \frac{e^{-m^2 T}}{\left(4\pi T\right)^{D/2}} \int_0^T d\tau_1 \int_0^T d\tau_2 \left(\Delta^{\bullet}_{12}\right)^2 e^{-K^2 \Delta_{12}} \tag{353}$$

$$= -\, q^2 \left(2\pi\right)^D \varepsilon_1^\mu \varepsilon_2^\nu \left(K^2 \delta_{\mu\nu} - K_\mu K_\nu\right) \tag{354}$$

$$\times \int_0^\infty \frac{dT}{T} \frac{e^{-m^2 T}}{\left(4\pi T\right)^{D/2}} \int_0^T d\tau_1 \int_0^T d\tau_2 \left(\Delta^{\bullet}_{12}\right)^2 e^{-K^2 \Delta_{12}} \tag{355}$$

The transversal projector $K^2 \delta_{\mu\nu} - K_\mu K_\nu$ has factored out so the amplitude has become *manifestly gauge invariant at the integral level.*

We now define $\tau_i = T u_i$ and use the rescaled worldline co-ordinates u_i of unit length. We also set u_1 at the origin so that $u_1 = 0$ and redefine $u_2 \equiv u$. The Green's functions then become:

$$\Delta_{12} = |\tau_1 - \tau_2| - \frac{(\tau_1 - \tau_2)^2}{T} \qquad \rightarrow \qquad \Delta_{12} = Tu(1-u) \tag{356}$$

$$\Delta^{\bullet}_{12} = \text{sign}\,(\tau_i - \tau_j) - \frac{2\,(\tau_i - \tau_j)}{T} \qquad \rightarrow \qquad \Delta^{\bullet}_{12} = 2u - 1 \tag{357}$$

$$\int d\tau_i \qquad \rightarrow \qquad T \int du_i \tag{358}$$

Including the factor of T^2 that is required to change the integral measures $d\tau_1 d\tau_1$ The full amplitude becomes:

$$\Gamma\left[K_1, \varepsilon_1; K_2, \varepsilon_2\right] = -q^2\,(2\pi)^D\,\varepsilon_1^\mu \varepsilon_2^\nu \left(K^2 \delta_{\mu\nu} - K_\mu K_\nu\right) \int_0^\infty \frac{dT}{T} \frac{e^{-m^2 T} T^2}{(4\pi T)^{D/2}} \int_0^1 du\,(2u-1)^2\, e^{Tu(u-1)K^2} \tag{359}$$

The integral over T can be evaluated using the Gamma function.

Box 8. *Consider the Gamma-function:*

$$\Gamma(\lambda) = \int_0^\infty \frac{dx}{x}\; x^\lambda e^{-x} \tag{360}$$

Rescaling $x \rightarrow ax$, we find:

$$\Gamma(\lambda) a^{-\lambda} = \int_0^\infty \frac{dx}{x}\; x^\lambda e^{-ax} \tag{361}$$

Re-writing the integral as follows:

$$\Gamma\left[K_1, \varepsilon_1; K_2, \varepsilon_2\right] = -q^2\,(\pi)^{D/2}\,\varepsilon_1^\mu \varepsilon_2^\nu \left(K^2 \delta_{\mu\nu} - K_\mu K_\nu\right) \int_0^1 du\,(2u-1)^2 \underbrace{\int_0^\infty \frac{dT}{T} T^{2-D/2} e^{-T\left(u(1-u)K^2 + m^2\right)}}$$

We see that the indicated integral can be identified with (361), identifying x with T, λ with $2 - D/2$ and a with $u(1-u)K^2 + m^2$:

Re-writing the integral as follows:

$$\Gamma\left[K_1, \varepsilon_1; K_2, \varepsilon_2\right] = -q^2\,(\pi)^{D/2}\,\Gamma\left(2 - \tfrac{D}{2}\right) \varepsilon_1^\mu \varepsilon_2^\nu \left(K^2 \delta_{\mu\nu} - K_\mu K_\nu\right) \int_0^1 du\,(2u-1)^2 \left(u(1-u)K^2 + m^2\right)^{D/2-2}$$

Continuing this expression to Minkowski space, it can be seen that this agrees with the result obtained from the usual methods (Chapter 10.3 of Peskin and Schroeder [30]).

We have shown the power of the worldline path integrals to generate multiple Feynman diagrams in a gauge invariant combination. In the usual approach, if one wanted to calculate one-loop diagrams of the type depicted in Figure 24, one would have to deduce the possible Feynman diagrams. The worldline approach deduces them for you and this only the simplest illustration of how this can be more efficient then the canonically quantized approach.

5.4 Tree Level Master Formula for N-Photons

We should now find the formula for the amplitude for a particle to propagate from X_a to X_b, interacting with N-photons with momenta and polarizations as shown in Figure 25. We will re-use the results of Section 5.2 (the N-Photon Loop) and Section 4.4 (the N-Scalar Propagator) as much as possible.

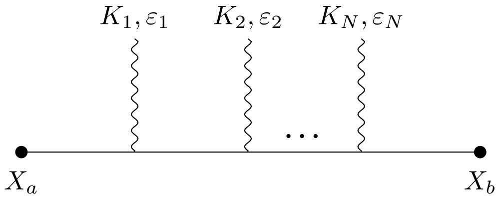


Figure 25: The particle propagates from X_a to X_b, interacting with N scalar fields along the way.

Repeating steps (305) to (315) for the worldline representation of the propagator as defined in Section 3.5:

$$\mathcal{A}\left[X_a, \varepsilon_1, \cdots, K_N, \varepsilon_N\right] = (-iq)^N \int_0^\infty dT e^{-m^2 T} \prod_{i=1}^{N} \int_0^T d\tau_i \underbrace{\int_{X(0)=X_a}^{X(T)=X_b} \mathcal{D}X e^{-\frac{1}{4}\int_0^T d\tau \dot{X}^2 + \sum_{i=1}^{N}\left(iK_i\cdot X + \varepsilon_i \cdot \dot{X}\right)}}\Bigg|_{\text{lin}\,\varepsilon_n} \tag{362}$$

Consider the indicated path integral:

$$\int_{X(0)=X_a}^{X(T)=X_b} \mathcal{D}X \exp\left(-\frac{1}{4}\int_0^T d\tau \dot{X}^2 + \sum_{i=1}^{N}\left(iK_i \cdot X + \varepsilon_i \cdot \dot{X}\right)\right) \tag{363}$$

Just as we did on line (252) for the N-Scalar Propagator, we redefine the co-ordinates according to $X(\tau) = X_a + \tau\left(X_b - X_a\right)/T + Q(\tau)$ and $Q(0) = Q(T) = 0$:

$$\begin{aligned}
&\int\limits_{Q(0)=Q(T)=0} \mathcal{D}Q \exp\left(-\frac{1}{4}\int_0^T d\tau\left(\dot{Q}^2 + \frac{\left(X_b - X_a\right)^2}{T^2}\right) + \sum_{i=1}^{N}\left\{iK_i \cdot \left(X_a + \frac{\tau_i}{T}\left(X_b - X_a\right) + Q(\tau)\right) + \frac{\varepsilon_i \cdot \left(X_b - X_a\right)}{T}\right\}\right) \\
&= e^{\sum_{i=1}^{N}\varepsilon_i\cdot\left(X_b - X_a\right)/T} \exp\left(i\sum_{i=1}^{N} K_i \cdot \left(X_a + \frac{\tau_i}{T}\left(X_b - X_a\right)\right) - \frac{\left(X_b - X_a\right)^2}{4T}\right) \underbrace{\int\limits_{Q(0)=Q(T)=0} \mathcal{D}Q e^{-\frac{1}{4}\int_0^T d\tau\ \dot{Q}^2 + i\sum_{i=1}^{N} K_i\cdot Q}}
\end{aligned} \tag{364}$$

We evaluated this path integral on the loop in Section 5.2 on line (326). We use these results to evaluate it for the propagator:

$$\frac{1}{\left(4\pi T\right)^{D/2}} \exp\left(\sum_{i,j=1}^{N}\left(K_i \cdot K_j \Delta_{ij} - 2iK_i \cdot \varepsilon_j {}^\bullet\Delta_{ij} - \varepsilon_i \cdot \varepsilon_j {}^\bullet\Delta^\bullet_{ij}\right)\right)\Bigg|_{\text{lin}\,\varepsilon_n} \tag{365}$$

There is a factor of 2 difference in the argument of the exponential because we define the Green's function as we did for the N-Scalar Propagator in Section 4.4, which was:

$$\Delta_{ij} = \left\langle \tau_i \left| \left(\frac{\partial}{\partial\tau}\right)^{-2} \right| \tau_j \right\rangle \tag{366}$$

We also saw in Section 4.4 that this is defined by:

$${}^{\bullet\bullet}\Delta_{ij} = \Delta^{\bullet\bullet}_{ij} = \delta\left(\tau_i - \tau_j\right) \tag{367}$$

We then saw that this and the appropriate boundary conditions are satisfied by:

$$\Delta_{ij} = \frac{\tau_i \tau_j}{T} + \frac{|\tau_i - \tau_j|}{2} - \frac{\tau_i + \tau_j}{2} \tag{368}$$

Now, we will also need the derivatives of Δ_{ij}:

$$^\bullet\Delta_{ij} = \frac{\tau_j}{T} + \tfrac{1}{2}\,\mathrm{sign}\,(\tau_j - \tau_i) - \tfrac{1}{2} \tag{369}$$

$$\Delta^\bullet_{ij} = \frac{\tau_i}{T} - \tfrac{1}{2}\,\mathrm{sign}\,(\tau_i - \tau_j) - \tfrac{1}{2} \tag{370}$$

$$^\bullet\Delta^\bullet_{ij} = \frac{1}{T} - \delta\,(\tau_i - \tau_j) \tag{371}$$

We turn our attention to the remaining part of line (364):

$$e^{\sum\limits_{i=1}^{N} \varepsilon_i \cdot (X_b - X_a)/T} \exp\left(i \sum_{i=1}^{N} K_i \cdot \Big(X_a + \frac{\tau_i}{T}(X_b - X_a) \Big) - \frac{(X_b - X_a)^2}{4T} \right) \tag{372}$$

Just as on line (268), we switch to centre-of-mass co-ordinate $X_+ = \frac{1}{2}(X_a + X_b)$ and distance co-ordinate $X_- = X_b - X_a$ and Fourier transform to momentum space:

$$e^{\sum\limits_{i=1}^{N} \varepsilon_i \cdot X_-/T} \exp\left(i \sum_{i=1}^{N} K_i \cdot \Big(X_+ - \tfrac{1}{2}X_- + \frac{\tau_i}{T} X_- \Big) - \frac{X_-^2}{4T} \right) \tag{373}$$

The only difference with the calculation in Section 4.4 is the term in polarization vectors ε_i, which has factored out. It is a function of X_-, so we can repeat steps (263) to (275) where we Fourier transform to momentum space and perform the integral over X_+ to find the delta function which enforces conservation of momentum.

$$\delta\left(P_a + P_b + \sum_{i=1}^{N} K_i \right) \underbrace{e^{\sum\limits_{i=1}^{N} \varepsilon_i \cdot X_-/T} \int dX_- \exp\left(i \left(P_b + \frac{1}{T} \sum_{i=1}^{N} K_i \tau_i \right) \cdot X_- - \frac{X_-^2}{4T} \right)} \tag{374}$$

We proceed to complete the square in X_- so that we may evaluate it as a Gaussian integral:

$$= \int dX_- \exp\left(\left(iP_b + \frac{1}{T} \sum_{i=1}^{N} (iK_i \tau_i + \varepsilon_i) \right) \cdot X_- - \frac{X_-^2}{4T} \right) \tag{375}$$

$$= \int dX_- \exp\left(-\frac{1}{4T} \left(X_- - 2T \left(iP_b + \frac{1}{T} \sum_{i=1}^{N} (iK_i \tau_i + \varepsilon_i) \right) \right)^2 + T \left(iP_b + \frac{1}{T} \sum_{i=1}^{N} (iK_i \tau_i + \varepsilon_i) \right)^2 \right) \tag{376}$$

$$= e^{T\left(iP_b + \frac{1}{T} \sum\limits_{i=1}^{N} (iK_i \tau_i + \varepsilon_i) \right)^2} \int dX_- \exp\left(-\frac{1}{4T} \left(X_- - 2T \left(iP_b + \frac{1}{T} \sum_{i=1}^{N} (iK_i \tau_i + \varepsilon_i) \right) \right)^2 \right) \tag{377}$$

$$= e^{T\left(iP_b + \frac{1}{T} \sum\limits_{i=1}^{N} (iK_i \tau_i + \varepsilon_i) \right)^2} \int dX_- \exp\left(-\frac{X_-^2}{4T} \right) \tag{378}$$

$$= e^{T\left(iP_b + \frac{1}{T} \sum\limits_{i=1}^{N} (iK_i \tau_i + \varepsilon_i) \right)^2} (4\pi T)^{D/2} \tag{379}$$

$$\tag{380}$$

Substituting this and line (365) into the line (364):

$$\delta\left(P_a + P_b + \sum_{i=1}^{N} K_i \right) \underbrace{e^{T\left(iP_b + \frac{1}{T} \sum\limits_{i=1}^{N} (iK_i \tau_i + \varepsilon_i) \right)^2}} e^{\sum\limits_{i,j=1}^{N} \left(K_i \cdot K_j \Delta_{ij} - 2iK_i \cdot \varepsilon_j {}^\bullet\Delta_{ij} - \varepsilon_i \cdot \varepsilon_j {}^\bullet\Delta^\bullet_{ij} \right)} \Bigg|_{\mathrm{lin}\,\varepsilon_n} \tag{381}$$

Expanding the indicated term:

$$\exp\left(T\left(iP_b+\frac{1}{T}\sum_{i=1}^{N}\left(iK_i\tau_i+\varepsilon_i\right)\right)^2\right) \tag{382}$$

$$=e^{-TP_b^2}\exp\left(-2P_b\cdot\sum_{i=1}^{N}K_i\tau_i+2iP_b\cdot\sum_{i=1}^{N}\varepsilon_i+\frac{1}{T}\sum_{i,j=1}^{N}\left(-K_i\cdot K_j\tau_i\tau_j+2i\tau_iK_i\cdot\varepsilon_j+\varepsilon_i\cdot\varepsilon_j\right)\right) \tag{383}$$

Next, we consider this term in (381):

$$\delta\left(P_a+P_b+\sum_{i=1}^{N}K_i\right)e^{T\left(iP_b+\frac{1}{T}\sum_{i=1}^{N}(iK_i\tau_i+\varepsilon_i)\right)^2}\underbrace{e^{\sum_{i,j=1}^{N}\left(K_i\cdot K_j\mathbb{\Delta}_{ij}-2iK_i\cdot\varepsilon_j{}^{\bullet}\mathbb{\Delta}_{ij}-\varepsilon_i\cdot\varepsilon_j{}^{\bullet}\overset{\bullet}{\mathbb{\Delta}}_{ij}\right)}}\Bigg|_{\operatorname{lin}\varepsilon_n} \tag{384}$$

We insert the expressions for $\mathbb{\Delta}_{ij}$ and its derivatives from lines (367) to (371):

$$\exp\sum_{i,j=1}^{N}\left(K_i\cdot K_j\mathbb{\Delta}_{ij}-2iK_i\cdot\varepsilon_j{}^{\bullet}\mathbb{\Delta}_{ij}-\varepsilon_i\cdot\varepsilon_j{}^{\bullet}\overset{\bullet}{\mathbb{\Delta}}_{ij}\right) \tag{385}$$

$$=\exp\sum_{i,j=1}^{N}\left(\begin{array}{l}K_i\cdot K_j\left(\frac{\tau_i\tau_j}{T}+\frac{|\tau_i-\tau_j|}{2}-\frac{\tau_i+\tau_j}{2}\right)-2iK_j\cdot\varepsilon_i\left(\frac{\tau_j}{T}+\frac{1}{2}\operatorname{sign}(\tau_j-\tau_i)-\frac{1}{2}\right)+\\-\varepsilon_i\cdot\varepsilon_j\left(\frac{1}{T}-\delta(\tau_i-\tau_j)\right)\end{array}\right) \tag{386}$$

Now that we have expanded the terms in (381), we combine them and see that cancellations occur:

$$\delta\left(P_a+P_b+\sum_{i=1}^{N}K_i\right)\underbrace{e^{T\left(iP_b+\frac{1}{T}\sum_{i=1}^{N}(iK_i\tau_i+\varepsilon_i)\right)^2}e^{\sum_{i,j=1}^{N}\left(K_i\cdot K_j\mathbb{\Delta}_{ij}-2iK_i\cdot\varepsilon_j{}^{\bullet}\mathbb{\Delta}_{ij}-\varepsilon_i\cdot\varepsilon_j{}^{\bullet}\overset{\bullet}{\mathbb{\Delta}}_{ij}\right)}}\Bigg|_{\operatorname{lin}\varepsilon_n} \tag{387}$$

$$e^{-TP_b^2}\exp\sum_{i,j=1}^{N}\left(\begin{array}{l}-2P_b\cdot\sum_{i=1}^{N}K_i\tau_i+2iP_b\cdot\sum_{i=1}^{N}\varepsilon_i+\\+\sum_{i,j=1}^{N}\left(\frac{1}{2}K_i\cdot K_j\left(|\tau_i-\tau_j|-(\tau_i+\tau_j)\right)-iK_j\cdot\varepsilon_i\left((\operatorname{sign}(\tau_i-\tau_j)-1)+\varepsilon_i\cdot\varepsilon_j\delta(\tau_i-\tau_j)\right)\right.\end{array}\right) \tag{388}$$

$$=e^{-TP_b^2}\exp\sum_{i,j=1}^{N}\left(\begin{array}{l}-2P_b\cdot\sum_{i=1}^{N}K_i\tau_i+2iP_b\cdot\sum_{i=1}^{N}\varepsilon_i+\boxed{-\sum_{i,j=1}^{N}\frac{1}{2}K_i\cdot K_j\tau_i-\sum_{i,j=1}^{N}\frac{1}{2}K_i\cdot K_j\tau_j+i\sum_{i,j=1}^{N}K_j\cdot\varepsilon_i}+\\+\sum_{i,j=1}^{N}\left(\frac{1}{2}K_i\cdot K_j|\tau_i-\tau_j|-iK_j\cdot\varepsilon_i\operatorname{sign}(\tau_i-\tau_j)+\varepsilon_i\cdot\varepsilon_j\delta(\tau_i-\tau_j)\right)\end{array}\right) \tag{389}$$

For the boxed terms, we use symmetry to relabel $i\leftrightarrow j$ in the second term so that it's equal to the first, and then we use momentum conservation to substitute $\sum_{j=1}^{N}K_j=-P_b-P_a$ in the box.

$$=e^{-TP_b^2}\exp\sum_{i,j=1}^{N}\left(\begin{array}{l} -P_b\cdot\sum_{i=1}^{N}K_i\tau_i+iP_b\cdot\sum_{i=1}^{N}\varepsilon_i+P_a\cdot\sum_{i=1}^{N}K_i\tau_i-iP_a\cdot\sum_{i=1}^{N}\varepsilon_i+ \\ +\sum_{i,j=1}^{N}\left(\frac{1}{2}K_i\cdot K_j\left|\tau_i-\tau_j\right|-iK_j\cdot\varepsilon_i\,\mathrm{sign}\left(\tau_i-\tau_j\right)+\varepsilon_i\cdot\varepsilon_j\delta\left(\tau_i-\tau_j\right)\right) \end{array}\right) \tag{390}$$

$$=e^{-TP_b^2}\exp\sum_{i,j=1}^{N}\left(\begin{array}{l} \left(P_a-P_b\right)\cdot\sum_{i=1}^{N}\left(K_i\tau_i-i\varepsilon_i\right)+ \\ +\sum_{i,j=1}^{N}\left(\frac{1}{2}K_i\cdot K_j\left|\tau_i-\tau_j\right|-iK_j\cdot\varepsilon_i\,\mathrm{sign}\left(\tau_i-\tau_j\right)+\varepsilon_i\cdot\varepsilon_j\delta\left(\tau_i-\tau_j\right)\right) \end{array}\right) \tag{391}$$

$$\tag{392}$$

We see that the Green's function and it's derivatives have reduced to the *translationally invariant parts*. In fact, we can define a new Green's function:

$$\mathbf{\Delta}_{ij}=\frac{1}{2}\left|\tau_i-\tau_j\right| \tag{393}$$

$$^{\bullet}\mathbf{\Delta}_{ij}=\frac{1}{2}\,\mathrm{sign}\left(\tau_i-\tau_j\right) \tag{394}$$

$$^{\bullet\bullet}\mathbf{\Delta}_{ij}=\delta\left(\tau_i-\tau_j\right) \tag{395}$$

We express our answer as:

$$=e^{-TP_b^2}\exp\left(\left(P_a-P_b\right)\cdot\sum_{i=1}^{N}\left(K_i\tau_i-i\varepsilon_i\right)+\sum_{i,j=1}^{N}\left(K_i\cdot K_j\mathbf{\Delta}_{ij}-2iK_j\cdot\varepsilon_i{}^{\bullet}\mathbf{\Delta}_{ij}+\varepsilon_i\cdot\varepsilon_j{}^{\bullet\bullet}\mathbf{\Delta}_{ij}\right)\right) \tag{396}$$

Substituting this into the the expression for the N-photon propagator (362):

$$\mathcal{A}\left[P_a,P_b;K_1,\varepsilon_1,\cdots,K_N,\varepsilon_N\right]=\left(-iq\right)^N\delta\left(P_a+P_b+\sum_{i=1}^{N}K_i\right)\int_0^\infty dTe^{-T\left(P_b^2+m^2\right)}\prod_{i=1}^{N}\int_0^T d\tau_i \tag{397}$$

$$\times\; e^{\left(P_a-P_b\right)\cdot\sum_{i=1}^{N}\left(K_i\tau_i-i\varepsilon_i\right)+\sum_{i,j=1}^{N}\left(K_i\cdot K_j\mathbf{\Delta}_{ij}-2iK_j\cdot\varepsilon_i{}^{\bullet}\mathbf{\Delta}_{ij}+\varepsilon_i\cdot\varepsilon_j{}^{\bullet\bullet}\mathbf{\Delta}_{ij}\right)}\Bigg|_{\mathrm{lin}\,\varepsilon_n} \tag{398}$$

We have arrived at the Tree Level Bern-Kosower Master Formula for Scalar QED.

5.5 Compton Scattering

First, we consider the Tree-Level Bern-Kosower Master Formula for $N=1$.

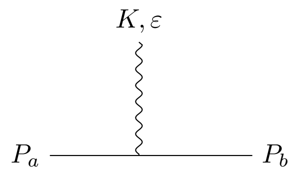


Figure 26: A partice interacting with a photon.

The Green's functions and it's derivatives are purely functions of $\tau_i - \tau_j$, and therefore vanish:

$$\mathcal{A}\left[P_a, P_b; K, \varepsilon\right] = (-iq)\,\delta\left(P_a + P_b + K\right) \int_0^\infty dT\; e^{-T\left(P_b^2+m^2\right)} \int_0^T d\tau\; e^{(P_a-P_b)\cdot(Ku-i\varepsilon)}\Big|_{\mathrm{lin}\,\varepsilon} \tag{399}$$

$$= e^{-i(P_a-P_b)\cdot\varepsilon}\Big|_{\mathrm{lin}\,\varepsilon} (-iq)\,\delta\left(P_a + P_b + K\right) \int_0^\infty dT\; e^{-T\left(P_b^2+m^2\right)} \int_0^T d\tau e^{(P_a-P_b)\cdot Ku} \tag{400}$$

$$= \underbrace{(P_a - P_b)\cdot\varepsilon}\,(-q)\,\delta\left(P_a + P_b + K\right) \int_0^\infty dT\; e^{-T\left(P_b^2+m^2\right)} \int_0^T d\tau e^{(P_a-P_b)\cdot Ku} \tag{401}$$

Taking terms linear in ε has resulted on the indicated term, the remainder of the integral is the same is that calculated at the end of Section 4.4.

$$(P_a - P_b)\cdot\varepsilon\,(-q)\,\delta\left(P_a + P_b + K\right) \tag{402}$$

This vanishes on-shell but, as we explain later, can be used to connect other diagrams. Now we consider $N=2$, which will give the amplitude of Compton Scattering on-shell.

$$\mathcal{A}\left[P_a, P_b; K_1, \varepsilon_1; K_2, \varepsilon_2\right] = (-iq)^2\,\delta\left(P_a + P_b + K_1 + K_2\right) \int_0^\infty dT e^{-T\left(P_b^2+m^2\right)} \int_0^T d\tau_1 \int_0^T d\tau_2 \tag{403}$$

$$\times\; \exp\left((P_a - P_b)\cdot \sum_{i=1}^{2} \left(K_i\tau_i - i\varepsilon_i\right) + \sum_{i,j=1}^{2} \left(K_i\cdot K_j \boldsymbol{\Delta}_{ij} - 2iK_j\cdot\varepsilon_i{}^{\bullet}\boldsymbol{\Delta}_{ij} + \varepsilon_i\cdot\varepsilon_j{}^{\bullet\bullet}\boldsymbol{\Delta}_{ij}\right)\right)\Bigg|_{\mathrm{lin}\,\varepsilon_n} \tag{404}$$

$$=\mathcal{A}\left[P_a, P_b; K_1, \varepsilon_1; K_2, \varepsilon_2\right] = (-iq)^2\,\delta\left(P_a + P_b + K_1 + K_2\right) \int_0^\infty dT e^{-T\left(P_b^2+m^2\right)} \int_0^T d\tau_1 \int_0^T d\tau_2 \tag{405}$$

$$\times\; \underbrace{\exp\left(-i\left(P_a - P_b\right)\cdot \sum_{i=1}^{2} \varepsilon_i + \sum_{i,j=1}^{2} \left(-2iK_j\cdot\varepsilon_i{}^{\bullet}\boldsymbol{\Delta}_{ij} + \varepsilon_i\cdot\varepsilon_j{}^{\bullet\bullet}\boldsymbol{\Delta}_{ij}\right)\right)}\Bigg|_{\mathrm{lin}\,\varepsilon_n} e^{(P_a-P_b)\cdot\sum_{i=1}^{2}K_i\tau_i + \sum_{i,j=1}^{2}K_i\cdot K_j\boldsymbol{\Delta}_{ij}} \tag{406}$$

Considering the polarization vectors:

$$\exp\left(-i\left(P_a - P_b\right)\cdot\left(\varepsilon_1 + \varepsilon_2\right) + 2i\left(K_2\cdot\varepsilon_1 - K_1\cdot\varepsilon_2\right){}^{\bullet}\boldsymbol{\Delta}_{12} + 2\varepsilon_1\cdot\varepsilon_2{}^{\bullet\bullet}\boldsymbol{\Delta}_{12}\right)\big|_{\mathrm{lin}\,\varepsilon_n} \tag{407}$$

$$= \exp\left(-i\left(P_a - P_b\right)\cdot\left(\varepsilon_1 + \varepsilon_2\right) + i\left(K_2\cdot\varepsilon_1 - K_1\cdot\varepsilon_2\right)\operatorname{sign}\left(\tau_1 - \tau_2\right) + 2\varepsilon_1\cdot\varepsilon_2\delta\left(\tau_1 - \tau_2\right)\right)\big|_{\mathrm{lin}\,\varepsilon_n} \tag{408}$$

The linear term of the Taylor expansion contributes the term $2\varepsilon_1\cdot\varepsilon_2\delta\left(\tau_1 - \tau_2\right)$ to the amplitude. As for the quadratic term:

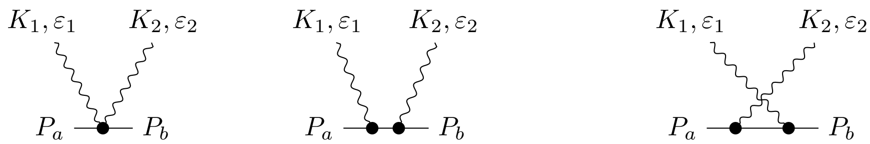
$$\tfrac{1}{2}\left(\left(P_b - P_a + K_2\operatorname{sign}\left(\tau_1 - \tau_2\right)\right)\cdot\varepsilon_1 + \left(P_b - P_a - K_1\operatorname{sign}\left(\tau_1 - \tau_2\right)\right)\cdot\varepsilon_2\right)^2\Big|_{\mathrm{lin}\,\varepsilon_n} \tag{409}$$

$$= \left(P_b - P_a + K_2\operatorname{sign}\left(\tau_1 - \tau_2\right)\right)\cdot\varepsilon_1\left(P_b - P_a - K_1\operatorname{sign}\left(\tau_1 - \tau_2\right)\right)\cdot\varepsilon_2 \tag{410}$$

$$\mathcal{A}\left[P_a,P_b;K_1,\varepsilon_1;K_2,\varepsilon_2\right]=(-iq)^2\,\delta\left(P_a+P_b+K_1+K_2\right)\int_0^\infty dTe^{-T\left(P_b^2+m^2\right)}\int_0^T d\tau_1\int_0^T d\tau_2 \tag{411}$$

$$\times\exp\left(\left(P_a-P_b\right)\cdot\left(K_1\tau_1+K_2\tau_2\right)+K_1\cdot K_2\left|\tau_1-\tau_2\right|\right) \tag{412}$$

$$\times\left(2\underbrace{\varepsilon_1\cdot\varepsilon_2\delta\left(\tau_1-\tau_2\right)}+\underbrace{\left(P_a-P_b+K_2\,\mathrm{sign}\left(\tau_1-\tau_2\right)\right)\cdot\varepsilon_1}\;\underbrace{\left(P_a-P_b-K_1\,\mathrm{sign}\left(\tau_1-\tau_2\right)\right)\cdot\varepsilon_2}\right) \tag{413}$$



As we saw in the one loop correction to the photon propagator in Section 5.3, the delta function generates the sea-gull vertex. The 3-point vertex terms correspond as shown to the Feynman diagrams when $\tau_2>\tau_1$. When $\tau_1>\tau_2$, the correspondence is reversed and this is enforced by the sign function. We also have the freedom to calculate the amplitudes for specific diagrams by deleting terms appropriately.

Another powerful application of the worldline formulation is the possibility of 'sewing' external legs together to create loops. For example, we can sew together the photons of the above Feynman diagrams with the replacement $\varepsilon_1^\mu\varepsilon_2^\nu\to\delta^{\mu\nu}/q^2$ to calculate Figure 27.

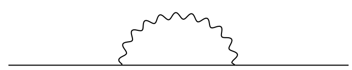


Figure 27: One-Loop correction to the scalar propagator.

This calculation is detailed in [1]. This can also be applied to the Bern-Kosower Master Formulae we derived for the loop in Section 4.3 and 5.2. For example, the external legs of the one-particle irreducible diagrams given by $\Gamma\left[P_1,\varepsilon_1;\cdots;P_4,\varepsilon_4\right]$ can be sewn together to calculate diagrams such as:

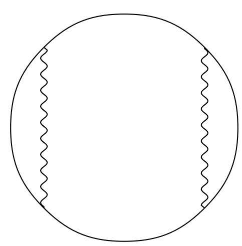


Figure 28: Connecting two pairs of external legs in the 4-photon, one-loop diagram to create a 3-loop diagram.

Combined with the fact that the Master Formulae calculate amplitudes for all permutations of the external legs, it's clear that the worldline formulation can be a highly versatile and efficient method for calculating scattering amplitudes.

6 Spinor QED

6.1 The Supespr Lagrangian

So far, we have been able to calculate the amplitude for bosonic particles to interact with scalar fields and electromagnetic fields. We would now like to extend this approach to fermions and their interactions.

However, the worldline fields X^μ and e are *bosonic*, and cannot account for spin degrees of freedom. Therefore, we need to introduce a *fermionic counterpart* of X^μ, which we will denote as ψ^μ. This *fermionic position variable* is also a Lorentz 4-vector and will therefore transform according to $\delta\psi = \eta\frac{d\psi}{d\tau}$. However, ψ^μ obeys the Grassman algebra:

$$\psi^\mu\psi^\nu = -\psi^\nu\psi^\mu \tag{414}$$

In Section 2, we deduced the Brink-di Vecchia-Howe Lagrangian $\mathcal{L}[X, e]$ on line 50, by imposing reparameterization invariance. We now wish to find $\mathcal{L}[X^\mu, \psi^\mu, e]$ and, as before, we will find that the form of this Lagrangian is contrained by reparameterization inveriance. We demonstrate in Appendix A.2 that the massless form of this Lagrangian will take the form:

$$\mathcal{L}[X^\mu, \psi^\mu, e] = \frac{1}{2}\left(\frac{\dot{X}^2}{e} - i\psi^\mu\dot{\psi}_\mu\right) \tag{415}$$

However, it can be shown that this form of the Lagrangian results in negative norm-states:

- Insert argument for negative norm states
- $\left|a^\dagger \left|0\right\rangle\right|^2$, $\{\psi^\mu, \psi^\nu\} = \eta^{\mu\nu}$, negative number in minkowsky spacetime metric causes negative norm states.

The issue is resolved by introducing *supersymmetry* on the worldline. The bosonic position variable X^μ and fermionic position variable ψ^μ are then related by a *supersymmetry generator*. This allows us to transform some of the creation and annihilation operators into each other, and this newfound redundancy gives us the freedom to combine variables in such a way that preserves the positive norm condition.

Supersymmetry also results in a fermionic counterpart for e, which we will denote as χ. This field will transform in the same way as e, such that $\delta\chi = \frac{d}{d\tau}(\eta\chi)$, but obeys the Grassman algebra. We demonstrate in Appendix A.3 that the reparametrization invariant form of $\mathcal{L}[X^\mu, \psi^\mu, e, \chi]$ is:

$$\mathcal{L}[X^\mu, \psi^\mu, e, \chi] = \frac{1}{2}\left(\frac{\dot{X}^2}{e} - i\psi^\mu\dot{\psi}_\mu - i\frac{\chi\dot{X}^\mu\psi_\mu}{e}\right) \tag{416}$$

Furthermore, we demonstrate in Appendix B that this Lagrangian is invariant under the following *supersymmetry* transformations:

$$\begin{aligned}
X^\mu &\to X^\mu + i\alpha\psi^\mu \\
\psi^\mu &\to \psi^\mu + \alpha\left(\frac{\dot{X}^\mu}{e} - \frac{i\chi\psi^\mu}{2e}\right) \\
e &\to e + i\alpha\chi \\
\chi &\to \chi + 2\dot{\alpha}
\end{aligned} \tag{417}$$

6.2 Dirac Equation

The Euler-Lagrange equation for χ gives a constraint equation:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \chi} &= \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{\chi}}\right) && \text{Using the Lagrangian: } \mathcal{L} = \tfrac{1}{2}\left(\tfrac{\dot{X}^2}{e} - i\psi^\mu \dot{\psi}_\mu - \tfrac{i\chi \dot{X}^\mu \psi_\mu}{e}\right) \\
\frac{\dot{X}^\mu \psi_\mu}{e} &= 0 && \text{Reparametrization invariance allows us to set } e = 1 \qquad (418) \\
\dot{X}^\mu \psi_\mu &= 0 && \text{This constraint corresponds to the orthogonality of velocity and spin} \qquad (419)
\end{aligned}
$$

The Euler-Lagrange equation for e gives the other constraint equation:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial e} &= \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{e}}\right) \\
-\frac{\left(\dot{X}^2 - i\chi \dot{X}^\mu \psi_\mu\right)\dot{e}}{e^2} &= 0 \\
\dot{X}^2 &= 0 && \text{Here, we used line (419). we arrive at the mass-shell condition for a massless particle} \qquad (420)
\end{aligned}
$$

The Euler-Lagrange equation for X gives the equation of motion:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial X_\mu} &= \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{X}_\mu}\right) \\
0 &= \frac{d}{d\tau}\left(\frac{2\dot{X}^\mu}{e} - \frac{i\chi\psi^\mu}{e}\right) && \text{On line (418) we set } e = 1\text{. We also stated on line (417) the supergauge transformation } \delta e = i\alpha\chi\text{. It then follows that } \chi = 0. \qquad (421) \\
\ddot{X}^\mu &= 0 && \text{The equation of motion is the same as what we derived in Section 2.5} \qquad (422)
\end{aligned}
$$

Euler-Lagrange equation for ψ gives the other equation of motion:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \psi_\mu} &= \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}_\mu}\right) \\
\dot{\psi}^\mu &= 0 \\
\psi^\mu &= \text{constant Grassman 4-vector} \\
\{\psi_\mu, \psi_\nu\} &= g^{\mu\nu} \\
\psi^\mu &= \tfrac{1}{\sqrt{2}}\gamma^\mu && \text{We solve this anti-commutator relation by expressing } \psi \text{ in terms of the Gamma matrices } \gamma^\mu\text{, so that the above anti-commutator relation becomes the defining equation of the Dirac-Clifford Algebra.} \\
|\psi\rangle &= |p\rangle\, u(p) && \text{With } |p\rangle = e^{-ip\not{x}}|0\rangle.\ u(p) \text{ is a Dirac spinor and } |p\rangle \text{ are the momentum eigenstates.}
\end{aligned}
$$

The constraint (420) implies masslessness and we can re-write (419): $\psi^\mu \dot{X}_\mu$ as $\gamma^\mu \dot{X}_\mu$ to show that it's equivalent to the massless Dirac equation. This is often written as $\gamma^\mu \partial_\mu \psi = \not{\partial}\psi$, where the field ψ is not to be confused with our fermionic position variable.

6.3 Minimal Gauge Coupling

$$
\begin{aligned}
H[X^\mu, \psi^\mu, e, \chi] = P \cdot \dot{X} - \mathcal{L}[X^\mu, \psi^\mu, e, \chi] &= P \cdot \dot{X} - \frac{1}{2}\left(\frac{\dot{X}^2}{e} - i\psi^\mu \dot{\psi}_\mu\right) \\
&= \frac{1}{2}\left(\frac{\dot{X}^2}{e} + i\psi^\mu \dot{\psi}_\mu\right) \\
&= \frac{e}{2}\pi^2 + \frac{i}{2}\psi^\mu \dot{\psi}_\mu
\end{aligned}
$$

As in Section 5.1, we denote the conjugate momentum $P^\mu = \dot{X}^\mu/e$ as π, which is the momentum that we must covariantize.

$$
\begin{aligned}
\not{P} \to \pi = \left(\not{P} + q\not{A}\right)^2 &= (\gamma \cdot (\partial + qA))^2 \\
&\propto (\psi \cdot (P + qA))^2 \\
&= (\psi_\mu (P^\mu + qA^\mu))(\psi_\nu (P^\nu + qA^\nu)) \\
&= \psi_\mu \psi_\nu P^\mu P^\nu + q(\psi_\mu \psi_\nu P^\mu A^\nu + \psi_\mu \psi_\nu P^\nu A^\mu) && \text{As per minimal coupling, we discard higher multipole moments.} \\
&= \psi_\mu \psi_\nu P^\mu P^\nu + q(\psi_\mu \psi_\nu P^\mu A^\nu + \psi_\nu \psi_\mu P^\mu A^\nu) && \text{Swapping indices on the last term} \\
&= \psi_\mu \psi_\nu P^\mu P^\nu + q(\psi_\mu \psi_\nu P^\mu A^\nu - \psi_\mu \psi_\nu P^\nu A^\mu) && \text{Using 414} \\
&= \psi_\mu \psi_\nu P^\mu P^\nu + q\psi_\mu \psi_\nu (P^\mu A^\nu - P^\nu A^\mu)
\end{aligned}
$$

The final term corresponds to the electromagnetic field strength tensor, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. Therefore, the correct gauge invariant expression for the Hamiltonian is:

The Hamiltonian of the particle becomes:

$$
H = \frac{e}{2}\left(\pi^2 + m^2\right) \tag{423}
$$

Through a Legendre transform of this Hamiltonian, we deduce the form of the Lagrangian in Minkowski space:

$$
\begin{aligned}
\mathcal{L}[X^\mu, e, A^\mu] &= P \cdot \dot{X} - \frac{e}{2}H && (424) \\
&= P \cdot \dot{X} - \frac{e}{2}\left(\pi^2 + m^2\right) && (425)
\end{aligned}
$$

We eliminate P^μ using (423) and requiring that the gauge-transformed momentum is equal to the conjugate momentum for the Brink-di Vecchia-Howe action (which see derived on line (51)), so that $\pi^\mu = \dot{X}^\mu/e$.

$$
\begin{aligned}
\mathcal{L}[X^\mu, e, A^\mu] &= (\pi^\mu - qA^\mu)\dot{X}_\mu - \frac{e}{2}\left(\pi^2 - m^2\right) && (426) \\
&= \left(\frac{\dot{X}^\mu}{e} - qA^\mu\right)\dot{X}_\mu - \frac{e}{2}\left(\frac{\dot{X}^2}{e^2} - m^2\right) && (427) \\
&= \frac{1}{2}\left(\frac{\dot{X}^2}{e} - m^2 e\right) + qA \cdot \dot{X} && (428)
\end{aligned}
$$

We perform a Wick rotation to express the Lagrangian in Euclidean space:

$$\mathcal{L}\left[X^{\mu}, e, A^{\mu}\right] = \frac{1}{2}\left(\frac{\dot{X}^2}{e} + m^2 e\right) + iqA \cdot \dot{X} \tag{429}$$

We have derived the effective Lagrangian of a relativistic particle of charge q coupled to an Abelian vector field A^{μ}. We use this to replace the action in (198), thus finding the effective action for Scalar QED:

$$\Gamma[A] = \int_0^{\infty} \frac{dT}{T}\, e^{-m^2 T} \int\limits_{X(0)=X(T)=X_0} \mathcal{D}X e^{-\int_0^T d\tau \left(\frac{\dot{X}^2}{4} + iqA \cdot \dot{X}\right)} \tag{430}$$

7 Conclusion

7.1 Summary

Through our exploration of the Worldline Formalism, we have learnt an approach to Quantum Field Theory which is strongly rooted in *geometry* and can be seen as the *infinite tension limit* of String Theory. We saw that, as the tension of the string goes to infinity, the string contricts to a particle and the Riemannian manifold transforms from a worldsheet to a world*line*. WL formulates the dynamics of the particle in terms of the *1-dimensional, relativistic, quantum mechanical field theory* that lives on the Riemannian manifold.

We found that the *Sigma-model* enables us to formulate a manifestly Lorentz invariant Lagrangian for a relativistic particle, by introducing the worldline parameter τ. Quantities that are usually functions of spacetime variables are seen as functions of τ, as *fields on the Riemannian manifold*. We derived the symmetries and transformation laws of these fields, particularly *reparametrization invariance* which exists because we have introduced a redundancy to our system by parametrizing it in terms τ.

Analogously to ST, we formulated the the Brink-di Vecchia-Howe action so that we may *quantize* the particle, and in doing so we discovered a *worldline metric* whose 'square root' is a field that lives on the worldline, the Einbein. We also saw that the action of the relativistic particle is equivalent to that of 1-dimensional general relativity.

In quantizing the particle, we created a quantum mechanical field theory which is a function of the dynamical worldline quantities, and this enabled us to compute the dynamics of the relativistic particles through *quantum mechanical correlation functions*.

We computed path integrals and compared them with the canonically quantized theory to deduce the action which corresponds to the correct amplitude of propagation for a free particle. We later found that this action emerged from a gauge-fixed path integral for a relativistic particle, whose worldline satisfies the boundary conditions of the transition.

We found the form of these path integral for two key topologies: the interval and the loop. We later modified the Brink-di Vecchia-Howe action such that it encapsulates the behaviour of a relativsitic particle coupled to external fields. Like ST, these fields are treated as classical, background fields while the particle is treated as a quantum object.

In these interacting theories, we brought the path integral into a form which readily gives the amplitude of the particle interacting N times with the external field. These are the Bern-Kosower Master Formulae (BKMF) and they were originally derived through the infinite tension limimt of String Theory.

For Abelian fields such as the ones we studied, we saw that the BKMF gives the amplitude for an interaction with N vertices, and integrates the vertex around all positions along the worldline. In doing so, it gives all possible permutations of the external legs. For Scalar QED, we saw how it generates both 3-point and 4-point vertices, and we compared the computations with standard quantum field theoretic results.

Furthermore, we saw that the BKMF computes results such that they are *gauge-invariant at the integral level*. This results in the compactification of terms and we saw examples of this with the 1-loop correction to the photon propagator and Compton scattering.

7.2 Further Topics

Having covered the basics of the worldline formalism, we present an outline of advanced topics. While. we restricted our discussion to the dynamics of a *scalar* particle, the worldline formalism accounts for spin degrees of freedom by introducing a partner variable $\psi^\mu(\tau)$ for the worldline position variable $X^\mu(\tau)$. ψ transforms in the same way as X: as a set of 4 scalar fields. However, ψ is an anti-commuting variable such that $\psi^\mu\psi^\nu = -\psi^\nu\psi^\mu$. It's proven in Appendix A.2 that the following Lagrangian (for a massles particle) satisfies the requirement of reparametrization invariance:

$$\mathcal{L} = \frac{1}{2}\left(\frac{\dot{X}^2}{e} - i\psi^\mu\dot{\psi}_\mu\right) \tag{431}$$

However, the introduction of ψ results in negative norm states and these can be cured if we also introduce an anti-commuting partner variable for the einbein field, which we call $\chi(\tau)$. It's shown in Appendix A.3 that the following Lagrangian is reparametrization invariant:

$$\mathcal{L} = \frac{1}{2}\left(\frac{\dot{X}^2}{e} - i\psi^\mu\dot{\psi}_\mu - i\frac{\chi\dot{X}^\mu\psi_\mu}{e}\right) \tag{432}$$

We have introduced a *local supersymmetry on the worldline*, and we show the supersymmetric transformation of this Lagrangian in Appendix B.

We quantize the spinor particle through the quantum mechanical anti-commutation relation $\{\psi^\mu, \psi^\nu\} = \eta^{\mu\nu}$. This is solved by the *gamma matrices* so we deduce $\sqrt{2}\psi = \gamma$. The equation of motion for χ is $\psi P = 0$, where P is the conjugate momentum $\dot{X}/e$. We can substitute P for the quantum mechanical operator $i\partial$ and use the solution for ψ so that this equation of motion becomes $\gamma\partial = 0$. When imposed as a constraint on the Hilbert space, this is the *massless Dirac equation.*

This Lagrangian is then used to formulate the worldline path integral representation and we may gauge-fix χ, as we did for the einbein. We may use supersymmetry on the worldline to derive Bern-Kosower Master Formulae for the spinor particles [37].

We studied interactions with electromagnetic fields that can be characterized in terms of momentum and polarization. However, to account for strong interactions, we must also include the *colour factor.* Also, the BKMF for Abelian theories allows external legs to commute but a non-Abelian theory should only include all *cyclic permutations* of the external legs, which we achieve through *path-ordering.*

$$\int_0^T d\tau_1 \cdots \int_0^T d\tau_N \rightarrow \int_0^{\tau_2} d\tau_1 \int_{\tau_1}^{\tau_2} d\tau_2 \cdots \int_{\tau_N - 1}^{\tau_N} d\tau_N \tag{433}$$

There is ongoing work on the worldline formulation, so that it's power and versatility can be applied in new contexts. Active fields of research are benefiting from the unique insight of the worldline approach and we conclude by highlighting some recent and significant applications.

- Calculation of Graviton amplitudes [6][11][12].
- Gravitational and Axial Anomalies [3]
- The AdS/CFT correspondence [26][17]
- Higher Spin Fields and Differential Forms [5][4].
- The use of worldline instantons for semiclassical nonperturbative computation [13].
- QFT in non-commutative space-time [2][8].
- Montecarlo simulations for Casimir energies based on the worldline approach [24].
- Application to the standard model to study group representations and chiralities that appear in nature [25].

Appendices

A Reparamaterization Invariance

A.1 $\mathcal{L}[X,e]$

We showed in Section 2 that under reparametrization $\tau \to \tau + \eta(\tau)$ the fields transform as follows:

$$X^\mu \to X^\mu + \eta \frac{dX^\mu}{d\tau}, \quad e \to e + \frac{d}{d\tau}\left(\eta(\tau)e\right) \tag{434}$$

The Brink-di Vecchia-Howe action transforms as:

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\left(\frac{\dot{X}^2}{e} - m^2 e\right) &(435)\\
&\to \frac{1}{2}\left(\frac{\left(\dot{X}^\mu + \frac{d}{d\tau}\left(\eta \dot{X}\right)\right)^2}{e + \frac{d}{d\tau}\left(\eta e\right)} - m^2\left(e + \frac{d}{d\tau}\left(\eta e\right)\right)\right) &(436)\\
&= \frac{1}{2}\left(\frac{\left(\dot{X}^\mu + \eta \ddot{X}^\mu + \dot{\eta}\dot{X}^\mu\right)^2}{e + \dot{\eta}e + \eta\dot{e}} - m^2\left(e + \eta\dot{e} + \dot{\eta}e\right)\right) &(437)\\
&= \frac{1}{2}\left(\frac{\dot{X}^2 + 2\eta\dot{X}^\mu\ddot{X}_\mu + 2\dot{\eta}\dot{X}^2}{e + \dot{\eta}e + \eta\dot{e}} - m^2\left(e + \eta\dot{e} + \dot{\eta}e\right)\right) &(438)\\
&= \frac{1}{2}\left(\frac{\dot{X}^2 + 2\eta\dot{X}^\mu\ddot{X}_\mu + 2\dot{\eta}\dot{X}^2}{e\left(1 + \dot{\eta} + \eta\frac{\dot{e}}{e}\right)} - m^2\left(e + \eta\dot{e} + \dot{\eta}e\right)\right) &(439)\\
&= \frac{1}{2}\left(\frac{\dot{X}^2 + 2\eta\dot{X}^\mu\ddot{X}_\mu + 2\dot{\eta}\dot{X}^2}{e}\left(1 - \dot{\eta} - \eta\frac{\dot{e}}{e}\right) - m^2\left(e + \eta\dot{e} + \dot{\eta}e\right)\right) &(440)\\
&= \frac{1}{2}\left(\frac{\dot{X}^2 + 2\eta\dot{X}^\mu\ddot{X}_\mu + 2\dot{\eta}\dot{X}^2}{e} - \frac{\dot{\eta}\dot{X}^2}{e} - \frac{\eta\dot{X}^2\dot{e}}{e^2} - m^2\left(e + \eta\dot{e} + \dot{\eta}e\right)\right) &(441)\\
&= \frac{1}{2}\left(\frac{\dot{X}^2 + 2\eta\dot{X}^\mu\ddot{X}_\mu + \dot{\eta}\dot{X}^2}{e} - \frac{\eta\dot{X}^2\dot{e}}{e^2} - m^2\left(e + \eta\dot{e} + \dot{\eta}e\right)\right) &(442)\\
&= \mathcal{L} + \frac{1}{2}\left(\frac{2\eta\dot{X}^\mu\ddot{X}_\mu + \dot{\eta}\dot{X}^2}{e} - \frac{\eta\dot{X}^2\dot{e}}{e^2} - m^2\left(\eta\dot{e} + \dot{\eta}e\right)\right) &(443)\\
&= \mathcal{L} + \eta\left(\frac{1}{2}\left(\frac{2\dot{X}^\mu\ddot{X}_\mu}{e} - \frac{\dot{X}^2\dot{e}}{e^2} - m^2\dot{e}\right)\right) + \dot{\eta}\left(\frac{1}{2}\left(\frac{\dot{X}^2}{e} - m^2 e\right)\right) &(444)\\
&= \mathcal{L} + \eta\frac{d}{d\tau}\left(\frac{1}{2}\left(\frac{\dot{X}^2}{e} - m^2 e\right)\right) + \dot{\eta}\left(\frac{1}{2}\left(\frac{\dot{X}^2}{e} - m^2 e\right)\right) &(445)\\
&= \mathcal{L} + \eta\dot{\mathcal{L}} + \dot{\eta}\mathcal{L} &(446)\\
&= \mathcal{L} + \frac{d}{d\tau}\left(\eta\mathcal{L}\right) &(447)
\end{aligned}$$

The Lagrangian transforms as a total derivative, which vanishes upon integration by parts and imposition of boundary conditions. Therefore, the Lagrangian is reparametrization invariant.

A.2 $\mathcal{L}[X, e, \psi]$

Applying the reparaemtrization:

$$X^\mu \to X^\mu + \eta \frac{dX^\mu}{d\tau}, \quad \psi^\mu \to \psi^\mu + \eta \dot{\psi}^\mu \quad e \to e + \frac{d}{d\tau}(\eta(\tau)) \tag{448}$$

The Lagrangian transforms as:

$$\mathcal{L} \to \frac{1}{2}\left(\frac{\dot{X}^2}{e} - i\psi^\mu \dot{\psi}_\mu\right) \tag{449}$$

$$= \frac{1}{2}\left(\frac{\left[\frac{d}{dt}\left(X^\mu + \eta \dot{X}^\mu\right)\right]^2}{e + \frac{d}{d\tau}(\eta e)} - i\left[\left(\psi^\mu + \eta\dot{\psi}^\mu\right)\frac{d}{dt}\left(\psi_\mu + \eta\dot{\psi}_\mu\right)\right]\right) \tag{450}$$

$$= \frac{1}{2}\left(\frac{\dot{X}^2 + \dot{\eta}\dot{X}^2 + 2\eta\dot{X}^\mu\ddot{X}_\mu}{e} - \frac{\eta\dot{X}^2\dot{e}}{e^2} - i\left[\left(\psi^\mu + \eta\dot{\psi}^\mu\right)\frac{d}{dt}\left(\psi_\mu + \eta\dot{\psi}_\mu\right)\right]\right) \quad \text{Repeating steps (437) to (442)} \tag{451}$$

$$= \frac{1}{2}\left(\frac{\dot{X}^2 + \dot{\eta}\dot{X}^2 + 2\eta\dot{X}^\mu\ddot{X}_\mu}{e} - \frac{\eta\dot{X}^2\dot{e}}{e^2} - i\left[\left(\psi^\mu + \eta\dot{\psi}^\mu\right)\left(\dot{\psi}_\mu + \eta\ddot{\psi}_\mu + \dot{\eta}\dot{\psi}_\mu\right)\right]\right) \tag{452}$$

$$= \frac{1}{2}\left(\frac{\dot{X}^2 + \dot{\eta}\dot{X}^2 + 2\eta\dot{X}^\mu\ddot{X}_\mu}{e} - \frac{\eta\dot{X}^2\dot{e}}{e^2} - i\left[\psi^\mu\dot{\psi}_\mu + \eta\psi^\mu\ddot{\psi}_\mu + \dot{\eta}\psi^\mu\dot{\psi}_\mu\right]\right) \quad \text{Using } \dot{\psi}^2 = 0 \tag{453}$$

$$= \mathcal{L} + \frac{1}{2}\left(\frac{\dot{\eta}\dot{X}^2 + 2\eta\dot{X}^\mu\ddot{X}_\mu}{e} - \frac{\eta\dot{X}^2\dot{e}}{e} - i\left[\eta\psi^\mu\ddot{\psi}_\mu + \dot{\eta}\psi^\mu\dot{\psi}_\mu\right]\right) \tag{454}$$

$$= \mathcal{L} + \dot{\eta}\left\{\frac{1}{2}\left(\frac{\dot{X}^2}{e} - i\psi^\mu\dot{\psi}_\mu\right)\right\} + \eta\left\{\frac{1}{2}\left(\frac{2\dot{X}^\mu\ddot{X}_\mu}{e} - \frac{\dot{X}^2\dot{e}}{e^2} - i\psi^\mu\ddot{\psi}_\mu\right)\right\} \tag{455}$$

$$= \mathcal{L} + \dot{\eta}\left\{\frac{1}{2}\left(\frac{\dot{X}^2}{e} - i\psi^\mu\dot{\psi}_\mu\right)\right\} + \eta\frac{d}{dt}\left\{\frac{1}{2}\left(\frac{\dot{X}^2}{e} - i\psi^\mu\dot{\psi}_\mu\right)\right\} \tag{456}$$

$$= \mathcal{L} + \dot{\eta}\mathcal{L} + \eta\dot{\mathcal{L}} \tag{457}$$

$$= \mathcal{L} + \frac{d}{d\tau}(\eta\mathcal{L}) \tag{458}$$

The reparametrization introduces a total derivative to the Lagrangian, therefore it's reparametrization invariant.

A.3 $\mathcal{L}\left[X,e,\psi,\chi\right]$

$$\mathcal{L}\left[X,e,\psi,\chi\right]=\frac{1}{2}\left(\frac{\dot{X}^2}{e}-i\psi^\mu\dot{\psi}_\mu-i\frac{\chi\dot{X}^\mu\psi_\mu}{e}\right)=\underbrace{\frac{1}{2}\frac{\dot{X}^2}{e}-\frac{i}{2}\psi^\mu\dot{\psi}_\mu}_{\mathcal{L}_A}\underbrace{-\frac{i}{2}\frac{\chi\dot{X}^\mu\psi_\mu}{e}}_{\mathcal{L}_B}=\mathcal{L}_A+\mathcal{L}_B$$

$\mathcal{L}_A\to\mathcal{L}_A+\frac{d}{d\tau}(\eta\mathcal{L}_A)$ is proven in A.2, so we consider the reparametrization of $\mathcal{L}_B$: Reparametrization causes the fields to transform in the following way:

$$X^\mu\to X^\mu+\eta\dot{X},\quad \psi^\mu\to\psi^\mu+\eta\dot{\psi},\quad e\to e+\frac{d}{d\tau}\left(\eta e\right),\quad \chi\to\chi+\frac{d}{d\tau}\left(\eta\chi\right)\tag{459}$$

$$\mathcal{L}_B\to\frac{i\frac{i}{2}\left(\chi+\frac{d}{d\tau}(\eta\chi)\right)\left(\dot{X}^\mu+\frac{d}{dt}(\eta\dot{X})\right)\left(\psi_\mu+\eta\dot{\psi}_\mu\right)}{e+\dot{\eta}e+\eta\dot{e}}\tag{460}$$

$$=\frac{-\frac{i}{2}\left\{\chi\dot{X}^\mu\psi_\mu+\frac{d}{d\tau}(\eta\chi)\dot{X}^\mu\psi_\mu+\chi\frac{d}{d\tau}(\eta\dot{X})\psi_\mu+\eta\chi\dot{X}^\mu\dot{\psi}_\mu\right\}}{e+\dot{\eta}e+\eta\dot{e}}\tag{461}$$

$$=\frac{-\frac{i}{2}\left\{\chi\dot{X}^\mu\psi_\mu+\frac{d}{d\tau}(\eta\chi)\dot{X}^\mu\psi_\mu+\chi\frac{d}{d\tau}(\eta\dot{X})\psi_\mu+\eta\chi\dot{X}^\mu\dot{\psi}_\mu\right\}}{e}\left(1-\dot{\eta}-\eta\frac{\dot{e}}{e}\right)\tag{462}$$

$$=\left\{\mathcal{L}_B+\frac{-\frac{i}{2}\left\{\frac{d}{d\tau}(\eta\chi)\dot{X}^\mu\psi_\mu+\chi\frac{d}{d\tau}(\eta\dot{X})\psi_\mu+\eta\chi\dot{X}^\mu\dot{\psi}_\mu\right\}}{e}\right\}\left(1-\dot{\eta}-\eta\frac{\dot{e}}{e}\right)\tag{463}$$

$$=\left(1-\dot{\eta}-\eta\frac{\dot{e}}{e}\right)\mathcal{L}_B+\frac{-\frac{i}{2}\left\{\frac{d}{d\tau}(\eta\chi)\dot{X}^\mu\psi_\mu+\chi\frac{d}{d\tau}(\eta\dot{X})\psi_\mu+\eta\chi\dot{X}^\mu\dot{\psi}_\mu\right\}}{e}\tag{464}$$

$$=\left(1-\dot{\eta}-\eta\frac{\dot{e}}{e}\right)\mathcal{L}_B+\frac{-\frac{i}{2}\left\{\frac{d}{d\tau}(\eta\chi)\dot{X}^\mu\psi_\mu+\chi\frac{d}{d\tau}(\eta\dot{X})\psi_\mu+\chi\dot{X}^\mu\frac{d}{d\tau}\left(\eta\psi_\mu\right)-\dot{\eta}\chi\dot{X}^\mu\psi_\mu\right\}}{e}\tag{465}$$

$$=\left(1-\dot{\eta}-\eta\frac{\dot{e}}{e}\right)\mathcal{L}_B+\frac{-\frac{i}{2}\left\{\frac{d}{d\tau}(\eta\chi)\dot{X}^\mu\psi_\mu+\chi\frac{d}{d\tau}(\eta\dot{X})\psi_\mu+\chi\dot{X}^\mu\frac{d}{d\tau}\left(\eta\psi_\mu\right)\right\}}{e}-\dot{\eta}\mathcal{L}_B\tag{466}$$

$$=\left(1-2\dot{\eta}-\eta\frac{\dot{e}}{e}\right)\mathcal{L}_B+\frac{-\frac{i}{2}\left\{\frac{d}{d\tau}(\eta\chi)\dot{X}^\mu\psi_\mu+\chi\frac{d}{d\tau}(\eta\dot{X})\psi_\mu+\chi\dot{X}^\mu\frac{d}{d\tau}\left(\eta\psi_\mu\right)\right\}}{e}\tag{467}$$

$$=\left(1-2\dot{\eta}-\eta\frac{\dot{e}}{e}\right)\mathcal{L}_B-\frac{i}{2}\left\{\frac{3\dot{\eta}\chi\dot{X}^\mu\psi_\mu}{e}+\frac{\eta\frac{d}{d\tau}\left(\chi\dot{X}^\mu\psi_\mu\right)}{e}\right\}\tag{468}$$

$$=\left(1-2\dot{\eta}-\eta\frac{\dot{e}}{e}\right)\mathcal{L}_B+3\dot{\eta}\mathcal{L}_B-\frac{i}{2}\left\{\frac{\eta\frac{d}{d\tau}\left(\chi\dot{X}^\mu\psi_\mu\right)}{e}\right\}\tag{469}$$

$$=\left(1+\dot{\eta}-\eta\frac{\dot{e}}{e}\right)\mathcal{L}_B-\frac{i}{2}\left\{\frac{\eta\frac{d}{d\tau}\left(\chi\dot{X}^\mu\psi_\mu\right)}{e}\right\}\tag{470}$$

$$=\mathcal{L}_B+\dot{\eta}\mathcal{L}_B-\frac{\eta\dot{e}}{e}\mathcal{L}_B-\frac{i}{2}\left\{\frac{\eta\frac{d}{d\tau}\left(\chi\dot{X}^\mu\psi_\mu\right)}{e}\right\}\tag{471}$$

$$=\mathcal{L}_B+\dot{\eta}\mathcal{L}_B-\frac{i\eta}{2}\left\{-\frac{\chi\dot{X}^\mu\psi_\mu\dot{e}}{e^2}+\frac{\frac{d}{d\tau}\left(\chi\dot{X}^\mu\psi_\mu\right)}{e}\right\}\tag{472}$$

$$= \mathcal{L}_B + \dot{\eta}\mathcal{L}_B + \eta \frac{d}{d\tau}\left(-\frac{i}{2}\frac{\chi \dot{X}^\mu \psi_\mu}{e}\right) \tag{473}$$

$$= \mathcal{L}_B + \dot{\eta}\mathcal{L}_B + \eta\dot{\mathcal{L}}_B \tag{474}$$

$$= \mathcal{L}_B + \frac{d}{d\tau}\left(\eta\mathcal{L}_B\right) \tag{475}$$

$$\mathcal{L} = \mathcal{L}_A + \mathcal{L}_B \to \mathcal{L}_A + \mathcal{L}_B + \frac{d}{d\tau}\left(\eta\mathcal{L}_A + \eta\mathcal{L}_B\right) \tag{476}$$

$$= \mathcal{L} + \frac{d}{d\tau}\left(\eta\mathcal{L}\right) \tag{477}$$

Therefore this is the correct reparamterization invariant form of the Lagrangian.

B Supersymmetric Transformation

$$\mathcal{L} = \underbrace{\frac{\dot{X}^2}{2e}} - \underbrace{\frac{i\psi^\mu\dot{\psi}_\mu}{2}} - \underbrace{\frac{i\chi\dot{X}^\mu\psi_\mu}{2e}} \qquad \text{Supersymmetry Transformations:} \left(\begin{array}{ll} X^\mu & \to X^\mu + i\alpha\psi^\mu \\ \psi^\mu & \to \psi^\mu + \alpha\left(\frac{\dot{X}^\mu}{e} - \frac{i\chi\psi^\mu}{2e}\right) \\ e & \to e + i\alpha\chi \\ \chi & \to \chi + 2\dot{\alpha} \end{array} \right)$$

α is an infinitesimal supergauge parameter and is a Grassman odd variable. We will use that α, ψ and χ anti-commute amongst themselves, which implies that $\alpha^2 = \psi^2 = \chi^2 = 0$. We will also use that $\alpha\dot{\alpha} = \mathcal{O}\left(\alpha^2\right) = 0$.

$$\begin{aligned}
\mathcal{L}_1 \to & \frac{1}{2}\left\{\frac{\left(\dot{X}^\mu + i\frac{d}{d\tau}\left(\alpha\psi^\mu\right)\right)^2}{e + i\alpha\chi}\right\} \\
&= \frac{1}{2}\left\{\frac{\dot{X}^2 + 2i\dot{X}^\mu\left(\dot{\alpha}\psi_\mu + \alpha\dot{\psi}_\mu\right)}{e + i\alpha\chi}\right\} \\
&= \frac{1}{2}\left\{\frac{\dot{X}^2 + 2i\dot{X}^\mu\left(\dot{\alpha}\psi_\mu + \alpha\dot{\psi}_\mu\right)}{e}\left(1 - \frac{i\alpha\chi}{e}\right)\right\} \\
&= \mathcal{L}_1 + \underbrace{\frac{1}{2}\left\{\frac{2i\dot{X}^\mu\left(\dot{\alpha}\psi_\mu + \alpha\dot{\psi}_\mu\right)}{e} - \frac{i\dot{X}^2\alpha\chi}{e^2}\right\}}_{\delta\mathcal{L}_1}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_2 \to & -\frac{i}{2}\left(\psi^\mu + \alpha\left(\frac{\dot{X}^\mu}{e} - \frac{i\chi\psi^\mu}{2e}\right)\right)\frac{d}{d\tau}\left(\psi_\mu + \alpha\left(\frac{\dot{X}_\mu}{e} - \frac{i\chi\psi_\mu}{2e}\right)\right) \\
&= -\frac{i}{2}\left(\psi^\mu + \alpha\left(\frac{\dot{X}^\mu}{e} - \frac{i\chi\psi^\mu}{2e}\right)\right)\left(\dot{\psi}_\mu + \frac{d}{d\tau}\left\{\alpha\left(\frac{\dot{X}_\mu}{e} - \frac{i\chi\psi_\mu}{2e}\right)\right\}\right) \\
&= \mathcal{L}_2 \underbrace{- \frac{i\psi^\mu}{2}\frac{d}{d\tau}\left\{\alpha\left(\frac{\dot{X}_\mu}{e} - \frac{i\chi\psi_\mu}{2e}\right)\right\} - \frac{i\alpha}{2}\left(\frac{\dot{X}^\mu}{e} - \frac{i\chi\psi^\mu}{2e}\right)\dot{\psi}_\mu}_{\delta\mathcal{L}_2}
\end{aligned}$$

$$\begin{aligned}
\delta\mathcal{L}_2 = & -\frac{i\psi^\mu}{2}\frac{d}{d\tau}\left\{\alpha\left(\frac{\dot{X}_\mu}{e} - \frac{i\chi\psi_\mu}{2e}\right)\right\} + \\
& -\frac{i\alpha}{2}\left(\frac{\dot{X}^\mu}{e} - \frac{i\chi\psi^\mu}{2e}\right)\dot{\psi}_\mu \\
= & -\frac{i\psi^\mu}{2}\left\{\dot{\alpha}\left(\frac{\dot{X}_\mu}{e} - \frac{i\chi\psi_\mu}{2e}\right)\right\} - \frac{i\psi^\mu}{2}\left\{\alpha\left(\frac{\ddot{X}_\mu}{e} - \frac{\dot{X}_\mu\dot{e}}{e^2} - \frac{i\dot{\chi}\psi_\mu}{2e} - \frac{i\chi\dot{\psi}_\mu}{2e} + \frac{i\chi\psi_\mu\dot{e}}{2e^2}\right)\right\} \\
& -\frac{i\alpha}{2}\left(\frac{\dot{X}^\mu}{e} - \frac{i\chi\psi^\mu}{2e}\right)\dot{\psi}_\mu \\
= & -\frac{i\dot{X}_\mu\psi^\mu\dot{\alpha}}{2e} - \frac{i\ddot{X}_\mu\psi^\mu\alpha}{2e} + \frac{i\dot{X}_\mu\dot{e}\psi^\mu\alpha}{2e^2} - \frac{\psi^\mu\alpha\chi\dot{\psi}_\mu}{4e} - \frac{i\dot{X}^\mu\alpha\dot{\psi}_\mu}{2e} - \frac{\alpha\chi\psi^\mu\dot{\psi}_\mu}{4e}
\end{aligned}$$

$$\mathcal{L}_3 \to \frac{-\frac{i}{2}\left(\chi+2\dot{\alpha}\right)\left(\dot{X}^\mu + i\frac{d}{d\tau}\left(\alpha\psi^\mu\right)\right)\left(\psi_\mu + \alpha\left(\frac{\dot{X}_\mu}{e} - \frac{i\chi\psi_\mu}{2e}\right)\right)}{e+i\alpha\chi}$$

$$= \frac{-\frac{i}{2}\left(\chi+2\dot{\alpha}\right)\left(\dot{X}^\mu + i\dot{\alpha}\psi^\mu + i\alpha\dot{\psi}^\mu\right)\left(\psi_\mu + \frac{\dot{X}_\mu\alpha}{e} - \frac{i\alpha\chi\psi_\mu}{2e}\right)}{e+i\alpha\chi}$$

$$= \frac{-\frac{i}{2}\left\{\dot{X}^\mu\chi\psi_\mu + \frac{\dot{X}^2\chi\alpha}{e} + i\chi\alpha\dot{\psi}^\mu\psi_\mu + 2\dot{X}^\mu\dot{\alpha}\psi_\mu\right\}}{e+i\alpha\chi}$$

$$= \frac{-\frac{i}{2}\left\{\dot{X}^\mu\chi\psi_\mu + \frac{\dot{X}^2\chi\alpha}{e} + i\chi\alpha\dot{\psi}^\mu\psi_\mu + 2\dot{X}^\mu\dot{\alpha}\psi_\mu\right\}\left(1-\frac{i\alpha\chi}{e}\right)}{e}$$

$$= \frac{-\frac{i}{2}\left\{\dot{X}^\mu\chi\psi_\mu + \frac{\dot{X}^2\chi\alpha}{e} + i\chi\alpha\dot{\psi}^\mu\psi_\mu + 2\dot{X}^\mu\dot{\alpha}\psi_\mu\right\}}{e}$$

$$= \mathcal{L}_3 \underbrace{-\frac{i}{2}\left\{\frac{\dot{X}^2\chi\alpha}{e^2} + \frac{i\chi\alpha\dot{\psi}^\mu\psi_\mu}{e} + \frac{2\dot{X}^\mu\dot{\alpha}\psi_\mu}{e}\right\}}_{\delta\mathcal{L}_3}$$

$$\delta\mathcal{L}_3 = -\frac{i\dot{X}^2\chi\alpha}{2e^2} + \frac{\chi\alpha\dot{\psi}^\mu\psi_\mu}{2e} - \frac{i\dot{X}^\mu\dot{\alpha}\psi_\mu}{e}$$

$$\begin{aligned}
\delta\mathcal{L} =& \delta\mathcal{L}_1 + \delta\mathcal{L}_2 + \delta\mathcal{L}_3 \\
=& \frac{i\dot{X}^\mu\dot{\alpha}\psi_\mu}{e} + \frac{i\dot{X}^\mu\alpha\dot{\psi}_\mu}{e} - \frac{i\dot{X}^2\alpha\chi}{2e^2} - \frac{i\dot{X}_\mu\psi^\mu\dot{\alpha}}{2e} - \frac{i\ddot{X}_\mu\psi^\mu\alpha}{2e} + \frac{i\dot{X}_\mu\dot{e}\psi^\mu\alpha}{2e^2} - \frac{\psi^\mu\alpha\chi\dot{\psi}_\mu}{2e} \\
&- \frac{i\dot{X}^\mu\alpha\dot{\psi}_\mu}{2e} - \frac{i\dot{X}^2\chi\alpha}{2e^2} + \frac{\chi\alpha\dot{\psi}^\mu\psi_\mu}{2e} - \frac{i\dot{X}^\mu\dot{\alpha}\psi_\mu}{e} \\
=& \frac{i\dot{X}^\mu\alpha\dot{\psi}_\mu}{2e} - \frac{i\dot{X}_\mu\psi^\mu\dot{\alpha}}{2e} - \frac{i\ddot{X}_\mu\psi^\mu\alpha}{2e} + \frac{i\dot{X}_\mu\dot{e}\psi^\mu\alpha}{2e^2} \\
=& \frac{i\dot{X}^\mu\alpha\dot{\psi}_\mu}{2e} + \frac{i\dot{X}^\mu\dot{\alpha}\psi_\mu}{2e} + \frac{i\ddot{X}^\mu\alpha\psi_\mu}{2e} - \frac{i\dot{X}^\mu\dot{e}\alpha\psi_\mu}{2e^2} \\
=& \frac{d}{d\tau}\left(\frac{i\alpha\dot{X}^\mu\psi_\mu}{2e}\right)
\end{aligned}$$

C Zeta-Function Regularization

To compute the path integral with periodic boundary conditions, we first perform integration by parts on the action:

$$\int_{Q(0)=Q(1)} \mathcal{D}Q \exp\left(-\frac{1}{4T}\int_0^1 d\tau \left(\frac{dQ}{d\tau}\right)^2\right) = \int_{Q(0)=Q(1)} \mathcal{D}Q \exp\left(-\frac{1}{4T}\int_0^1 d\tau\ Q\left(-\frac{d}{d\tau^2}\right)Q\right) \tag{478}$$

We can re-express this integral in Gaussian form if we expand $Q^\mu(\tau)$ on a basis of orthogonal eigenfunctions of the operator $-\frac{d}{dt^2}$, which are periodic on the loop and obey the boundary conditions $Q(0) = Q(1)$:

$$Q^\mu(\tau) = \sum_{n=1}^{\infty} R_n^\mu \sin(n\pi\tau) \tag{479}$$

For some constant 4-vector R_n^μ.

To see how the functional integral measure changes under a change of co-ordinates, we consider the norm of Q^2 along the worldline. The norm is defined in terms of the metric on the worldline, which we saw in Section 2.6 is $g = -e^2$:

$$||Q||_g = \int d\tau \sqrt{-gQ\cdot Q} \tag{480}$$

The norm we need to consider is

$$\left|\left|Q^2\right|\right|_g = -\int_0^1 d\tau g\ Q^2 \tag{481}$$

$$\left|\left|Q^2\right|\right|_g = \int_0^1 d\tau e\ Q^2 \tag{482}$$

$$\left|\left|Q^2\right|\right|_g = 2T\int_0^1 d\tau\ Q^2 \tag{483}$$

$$= 2TR_n^2 \sum_{n=1}^{\infty}\int_0^1 d\tau\ \sin^2(n\pi\tau) \tag{484}$$

$$= \sum_{n=1}^{\infty} R_n^2 T \tag{485}$$

Now redefine the constant used to define the basis of eigenfunctions such that:

$$R_n \to S_n\sqrt{\frac{1}{T}} \tag{486}$$

$$Q^\mu(\tau) = \sqrt{\frac{1}{T}}\sum_{n=1}^{\infty} S_n^\mu \sin(n\pi\tau) \tag{487}$$

$$\left|\left|Q^2\right|\right|_g = \sum_{n=1}^{\infty} S_n^2 \tag{488}$$

This basis allows us to discretize the functional integral measure into an infinite product of ordinary integrals over S_n:

$$\int \mathcal{D}Q(\tau) \to \prod_{n=1}^{\infty}\int dS_n \tag{489}$$

We now substitute line the expansion for $Q^\mu(\tau)$ on line (487) into the quantum fluctuation path integral (478):

$$\int\limits_{Q(0)=Q(1)} \mathcal{D}Q \exp\left(-\frac{1}{4T}\int_0^1 d\tau \left(\frac{dQ}{d\tau}\right)^2\right) = \int \mathcal{D}Q \exp\left(-\frac{1}{4T}\int_0^1 d\tau\ Q\left(-\frac{d}{d\tau^2}\right)Q\right) \tag{490}$$

$$= \prod_{n=1}^{\infty}\int dS_n \exp\left(-\frac{1}{4T}\int_0^1 d\tau\ Q\left(-\frac{d}{d\tau^2}\right)Q\right) \tag{491}$$

$$= \prod_{n=1}^{\infty}\int dS_n \exp\left(-\sum_{n=1}^{\infty}\frac{(\pi n)^2}{4T^2}S_n^2\int_0^1 d\tau \sin^2(n\pi\tau)\right) \tag{492}$$

$$= \prod_{n=1}^{\infty}\int dS_n \exp\left(-\sum_{n=1}^{\infty}\frac{(\pi n)^2}{8T^2}S_n^2\right) \tag{493}$$

$$= \prod_{n=1}^{\infty}\int dS_n \exp\left(-\underbrace{\frac{(\pi n)^2}{8T^2}}S_n^2\right) \tag{494}$$

$$= \prod_{n=1}^{\infty}\sqrt{\frac{\pi}{\frac{(\pi n)^2}{8T^2}}} \tag{495}$$

$$= \left(\prod_{n=1}^{\infty}\frac{\pi n^2}{8T^2}\right)^{-1/2} \tag{496}$$

The indicated terms can also be considered eigenvalues, which we will denote as:

$$\lambda_n = \frac{\pi n^2}{8T^2} \tag{497}$$

We will consider these to be eigenvalues of an operator that we will denote as D. As we have an infinite product of eigenvalues, the result can be expressed as a functional determinant, which can be represented by the determinant of a diagonal matrix:

$$\prod_{n=1}^{\infty}\lambda_n \to \det\begin{pmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \ddots\end{pmatrix} \to \mathrm{Det}(D) \tag{498}$$

Note that we use Det for the infinite dimensional operators and det for the finite dimensional matrix that represents it. The quantum fluctuation path integral can now be expressed as:

$$\int\limits_{Q(0)=Q(1)} \mathcal{D}Q \exp\left(-\frac{1}{4T}\int_0^1 d\tau \left(\frac{dQ}{d\tau}\right)^2\right) = (\mathrm{Det}(D))^{-1/2} \tag{499}$$

The determinant contains a *divergent* infinite product but we can regularize this through the *zeta-function regularization* method, in which we relate the determinant to the Riemann-zeta function:

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z} \qquad \text{Convergent for } \Re(z) > 0 \tag{500}$$

For an operator D whose eigenvalues λ_n satisfy $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$, it's possible to define the *spectral zeta-function* $\zeta_D(z)$, which replaces the sum over integers in the Riemann-zeta function with a sum over the eigenvalues λ_n:

$$\zeta_D(z) = \sum_{n=1}^{\infty} \lambda_n^{-z} \qquad \text{Convergent for sufficiently large } \Re(z) \tag{501}$$

$$= \sum_{n=1}^{\infty} \left(\frac{\pi n^2}{8T^2}\right)^{-z} \tag{502}$$

$$= \left(\frac{8T^2}{\pi}\right)^{z} \zeta(2z) \tag{503}$$

It is known that the spectral-zeta function can be analytically continued to the entire complex plane (except possibly at a finite set of points) if D is an elliptic differential operator [35]. Next, we manipulate the determinant (498) as follows:

$$\mathrm{Det}(D) = \prod_{n=1}^{\infty} \lambda_n = \exp\left(\log\left(\prod_{n=1}^{\infty} \lambda_n\right)\right) \tag{504}$$

$$= \exp\left(\sum_{n=1}^{\infty} \log(\lambda_n)\right) \tag{505}$$

The sum of log of eigenvalues of D can be related to the derivative of its spectral zeta-function:

Box 9. *Derivative of the spectral zeta-function:*

$$\zeta_D(z) = \sum_{n=1}^{\infty} \lambda_n^{-z} \tag{506}$$

$$\frac{d\zeta_D(z)}{dz} = \sum_{z=1}^{\infty} \frac{d}{dz} \lambda_n^{-z} \tag{507}$$

$$= \sum_{z=1}^{\infty} \frac{d}{dz} e^{\ln\left(\lambda_n^{-z}\right)} \tag{508}$$

$$= \sum_{z=1}^{\infty} e^{\ln\left(\lambda_n^{-z}\right)} \frac{d}{dz} \ln \lambda_n^{-z} \tag{509}$$

$$= \sum_{z=1}^{\infty} \lambda_n^{-z} \frac{d}{dz} \left(-z \ln(\lambda_n)\right) \tag{510}$$

$$= \sum_{z=1}^{\infty} -\lambda_n^{-z} \ln \lambda_n \tag{511}$$

Using Box 9, we may express line (505) as:

$$\mathrm{Det}(D) = \exp\left(\sum_{n=1}^{\infty} \log \lambda_n\right) = \exp\left(-\left.\frac{d\zeta_D(z)}{dz}\right|_{z=0}\right) \qquad \text{We use the fact that the spectral zeta-function for an elliptic differential operator is regular at } z = 0 \text{ [35]} \tag{512}$$

We now differentiate our expression for $\zeta_D(z)$ on line (503):

$$\frac{d\zeta_D(z)}{dz} = \frac{d}{dz}\left(\left(\frac{8T^2}{\pi}\right)^z \zeta(2z)\right) \tag{513}$$

$$= \zeta(2z)\frac{d}{dz}\left(\frac{8T^2}{\pi}\right)^z + \left(\frac{8T^2}{\pi}\right)^z \frac{d\zeta(2z)}{dz} \tag{514}$$

$$= \zeta(2z)\frac{d}{dz}e^{z\log\left(\frac{8T^2}{\pi}\right)} + \left(\frac{8T^2}{\pi}\right)^z \frac{d\zeta(2z)}{dz} \tag{515}$$

$$= \left(\frac{8T^2}{\pi}\right)^z \log\left(\frac{8T^2}{\pi}\right)\zeta(2z) + \left(\frac{8T^2}{\pi}\right)^z \frac{d\zeta(2z)}{dz} \tag{516}$$

The determinant (512) can now be expressed as:

$$\mathrm{Det}(D) = \exp\left(-\frac{d\zeta_D(z)}{dz}\bigg|_{z=0}\right) = \exp\left(\log\left(\frac{8T^2}{\pi}\right)\zeta(0) + \frac{d\zeta(2z)}{dz}\bigg|_{z=0}\right) \tag{517}$$

We insert the known values of the Riemann-zeta function and its derivative at $z = 0$, which are $\zeta(0) = -\frac{1}{2}$ and $\frac{d\zeta(2z)}{dz}\Big|_{z=0} = -\frac{1}{2}\log(2\pi)$.

$$\mathrm{Det}(D) = \exp\left(-\frac{d\zeta_D(z)}{dz}\bigg|_{z=0}\right) = \exp\left(-\frac{1}{2}\log\left(\frac{8T^2}{\pi}\right) - \frac{1}{2}\log(2\pi)\right) \tag{518}$$

$$= \frac{1}{4T} \tag{519}$$

Using (499), we conclude:

$$\int_{Q(0)=Q(1)} \mathcal{D}Q \exp\left(-\frac{1}{4T}\int_0^1 d\tau \left(\frac{dQ}{d\tau}\right)^2\right) = (4T)^{-1/2} \tag{520}$$

In D-dimensions, the result of the integral is $(4T)^{-D/2}$

4 Higher-dimension Gaussian Integrals

4.1 Integral over real co-ordinates

We wish to evaluate path integrals of the form:

$$\int \mathcal{D}X \exp\left(-\int_0^T d\tau \ \dot{X}^2\right) \tag{521}$$

First, we perform integration by parts as the boundary conditions will always be such that we can dispose of the surface term:

$$\int \mathcal{D}X \exp\left(-\int_0^T d\tau \ X\left(-\frac{d^2}{dt^2}\right)X\right) \tag{522}$$

We can *discretize* the integral by dividing the integration range into n intervals. We identify X as a vector whose elements are the discretized co-ordinates:

$$X \to \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \equiv x_i \equiv \mathbf{x} \tag{523}$$

We also identify the differential operator in the path integral with an n-dimensional matrix:

$$-\frac{d^2}{dt^2} \to M_{ij} \equiv \mathbf{M} \tag{524}$$

The action can then be expressed as a sum in terms of the elements of $\mathbf{x}$ and $\mathbf{M}$, and the path integral can be expressed (up to a normalization constant) as a product of n ordinary integrals:

$$\int \mathcal{D}X \exp\left(-\int_0^T d\tau \ \dot{X}^2\right) \propto I_1 = \int\limits_{-\infty}^{\infty} \prod_{i=1}^{n} dx_i \exp\left(-\sum_{i,j} x_i M^{ij} x_j\right) = \int\limits_{-\infty}^{\infty} \prod_{i=1}^{n} dx_i \exp\left(-\mathbf{x}^T\mathbf{M}\mathbf{x}\right) \tag{525}$$

We can take $\mathbf{M}$ to be symmetric, as any anti-symmetric part of the $\mathbf{M}$ won't contribute to a Gaussian integral. We also take $\mathbf{M}$ as real. This allows us to diagonalize $\mathbf{M}$ to $\mathbf{D}$ using a rotation matrix $\mathbf{R} \in O(n)$:

$$\mathbf{M} = \mathbf{R}^T\mathbf{D}\mathbf{R} \tag{526}$$

$$\mathbf{D} = \mathrm{diag}\left(m_1, m_2, \cdots, m_n\right) \tag{527}$$

Where m_i are the eigenvalues of $\mathbf{M}$. We substitute this into (525):

$$I_1 = \int\limits_{-\infty}^{\infty} \prod_{i=1}^{n} dx_i \exp\left(-\mathbf{x}^T\mathbf{R}^T\mathbf{D}\mathbf{R}\mathbf{x}\right) \tag{528}$$

$$= \int\limits_{-\infty}^{\infty} \prod_{i=1}^{n} dx_i \exp\left(-\left(\mathbf{R}\mathbf{x}\right)^T \mathbf{D}\left(\mathbf{R}\mathbf{x}\right)\right) \tag{529}$$

We can define the rotated co-ordinate system such that $\mathbf{y} = \mathbf{Rx}$. To change the integration variables, we introduce the Jacobian factor $|\det(\mathbf{R})^T|$ but the determinant of an O(n) matrix is 1.

$$I_1 = \int_{-\infty}^{\infty} \prod_{i=1}^{n} dy_i \left|\det\left(\mathbf{R}^T\right)\right| \exp\left(-\mathbf{y}^T\mathbf{D}\mathbf{y}\right) \tag{530}$$

$$= \int_{-\infty}^{\infty} \prod_{i=1}^{n} dy_i \exp\left(-\mathbf{y}^T\mathbf{D}\mathbf{y}\right) \tag{531}$$

$$= \int_{-\infty}^{\infty} \prod_{i=1}^{n} dy_i \exp\left(-\sum_{j=1}^{n} m_j y_j^2\right) \tag{532}$$

$$= \int_{-\infty}^{\infty} \prod_{i=1}^{n} dy_i \prod_{j=1}^{n} \exp\left(-m_j y_j^2\right) \tag{533}$$

$$= \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} dy_i \exp\left(-m_j y_j^2\right) \right) \tag{534}$$

$$= \prod_{i=1}^{n} \sqrt{\frac{\pi}{m_i}} \tag{535}$$

$$= \sqrt{\frac{\pi^n}{\prod_{i=1}^{n} m_i}} \tag{536}$$

$$= \sqrt{\frac{\pi^n}{\det\left(\mathbf{D}\right)}} \tag{537}$$

$$= \sqrt{\frac{\pi^n}{\det\left(\mathbf{R}^T\mathbf{M}\mathbf{R}\right)}} \tag{538}$$

$$= \sqrt{\frac{\pi^n}{\left(\det\mathbf{R}^T\right)\left(\det\mathbf{M}\right)\left(\det\mathbf{R}\right)}} \tag{539}$$

$$= \sqrt{\frac{\pi^n}{\left(\det\mathbf{M}\right)}} \tag{540}$$

$$= \sqrt{\frac{\pi^n}{\mathrm{Det}\left(-\frac{d^2}{dt^2}\right)}} \tag{541}$$

In the final line, the finite-dimensional matrix trace det became an infinite-dimensional operator trace Det. In D-dimensions, the Gaussian integral on line (535) is raised to the power of D so the result of the integral is:

$$I_1 = \left(\frac{\pi^n}{\mathrm{Det}\left(-\frac{d^2}{dt^2}\right)} \right)^{D/2} \tag{542}$$

4.2 Integral over complex co-ordinates

We evaluate the Gaussian intergral:

$$I_2 = \left(\int_{-\infty}^{\infty} \prod_{i=1}^{n} dz_i\right) \left(\int_{\infty}^{\infty} \prod_{j=1}^{n} dz_j^*\right) \exp\left(-\mathbf{z}^\dagger \mathbf{H} \mathbf{z}\right) \tag{543}$$

Where z_i are complex variables and $\mathbf{H}$ is a Hermitian matrix. This is similar to the integral in Appendix 4.1, but now we use a unitary matrix $\mathbf{U}$ to diagonalize $\mathbf{H}$ such that $\mathbf{U}^\dagger \mathbf{H} \mathbf{U}$ and $\mathbf{E} = \operatorname{diag}(h_1, h_2, \cdots, h_n)$ where h_i are the eigenvalues of $\mathbf{H}$. We substitute this into (543) and perform the same procedure as in Appendix 4.1, changing co-ordinates to $\mathbf{w} = \mathbf{U}\mathbf{z}$:

$$I_2 = \int_{-\infty}^{\infty} \prod_{i=1}^{n} dz_i \prod_{j=1}^{n} dz_j^* \exp\left(-\mathbf{z}^\dagger \mathbf{U}^\dagger \mathbf{E} \mathbf{U} \mathbf{z}\right) \tag{544}$$

$$= \int_{-\infty}^{\infty} \prod_{i=1}^{n} dw_i \prod_{j=1}^{n} dw_j^* \left|\det\left(\mathbf{U}^\dagger\right)\right| \left|\det\left(\mathbf{U}\right)\right| \exp\left(-\mathbf{w}^\dagger \mathbf{E} \mathbf{z}\right) \tag{545}$$

$$= \int_{-\infty}^{\infty} \prod_{i=1}^{n} dw_i \prod_{j=1}^{n} dw_j^* \exp\left(-\mathbf{w}^\dagger \mathbf{E} \mathbf{z}\right) \tag{546}$$

$$= \int_{-\infty}^{\infty} \prod_{i=1}^{n} dw_i \prod_{j=1}^{n} dw_j^* \exp\left(-\sum_{k=1}^{n} h_k \left|w_j\right|^2\right) \tag{547}$$

We change variables once more, expressing w in terms of its real and imaginary components $w = u + iv$. We also see from $dw^*\ dw = 2\ du\ dv$ that there is a Jacobian factor of 2 for *each* integral measure.

$$I_2 = 2^n \int_{-\infty}^{\infty} \prod_{i=1}^{n} du_i \prod_{j=1}^{n} dv_j \exp\left(-\sum_{k=1}^{n} h_k \left(u_k^2 + v_k^2\right)\right) \tag{548}$$

$$= 2^n \left(\int_{-\infty}^{\infty} \prod_{i=1}^{n} du_i \exp\left(-\sum_{k=1}^{n} h_k u_k^2\right)\right) \left(\int_{\infty}^{\infty} \prod_{j=1}^{n} dv_j \exp\left(-\sum_{k=1}^{n} h_k v_k^2\right)\right) \tag{549}$$

$$= 2^n \prod_{i=1}^{n} \left(\int_{-\infty}^{\infty} du_i \exp\left(-h_i u_k^2\right)\right) \prod_{j=1}^{n} \left(\int_{-\infty}^{\infty} dv_j \exp\left(-h_j v_k^2\right)\right) \tag{550}$$

$$= 2^n \left(\prod_{i=1}^{n} \sqrt{\frac{\pi}{h_i}}\right) \left(\prod_{j=1}^{n} \sqrt{\frac{\pi}{h_j}}\right) \tag{551}$$

$$= \frac{(2\pi)^n}{\det\left(\mathbf{H}\right)} \tag{552}$$

4.3 Integral with source term

A path integral involving a source term takes the form:

$$\int \mathcal{D}X \exp \int_0^T d\tau \left(-\frac{\dot{X}^2}{4} + J(\tau) \cdot X \right) = \int \mathcal{D}X \exp \left(-\int_0^T d\tau \left(-X \cdot M \cdot X - J(\tau) \cdot X \right) \right) \tag{553}$$

We have integrated by parts and defined the operator M:

$$M = \left(\frac{1}{2} \frac{\partial}{\partial t} \right)^2 \tag{554}$$

We represent the terms of this expression as vectors and matrices as we did in Appendix 4.3:

$$I_3 = \int\limits_{-\infty}^{\infty} \prod_{i=1}^{n} dx_i \exp \left(-\mathbf{x}^T \mathbf{M} \mathbf{x} + \mathbf{J}^T \mathbf{x} \right) \tag{555}$$

We can complete the square:

$$-\mathbf{x}^T \mathbf{M} \mathbf{x} + \mathbf{J}^T \mathbf{x} = -\left(\mathbf{x} - \frac{1}{2} \mathbf{M}^{-1} \mathbf{J} \right)^T \mathbf{M} \left(\mathbf{x} - \frac{1}{2} \mathbf{M}^{-1} \mathbf{J} \right) + \frac{1}{4} \mathbf{J}^T \mathbf{M}^{-1} \mathbf{J} \tag{556}$$

In a Gaussian integral, we may perform a linear shift on the co-ordinates $\tilde{\mathbf{x}} \equiv \mathbf{x} - \frac{1}{2}\mathbf{M}^{-1}\mathbf{J}$ without introducing a Jacobian term in transforming the integral measures. The integral can now be expressed as:

$$I_3 = \int\limits_{-\infty}^{\infty} \prod_{i=1}^{n} d\tilde{x}_i \exp \left(-\left(\mathbf{x} - \frac{1}{2} \mathbf{M}^{-1} \mathbf{J} \right)^T \mathbf{M} \left(\mathbf{x} - \frac{1}{2} \mathbf{M}^{-1} \mathbf{J} \right) + \frac{1}{4} \mathbf{J}^T \mathbf{M}^{-1} \mathbf{J} \right) \tag{557}$$

$$= \exp \left(\frac{1}{4} \mathbf{J}^T \mathbf{M}^{-1} \mathbf{J} \right) \underbrace{\int\limits_{-\infty}^{\infty} \prod_{i=1}^{n} d\tilde{x}_i \exp \left(-\tilde{\mathbf{x}}^T \mathbf{M} \tilde{\mathbf{x}} \right)} \tag{558}$$

This indicated term has the exact form of the integral I_1 which we computed in Appendix 4.1. We can therefore the result of this integral as:

$$\frac{\int \mathcal{D}X \exp \int_0^T d\tau \left(-\frac{\dot{X}^2}{4} + J(\tau) \cdot X \right)}{\int \mathcal{D}X \exp \int_0^T d\tau \left(-\frac{\dot{X}^2}{4} \right)} \rightarrow \frac{\int_{-\infty}^{\infty} \prod_{i=1}^{n} dx_i \exp \left(-\mathbf{x}^T \mathbf{M} \mathbf{x} + \mathbf{J}^T \mathbf{x} \right)}{\int_{-\infty}^{\infty} \prod_{i=1}^{n} dx_i \exp \left(-\mathbf{x}^T \mathbf{M} \mathbf{x} \right)} \tag{559}$$

$$= \exp \left(\frac{1}{4} \mathbf{J}^T \mathbf{M}^{-1} \mathbf{J} \right) \tag{560}$$

$$\rightarrow \exp \left(-\int_0^T d\tau \int_0^T d\tau' J(\tau) \cdot \left\langle \tau_i \left| \left(\frac{\partial}{\partial \tau} \right)^{-2} \right| \tau_j \right\rangle \cdot J(\tau') \right) \tag{561}$$

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