Math For Artificial Intelligence

"Set Theory, Mathematical Logic: Building Blocks for AI Mathematics"

SECTION 1: FUNDAMENTALS OF SETS

What is a Set?

A set is a collection of well-defined and distinct objects.

- Well-defined means: You can clearly decide whether an object belongs to the set or not.
- The objects must be different from each other. A set does not include duplicates.

Think of a set like a box or a list that contains selected items that follow a clear rule.

Examples of Sets:

Set of vowels in English:
A = {a, e, i, o, u}

Set of even natural numbers:
E = {2, 4, 6, 8, 10, 12, ...}

Set of months with 31 days:
M = {January, March, May, July, August, October, December}

Examples that are NOT Sets:

"Set of beautiful flowers"
→ Not well-defined. Different people have different opinions about what is beautiful.

"Set of smart students"
→ Not clear who is included. The meaning of "smart" is subjective.

Elements or Members of a Set

Each object inside a set is called an element or member of the set.

Symbol Representation:

If an element belongs to a set, we write:
$a \in A$ (read as: "a belongs to A")

If an element does not belong to a set, we write:
$b \notin A$ (read as: "b does not belong to A")

Example:

Let $A = \{1, 2, 3, 4\}$

- $2 \in A$
- $5 \notin A$

Set Notation

In mathematics, we follow standard rules when naming sets and their elements:

Type	Symbol Example
Set name	**A, B, C, D**
Elements	**a, b, c, x, y**

Example:

Let $B = \{\text{apple, banana, mango}\}$

- "banana" is an element of B → banana $\in B$
- "grape" is not in the set → grape $\notin B$

Methods of Describing Sets

There are **three common ways** to describe or write sets:

1. Descriptive Method (Descriptive Form)

In this method, we **describe the elements of the set in words**.
It tells what kind of elements the set has, **without listing them one by one**.

Example:
Set of all even numbers less than 10.
This means the set includes 2, 4, 6, and 8 — but we are not writing them. We are just **describing** them.

2. Tabular Method (Roster Form)

In this method, we **list all elements** of the set inside curly brackets {}.
All elements are **separated by commas**.

Example:
A = {2, 4, 6, 8}
Here, set A contains all even numbers less than 10.
Each element is written clearly.

Note: If the set has too many elements, we may use "..." to show that it continues.
Example: B = {1, 2, 3, ..., 100}

3. Set Builder Method (Set Builder Notation)

In this method, we use **a rule or condition** to define all the elements in the set.
We use a **variable (like x)** and **write the condition it must satisfy**.

Format:
A = {x | condition about x}

This is read as:
"A is the set of all x such that (the condition goes here)."

Example:

A = {x | x is an even number, and x < 10}

This means the same as A = {2, 4, 6, 8}, but it is written using a rule.

Set A is the set of all x **such that** x is an even number and x is less than 10.

Some Important Standard Sets

In mathematics, certain sets are used frequently. These are known as **standard sets**.

$\mathbb{N}$ – Natural Numbers

The set of all positive counting numbers starting from 1.

$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$

These are the numbers we naturally count with, like 1, 2, 3, and so on. Zero is not included in this set.

$\mathbb{W}$ – Whole Numbers

The set of all natural numbers **including 0**.

$\mathbb{W} = \{0, 1, 2, 3, 4, 5, \ldots\}$

It is like $\mathbb{N}$ but also contains 0.

$\mathbb{Z}$ – Integers

The set of all whole numbers and their negatives.

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$\mathbb{Z}$ includes both positive and negative numbers, as well as zero.

$\mathbb{Z}^+$ – Positive Integers

The set of all **positive** whole numbers (natural numbers).

$\mathbb{Z}^+ = \{1, 2, 3, 4, 5, \ldots\}$

Same as $\mathbb{N}$. Sometimes $\mathbb{Z}^+$ is used to emphasize positivity in equations.

$\mathbb{O}$ – Odd Numbers

The set of all positive odd numbers.

$\mathbb{O} = \{1, 3, 5, 7, 9, \ldots\}$

These are numbers that are not divisible by 2. They leave a remainder of 1.

$\mathbb{E}$ – Even Numbers

The set of all positive even numbers.

$\mathbb{E} = \{2, 4, 6, 8, 10, \ldots\}$

Even numbers are divisible by 2. No remainder.

$\mathbb{Q}$ – Rational Numbers

The set of numbers that can be expressed as a fraction.

$\mathbb{Q} = \{ p/q \mid p, q \in \mathbb{Z}, q \neq 0 \}$

$\mathbb{Q}^c$ – Irrational Numbers

The set of real numbers that **cannot** be written as fractions.

$\mathbb{Q}^c = \{x \in \mathbb{R} \mid x \notin \mathbb{Q}\}$

- These are numbers that go on forever without repeating.
- Examples: $\sqrt{2}$, π (pi), e (Euler's number).
- The superscript **c** stands for "complement."
  So $\mathbb{Q}^c$ means "all real numbers **not** in $\mathbb{Q}$."

$\mathbb{R}$ – Real Numbers

The set of all rational **and** irrational numbers.

$\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$

- $\mathbb{R}$ includes every number that can be placed on the number line.
- This set includes fractions, whole numbers, negative numbers, irrational numbers, and decimals.
- The symbol ∪ means "union," i.e., both sets combined.

SECTION 2: TYPES AND CLASSIFICATIONS OF SETS

Classification of Sets by Number of Elements

1. Empty Set (Null Set or Void Set)

Notation: $\varnothing$ or $\{\}$

An **empty set** is a set that contains **no elements at all**. In other words, there is **nothing** inside it.

- The empty set is **a subset of every set**.
- The order (number of elements) is **0**.

The empty set is **unique**; there is only **one** empty set in mathematics.

2. Singleton Set (Unit Set)

A **singleton set** is a set that contains **exactly one element**.

- **Example:**

$$\{5\}$$

This set contains only one number, 5, so it is a singleton.

A singleton set is the smallest possible **non-empty** set.

3. Finite Set

A **finite set** is a set where we can **count all the elements**, and the counting will **come to an end**.

Example:

$$\{1, 2, 3, 4, 5\}$$

This set has 5 elements, and we can count them easily.

Finite sets can be small or large, but they must have a fixed number of elements.

4. Infinite Set

An **infinite set** is a set that has **endless elements**. The counting will never stop.

Example:

$$\{1, 2, 3, 4, 5, \ldots\}$$

This set represents all natural numbers, and it never ends.

: {..., -2, -1, 0, 1, 2, ...}.

Infinite sets can be numbers, shapes, or any other objects that go on forever.

Set Relationships

1. Equal Sets

Two sets are **equal** if they contain **exactly the same elements**.
The order of elements does not matter, and repeating an element does not change the set.

$A = B$ if and only if every element of A is also in B, and every element of B is also in A.

- **Example:**

$$A = \{1, 2, 3\}, \quad B = \{3, 2, 1\}$$

Here $A = B$ because both sets have the same elements.

- Sometimes we think of this as a **perfect match** between elements in both sets — each element in one set has a matching element in the other.

2. Equivalent Sets

Two sets are **equivalent** if they have the **same number of elements**, even if the elements themselves are different.

Example:

$$\{a, b, c\} \text{ and } \{1, 2, 3\}$$

Both have 3 elements, so they are equivalent.

Equal sets are always equivalent, but equivalent sets are **not always equal**. For example:

$$\{1, 2, 3\} \text{ and } \{4, 5, 6\}$$

are equivalent but **not equal**, because the elements are different.

3. Cardinal Number of a Set

The **cardinal number** (or size) of a set is simply the **number of elements** in that set.

- **Notation:**

  We write it as $n(A)$ or $|A|$.

- **Example:**

$$A = \{2, 4, 6, 8\}, \quad n(A) = 4$$

because there are 4 elements in A.

SECTION 3: SUBSET RELATIONSHIPS

Subset

If every element of set **A** is also an element of set **B**, then we say **A** is a subset of **B**.

Notation:
A ⊆ B → Means "A is contained in B" or "A is a part of B."
(Symbol ⊆ means "is a subset of" or "is contained in.")

Example:
A = {1, 2}, B = {1, 2, 3, 4}
Since 1 and 2 are in **B**, we write:
A ⊆ B

Superset

If **B** contains all elements of **A**, then **B** is called a superset of **A**.

Notation:
B ⊇ A → Means "B contains A" or "B is a superset of A."
(Symbol ⊇ means "is a superset of.")

Example:
B = {1, 2, 3, 4}, A = {1, 2}
Here **B** ⊇ **A**.

Types of Subsets

1. Proper Subset

A set **A** is a proper subset of **B** if:

1. Every element of **A** is in **B**, and
2. **A** is not equal to **B**.

Notation:
A ⊂ B → Means "A is a proper subset of B."
(Symbol ⊂ means "is a proper subset of" — contained but not equal.)

Example:
A = {1, 2}, B = {1, 2, 3}
Here **A ⊂ B** because **A** is part of **B**, but not the same as **B**.

2. Improper Subset

An improper subset of a set is either:

- The set itself: **A ⊆ A**
- The empty set: **∅ ⊆ A**

(Symbol **∅** means "empty set" — a set with no elements.)

Every set is an improper subset of itself. The empty set is a subset of every set.

Example:
{1, 2, 3} ⊆ {1, 2, 3}
and
∅ ⊆ {1, 2, 3}

Power Set

The power set of a set **A** is the set of all possible subsets of **A**, including the empty set and **A** itself.

Notation:

P(A) or 2^A → Means "power set of A."

Formula:

If **A** has **n** elements, then the power set will have:

$|P(A)| = 2^n$ elements.

(Symbol $|P(A)|$ means "number of elements in the power set.")

Example:
If A = {1, 2}, then:
P(A) = {∅, {1}, {2}, {1, 2}}
Here n = 2, so 2^2 = 4 subsets.

3.4 Universal Set

A universal set is the set that contains all objects under discussion for a particular problem or situation.

Usually written as **U** or **Ω** (Greek letter Omega).

All sets being discussed are subsets of the universal set.

Example:
If we are talking about natural numbers less than 10:
U = {1, 2, 3, 4, 5, 6, 7, 8, 9}
and if:
A = {2, 4, 6, 8}
then $A \subseteq U$.

SECTION 4: OPERATIONS ON SETS

Basic Set Operations

1. Union of Two Sets

The **union** of two sets A and B is the set that contains **all elements** that are in A, or in B, or in both.

Notation:

$$A \cup B$$

Read as: “A union B.”

Set-builder form:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

Example:

A = {1, 2, 3}

B = {3, 4, 5}

$$A \cup B = \{1, 2, 3, 4, 5\}$$

2. Intersection of Two Sets

The **intersection** of two sets A and B is the set that contains only the **elements common** to both A and B.

Notation:

$$A \cap B$$

Read as: “A intersection B.”

Set-builder form:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

Example:

A = {1, 2, 3}

B = {3, 4, 5}

$$A \cap B = \{3\}$$

3. Difference of Two Sets

The **difference** of two sets A and B is the set of elements that are in A but **not** in B.

Notation:

$$A - B \quad \text{or} \quad A \setminus B$$

Read as: “A minus B.”

Set-builder form:

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}$$

Example:

A = {1, 2, 3}

B = {3, 4, 5}

$$A - B = \{1, 2\}$$

4. Complement of a Set

The **complement** of a set A is the set of all elements in the **universal set U** that are **not** in A.

Notation:

$$A' \quad \text{or} \quad A^c$$

Read as: "A complement."

Set-builder form:

$$A' = U - A = \{x \mid x \in U \text{ and } x \notin A\}$$

Example:

U = {1, 2, 3, 4, 5}

A = {2, 4}

$$A' = \{1, 3, 5\}$$

SECTION 5: VISUAL REPRESENTATION

Venn Diagrams

A Venn diagram is a simple and powerful way to visually represent **sets** and the relationships between them.
It is often used in mathematics, logic, statistics, and problem-solving to clearly see how sets overlap, share elements, or differ.

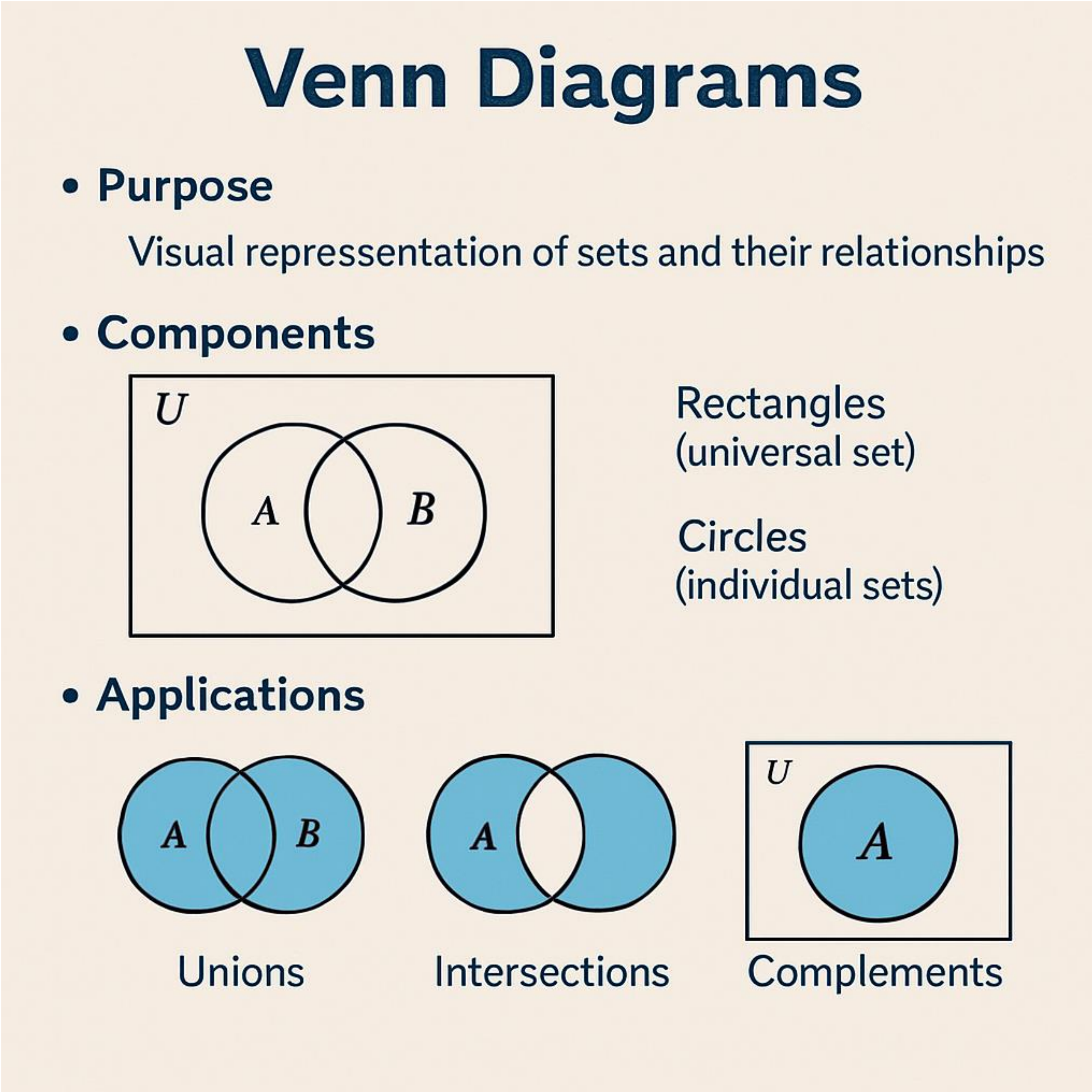
Example:

If:

- Universal Set U = {1, 2, 3, 4, 5, 6}
- A = {1, 2, 3}
- B = {3, 4, 5}

Then:

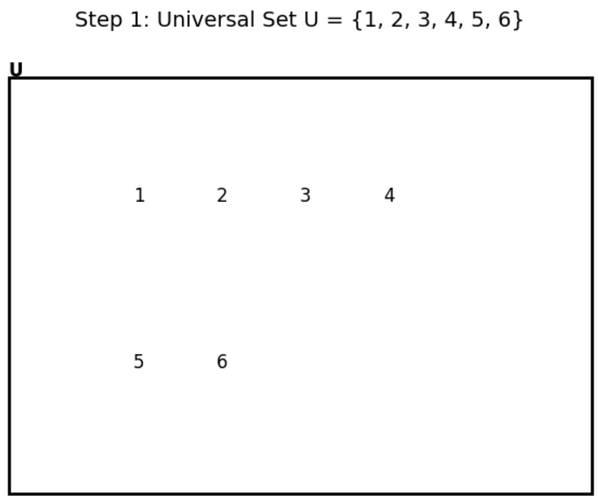
- **A ∪ B** = {1, 2, 3, 4, 5} (everything in A or B)
- **A ∩ B** = {3} (common to both)
- **A′** = {4, 5, 6} (in U but not in A)



Main Components:

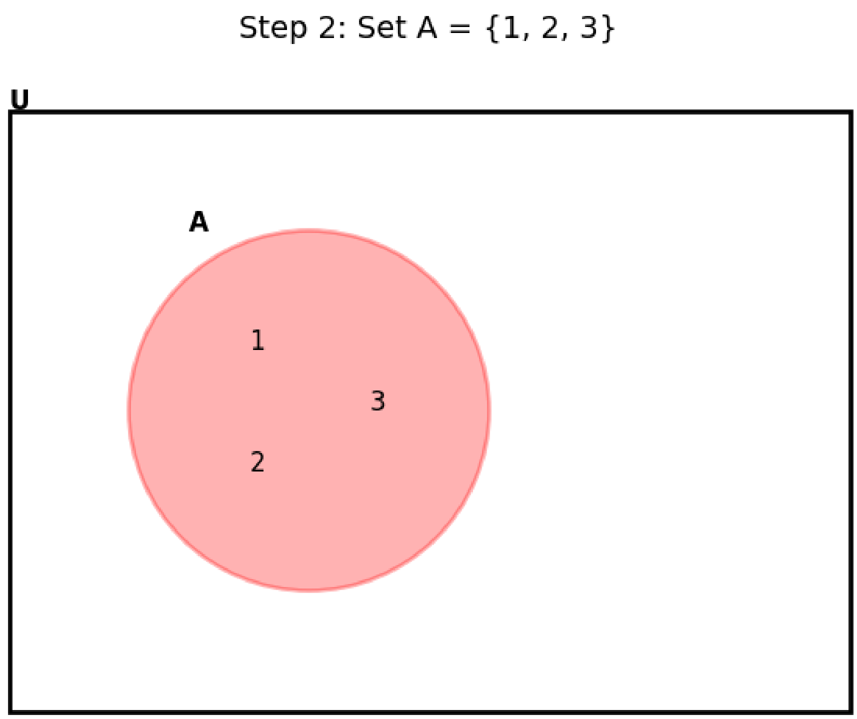
Universal Set (U)

- Represented by a **rectangle**.
- Contains **all elements** under discussion.
- Every set drawn is inside this rectangle.



Sets

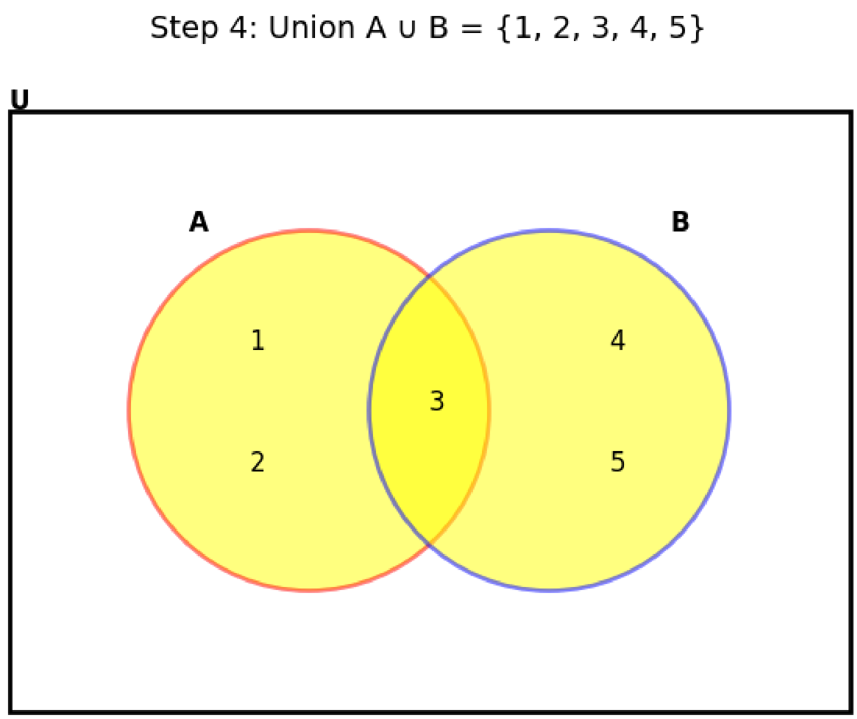
- Represented by **circles** inside the rectangle.
- Each circle contains the elements of that particular set.
- Circles may overlap to show common elements (intersection).



Applications in Sets:

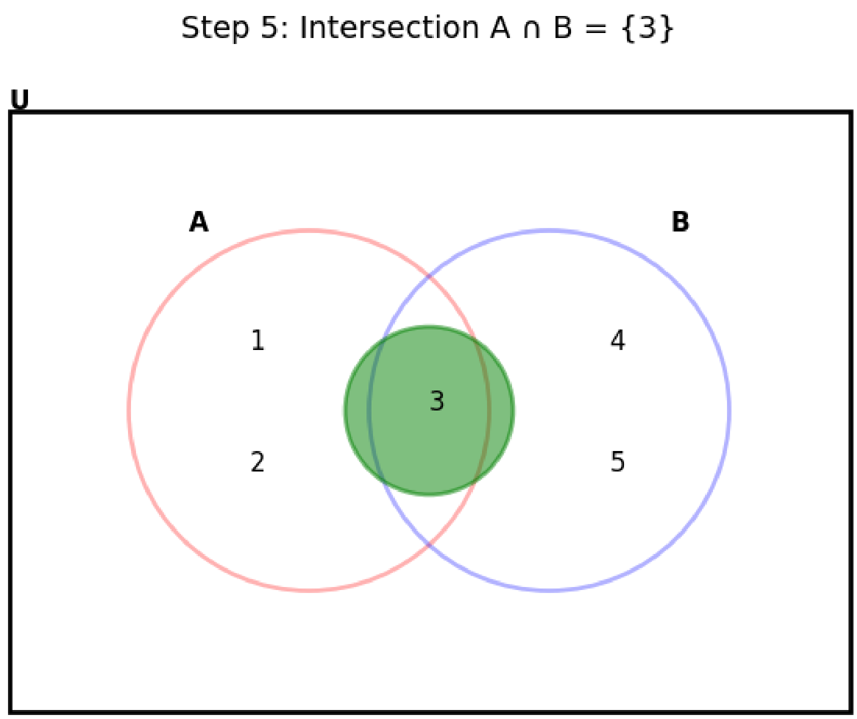
Union ($A \cup B$)

- All elements that are in **A**, or in **B**, or in both.
- Shaded region covers the entire area of both circles.



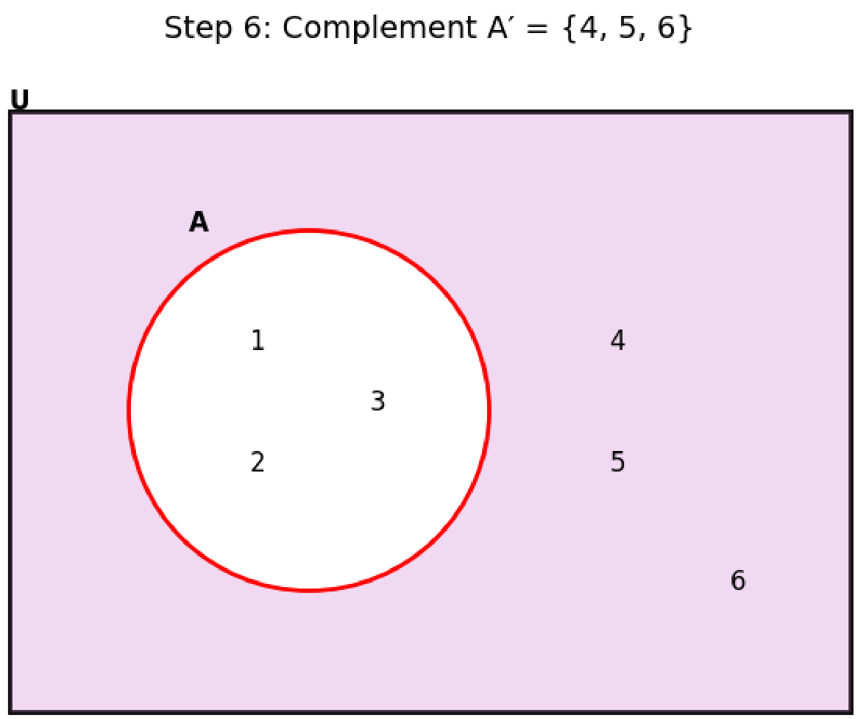
Intersection (A ∩ B)

- Only elements **common** to both sets.
- Shaded region is where the circles overlap.



Complement (A')

- All elements in the **universal set** that are **not** in A.
- Shaded region is outside circle A but inside the rectangle.



SECTION 6: SET RELATIONSHIPS AND CLASSIFICATIONS

Set Interactions

Disjoint Sets

Two sets are called **disjoint** if they have **no elements in common**. This means that their intersection is the empty set.

Notation:

$$A \cap B = \varnothing$$

Read as: "A intersection B is empty" or "A and B have nothing in common."

Example:

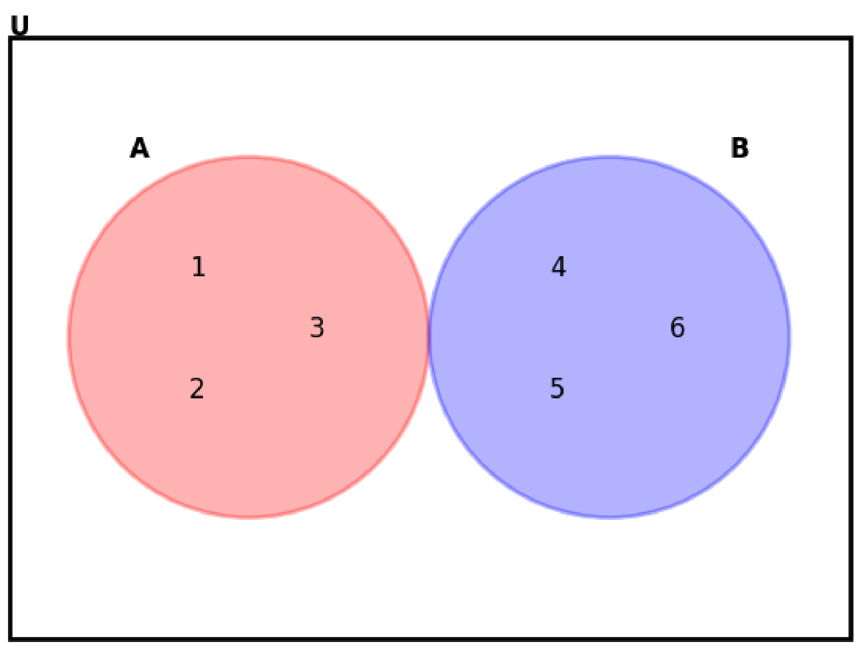
A = {1, 2, 3}

B = {4, 5, 6}

Since there are no numbers that appear in both A and B, we write:

$$A \cap B = \varnothing$$

Disjoint Sets: A ∩ B = ∅ (A = {1, 2, 3}, B = {4, 5, 6})



2. Overlapping Sets

Two sets are called **overlapping** if they have **at least one element in common**. This means their intersection is **not** empty.

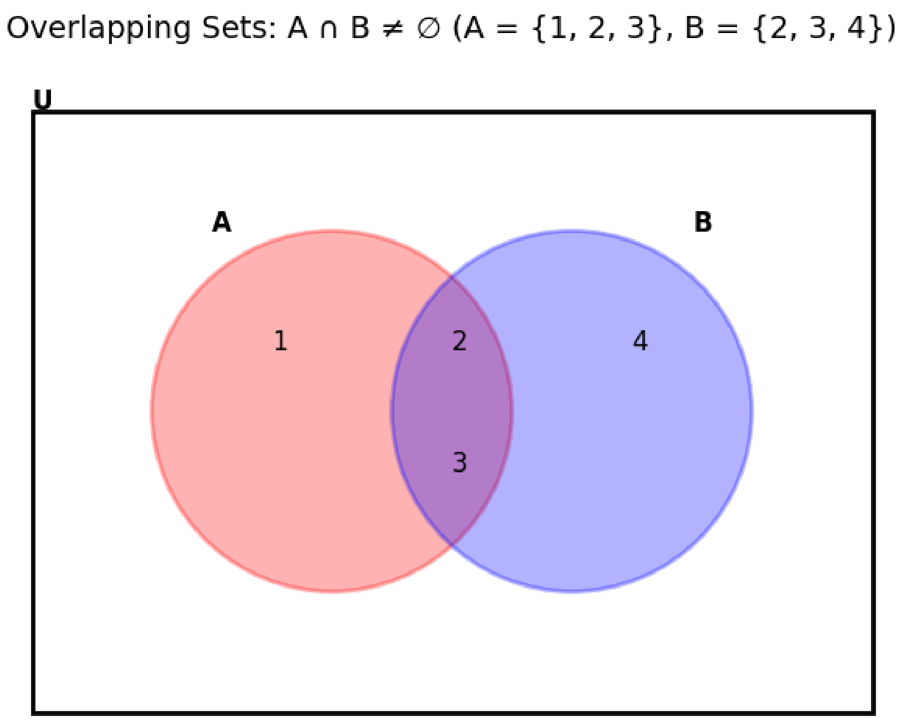
Notation:

$$A \cap B \neq \varnothing$$

Read as: “A intersection B is not empty” or “A and B share some elements.”

Example:
A = {1, 2, 3}
B = {2, 3, 4}
Here, 2 and 3 appear in both A and B, so:

$$A \cap B = \{2, 3\}$$



6.2 Properties of Set Operations

Properties of Union:

The **union** operation follows certain rules, called **laws**, that are always true for any sets.

1. Commutative Law

$$A \cup B = B \cup A$$

Read as: "A union B is the same as B union A."
The order of the sets does not matter when taking their union — you'll get the same result.

Example:
A = {1, 2}, B = {2, 3}
A ∪ B = {1, 2, 3}
B ∪ A = {1, 2, 3}

2. Associative Law

$$(A \cup B) \cup C = A \cup (B \cup C)$$

Read as: "When taking the union of three sets, it doesn't matter how you group them."
You can join sets in any grouping — the result is the same.

Example:
A = {1, 2}, B = {2, 3}, C = {3, 4}
(A ∪ B) ∪ C = {1, 2, 3, 4}
A ∪ (B ∪ C) = {1, 2, 3, 4}

3. Identity Law

$$A \cup \varnothing = A \quad \text{and} \quad A \cup U = U$$

Read as:

- "A union empty set is A."
- "A union universal set is the universal set."

Meaning:

- Adding nothing to A leaves A unchanged.
- Adding everything (U) covers all elements, so you get U.

Example:
A = {1, 2}, U = {1, 2, 3, 4}

A ∪ ∅ = {1, 2}
A ∪ U = {1, 2, 3, 4}

4. Idempotent Law

$$A \cup A = A$$

Read as: “A union A is just A.”

Joining a set with itself doesn’t change it.

Example:
A = {1, 2, 3}
A ∪ A = {1, 2, 3}

Properties of Intersection

1. Commutative Law

$$A \cap B = B \cap A$$

Read as: “A intersection B is the same as B intersection A.”

The order of the sets does not matter when finding common elements — the result will be the same.

Example:
A = {1, 2}, B = {2, 3}
A ∩ B = {2}
B ∩ A = {2}

2. Associative Law

$$(A \cap B) \cap C = A \cap (B \cap C)$$

Read as: “When finding the intersection of three sets, it doesn’t matter how you group them.”

You can check for common elements in any grouping — the result will be the same.

Example:
A = {1, 2, 3}, B = {2, 3, 4}, C = {3, 4, 5}
(A ∩ B) ∩ C = {3}
A ∩ (B ∩ C) = {3}

3. Identity Law

$$A \cap U = A \quad \text{and} \quad A \cap \varnothing = \varnothing$$

- “A intersection universal set is A.”
- “A intersection empty set is the empty set.”

If you compare A with everything (U), you just get A.

If you compare A with nothing (∅), you get nothing.

Example:
A = {1, 2, 3}, U = {1, 2, 3, 4, 5}
A ∩ U = {1, 2, 3}
A ∩ ∅ = ∅

4. Idempotent Law

Rule:

$$A \cap A = A$$

Read as: “A intersection A is just A.”

Comparing a set with itself gives the same set back.

Example:
A = {1, 2, 3}
A ∩ A = {1, 2, 3}

Distributive Laws

1. Union over Intersection

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Read as: "A union (B intersection C) equals (A union B) intersection (A union C)."
When combining union and intersection, you can "distribute" A across the intersection inside the brackets.

Example:
A = {1, 2}, B = {2, 3}, C = {2, 4}
B ∩ C = {2}
A ∪ (B ∩ C) = {1, 2}
A ∪ B = {1, 2, 3}
A ∪ C = {1, 2, 4}
(A ∪ B) ∩ (A ∪ C) = {1, 2}

2. Intersection over Union

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Read as: "A intersection (B union C) equals (A intersection B) union (A intersection C)."

When combining intersection and union, you can "distribute" A across the union inside the brackets.

Example:
A = {1, 2, 3}, B = {2, 4}, C = {3, 5}
B ∪ C = {2, 3, 4, 5}
A ∩ (B ∪ C) = {2, 3}
A ∩ B = {2}
A ∩ C = {3}
(A ∩ B) ∪ (A ∩ C) = {2, 3}

Properties of Complement

The **complement** of a set A (written A' or A^c) contains everything in the **universal set U** that is **not in A**.

1. Complement of Universal Set

$$U' = \varnothing$$

Read as: “The complement of the universal set is the empty set.”

Meaning:
There’s nothing outside the universal set.

Rule: $U' = \varnothing$

Example:

Let $U = \{1, 2, 3, 4, 5\}$

Everything outside U = nothing

So, $U' = \varnothing$

2. Complement of Empty Set

$$\varnothing' = U$$

Read as: “The complement of the empty set is the universal set.”

Everything is outside the empty set.

Rule: $\varnothing' = U$

Example:

Let $U = \{1, 2, 3, 4, 5\}$

Everything outside the empty set is all elements in U

So, $\varnothing' = \{1, 2, 3, 4, 5\}$

3. Double Complement

$$(A')' = A$$

Read as: "The complement of the complement of A is A."

If you take the opposite of A twice, you end up back with A.

Rule: $(A')' = A$

Example:

Let $U = \{1, 2, 3, 4, 5\}$, $A = \{1, 3, 5\}$

First complement: $A' = \{2, 4\}$

Second complement: $(A')' = \{1, 3, 5\}$ = A

4. Complement Laws

$$A \cup A' = U$$

Read as: "A union its complement equals the universal set."

a) $A \cup A' = U$

Example:

Let $U = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3\}$

$A' = \{4, 5\}$

$A \cup A' = \{1, 2, 3, 4, 5\} = U$

$$A \cap A' = \varnothing$$

Read as: "A intersection its complement equals the empty set."

b) $A \cap A' = \varnothing$

Example:

$A = \{1, 2, 3\}$, $A' = \{4, 5\}$

$A \cap A' = \varnothing$ (no elements in common)

- A set and its opposite together make everything (U).
- A set and its opposite have no common elements.

De Morgan's Laws

1. First Law:

$$(A \cup B)' = A' \cap B'$$

Read as: "The complement of the union of two sets is the intersection of their complements."

If something is **not** in A or B, then it must be outside both A **and** B at the same time.

Example:

Let $U = \{1, 2, 3, 4, 5, 6\}$

$A = \{1, 2, 3\}$, $B = \{3, 4\}$

- $A \cup B = \{1, 2, 3, 4\}$
- $(A \cup B)' = \{5, 6\}$
- $A' = \{4, 5, 6\}$, $B' = \{1, 2, 5, 6\}$

A'∩B'={5,6} matches $(A \cup B)'$

2. Second Law:

$$(A \cap B)' = A' \cup B'$$

Read as: “The complement of the intersection of two sets is the union of their complements.”

Example:

Let $U = \{1, 2, 3, 4, 5, 6\}$

$A = \{1, 2, 3\}$, $B = \{3, 4\}$

- $A \cap B = \{3\}$
- $(A \cap B)' = \{1, 2, 4, 5, 6\}$
- $A' = \{4, 5, 6\}$, $B' = \{1, 2, 5, 6\}$
- $A' \cup B' = \{1, 2, 4, 5, 6\}$ matches $(A \cap B)'$

SECTION 7: MATHEMATICAL LOGIC FOUNDATIONS

Types of Logic Systems

1. Inductive Logic

A way of reasoning where we move from specific examples or observations to a general rule or conclusion.

- We look for patterns and make predictions.

Example:

- Observation: The sun rises in the east every morning we've seen.
- Conclusion: The sun always rises in the east.

1. Mango, banana, and apple are all sweet → *All fruits are sweet.*
2. Every time I drop an object, it falls → *Things always fall towards the ground.*

2. Deductive Logic

A way of reasoning where we start with a general rule or statement and apply it to specific situations to reach a conclusion.

- If the starting statements are true, the conclusion must also be true.

Example:

- Rule: All squares have four sides.
- Case: Shape X is a square.
- Conclusion: Shape X has four sides.

1. All dogs bark. Max is a dog → *Max barks.*
2. All triangles have three sides. Shape Y is a triangle → *Shape Y has three sides.*

3. Aristotelian Logic

Classical logic developed by Aristotle, based on syllogisms (logical arguments with two premises and a conclusion).

- **Structure:**
  - Premise 1: All humans are mortal.
  - Premise 2: Socrates is a human.
  - Conclusion: Socrates is mortal.

1. Premise 1: All birds have feathers.
   Premise 2: A parrot is a bird.
   Conclusion: *A parrot has feathers.*
2. Premise 1: All students must take exams.
   Premise 2: Sara is a student.
   Conclusion: *Sara must take exams.*

4. Non-Aristotelian Logic

Modern logic systems that go beyond classical syllogisms. They often use symbols, truth tables, and advanced rules.

- **Examples:** Propositional logic, predicate logic, fuzzy logic, modal logic.

Example (Propositional Logic):

- Statement: If it rains, the ground will be wet.
- Symbolic form: $p \rightarrow q$
- If p ("it rains") is true, then q ("the ground is wet") must be true.

Propositional logic:

- Rule: If it's cloudy, it might rain.
- Symbol: $p \rightarrow q$ (if cloudy, then rain)

Propositions and Symbolic Logic

1. Proposition

A **proposition** is a sentence or statement that can only be **true** or **false**, but not both at the same time.

- **Example (True):** "The Earth orbits the Sun."
- **Example (False):** "2 + 2 = 5."
- **Not a proposition:** "Close the door." (This is a command, not something true/false.)

2. Symbolic Logic

Symbolic logic is a way of using **symbols** to represent statements and logical operations so they're easier to work with in mathematics and computer science.

Example: Let p = "It is raining"

- Instead of writing the full sentence every time, we can use **p** in logic rules.

3. Truth Values

Every proposition has a **truth value**:

- **T (True)** or **F (False)**
- Sometimes also written as **1 (True)** and **0 (False)** in digital logic or computer science.
- Example:
  - p: "Cats have tails" → **T** (True)
  - q: "The Moon is made of cheese" → **F** (False)

SECTION 8: LOGICAL CONNECTIVES AND OPERATIONS

Basic Logical Operations

1. Negation (NOT) $\neg p$ or $\sim p$

"NOT p" or "It is not the case that p."

This changes the truth value of a statement. If p is True, NOT p is False, and vice versa.

$\neg$ or $\sim$ means "opposite" or "reverse the truth value."

Truth Table:

Table 1: Negation (NOT): $\neg p$ or $\sim p$ — "It is not the case that p." Example: p: "It is sunny." (T) $\to$ $\neg p$: "It is not sunny." (F)

p	$\neg p$
T	F
F	T

Example:

p: "It is sunny." → T

$\neg p$: "It is not sunny." → F

2. Conjunction (AND) $p \wedge q$

"p AND q."

The result is True **only** if **both** p and q are True.

Symbol: $p \wedge q$

$\wedge$ looks like an upside-down "V" and means "both must happen."

- p: "I have a ticket." (T)
- q: "I have an ID." (T)
- $p \wedge q$: "I can enter the event." → T

Table 2: Conjunction (AND): $p \wedge q$ — "p and q are both true."
Example: p: "I have a ticket." (T), q: "I have an ID." (T) → $p \wedge q$: "I can enter the event." (T)

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

3. Disjunction (OR) $p \vee q$

"p OR q" (inclusive OR — meaning one or both can be true).

The result is False only if **both** are False.

$\vee$ looks like a "V" and means "at least one must happen."

Example:

- p: "I will go to the park." (T)
- q: "I will go to the beach." (F)
- $p \vee q$: "I will go to the park or beach." → T

Table 3: Disjunction (OR): $p \vee q$ — "p or q or both are true."
Example: p: "I will go to the park." (T), q: "I will go to the beach." (F) → $p \vee q$: "I will go to the park or beach." (T)

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

4. Implication (IF–THEN) $p \rightarrow q$

"If p, then q" or "p implies q."

If p is True, q must also be True for the statement to hold. It's only False when p is True but q is False.

$\rightarrow$ means “leads to” or “results in.”

Example:

- p: “If it rains” →
- q: “The ground gets wet.”
- T (rain) → T (wet ground) ✅ True
- T (rain) → F (dry ground) ❌ False

Table 4: Implication (IF–THEN): $p \to q$ — “If p is true, then q must also be true.”
Example: p: “It rains.” (T) $\to$ q: “The ground gets wet.” (T) $\to$ True; (T, F) $\to$ False

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

5. Biconditional (IF AND ONLY IF) $p \leftrightarrow q$ or $p \equiv q$

“p if and only if q.”

True if p and q are either both True or both False.

$\leftrightarrow$ means "both ways" — p implies q, and q implies p.

Example:

- p: "The light is on."
- q: "The switch is up."
- $p \leftrightarrow q$: True if both match (on & up, or off & down).

Table 5: Biconditional (IF AND ONLY IF): $p \leftrightarrow q$ — "p and q are either both true or both false."
Example: p: "The light is on." q: "The switch is up." $\rightarrow$ True only if both match (on & up, or off & down).

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

SECTION 9: CONDITIONALS AND RELATED STATEMENTS

What is a Conditional?

A conditional is like a **promise or rule** that says:

> If **p** happens, then **q** will happen.

In symbols: **p → q**

Example:

> If it rains (**p**), then the ground gets wet (**q**).

The key thing: This rule **only makes a promise** about situations where p is true. It does *not* promise what happens when p is false.

1. Original (Direct) – $p \rightarrow q$

- **What it does:** States a rule: *When p happens, q must happen.*
- **How it works:** Looks only at cases where p is true, and checks if q follows.
- **Example:**
  - Rule: If it rains, the ground gets wet.
  - Scenario: If rain happens, you must check—did the ground get wet?
- **Important:** This rule says nothing about what happens if it does *not* rain.

2. Converse – $q \to p$

- **What it does: Flips** the original rule's order. Now we're saying:

  > If q happens, then p must have happened.

- **Why it's different:** Just because q is true doesn't mean p is true.
- **Example:**
  - Converse: If the ground is wet, it rained.
  - Problem: The ground could be wet because someone watered the garden, so the converse is not always true.
- **Summary:** Converse checks if q can "guarantee" p, but that's not always the case.

3. Inverse – $\neg p \to \neg q$

- **What it does: Negates** both parts of the original. Now it's saying:

  > If p does *not* happen, then q does *not* happen.

- **Why it's different:** This is about the *absence* of p and q.
- **Example:**
  - Inverse: If it does not rain, the ground is not wet.
  - Problem: Even if it doesn't rain, sprinklers or a bucket of water could still make the ground wet.
- **Summary:** Inverse checks if *not p* means *not q*, but that's not automatically true.

4. Contrapositive – $\neg q \to \neg p$

- **What it does: Flips** the order *and* **negates** both parts.

  > If q is not true, then p is not true.

- **Why it's special:** This is **logically equivalent** to the original.
- **Example:**
  - Contrapositive: If the ground is not wet, then it did not rain.
  - This matches the original perfectly—if rain always makes the ground wet, then a dry ground means no rain.
- **Summary:** Contrapositive is always true whenever the original is true.

Note:

- Original and Contrapositive are **always the same in truth value**.
- Converse and Inverse are **always the same in truth value**, but **not** guaranteed to match the original.

SECTION 10: TRUTH TABLE CLASSIFICATIONS

1. Tautology

A statement that is **true in every possible situation**, no matter what truth values p and q take.
In other words: It's **impossible** for it to be false.

Symbol example: $p \lor \neg p$

- Read: "p or not p"
- Why always true? Either p is true, or it's not — there's no other option, so one side of the "OR" is always true.

Truth table for $p \lor \neg p$**:**

p	¬p	p ∨ ¬p
T	F	T
F	T	T

Every row is True → Tautology.

example:

"It will either rain today or it won't."
No matter what happens, the statement is true.

2. Contradiction

A statement that is **false in every possible situation** — it can never be true.
It's the **opposite** of a tautology.

Symbol example: $p \wedge \neg p$

- Read: "p and not p"
- Why always false? Something can't be both true **and** false at the same time.

Truth table for $p \wedge \neg p$**:**

p	¬p	p ∧ ¬p
T	F	F
F	T	F

Every row is False → Contradiction.

example:

"It's raining and it's not raining at the same time."
That's impossible, so it's always false.

3. Contingency

A statement that is **true in some cases and false in others**.
Its truth value depends on the specific truth values of its components.

Symbol example: $p \wedge q$

- Read: "p and q"
- Truth depends on whether both p and q are true.

Truth table for $p \wedge q$**:**

p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

Mixed T/F results → Contingency.

example:

"It is raining and I have my umbrella."
This can be true sometimes (when both happen) and false other times.

SECTION 11: QUANTIFIERS

Types of Quantifiers

1. Universal Quantifier (∀)

"For all…" or "For every…"
It claims that a statement is true **for every element** in a given set.
If there's even **one counterexample**, the statement is false.

Symbol form:

$$\forall x \, P(x)$$

- Read: "For all x, P(x) is true."
- Example:

$$\forall x \in \mathbb{R}, \; x^2 \geq 0$$

Translation: "For every real number x, x^2 is greater than or equal to zero."

True because squaring any real number gives a non-negative result.

Real-life analogy:

"Every student in the class passed the test."
If even one student failed, the statement is false.

2. Existential Quantifier (∃)

"There exists…" or "There is at least one…"
It claims that the statement is true **for at least one** element in the set.
It does **not** require it to be true for all.

Symbol form:

$$\exists x\, P(x)$$

- Read: “There exists an x such that P(x) is true.”
- Example:

$$\exists x \in \mathbb{R},\ x^2 = 4$$

Translation: “There exists a real number x whose square equals 4.”

True because x could be 2 or -2.

Real-life analogy:

“There is at least one student in the class who got a perfect score.”
It doesn’t matter if most didn’t — as long as one did, the statement is true.

Quantifier Relationships (Negations)

These show how to **flip** statements from true to false or vice versa.

Negation of Universal

$$\neg(\forall x\, P(x)) \equiv \exists x\, \neg P(x)$$

“It’s not true that P(x) is true for all x” is equivalent to “There exists at least one x for which P(x) is false.”

Example:
Original universal:

> "All birds can fly." ($\forall x$, if x is a bird → x can fly)
> Negation:
> "There exists at least one bird that cannot fly." (e.g., penguins)

Negation of Existential

$$\neg(\exists x\, P(x)) \equiv \forall x\, \neg P(x)$$

"It's not true that there exists at least one x for which P(x) is true" is equivalent to "P(x) is false for all x."

Example (Existential & its Negation):

- **Original existential:**

$$\exists x,\ x \text{ is a red car in the parking lot.}$$

Meaning: "There is at least one red car in the parking lot."

If even one red car is there, this is true.

- **Negation:**

$$\forall x,\ x \text{ is not a red car in the parking lot.}$$

Meaning: "There are no red cars in the parking lot."

Every single car is some other color — if even one is red, this is false.