Notes on transitions in atomic & semiconductor systems

Claude Cohen Tannoudji Atom-photon interactions:

Let $t_i = -T/2$, $t_f = T/2$, then we have $\tilde{U}(T) = e^{iH_0T/2}\, U(T)\, e^{iH_0T/2}$

$(\hbar = 1) \rightarrow$ in interaction representation, $\tilde{A}(t) = e^{iH_0t/\hbar} A e^{-iH_0t/\hbar}$

$$\Rightarrow i\hbar \frac{d}{dt}|\tilde{\psi}(t)\rangle = i\hbar \frac{d}{dt}\left(e^{iH_0t/\hbar}|\psi(t)\rangle\right)$$

$$= i\hbar\left(\frac{iH_0}{\hbar}|\tilde{\psi}(t)\rangle\right) + e^{iH_0t/\hbar}(H_0+V)|\psi(t)\rangle =$$

$$e^{iH_0t/\hbar} V e^{-iH_0t/\hbar} e^{iH_0t/\hbar} |\psi(t)\rangle =$$

$$\tilde{V}(t)|\tilde{\psi}(t)\rangle \Rightarrow i\hbar \frac{d}{dt}|\tilde{\psi}(t)\rangle = \tilde{V}(t)|\tilde{\psi}(t)\rangle$$

Now $|\psi(t_f)\rangle = U(t_f, t_i)|\psi(t_i)\rangle$, we know that $|\psi(t_f)\rangle = e^{-iH_0t_f/\hbar}|\tilde{\psi}(t_f)\rangle \Longrightarrow$

$$e^{-iH_0t_f/\hbar}|\tilde{\psi}(t_f)\rangle = U(t_f, t_i)\, e^{-iH_0t_i/\hbar}|\tilde{\psi}(t_i)\rangle$$

$$\Rightarrow |\tilde{\psi}(t_f)\rangle = e^{iH_0t_f/\hbar}\, U(t_f, t_i)\, e^{-iH_0t_i/\hbar}|\tilde{\psi}(t_i)\rangle$$

$$U(t_f, t_i) = U_0(t_f, t_i) + \frac{1}{i\hbar}\int_{t_i}^{t_f} U_0(t_f, t)\, V\, U(t, t_i)$$

Check: $U(t_i, t_i) = U_0(t_i, t_i)^{t_i} = \mathbb{1}$ ✓

$i\hbar \frac{d}{dt} U(t_f, t_i) = (H_0 + V) U(t_f, t_i)$: Check $\Rightarrow$

$$i\hbar \frac{d}{dt} U_0(t_f, t_i) = \left(-i\frac{H_0}{\hbar}\right)(i\hbar) U_0(t_f, t_i) = H_0 U_0(t_f, t_i)$$

$$i\hbar \frac{d}{dt_f}\left(\frac{1}{i\hbar}\int_{t_i}^{t_f} dt\, U_0(t_f, t) V U(t, t_i)\right) =$$

$$\frac{d}{dt_f}\int_{t_i}^{t_f} dt\, U_0(t_f, t) V U(t, t_i) = \overset{=1}{U_0(t_f, t_f)} V U(t_f, t_i) +$$

$$\int_{t_i}^{t_f} -i\frac{H_0}{\hbar} U_0(t_f, t) V U(t, t_i) dt = V U(t_f, t_i) +$$

$H_0 \left(U(t_f, t_i) - U_0(t_f, t_i)\right) \Rightarrow$ all terms combined

gives us $(H_0 + V) U(t_f, t_i)$

<u>First order transition amplitudes:</u>

$$\frac{1}{i\hbar}\int_{t_i}^{t_f} d\tau_1\, V_{fi}\, e^{i(E_i - E_f)\tau/\hbar} = \frac{V_{fi}}{i\hbar}\int_{-T}^{T} e^{i(E_i - E_f)\tau/\hbar} \Rightarrow$$

as $T \to \infty$, $\frac{1}{} \delta(\omega_i - \omega_f) V_{fi} \times 2\pi =$

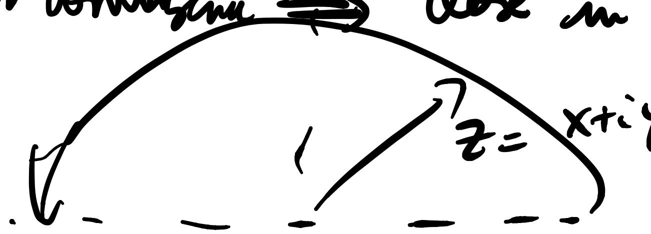
$- i\,\delta(E_i - E_f)(2\pi)\,V_{fi}$

Second order transition amplitudes:

Note that $\theta(t - t_0)\, e^{-iE_k(t_2 - t_1)/\hbar} = \lim_{\eta \to 0^+} \frac{-1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-iE(t_2 - t_1)/\hbar}}{E + i\eta - E_k}\, dE$, check: if $t > t_0$,

LHS is $e^{-iE_k(t_2 - t_1)/\hbar}$, $t_2 - t_1 > 0 \Rightarrow$

$E \sim E + i\delta \Rightarrow -i(i\delta)(t_2 - t_1) \Rightarrow \delta < 0$ for convergence $\Rightarrow$ close in ~~upper~~ lower half plane.



pole is at $E = E_k - i\eta \Rightarrow$ get $(2\pi i)$ Residue

× negative sign for path direction.

DOS: $\rho(E) = \sum_k \delta(E - E_k) = \frac{L^3}{(2\pi)^3} \int d^3k\, \delta(E - E_k)$

$= \frac{V}{(2\pi)^3} \int k^2 dk\, \underbrace{\sin\theta\, d\theta\, d\varphi}_{d\Omega}\, \delta(\varepsilon - \varepsilon_k)$,

$$\varepsilon_k = \frac{\hbar^2 k^2}{2m} = \frac{V}{(2\pi)^3}(4\pi)\int k^2 dk\, \delta\left(\varepsilon - \frac{\hbar^2 k^2}{2m}\right) \Rightarrow$$

let $x = \frac{\hbar^2 k^2}{2m} \Rightarrow k^2 = \frac{2m x}{\hbar^2},\ k = \sqrt{\frac{2m}{\hbar^2}}\sqrt{x} \Rightarrow$

$$\frac{dk}{dx} = \sqrt{\frac{2m}{\hbar^2}}\frac{1}{2\sqrt{x}} \Rightarrow dk = \sqrt{\frac{m}{2\hbar^2}}\frac{dx}{\sqrt{x}} \Rightarrow$$

$$\rho(\varepsilon) = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)\int_0^\infty \frac{\sqrt{x}}{\sqrt{2}}\,\delta(\varepsilon - x) =$$

$\frac{V}{\pi^2}\left(\frac{m^{3/2}}{\hbar^3}\right)\frac{\sqrt{\varepsilon}}{\sqrt{2}} \Rightarrow$ if we look at per energy per solid angle, we get instead $\frac{V}{4\pi^3}\frac{M\sqrt{M\varepsilon}}{\hbar^3}$ ✓

Two level atom coupled to Radiation Field.

$$A(r=0) = \sum_j \sqrt{\frac{\hbar}{2\varepsilon_0\omega_j L^3}}\,(a_j + a_j^\dagger)$$

$$E_\perp(r=0) = i\sum_j \sqrt{\frac{\hbar\omega_j}{2\varepsilon_0\omega_j L^3}}\,(a_j - a_j^\dagger)$$

$$V = -d\cdot E_\perp(0) = -i\, d\cdot\sum_i \sqrt{\frac{\hbar\omega_i}{2\varepsilon_0 V}}\,\varepsilon_i(a_i - a_i^\dagger)$$

$$\Gamma_{b\to a} = \frac{2\pi}{\hbar}\sum_{k,\varepsilon}|\langle a|V|b\rangle|^2\,\delta(\hbar\omega - \hbar\omega_{ba})$$

units! $\frac{1}{(\hbar)(eV)}(eV)^2 = \frac{eV}{\hbar} = \frac{eV}{eV\cdot s} = \frac{1}{s}$ ✓

Atoms in electromagnetic fields

David Tong Notes (4)

$$H = \frac{-\hbar^2}{2m}\nabla^2 - \frac{e^2}{4\pi\varepsilon_0 r} + e\mathcal{E}z \quad \text{(Stark effect)}$$

$$\Delta E_0 = e^2\mathcal{E}^2 \sum_{n=1}^{\infty} \sum_{l,m} \frac{|\langle 1,0,0|z|n,l,m\rangle|^2}{E_0 - E_n}$$

Rabi Oscillations

$$H_0|\psi_i\rangle = E_i|\psi_i\rangle$$

$$|\psi(t)\rangle = c_1(t)e^{-iE_1t/\hbar}|\psi_1\rangle + c_2(t)e^{-iE_2t/\hbar}|\psi_2\rangle$$

$$\Rightarrow i\hbar\dot{c}_1(t) = c_1(t)\langle\psi_1|H_1|\psi_1\rangle + c_2\langle\psi_1|H_1|\psi_2\rangle \times e^{-it\hbar(E_2-E_1)}$$

$$i\hbar\dot{c}_2(t) = c_2(t)\langle\psi_2|H_1|\psi_2\rangle + c_1\langle\psi_2|H_1|\psi_1\rangle \, e^{-it\hbar(E_1-E_2)}$$

$x\, e$

Now $\langle \psi_1 | x | \psi_1 \rangle = 0 = \langle \psi_2 | x | \psi_2 \rangle \Rightarrow$

$$i\hbar\, \dot{c}_1(t) = c_2 e^{-i\omega_0 t} \langle \psi_1 | H | \psi_2 \rangle$$

$$i\hbar\, \dot{c}_2(t) = c_1 e^{i\omega_0 t} \langle \psi_2 | H | \psi_1 \rangle \Rightarrow$$

$$\boxed{\begin{aligned} i\, \dot{c}_1(t) &= c_2 \Omega \cos\omega t\, e^{-i\omega_0 t} \\ i\, \dot{c}_2(t) &= c_1 \Omega \cos(\omega t)\, e^{i\omega_0 t} \end{aligned}}$$

Therefore we have:

$$i\dot{c}_1(t) = \frac{c_2 \Omega}{2} \left\{ e^{i(\omega-\omega_0)t} + e^{-i(\omega+\omega_0)t} \right\}$$

$$i\dot{c}_2(t) = \frac{c_1 \Omega}{2} \left\{ e^{i(\omega+\omega_0)t} + e^{-i(\omega-\omega_0)t} \right\}$$

$\Rightarrow$ Rotating wave approximation:

discard $\omega+\omega_0$, $-\omega-\omega_0$ oscillating

terms $\Rightarrow i\dot{c}_1(t) = \frac{\Omega}{2} e^{i\delta t} c_2,\ i\dot{c}_2(t) = \frac{\Omega}{2} e^{-i\delta t} c_1$

$\Rightarrow$ at resonance, we get $c_1 = \cos\left(\frac{\Omega t}{2}\right)$

$c_2 = -i \sin \frac{\Omega t}{2}$

Off resonance:

$$\ddot{c}_1 = -i\frac{\Omega}{2}\left(i\delta c_2 e^{i\delta t} + e^{i\delta t}\dot{c}_2(t) \right)$$

$$= -i\frac{\Omega}{2} e^{i\delta t}\left[\frac{i\delta(2i)}{\Omega}\dot{c}_1(t) e^{-i\delta t} + \right.$$

$$\left. -i\frac{\Omega}{2} e^{-i\delta t} c_1 \right] \longrightarrow$$

$$-\frac{i\Omega}{2}\left[-\frac{2\delta}{\Omega}\dot{c}_1(t) - i\frac{\Omega}{2} c_1 \right] =$$

$$-\frac{\Omega^2}{4} c_1 + \frac{i\delta}{2}\dot{c}_1(t) \Rightarrow$$

$$\ddot{c}_1 - i\delta \dot{c}_1(t) + \frac{\Omega^2}{4} c_1(t) = 0 \Rightarrow$$

$$\left[d - i\delta + i\sqrt{\Omega^2+\delta^2}\right]\left[d - i\delta - i\sqrt{\Omega^2+\delta^2}\right] c_1$$

$\left[dt \quad 2 \quad 2 \right] \left[dt \quad 2 \quad 2 \right]$

$\Rightarrow$ (proof: $\left(\frac{d}{dt} - \frac{i\delta}{2}\right)^2 + \left(\frac{\Omega^2+\delta^2}{4}\right) \Rightarrow$

$$\frac{\partial^2}{\partial t^2} - 2\frac{i\delta}{2}\frac{\partial}{\partial t} - \cancel{\frac{\delta^2}{4}} + \frac{\Omega^2}{4} + \cancel{\frac{\delta^2}{4}} \quad ✓$$

let $c_1 = e^{rt} \to r^2 - i\delta r + \frac{\Omega^2}{4} = 0$

$$r = \frac{i\delta \pm \sqrt{-\delta^2 - \Omega^2}}{2} = \frac{i\delta \pm i\sqrt{\Omega^2+\delta^2}}{2} =$$

$$e^{i\left(\frac{\delta \pm \sqrt{\delta^2+\Omega^2}}{2}\right)t} \Rightarrow$$

$$e^{i\frac{\delta t}{2}}\left\{A\cos\left(\frac{\sqrt{\delta^2+\Omega^2}}{2}t\right) + B\sin\left(\frac{\sqrt{\delta^2+\Omega^2}}{2}t\right)\right.$$

$\Rightarrow$ general rabi frequency is

$$\sqrt{\Omega^2+\delta^2}$$

Making a coherent state:

$$H = \hbar\omega(a^\dagger a + 1/2) + \hbar(f^*(t)a + f(t)a^\dagger)$$

Solve with interaction picture:

$$H_0 = \hbar\omega(a^\dagger a + 1/2)$$

$$|\psi_I\rangle = e^{iH_0 t/\hbar}|\psi_S\rangle \implies$$

Interaction picture interacting Hamiltonian is equal to

$$\hbar e^{iH_0 t/\hbar}\left(f^*(t)a + f(t)a^\dagger\right)e^{-iH_0 t/\hbar}$$

$$= \boxed{\hbar\left(e^{-i\omega t}f^*(t)a + e^{i\omega t}f(t)a^\dagger\right)}$$

$$\boxed{i\hbar\frac{\partial U_I}{\partial t} = H_I U_I}$$

$$U_I(t) = \exp\left(\alpha(t)a^\dagger - \alpha^*(t)a + i\varphi(t)\right)$$

$$\Rightarrow i\hbar\left(\alpha'(t)a^\dagger - \alpha^{*\prime}(t)a + i\varphi'(t)\right) =$$

$$\hbar\left(e^{-i\omega t}f^*(t)a + e^{i\omega t}f(t)a^\dagger\right) \implies$$

Check: let $\alpha(t) = -i\int_0^t dt' f(t') e^{i\omega t'}$

$\rightarrow \alpha'(t) = -i f(t) e^{i\omega t} \Rightarrow$

$f(t) e^{i\omega t} a^\dagger + f^*(t) e^{-i\omega t} a + i\varphi'(t)$

$|\psi_I(t)\rangle = e^{-if_0(a+a^\dagger)t}|0\rangle \Rightarrow$

$|\psi_S(t)\rangle = e^{-iH_0 t/\hbar} e^{-if_0(a+a^\dagger)t}|0\rangle$

Jaynes-Cummings Model

Two atomic states: $|\uparrow, \downarrow\rangle$

$H_{atom} = \frac{1}{2}\begin{pmatrix} \hbar\omega_0 & 0 \\ 0 & -\hbar\omega_0 \end{pmatrix}$

Place the atom in a cavity,

$\lambda = 2\pi c$, since $\lambda = 2\pi$.

$\overline{\omega}$, $\overline{q}$,

$e^{iq\cdot r - i\omega t} \Rightarrow \nabla^2 \psi = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} \Rightarrow$

$q^2 = \frac{\omega^2}{c^2} \Rightarrow q = \omega / c \Rightarrow$

$\lambda = \frac{2\pi c}{\omega}$

Let $|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$

$H_{JC} = \frac{\hbar}{2}\begin{pmatrix} \omega_0 & ga \\ ga^\dagger & -\omega_0 \end{pmatrix} + \hbar\omega a^\dagger a$

Look at $|n\rangle$ basis subset of

H_{JC}: $H_{JC}|n, \downarrow\rangle = \hbar\omega n |n, \downarrow\rangle$

$+ \frac{\hbar}{2} g \sqrt{n} \, |n-1, \uparrow\rangle - \frac{\hbar\omega_0}{2}|n, \downarrow\rangle$

$= \hbar\omega (n - \frac{1}{2}) |n, \downarrow\rangle + \frac{1}{2}(\hbar\omega - \hbar\omega_0)|n, \downarrow\rangle$

$+\frac{1}{2}\hbar g\sqrt{n}\,|n-1,\uparrow\rangle$

Now look at effect on $|n-1,\uparrow\rangle$

$\hbar\omega(n-1) + \frac{\hbar}{2}\omega_0\,|n-1,\uparrow\rangle +$

$\frac{\hbar}{2} g\sqrt{n}\;|n,\downarrow\rangle$

<u>Fully quantum Rabi oscillations</u>

$$H_n = \left(n-\tfrac{1}{2}\right)\omega\,\mathbb{1}_2 + \tfrac{1}{2}(\omega_0-\omega)\,\sigma_z +$$

$\frac{1}{2}g\sqrt{n} \Rightarrow$ find eigenstates:

$$\begin{pmatrix} \frac{1}{2}(\omega_0-\omega) & \frac{1}{2}g\sqrt{n} \\ \frac{1}{2}g\sqrt{n} & -\frac{1}{2}(\omega_0-\omega) \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} \omega_0-\omega & g\sqrt{n} \\ g\sqrt{n} & \omega-\omega_0 \end{pmatrix} \longrightarrow$$

$$(\omega_0 - \omega - \varepsilon)(\omega - \omega_0 - \varepsilon) - g^2 n = 0$$

$$\Rightarrow \varepsilon^2 - (\omega - \omega_0)^2 - g^2 n = 0 \Rightarrow$$

$$\varepsilon = \left(n - \frac{1}{2}\right)\omega \hbar \pm \hbar \sqrt{g^2 n + (\omega - \omega_0)^2}$$

$\delta = \omega - \omega_0$ (detuning parameter)

$$(\omega_0 - \omega - \varepsilon)\, a + g\sqrt{n}\, b = 0$$

$$a = \frac{-g\sqrt{n}}{\omega_0 - \omega - \varepsilon} = \frac{g\sqrt{n}}{\varepsilon - \omega_0 + \omega} \Rightarrow$$

$$\frac{g\sqrt{n}}{\sqrt{g^2 n + (\omega - \omega_0)^2} - \omega_0 + \omega} = -\tan\theta$$

$$\tan 2\theta = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} = \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$\frac{2\tan\theta}{(1+\tan\theta)(1-\tan\theta)} \Rightarrow$$

$$\frac{2}{\sqrt{g^2 n + (\omega - \omega_0)^2} - \omega_0 + \omega}$$

$\Rightarrow$ <u>fluctuates</u>.

$$P_p(t) = \frac{g^2 n}{g^2 n + \delta^2} \sin^2\left(\frac{\sqrt{g^2 n + \delta^2}}{2}\, t\right)$$

$\Rightarrow \underline{\Omega \sim g\sqrt{n}} \Rightarrow \sim \underline{g E}$

Death & Resurrection