On the Denoising of Cross-Spectral Matrices for (Aero)Acoustic Applications

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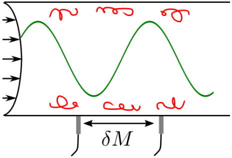
Context

- Unwanted random noise:
  - electronic, ambient, flow-induced,...
  - short correlation lengths



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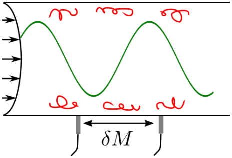

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  - physical removal : windscreen, mic recession, porous treatment, vibrating structure filtering...
  - use a background noise measurement $\rightarrow$ not always available or representative
  - wavenumber filtering $\rightarrow$ requires high spatial sampling
  - diagonal removal $\rightarrow$ underestimation of source level
  - exploit noise/signal properties & solve an optimization problem

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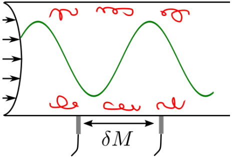

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Cross-Spectral Matrix (covariance of Fourier component):

$$\boldsymbol{S}_{pp} = \frac{1}{N_s} \sum_i \boldsymbol{p}_i \boldsymbol{p}_i^H$$

- Hermitian (conjugate symmetric)
- Positive semidefinite (nonnegative eigenvalues)

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- Rank of $\boldsymbol{S}_{aa}$ = number of equivalent uncorrelated sources

Context – CSM properties

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For $N_s \to \infty$

- Short correlation length : off-diagonal elements of $\boldsymbol{S}_{nn} \to 0$
- Independent signal/noise : cross-terms $\to 0$

How to separate signal from noise ?

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- Proposed method: Probabilistic Factor Analysis
- What is the influence on denoising performance of :
  - noise level,
  - number of snapshots,
  - number of sources ?

1. Diagonal Reconstruction
   - Comparison on a test case
2. Robust Principal Component Analysis
3. Probabilistic Factor Analysis
4. Comparison

Diagonal Reconstruction

"Remove as much noise as possible as long as denoised CSM remains non-negative"

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Convex optimization (Hald, 2017)

$$\text{maximize } \|\boldsymbol{\sigma}_n^2\|_1 \;\; \text{subject to} \;\; \boldsymbol{S}_{pp} - \mathrm{diag}\left(\boldsymbol{\sigma}_n^2\right) \geq 0$$

Problem solved with CVX Matlab toolbox

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Linear optimization (Dougherty, 2016)

$$\text{maximize } \|\boldsymbol{\sigma}_n^2\|_1 \text{ subject to } \boldsymbol{V}_{(k-1)}^H \left(\boldsymbol{S}_{pp} - \operatorname{diag}\left(\boldsymbol{\sigma}_n^2\right)_{(k)}\right) \boldsymbol{V}_{(k-1)} \geq 0$$

$\boldsymbol{V}_{(k-1)}$: eigenvectors of $\boldsymbol{S}_{pp} - \operatorname{diag}\left(\boldsymbol{\sigma}_n^2\right)_{(1,\ldots,k-1)}$

Solved with *linprog* Matlab function

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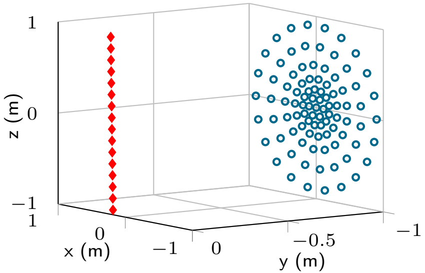
Alternating Projections (Leclère et al., 2015)

$$\boldsymbol{S}_{pp_{(k+1)}} := \bar{\boldsymbol{S}}_{pp_{(0)}} + \mathrm{diag}\left(\boldsymbol{V}_{(k)}^H \boldsymbol{s}_{(k)}^{+} \boldsymbol{V}_{(k)}\right)$$

$\boldsymbol{V}_{(k)}$ and $\boldsymbol{s}_{(k)}$: eigenvectors/values of $\boldsymbol{S}_{pp_{(k)}}$

Diagonal Reconstruction – Test case

- Default parameters:
  - 20 uncorrelated free field monopoles: ♦
  - 93 receivers: ○
  - SNR: 10 dB
  - 10^4 snapshots
  - frequency: 15 kHz



From a benchmark case provided by PSA3

- Varying parameters:
  - number of ♦ (rank of $\boldsymbol{S}_{aa}$)
  - SNR
  - number of snapshots (level of extra-diagonal terms)

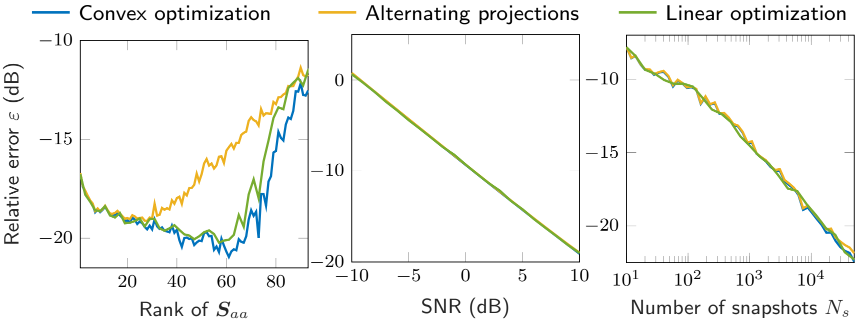
- Error on the signal CSM:

$$\varepsilon = \frac{\| \operatorname{diag}(\boldsymbol{S}_{aa}) - \operatorname{diag}\left(\hat{\boldsymbol{S}}_{aa}\right) \|_2}{\| \operatorname{diag}(\boldsymbol{S}_{aa}) \|_2}$$

Diagonal Reconstruction

Default values:

20 sources	93 receivers	SNR: 10 dB	10^4 snapshots	frequency: 15 kHz



Select Convex Optimization (DRec) for further comparison

✓ Fast, simple code
✓ Better performance

✗ Local optimization
✗ Denoises only auto-spectra

RPCA

"Search S_{aa} as a low rank matrix and S_{nn} as a sparse matrix"

$$\text{minimize } \|S_{aa}\|_* + \lambda \|S_{nn}\|_1 \quad \text{subject to} \quad S_{aa} + S_{nn} = S_{pp}$$

$\|\cdot\|_*$: nuclear norm (sum of eigenvalues: related to rank)
$\|\cdot\|_1$: ℓ_1-norm (related to sparsity)

Solved with a proximal gradient algorithm

RPCA (Wright et al., 2009)

✓ Modifies the whole CSM

✓ Widely used in image processing

✗ Local optimization

✗ Choose regularization parameter:

- L-curve criterion,
- Generalized cross validation method,
- Bayesian criterion, . . .

$\hookrightarrow$ For comparison : - optimal λ (unknown on real case)
- "universal" constant parameter $\lambda = M^{-\frac{1}{2}} = 0.1$

Probabilistic Factor Analysis

- **Latent variable model**

observable variables

$$\boldsymbol{p} = \boldsymbol{L}\boldsymbol{c} + \boldsymbol{n}$$

matrix of loading factors

latent factors

error (noise)

  - Capture dominant correlation with fewer parameters (close to PCA)
  - Extract anisotropic noise

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- **Statistical inference:** See parameters as random variables

$$\boldsymbol{L} \sim \mathcal{N}_{\mathbb{C}}(0, \boldsymbol{\gamma}^2) \qquad \boldsymbol{c} \sim \mathcal{N}_{\mathbb{C}}(0, \boldsymbol{I}\boldsymbol{\alpha}^2) \qquad \boldsymbol{n} \sim \mathcal{N}_{\mathbb{C}}(0, \boldsymbol{I}\boldsymbol{\sigma}^2)$$

$+$ non-informative priors : $\boldsymbol{\gamma}^2, \boldsymbol{\alpha}^2, \boldsymbol{\sigma}^2 \sim \mathcal{IG}(a_{\gamma,\alpha,\sigma}, b_{\gamma,\alpha,\sigma})$

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  Iterative draws in the marginal conditional distributions of each parameter

Probabilistic Factor Analysis

- **Latent variable model**

observable variables → $\boxed{\boldsymbol{p} = \boldsymbol{L}\boldsymbol{c} + \boldsymbol{n}}$ ← error (noise)

matrix of loading factors → $\boldsymbol{L}$; latent factors → $\boldsymbol{c}$

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- **Finally**, signal CSM: $$\hat{\boldsymbol{S}}_{aa} = \frac{1}{N_s} \sum_{i=1}^{N_s} \boldsymbol{L}\boldsymbol{c}_i\boldsymbol{c}_i^H\boldsymbol{L}^H$$

Probabilistic Factor Analysis

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PFA

✓ Global optimization

✓ Flexible model

✓ Cross-terms taken into account in the model

✗ Computationally expensive

Comparison

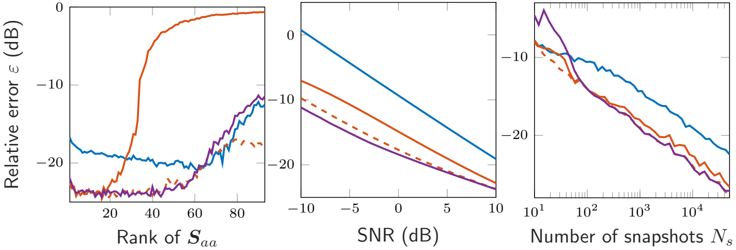
DRec

RPCA: λ_{opt}

$\lambda = M^{-\frac{1}{2}}$

PFA

- **Homogeneous noise**



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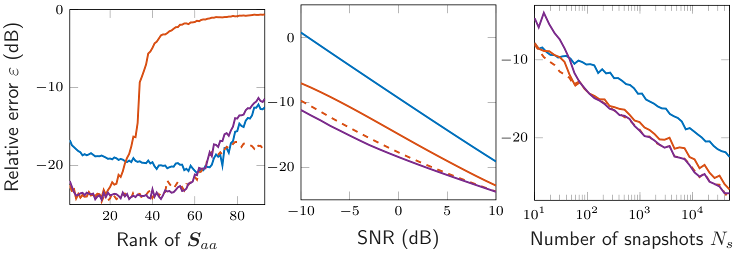

$\hookrightarrow$ For $N_{src} \geq 0.75M$: denoising problem becomes poorly conditioned

$\hookrightarrow$ Error linearly decreases with increasing SNR

$\hookrightarrow$ Error linearly decreases with logarithmically increasing N_s

Comparison

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PFA

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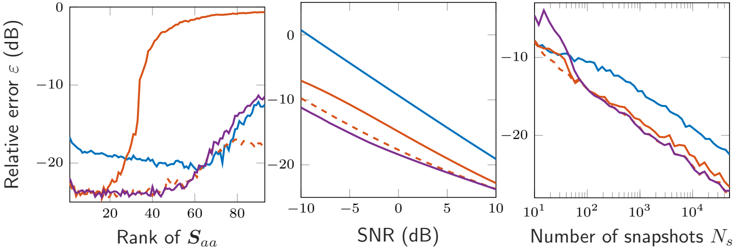
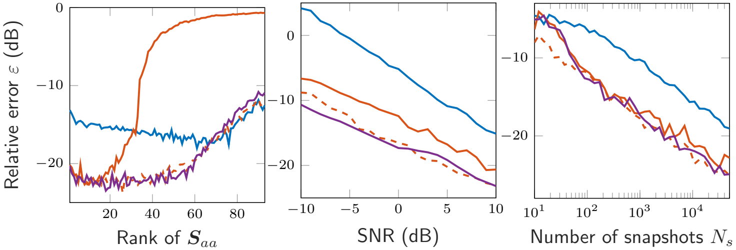


- **Heterogeneous noise**: SNR 10 dB lower on 10 random receivers



Conclusion

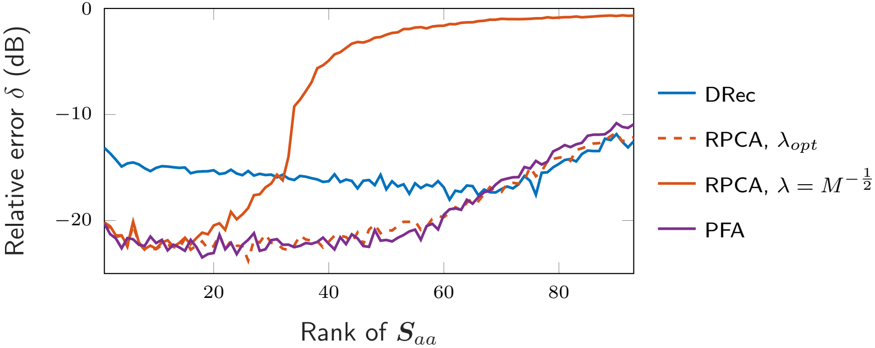
- Hard to denoise full rank CSM
- DRec: fast and simple but error 5 dB higher

- PFA
  - performance similar to RPCA using λ_{opt}
  - PFA and RPCA more robust to heterogeneous noise
  - can be solved using Expectation-Maximization algorithm
  - initialize with DRec to increase convergence speeds

- Future work :
  - denoising of the whole CSM
  - adapt PFA to correlated noise
  - effect of denoising on imaging ?

Conclusion

- Choose your denoising method according to the expected number of sources



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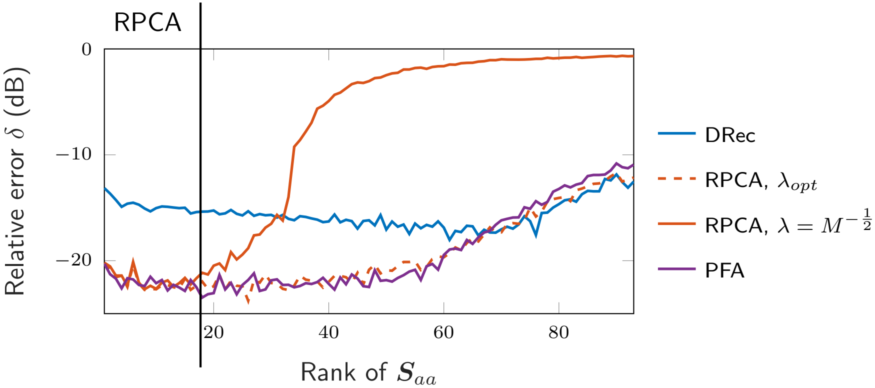

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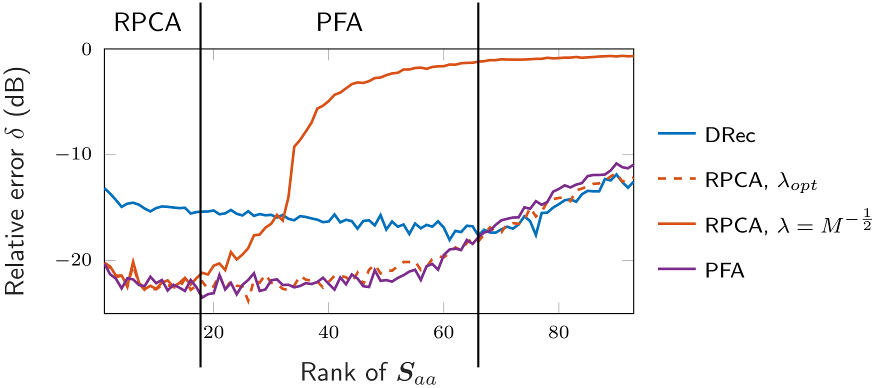

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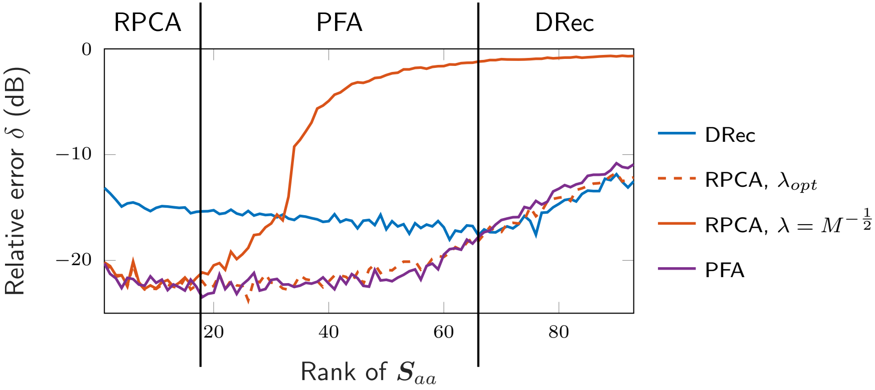


References

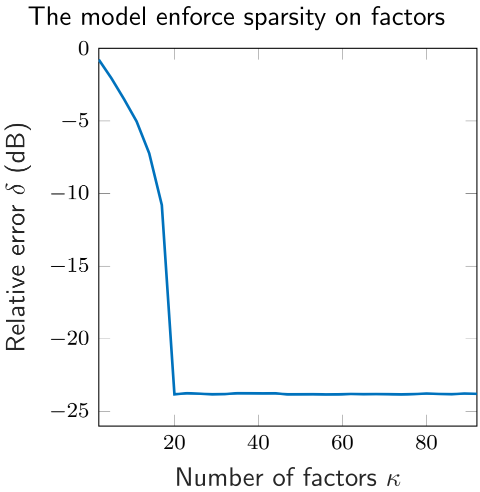
R. Dougherty. Cross spectral matrix diagonal optimization. In *6th Berlin Beamforming Conference*, 02 2016.

J. Hald. Removal of incoherent noise from an averaged cross-spectral matrix. *The Journal of the Acoustical Society of America*, 142(2): 846–854, 2017.

Q. Leclère, N. Totaro, C. Pézerat, F. Chevillotte, and P. Souchotte. Extraction of the acoustic part of a turbulent boundary layer from wall pressure and vibration measurements. In *Novem 2015 - Noise and vibration - Emerging technologies*, Proceedings of Novem 2015, page 49046, Dubrovnik, Croatia, Apr. 2015.

J. Wright, A. Ganesh, S. Rao, Y. Peng, and Y. Ma. Robust principal component analysis: Exact recovery of corrupted low-rank matrices via convex optimization. In *Advances in neural information processing systems*, pages 2080–2088, 2009.

PFA – Choosing the number of factor



Bayesian inference – MAP

$$\boldsymbol{p} = \boldsymbol{L}\boldsymbol{c}$$

- Bayes theorem: $[\boldsymbol{c}|\boldsymbol{p}] = \frac{[\boldsymbol{p}|\boldsymbol{c}][\boldsymbol{c}]}{[\boldsymbol{p}]}$
- Maximize a posteriori density:

$$\boldsymbol{c} = \arg\max_{\boldsymbol{c}} [\boldsymbol{c}|\boldsymbol{p}]$$
$$\arg\min_{\boldsymbol{c}} (-\log[\boldsymbol{p}|\boldsymbol{c}] - \log[\boldsymbol{c}])$$

- If you know which family your posterior is from
  $\hookrightarrow$ optimization problem
- If no analytical notion of the posterior
  $\hookrightarrow$ sampling: visit a collection of $\boldsymbol{c}$ with a Markov Chain
- MCMC : performs a biased random walk to explore the distribution (each sample is correlated with nearby samples).

Expectation-Maximization Algorithm

Deterministic algorithm for Bayesian inference

$$\boldsymbol{p} = \boldsymbol{L}\boldsymbol{c} + \boldsymbol{n}$$

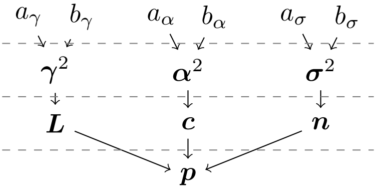
1. Calculate the expected value of the likelihood function

$$Q = \mathbb{E}([\boldsymbol{p} \mid \boldsymbol{c}, \boldsymbol{L}, \boldsymbol{n}])$$

2. Find $\boldsymbol{c}$, $\boldsymbol{L}$, $\boldsymbol{n}$ that maximize Q

Gibbs sampling

Hierarchical model:



Gibbs sampling: update successively each variable

Require: $\boldsymbol{p},\ a_\gamma^{(0)},\ b_\gamma^{(0)},\ a_\alpha^{(0)},\ b_\alpha^{(0)},\ a_\sigma^{(0)},\ b_\sigma^{(0)}$
for k **do**
  sample $\boldsymbol{c}$ in $[\boldsymbol{c} \mid \boldsymbol{p}, \boldsymbol{L}^{(k-1)}, \boldsymbol{\gamma}^{(k-1)}, \boldsymbol{\alpha}^{(k-1)}, \boldsymbol{\sigma}^{(k-1)}]$
  sample $\boldsymbol{L}$ in $[\boldsymbol{L} \mid \text{rest}]$
  sample $\boldsymbol{\gamma}^2$ in $[\boldsymbol{\gamma}^2 \mid \text{rest}]$
  sample $\boldsymbol{\alpha}^2$ in $[\boldsymbol{\alpha}^2 \mid \text{rest}]$
  sample $\boldsymbol{\sigma}^2$ in $[\boldsymbol{\sigma}^2 \mid \text{rest}]$
end for