Hierarchical Bayesian Learning for Noise Estimation in Electromagnetic Brain Source Imaging

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joint work with Chang Cai, Yijing Gao, Sanjay Ghosh,
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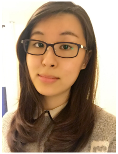
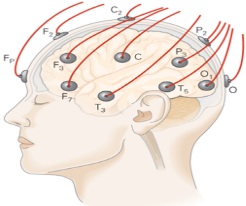
     

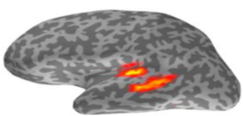
Electromagnetic Brain Source Imaging (BSI)

Electro-/Magnetoencephalography (E/MEG): A non-invasive brain imaging technique with high temporal resolution (order of ms).

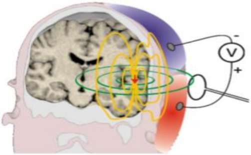
$\mathbf{Y} = \mathbf{L}\mathbf{X} + \mathbf{E}$	**Linear regression problem**
$\mathbf{Y} \in \mathbb{R}^{M \times T}$	M:#Sensors, T:#Samples,
$\mathbf{X} \in \mathbb{R}^{N \times T}$	N:#Sources, ($M \ll N$)
$\mathbf{L} \in \mathbb{R}^{M \times N}$	Lead Field Matrix (Known)



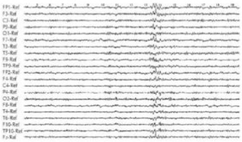
$\mathbf{X}$



$\mathbf{L}$

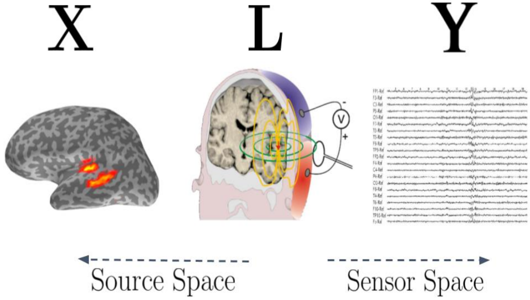


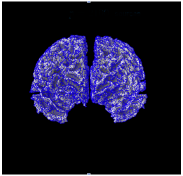
$\mathbf{Y}$



Source Space ⟵ - ⟶ Sensor Space

Electromagnetic Brain Source Imaging (BSI)





[Macsymenko '09][Massias '19]

Ill-posed inverse problem: (#Sensors$= 32 \sim 256$ vs #Sources$= 10^3 \sim 10^4$)

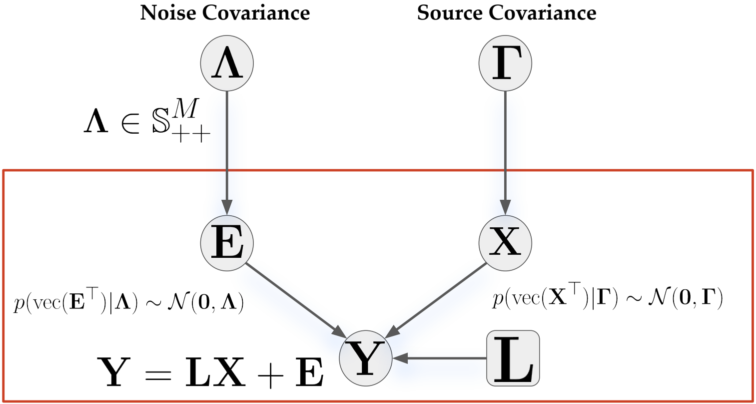
$$\mathbf{X}^* = \underset{\mathbf{X}}{\operatorname{argmin}} \; \underbrace{\|\mathbf{Y} - \mathbf{L}\mathbf{X}\|_F^2}_{\text{Likelihood:}p(\mathbf{Y}|\mathbf{X})} + \lambda \; \underbrace{\mathcal{R}(\mathbf{X})}_{\text{Prior:}p(\mathbf{X})}$$

1. Type-I MAP methods: ℓ_1, ℓ_2, $\ell_{1,2}$-norms, sparsity in transformed domains (Gabor).
   [Pascual-Marqui et al., '07][Haufe et al, '08, '11][Gramfort et al., '12, '13][Castaño-Candamil et al., '15]
2. Type-II ML approaches: different sparse Bayesian learning (SBL) variants ignoring the temporal dynamics. [Wipf et al., '09, '10, '11][Owen et al, '12][Cai et al., '17, '21]

Hierarchical Bayesian Learning

Probabilistic graphical model:

- Model the current activity of the brain sources as Gaussian scale mixtures
- $\mathbf{\Lambda}$: Noise covariance with full structure



Hierarchical Bayesian Inference and Type-II Loss

Posterior source distribution: $p(\mathbf{X}|\mathbf{Y}, \mathbf{\Gamma}, \mathbf{\Lambda}) = \prod_{t=1}^{T} \mathcal{N}(\bar{\mathbf{x}}(t), \mathbf{\Sigma}_{\mathbf{x}})$
where

$$\begin{aligned}
\bar{\mathbf{x}}(t) &= \mathbf{\Gamma}\mathbf{L}^{\top}(\mathbf{\Sigma}_{\mathbf{y}})^{-1}\mathbf{y}(t) \\
\mathbf{\Sigma}_{\mathbf{x}} &= \mathbf{\Gamma} - \mathbf{\Gamma}\mathbf{L}^{\top}(\mathbf{\Sigma}_{\mathbf{y}})^{-1}\mathbf{L}\mathbf{\Gamma} \\
\mathbf{\Sigma}_{\mathbf{y}} &= \mathbf{\Lambda} + \mathbf{L}\mathbf{\Gamma}\mathbf{L}^{\top} .
\end{aligned}$$

$\mathbf{\Gamma}$, and $\mathbf{\Lambda}$ are learned by minimizing the negative log marginal likelihood (Type-II) loss, $-\log p(\mathbf{Y}|\mathbf{\Gamma}, \mathbf{\Lambda})$:

$$\text{Type} - \text{II Loss} : \mathcal{L}^{\text{II}}(\mathbf{\Gamma}, \mathbf{\Lambda}) = \log|\mathbf{\Lambda} + \mathbf{L}\mathbf{\Gamma}\mathbf{L}^{\top}| + \frac{1}{T}\sum_{t=1}^{T} \mathbf{y}(t)^{\top} \left(\mathbf{\Lambda} + \mathbf{L}\mathbf{\Gamma}\mathbf{L}^{\top}\right)^{-1} \mathbf{y}(t) .$$

Challenges

$$\text{Type} - \text{II Loss}: \mathcal{L}^{\mathrm{II}}(\boldsymbol{\Gamma}, \boldsymbol{\Lambda}) = \log|\boldsymbol{\Lambda} + \mathbf{L}\boldsymbol{\Gamma}\mathbf{L}^{\top}| + \frac{1}{T}\sum_{t=1}^{T}\mathbf{y}(t)^{\top}\left(\boldsymbol{\Lambda} + \mathbf{L}\boldsymbol{\Gamma}\mathbf{L}^{\top}\right)^{-1}\mathbf{y}(t)\,.$$

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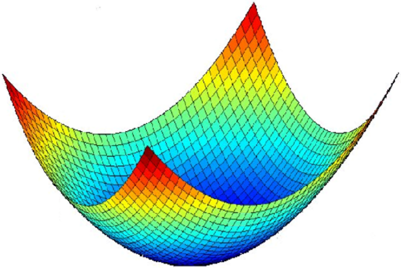
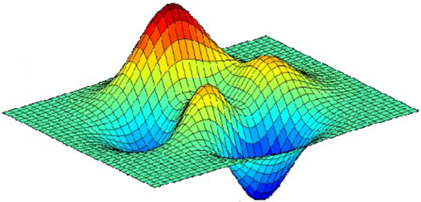
1. Non-convex Type-II ML loss function: Non-trivial to solve.



convex function

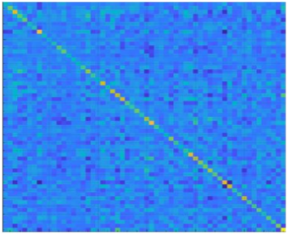


non-convex function

Challenges

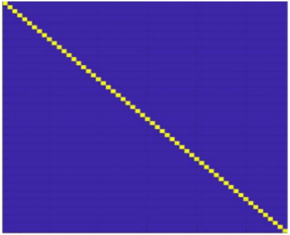
$$\text{Type} - \text{II Loss}: \mathcal{L}^{\text{II}}(\boldsymbol{\Gamma}, \boldsymbol{\Lambda}) = \log|\boldsymbol{\Lambda} + \mathbf{L}\boldsymbol{\Gamma}\mathbf{L}^\top| + \frac{1}{T}\sum_{t=1}^{T}\mathbf{y}(t)^\top \left(\boldsymbol{\Lambda} + \mathbf{L}\boldsymbol{\Gamma}\mathbf{L}^\top\right)^{-1}\mathbf{y}(t) .$$

1. Non-convex Type-II ML loss function: Non-trivial to solve.
2. Most contributions in the literature are limited to the estimation of only a diagonal noise covariance: Limiting assumption in practice.



Full structure

$\boldsymbol{\Lambda}$



Diagonal

Challenges

$$\text{Type} - \text{II Loss}: \mathcal{L}^{\text{II}}(\mathbf{\Gamma}, \mathbf{\Lambda}) = \log|\mathbf{\Lambda} + \mathbf{L}\mathbf{\Gamma}\mathbf{L}^\top| + \frac{1}{T}\sum_{t=1}^{T} \mathbf{y}(t)^\top \left(\mathbf{\Lambda} + \mathbf{L}\mathbf{\Gamma}\mathbf{L}^\top\right)^{-1} \mathbf{y}(t) \,.$$

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Our contribution: Joint estimation of Gaussian regression parameter distributions and Gaussian noise distributions with full covariance structure by solving the non-convex Type-II loss.

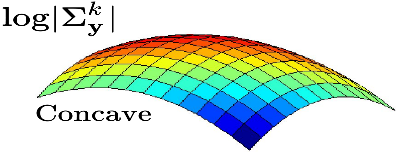
Convex Majorizing Functions

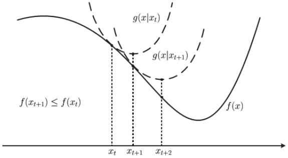
Theorem

Optimizing $\mathcal{L}^{\mathrm{II}}(\boldsymbol{\Gamma}, \boldsymbol{\Lambda})$ with respect to $\boldsymbol{\Lambda}$ is equivalent to optimizing the following majorzing function:

$$\mathcal{L}^{\mathrm{conv}}_{\mathrm{noise}}(\boldsymbol{\Gamma}^k, \boldsymbol{\Lambda}) = \mathrm{tr}\left(\left(\boldsymbol{\Sigma}^k_{\mathbf{y}}\right)^{-1} \boldsymbol{\Lambda} \right) + \mathrm{tr}(\mathbf{M}^k_{\mathrm{N}} \boldsymbol{\Lambda}^{-1}) \,,$$

where $\mathbf{M}^k_{\mathrm{N}} := \frac{1}{T} \sum_{t=1}^{T} (\mathbf{y}(t) - \mathbf{L}\bar{\mathbf{x}}^k(t))(\mathbf{y}(t) - \mathbf{L}\bar{\mathbf{x}}^k(t))^\top$ *and* $\boldsymbol{\Sigma}^k_{\mathbf{y}} = \boldsymbol{\Lambda}^k + \mathbf{L}\boldsymbol{\Gamma}^k\mathbf{L}^\top$.





[Sun et al., '17][Razzaviyan et al., '13]

$$\boxed{\mathcal{L}^{\mathrm{II}}(\boldsymbol{\Gamma}, \boldsymbol{\Lambda}) = \log|\boldsymbol{\Lambda} + \mathbf{L}\boldsymbol{\Gamma}\mathbf{L}^\top| + \frac{1}{T} \sum_{t=1}^{T} \mathbf{y}(t)^\top \left(\boldsymbol{\Lambda} + \mathbf{L}\boldsymbol{\Gamma}\mathbf{L}^\top\right)^{-1} \mathbf{y}(t)}$$

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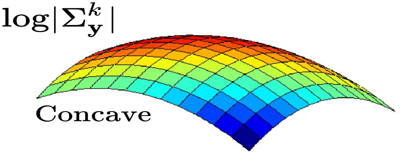

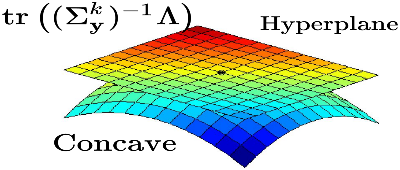

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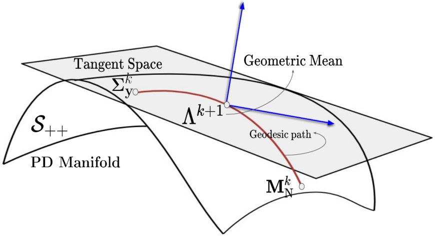
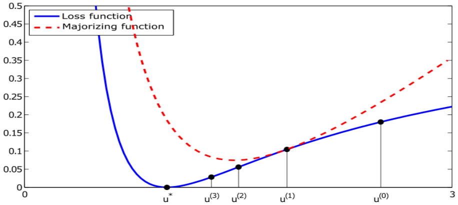
Theorem (Geometric mean)

$\mathcal{L}_{\text{noise}}^{\text{conv}}(\mathbf{\Gamma}^k, \mathbf{\Lambda})$ is geodesically convex with respect to the PD manifold, and its optimal solution with respect to $\mathbf{\Lambda}$ can be attained according to the following update rule:

$$\mathbf{\Lambda}^{k+1} \leftarrow (\mathbf{\Sigma}_{\mathbf{y}}^k)^{\frac{1}{2}} \left((\mathbf{\Sigma}_{\mathbf{y}}^k)^{-1/2} \mathbf{M}_{\mathrm{N}}^k (\mathbf{\Sigma}_{\mathbf{y}}^k)^{-1/2} \right)^{\frac{1}{2}} (\mathbf{\Sigma}_{\mathbf{y}}^k)^{\frac{1}{2}} \, .$$

which leads to a majorization-minimization (MM) algorithm with convergence guarantees $\rightsquigarrow$ Full-structural noise (FUN) learning algorithm.





Riemannian Update on the Manifold of PD Matrices

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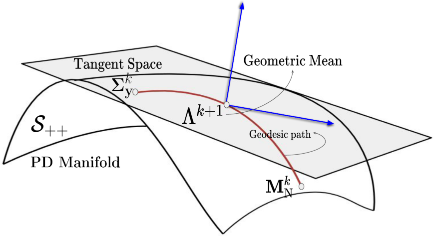

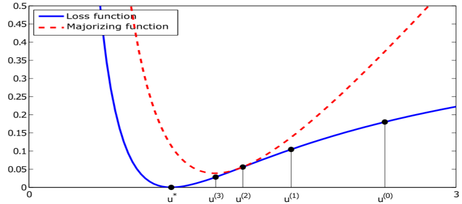

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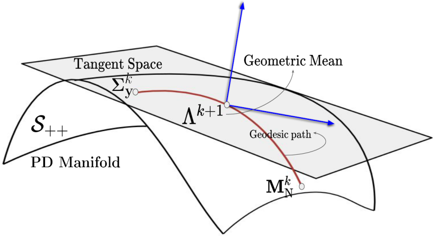
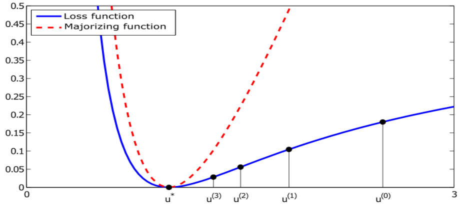
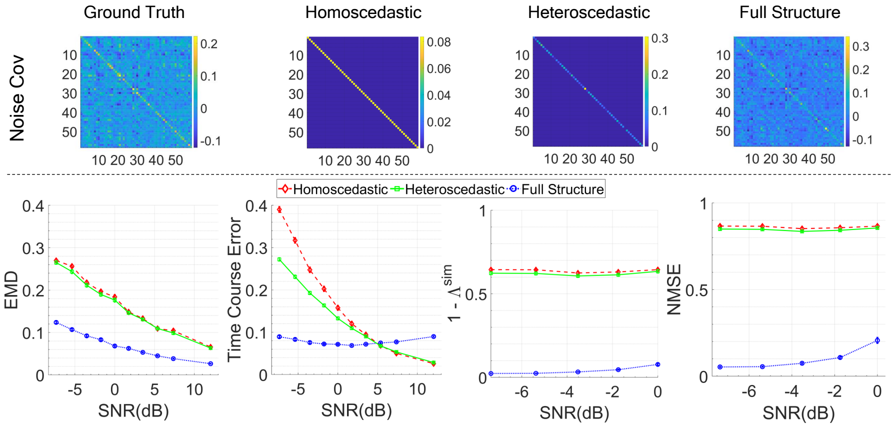




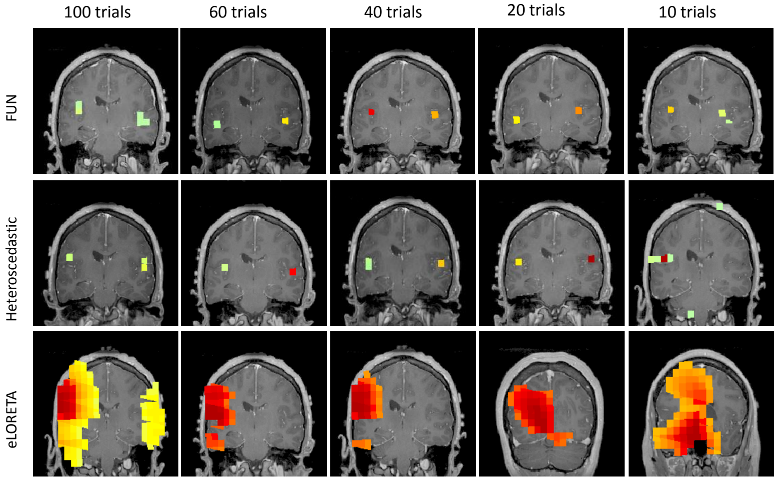
Numerical Results

Conclusion I: FUN learning consistently outperforms its homoscedastic and heteroscedastic counterparts according to all evaluation metrics.



Real Data Analysis of Auditory Evoked Fields (AEF)

Conclusion II: FUN learning can provide accurate reconstruction at the expected locations of the auditory cortex even under extreme SNR conditions - superior to benchmarks.



Thank you For yoUr attentioN!

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