$$\underset{(M\times1)}{Y} = \underset{(M\times k)}{L}\underset{(k,1)}{x} + \underset{(M\times1)}{\varepsilon}$$ (unobservable variable)

$$\left(\underline{PCA} \quad S_y W = \lambda W \Rightarrow x = W^T y\right)$$

$$\underset{(M\times M)}{C_y} = \mathbb{E}(Lxx^TL^T) + \mathbb{E}(\varepsilon\varepsilon^T) = \underset{(M\times k)(k\times M)}{LL^T} + \sigma^2 I \quad (=C)$$

$\bar{\bar{\Psi}}$ $(M\times M)$ diagonale.

$[x] \sim \mathcal{N}(0, I) \rightarrow$ convention

$[y] \sim \mathcal{N}(0, C)$

$M = L^TL + \sigma^2 I \rightarrow$ Rmq : impossible si $\sigma^2 \neq$ scalaire

$\bar{\bar{\Psi}} = \vec{d}\,\bar{\bar{I}}$ (

$$[y|x] \sim \mathcal{N}(Lx, \sigma^2 I)$$

≠ Analyse factorielle

$G = (I + L'\Psi^{-1}L)^{-1}$ $(k\times k)$ $= G^T$

$B = \frac{L'}{LL' + \Psi}$, $\Omega = I - BL$

$$\underset{\text{posterior}}{[x|y]} \sim \mathcal{N}(M^{-1}L^Ty, \; \sigma^2M^{-1})$$

$$[y, x] = \underset{\text{vraisemb.}}{[y|x]}\underset{\text{prior}}{[x]} = \frac{1}{(2\pi\sigma^2)^{d/2}} e^{-\frac{\|y-Lx\|^2}{2\sigma^2}} \times \frac{1}{(2\pi)^{q/2}} e^{-\frac{\|x\|^2}{2}}$$

$\mathcal{N}(Lx, \sigma^2 I)$

E-STEP

$$\mathbb{E}\left(\sum_n \ln[y_n, x_n, L, \sigma^2]\right) = -\sum_{n=1}^{N}\left(\frac{d}{2}\ln 2\pi\sigma^2 + \frac{1}{2\sigma^2}\|y\|^2 - \frac{1}{\sigma^2}\mathbb{E}(x)^T L^T y + \frac{1}{2\sigma^2}\mathrm{tr}(L^TL\,\mathbb{E}(xx^T)) + \frac{1}{2}\mathrm{tr}(\mathbb{E}(xx^T))\right)$$

avec $\mathbb{E}(x) = M^{-1}L^Ty$ $= GL^T\Psi^{-1}y$ et $\mathbb{E}(xx^T) = \sigma^2M^{-1} + \mathbb{E}(x)\mathbb{E}(x)^T \rightarrow$ OK

$= \underset{(k\times k)}{G} + \mathbb{E}(x)\mathbb{E}(x)^T$

M-STEP

$$\frac{\partial\mathbb{E}(\mathcal{L})}{\partial L} = 0 \Rightarrow \tilde{L} = \frac{\sum_n y\,\mathbb{E}(x)^T}{\sum_n \mathbb{E}(xx^T)}$$

$$\frac{\partial\mathbb{E}(\mathcal{L})}{\partial\sigma^2} = 0 \Rightarrow \tilde{\sigma}^2 = \frac{1}{ND}\sum_n\left(\|y\|^2 - 2\,\mathbb{E}(x)^T\tilde{L}y + \mathrm{tr}(\mathbb{E}(xx^T)\tilde{L}^T\tilde{L})\right)$$

$$\tilde{L} = \frac{\sum y(M^{-1}L^Ty)^T}{\sum \sigma^2M^{-1} + (M^{-1}L^Ty)(M^{-1}L^Ty)^T}$$

$$\tilde{L}_{Anafac} = \frac{S_yB'}{I - BL + BS_yB'}$$

$$\tilde{\sigma}^2 = \frac{1}{ND}\left(S_y - \frac{\Sigma}{N}2(M^{-1}L^Ty)^T\tilde{L}y + \frac{\Sigma}{N}(\sigma^2M^{-1} + (M^{-1}L^Ty)(M^{-1}L^Ty)^T)\tilde{L}^T\tilde{L}\right)$$

$$\frac{\partial x^TAy}{\partial A} = xy^T$$

$$\frac{\partial}{\partial A}\mathrm{Tr}(AB) = B^T$$

$$\frac{\partial \log|A|}{\partial A} = (A^{-1})^T$$

$$\tilde{\sigma}^2 = \frac{1}{I_{tot}}\sum_n \underbrace{(I - \tilde{L}B)S_y(I - \tilde{L}B)^T}_{C} + \tilde{L}(I - BL)\tilde{L}^T$$

I_{tot} : Sum(I_p)

Notation J : $GL^T\Psi^{-1} = M^{-1}L^T = \underline{B}$; $\mathbb{E}(x) = By$ $\quad \mathbb{E}(xx^T) = \Omega + BS_yB'$

$G = \Omega$ oui

$$\tilde{\sigma}_{bishop} = diag\left(S_y - \tilde{L} \frac{1}{N} \Sigma\, \mathbb{E}(y)\, y^T \right) = diag\left(S_y - \tilde{L} \left(\frac{L' \Psi^{-1} S_y}{I + L'\Psi^{-1} L} \right) \right) \quad (p. 586)$$

$$= diag\left(S_y - \tilde{L}\, B\, S_y \right)$$

$$\tilde{\sigma}_{jerome} = \frac{1}{N\beta^2} \sum_N \left(I - \tilde{L}B \right) S_y \left(I - \tilde{L}B \right)' + \tilde{L}\left(I - BL \right) \tilde{L}'$$

$$\tilde{L}' = \frac{B S_y}{I - L'B' + B S_y B' \; BL + B S_y B'}$$

$$= \frac{1}{N} \sum_N \left(S_y - \tilde{L} B S_y \right)\left(I - \tilde{L}B \right)' + \tilde{L}\left(I - BL \right) \tilde{L}'$$

$$= \frac{1}{N} \Sigma \left\{ S_y - \tilde{L} B S_y - S_y B' \cancel{B}^{\tilde{L}^T} - \tilde{L} B S_y B' \tilde{L}' + \tilde{L}\left(I - BL \right) \tilde{L}' \right.$$

$$\left(= \frac{1}{N} \Sigma \left\{ S_y - \tilde{L} B S_y - S_y B' \cancel{B}^{\tilde{L}^T} + \tilde{L} \cancel{\left(B S_y B' + I - BL \right)} \frac{B S_y}{\cancel{I - BL + B S_y B'}} \right)$$

$$\cancel{= \frac{1}{N} \Sigma\, S_y - S_y \tilde{L} B}$$

$$= \frac{1}{N} \Sigma\, S_y - \tilde{L} B S_y - S_y B' \tilde{L}' + \frac{S_y B'}{\cancel{I - BL + B S_y B'}} \cancel{\left(B S_y B' + I - BL \right)} \tilde{L}'$$

$$diag\left(= \frac{1}{N} \Sigma\, S_y - \underbrace{\tilde{L} B S_y}_{L\, S_{xx}\, L'} \right)$$

Mhr rang nitmax errmin Sye, L

EM-CSQL-fit (S_y , I , K , option , Ini)

EM → $\tilde{S}_{yy} = LL' + \sigma^2$

$$[y|x](x) = \frac{1}{\pi^M |\bar{\sigma}^2|^M} e^{-(y-HLx)^H \bar{\sigma}^{-2}(y-HLx)}$$

$$\frac{1}{\pi^K} e^{-x^H x}$$

moyenne

$$y_i|x_i \sim \mathcal{N}_{\mathbb{C}}(HLx_i, \bar{\sigma}^2)$$

$$x_i|y_i \sim \mathcal{N}_{\mathbb{C}}(By_i, \Omega)$$

$$\theta = \bar{\sigma}^2, L$$

$$\ln([y,x]) = \ln \sum_i [y_i, x_i] = \sum_i \underbrace{-\log \pi^M}_{\perp\!\!\perp(\theta)} - \log|\bar{\sigma}^2| - (y-HLx)^H \bar{\sigma}^{-2}(y-HLx) \underbrace{- \log \pi^K - x^H x}_{\perp\!\!\perp(\theta)}$$

$$\mathcal{L}_\theta = \sum_i -\log|\bar{\sigma}^2| - y_i^H \bar{\sigma}^{-2} y_i + x^H L^H H^H \bar{\sigma}^{-2} y + y^H \bar{\sigma}^{-2} HLx - \underbrace{x_i^H L^H H^H \bar{\sigma}^{-2} HLx_i}_{= \mathrm{Tr}(L^H H^H \bar{\sigma}^{-2} HL x_i x_i^H)}$$

$$\mathbb{E}(\mathcal{L}_\theta) = \sum_i -\log|\bar{\sigma}^2| - y_i^H \bar{\sigma}^{-2} y_i + \underbrace{\mathbb{E}(x_i)^H L^H H^H \bar{\sigma}^{-2} y_i}_{a^H X^H b} + \underbrace{y_i^H \bar{\sigma}^{-2} HL\,\mathbb{E}(x_i)}_{b^H X^H a} - \underbrace{\mathrm{Tr}(L^H H^H \bar{\sigma}^{-2} HL\,\mathbb{E}(x_i x_i^H))}_{X^H A X B}$$

$$\frac{\partial a^H X^H b}{\partial x} = \boxed{ba^H} \qquad \frac{\partial b^H X^H a}{\partial x} = \boxed{ba^H} \qquad \frac{\partial (X^H A X B)}{\partial x} = AXB + A^H X B^H$$

$$\frac{\partial \mathbb{E}(\mathcal{L}_\theta)}{\partial L} = \sum_i 2 H^H \bar{\sigma}^{-2} y_i\, \mathbb{E}(x_i)^H - H^H \bar{\sigma}^{-2} HL\, \mathbb{E}(x_i x_i^H) - H^H \bar{\sigma}^{-2} HL\, \mathbb{E}(x_i x_i^H)^H$$

$$= \sum_i 2H^H \bar{\sigma}^{-2} y_i\, \mathbb{E}(x_i)^H - 2H^H \bar{\sigma}^{-2} HL\, \mathbb{E}(x_i x_i^H)$$

$$\boxed{\text{or: } \mathbb{E}(x_i) = By_i \quad \text{et} \quad \mathbb{E}(x_i x_i^H) = \Omega + By_i y_i^H B^H}$$

$$\frac{\partial \mathbb{E}(\mathcal{L}_\theta)}{\partial L} = 2H^H \bar{\sigma}^{-2} S_y B^H - 2H^H \bar{\sigma}^{-2} HL(\Omega + BS_y B^H)$$

$$= 0 \Leftrightarrow \underbrace{H^H \bar{\sigma}^{-2} H}_{N \times N} L \underbrace{(\Omega + BS_y B^H)}_{K \times K} = H^H \bar{\sigma}^{-2} S_y B^H$$

$$\boxed{\tilde{L} = (H^H \bar{\sigma}^{-2} H)^{-1} H^H \bar{\sigma}^{-2} S_y B^H (\Omega + BS_y B^H)^{-1}}$$

$$\frac{\partial \mathbb{E}(\mathcal{L}_\theta)}{\partial \bar{\sigma}^{-2}} = \sum_i + \bar{\sigma}^{+2} - y_i y_i^H + 2H\tilde{L}\,\mathbb{E}(x_i) y_i^H - H\tilde{L}\,\mathbb{E}(x_i x_i^H)^H \tilde{L}^H H^H$$

$$\frac{\partial \log A}{\partial A} = A^{-T} \quad \Big| \quad \frac{\partial x^H A y}{\partial A} = xy^H \qquad \frac{\partial (A X B)}{\partial x} = A^T B^T$$

$$\text{et } \log|\sigma^2| = -\log|\sigma^{-2}|$$

$$\bar{\sigma}^2 = \sum_i y_i y_i^H - \sum_i 2H\tilde{L}\,\mathbb{E}(x_i) y_i^H + \sum_i H\tilde{L}\,\mathbb{E}(x_i x_i^H)^H \tilde{L}^H H^H$$

$$= S_y - 2H\tilde{L}BS_y + H\tilde{L}(\Omega + BS_y B^H)\tilde{L}^H H^H$$

$$\bar{\sigma}^2 = (I - H\tilde{L}B) S_y (I - H\tilde{L}B)^H + H\tilde{L}(\overbrace{I - BHL}^{\Omega})\tilde{L}^H H^H$$