On the Denoising of Cross-Spectral Matrices for (Aero)Acoustic Applications

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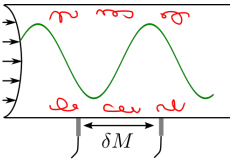
Context

- Unwanted random noise:
  - electronic, ambient, flow-induced,...
  - short correlation lengths



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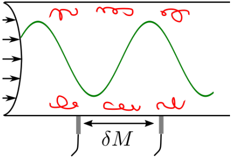

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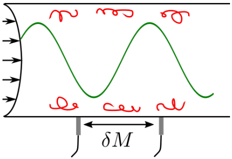

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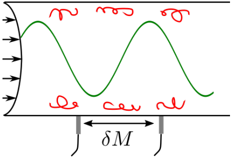

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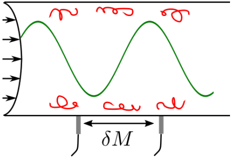

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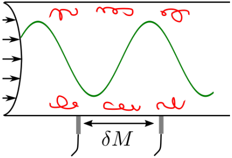

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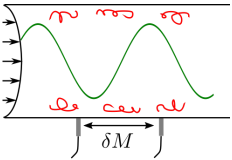

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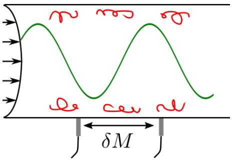

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$$\left\langle \underbrace{\boldsymbol{p}}_{\text{measured spectra}} = \underbrace{\color{green}{\boldsymbol{a}}}_{\text{source spectrum}} + \underbrace{\color{red}{\boldsymbol{n}}}_{\text{Gaussian noise}} \right\rangle_{N_s} \text{ snapshots}$$

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- Rank of $\boldsymbol{S}_{aa}$ = number of uncorrelated monopoles

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For $N_s \to \infty$

- Short correlation length : off-diagonal elements of $S_{nn} \to 0$
- Independent signal/noise : cross-terms $\to 0$

How to separate signal part from noise ?

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- What is the influence on denoising performance of :
  - noise level,
  - number of snapshots,
  - number of sources ?

1 Diagonal Reconstruction
Comparison on a test case

2 RPCA

3 Probabilistic Factor Analysis

4 Comparison

Diagonal Reconstruction

"Remove as much noise as possible as long as denoised CSM remains positive"

Convex optimization (Hald, 2017)

$$\text{maximize } \|\boldsymbol{\sigma}_n^2\|_1 \quad \text{subject to} \quad \boldsymbol{S}_{pp} - \mathrm{diag}\left(\boldsymbol{\sigma}_n^2\right) \geq 0$$

Problem solved with CVX Matlab toolbox.

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Linear optimization (Dougherty, 2016)

$$\text{maximize } \|\boldsymbol{\sigma}_n^2\|_1 \text{ subject to } \boldsymbol{V}_{(k-1)}^H \left(\boldsymbol{S}_{pp} - \operatorname{diag}\left(\boldsymbol{\sigma}_n^2\right)_{(k)}\right) \boldsymbol{V}_{(k-1)} \geq 0$$

$\boldsymbol{V}_{(k-1)}$: eigenvectors of $\boldsymbol{S}_{pp} - \operatorname{diag}\left(\boldsymbol{\sigma}_n^2\right)_{(1,\ldots,k-1)}$

Solved with *linprog* Matlab function .

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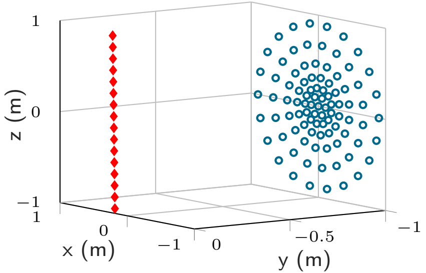
Alternating Projections (Leclère et al., 2015)

$$\boldsymbol{S}_{pp_{(k+1)}} := \bar{\boldsymbol{S}}_{pp_{(0)}} + \boldsymbol{V}_{(k)}^H \boldsymbol{s}_{(k)}^{+} \boldsymbol{V}_{(k)}$$

$\boldsymbol{V}_{(k)}^H$ and $\boldsymbol{s}_{(k)}$: eigenvectors/values of $\boldsymbol{S}_{pp_{(k)}}$.

Diagonal Reconstruction

- Default parameters:
  - 20 uncorrelated free field monopoles: ♦
  - 93 receivers: ○
  - SNR: 10 dB
  - 10^4 snapshots
  - frequency: 15 kHz



- Varying parameters:
  - number of ♦ (rank of $\boldsymbol{S}_{aa}$) : from 1 to 93
  - SNR from -10 to 10 dB
  - Number of snapshots (level of extra-diagonal terms): from 10 to 5.10^4
- Error on the signal CSM:

$$\delta = \frac{\| \operatorname{diag}(\boldsymbol{S}_{aa}) - \operatorname{diag}\left(\hat{\boldsymbol{S}}_{aa}\right) \|_2}{\| \operatorname{diag}(\boldsymbol{S}_{aa}) \|_2}$$

Diagonal Reconstruction

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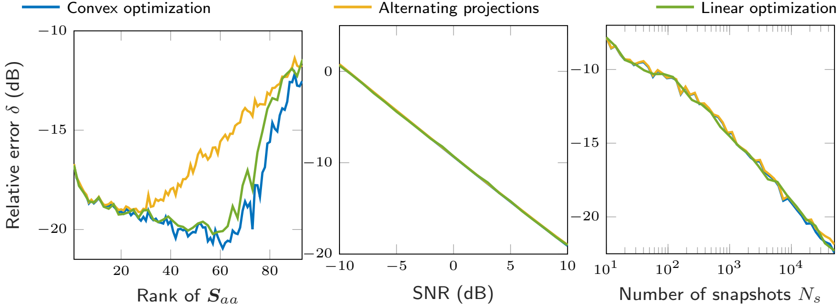

Select Convex Optimization (DRec) for further comparison

✓ Fast, simple code
✓ Better performance

✗ Local optimization
✗ Denoises only auto-spectra

RPCA

"Search $\boldsymbol{S}_{aa}$ as a low rank matrix and $\boldsymbol{S}_{nn}$ as a sparse matrix"

$$\text{minimize } \|\boldsymbol{S}_{aa}\|_* + \lambda \|\boldsymbol{S}_{nn}\|_1 \quad \text{subject to} \quad \boldsymbol{S}_{aa} + \boldsymbol{S}_{nn} = \boldsymbol{S}_{pp}$$

$\|\cdot\|_*$: nuclear norm (related to rank)
$\|\cdot\|_1$: ℓ_1-norm (related to sparsity)

Solved with a proximal gradient algorithm.

RPCA (Wright et al., 2009)

✓ Modifies the whole CSM

✗ Local optimization

✗ Choose regularization parameter:

- L-curve criterion,
- Generalized cross validation method,
- Bayesian criterion, ...

↪ For comparison : - optimal λ (unknown on real case)
- "universal" constant parameter $\lambda = M^{-\frac{1}{2}} = 0.1$

Probabilistic Factor Analysis

- **Latent variable model**

$$\underbrace{\boldsymbol{p}}_{\text{observable variables}} = \underbrace{\boldsymbol{L}}_{\text{matrix of loading factors}}\underbrace{\boldsymbol{c}}_{\text{latent factors}} + \underbrace{\boldsymbol{n}}_{\text{error (noise)}}$$

Probabilistic Factor Analysis

- **Latent variable model**

observable variables $\longrightarrow$ $\boxed{\boldsymbol{p} = \boldsymbol{L}\boldsymbol{c} + \boldsymbol{n}}$

matrix of loading factors

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error (noise)

  - Capture dominant correlation with fewer parameters (close to PCA)

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- **Finally**, signal CSM:

$$\hat{\boldsymbol{S}}_{aa} = \frac{1}{N_s} \sum_{i=1}^{N_s} \boldsymbol{L}\boldsymbol{c}_i\boldsymbol{c}_i^H \boldsymbol{L}^H$$

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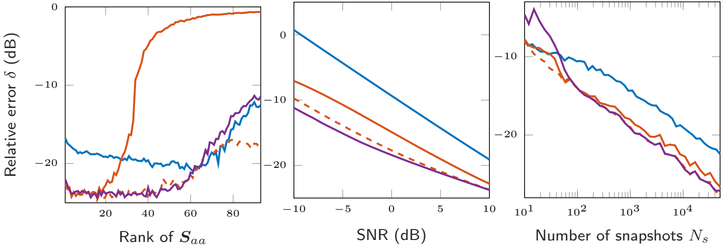
PFA

✓ Global optimization
✓ Flexible model

✗ Computationally expensive

Comparison

- **Homogeneous noise**



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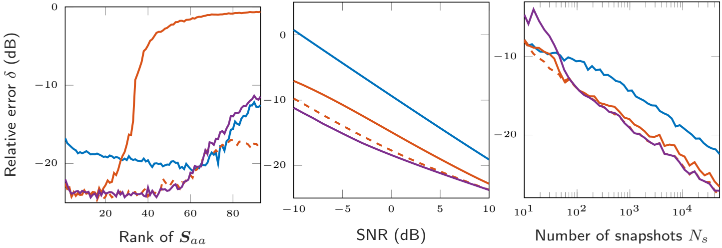

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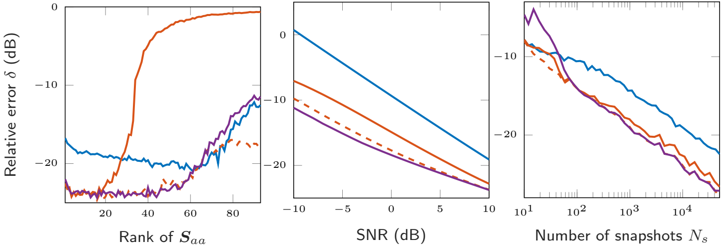

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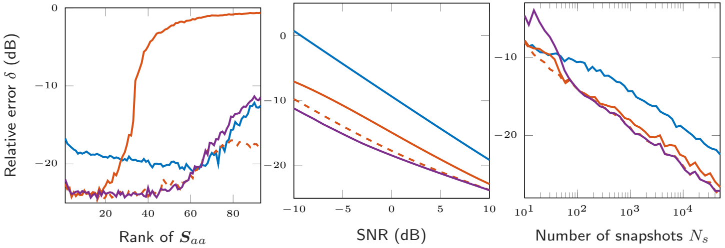

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↪ Error linearly decreases with logarithmically increasing N_s

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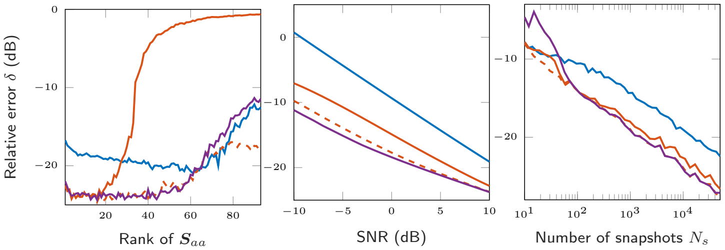
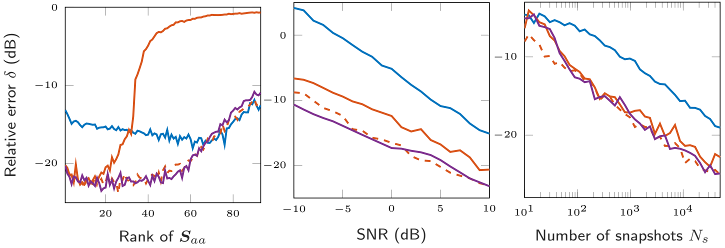


- **Heterogeneous noise**: SNR 10 dB lower on 10 random receivers



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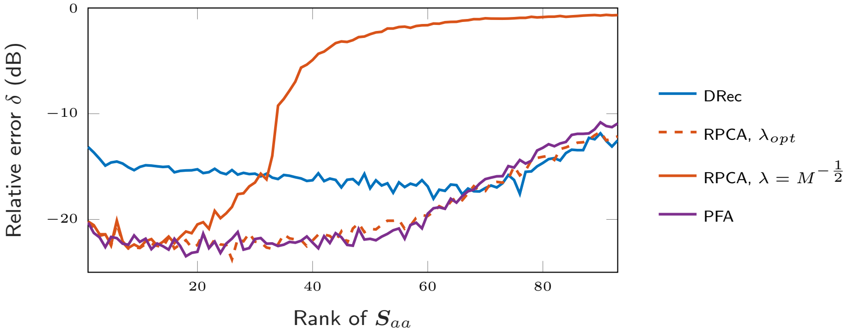

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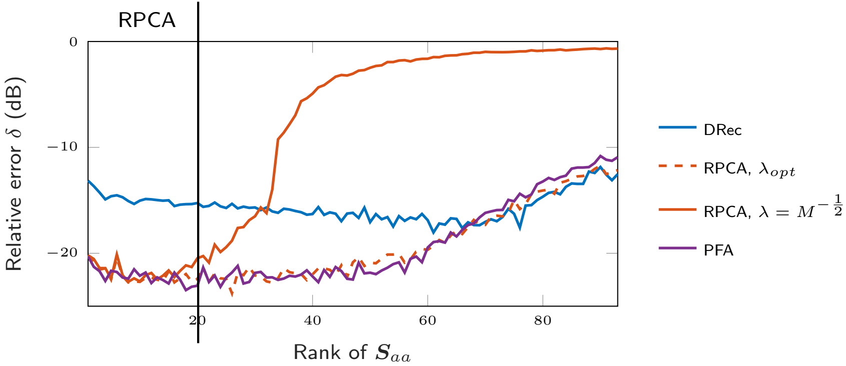

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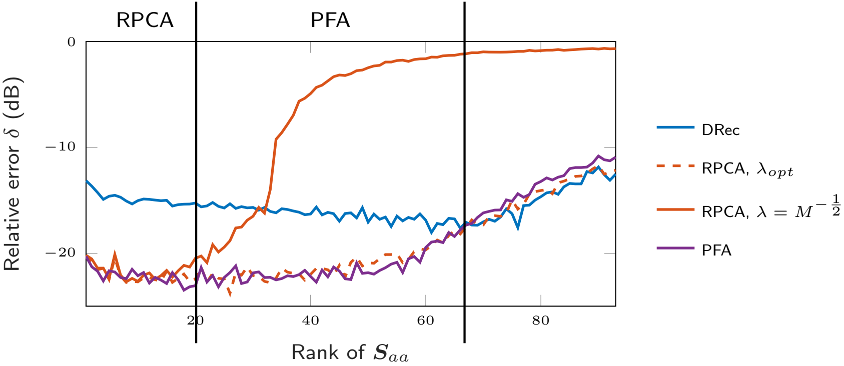

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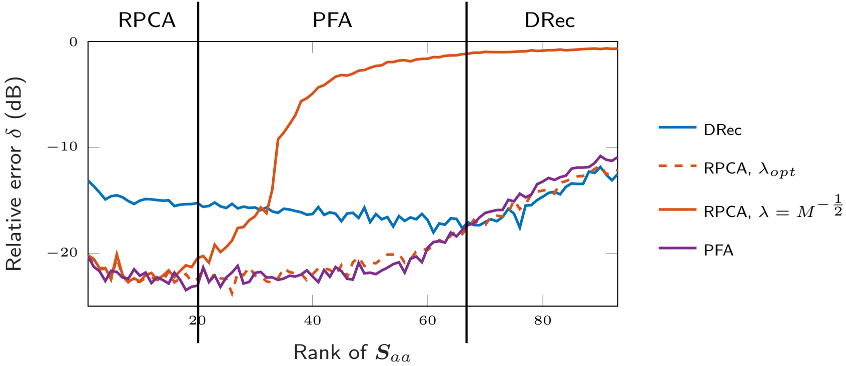

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