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Exercices supplémentaires : Calcul intégral

Exercice 1: Primitives

Donner les primitives des fonctions suivantes :

a) $f(x) = 3x^2 + 2x + 1$
$F(x) = x^3 + x^2 + x + c$

b) $f(x) = \sin(x)$
$F(x) = -\cos(x) + c$

c) $f(x) = \dfrac{1}{x}$
$F(x) = \ln(|x|) + c$

d) $f(x) = x - \dfrac{1}{x^2}$
$F(x) = \dfrac{x^2}{2} + \dfrac{1}{x} + c$

e) $f(x) = -x^2 + x$
$F(x) = -\dfrac{x^3}{3} + \dfrac{x^2}{2} + c$

f) $f(x) = \dfrac{1}{x^3}$
$F(x) = -\dfrac{1}{2x^2} + c$

g) $f(x) = \dfrac{x^4 + 1}{x^2}$
$F(x) = \dfrac{x^4 - 3}{3x} + c$

h) $f(x) = 3\sin(x) + 2\cos(x)$
$F(x) = -3\cos x + 2\sin x + c$

i) $f(x) = 2(2x+1)^3$
$F(x) = \dfrac{(2x+1)^4}{4} + c$

j) $f(x) = (3x+1)^{-5}$
$F(x) = -\dfrac{(3x+1)^{-4}}{12} + c$

k) $f(x) = (-2x+1)^5$
$F(x) = -\dfrac{(-2x+1)^4}{8} + c$

l) $f(x) = \dfrac{2x+1}{(x^2+x+1)^4}$
$F(x) = -\dfrac{1}{3(x^2+x+1)^3} + c$

m) $f(x) = \sin(x)\cos^3(x)$
$F(x) = -\dfrac{\cos^4(x)}{4} + c$

n) $f(x) = \dfrac{\ln^2(x)}{x}$
$F(x) = \dfrac{\ln^3(x)}{3} + c$

o) $f(x) = \dfrac{1}{\sqrt{x+1}}$
$F(x) = 2\sqrt{x+1} + c$

p) $f(x) = \dfrac{3x}{\sqrt{x^2+1}}$
$F(x) = 3\sqrt{x^2+1} + c$

q) $f(x) = \dfrac{1}{x^2\sqrt{1+\frac{1}{x}}}$
$F(x) = -2\sqrt{1+\dfrac{1}{x}} + c$

r) $f(x) = 3\sin(3x + \dfrac{\pi}{2})$
$F(x) = \sin(3x) + c$

s) $f(x) = x\cos(x^2 + \pi)$
$F(x) = -\dfrac{1}{2}\sin(x^2) + c$

t) $f(x) = \dfrac{\sin(\sqrt{x})}{\sqrt{x}}$
$F(x) = -2\cos(\sqrt{x}) + c$

u) $f(x) = \dfrac{2x^2 + 3x + 5}{x}$
$F(x) = x^2 + 3x + 5\ln(x) + c$

v) $f(x) = \dfrac{\ln(x)}{x}$
$F(x) = \dfrac{\ln^2(x)}{2} + c$

w) $f(x) = \dfrac{\mathrm{e}^x}{\mathrm{e}^x + 1}$
$F(x) = \ln(\mathrm{e}^x + 1) + c$

x) $f(x) = \dfrac{1}{\mathrm{e}^{2x}}$
$F(x) = -\dfrac{1}{2\mathrm{e}^{2x}} + c$

y) $f(x) = \dfrac{\sin(x)}{2 + \cos(x)}$
$F(x) = -\ln(2 + \cos(x)) + c$

z) $f(x) = \dfrac{x^3}{1 + x^2}$
$F(x) = \dfrac{x^2}{2} - \dfrac{1}{2}\ln(1 + x^2) + c$

Exercice 2: Calcul d'intégrales

Calculer les intégrales suivantes :

a) $I = \int_0^3 (x+4)\mathrm{d}x$

$$I = \left[\frac{x^2}{2} + 4x\right]_0^3 = \frac{9}{2} + 12$$

b) $I = \int_{-1}^1 (2t^2 - 1)\mathrm{d}t$

$$I = \left[\frac{2t^3}{3} - t\right]_{-1}^1 = \frac{4}{3} - 2$$

c) $I = \int_1^2 \frac{3}{\sqrt{t}}\mathrm{d}t$

$$I = \left[6\sqrt{t}\right]_1^2 = 6(\sqrt{2} - 1)$$

d) $I = \int_0^\pi \sin(t)\mathrm{d}t$

$$I = [-\cos(t)]_0^\pi = 2$$

e) $I = \int_0^1 (2x+3)(x^2+3x-5)\mathrm{d}x$

$$I = \frac{1}{2}\left[(x^2+3x-5)^2\right]_0^1 = -12$$

f) $I = \int_{-1}^1 \frac{2t+1}{(t^2+t+1)^2}\mathrm{d}t$

$$I = \left[\frac{-1}{t^2+t+1}\right]_{-1}^1 = \frac{2}{3}$$

g) $I = \int_0^\pi \sin^2(t)\mathrm{d}t$

$$I = \frac{1}{2}\left[x - \frac{\sin(2x)}{2}\right]_0^\pi = \frac{\pi}{2}$$

h) $I = \int_{-1}^0 \frac{2t}{t^2+1}\mathrm{d}t$

$$I = \left[\frac{-1}{t^2+1}\right]_{-1}^0 = -\frac{1}{2}$$

i) $I = \int_1^2 x\ln x\mathrm{d}x$

Poser $u' = x$ et $v = \ln x$. $I = 2\ln 2 - \frac{3}{4}$

j) $I = \int_0^1 (2x+1)\mathrm{e}^x\mathrm{d}x$

Poser $u = 2x+1$ et $v' = \mathrm{e}^x$. $I = 1 + \mathrm{e}$

k) $I = \int_1^{\mathrm{e}} \frac{\ln x}{x^2}\mathrm{d}x$

Poser $u = \ln x$ et $v' = \frac{1}{x^2}$. $I = \frac{\mathrm{e}-2}{\mathrm{e}}$

l) $I = \int_1^x \ln t\mathrm{d}t$

$$I = [t\ln t - t]_1^x = x\ln x - x + 1$$

Exercice 3: Primitives

Donner les primitives des fonctions suivantes :

a) $f(x) = \frac{1}{x^4 - x}$

$$F(x) = \int \frac{x^2}{x^3-1} - \frac{1}{x}\mathrm{d}x$$
$$= \frac{1}{3}\ln(x^3-1) - \ln(x) + c$$

b) $f(x) = \frac{3x+1}{x^2-1}$

$$F(x) = \int \frac{1}{1+x} + \frac{2}{x-1}\mathrm{d}x$$
$$= 2\ln(x-1) + \ln(x+1) + c$$

Exercice 4: Changement de variables

Calculer les intégrales suivantes :

a) $I = \int_0^1 \sqrt{1-x^2}\mathrm{d}x$

Poser le changement $x = \sin(u)$. Alors $I = \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 u}\cos u\ \mathrm{d}u = \left[\frac{\sin(2u)}{2} + \frac{u}{2}\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$

b) $I = \int_0^{\frac{\pi}{4}} \frac{\tan x}{\cos x\,(\cos x + \sin x)}\mathrm{d}x$

Poser le changement $u = \tan x$. Alors $I = \int_0^1 \frac{u}{u+1}\mathrm{d}u = 1 - \ln 2$