Joint Hierarchical Bayesian Learning of Full-structure Noise for Brain Source Imaging

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joint work with Chang Cai, Klaus-Robert Müller, Srikantan S. Nagarajan, and Stefan Haufe

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Electromagnetic Brain Source Imaging (BSI)

$\mathbf{Y} = \mathbf{L}\mathbf{X} + \mathbf{E}$	**Linear regression problem**
$\mathbf{Y} \in \mathbb{R}^{M \times T}$	M:#Sensors, T:#Samples,
$\mathbf{X} \in \mathbb{R}^{N \times T}$	N:#Sources, ($M \ll N$)
$\mathbf{L} \in \mathbb{R}^{M \times N}$	Lead Field Matrix (Known)

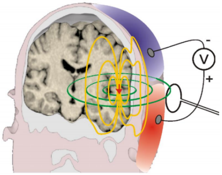


Figure by Lauri Parkkonen

Ill-posed inverse problem:

1. Type-I MAP methods: ℓ_1, ℓ_2, $\ell_{1,2}$-norms, sparsity in transformed domains (Gabor).
2. Type-II ML approaches: different sparse Bayesian learning (SBL) variants.

Hierarchical Bayesian regression:

- Source distribution: $p(\mathbf{X}|\boldsymbol{\Gamma}) = \prod_{t=1}^{T} \mathcal{N}(0, \boldsymbol{\Gamma})$, $\boldsymbol{\Gamma}$: Hyper-parameters
- Noise distribution: $\mathbf{E} = [\mathbf{e}(1), \ldots, \mathbf{e}(T)]$, $\mathbf{e}(t) \sim \mathcal{N}(0, \boldsymbol{\Lambda})$, $t = 1, \cdots, T$
  $\boldsymbol{\Lambda}$: Noise covariance with full structure
- Measurement distribution: $p(\mathbf{Y}|\mathbf{X}) = \prod_{t=1}^{T} \mathcal{N}(\mathbf{L}\mathbf{x}(t), \boldsymbol{\Lambda})$

Type-II Bayesian Learning

Posterior source distribution: $p(\mathbf{X}|\mathbf{Y},\boldsymbol{\Gamma}) = \prod_{t=1}^{T} \mathcal{N}(\bar{\mathbf{x}}(t), \boldsymbol{\Sigma}_{\mathbf{x}})$, with

$$\bar{\mathbf{x}}(t) = \boldsymbol{\Gamma}\mathbf{L}^{\top}(\boldsymbol{\Sigma}_{\mathbf{y}})^{-1}\mathbf{y}(t) \quad \boldsymbol{\Sigma}_{\mathbf{x}} = \boldsymbol{\Gamma} - \boldsymbol{\Gamma}\mathbf{L}^{\top}(\boldsymbol{\Sigma}_{\mathbf{y}})^{-1}\mathbf{L}\boldsymbol{\Gamma} \quad \boldsymbol{\Sigma}_{\mathbf{y}} = \boldsymbol{\Lambda} + \mathbf{L}\boldsymbol{\Gamma}\mathbf{L}^{\top},$$

as a result of learning $\boldsymbol{\Gamma}$ and $\boldsymbol{\Lambda}$ through minimizing $-\log p(\mathbf{Y}|\boldsymbol{\Gamma},\boldsymbol{\Lambda})$:

$$\mathrm{Type-II\ Loss}: \mathcal{L}^{\mathrm{II}}(\boldsymbol{\Gamma},\boldsymbol{\Lambda}) = \log|\boldsymbol{\Lambda} + \mathbf{L}\boldsymbol{\Gamma}\mathbf{L}^{\top}| + \frac{1}{T}\sum_{t=1}^{T}\mathbf{y}(t)^{\top}\left(\boldsymbol{\Lambda} + \mathbf{L}\boldsymbol{\Gamma}\mathbf{L}^{\top}\right)^{-1}\mathbf{y}(t)\ .$$

1. Non-convex Type-II ML loss function: Non-trivial to solve.
2. Most contributions in the literature are limited to the estimation of only a diagonal noise covariance: Limiting assumption in practice.

Our contribution: Joint estimation of Gaussian regression parameter distributions and Gaussian noise distributions with full covariance structure.

Convex Majorizing Functions

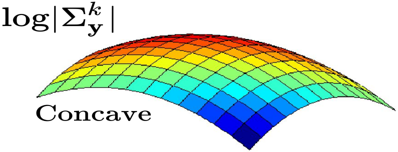
Theorem

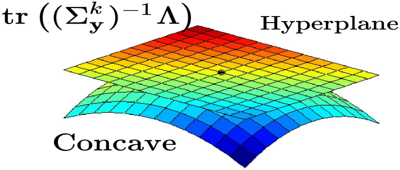
Optimizing $\mathcal{L}^{\mathrm{II}}(\mathbf{\Gamma}, \mathbf{\Lambda})$ *with respect to* $\mathbf{\Lambda}$ *is equivalent to optimizing the following majorzing function:*

$$\mathcal{L}_{\mathrm{noise}}^{\mathrm{conv}}(\mathbf{\Gamma}^k, \mathbf{\Lambda}) = \mathrm{tr}\big(\left(\mathbf{C}_{\mathrm{N}}^k\right)^{-1} \mathbf{\Lambda}\big) + \mathrm{tr}(\mathbf{M}_{\mathrm{N}}^k \mathbf{\Lambda}^{-1}) \ ,$$

where $\mathbf{C}_{\mathrm{N}}^k$ *and* $\mathbf{M}_{\mathrm{N}}^k$ *are defined as:*

$$\mathbf{C}_{\mathrm{N}}^k := \left(\mathbf{\Sigma}_{\mathbf{y}}^k\right) \ , \quad \mathbf{M}_{\mathrm{N}}^k := \frac{1}{T}\sum_{t=1}^{T}(\mathbf{y}(t) - \mathbf{L}\bar{\mathbf{x}}^k(t))(\mathbf{y}(t) - \mathbf{L}\bar{\mathbf{x}}^k(t))^\top \ .$$





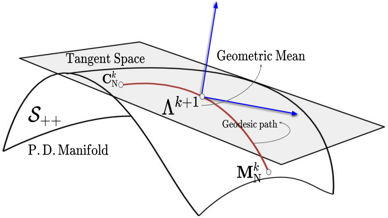
MM Framework with Convergence Guarantees

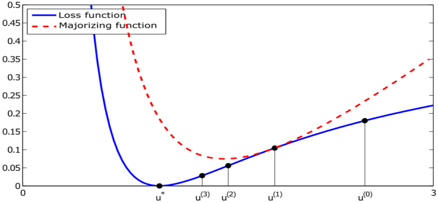
Theorem

$\mathcal{L}^{\text{conv}}_{\text{noise}}(\boldsymbol{\Gamma}^k, \boldsymbol{\Lambda})$ is geodesically convex with respect to the P.D. manifold, and its optimal solution with respect to $\boldsymbol{\Lambda}$ can be attained according to the following update rule:

$$\boldsymbol{\Lambda}^{k+1} \leftarrow (\mathbf{C}^k_{\mathrm{N}})^{\frac{1}{2}} \left((\mathbf{C}^k_{\mathrm{N}})^{-1/2} \mathbf{M}^k_{\mathrm{N}} (\mathbf{C}^k_{\mathrm{N}})^{-1/2} \right)^{\frac{1}{2}} (\mathbf{C}^k_{\mathrm{N}})^{\frac{1}{2}} \ ,$$

which leads to a majorization-minimization (MM) algorithm with convergence guarantees $\rightsquigarrow$ Full-structural noise (FUN) learning algorithm.





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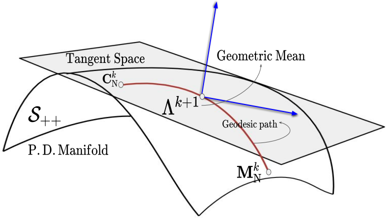

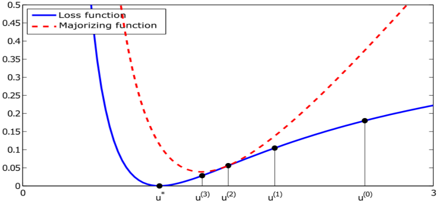

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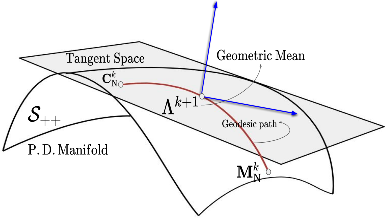
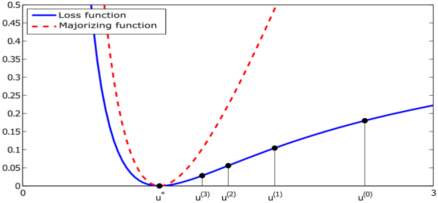
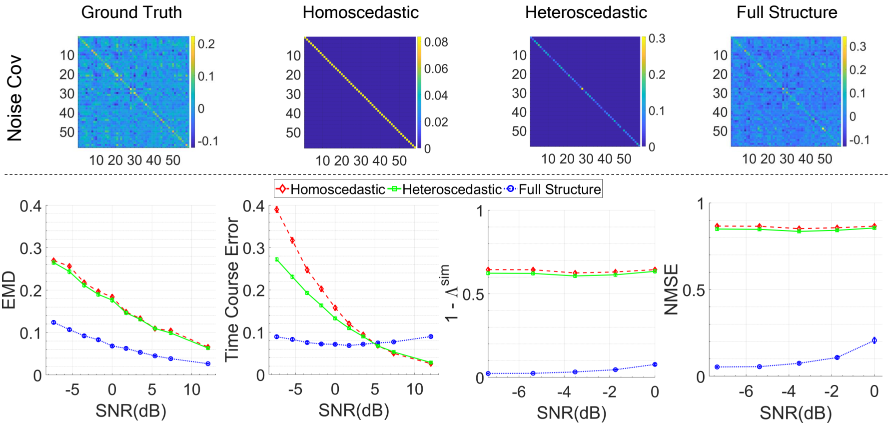


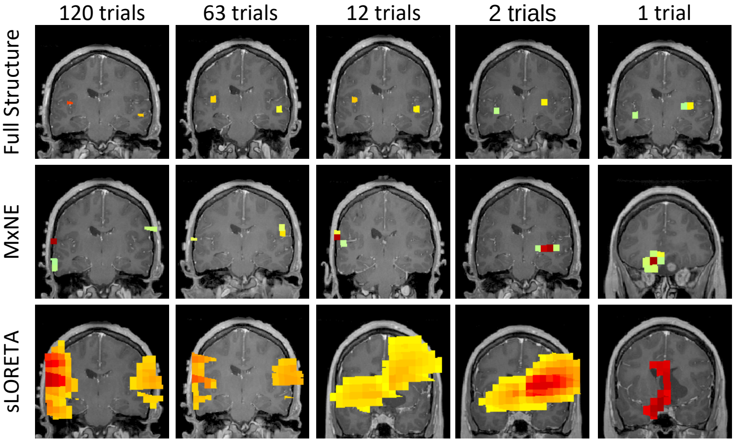


Numerical Results

Conclusion I: FUN learning consistently outperforms its homoscedastic and heteroscedastic counterparts according to all evaluation metrics.



Conclusion II: FUN learning can provide accurate reconstruction even under extreme SNR conditions - superior to benchmarks.



Thank you For yoUr attentioN!

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