Math for AI (Artificial Intelligence)

Lecture 1: Complete Number System & Operations – Real, Rational, Complex, Logarithms, Exponents & More for AI (Math Basics 1)

SECTION 1: FUNDAMENTAL NUMBER SYSTEMS

1. Natural Numbers

Natural Numbers are the numbers we use for counting objects. They start from 1 and go on infinitely in the positive direction.

Set Notation:
N = {1, 2, 3, 4, 5, ...}

- 0 is not a natural number.
- They are sometimes called Counting Numbers.

2. Whole Numbers

Whole Numbers are like Natural Numbers but include 0 as well. They represent all non-negative integers starting from 0.

Set Notation:
W = {0, 1, 2, 3, 4, ...}

- Every natural number is a whole number.
- Whole Numbers = Natural Numbers ∪ {0}

3. Integers

Integers are all numbers without fractional or decimal parts. They include:

- All positive numbers,
- All negative numbers,
- and zero.

What is a Decimal Part?

A **decimal part** means numbers that come after the decimal point.

- In **5.7**, the **.7** is the decimal part.

- Integers **do not have** any numbers after the decimal point.
  Examples of integers: 5, -2, 0
  Not integers: 5.7, -3.5

What is a Fractional Part?

A **fractional part** means numbers expressed as fractions (p/q form) where $q \neq 1$.

- **1/2**, **3/4**, **-5/6** are fractions.
- Integers **do not include** fractions.
- However, numbers like **4/1 = 4** are integers because the denominator is 1.

Set Notation:
Z = {..., -3, -2, -1, 0, 1, 2, 3, ...}

Types of Integers:

- Positive Integers: Numbers greater than 0 (1, 2, 3, ...)
- Negative Integers: Numbers less than 0 (-1, -2, -3, ...)
- Zero (0): Neither positive nor negative.

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4. Even Numbers

An Even Number is an integer that is exactly divisible by 2, meaning it leaves no remainder when divided by 2.

General Form:
Even Number = $2n$, where $n \in Z$ (an integer)

Examples:
2, 4, 6, 8, 10, ...

- Sum of two even numbers is even.
- Product of two even numbers is even.

5. Odd Numbers

An Odd Number is an integer that is not exactly divisible by 2. When divided by 2, it leaves a remainder of 1.

General Form:
Odd Number = $2n + 1$, where $n \in Z$

Examples:
1, 3, 5, 7, 9, ...

Key Property:

- Sum of two odd numbers is even.
- Product of two odd numbers is odd.

6. Prime Numbers

A **Prime Number** is a number that can only be divided by **1** and **itself** without leaving a remainder.

Condition for Prime Numbers:

1. It must be a **natural number** (1, 2, 3, 4, 5, ...).
2. It must be **greater than 1**.
3. It should have **only two divisors (factors)**:
   - The number **1**.
   - The number **itself**.

Examples:
2, 3, 5, 7, 11, 13, ...

What is a Divisor (or Factor)?

A **divisor** of a number is a number that divides it **completely without remainder**.

For example:

- Divisors of 6 are: **1, 2, 3, 6**
  - Because $6 \div 1 = 6$
  - $6 \div 2 = 3$
  - $6 \div 3 = 2$
  - $6 \div 6 = 1$

Important Note:

- 2 is the only even prime number.

- Prime numbers cannot be divided evenly by any number other than 1 and itself.

7. Composite Numbers

A Composite Number is a natural number greater than 1 that has more than two divisors.
4, 6, 8, 9, 10, 12, ...

- 1 is neither prime nor composite.

What is Absolute Value?

Absolute Value of a number means its **distance from zero on a number line**, without considering whether it's positive or negative.

- It is always a **non-negative number (≥ 0)**.

symbol **|x|** "absolute value of x". **vertical bars** around the number:

Examples:

- **|5| = 5** (because 5 is already positive)
- **|-5| = 5** (because -5 is 5 units away from 0)

Formal Definition of Absolute Value (Piecewise Form):

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

- If x is **positive** or **zero**, the absolute value is just **x**.
- If x is **negative**, the absolute value is **–x**, which makes it positive.

Properties of Absolute Value:

1. Non-Negative Property

$$|x| \geq 0$$

- The absolute value of any number is **never negative**.
- No matter how negative a number is, its absolute value is **always zero or positive**.
- Example:

- |7| = 7
- |-7| = 7
- |0| = 0

2. Identity Property

$$|x| = x \quad \text{if } x \geq 0$$

- If a number is already **positive or zero**, its absolute value is the number itself.
- Example:
  - |4| = 4
  - |0| = 0

3. Opposite Numbers Have Same Absolute Value

$$|x| = |-x|$$

- The absolute value of a number and its negative (opposite) is always the same.
- Both numbers are at the same distance from zero but on opposite sides.
- Example:
  - |5| = 5
  - |-5| = 5

4. Multiplicative Property

$$|a \times b| = |a| \times |b|$$

- You can first multiply the numbers and then take the absolute value.
- Or you can take the absolute values first and then multiply them. The result will be the same.

Example:

- |-2 × 3| = |-6| = 6

- |–2| × |3| = 2 × 3 = 6

5. Division Property

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|} \quad \text{where } b \neq 0$$

- **b ≠ 0** (because division by zero is undefined).
- The absolute value of a division is equal to the division of the absolute values.

Example:

- |-8 ÷ 2| = |-4| = 4
- |–8| ÷ |2| = 8 ÷ 2 = 4

6. Additive (Triangle Inequality) Property

$$|a + b| \leq |a| + |b|$$

- The absolute value of a sum is **always less than or equal** to the sum of the absolute values.
- This is known as the **Triangle Inequality** in mathematics.

Example:

- a = 3, b = –4
- |3 + (–4)| = |–1| = 1
- |3| + |–4| = 3 + 4 = 7
- So, |a + b| = 1 ≤ 7 = |a| + |b|

7. Distance Between Two Numbers

$$|a - b|$$

- The absolute value of (a - b) gives the **distance between a and b** on a number line.

Example:

- Distance between 5 and 2 = |5 – 2| = |3| = 3
- Distance between –3 and 4 = |–3 – 4| = |–7| = 7

SECTION 2: LAWS AND PROPERTIES OF INTEGERS

Integers are all numbers without fractional or decimal parts. They include:

- All positive numbers,
- All negative numbers,
- and zero.

Addition of Integers

Rules for Adding Integers:

Case 1: Adding Two Positive Integers

Add their values normally; the result is **positive**.

$$(+a) + (+b) = +(a + b)$$

$$(+4) + (+7) = +11$$

Case 2: Adding Two Negative Integers

Add their values, keep the **negative sign**.

$$(-a) + (-b) = -(a + b)$$

$$(-3) + (-5) = -8$$

Case 3: Adding a Positive Integer and a Negative Integer

1. Subtract the **smaller absolute value** from the **larger absolute value**.

2. Keep the sign of the **larger absolute value**.

$$(+a) + (-b) = \begin{cases} +(a-b), & \text{if } a > b \\ -(b-a), & \text{if } b > a \end{cases}$$

Example 1 (Positive is bigger):

$$(+7) + (-4) = +3$$

Because 7 – 4 = 3 and 7 (positive) is bigger.

Example 2 (Negative is bigger):

$$(+5) + (-8) = -3$$

Because 8 – 5 = -3 and 8 (negative) is bigger.

Case 4: Adding a Number with Zero

Adding zero does not change the number.

$$a + 0 = a$$

Example:

$$(-6) + 0 = -6$$

$$(+9) + 0 = +9$$

Subtraction of Integers

Rules for Subtracting Integers:

Rule 1: Subtracting a Positive Integer

$$a - (+b) = a + (-b)$$

- Subtracting a positive is the same as **adding a negative**.
- Move **to the left** on the number line.

Example 1:

$$5 - (+3) = 5 + (-3) = 2$$

Example 2:

$$-2 - (+4) = -2 + (-4) = -6$$

Rule 2: Subtracting a Negative Integer

$$a - (-b) = a + (+b)$$

- Subtracting a negative is the same as **adding a positive**.
- Move **to the right** on the number line.

Example 1:

$$7 - (-2) = 7 + (+2) = 9$$

Example 2:

$$-5 - (-3) = -5 + (+3) = -2$$

Rule 3: Subtracting Zero

$$a - 0 = a$$

Subtracting zero does not change the number.

Example:

$$9 - 0 = 9$$

$$-6 - 0 = -6$$

Rule 4: Zero Minus Any Integer

$$0 - a = -a$$

When you subtract a number from 0, you get its **additive inverse** (change its sign).

Example:

$$0 - 4 = -4$$

$$0 - (-5) = +5$$

Multiplication of Integers

Sign Rules for Multiplication of Integers:

Multiplying	Rule	Example
Positive × Positive	Result is Positive	(+3) × (+4) = +12
Negative × Negative	Result is Positive (two negatives make a positive)	(–5) × (–2) = +10
Positive × Negative	Result is Negative	(+6) × (–3) = –18
Negative × Positive	Result is Negative	(–7) × (+2) = –14

Zero Multiplication Rule:

- Any number multiplied by **0** is **always 0**.
- a × 0 = 0 and 0 × a = 0

Example:

- 5 × 0 = 0
- (–7) × 0 = 0

Division of Integers

a / b: Division where a is **dividend**, b is **divisor**.

b ≠ 0: Division by zero is **not allowed (undefined)**.

Rules and Conditions for Integer Division:

Dividing	Rule	Example
Positive ÷ Positive	Result is Positive	(+12) ÷ (+3) = +4
Negative ÷ Negative	Result is Positive (two negatives cancel out)	(–20) ÷ (–4) = +5
Positive ÷ Negative	Result is Negative	(+15) ÷ (–5) = –3
Negative ÷ Positive	Result is Negative	(–18) ÷ (+6) = –3

Zero Division Rule:

- **Zero divided by any non-zero number is 0:**

$$0 \div a = 0 \quad (a \neq 0)$$

- **Division by zero is undefined:**

$$a \div 0 = \text{undefined}$$

Examples:

- 0 ÷ 7 = 0
- 9 ÷ 0 = undefined (impossible)

Fundamental Laws (Binary Operations)

Addition Laws:

1. Commutative Law of Addition

The **order of numbers in addition does not affect the sum**.

$$a + b = b + a$$

Swapping the numbers you are adding doesn't change the result.

Example:

$$5 + 3 = 3 + 5 = 8$$

$$(-7) + 4 = 4 + (-7) = -3$$

2. Associative Law of Addition

When adding three or more numbers, **grouping (parentheses) doesn't affect the result**.

$$(a + b) + c = a + (b + c)$$

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You can group numbers in any way during addition; the sum remains the same.

Example:

$$(2 + 3) + 5 = 5 + 5 = 10$$

$$2 + (3 + 5) = 2 + 8 = 10$$

For Negative Numbers:

$$(-4 + 6) + (-2) = 2 + (-2) = 0$$

$$-4 + (6 + (-2)) = -4 + 4 = 0$$

3. Additive Identity

Adding **0** to any number **does not change** the number.

$$a + 0 = a$$

Zero is called the **additive identity** because adding it leaves the number unchanged.

Example:

$$7 + 0 = 7$$

$$-9 + 0 = -9$$

4. Additive Inverse

Adding a number and its **opposite (negative of that number)** always gives **0**.

$$a + (-a) = 0$$

–a: Additive inverse (opposite sign of a).

A number’s additive inverse cancels it out, resulting in zero.

Example:

$$8 + (-8) = 0$$

$$-5 + (+5) = 0$$

5. Closure Property of Addition

The sum of **two integers is always an integer**.

$$a + b \in \mathbb{Z} \quad (\text{where } \mathbb{Z} = \text{set of integers})$$

- **∈**: "belongs to".
- **ℤ**: Symbol for set of integers.

When you add any two integers, the result is also an integer. This is called **closure under addition**.

Example:

$$3 + 5 = 8 \quad (Integer)$$

$$-4 + 9 = 5 \quad (Integer)$$

$$-7 + (-2) = -9 \quad (Integer)$$

Multiplication Laws (Binary Operations)

1. Commutative Law of Multiplication

The **order of numbers** in multiplication **does not affect the product**.

$$a \times b = b \times a$$

Swapping the order of factors (numbers being multiplied) doesn't change the result.

Example:

$$3 \times 7 = 7 \times 3 = 21$$

$$(-5) \times 4 = 4 \times (-5) = -20$$

2. Associative Law of Multiplication

When multiplying three or more numbers, the way you **group them (using brackets)** doesn't affect the product.

$$(a \times b) \times c = a \times (b \times c)$$

You can multiply in any grouping order, and the result will remain the same.

Example:

$$(2 \times 3) \times 5 = 6 \times 5 = 30$$

$$2 \times (3 \times 5) = 2 \times 15 = 30$$

For Negative Numbers:

$$(-2 \times 4) \times 3 = (-8) \times 3 = -24$$

$$-2 \times (4 \times 3) = -2 \times 12 = -24$$

3. Multiplicative Identity

Multiplying any number by **1** gives the number itself.

$$a \times 1 = a$$

1 is called the **Multiplicative Identity** because multiplying by 1 leaves a number unchanged.

Example:

$$7 \times 1 = 7$$

$$-12 \times 1 = -12$$

4. Multiplicative Inverse (Reciprocal Law)

Multiplying a number by its **reciprocal (1 divided by that number)** always gives **1**.

$$a \times \frac{1}{a} = 1 \quad (\text{where } a \neq 0)$$

- **1/a**: Reciprocal of a.
- **a ≠ 0**: a cannot be zero because division by zero is undefined.

Every number (except 0) has a **multiplicative inverse** which, when multiplied, results in 1.

- Reciprocal of 5 is 1/5.
- Reciprocal of -3 is -1/3.

Examples:

$$5 \times \frac{1}{5} = 1$$

$$-4 \times \frac{1}{-4} = 1$$

$$\frac{2}{3} \times \frac{3}{2} = 1$$

Important Note:

- **0 does not have a multiplicative inverse** because division by 0 is undefined.

5. Closure Property of Multiplication

Multiplying two integers always results in another integer.

$$a \times b \in \mathbb{Z} \quad \text{(Set of Integers)}$$

When you multiply any two integers, you always get an integer. This is called **closure under multiplication**.

Examples:

$$4 \times 5 = 20 \quad \text{(Integer)}$$

$$-3 \times 7 = -21 \quad \text{(Integer)}$$

$$-6 \times -2 = 12 \quad \text{(Integer)}$$

Distributive Law of Multiplication Over Addition

Formula:

$$a \times (b + c) = (a \times b) + (a \times c)$$

Multiplying a number by a sum is the **same as multiplying that number by each addend separately**, and then **adding the results**.

Let's take:
a = 3, b = 4, c = 5

Compute both sides of the equation:

Left-Hand Side (LHS):

$$a \times (b + c) = 3 \times (4 + 5) = 3 \times 9 = 27$$

Right-Hand Side (RHS):

$$(a \times b) + (a \times c) = (3 \times 4) + (3 \times 5) = 12 + 15 = 27$$

Both sides give the same result: **27**
Thus, the distributive law is valid.

SECTION 3: ORDER OF OPERATIONS AND BRACKETS

Order of Mathematical Operations

BODMAS/PEMDAS Rule

1. Parentheses ( )

2. Square Brackets [ ]

3. Braces { }

4. Bar / Vinculum — Overline (Fraction Bar)

- Horizontal line used for fractions and long expressions.

$$\frac{3+5}{2}$$

5. Angle Brackets ⟨ ⟩

Priority	What to Solve	Bracket Symbol
1	Parentheses	( )
2	Square Brackets	[ ]
3	Braces	{ }
4	Overline / Vinculum	—

BODMAS / PEMDAS Rule:

Acronym	Full Form	Sequence
B	Brackets (Parentheses)	1st
O / E	Orders (Exponents, Powers, Roots)	2nd
D	Division	3rd
M	Multiplication	3rd
A	Addition	4th
S	Subtraction	4th

Note: Division and Multiplication are performed **from left to right**, same with Addition and Subtraction.

SECTION 4: RATIONAL AND IRRATIONAL NUMBERS

Rational Numbers

A **Rational Number** is any number that can be written in the form of a **fraction (p/q)** where:

- **p** and **q** are integers (whole numbers, positive or negative)

- **q ≠ 0** (denominator cannot be zero)

$\mathbb{Q}$ = Set of Rational Numbers

Mathematical Form:

$$\frac{p}{q} \quad \text{where} \quad p, q \in \mathbb{Z} \quad \text{and} \quad q \neq 0$$

Decimal Representation of Rational Numbers:

Rational numbers can be expressed in **two decimal forms**:

1. Terminating Decimals:

- Decimal expansion **stops** after a finite number of digits.
- Example:

$$\frac{1}{2} = 0.5 \quad \text{(terminates after 1 digit)}$$

$$\frac{3}{4} = 0.75$$

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2. Non-Terminating Recurring Decimals (Repeating Decimals):

- Decimal expansion **never ends**, but follows a **repeating pattern**.
- Example:

$$\frac{1}{3} = 0.3333... = 0.\overline{3} \quad \text{(3 keeps repeating)}$$

$$\frac{22}{7} = 3.142857142857... = 3.\overline{142857}$$

Examples of Rational Numbers:

- Positive: $\frac{5}{2}$, 3, 0.75
- Negative: $-\frac{4}{7}$, -2
- Zero: 0 (because $\frac{0}{1} = 0$)

Irrational Numbers

An **Irrational Number** is a number that **cannot be expressed as a fraction (p/q)**, where p and q are integers.

Its decimal expansion:

- **Never terminates** (goes on forever)
- **Never repeats** (no recurring pattern)
- They are **non-terminating, non-repeating decimals**.

ℚ' = Set of Irrational Numbers

Number	Decimal Form	Why Irrational?
√2	1.4142135... (never ends, no pattern)	Cannot be expressed as exact fraction
√3	1.7320508...	Non-terminating, non-repeating
π (pi)	3.141592653...	Famous irrational number used in circles
e (Euler's Number)	2.718281828...	Used in logarithms and continuous growth in AI

SECTION 5: REAL NUMBERS AND THEIR PROPERTIES

Real Number System

The **Real Number System** is the set of all numbers that can be represented on the number line. It includes both:

1. **Rational Numbers** (can be expressed as fractions)

2. **Irrational Numbers** (cannot be expressed as fractions)

Mathematical Representation:

$\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$

Where:

- $\mathbb{R}$ = Set of Real Numbers
- $\mathbb{Q}$ = Set of Rational Numbers
- $\mathbb{Q}'$ = Set of Irrational Numbers
- $\cup$ = Union (combines both sets into one

Breakdown of Real Numbers:

Category	Examples
Rational Numbers ($\mathbb{Q}$)	1/2, -3, 4.75, 0, 7/3, 0.333...
Irrational Numbers ($\mathbb{Q}'$)	$\sqrt{2}$, π, e, $\sqrt{5}$

Properties of Real Numbers

1. Closure Property (Addition)

When you add any two real numbers, the result will also be a real number.

$a + b \in \mathbb{R}$

(where $a, b \in \mathbb{R}$)

Symbols Explained:

$\in \mathbb{R}$ "belongs to the set of Real Numbers"

Example:

$5 + 3 = 8$ ($8 \in \mathbb{R}$)

2. Closure Property (Multiplication)

When you multiply any two real numbers, the result is always a real number.

Symbolically:

$$a \times b \in \mathbb{R}$$

$$(\text{where } a, b \in \mathbb{R})$$

Example:
$2 \times 3 = 6$

3. Commutative Property (Addition)

The order in which you add two real numbers does not affect the sum.

$$a + b = b + a$$

Example:
$5 + 7 = 7 + 5 = 12$

4. Commutative Property (Multiplication)

The order in which you multiply two real numbers does not affect the product.

$$a \times b = b \times a$$

Example:
$4 \times 9 = 9 \times 4 = 36$

5. Associative Property (Addition)

The way numbers are grouped in addition does not affect the sum.

$$(a + b) + c = a + (b + c)$$

Example:
$(2 + 3) + 4 = 2 + (3 + 4) = 9$

6. Associative Property (Multiplication)

The way numbers are grouped in multiplication does not affect the product.

$$(a \times b) \times c = a \times (b \times c)$$

Example:
$(2 \times 3) \times 5 = 2 \times (3 \times 5) = 30$

7. Distributive Property

Multiplication distributes over addition.

$$a \times (b + c) = (a \times b) + (a \times c)$$

Example:

$$2 \times (3 + 4) = (2 \times 3) + (2 \times 4)$$

$$2 \times 7 = 6 + 8 = 14$$

8. Additive Identity

Adding 0 to any real number does not change its value.

$$a + 0 = a$$

Example:

7 + 0 = 7

9. Multiplicative Identity

Multiplying any real number by 1 does not change its value.

$$a \times 1 = a$$

Example:

15 × 1 = 15

10. Additive Inverse

For every real number a, there exists -a, which is its additive inverse. Their sum is 0.

$$a + (-a) = 0$$

Example:

5 + (-5) = 0

11. Multiplicative Inverse

For every non-zero real number a, there exists 1/a, which is its multiplicative inverse. Their product is 1.

Symbolically:

$$a \times (1/a) = 1, \text{ where } a \neq 0$$

Example:
$4 \times (1/4) = 1$

Properties of Equality

1. Reflexive Property

Any number is always equal to itself.

$$a = a$$

Example:
$5 = 5$
$\pi = \pi$

2. Symmetric Property

If one number equals another, then the second number equals the first.

$$\text{If } a = b, \text{ then } b = a$$

Example:

$$\text{If } 7 = x, \text{ then } x = 7$$

$$\text{If } a = \sqrt{2}, \text{ then } \sqrt{2} = a$$

3. Transitive Property

If a number equals a second number, and that second number equals a third number, then the first and third numbers are also equal.

$$\text{If } a = b \text{ and } b = c, \text{ then } a = c$$

Example:

$$\text{If } a = 5 \text{ and } 5 = c, \text{ then } a = c$$

$$\text{If } x = 2, \text{ and } 2 = y, \text{ then } x = y$$

4. Addition Property of Equality

You can add the same number to both sides of an equation without changing its truth.

If $a = b$, then $a + c = b + c$

Example:

If $3 = 3$, then $3 + 5 = 3 + 5 \rightarrow 8 = 8$

5. Subtraction Property of Equality

You can subtract the same number from both sides of an equation without changing its truth.

If $a = b$, then $a - c = b - c$

Example:

If $x = 10$, then $x - 4 = 10 - 4 \rightarrow x - 4 = 6$

6. Multiplication Property of Equality

You can multiply both sides of an equation by the same number, and the equation remains true.

If $a = b$, then $a \times c = b \times c$

Example:

If $y = 3$, then $y \times 2 = 3 \times 2 \rightarrow y \times 2 = 6$

7. Division Property of Equality

You can divide both sides of an equation by the same non-zero number, and the equation remains true.

If $a = b$ and $c \neq 0$, then $a \div c = b \div c$ or $a/c = b/c$

Example:

If $m = 12$, then $m \div 3 = 12 \div 3 \rightarrow m \div 3 = 4$

Properties of Inequalities

1. Trichotomy Property

For any two real numbers **a** and **b**, exactly **one** of these three statements is true:

1. $a < b$ (a is less than b)
2. $a = b$ (a is equal to b)
3. $a > b$ (a is greater than b)

For any $a, b \in \mathbb{R}$, only one of $a < b$, $a = b$, or $a > b$ is true.

Example:

If $a = 3$ and $b = 7 \rightarrow a < b$

If $a = 5$ and $b = 5 \rightarrow a = b$

If $a = 10$ and $b = 2 \rightarrow a > b$

2. Transitive Property of Inequality

If one number is less than a second, and the second is less than a third, then the first is less than the third.

If $a < b$ and $b < c$, then $a < c$

Example:

If $a = 2$, $b = 4$, and $c = 6$,

Since $2 < 4$ and $4 < 6 \rightarrow$ we conclude $2 < 6$

3. Addition Property of Inequality

You can add the same number to both sides of an inequality, and the inequality remains true.

If $a < b$, then $a + c < b + c$

Example:

If $a = 3$ and $b = 5 \rightarrow 3 < 5$

Adding $c = 2$ to both sides:

$3 + 2 < 5 + 2 \rightarrow 5 < 7$ (True)

4. Subtraction Property of Inequality

You can subtract the same number from both sides of an inequality, and the inequality remains true.

If $a < b$, then $a - c < b - c$

Example:

If $a = 8$ and $b = 10 \rightarrow 8 < 10$

Subtracting $c = 3$ from both sides:

$8 - 3 < 10 - 3 \rightarrow 5 < 7$ (True)

5. Multiplication Property (Positive)

If you multiply both sides of an inequality by the same positive number, the direction of the inequality remains the same.

If $a < b$ and $c > 0$, then $a \times c < b \times c$

Example:

If $a = 2$, $b = 4$, and $c = 3 \rightarrow 2 < 4$

Multiplying both sides by 3:

$2 \times 3 < 4 \times 3 \rightarrow 6 < 12$ (True)

6. Multiplication Property (Negative)

If you multiply both sides of an inequality by the same negative number, the direction of the inequality **reverses**.

$$\text{If } a < b \text{ and } c < 0, \text{ then } a \times c > b \times c$$

$c < 0$ c is a negative number

Example:

If $a = 2$, $b = 5$, and $c = -2 \rightarrow 2 < 5$

Multiplying both sides by -2:

$2 \times (-2) > 5 \times (-2) \rightarrow -4 > -10$ (True)

7. Division Property of Inequality

If you divide both sides of an inequality by the same positive number, the inequality remains true. (Dividing by a negative number reverses the inequality, but that is similar to multiplication.)

$$\text{If } a < b \text{ and } c > 0, \text{ then } a \div c < b \div c \text{ or } a/c < b/c$$

$c > 0$ c is a positive number

Example:

If $a = 6$ and $b = 12$, and $c = 3 \rightarrow 6 < 12$

Dividing both sides by 3:

$6 \div 3 < 12 \div 3 \rightarrow 2 < 4$ (True)

SECTION 6: POWERS, ROOTS, AND RADICALS

Squares and Square Roots

Perfect Squares

A **Perfect Square** is a number that is obtained by multiplying an integer by itself.

$n \times n = n^2$ (read as "n squared")

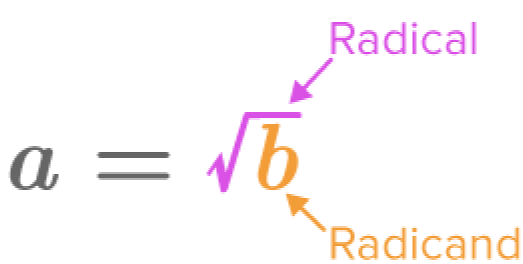
Examples of Perfect Squares:

Integer (n)	Perfect Square (n^2)
1	1 × 1 = 1
2	2 × 2 = 4
3	3 × 3 = 9
4	4 × 4 = 16
5	5 × 5 = 25
...	...

Square Roots (√a)

The **Square Root** of a number **a** is a number which, when multiplied by itself, gives **a**.

Symbolically:
$\sqrt{a} = b \rightarrow$ where $b \times b = a$



√a Square root of 'a'

Examples:

Number (a)	Square Root (√a)
4	√4 = 2 (because 2 × 2 = 4)
9	√9 = 3 (because 3 × 3 = 9)

Number (a)	Square Root (√a)
16	√16 = 4
25	√25 = 5

Properties of Square Roots

1. Square Root of a Perfect Square is an Integer

If **a** is a perfect square, then √a is a whole number.

- Example: √36 = 6 (because 6 × 6 = 36)

---

2. √(a × b) = √a × √b (Product Property)

The square root of a product equals the product of their square roots.
√(a × b) = √a × √b

Example:
√(4 × 9) = √4 × √9 → 2 × 3 = 6

3. √(a / b) = √a / √b (Quotient Property)

The square root of a fraction equals the square root of numerator divided by the square root of denominator. (b ≠ 0)

Example:
√(16 / 25) = √16 / √25 → 4 / 5

4. Square Root of Zero

√0 = 0
(Because 0 × 0 = 0)

5. Square Root of 1

√1 = 1
(Because 1 × 1 = 1)

6. Square Root of a Negative Number is Not a Real Number

$\sqrt{(-a)}$ is **not** a real number.

- For example: $\sqrt{(-4)}$ does not exist in Real Numbers ($\mathbb{R}$).
- Such roots are part of **Complex Numbers**, represented using **i (imaginary unit)**.

7. Simplifying Square Roots

If a number inside the square root is not a perfect square, it can be simplified by factoring out perfect squares.

Example:

$\sqrt{50} = \sqrt{(25 \times 2)} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}$

Cubes and Cube Roots

Perfect Cubes

A **Perfect Cube** is a number that is obtained by multiplying an integer by itself three times.

$$n \times n \times n = n^3 \text{ (read as "n cubed")}$$

Examples of Perfect Cubes:

Integer (n)	Perfect Cube (n^3)
1	$1 \times 1 \times 1 = 1$
2	$2 \times 2 \times 2 = 8$
3	$3 \times 3 \times 3 = 27$
4	$4 \times 4 \times 4 = 64$
5	$5 \times 5 \times 5 = 125$
6	$6 \times 6 \times 6 = 216$
...	...

Cube Roots ($\sqrt[3]{a}$)

The **Cube Root** of a number **a** is a number which, when multiplied by itself three times, gives **a**.

$$\sqrt[3]{a} = b \rightarrow \text{where } b \times b \times b = a$$

$\sqrt[3]{a}$ Cube root of 'a'

Examples:

Number (a)	Cube Root ($\sqrt[3]{a}$)
8	$\sqrt[3]{8} = 2$ (because $2 \times 2 \times 2 = 8$)
27	$\sqrt[3]{27} = 3$
64	$\sqrt[3]{64} = 4$
125	$\sqrt[3]{125} = 5$
1	$\sqrt[3]{1} = 1$

Properties of Cube Roots

1. Cube Root of a Perfect Cube is an Integer

If **a** is a perfect cube, then $\sqrt[3]{a}$ is a whole number.

- Example: $\sqrt[3]{216} = 6$ (because $6 \times 6 \times 6 = 216$)

2. $\sqrt[3]{(a \times b)} = \sqrt[3]{a} \times \sqrt[3]{b}$ (Product Property)

The cube root of a product equals the product of their cube roots.

$\sqrt[3]{(a \times b)} = \sqrt[3]{a} \times \sqrt[3]{b}$

Example:

$\sqrt[3]{(8 \times 27)} = \sqrt[3]{8} \times \sqrt[3]{27} \rightarrow 2 \times 3 = 6$

3. $\sqrt[3]{(a / b)} = \sqrt[3]{a} / \sqrt[3]{b}$ (Quotient Property)

The cube root of a fraction equals the cube root of numerator divided by the cube root of denominator. ($b \neq 0$)

Example:
$\sqrt[3]{(64 / 125)} = \sqrt[3]{64} / \sqrt[3]{125} \rightarrow 4 / 5$

4. Cube Root of Zero

$\sqrt[3]{0} = 0$

(Because $0 \times 0 \times 0 = 0$)

5. Cube Root of 1

$\sqrt[3]{1} = 1$

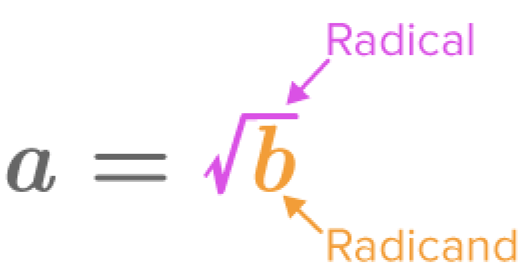
(Because $1 \times 1 \times 1 = 1$)

6. Cube Roots of Negative Numbers are Real Numbers

Unlike square roots, the cube root of a negative number is also a real number.

- Example: $\sqrt[3]{(-8)} = -2$ (because $-2 \times -2 \times -2 = -8$)

Radicals and Radicands



Radical (√ Symbol)

The **Radical Symbol (√)** is used to represent a root of a number. Most commonly, it represents a **square root**, but it can also be used for cube roots, fourth roots, etc.

Example:
$\sqrt{9}$ means "the square root of 9" → Answer: 3 (because 3 × 3 = 9)

Radicand

The **Radicand** is the **number or expression placed under the radical symbol (√)**. It is the number you are finding the root of.

Example:
In $\sqrt{16}$,

- Radical symbol: √
- **Radicand**: 16 → We're finding the square root of 16 (Answer: 4)

Index (or Degree of the Root)

The **Index** is the small number written just above and to the left of the radical symbol (√), indicating **which root** is being taken.

Index	Meaning	Example	Answer
2	Square Root (default if not written)	$\sqrt{9}$	3
3	Cube Root	$\sqrt[3]{27}$	3
4	Fourth Root	$\sqrt[4]{16}$	2
n	n-th Root	$\sqrt[n]{a}$	Depends on 'n' and 'a'

Radical Exponential Form

Converting Radicals to Exponential Form

Radical expressions (like square roots and cube roots) can be rewritten using exponents with **fractional powers**.

Basic Conversion Rules:

Radical Form	Exponential Form
$\sqrt{a}$	a^(1/2)
$\sqrt[3]{a}$	a^(1/3)

Radical Form Exponential Form

$\sqrt[n]{a}$ $a^{(1/n)}$

$a^{(1/n)}$ 'a' raised to the power of 1/n (fractional exponent)

Examples:

- $\sqrt{16} = 16^{(1/2)} = 4$
- $\sqrt[3]{27} = 27^{(1/3)} = 3$
- $\sqrt[4]{81} = 81^{(1/4)} = 3$

Why is this Useful?

Fractional exponents are used in algebra and calculus because they make it easier to apply power rules and simplify complex expressions.

Properties of Radicals

1. Product Property of Radicals

Rule: $\sqrt{(a \times b)} = \sqrt{a} \times \sqrt{b}$
The square root of a product equals the product of the square roots.

Example:
$\sqrt{(4 \times 9)} = \sqrt{4} \times \sqrt{9} \rightarrow 2 \times 3 = 6$

2. Quotient Property of Radicals

Rule: $\sqrt{(a / b)} = \sqrt{a} / \sqrt{b}$ $(b \neq 0)$
The square root of a division is equal to the division of the square roots.

Example:
$\sqrt{(25 / 4)} = \sqrt{25} / \sqrt{4} \rightarrow 5 / 2 = 2.5$

3. Power Property of Radicals

$$\sqrt{a^n} = \left(\sqrt{a}\right)^n$$

Rule:

Explanation:
You can take the root first and then raise to the power, or vice versa depending on context.

Example:

$$\sqrt{9^2} = \left(\sqrt{9}\right)^2 = 3^2 = 9$$

4. Root of a Root Property

Rule: $\sqrt[n]{\sqrt[m]{a}} = \sqrt[mn]{a}$

(It means, if you take a root inside another root, you can multiply the indices.)

Symbols Explained:

- $\sqrt[n]{\ }$ means n-th root.
- $\sqrt[m]{\ }$ means m-th root.
- $\sqrt[mn]{a}$ means take the (m×n)-th root of a.

Example:

$\sqrt{\sqrt[3]{8}} = \sqrt[6]{8}$

(Reason: Square root has index 2, cube root has index 3 → 2 × 3 = 6)

5. Simplifying Property

Rule: $\sqrt{(a^2)} = |a|$

Explanation:

The square root of a square returns the absolute value of 'a'.

- $|a|$ means absolute value, which is always non-negative.

Examples:

- $\sqrt{(5^2)} = \sqrt{25} = 5$
- $\sqrt{((-7)^2)} = \sqrt{49} = 7$ (because we take absolute value, not -7)

SECTION 7: EXPONENTS AND INDICES

Exponent

n **Exponent** tells you how many times to **multiply a number (called the base)** by itself.

Example:

$$2^3 = 2 \times 2 \times 2 = 8$$

Here:

- **2** is the **base**.
- **3** is the **exponent** (it means multiply 2 by itself 3 times).

Mathematical Form:

$$a^n$$

where:

- a = **Base** (any real number)
- n = **Exponent** (can be a whole number, negative, or a fraction)

Expression	Meaning	Result
3^2	3 × 3	9
5^3	5 × 5 × 5	125
2^4	2 × 2 × 2 × 2	16

Exponents?

Exponents (Plural of Exponent):

The term **Exponents** refers to the general concept of powers or indices in mathematics — it includes:

- Positive exponents (e.g., 2^3)
- Zero exponent (e.g., $5^0 = 1$)
- Negative exponents (e.g., $2^{-2} = \frac{1}{4}$)
- Fractional exponents (e.g., $16^{1/2} = 4$)

Exponents represent **repeated multiplication**.
When the exponent is:

- **Positive Integer** → Multiply repeatedly.
- **Zero** → Always equals 1 (except for 0^0).
- **Negative Integer** → Represents reciprocal.
- **Fraction** → Represents roots (e.g., square roots, cube roots).

Exponent Type	Expression	Meaning	Result
Positive	2^3	2 × 2 × 2	8
Zero	7^0	Always equals 1	1
Negative	3^{-2}	1 / (3 × 3)	1/9
Fractional	$16^{1/2}$	Square root of 16	4

Laws of Exponents

1. Product Rule (Multiplying Same Base)

For any real number a and integers m and n:

$$a^m \times a^n = a^{m+n}$$

When multiplying powers with the same base, add their exponents.

Example:

$$2^3 \times 2^4 = 2^{3+4} = 2^7 = 128$$

2. Quotient Rule (Dividing Same Base)

For any real number a (where $a \neq 0$) and integers m and n:

$$\frac{a^m}{a^n} = a^{m-n}$$

When dividing powers with the same base, subtract the exponent of the denominator from the exponent of the numerator.

Example:

$$\frac{5^6}{5^2} = 5^{6-2} = 5^4 = 625$$

3. Power of a Power Rule

For any real number a and integers m and n:

$$(a^m)^n = a^{m \cdot n}$$

Explanation:

When raising a power to another power, multiply the exponents.

Example:

$$(3^2)^4 = 3^{2\times 4} = 3^8 = 6561$$

4. Product to Power Rule

For any real numbers a and b, and integer n:

$$(ab)^n = a^n \times b^n$$

When raising a product to a power, apply the exponent to each factor.

Example:

$$(2 \times 5)^3 = 2^3 \times 5^3 = 8 \times 125 = 1000$$

5. Quotient to Power Rule

For any real numbers a and b (where $b \neq 0$), and integer n:

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

When raising a quotient to a power, apply the exponent to both numerator and denominator.

Example:

$$\left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2} = \frac{9}{16}$$

6. Zero Exponent Rule

For any real number a (where $a \neq 0$):

$$a^0 = 1$$

Any non-zero number raised to the power of zero equals 1.

Example:

$$7^0 = 1$$

7. Negative Exponent Rule

For any real number a (where $a \neq 0$) and integer n:

$$a^{-n} = \frac{1}{a^n}$$

A negative exponent indicates reciprocal of the base raised to the positive exponent.

Example:

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$$

8. Fractional Exponent Rule

For any real number a (where $a \geq 0$) and integers m, n (where $n \neq 0$):

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

Fractional exponents represent roots. The denominator of the exponent indicates the root, and the numerator indicates the power.

Example:

$$8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$$

SECTION 8: LOGARITHMS

Logarithm?

A **logarithm** is the **inverse operation of exponentiation**.

- **Exponentiation** is when you raise a number to a power:

$$a^x = y \quad 2^3 = 8,$$

- **Logarithm** answers the question:
  "**To what exponent must we raise a (base) to get y?**"

$$\log_a(y) = x \quad \log_2(8) = 3$$

Mathematical Definition:

If:

$$a^x = y$$

then:

$$\log_a(y) = x$$

Where:

Symbol	Meaning
a	**Base** of the logarithm (a > 0, a ≠ 1)
x	**Exponent** (the answer we're finding)
y	**Result** (the number we take the log of)
$\log_a(y)$	Logarithm of y with base a

Exponential Form	Logarithmic Form
$10^2 = 100$	$\log_{10}(100) = 2$
$3^4 = 81$	$\log_3(81) = 4$
$5^0 = 1$	$\log_5(1) = 0$

- The base a must always be **greater than 0** and **cannot be 1**.
- Logarithms **undo exponentiation** (just like subtraction undoes addition).

Exponent: Tells **how many times to multiply**.

Logarithm: Tells **what exponent was used to get a number**.

Two Important Bases of Logarithms

1. Common Logarithm (Base 10)(Logarithm of x with base 10)

- The **Common Logarithm** has a **base of 10**.
- It is written as:
- $$\log_{10}(x) \quad \text{or simply} \quad \log(x)$$
- **If no base is mentioned**, it is understood to be base 10.

Example:

$$\log_{10}(1000) = 3 \quad \text{because} \quad 10^3 = 1000$$

2. Natural Logarithm (Base e)(Logarithm of x with base e)

- The **Natural Logarithm** has a **base of Euler's Number (e)**.
- It is written as:

- $\log_e(x) \quad \text{or simply} \quad \ln(x)$
- **Euler's Constant (e)** is an **irrational number** approximately equal to:

$$e \approx 2.71828...$$

Example:

$$\ln(e) = 1 \quad \text{because} \quad e^1 = e$$

Laws and Properties of Logarithms

1. Product Property (Multiplication Rule)

$$\log_a(x \cdot y) = \log_a(x) + \log_a(y)$$

Explanation:

- The logarithm of a product equals the sum of logarithms.
- Multiply inside becomes addition outside.

Example:

$$\log_2(8 \cdot 4) = \log_2(8) + \log_2(4)$$

$$\log_2(32) = 3 + 2 = 5$$

(Since $2^3 = 8$ and $2^2 = 4$)

2. Quotient Property (Division Rule)

$$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$$

- The logarithm of a quotient equals the difference of logarithms.
- Division inside becomes subtraction outside.

Example:

$$\log_3\left(\frac{81}{9}\right) = \log_3(81) - \log_3(9)$$

$$4 - 2 = 2$$

(Since $3^4 = 81$ and $3^2 = 9$)

3. Power Property (Exponent Rule)

$$\log_a(x^n) = n \cdot \log_a(x)$$

- The exponent n can be brought in front as a multiplier.
- Powers inside become multiplication outside.

Example:

$$\log_5(25^3) = 3 \cdot \log_5(25)$$

$$3 \cdot 2 = 6$$

(Since $5^2 = 25$)

4. Base Change Formula

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

- Allows changing the base of a logarithm from a to b.
- Used when calculator only has base 10 (log) or base e (ln).

Example:

$$\log_2(8) = \frac{\log_{10}(8)}{\log_{10}(2)}$$

Compute using calculator:

$$\frac{0.9031}{0.3010} = 3$$

5. Identity Property

$$\log_a(a) = 1$$

- The log of a number to its own base is always 1.

Example:

$$\log_7(7) = 1$$

(Because $7^1 = 7$)

6. Zero Property

$$\log_a(1) = 0$$

- The logarithm of 1 in any base is always 0.

Example:

$$\log_{10}(1) = 0$$

(Because any number raised to power 0 is 1)

7. Inverse Property (Exponential Form)

$$a^{\log_a(x)} = x$$

- Exponentials and logarithms are inverse functions.
- Raising the base to its own log brings back the original number.

Example:

$$2^{\log_2(16)} = 16$$

(Because $\log_2(16) = 4$ and $2^4 = 16$)

8. Inverse Property (Logarithmic Form)

$$\log_a(a^x) = x$$

- Logarithm of a number raised to its base exponent equals the exponent.
- Logs "cancel out" exponents.

Example:

$$\log_5(5^3) = 3$$

(Because log and exponential cancel each other out)

9. Domain Restriction

$$x > 0$$

- Logarithms are **only defined for positive numbers**.
- You cannot take log of 0 or negative numbers in real numbers.

Example:

$$\log(0) \text{ is undefined}, \quad \log(-5) \text{ is undefined}$$

You can only take log of positive numbers.

10. Negative Property (Reciprocal Rule)

$$\log_a\left(\frac{1}{x}\right) = -\log_a(x)$$

Explanation:

- The log of a reciprocal becomes negative.

Example:

$$\log_3\left(\frac{1}{9}\right) = -\log_3(9)$$

$$= -2$$

(Since $3^2 = 9$)

11. Root Property (Radical Rule)

$$\log_a(\sqrt[n]{x}) = \frac{1}{n} \cdot \log_a(x)$$

- Roots inside the log can be expressed as fractions outside.

Example:

$$\log_2(\sqrt{8}) = \frac{1}{2} \cdot \log_2(8)$$

$$= \frac{1}{2} \cdot 3 = 1.5$$

(Since $\sqrt{8} = 2.828$, but using properties we get exact log value)

Antilogarithms

The **Antilogarithm** is the **inverse operation of a logarithm**.

In other words:

$$\text{If} \quad \log_a(x) = y, \quad \text{then} \quad \text{antilog}_a(y) = x$$

Or simply:

$$x = a^y$$

Formula of Antilogarithms:

$$\text{antilog}_a(y) = a^y$$

Where:

- a = Base of the logarithm (a > 0, a ≠ 1)
- y = Logarithm value (Exponent)
- x = Original number (Result)

$\text{antilog}_a(y)$	Antilog of y with base a (means a^y)

Logarithms and Antilogarithms are **inverse functions**.

- Logarithm: **Exponential → Log Form**

$$a^y = x \quad \Longrightarrow \quad \log_a(x) = y$$

- Antilogarithm: **Log Form → Exponential**

$$\log_a(x) = y \quad \Longrightarrow \quad x = a^y$$

Thus:

- **Logarithm "extracts" the exponent.**
- **Antilogarithm "rebuilds" the original number** from the exponent.

Examples of Antilogarithms:

Example 1 (Base 10 – Common Antilog):

$$\log_{10}(x) = 2$$

Find x:

$$x = \text{antilog}_{10}(2) = 10^2 = 100$$

Example 2 (Base e – Natural Antilog):

$$\ln(x) = 1$$

Find x:

$$x = e^1 = 2.71828$$

SECTION 9: RATIOS, RATES, AND PROPORTIONS

1. Ratio

A **Ratio** is a comparison of two quantities that have the **same unit**.

$$\text{Ratio} = \frac{a}{b} \quad \text{or} \quad a : b$$

Where:

- a and b are quantities of the same unit.
- $b \neq 0$

a	First quantity (same unit)
b	Second quantity (same unit, non-zero)
$:$	Ratio symbol (read as "to")
$\frac{a}{b}$	Fraction form of ratio

Example:

If a box has **3 red balls** and **5 blue balls**, the ratio of red to blue is:

$$\frac{3}{5} \quad \text{or} \quad 3:5$$

2. Rate

A **Rate** is a comparison of two quantities that have **different units**.

$$\text{Rate} = \frac{a\,(\text{unit 1})}{b\,(\text{unit 2})}$$

Example:

If a car travels **120 km in 2 hours**, the rate is:

$$\frac{120\,\text{km}}{2\,\text{hours}} = 60\,\text{km}/\text{hour}$$

3. Percentage

Definition (Simple Words):

A **Percentage** is a ratio that compares a quantity to **100**.

$$\text{Percentage} = \frac{a}{100} \quad \text{or} \quad a\%$$

Example:

If you score **45 marks out of 50**, your percentage is:

$$\frac{45}{50} \times 100 = 90\%$$

TIME AND AVERAGE RATE, TYPES OF PROPORTIONS

1. Average Rate

The **Average Rate** is the total change in quantity divided by the total time taken.

Mathematical Formula:

$$\text{Average Rate} = \frac{\text{Total Change}}{\text{Total Time}}$$

Total Change	The total quantity increased or decreased
Total Time	The total time taken for that change
·	Division (per unit time)

If a car travels **300 km** in **5 hours**, then:

$$\text{Average Speed} = \frac{300\,\text{km}}{5\,\text{hours}} = 60\,\text{km/h}$$

Example (Growth Rate):

If a company's profit increases by **$5000** over **2 years**, then:

$$\text{Average Growth Rate} = \frac{5000\,\text{dollars}}{2\,\text{years}} = 2500\,\text{dollars/year}$$

2. Types of Proportions

2.1 Direct Proportion

Two quantities are in **Direct Proportion** if an increase in one results in a proportional increase in the other.

Mathematical Form:

$$y = kx \quad \text{or} \quad y \propto x$$

Where:

- k is the constant of proportionality.
- x and y increase or decrease together.

y	Dependent Variable
x	Independent Variable
k	Constant Ratio (Proportionality Constant)
$\propto$	Proportional To

Example:

If the cost of 1 pen is $2, then the cost of x pens is:

$$\text{Cost} = 2x$$

2.2 Inverse Proportion

Definition:

Two quantities are in **Inverse Proportion** if an increase in one results in a proportional decrease in the other.

Mathematical Form:

$$y = \frac{k}{x} \quad \text{or} \quad y \propto \frac{1}{x}$$

Where:

- k is a constant.
- As x increases, y decreases proportionally.

y	Dependent Variable
x	Independent Variable
k	Constant Product (x · y = k)
$\propto$	Proportional To

Example:

If 5 workers complete a task in **10 days**, how long will 10 workers take?

$$\text{Work Time} \propto \frac{1}{\text{Number of Workers}}$$

Doubling the workers halves the time:

$$\frac{10 \text{ days}}{2} = 5 \text{ days}$$

2.3 Compound Proportion

A **Compound Proportion** involves more than two quantities related through **multiple direct or inverse proportions** simultaneously.

Example:

If 4 machines produce 100 items in 5 hours, how many items will **8 machines** produce in **10 hours**?

- **Machines ↔ Items (Direct Proportion)**
- **Time ↔ Items (Direct Proportion)**

Solution:

$$\text{New Items} = 100 \times \frac{8}{4} \times \frac{10}{5} = 100 \times 2 \times 2 = 400$$

We have **two factors** affecting production:

1. **Number of Machines** (More machines → More items) → **Direct Proportion**
2. **Time** (More time → More items) → **Direct Proportion**

The formula for Compound Proportion is:

$$\text{New Quantity} = \text{Old Quantity} \times \frac{\text{New Factor 1}}{\text{Old Factor 1}} \times \frac{\text{New Factor 2}}{\text{Old Factor 2}} \times \dots$$

For this problem:

$$\text{New Items} = 100 \times \frac{8}{4} \times \frac{10}{5}$$

Step 3: Understand What's Happening

Factor	Old Value	New Value	Ratio	Effect
Machines	4	8	$\frac{8}{4} = 2$	Double the output
Time	5 hours	10 hours	$\frac{10}{5} = 2$	Double the output

$$\text{New Items} = 100 \times 2 \times 2$$

$$\boxed{\text{New Items} = 400}$$

Thus, **8 machines in 10 hours will produce 400 items**.

Distance, Speed, and Time

1. Uniform Speed (Constant Speed)

Uniform speed means an object moves **at the same speed** without any change in its velocity over time.

Formula:

$$\text{Distance} = \text{Speed} \times \text{Time}$$

$$d = s \times t$$

d	Distance traveled (meters, km, etc.)
s	Speed or velocity (m/s, km/h, etc.)
t	Time taken (seconds, hours, etc.)

A car is moving at a **uniform speed of 60 km/h**. How far will it travel in **3 hours**?

Solution:

Given:

- Speed (s) = 60 km/h
- Time (t) = 3 hours

Using formula:

$$d = s \times t = 60 \times 3 = 180\,\text{km}$$

Answer: The car will travel **180 km**.

2. Average Speed

Average speed is calculated when an object's speed **is not constant**. It is the **total distance covered divided by the total time taken**.

Formula:

$$\text{Average Speed} = \frac{\text{Total Distance}}{\text{Total Time}}$$

$$s_{\text{avg}} = \frac{d_{\text{total}}}{t_{\text{total}}}$$

Example:

- A car covers 100 km in 2 hours, then 150 km in 3 hours.
- Total Distance = 100 km + 150 km = 250 km
- Total Time = 2 hr + 3 hr = 5 hr

$$s_{\text{avg}} = \frac{250}{5} = 50\,\text{km/h}$$

3. Speed-Distance-Time Relationship

These three quantities are always connected through the basic formula:

$$d = s \times t$$

You can rearrange this formula based on what you need to find:

Find	Formula
Speed (s)	$s = \frac{d}{t}$
Distance (d)	$d = s \times t$
Time (t)	$t = \frac{d}{s}$

Example Problem 1 (Finding Distance):

A cyclist moves at a uniform speed of 15 km/h for 2 hours.

$$d = s \times t = 15 \times 2 = 30\,\text{km}$$

Example Problem 2 (Finding Time):

A train travels 300 km at a constant speed of 60 km/h. How long does it take?

$$t = \frac{d}{s} = \frac{300}{60} = 5\,\text{hours}$$

Example Problem 3 (Finding Speed):

A car covers 180 km in 3 hours. What is its speed?

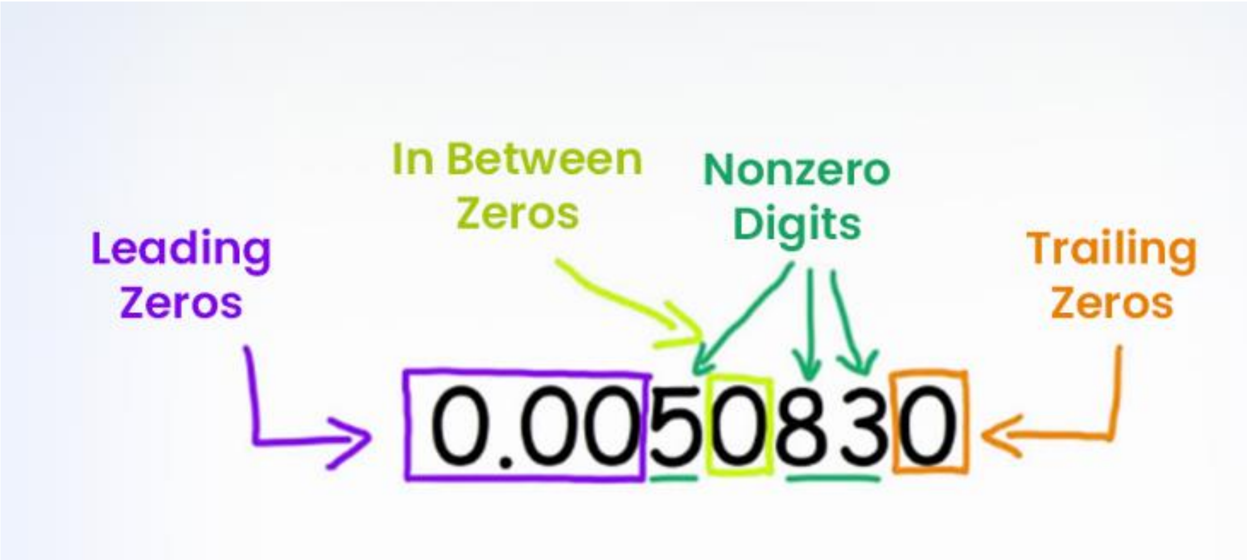
$$s = \frac{d}{t} = \frac{180}{3} = 60\,\mathrm{km/h}$$

Estimation and Approximation

1. Significant Figures (Sig. Figs)

Significant figures are the digits in a number that contribute to its precision. They include:

- All non-zero digits.
- Zeros **between** non-zero digits.
- Trailing zeros **only if** there's a decimal point.



Rules for Counting Significant Figures:

Rule	Example	Significant Figures
Non-zero digits are always significant	1234	4
Zeros between non-zero digits are significant	1025	4

Rule	Example	Significant Figures
Leading zeros are **not significant**	0.0045	2
Trailing zeros are significant **if decimal exists**	23.400	5
Trailing zeros without decimal are **not significant**	1500	2

Example:

How many significant figures are there in **0.005060**?

- Ignore leading zeros.
- Count: 5, 0, 6, 0 → **4 significant figures**.

2. Approximation Error

Approximation Error is the difference between the **exact value** and the **approximate value**.

Formula:

$$\text{Approximation Error} = |\text{Exact Value} - \text{Approximate Value}|$$

Example:

The exact value of π = 3.14159, approximate value is taken as 3.14.

Approximation Error:

$$|3.14159 - 3.14| = 0.00159$$

3. Rounding

Rounding is simplifying a number to make it easier to work with while keeping it close to its original value.

Rules of Rounding:

Situation	What to Do
If digit after rounding place is **less than 5**	Keep the digit same
If digit after rounding place is **5 or more**	Increase the digit by 1

Example 1: Rounding to Nearest Tenth

Round 4.76 to the nearest tenth:

- Look at hundredth digit → 6 (which is ≥ 5)
- Increase tenth digit: 4.7 → **4.8**

Example 2: Rounding to Nearest Whole Number

Round 5.43 to nearest whole number:

- Look at tenths digit → 4 (which is < 5)
- Keep unit digit same: **5**

Rounding 5.5 (Nearest Whole Number)

Situation:

You are rounding **5.5** to the nearest whole number.

Rule for Rounding 5:

- **If the digit after the rounding place is exactly 5:**
  - Most common rule: **Round Up**.
  - This is known as the **"Round Half Up" Rule**.
  - So, **5.5** → **6**.

Ordinary Numbers to Scientific Notation

Scientific Notation is a way of writing **very large** or **very small numbers** in a compact form as:

$$N \times 10^n$$

Where:

- N = A number called **Mantissa** ($1 \leq |N| < 10$)
- 10^n = Power of 10 (Exponent)

General Form:

$$a \times 10^n \quad \text{where} \quad 1 \leq |a| < 10, \quad n \in \mathbb{Z}$$

- a = Significant digits (Mantissa)
- n = Exponent (Integer)

Conversion Steps:

Case 1: Large Numbers (Positive Exponent)

1. Move the **decimal point** to the **left** until only **one non-zero digit** is to its left.
2. Count how many places you moved the decimal.
3. The number becomes:

$$N \times 10^n \quad \text{where} \quad n = \text{number of decimal shifts to the left}$$

Example 1:

Convert 45,000,000 to scientific notation.

- Step 1: Move decimal 7 places left → 4.5
- Step 2: Exponent n = 7
- Result:

$$4.5 \times 10^7$$

Case 2: Small Numbers (Negative Exponent)

1. Move the **decimal point** to the **right** until only **one non-zero digit** is to its left.
2. Count how many places you moved the decimal.
3. The number becomes:

$$N \times 10^{-n} \quad \text{where} \quad n = \text{number of decimal shifts to the right}$$

Example 2:

Convert 0.00036 to scientific notation.

- Step 1: Move decimal 4 places right → 3.6
- Step 2: Exponent n = -4
- Result:

$$3.6 \times 10^{-4}$$

Ordinary Number	Scientific Notation
7,500,000	7.5×10^{6}
0.0025	2.5×10^{-3}
602,000,000,000,000,000,000,000 (Avogadro's Number)	6.02×10^{23}
0.000000000089	8.9×10^{-11}

Converting Scientific Notation to Ordinary Numbers

Scientific Notation is written in the form:

$$N \times 10^n$$

Where:

- N = Mantissa (a number between 1 and 10)
- n = Exponent (an integer)
- n tells how many times to multiply (or divide) by 10.

Conversion Rules:

Case 1: Positive Exponent (n > 0)

- Move the **decimal point to the right** by **n places**.
- Fill empty places with **zeros**.

Example 1:

Convert 3.5×10^4 to ordinary number.

- Exponent is +4 → Move decimal 4 places right.
- Steps:

$$3.5 \quad \rightarrow \quad 35 \quad \rightarrow \quad 350 \quad \rightarrow \quad 3,500 \quad \rightarrow \quad 35,000$$

- Final Answer:

$$3.5 \times 10^4 = 35,000$$

Case 2: Negative Exponent (n < 0)

- Move the **decimal point to the left** by **n places**.
- Add **leading zeros** if needed.

Example 2:

Convert 4.2×10^{-3} to ordinary number.

- Exponent is -3 → Move decimal 3 places left.
- Steps:

$$4.2 \quad \rightarrow \quad 0.42 \quad \rightarrow \quad 0.042 \quad \rightarrow \quad 0.0042$$

- Final Answer:

$$4.2 \times 10^{-3} = 0.0042$$

SECTION 10: COMPLEX NUMBERS

Introduction to Complex Numbers

Why Do We Need Complex Numbers?

In real numbers, some equations have **no solution**.

Example:

Solve **x^2 + 1 = 0**

Step-by-Step:

x^2 = -1

But no real number squared equals -1, because the square of any real number is non-negative.

Solution: Introducing Imaginary Numbers

We define a special number:

$$i = \sqrt{-1}$$

This **imaginary unit (i)** allows us to take the square root of negative numbers.

Complex Number

A **Complex Number** is written as:

$$z = a + bi$$

where:

- a = **Real Part** (belongs to Real Numbers, $a \in \mathbb{R}$)
- b = **Imaginary Part** (also a Real Number, but multiplied with i)

Example:

$$z = 3 + 2i \quad (\text{Real part: 3, Imaginary part: 2i})$$

Integral Powers of i (Imaginary Unit Powers Pattern)

Power of i	Value	Explanation
i^1	i	By definition
i^2	-1	Because $i \times i = \sqrt{-1} \times \sqrt{-1} = -1$
i^3	-i	$i^3 = i^2 \times i = (-1) \times i = -i$
i^4	1	$i^4 = i^2 \times i^2 = (-1) \times (-1) = 1$

Pattern Repeats Every 4 Powers

After every 4 powers, the values repeat:

$$i^5 = i \quad (same as i^1)$$

$$i^6 = -1 \quad (same as i^2)$$

$$i^7 = -i \quad (same as i^3)$$

$$i^8 = 1 \quad (same as i^4)$$

Example:

Compute i^{11}

$$i^{11} = i^{4\times2+3} = (i^4)^2 \times i^3 = 1^2 \times (-i) = -i$$

Set of Complex Numbers

Notation of Complex Numbers Set:

The set of all complex numbers is denoted by:

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

Where:

- a = Real part of the complex number
- b = Imaginary coefficient (real number multiplied with i)
- i = Imaginary unit, $i = \sqrt{-1}$
- $\mathbb{R}$ = Set of real numbers
- $\mid$ = "such that"

Example:

For $z = 4 - 3i$,

$$\mathrm{Re}(z) = 4 \quad \text{and} \quad \mathrm{Im}(z) = -3$$

Equality of Complex Numbers

Two complex numbers are equal if and only if their corresponding real and imaginary parts are equal.

If:

$$a + bi = c + di$$

Then:

$$a = c \quad \text{and} \quad b = d$$

Example:

Given $z_1 = 3 + 4i$ and $z_2 = 3 + 4i$,

$$\mathrm{Re}(z_1) = \mathrm{Re}(z_2) = 3 \quad \text{and} \quad \mathrm{Im}(z_1) = \mathrm{Im}(z_2) = 4$$

Hence, $z_1 = z_2$.

Properties of Equality (Same as Real Numbers):

1. **Reflexive Property:** $z = z$
2. **Symmetric Property:** If $z_1 = z_2$, then $z_2 = z_1$
3. **Transitive Property:** If $z_1 = z_2$ and $z_2 = z_3$, then $z_1 = z_3$

Operations on Complex Numbers

1. Addition of Complex Numbers

Given:

$$z_1 = a + bi \quad \text{and} \quad z_2 = c + di$$

Addition Rule:

$$z_1 + z_2 = (a + c) + (b + d)i$$

Example:

$$(2 + 3i) + (4 + 5i) = (2 + 4) + (3 + 5)i = 6 + 8i$$

2. Subtraction of Complex Numbers

Given:

$$z_1 = a + bi \quad \text{and} \quad z_2 = c + di$$

Subtraction Rule:

$$z_1 - z_2 = (a - c) + (b - d)i$$

Example:

$$(7 + 6i) - (3 + 2i) = (7 - 3) + (6 - 2)i = 4 + 4i$$

3. Multiplication of Complex Numbers

Given:

$$z_1 = a + bi \quad \text{and} \quad z_2 = c + di$$

Multiplication Rule:

$$z_1 \times z_2 = (ac - bd) + (ad + bc)i$$

Example:

$$(1 + 2i)(3 + 4i) = (1 \times 3 - 2 \times 4) + (1 \times 4 + 2 \times 3)i$$

$$= (3 - 8) + (4 + 6)i = -5 + 10i$$

4. Division of Complex Numbers (Using Conjugates)

Given:

$$z_1 = a + bi \quad \text{and} \quad z_2 = c + di \quad (z_2 \neq 0)$$

Division Rule:

Multiply numerator and denominator by the conjugate of the denominator:

$$\frac{z_1}{z_2} = \frac{a + bi}{c + di} \times \frac{c - di}{c - di}$$

Simplification:

$$\frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}$$

Example:

$$\frac{5 + 3i}{2 + i} = \frac{5 + 3i}{2 + i} \times \frac{2 - i}{2 - i}$$

Numerator:

$$(5 \times 2 + 3 \times 1) + (3 \times 2 - 5 \times 1)i = (10 + 3) + (6 - 5)i = 13 + i$$

Denominator:

$$2^2 + 1^2 = 4 + 1 = 5$$

Final Answer:

$$\frac{13 + i}{5} = \frac{13}{5} + \frac{1}{5}i$$

Properties of Fundamental Operations on Complex Numbers

1. Closure Property

- **Addition:** The sum of any two complex numbers is always a complex number.
- **Subtraction:** The difference of any two complex numbers is also a complex number.
- **Multiplication:** The product of any two complex numbers is also a complex number.

Example:

Given $z_1 = 2 + 3i$, $z_2 = 4 + 5i$

- $z_1 + z_2 = (2 + 4) + (3 + 5)i = 6 + 8i$ → Complex
- $z_1 - z_2 = (2 - 4) + (3 - 5)i = -2 - 2i$ → Complex
- $z_1 \times z_2 = (2 \times 4 - 3 \times 5) + (2 \times 5 + 3 \times 4)i = (8 - 15) + (10 + 12)i = -7 + 22i$ → Complex

2. Commutative Property

- **Addition Commutativity:** $z_1 + z_2 = z_2 + z_1$
- **Multiplication Commutativity:** $z_1 \times z_2 = z_2 \times z_1$

Example:

Addition:

$$(1 + 2i) + (3 + 4i) = 4 + 6i = (3 + 4i) + (1 + 2i)$$

Multiplication:

$$(1 + i)(2 + 3i) = (2 + 3i + 2i + 3i^2) = (2 + 5i - 3) = -1 + 5i$$

$$(2 + 3i)(1 + i) = (2 + 2i + 3i + 3i^2) = (2 + 5i - 3) = -1 + 5i$$

3. Associative Property

- **Addition Associativity:** $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
- **Multiplication Associativity:** $(z_1 \times z_2) \times z_3 = z_1 \times (z_2 \times z_3)$

Example:

Addition:

$$[(1+i)+(2+3i)]+(4+i)=(3+4i)+(4+i)=7+5i$$

$$(1+i)+[(2+3i)+(4+i)]=(1+i)+(6+4i)=7+5i$$

4. Distributive Property

- Multiplication distributes over addition:

$$z_1 \times (z_2 + z_3) = z_1 \times z_2 + z_1 \times z_3$$

Example:

Given $z_1 = 1 + i$, $z_2 = 2 + i$, $z_3 = 3 + 2i$,

Compute:

$$(1+i) \times [(2+i)+(3+2i)] = (1+i) \times (5+3i)$$

Multiplying:

$$(1+i)(5+3i) = 5+3i+5i+3i^2 = 5+8i-3 = 2+8i$$

Separately compute:

$$(1+i)(2+i) = 2+i+2i+i^2 = 2+3i-1 = 1+3i$$

$$(1+i)(3+2i) = 3+2i+3i+2i^2 = 3+5i-2 = 1+5i$$

Sum:

$$(1+3i)+(1+5i)=2+8i$$

Both sides are equal → Distributive law holds.

5. Identity Elements

An **identity element** is a special number that, when used in an operation (addition or multiplication) with another number, does not change that number's value.

In complex numbers, there are **two types of identity elements**:

1. **Additive Identity**
2. **Multiplicative Identity**

1. Additive Identity

The **additive identity** is the number that, when added to any complex number, leaves it unchanged.

For complex numbers, the additive identity is:

$$0+0i$$

General Form:

$$z+0=z \quad \forall z \in \mathbb{C}$$

Where:

- $z=a+bi$
- $0=0+0i$

When you add **0 + 0i** to any complex number $a+bi$, the result remains $a+bi$.

∀	**For all**	This is called the **universal quantifier**. It means "for every" or "for all".
z	**Variable z**	Here, **z** represents any complex number.
∈	**Belongs to** or **is an element of**	This symbol denotes **membership** in a set.
ℂ (C)	**Set of Complex Numbers**	ℂ represents the **set of all complex numbers**.

'**For all z in the set of complex numbers.**'

Example:

$$(3+4i)+(0+0i)=(3+0)+(4+0)i=3+4i$$

Thus, $0+0i$ is the **additive identity**.

2. Multiplicative Identity

The **multiplicative identity** is the number that, when multiplied by any complex number, leaves it unchanged.

For complex numbers, the multiplicative identity is:

$$1 + 0i$$

General Form:

$$z \times 1 = z \quad \forall z \in \mathbb{C}$$

Where:

- $z = a + bi$
- $1 = 1 + 0i$

When you multiply **1 + 0i** with any complex number $a + bi$, the result remains $a + bi$.

Example:

$$(3 + 4i) \times (1 + 0i) = (3 \times 1 - 4 \times 0) + (3 \times 0 + 4 \times 1)i = 3 + 4i$$

Thus, $1 + 0i$ is the **multiplicative identity**.

6. Inverse Elements

- **Additive Inverse:** For every $z = a + bi$, the additive inverse is $-a - bi$.

$$z + (-z) = 0$$

  Example: $(3 + 4i) + (-3 - 4i) = 0$

- **Multiplicative Inverse:** For every non-zero $z = a + bi$, the multiplicative inverse is:

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{a - bi}{a^2 + b^2}$$

Conjugate Complex Numbers

For a complex number:

$$z = a + bi$$

The **conjugate** of z is denoted by:

$$\bar{z} = a - bi$$

- a: Real part
- b: Imaginary part
- **Conjugate** simply means changing the **sign of the imaginary part**.

Properties of Conjugates:

1. Sum of a Complex Number and its Conjugate:

$$z + \bar{z} = (a + bi) + (a - bi) = 2a$$

- The sum is always a **real number**.

Example:

If $z = 3 + 4i$, then:

$$z + \bar{z} = (3 + 4i) + (3 - 4i) = 6$$

2. Product of a Complex Number and its Conjugate:

$$z \cdot \bar{z} = (a + bi)(a - bi) = a^2 + b^2$$

- The product is always a **real, positive number**.

Example:
If $z = 3 + 4i$, then:

$$z \cdot \bar{z} = (3 + 4i)(3 - 4i) = 3^2 + 4^2 = 9 + 16 = 25$$

3. Double Conjugate:

$$\bar{\bar{z}} = z$$

- Taking the conjugate twice returns the original number.

Finite Numbers and Infinite Numbers

Finite Numbers

A **Finite Number** is any number that has a **definite, countable value**.
It can be **measured, counted, and expressed exactly**.

Key Characteristics:

- Can be **positive**, **negative**, or **zero**.
- Has a **fixed numerical value**.
- Can be written in **decimal, fraction, percentage**, or any standard number form.

Mathematical Representation:

.

$$x \in \mathbb{R} \quad \text{where} \quad |x| < \infty$$

Here:

- x = finite number.
- $\mathbb{R}$ = Set of Real Numbers.
- $|x|$ = Absolute value of x.
- ∞ = Infinity symbol (meaning unbounded).

Examples:

- 7 (finite)
- -13 (finite)
- 0 (finite)
- 3.1415 (finite approximation of π)
- $\frac{2}{5}$ (finite fraction)

Infinite Numbers

An **Infinite Number** refers to a quantity that is **unbounded** or **limitless**.
It is not a fixed, countable number but represents a **concept of endlessness**.

Symbol:

$$\infty$$

(Infinity symbol)

Key Characteristics:

- Cannot be measured exactly.
- It goes on **forever without end**.

- Infinity is **not a specific number**, but a **concept**.

Types of Infinity:

1. **Positive Infinity (+∞):**
   - Numbers increasing without bound.
   - Example: $x \to +\infty$ means x increases endlessly.
2. **Negative Infinity (-∞):**
   - Numbers decreasing without bound.
   - Example: $x \to -\infty$ means x decreases endlessly.

Mathematical Usage:

1. **Limits:**

limx→∞1x=0\lim_{x \to \infty} \frac{1}{x} = 0x→∞limx1=0

As x approaches infinity, 1x\frac{1}{x}x1 gets closer to 0.

2. **Unbounded Sets:**
   - The set of **Natural Numbers (N)** is infinite.

N={1,2,3,4,5,… }\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}N={1,2,3,4,5,…}

3. **Infinity in Calculus:**
   - Infinite series like:

1+12+13+14+…(Harmonic Series)1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \quad \text{(Harmonic Series)}1+21+31+41+…(Harmonic Series)

goes on infinitely.

Examples of Infinite Quantities:

- Number of natural numbers.
- Points on a number line.

- Limit of a function approaching infinity.

Important Note:

- **Infinity (∞)** is a **concept**, **not a finite number**.
- You **cannot perform arithmetic** with infinity like with normal numbers (e.g., ∞ - ∞ is undefined).