Laser Theory: (from Scully & Zubairy)

Schrodinger equation: $i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$

$\Rightarrow \frac{\partial}{\partial t}|\psi(t)\rangle = \frac{-i}{\hbar} H|\psi(t)\rangle$

$|\psi(t)\rangle = U(t)|\psi(0)\rangle \Rightarrow \dot{U}(t) = \frac{-i}{\hbar} H U(t)$

$\boxed{U(0) = 1}$

It is in general useful to move to the interaction picture, where the time evolution is the same as the Heisenberg picture if we only look at H_0

$$|\psi^I(t)\rangle = e^{iH_0 t} U(t)|\psi(0)\rangle$$

$$A(t) = e^{iH_0 t} A_s e^{-iH_0 t}$$

So, we have that $|\psi^I(t)\rangle = e^{iH_0 t}|\psi^S(t)\rangle$

Note that for $H = H_0$, $|\psi^I(t)\rangle = |\psi^S(0)\rangle = |\psi^I(0)\rangle$

$\Rightarrow$ same as Heisenberg picture evolution.

$$\frac{\partial}{\partial t}|\psi_I(t)\rangle = \frac{iH_0}{\hbar}|\psi_I(t)\rangle + e^{iH_0 t}\left(\frac{-i}{\hbar} H|\psi_s(t)\rangle\right)$$

$$= \frac{iH_0}{\hbar}|\psi_I(t)\rangle \frac{-i}{\hbar} H_0|\psi_I(t)\rangle - \frac{i}{\hbar} e^{iH_0 t} V|\psi_S(t)\rangle$$

$iH_0 t \quad -iH_0 t \quad iH_0 t$

$= \frac{-i}{\hbar} e \quad V e \quad e \quad |\psi_c(t)\rangle =$

$$\boxed{\frac{-i}{\hbar} V_I(t) \, |\psi_I(t)\rangle}$$

Therefore $\boxed{\frac{\partial}{\partial t} |\psi_I(t)\rangle = \frac{-i}{\hbar} \hat{V}(t) \, |\psi_I(t)\rangle}$ (1)

Formal solution of (1) is given by:

$$|\psi_I(t)\rangle = U_I(t) |\psi_I(0)\rangle$$

Where $U_I(t) = T\exp\left(\frac{-i}{\hbar}\int_0^t \hat{V}(t)\, dt\right)$

Check: $\boxed{\dot{U}(t) = \frac{-i}{\hbar} \hat{V}(t) \hat{U}(t)}$

$$T\exp\left(\frac{-i}{\hbar}\int_0^t \hat{V}(t)\, dt\right) = \mathbb{1} \frac{-i}{\hbar} \int_0^t dt_1 \hat{V}(t_1) + \left(\frac{-i}{\hbar}\right)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \, \hat{V}(t_1) \hat{V}(t_2) + \ldots$$

$$\dot{U}(t) = \frac{-i}{\hbar} \hat{V}(t) + \left(\frac{-i}{\hbar}\right)^2 \int_0^{t_1} dt_2 \, \hat{V}(t) \hat{V}(t_2)\, dt_2$$

$$\ldots = \boxed{\frac{-i}{\hbar} \hat{V}(t) \, \hat{U}(t)}$$

Consider a two level atom interactions w/

a monochromatic field of frequency $\underline{\omega^0}$.

$$H_0^n = (\hbar\omega_a)^n |a\rangle\langle a| + (\hbar\omega_b)^n |b\rangle\langle b|$$

Since $H_0 = \begin{array}{cc} \begin{pmatrix} \hbar\omega_a & 0 \\ 0 & \hbar\omega_b \end{pmatrix} & \begin{array}{c} |a\rangle \\ |b\rangle \end{array} \\ \begin{array}{cc} \langle a| & \langle b| \end{array} & \end{array} \Rightarrow$

$$U_0(t) = \exp\left(-\frac{i}{\hbar} H_0 t\right) = e^{-i\omega_a t}|a\rangle\langle a| + e^{-i\omega_b t}|b\rangle\langle b|$$

We take an atom at $z=0$

$H_1 = -exE(t) = -e\big(\langle a|x|b\rangle |a\rangle\langle b| +$
$\langle b|x|a\rangle |b\rangle\langle a|\big) E(t)$ (in the two level approxi-
-mation (since $\langle a|H_1|b\rangle = -e\langle a|x|b\rangle E(t)$ ✓
and $\boxed{\langle b|H_1|a\rangle = -e\langle b|x|a\rangle E(t)}$) ✓

$$\hat{V}(t) = e^{iH_0 t} H_1 e^{-iH_0 t} \Rightarrow$$

$$\left(e^{i\omega_a t}|a\rangle\langle a| + e^{i\omega_b t}|b\rangle\langle b|\right) H_1 \left(e^{-i\omega_a t}|a\rangle\langle a| + e^{-i\omega_b t}|b\rangle\langle b|\right)$$

$$\boxed{\Omega_R = \frac{e\langle a|x|b\rangle E}{\hbar}}$$

$$V(t) = e^{i(\omega_a-\omega_b)t}\langle a|H_1|b\rangle |a\rangle\langle b| + e^{i(\omega_b-\omega_a)t}\langle b|H_1|a\rangle$$

$$\Rightarrow \frac{\Omega_R}{2}\left(e^{i\nu t}+e^{-i\nu t}\right)\left[e^{i\omega t}e^{-i\phi}|a\rangle\langle b| + e^{-i\omega t}e^{i\phi}|b\rangle\langle a|\right]$$

(where ϕ is phase of the dipole matrix element). Now we get terms that go as $\omega+\nu$ and $\omega-\nu$ $\Rightarrow$

$$\frac{\Omega_R}{2}\left[e^{i(\omega+\nu)t}e^{-i\phi}|a\rangle\langle b| + e^{i(\omega-\nu)t}e^{-i\phi}|a\rangle\langle b| + e^{i(\nu-\omega)t}e^{i\phi}|b\rangle\langle a| + e^{i\phi}e^{-i(\omega+\nu)t}|b\rangle\langle a|\right)$$

Now, terms that go as $\omega+\nu$ or $-\omega-\nu$ oscillate very rapidly and give negligible contributions. Therefore, the interaction picture Interacting Hamiltonian is equal to:

$$\hat{V}(t) = -\frac{\hbar\Omega_R}{2}\left(e^{-i\phi}|a\rangle\langle b| + e^{i\phi}|b\rangle\langle a|\right)$$

(Resonance assumption $\boxed{\omega \equiv \omega_a-\omega_b = \nu}$)

Therefore, $U_I(t) \equiv T\exp\left(\frac{-i}{\hbar}\int_0^t \hat{V}(t)\right) =$

$$T\exp\left(\frac{i\Omega_R}{2}\int_0^t \left(e^{-i\phi}|a\rangle\langle b| + e^{i\phi}|b\rangle\langle a|\right)dt'\right)$$

$$\cdots \left(\frac{\cdots}{2} \right) \cdots \Big)$$

$\Rightarrow$ The time ordering operator can be neglected because $\hat{J}(t)$ is time independent. Therefore we have

$$\exp\begin{pmatrix} 0 & \frac{i\Omega_R t}{2}e^{-i\phi} \\ \frac{i\Omega_R t}{2}e^{i\phi} & 0 \end{pmatrix} = \exp\left(\frac{i\Omega_R t}{2}(\cos\phi, \sin\phi, 0)\cdot(\sigma_x, \sigma_y, \sigma_z) \right) = \mathbb{1}\cos\left(\frac{\Omega_R t}{2}\right) + i\sin\left(\frac{\Omega_R t}{2}\right)\begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{\Omega_R t}{2} & i\sin\frac{\Omega_R t}{2}e^{-i\phi} \\ i\sin\frac{\Omega_R t}{2}e^{i\phi} & \cos\frac{\Omega_R t}{2} \end{pmatrix}$$

Now, if $|\psi(0)\rangle = |a\rangle$, $|\psi_I(t)\rangle = U_I(t)|a\rangle =$

$\cos\left(\frac{\Omega_R t}{2}\right)|a\rangle + i\sin\left(\frac{\Omega_R t}{2}\right)e^{i\phi}|b\rangle$, $|\psi_S(t)\rangle =$

$$e^{-iH_0 t}|\psi_I(t)\rangle = e^{-i\omega_a t}\cos\frac{\Omega_R t}{2}|a\rangle + ie^{-i\omega_b t}\sin\frac{\Omega_R t}{2}e^{i\phi}|b\rangle$$

Note that if dipole moment $\langle a|X|b\rangle$ is zero, we recover the free evolution of $|\psi(t)\rangle$.

$$V = er(t)\cdot E(t) = -e\,e^{iH_0 t}\, r\, e^{-iH_0 t}\cdot E(t) \Rightarrow$$

$$V_{ab} = -e\, r_{ab} \cdot E(t)\, e^{i\omega t}, \quad V_{ba}(t) = -e\, r_{ba} \cdot E(t)\, e^{-i\omega t}$$

Take linear polarization: $E(t) = \hat{x} E_0 \cos(\nu t) \Rightarrow$

$$V_{ab}(t) = -\frac{e(x_{ab})}{2} E_0 \left(e^{i(\omega-\nu)t} + e^{i(\omega+\nu)t} \right)$$

$\Rightarrow$ Rotating wave approximation neglect $\omega+\nu$, $-\omega-\nu$ components.

For LCP (left handed polarization of light)

$$E(t) = E_0 (\hat{e}_x \cos\nu t - \hat{e}_y \sin\nu t) \Longrightarrow$$

$$V_{ab}(t) = -e E_0 \left(x_{ab} \cos\nu t\, e^{i\omega t} - y_{ab} \sin\nu t\, e^{i\omega t} \right)$$

$$x_{ab} = \int \psi_a^*(r)\, x\, \psi_b(r)\, dr = D,$$

$$y_{ab} = \int \psi_a^*(r)\, y\, \psi_b(r)\, dr = -iD \quad \text{check:}$$

$$x_{ab} \propto \int (x - iy)\, x\, e^{-r/2a_0} e^{-r/a_0}$$

$$y_{ab} \propto \int (x - iy)\, y\, e^{-r/2a_0} e^{-r/a_0} \Longrightarrow$$

$$x_{ab} \propto \int x^2 e^{-3r/2a_0} r^2\, d\Omega\, dr$$

$$y_{ab} \propto \int -iy^2 e^{-3r/2a_0} r^2\, d\Omega\, dr \Rightarrow y_{ab} = -i\, x_{ab}$$

Therefore $V_{ab}(t) = -eE_0 X_{ab}(\cos\nu t - i\sin\nu t)e^{i\omega t}$

$V_{ba}(t) = -eE_0 X_{ba}(\cos\nu t + i\sin\nu t)\,e^{-i\omega t}$

$\Rightarrow$ $\omega+\nu$ and $-\omega-\nu$ terms do not appear and rotating wave approximation is exact.

Atom-field interaction Hamitonians

$$H = H_A + H_F - e\mathbf{r}\cdot\mathbf{E}, \quad H_F = \sum_k \hbar\omega_k\left(a_k^\dagger a_k + \tfrac{1}{2}\right)$$

$$E = \sum_k \hat{\varepsilon}_k \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\left(a_k + a_k^\dagger\right) \Rightarrow$$

$$H = \sum_k \hbar\omega_k\left(a_k^\dagger a_k + \tfrac{1}{2}\right) + \sum_i E_i |i\rangle\langle i| +$$
$$\hbar \sum_{ij}\sum_k \frac{e(r_{ij}\cdot\hat{\varepsilon}_k)}{\hbar}\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\left(a_k + a_k^\dagger\right)$$

Check that E has units of electric field:

$$E^2 \sim \frac{eV}{\varepsilon_0 L^3} \rightarrow \varepsilon_0 |E|^2 \text{ is an energy density}$$

Therefore, QED.

In two level approximation, we have that

[illegible]

$H = \sum_i \hbar\omega_k a_k^\dagger a_k + \frac{1}{2}\hbar\omega\sigma_z + \sum_i \hbar g_k(\sigma_+ + \sigma_-)(a_k + a_k^\dagger)$, Note that $\sigma_+ a_k^\dagger$ and $\sigma_- a_k$ correspond to energy non-conserving terms $\Rightarrow$ dropping them corresponds to a rotating wave approximation.

Therefore, $H = \sum_k \hbar\omega_k a_k^\dagger a_k + \frac{1}{2}\hbar\omega\sigma_z + \hbar\sum_i g_k(\sigma_+ a_k + \sigma_- a_k^\dagger)$

Single mode optics in interaction picture then gives us: $\hat{V}^I(t) = \hbar g\left(\sigma_+ a\, e^{i(\omega-\nu)t} + a^\dagger \sigma_- e^{-i(\omega-\nu)t}\right)$

$U_I(t) = T\exp\left(\frac{-i}{\hbar}\int_0^t V_I(t)\,dt\right) \Rightarrow$ at resonance, we have, $U(t) = \exp\left(-\frac{it}{\hbar}\hat{V}_I\right)$

$= \exp\left[\begin{pmatrix} 0 & -\frac{it}{\hbar}a \\ -\frac{it}{\hbar}a_k^\dagger & 0 \end{pmatrix}\right]$

Note that $\dot{U}_I(t) = \frac{-i}{\hbar} V_I(t) U_I(t) \longrightarrow$

$$\int_0^t \dot{U}_I(t')\,dt' = \frac{-i}{\hbar}\int_0^t V_I(t') U_I(t') \Rightarrow$$

$$U_I(t) = U_I(0) - \frac{i}{\hbar}\int_0^t V_I(t')\, U_I(t')\, dt'$$

$\Rightarrow$ to second order we have

$$\boxed{U_I(0) = I} + \left(\frac{-i}{\hbar}\right)\int_0^t V_I(t') U_I(t') dt' +$$

$$\left(\frac{-i}{\hbar}\right)^2 \int_0^t \left[\int_0^{t_1} V_I(t_1)\, V_I(t_2)\, U_I(t_2)\, dt_2\right] dt_1 \Rightarrow$$

QED

Reservoir Theory:

Heisenberg - Langevin Approach

pg. 362 of Scully & Zubairy.

At resonance, $\boxed{\nu = \omega \equiv \omega_a - \omega_b}$,
the interaction picture interacting
Hamiltonian is given by $\hbar g \sum_i f(t, t_i)$

$(a^\dagger \sigma_-^j + \sigma_+^j a)$, $f(t, t_j) = \theta(t - t_j) e^{-\gamma(t-t_j)/2}$

where $\theta(t - t_j)$ maintains causality.

Now, we know that $A(t) = e^{+iH_0 t/\hbar} A_s e^{-iH_0 t/\hbar}$

$\Rightarrow$ for some reason, Scully and Zubairy look at Heisenberg picture? . . .

$\dot{a} = \frac{i}{\hbar}[H, a] \Rightarrow \sim \sum_t \dot{f}(t, t_j)\sigma_-^j$

$\sigma_- \sigma_+ - \sigma_+ \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} -$

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \propto \boxed{\sigma_z}$

$i\hbar \frac{\partial}{\partial t}|\psi^I\rangle = V^I |\psi^I\rangle$

I think what Scully is trying to get across is that if we have

$a|\psi^0\rangle$ $i\hbar \frac{\partial}{\partial t}(a|\psi^0\rangle) = V^I a|\psi^0\rangle$

actually, this is different.

Regardless, following Scully, we get:

$$\sigma_-^j = \sigma_-^j(t_j) + ig\int_{t_j}^{t} dt' f(t', t_j)\sigma_z^j(t')\, a(t')$$

A (linear gain coefficient) $= \dfrac{2g^2 r_a}{\gamma^2}$

Back to density Matrix Approach:

Markovian White Noise

$$H = \hbar\nu a^\dagger a + \sum_k \hbar\nu_k b_k^\dagger b_k + \hbar\sum_k g_k(b_k^\dagger a + a^\dagger b_k), \quad \dot{a} = \frac{i}{\hbar}[H, a] = -i\nu a - i\sum_k g_k b_k(t)$$

$\Rightarrow$ Equation reservoir operators

$$\int_0^t [\dot{b}_k + i\nu_k b_k(t')]\, dt' = -ig_k\int_0^t a(t')\, dt'$$

$$b_k(t) = b_k(0)\, e^{-i\nu_k t} - i g_k \int_0^t dt'\, a(t')\, e^{-i\nu_k(t-t')}$$

Check: $\dot{b}_k(t) \underset{t}{=} -i\nu_k b_k(0) e^{-i\nu_k t} - i g_k \Big[a(t) - i\nu_k \int_0^t dt'\, a(t')\, e^{-i\nu_k(t-t')} \Big]$

$= -i\nu_k b_k - i g_k a(t)$ ✓ <u>QED</u>

$$\frac{da}{dt} = -i\nu a(t) - \sum_k g_k^2 \int_0^t dt'\, a(t')\, e^{-i\nu_k(t-t')} + $$

$$- i \sum_k g_k b_k(0) e^{-i\nu_k t}$$

<u>Slowly varying annihilation operators:</u> $\underline{\tilde{a}(t) = a(t)\, e^{i\nu t}}$

$$\Rightarrow \dot{\tilde{a}}(t) e^{-i\nu t} = -\sum_k g_k^2 \int_0^t dt'\, \tilde{a}(t')\, e^{-i\nu t'} \times e^{-i\nu_k(t-t')}$$

e

$$\Rightarrow \dot{\tilde{a}}(t) = -\sum_k g_k^2 \int_0^t dt' \, \tilde{a}(t') e^{i\nu(t-t')} \times$$

$$e^{-i\nu_k(t-t')} \Big]$$

$$\text{Also, } \dot{\tilde{a}} = -\sum_k g_k^2 \int_0^t dt' \, \tilde{a}(t') e^{-i(\nu_k-\nu)(t-t')}$$

$$+ F_{\tilde{a}}(t), \quad F_{\tilde{a}}(t) = \underline{e^{i\nu t} f_a(t)}$$

$$\Rightarrow \int_0^t F_{\tilde{a}}(t') \, dt' = -i \int \sum_k g_k b_k(0) \times$$

$$e^{-i(\nu_k-\nu)t} \Rightarrow \int_0^t dt' \, e^{-i(\nu_k-\nu)t'}$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{i\omega t} = 2\pi \delta(\omega) \Longrightarrow \text{for large}$$

times we have $\int_0^t F_{\tilde{a}}(t') dt' =$

$$\boxed{\pi \sum_i' g_k b_k(0)\, \delta(\nu_k - \nu)}$$

$$\sum_i' g_k^2 \int_0^t dt' \tilde{a}(t') e^{-i(\nu_k-\nu)(t-t')}$$

$$D(\nu) = \sum_i' \delta(\nu - ck) =$$

$$\frac{2V}{(2\pi)^3} \int 4\pi k^2 \delta(\nu - ck)\, dk =$$

$$\delta(f(k)) = \sum_{k_0} \frac{\delta(k-k_0)}{|f'(k_0)|}$$

$$\boxed{\frac{V g_s}{2\pi^2} \frac{\nu^2}{c^3}}$$, answer in

$$D(\nu)\, \pi\, g^2(\nu/c)\, \tilde{a}(t) \quad (9.1.13)$$

Check:

$$\sum_k' \longrightarrow \int D(\nu_k)\, d\nu_k \Longrightarrow$$

$$\int D(\nu_k)\, g_k^2 \int_0^t dt' \tilde{a}(t') e^{-i(\nu_k - \nu)(t-t')}$$

$\Rightarrow D(\nu_k)$ varies little around $\nu \Rightarrow$

$$\int_{-\infty}^{\infty} d\nu_k \, e^{i(\nu-\nu_k)(t-t')} = 2\pi\delta(t-t')$$

$$= \boxed{e^{i\nu(t-t')}\delta(t-t') = \delta(t-t')}$$

Therefore, we get

$$\boxed{D(\nu)\, g^2_{\nu/c}\, \pi \tilde{a}(t)}$$

$$\sum_k \to \int D(\nu_k)\, d\nu_k$$

Prove equivalence: let

$$\int f(\varepsilon) D(\varepsilon)\, d\varepsilon = \int f(\varepsilon) \sum_k \delta(\varepsilon-\varepsilon_k)\, d\varepsilon$$

$$= \sum f(\varepsilon_k) \Rightarrow \underline{QED}$$

$\bar{K}$

Spontaneous emission enhancement

$\Gamma = 2\pi \langle |g(\omega)|^2 \rangle D(\omega) \rightarrow$

$\frac{\Gamma_{cavity}}{\Gamma}$ (for ν (cavity resonance) $\simeq$

ω_{ab} (atomic transition) $\Rightarrow$

$2\pi \langle |g(\omega)|^2 \rangle \frac{2Q}{\pi\omega}$,

$|g(\omega)|^2 \sim |d \cdot E|^2 = |d_{ab}|^2 \left(\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \right)^2$

$= |d_{ab}|^2 \frac{\hbar\omega}{2\varepsilon_0} \Rightarrow$ I believe

Scully includes factor of $1/3$

since $|x_{ab}|^2 = \frac{1}{3} |r_{ab}|^2$