On the Denoising of Cross-Spectral Matrices for (Aero)Acoustic Applications

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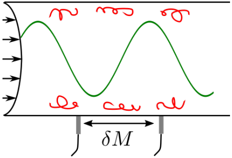
Context

- Unwanted random noise:
    - electronic, ambient, flow-induced,...
    - short correlation lengths



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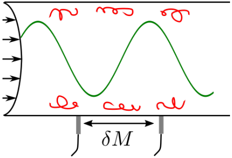

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    - physical removal : mic recession, porous treatment, vibrating structure filtering. . .
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    - wavenumber filtering → needs high spatial sampling
    - diagonal removal → underestimation of source level
    - exploit noise/signal properties & solve an optimization problem

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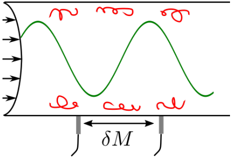

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Cross-Spectral Matrix (covariance of Fourier component):

$$\boldsymbol{S}_{pp} = \frac{1}{N_s} \sum_i \boldsymbol{p}_i \boldsymbol{p}_i^H$$

- Hermitian (conjugate symmetric)
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- Rank of $\boldsymbol{S}_{aa}$ = number of uncorrelated monopoles

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For $N_s \to \infty$

- Short correlation length : off-diagonal elements of $S_{nn} \to 0$
- Independent signal/noise : cross-terms $\to 0$

How to separate signal part from noise ?

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- What is the influence on denoising performance of :
  - noise level,
  - number of snapshots,
  - number of sources ?

Diagonal Reconstruction

"Remove as much noise as possible as long as denoised CSM remains positive"

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Convex optimization (Hald, 2017)

$$\text{maximize } \|\boldsymbol{\sigma}_n^2\|_1 \;\; \text{subject to } \; \boldsymbol{S}_{pp} - \mathrm{diag}\left(\boldsymbol{\sigma}_n^2\right) \geq 0$$

Problem solved with CVX Matlab toolbox

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$$\text{maximize } \|\boldsymbol{\sigma}_n^2\|_1 \text{ subject to } \boldsymbol{V}_{(k-1)}^H \left(\boldsymbol{S}_{pp} - \operatorname{diag}\left(\boldsymbol{\sigma}_n^2\right)_{(k)}\right) \boldsymbol{V}_{(k-1)} \geq 0$$

$\boldsymbol{V}_{(k-1)}$: eigenvectors of $\boldsymbol{S}_{pp} - \operatorname{diag}\left(\boldsymbol{\sigma}_n^2\right)_{(1,\ldots,k-1)}$

Solved with *linprog* Matlab function

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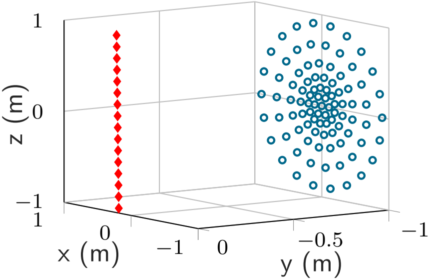
Alternating Projections (Leclère et al., 2015)

$$\boldsymbol{S}_{pp_{(k+1)}} := \bar{\boldsymbol{S}}_{pp_{(0)}} + \boldsymbol{V}_{(k)}^H \boldsymbol{s}_{(k)}^{+} \boldsymbol{V}_{(k)}$$

$\boldsymbol{V}_{(k)}^H$ and $\boldsymbol{s}_{(k)}$: eigenvectors/values of $\boldsymbol{S}_{pp_{(k)}}$

Diagonal Reconstruction – Test case

- Default parameters:
  - 20 uncorrelated free field monopoles: ♦
  - 93 receivers: ○
  - SNR: 10 dB
  - 10^4 snapshots
  - frequency: 15 kHz



From a benchmark case provided by PSA3

- Varying parameters:
  - number of ♦ (rank of $\boldsymbol{S}_{aa}$)
  - SNR
  - number of snapshots (level of extra-diagonal terms)

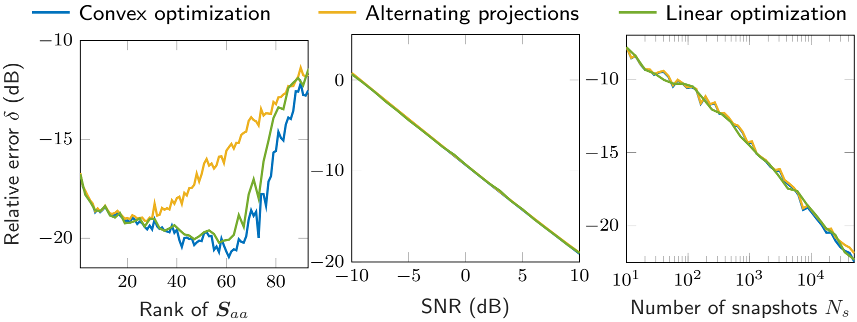
- Error on the signal CSM:

$$\delta = \frac{\| \operatorname{diag}(\boldsymbol{S}_{aa}) - \operatorname{diag}\left(\hat{\boldsymbol{S}}_{aa}\right) \|_2}{\| \operatorname{diag}(\boldsymbol{S}_{aa}) \|_2}$$

Diagonal Reconstruction

Default values:

20 sources	93 receivers	SNR: 10 dB	10^4 snapshots	frequency: 15 kHz



Select Convex Optimization (DRec) for further comparison

✓ Fast, simple code

✓ Better performance

✗ Local optimization

✗ Denoises only auto-spectra

RPCA

"Search $\boldsymbol{S}_{aa}$ as a low rank matrix and $\boldsymbol{S}_{nn}$ as a sparse matrix"

$$\text{minimize } \|\boldsymbol{S}_{aa}\|_* + \lambda\|\boldsymbol{S}_{nn}\|_1 \quad \text{subject to} \quad \boldsymbol{S}_{aa} + \boldsymbol{S}_{nn} = \boldsymbol{S}_{pp}$$

$\|\cdot\|_*$: nuclear norm (related to rank)
$\|\cdot\|_1$: ℓ_1-norm (related to sparsity)

Solved with a proximal gradient algorithm

RPCA (Wright et al., 2009)

✓ Modifies the whole CSM

✓ Widely used in image processing

✗ Local optimization

✗ Choose regularization parameter:

- L-curve criterion,
- Generalized cross validation method,
- Bayesian criterion, . . .

↪ For comparison :

- optimal λ (unknown on real case)
- "universal" constant parameter $\lambda = M^{-\frac{1}{2}} = 0.1$

1 Diagonal Reconstruction

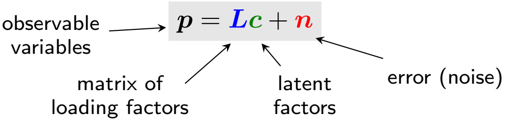
2 Robust Principal Component Analysis

3 Probabilistic Factor Analysis

4 Comparison

Probabilistic Factor Analysis

- **Latent variable model**



$$\boldsymbol{p} = \boldsymbol{L}\boldsymbol{c} + \boldsymbol{n}$$

observable variables

matrix of loading factors

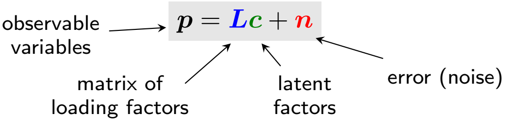
latent factors

error (noise)

  - Capture dominant correlation with fewer parameters (close to PCA)
  - Extract anisotropic noise

Probabilistic Factor Analysis

- **Latent variable model**



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$$\boldsymbol{L} \sim \mathcal{N}_{\mathbb{C}}(0, \boldsymbol{\gamma}^2) \qquad \boldsymbol{c} \sim \mathcal{N}_{\mathbb{C}}(0, \boldsymbol{I}\boldsymbol{\alpha}^2) \qquad \boldsymbol{n} \sim \mathcal{N}_{\mathbb{C}}(0, \boldsymbol{I}\boldsymbol{\sigma}^2)$$

$+$ non-informative priors : $\boldsymbol{\gamma}^2, \boldsymbol{\alpha}^2, \boldsymbol{\sigma}^2 \sim \mathcal{IG}(a_{\gamma,\alpha,\sigma}, b_{\gamma,\alpha,\sigma})$

Probabilistic Factor Analysis

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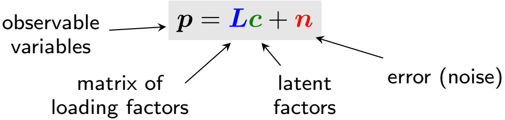

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  Iterative draws in the marginal conditional distributions of each parameter

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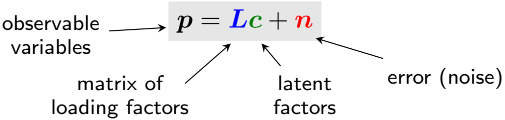

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- **Finally**, signal CSM: $$\hat{\boldsymbol{S}}_{aa} = \frac{1}{N_s} \sum_{i=1}^{N_s} \boldsymbol{L}\boldsymbol{c}_i\boldsymbol{c}_i^H\boldsymbol{L}^H$$

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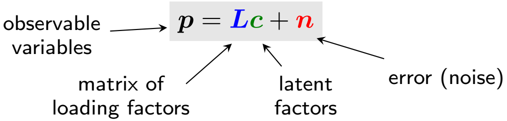

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$$p = Lc + n$$

matrix of loading factors

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PFA

✓ Global optimization

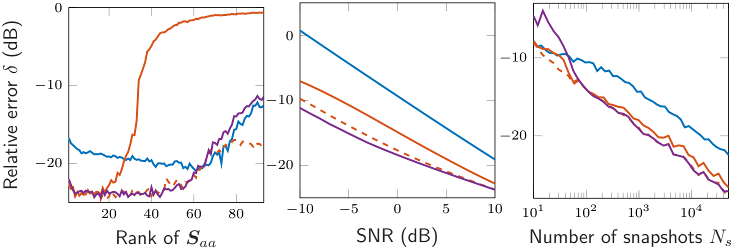
✓ Flexible model

✗ Computationally expensive

Comparison

— DRec - - RPCA, λ_{opt} — RPCA, $\lambda = M^{-\frac{1}{2}}$ — PFA

- **Homogeneous noise**



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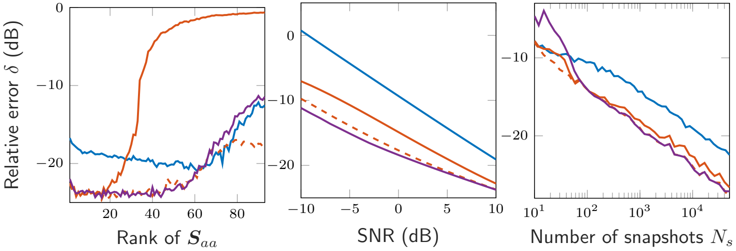

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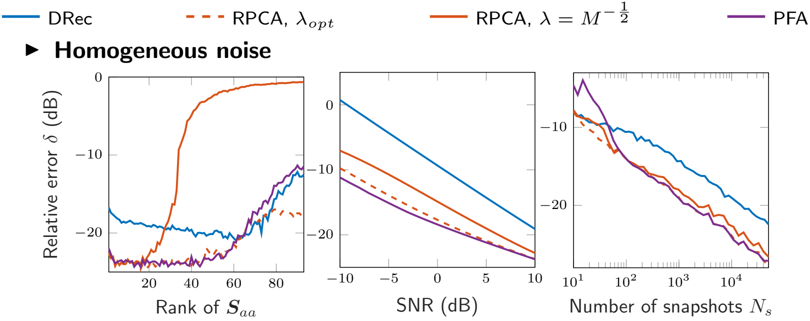

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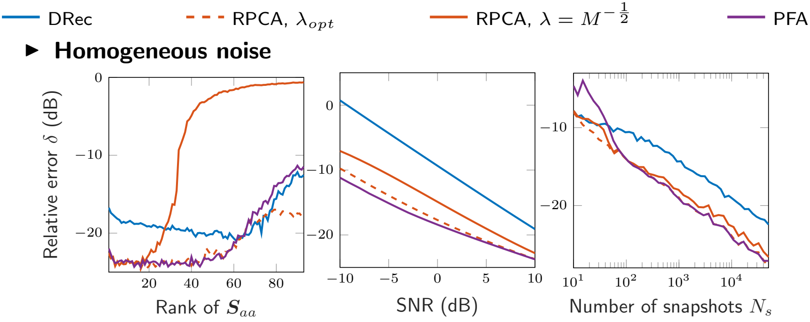

↪ For $N_{src} \geq 0.75M$: denoising problem becomes poorly conditioned

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↪ Error linearly decreases with logarithmically increasing N_s

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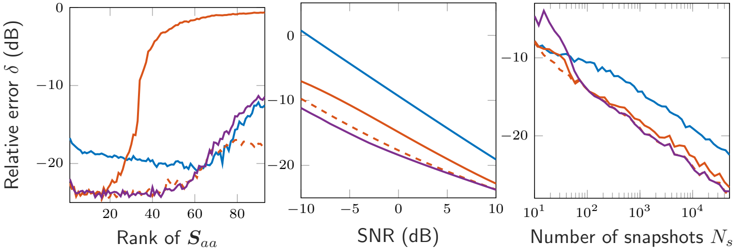
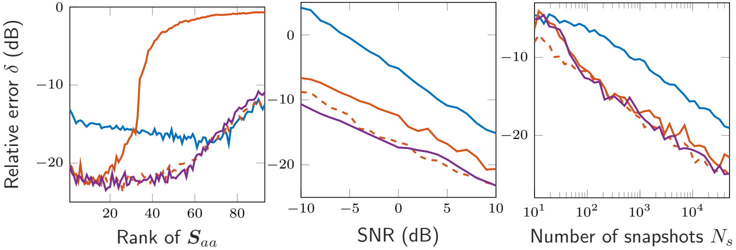


- **Heterogeneous noise**: SNR 10 dB lower on 10 random receivers



Conclusion

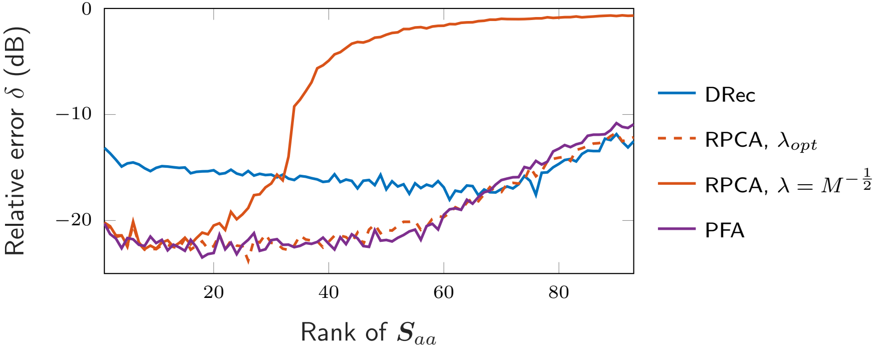
- DRec: fast and simple but error 5 dB higher for rank under 70

- PFA
  - performance similar to RPCA using λ_{opt}
  - PFA and RPCA more robust to heterogeneous noise
  - flexible model → to be adapted for correlated noise

- Future work :
  - denoising of the whole CSM
  - effect of denoising on imaging ?

Conclusion

- Choose your denoising method according to the expected number of sources



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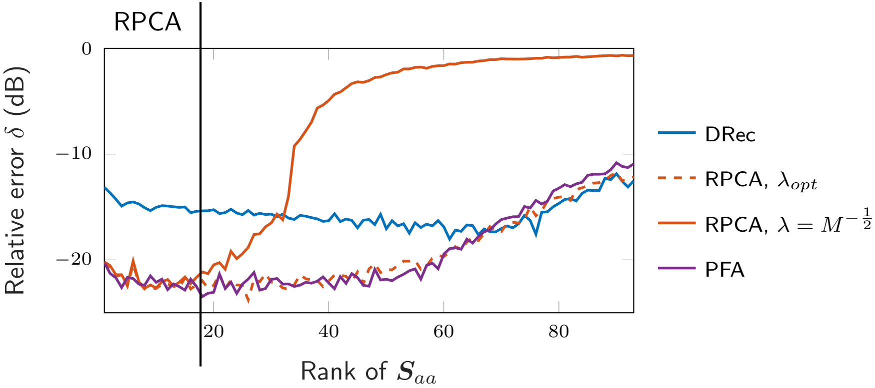

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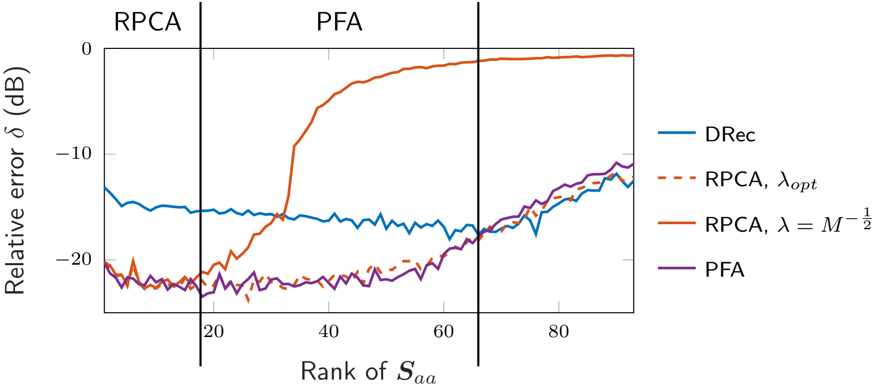

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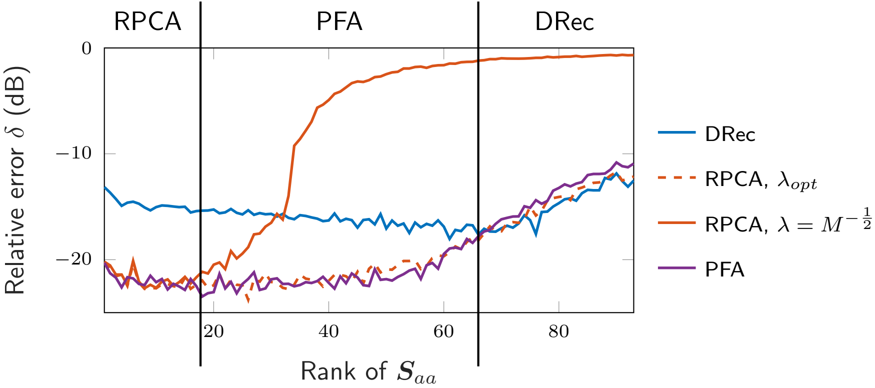


References

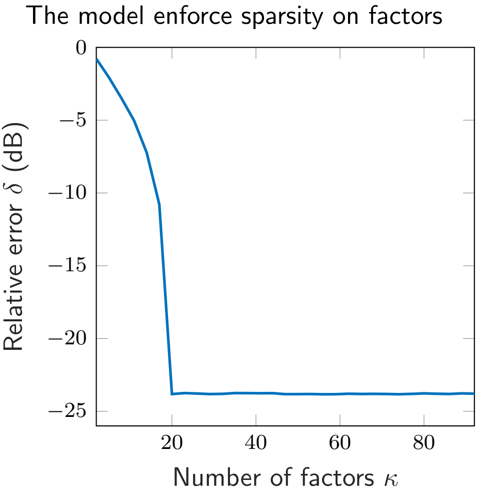
R. Dougherty. Cross spectral matrix diagonal optimization. In *6th Berlin Beamforming Conference*, 02 2016.

J. Hald. Removal of incoherent noise from an averaged cross-spectral matrix. *The Journal of the Acoustical Society of America*, 142(2): 846–854, 2017.

Q. Leclère, N. Totaro, C. Pézerat, F. Chevillotte, and P. Souchotte. Extraction of the acoustic part of a turbulent boundary layer from wall pressure and vibration measurements. In *Novem 2015 - Noise and vibration - Emerging technologies*, Proceedings of Novem 2015, page 49046, Dubrovnik, Croatia, Apr. 2015.

J. Wright, A. Ganesh, S. Rao, Y. Peng, and Y. Ma. Robust principal component analysis: Exact recovery of corrupted low-rank matrices via convex optimization. In *Advances in neural information processing systems*, pages 2080–2088, 2009.

PFA – Choosing the number of factor



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Bayésien
MCMC