Math For Artificial Intelligence

"Complete Matrix Theory & Linear Systems: From Basics to AI Applications"

SECTION 1: MATRIX FUNDAMENTALS

1.1 Introduction to Matrices

What is a Matrix?

A matrix is a rectangular arrangement of numbers (or objects) in rows and columns. Think of it like a grid or a table used to organize numbers neatly.

- **Rows go horizontally (left to right).**
- **Columns go vertically (top to bottom).**

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Here, matrix A has **2 rows** and **3 columns**.

Notation

- A matrix is usually represented by a **capital letter** (A, B, C, ...).
- The element in the i-th row and j-th column is written as a_{ij}.

Example:

If

$$A = \begin{bmatrix} 1 & 7 \\ 3 & 5 \end{bmatrix}$$

- $a_{11} = 1$ (row 1, column 1)
- $a_{12} = 7$ (row 1, column 2)
- $a_{21} = 3$ (row 2, column 1)

i = row index

j = column index

1.2 Order of a Matrix

The order of a matrix tells us its size:

$$m \times n$$

where m = number of rows, n = number of columns.

(Read as "m by n matrix").

1.

$$\begin{bmatrix} 2 & 5 & 7 \\ 1 & 0 & 9 \end{bmatrix}$$

Order: 2×3

2.

$$\begin{bmatrix} 1 & 4 \\ 2 & 6 \\ 3 & 8 \end{bmatrix}$$

Order: 3×2

3.

$$\begin{bmatrix} 7 \\ 9 \\ 11 \\ 15 \end{bmatrix}$$

Order: 4×1 (a **column matrix**)

1.3 Matrix Components

1. Row

A **row** is a horizontal line of elements in a matrix.

Example:

In

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

- First row = [1, 2, 3]
- Second row = [4, 5, 6]

2. Column

A column is a vertical line of elements in a matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

- First column = [1, 4]
- Second column = [2, 5]
- Third column = [3, 6]

3. Main (Principal) Diagonal

The **main diagonal** (also called **principal diagonal**) contains elements from **top-left to bottom-right** of a square matrix.

Example:

$$\begin{bmatrix} 2 & 5 & 7 \\ 4 & 9 & 1 \\ 6 & 3 & 8 \end{bmatrix}$$

Main (Principal) Diagonal = {2, 9, 8}

4. Secondary Diagonal

The **secondary diagonal** runs from **top-right to bottom-left**.

$$\begin{bmatrix} 2 & 5 & 7 \\ 4 & 9 & 1 \\ 6 & 3 & 8 \end{bmatrix}$$

Secondary Diagonal = {7, 9, 6}

SECTION 2: BASIC TYPES OF MATRICES

2.1 Classification by Shape

1. Row Matrix

Definition

A **Row Matrix** has only **one row** and multiple columns.

Its order is $1 \times n$.

Example

$$R = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

Order: 1×4

2. Column Matrix

Definition

A **Column Matrix** has only **one column** and multiple rows.

Its order is $m \times 1$.

Example

$$C = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Order: 3×1

3. Square Matrix

Definition

A **Square Matrix** has the **same number of rows and columns.**

Order: $m \times m$.

Example

$$S = \begin{bmatrix} 1 & 0 & 2 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Order: 3×3

4. Rectangular Matrix

Definition

A **Rectangular Matrix** has a different number of rows and columns.

Order: $m \times n$, where $m \neq n$.

Example

$$R = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \end{bmatrix}$$

Order: 2×3

SECTION 2.2: SPECIAL SQUARE MATRICES

1. Null Matrix (Zero Matrix)

Definition

A **Null Matrix** (or Zero Matrix) is a matrix in which **all elements are zero**.

Notation

$\mathbf{O}$ or 0

Example

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

2. Identity Matrix

Definition

An **Identity Matrix** is a **square matrix** with:

- **1's on the principal (main) diagonal,**
- **0's elsewhere.**

Notation

$\mathbf{I}$ or I_n (for an $n \times n$ identity matrix)

Example

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Scalar Matrix

Definition

A **Scalar Matrix** is a diagonal matrix where **all diagonal elements are equal**. It can be written as kI, where k is a scalar.

Example

$$S = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = 3I_2$$

4. Diagonal Matrix

Definition

A **Diagonal Matrix** is a square matrix in which **all non-diagonal elements are zero**, while diagonal elements may be **non-zero**.

Example

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

SECTION 2.3: ADVANCED MATRIX TYPES

1. Upper Triangular Matrix

Definition

A **Upper Triangular Matrix** is a **square matrix** in which **all elements below the principal diagonal are zero**.

Example

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

2. Lower Triangular Matrix

Definition

A **Lower Triangular Matrix** is a **square matrix** in which **all elements above the principal diagonal are zero**.

Example

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$$

3. Triangular Matrix

Definition

A **Triangular Matrix** is a general term for a matrix that is either:

- **Upper Triangular**, or
- **Lower Triangular**.

So every triangular matrix is a **special square matrix** with zeros either above or below the diagonal.

SECTION 3: MATRIX RELATIONSHIPS AND BASIC OPERATIONS

3.1 Matrix Equality

Two matrices are said to be **Equal** if:

1. They have the **same order** (same number of rows and columns).
2. All their **corresponding elements** are equal.

Condition

$$A = B \quad \text{if and only if} \quad a_{ij} = b_{ij} \quad \forall\, i, j$$

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Here, $A = B$ because:

- Both are 2×3 matrices
- Each element a_{ij} equals the corresponding b_{ij}.

But, if even a single element differs or the sizes differ, then $A \neq B$.

3.2 Basic Matrix Operations

1. Negative of a Matrix

Definition

The **negative of a matrix** A is obtained by changing the sign of every element in A.

Formula

$$-A = [-a_{ij}]$$

Example

$$A = \begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}, \quad -A = \begin{bmatrix} -2 & 3 \\ -4 & -5 \end{bmatrix}$$

2. Addition of Matrices

Definition

Two matrices of the **same order** can be added by adding their corresponding elements.

Formula

$$(A + B)_{ij} = a_{ij} + b_{ij}$$

Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

3. Subtraction of Matrices

Definition

Subtraction is defined as **adding the negative** of the second matrix.

Formula

$$A - B = A + (-B), \quad (A - B)_{ij} = a_{ij} - b_{ij}$$

Example

$$A = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 4-1 & 7-3 \\ 2-5 & 6-2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -3 & 4 \end{bmatrix}$$

4. Scalar Multiplication

Definition

If k is a scalar (a real number), multiplying a matrix A by k means multiplying **every element of** A by k.

Formula

$$(kA)_{ij} = k \times a_{ij}$$

Example

$$A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}, \quad k = 4$$

$$4A = \begin{bmatrix} 4 \times 2 & 4 \times (-1) \\ 4 \times 0 & 4 \times 3 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ 0 & 12 \end{bmatrix}$$

3.3 Properties of Matrix Addition

Matrix addition follows several fundamental properties that make it a well-defined algebraic operation.

1. Closure Property

Definition

If A and B are two matrices of the same order, then their sum $A + B$ is also a matrix of the same order.

Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

👉 Both A and B are 2×2, and their sum is also 2×2.

2. Commutative Property

Definition

The order of addition does not matter:

$$A + B = B + A$$

Example

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 6 & 10 \\ 8 & 12 \end{bmatrix}, \quad B + A = \begin{bmatrix} 6 & 10 \\ 8 & 12 \end{bmatrix}$$

✅ Clearly, $A + B = B + A$.

3. Associative Property

Definition

For any three matrices of the same order:

$$(A+B)+C=A+(B+C)$$

Example

$$A=\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix},\quad B=\begin{bmatrix}2 & 3\\ 1 & 4\end{bmatrix},\quad C=\begin{bmatrix}0 & 5\\ 6 & 7\end{bmatrix}$$

$$(A+B)+C=\begin{bmatrix}3 & 3\\ 3 & 5\end{bmatrix}+\begin{bmatrix}0 & 5\\ 6 & 7\end{bmatrix}=\begin{bmatrix}3 & 8\\ 9 & 12\end{bmatrix}$$

$$A+(B+C)=\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}+\begin{bmatrix}2 & 8\\ 7 & 11\end{bmatrix}=\begin{bmatrix}3 & 8\\ 9 & 12\end{bmatrix}$$

✅ Both sides are equal.

4. Existence of Additive Identity

Definition

There exists a **zero matrix** O such that:

$$A+O=A$$

Example

$$A=\begin{bmatrix}4 & -2\\ 1 & 3\end{bmatrix},\quad O=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$A+O=\begin{bmatrix}4 & -2\\ 1 & 3\end{bmatrix}=A$$

5. Existence of Additive Inverse

Definition

For every matrix A, there exists a matrix $-A$ such that:

$$A + (-A) = O$$

Example

$$A = \begin{bmatrix} 2 & -5 \\ 7 & 1 \end{bmatrix}, \quad -A = \begin{bmatrix} -2 & 5 \\ -7 & -1 \end{bmatrix}$$

$$A + (-A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

✅ Hence, additive inverse exists for every matrix.

3.4 Properties of Scalar Multiplication

Scalar multiplication in matrices means multiplying every element of a matrix by a constant (called a scalar). It follows these important properties:

1. Associativity of Scalars

Definition

For scalars k and l and matrix A:

$$k(lA) = (kl)A$$

Example

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}, \quad k = 2, \quad l = 3$$

$$lA = 3A = \begin{bmatrix} 6 & 9 \\ 12 & 15 \end{bmatrix}, \quad k(lA) = 2 \times \begin{bmatrix} 6 & 9 \\ 12 & 15 \end{bmatrix} = \begin{bmatrix} 12 & 18 \\ 24 & 30 \end{bmatrix}$$

$$(kl)A = (2 \times 3)A = 6A = \begin{bmatrix} 12 & 18 \\ 24 & 30 \end{bmatrix}$$

✅ Hence, $k(lA) = (kl)A$.

2. Distributivity over Matrix Addition

Definition

$$k(A+B) = kA + kB$$

Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}, \quad k = 2$$

$$A + B = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}, \quad k(A+B) = 2 \times \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix} = \begin{bmatrix} 12 & 16 \\ 20 & 24 \end{bmatrix}$$

$$kA + kB = 2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 2\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} + \begin{bmatrix} 10 & 12 \\ 14 & 16 \end{bmatrix} = \begin{bmatrix} 12 & 16 \\ 20 & 24 \end{bmatrix}$$

✅ Hence, distributivity holds.

3. Distributivity over Scalar Addition

Definition

$$(k+l)A = kA + lA$$

Example

$$A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}, \quad k = 2, \quad l = 5$$

$$(k+l)A = 7A = \begin{bmatrix} 14 & -7 \\ 0 & 21 \end{bmatrix}$$

$$kA + lA = 2A + 5A = \begin{bmatrix} 4 & -2 \\ 0 & 6 \end{bmatrix} + \begin{bmatrix} 10 & -5 \\ 0 & 15 \end{bmatrix} = \begin{bmatrix} 14 & -7 \\ 0 & 21 \end{bmatrix}$$

✅ Hence, distributivity over scalar addition holds.

4. Identity Property

Definition

Multiplying a matrix by scalar **1** gives the same matrix:

$$1 \times A = A$$

Example

$$A = \begin{bmatrix} 4 & 7 \\ -2 & 5 \end{bmatrix}$$

$$1 \times A = \begin{bmatrix} 4 & 7 \\ -2 & 5 \end{bmatrix} = A$$

✅ Hence, identity property holds.

SECTION 4: MATRIX MULTIPLICATION AND ADVANCED OPERATIONS

4.1 Matrix Multiplication

Matrix multiplication is one of the most fundamental and widely used operations in linear algebra. Unlike scalar multiplication or matrix addition, it follows specific rules and conditions.

Condition for Multiplication

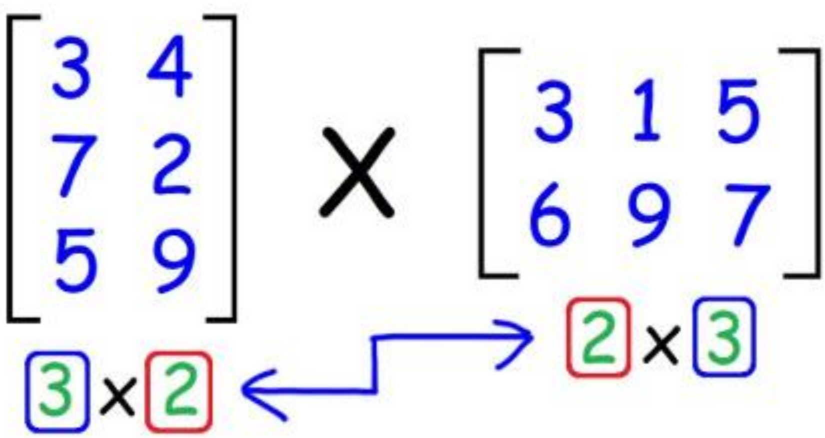
If

$$A \text{ is of order } (m \times n), \quad B \text{ is of order } (n \times p)$$

then their product

$$C = AB \text{ will be of order } (m \times p).$$

That means the number of **columns of the first matrix** must equal the number of **rows of the second matrix**.



Rule of Multiplication

Each element of the product matrix C is computed as:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

This means:

- Take the i^{th} row of matrix A
- Multiply it element-wise with the j^{th} column of matrix B
- Add the results

- c_{ij}: The entry in the **i-th row and j-th column** of the resulting matrix C.
- a_{ik}: The entry in the **i-th row and k-th column** of matrix A.
- b_{kj}: The entry in the **k-th row and j-th column** of matrix B.
- k: A counter that goes from **1 to n** (where n is the number of columns in A, or equivalently, the number of rows in B).
- $\sum$: The summation symbol, meaning "add everything together."

Example

Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Here, A is 2×2, B is 2×2. Multiplication is possible, and the product will also be 2×2

.

$$AB = \begin{bmatrix} (1 \times 5 + 2 \times 7) & (1 \times 6 + 2 \times 8) \\ (3 \times 5 + 4 \times 7) & (3 \times 6 + 4 \times 8) \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

Non-Commutativity of Matrix Multiplication

In general,

$$AB \neq BA$$

Example

Using the same matrices A and B:

$$BA = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} (5 \times 1 + 6 \times 3) & (5 \times 2 + 6 \times 4) \\ (7 \times 1 + 8 \times 3) & (7 \times 2 + 8 \times 4) \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$$

Clearly,

$$AB = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix} \quad \neq \quad BA = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$$

Thus, matrix multiplication is **not commutative**.

4.2 Properties of Matrix Multiplication

Matrix multiplication follows certain algebraic properties that are essential in linear algebra, computer science, and AI applications. However, unlike addition, multiplication does not obey all standard arithmetic rules (e.g., commutativity).

1. Associativity

For three conformable matrices A, B, C:

$$(AB)C = A(BC)$$

This means when multiplying three matrices, the grouping does not matter.

Example

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Compute $(AB)C$ and $A(BC)$; both will yield the same result.

2. Distributivity (Right)

$$A(B + C) = AB + AC$$

A single matrix multiplied by the sum of two matrices on the right distributes over addition.

Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

1. Compute $B + C = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$
2. Then $A(B + C) = \begin{bmatrix} 5 & 7 \\ 13 & 19 \end{bmatrix}$
3. Separately compute $AB = \begin{bmatrix} 4 & 2 \\ 10 & 4 \end{bmatrix}$, and $AC = \begin{bmatrix} 1 & 5 \\ 3 & 11 \end{bmatrix}$.
4. Add: $AB + AC = \begin{bmatrix} 5 & 7 \\ 13 & 19 \end{bmatrix}$.

Hence, $A(B + C) = AB + AC$.

3. Distributivity (Left)

$$(A + B)C = AC + BC$$

A single matrix multiplied by the sum of two matrices on the left also distributes over addition.

Example

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

1. Compute $A + B = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$.
2. Then $(A + B)C = \begin{bmatrix} 4 & 6 \\ 6 & 10 \end{bmatrix}$.
3. Separately compute $AC = \begin{bmatrix} 1 & 2 \\ 5 & 8 \end{bmatrix}$, and $BC = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$.
4. Add: $AC + BC = \begin{bmatrix} 4 & 6 \\ 6 & 10 \end{bmatrix}$.

Hence, $(A + B)C = AC + BC$.

4. Identity Property

For an identity matrix I of suitable order:

$$AI = IA = A$$

The identity matrix acts as the multiplicative identity in matrix multiplication.

Example

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AI = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}, \quad IA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

Thus, $AI = IA = A$.

5. Non-Commutativity

In general,

$$AB \neq BA$$

Even if both AB and BA exist, they may not be equal. This property distinguishes matrix multiplication from real number multiplication.

Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}, \quad BA = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

Since $AB \neq BA$, matrix multiplication is not commutative.

4.3 Transpose of Matrix

The transpose of a matrix A, denoted by A^T or A', is obtained by interchanging rows and columns.

$$(A^T)_{ij} = a_{ji}$$

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

1. Property: $(A + B)^T = A^T + B^T$

The transpose of a sum is equal to the sum of transposes.

Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

$$(A + B)^T = \begin{bmatrix} 6 & 10 \\ 8 & 12 \end{bmatrix}$$

Separately,

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \quad B^T = \begin{bmatrix} 5 & 7 \\ 6 & 8 \end{bmatrix}$$

$$A^T + B^T = \begin{bmatrix} 6 & 10 \\ 8 & 12 \end{bmatrix}$$

Hence proved:

$$(A + B)^T = A^T + B^T$$

2. Property: $(AB)^T = B^T A^T$

The transpose of a product is the product of transposes in reverse order.

Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$$

Now separately,

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \quad B^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$$

Hence proved:

$$(AB)^T = B^T A^T$$

3. Property: $(A^T)^T = A$

The transpose of the transpose of a matrix is the matrix itself.

Example

$$A = \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 7 & 9 \\ 8 & 10 \end{bmatrix}$$

$$(A^T)^T = \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix} = A$$

SECTION 5: SPECIAL MATRIX TYPES AND PROPERTIES

5.1 Special Matrix Categories

1. Symmetric Matrix

Definition:

A matrix A is **symmetric** if

$$A = A^T$$

That is, the matrix equals its transpose. Equivalently,

$$a_{ij} = a_{ji}$$

for all i, j.

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Transpose:

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Since $A = A^T$, the matrix is **symmetric**.

Skew-Symmetric Matrix

Definition

A square matrix A is **skew-symmetric** if

$$A^T = -A.$$

Component form

$$a_{ij} = -a_{ji} \quad \text{for all } i, j,$$

That is, the diagonal elements are always zero.

Example:

$$A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 4 \\ 3 & -4 & 0 \end{bmatrix}$$

Transpose:

$$A^T = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix}$$

Clearly, $A^T = -A$, so the matrix is **skew-symmetric**.

3. Orthogonal Matrix

Definition:

A square matrix A is **orthogonal** if

$$AA^T = A^TA = I$$

Example:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Transpose:

$$A^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Now compute AA^T:

$$AA^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Thus, A is an **orthogonal matrix**.

5.2 Complex Matrix Types

Matrices can also have **complex numbers** as their elements. Complex numbers are of the form

$$z = x + iy$$

where x is the **real part**, y is the **imaginary part**, and $i = \sqrt{-1}$.

The **complex conjugate** of a number $z = x + iy$ is

$$\overline{z} = x - iy$$

(we just change the sign of the imaginary part).

Using this, we can define different types of complex matrices.

1. Conjugate Matrix

Definition:

If $A = [a_{ij}]$ is a complex matrix, then its **conjugate matrix** is

$$\overline{A} = [\overline{a_{ij}}],$$

where every element is replaced with its complex conjugate.

Step-by-step:

- Take each entry of A.
- If it has an imaginary part, change the sign of the imaginary part.
- Keep real numbers unchanged.

Example:

$$A = \begin{bmatrix} 1+i & 2-3i \\ -4i & 5 \end{bmatrix}$$

- Conjugate of $1+i$ is $1-i$.
- Conjugate of $2-3i$ is $2+3i$.
- Conjugate of $-4i$ is $+4i$.
- Conjugate of 5 is still 5.

So,

$$\overline{A} = \begin{bmatrix} 1-i & 2+3i \\ 4i & 5 \end{bmatrix}.$$

2. Hermitian Matrix

Definition:

A square matrix A is **Hermitian** if it equals its **conjugate transpose**:

$$A^* = A$$

where $A^* = \overline{A}^T$.

Step-by-step:

1. First take the **transpose** of A (swap rows and columns).
2. Then take the **complex conjugate** of each entry.
3. If the resulting matrix equals the original matrix, then A is Hermitian.

Why it matters:

This is the complex analog of a **symmetric matrix**.

Example:

$$A = \begin{bmatrix} 2 & 3+i \\ 3-i & 5 \end{bmatrix}$$

- Transpose:

$$A^T = \begin{bmatrix} 2 & 3-i \\ 3+i & 5 \end{bmatrix}$$

- Conjugate of transpose:

$$A^* = \begin{bmatrix} 2 & 3+i \\ 3-i & 5 \end{bmatrix}$$

Since $A^* = A$, this is a **Hermitian matrix**.

3. Skew-Hermitian Matrix

Definition:

A square matrix A is **Skew-Hermitian** if

$$A^* = -A$$

where $A^* = \overline{A}^T$.

Step-by-step:

1. Take transpose of A.
2. Take conjugate of each entry.
3. If this result is equal to the **negative** of the original matrix, then A is skew-Hermitian.

Important note:

- The diagonal entries of skew-Hermitian matrices are always **purely imaginary** (like $i, -i, 2i, 0$).

Example:

$$A = \begin{bmatrix} i & 2+i \\ -2+i & -i \end{bmatrix}$$

- Transpose:

$$A^T = \begin{bmatrix} i & -2+i \\ 2+i & -i \end{bmatrix}$$

- Conjugate:

$$A^* = \begin{bmatrix} -i & -2-i \\ 2-i & i \end{bmatrix}$$

Now check:

$$-A = \begin{bmatrix} -i & -2-i \\ 2-i & i \end{bmatrix} = A^*.$$

Thus, A is **skew-Hermitian**.

5.3 Power-Related Matrix Types

These are special types of matrices that are defined based on how they behave under **repeated multiplication (powers of a matrix)**.
That means we look at A^2, A^3, A^k, etc., and check for specific properties.

1. Idempotent Matrix

Definition:

A matrix A is called **idempotent** if

$$A^2 = A$$

That means, when you multiply the matrix by itself, you get the same matrix back.

Step-by-step:

- Take the matrix A.
- Multiply it with itself.
- If the result equals the original A, then A is **idempotent**.

Example:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A$$

So, this is an **idempotent matrix**.

2. Involutary Matrix

Definition:

A matrix A is called **involutary** if

$$A^2 = I$$

where I is the identity matrix.

This means A is its own inverse, because multiplying it by itself gives the identity.

Step-by-step:

- Multiply A with itself.
- If the result is I, then A is involutary.

Example:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

So, A is **involutary**.

3. Nilpotent Matrix

Definition:

A matrix A is called **nilpotent** if there exists a positive integer k such that

$$A^k = O$$

where O is the **zero matrix**.

Step-by-step:

- Multiply A by itself repeatedly.
- If after some number of multiplications, the result becomes the zero matrix, then A is nilpotent.

Example:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

So, A is **nilpotent** with $k = 2$.

4. Periodic Matrix

Definition:

A matrix A is called **periodic** if there exists a positive integer $k > 1$ such that

$$A^k = A$$

This means the matrix returns to itself after multiplying it by itself multiple times.

Step-by-step:

- Compute powers of A.
- If for some $k > 1$, the result comes back to A, then A is periodic.

Example:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- First, calculate $A^2 = I$.
- Then $A^3 = AI = A$.

Here, $A^3 = A$, so this matrix is **periodic with period 3**.

SECTION 6: DETERMINANTS - BASIC CONCEPTS

6.1 Determinant of 2×2 Matrix

The determinant of a matrix is a single number that we calculate from a square matrix. It plays a very important role in linear algebra: it tells us about area, orientation, invertibility, and more.

Formula

For a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The determinant is calculated as:

$$\det(A) = |A| = ad - bc$$

That means:

- Multiply the diagonal from top-left to bottom-right ($a \times d$)
- Multiply the diagonal from top-right to bottom-left ($b \times c$)
- Subtract them: $(ad - bc)$

Step-by-Step Example

Let

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

$$\det(A) = (3)(4) - (2)(1) = 12 - 2 = 10$$

So the determinant is **10**.

6.2 Matrix Classifications by Determinant

The determinant of a matrix tells us a lot about the matrix's properties — especially whether it is invertible or not. Based on the determinant, we classify matrices into Singular and Non-Singular.

1. Singular Matrix

A **Singular Matrix** is a square matrix with

$$\det(A) = 0$$

The matrix has no inverse.

Example

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$$

$$\det(A) = (2)(2) - (4)(1) = 4 - 4 = 0$$

So A is **singular**.

2. Non-Singular Matrix

A **Non-Singular Matrix** is a square matrix with

$$\det(A) \neq 0$$

Example

$$B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

$$\det(B) = (3)(4) - (2)(1) = 12 - 2 = 10 \neq 0$$

So B is **non-singular**.

SECTION 7: DETERMINANTS - ADVANCED CONCEPTS

7.1 Determinants for Larger Matrices

When matrices become larger than 2×2, computing determinants directly becomes more complex. To handle this, we use the ideas of **minors** and **cofactors**, which break down a determinant into smaller determinants.

1. Minor of an Element

The **minor** of an element a_{ij} in a square matrix A is the determinant of the submatrix obtained by **deleting the i-th row and j-th column** from A.

- **Notation:**
  The minor of a_{ij} is written as M_{ij}.

Example

Consider the 3×3 matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Find the **minor of element** a_{12} (the element in row 1, column 2, which is 2).

1. Remove row 1 and column 2:

$$\text{Remaining submatrix} = \begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}$$

2. Compute its determinant:

$$M_{12} = (4)(9) - (6)(7) = 36 - 42 = -6$$

So, the **minor of** a_{12} **is** -6.

Example 2: **Minor of $a_{11} = 1$**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

1. Delete **row 1** and **column 1**.

   Remaining submatrix:

$$\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}$$

2. Determinant:

$$M_{11} = (5)(9) - (6)(8) = 45 - 48 = -3$$

✅ Minor of a_{11} is -3.

Example3.

Minor of $a_{23} = 6$

1. Delete **row 2** and **column 3**.

   Remaining submatrix:

$$\begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$$

2. Determinant:

$$M_{23} = (1)(8) - (2)(7) = 8 - 14 = -6$$

✅ Minor of a_{23} is -6.

Example 3: Minor of $a_{32} = 8$

1. Delete **row 3** and **column 2**.

   Remaining submatrix:

$$\begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$$

2. Determinant:

$$M_{32} = (1)(6) - (3)(4) = 6 - 12 = -6$$

✅ Minor of a_{32} is -6.

2.Cofactor of an Element

The **cofactor** of an element a_{ij} in a square matrix A is the **signed minor**. It is denoted by C_{ij} and defined as

$$C_{ij} = (-1)^{i+j} M_{ij},$$

where M_{ij} is the **minor** of a_{ij} (the determinant of the submatrix formed by deleting row i and column j).
The factor $(-1)^{i+j}$ assigns a plus or minus sign depending on the position (i, j) (the sign pattern alternates like a checkerboard).

3. To compute a cofactor C_{ij} you first remove row i and column j from A.
4. Compute the determinant of the remaining submatrix — that is the **minor** M_{ij}.
5. Multiply that minor by $(-1)^{i+j}$ to get the **cofactor** (this sign ensures the correct orientation when expanding determinants and when building the adjugate matrix used for inverses).

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

We previously computed a few minors. Here are several minors and their cofactors, computed step by step.

Minor and cofactor for $a_{11} = 1$

- Delete row 1, column 1 → submatrix $\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}$.
- Minor: $M_{11} = 5 \cdot 9 - 6 \cdot 8 = 45 - 48 = -3.$
- Sign: $(-1)^{1+1} = +1.$
- Cofactor: $C_{11} = +1 \cdot (-3) = -3.$

Minor and cofactor for $a_{12} = 2$

- Delete row 1, column 2 → submatrix $\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}$.
- Minor: $M_{12} = 4 \cdot 9 - 6 \cdot 7 = 36 - 42 = -6.$
- Sign: $(-1)^{1+2} = -1.$
- Cofactor: $C_{12} = -1 \cdot (-6) = +6.$

Minor and cofactor for $a_{13} = 3$

- Delete row 1, column 3 → submatrix $\begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}$.
- Minor: $M_{13} = 4 \cdot 8 - 5 \cdot 7 = 32 - 35 = -3$.
- Sign: $(-1)^{1+3} = +1$.
- Cofactor: $C_{13} = +1 \cdot (-3) = -3$.

Minor and cofactor for $a_{21} = 4$

- Delete row 2, column 1 → submatrix $\begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix}$.
- Minor: $M_{21} = 2 \cdot 9 - 3 \cdot 8 = 18 - 24 = -6$.
- Sign: $(-1)^{2+1} = -1$.
- Cofactor: $C_{21} = -1 \cdot (-6) = +6$.

Minor and cofactor for $a_{22} = 5$

- Delete row 2, column 2 → submatrix $\begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix}$.
- Minor: $M_{22} = 1 \cdot 9 - 3 \cdot 7 = 9 - 21 = -12$.
- Sign: $(-1)^{2+2} = +1$.
- Cofactor: $C_{22} = +1 \cdot (-12) = -12$.

Minor and cofactor for $a_{23} = 6$

- Delete row 2, column 3 → submatrix $\begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$.
- Minor: $M_{23} = 1 \cdot 8 - 2 \cdot 7 = 8 - 14 = -6$.
- Sign: $(-1)^{2+3} = -1$.
- Cofactor: $C_{23} = -1 \cdot (-6) = +6$.

Minor and cofactor for $a_{31} = 7$

- Delete row 3, column 1 → submatrix $\begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$.
- Minor: $M_{31} = 2 \cdot 6 - 3 \cdot 5 = 12 - 15 = -3$.
- Sign: $(-1)^{3+1} = +1$.
- Cofactor: $C_{31} = +1 \cdot (-3) = -3$.

Minor and cofactor for $a_{32} = 8$

- Delete row 3, column 2 → submatrix $\begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$.
- Minor: $M_{32} = 1 \cdot 6 - 3 \cdot 4 = 6 - 12 = -6$.
- Sign: $(-1)^{3+2} = -1$.
- Cofactor: $C_{32} = -1 \cdot (-6) = +6$.

Minor and cofactor for $a_{33} = 9$

- Delete row 3, column 3 → submatrix $\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$.
- Minor: $M_{33} = 1 \cdot 5 - 2 \cdot 4 = 5 - 8 = -3$.
- Sign: $(-1)^{3+3} = +1$.
- Cofactor: $C_{33} = +1 \cdot (-3) = -3$.

Cofactor matrix (matrix of all C_{ij})

Collect the cofactors in the same positions to get the **cofactor matrix**:

$$[C_{ij}] = \begin{bmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{bmatrix}.$$

The **adjugate** (adjoint) of A is the transpose of the cofactor matrix:

$$\mathrm{adj}(A) = [C_{ij}]^T = \begin{bmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{bmatrix}$$

(in this specific case the cofactor matrix is symmetric so its transpose is the same).

7.3 Determinant of an n×n Matrix (Cofactor Expansion Method)

For a square matrix $A = [a_{ij}]$ of order n:

- The **determinant** can be computed by expanding along any row or any column.
- It uses the **cofactor** C_{ij}, which is defined as:

$$C_{ij} = (-1)^{i+j} M_{ij}$$

where M_{ij} is the **minor** of element a_{ij}.

Formula

1. Expansion along the i^{th} row:

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij}$$

2. Expansion along the j^{th} column:

$$\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij}$$

This means we can pick **any row** or **any column** to expand and calculate the determinant.

Example

Take the 3×3 matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Expansion along the first row (i = 1):

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

- $a_{11} = 1$,
  $M_{11} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = (5)(9) - (6)(8) = 45 - 48 = -3$
  $C_{11} = (-1)^{1+1} M_{11} = (+1)(-3) = -3$
- $a_{12} = 2$,
  $M_{12} = \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = (4)(9) - (6)(7) = 36 - 42 = -6$
  $C_{12} = (-1)^{1+2} M_{12} = (-1)(-6) = 6$
- $a_{13} = 3$,
  $M_{13} = \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = (4)(8) - (5)(7) = 32 - 35 = -3$
  $C_{13} = (-1)^{1+3} M_{13} = (+1)(-3) = -3$

Now substitute:

$$\det(A) = (1)(-3) + (2)(6) + (3)(-3) = -3 + 12 - 9 = 0$$

So,

$$\det(A) = 0$$

This shows how **cofactor expansion** works.

You can expand along any row or column, but the result will always be the same.

7.2 Properties of Determinants

Determinants obey several important rules that help simplify calculations.

1. Row/Column Interchange

If you swap two rows (or two columns) of a matrix, the determinant changes its sign.

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \det(A) = (1)(4) - (2)(3) = 4 - 6 = -2$$

Swap rows:

$$A' = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}, \quad \det(A') = (3)(2) - (4)(1) = 6 - 4 = 2$$

Notice: $\det(A') = -\det(A)$.

2. Scalar Multiplication

If every entry of a row (or column) is multiplied by a scalar k, then the determinant is also multiplied by k.

For an $n \times n$ matrix:

$$\det(kA) = k^n \det(A)$$

Example:
For a 2×2 matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \det(A) = -2$$

Multiply matrix by $k = 2$:

$$2A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}, \quad \det(2A) = 2^2 \cdot \det(A) = 4(-2) = -8$$

Since the determinant scales by k for each row, and there are n rows in an $n \times n$ matrix, the overall scaling factor is k^n.

So:

$$\det(kA) = k^n \det(A)$$

3. Identical Rows/Columns

If two rows (or two columns) of a matrix are identical, the determinant is zero.

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}, \quad \det(A) = (1)(2) - (2)(1) = 0$$

4. Row/Column Addition

If we add a multiple of one row to another row (or a multiple of one column to another), the determinant does not change.

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \det(A) = -2$$

Replace $R_2 \to R_2 - 3R_1$:

$$A' = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}, \quad \det(A') = (1)(-2) - (2)(0) = -2$$

So, determinant unchanged.

5. Triangular Matrix

For an upper or lower triangular matrix, the determinant is the product of its diagonal elements.

Example:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\det(A) = (2)(4)(6) = 48$$

6. Product Rule

The determinant of a product of two square matrices is equal to the product of their determinants:

$$\det(AB) = \det(A) \cdot \det(B)$$

This is a very important property in linear algebra.

7. Transpose Rule

The determinant of a matrix equals the determinant of its transpose:

$$\det(A^T) = \det(A)$$

This follows from the definition, because transposing does not change the minors or cofactors used in determinant calculation.

SECTION 8: MATRIX INVERSE AND ADJOINT

8.1 Adjoint of Matrix

The **adjoint (or adjugate)** of a matrix is an important concept used to calculate the **inverse** of a square matrix. It is defined as the **transpose of the cofactor matrix**.

1. For a 2×2 Matrix

If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then the **adjoint of A** is given by:

$$\mathrm{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Explanation:

- We swap the diagonal elements $a \leftrightarrow d$.
- We change the signs of the off-diagonal elements b and c.

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\mathrm{adj}(A) = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}.$$

2. For an $n \times n$ Matrix (n ≥ 3)

The process is more general:

1. **Find cofactors:** For each element a_{ij}, compute the **cofactor**

$$C_{ij} = (-1)^{i+j} M_{ij},$$

   where M_{ij} is the **minor** (determinant of the submatrix obtained by deleting the i-th row and j-th column).

2. **Form the cofactor matrix:** Arrange all cofactors in the same positions as the corresponding elements of A.

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

3. **Transpose the cofactor matrix:**

$$\text{adj}(A) = C^T$$

Example (3×3 Matrix):

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}$$

Step 1: Compute cofactors C_{ij}. For example:

- $C_{11} = \det \begin{bmatrix} 4 & 5 \\ 0 & 6 \end{bmatrix} = (4)(6) - (5)(0) = 24$
- $C_{12} = -\det \begin{bmatrix} 0 & 5 \\ 1 & 6 \end{bmatrix} = -((0)(6) - (5)(1)) = -(-5) = 5$
- $C_{13} = \det \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} = (0)(0) - (4)(1) = -4$

(Similarly, compute all 9 cofactors.)

Step 2: Form the cofactor matrix.

Step 3: Take its transpose → that is $\text{adj}(A)$.

The adjoint is crucial because it is directly used in the formula for the **inverse of a matrix**:

$$A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)}, \quad \text{if } \det(A) \neq 0.$$

8.2 Inverse of Matrix

The **inverse of a matrix** A (denoted A^{-1}) is the matrix that satisfies:

$$A \cdot A^{-1} = A^{-1} \cdot A = I,$$

where I is the **identity matrix**.

An inverse exists **only if** $\det(A) \neq 0$ (non-singular matrix).

1. For a 2×2 Matrix

If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A),$$

where

$$\det(A) = ad - bc,$$

and

$$\operatorname{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Example (2×2)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

1. Determinant:

$$\det(A) = (1)(4) - (2)(3) = 4 - 6 = -2$$

2. Adjoint:

$$\operatorname{adj}(A) = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

3. Inverse:

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

2. For an $n \times n$ Matrix (n ≥ 3)

The same rule applies, but the computation is more complex.

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A), \quad \text{if } \det(A) \neq 0$$

The **inverse of a matrix** A (denoted A^{-1}) is the matrix that satisfies:

$$A \cdot A^{-1} = A^{-1} \cdot A = I,$$

where I is the **identity matrix**.

An inverse exists **only if** $\det(A) \neq 0$ (non-singular matrix).

Example matrix

Take

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 0 & 6 \end{bmatrix}.$$

We will compute $\det(A)$, all minors M_{ij}, cofactors $C_{ij} = (-1)^{i+j} M_{ij}$, the cofactor matrix, the adjugate (transpose of cofactor matrix), and finally A^{-1}.

1) Compute minors and cofactors

For each element a_{ij} remove row i and column j, compute the 2×2 determinant (that M_{ij}), then apply the sign $(-1)^{i+j}$.

Row 1:

- $a_{11} = 1$: remove row1,col1 → submatrix $\begin{bmatrix} 4 & 5 \\ 0 & 6 \end{bmatrix}$.

$$M_{11} = 4 \cdot 6 - 5 \cdot 0 = 24, \qquad C_{11} = (-1)^{1+1} M_{11} = +24.$$

- $a_{12} = 2$: remove row1,col2 → submatrix $\begin{bmatrix} 0 & 5 \\ 1 & 6 \end{bmatrix}$.

$$M_{12} = 0 \cdot 6 - 5 \cdot 1 = -5, \qquad C_{12} = (-1)^{1+2} M_{12} = -1 \cdot (-5) = +5.$$

- $a_{13} = 3$: remove row1,col3 → submatrix $\begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$.

$$M_{13} = 0 \cdot 0 - 4 \cdot 1 = -4, \qquad C_{13} = (-1)^{1+3} M_{13} = +1 \cdot (-4) = -4.$$

Row 2:

- $a_{21} = 0$: remove row2,col1 → submatrix $\begin{bmatrix} 2 & 3 \\ 0 & 6 \end{bmatrix}$.

$$M_{21} = 2 \cdot 6 - 3 \cdot 0 = 12, \qquad C_{21} = (-1)^{2+1} M_{21} = -1 \cdot 12 = -12.$$

- $a_{22} = 4$: remove row2,col2 → submatrix $\begin{bmatrix} 1 & 3 \\ 1 & 6 \end{bmatrix}$.

$$M_{22} = 1 \cdot 6 - 3 \cdot 1 = 6 - 3 = 3, \qquad C_{22} = (-1)^{2+2} M_{22} = +3.$$

- $a_{23} = 5$: remove row2,col3 → submatrix $\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$.

$$M_{23} = 1 \cdot 0 - 2 \cdot 1 = 0 - 2 = -2, \qquad C_{23} = (-1)^{2+3} M_{23} = -1 \cdot (-2) = +2.$$

Row 3:

- $a_{31} = 1$: remove row3,col1 → submatrix $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$.

$$M_{31} = 2 \cdot 5 - 3 \cdot 4 = 10 - 12 = -2, \qquad C_{31} = (-1)^{3+1} M_{31} = +(-2) = -2.$$

- $a_{32} = 0$: remove row3,col2 → submatrix $\begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix}$.

$$M_{32} = 1 \cdot 5 - 3 \cdot 0 = 5, \qquad C_{32} = (-1)^{3+2} M_{32} = -1 \cdot 5 = -5.$$

- $a_{33} = 6$: remove row3,col3 → submatrix $\begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}$.

$$M_{33} = 1 \cdot 4 - 2 \cdot 0 = 4, \qquad C_{33} = (-1)^{3+3} M_{33} = +4.$$

Summary: the cofactor values C_{ij} are

$$[C_{ij}] = \begin{bmatrix} 24 & 5 & -4 \\ -12 & 3 & 2 \\ -2 & -5 & 4 \end{bmatrix}.$$

2) Determinant (cofactor expansion)

Expand along the first row (for example):

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = 1 \cdot 24 + 2 \cdot 5 + 3 \cdot (-4) = 24 + 10 - 12 = 22.$$

Because $\det(A) = 22 \neq 0$, A is invertible.

3) Adjugate (adjoint)

The adjugate $\mathrm{adj}(A)$ is the **transpose of the cofactor matrix**:

$$\mathrm{adj}(A) = [C_{ij}]^T = \begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix}.$$

(Each column of $\mathrm{adj}(A)$ is the corresponding cofactor row.)

4) Inverse using adjugate formula

$$A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A) = \frac{1}{22}\begin{bmatrix} 24 & -12 & -2 \\ 5 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix}.$$

You can simplify each entry:

$$A^{-1} = \begin{bmatrix} \frac{24}{22} & -\frac{12}{22} & -\frac{2}{22} \\ \frac{5}{22} & \frac{3}{22} & -\frac{5}{22} \\ -\frac{4}{22} & \frac{2}{22} & \frac{4}{22} \end{bmatrix} = \begin{bmatrix} \frac{12}{11} & -\frac{6}{11} & -\frac{1}{11} \\ \frac{5}{22} & \frac{3}{22} & -\frac{5}{22} \\ -\frac{2}{11} & \frac{1}{11} & \frac{2}{11} \end{bmatrix}.$$

You may verify $A^{-1}A = I$ (direct multiplication yields the identity).

$$A^{-1} \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

8.3 Properties of Matrix Inverse

1. Existence

- A matrix A^{-1} exists **if and only if** $\det(A) \neq 0$.
- If $\det(A) = 0$, the matrix is called **singular** and has no inverse.

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad \det(A) = 1 \cdot 4 - 2 \cdot 2 = 0$$

So, A^{-1} does **not** exist.

2. Uniqueness

- If the inverse of a matrix exists, it is always **unique**.
- There cannot be two different inverses for the same matrix.

Example:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad \det(A) = 1$$

Inverse:

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

This is the only inverse of A.

3. Inverse of Inverse

- $(A^{-1})^{-1} = A$
- Taking the inverse twice returns the original matrix.

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$$

Now, taking the inverse again:

$$(A^{-1})^{-1} = A$$

4. Product Rule

$$(AB)^{-1} = B^{-1}A^{-1}$$

(Notice the order is reversed)

Example:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}, \quad (AB)^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{6} \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$B^{-1}A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{6} \end{bmatrix} = (AB)^{-1} \quad \checkmark$$

5. Transpose Rule

$$(A^T)^{-1} = (A^{-1})^T$$

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$$

$$(A^{-1})^T = \begin{bmatrix} -2 & 1.5 \\ 1 & -0.5 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \quad (A^T)^{-1} = \begin{bmatrix} -2 & 1.5 \\ 1 & -0.5 \end{bmatrix}$$

$$(A^T)^{-1} = (A^{-1})^T \quad \checkmark$$

SECTION 9: ELEMENTARY OPERATIONS AND MATRIX FORMS

9.1 Elementary Row Operations

1. Row Interchange

$$R_i \leftrightarrow R_j$$

- **Meaning:** Swap row i with row j.
- **Purpose:** Used when we want a **nonzero pivot element** in a particular position, or to rearrange equations.
- **Important:** This operation does **not change the solution** of a system of equations.

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Swap $R_1 \leftrightarrow R_2$:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

Here, the first and second rows are interchanged.

2. Row Scaling

$$R_i \to kR_i, \quad (k \neq 0)$$

- **Meaning:** Multiply all elements of row i by a nonzero constant k.
- **Purpose:** Used to make a **pivot element equal to 1** (normalization step).
- **Important:** Multiplying by 0 is not allowed (it would destroy information).

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Multiply $R_2 \to 2R_2$:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_2 \to 2R_2} \begin{bmatrix} 1 & 2 \\ 6 & 8 \end{bmatrix}$$

Here, every entry in the second row was doubled.

3. Row Addition (Replacement)

$$R_i \to R_i + kR_j, \quad (i \neq j)$$

- **Meaning:** Replace row i with the sum of itself and k times another row j.
- **Purpose:** Used to **eliminate variables** and create zeros in specific positions (key step in Gaussian elimination).
- **Important:** The row R_j used for addition is not changed.

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Operation: $R_2 \to R_2 - 3R_1$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$$

Here, we eliminated the first entry of the second row by subtracting 3 times the first row.

Let see what happens here

Operation: $R_2 \to R_2 - 3R_1$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Step 1: Multiply Row 1 by 3

$$3R_1 = 3 \times [1 \;\; 2] = [3 \;\; 6]$$

Step 2: Subtract from Row 2

$$R_2 - 3R_1 = [3 \;\; 4] - [3 \;\; 6]$$

Do subtraction element by element:

- First element: $3 - 3 = 0$
- Second element: $4 - 6 = -2$

So, new Row 2 becomes:

$$R_2' = [0 \quad -2]$$

Step 3: Write the New Matrix

Now replace old R_2 with new R_2':

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$$

9.2 Elementary Column Operations

Just like we can operate on rows, we can also operate on columns.

1. Column Interchange ($C_i \leftrightarrow C_j$)

Swap two columns of a matrix.

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Operation: $C_1 \leftrightarrow C_2$

Swap column 1 with column 2:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{C_1 \leftrightarrow C_2} \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

✅ Output: Columns exchanged.

2. Column Scaling ($C_i \to kC_i$, $k \neq 0$)

Multiply an entire column by a nonzero constant.

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Operation: $C_2 \to 2C_2$

- Column 2 = $[2, 4]^T$
- Multiply by 2 → $[4, 8]^T$

New matrix:

$$\begin{bmatrix} 1 & 4 \\ 3 & 8 \end{bmatrix}$$

✅ Output: Only column 2 scaled.

3. Column Addition ($C_i \rightarrow C_i + kC_j$)

Replace one column with itself plus a multiple of another column.

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Operation: $C_2 \rightarrow C_2 + 3C_1$

- Column 1 = $[1, 3]^T$
- $3C_1 = [3, 9]^T$
- Column 2 = $[2, 4]^T$
- Add → $[2, 4] + [3, 9] = [5, 13]$

New matrix:

$$\begin{bmatrix} 1 & 5 \\ 3 & 13 \end{bmatrix}$$

✅ Output: Column 2 updated.

9.3 Matrix Forms

Row Echelon Form:

A matrix is said to be in row echelon form if it satisfies the following conditions:

1. Zero Rows at Bottom

Any row that is entirely zero must be written below all the nonzero rows.

Example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

✅ The zero row is at the bottom.

2. Leading Entry (Pivot Element)

The **first nonzero entry** in each nonzero row is called the **pivot element** (or row leader).

Example:

Row 1: first nonzero entry = 1

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

Row 2: first nonzero entry = 4

So, pivots = **1 and 4**.

3. Leading Zeros Increase Down Rows

As you move **down the rows**, the number of leading zeros increases.

Example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

- Row 1 → no leading zeros
- Row 2 → 1 leading zero
- Row 3 → 2 leading zeros

✅ Condition satisfied.

4. Pivot Element (Row Leader)

Each row's first nonzero entry = **pivot element**.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

Example above: pivots = **1, 4, 6**.

5. Pivot Column

The column that contains a pivot element is called a **pivot column**.

Example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

- Pivot 1 in column 1 → pivot column 1
- Pivot 4 in column 2 → pivot column 2
- Pivot 6 in column 3 → pivot column 3

So, pivot columns = {1, 2, 3}.

✅ Final Example (REF Matrix):

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

- Zero rows at bottom (none here).
- Pivots = 1, 3, 5.
- Leading zeros increase as you go down.
- Pivot columns = 1, 2, 3.

This matrix is in **Row Echelon Form**.

Reduced Row Echelon Form (RREF):

A matrix is said to be in **RREF** if it satisfies:

1. All the conditions of Row Echelon Form hold:
   - All zero rows (if any) are at the bottom.
   - Each row's leading entry (pivot) is to the right of the pivot in the row above.
   - Number of leading zeros increases as we move down.

   Example:

   $$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

   This is in Row Echelon Form (REF).

2. Each pivot element is equal to 1.

   Continuing the above example, divide row 2 by 3 and row 3 by 5:

   $$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & \frac{4}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

   Now all pivots are 1.

3. Each pivot is the only nonzero entry in its column (entries above and below the pivot are zero).

   Make column 2 and column 3 entries zero above pivots:

   - $R_1 \to R_1 - 2R_2$
   - $R_1 \to R_1 - R_3$
   - $R_2 \to R_2 - \frac{4}{3}R_3$

   Result:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   This is in **RREF**.

Another example (not identity but still RREF):

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

- Pivot in row 1 = 1 (column 1), rest of column = 0.
- Pivot in row 2 = 1 (column 2), rest of column = 0.
- Pivot columns contain only 1 as nonzero.
- Zero row is at bottom.

  This is also RREF.

9.4 Rank of Matrix

The **rank** of a matrix is the number of **non-zero rows** in its **row echelon form (REF)**.

Equivalent Meaning:

It is also equal to the number of **linearly independent rows or columns** in the matrix.

Notation:

$\text{rank}(A)$ or $r(A)$.

Example 1:

Take

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix}$$

Step 1: Convert to Row Echelon Form.

$$R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - R_1$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & -1 & -2 \end{bmatrix}$$

Step 2: Make pivot clear.

Swap $R_2 \leftrightarrow R_3$:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Here, there are **2 non-zero rows**.

So, $\text{rank}(A) = 2$.

Example 2:

$$B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

This is already in row echelon form.

Number of non-zero rows = 2.

So, $\text{rank}(B) = 2$.

SECTION 10: SYSTEMS OF LINEAR EQUATIONS - FUNDAMENTALS

10.1 Linear Equation Basics

Linear Equation:

A linear equation is an equation of **first degree** (no squares, cubes, etc.) in one or more variables.

General Form:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n$ are constants (coefficients), $x_1, x_2, \ldots, x_n$ are variables, and b is a constant term.

Example 1 (One Variable):

$$3x = 9$$

This is linear because the variable x has degree 1.

Solution: $x = 3$.

Example 2 (Two Variables):

$$2x + 3y = 6$$

This is linear in x and y. Its graph is a straight line in the xy-plane.

System of Linear Equations:

A system of linear equations is a **set of two or more linear equations** involving the same variables.

Example 3 (System of Two Equations in Two Variables):

$$x + y = 5$$

$$2x - y = 1$$

This system has two equations with the same variables x and y. Solving them together gives $x = 2, y = 3$.

10.2 Types of Linear Systems

1. Homogeneous System

A system of equations is called **homogeneous** if **all constant terms are zero**.
General form:

$$Ax = 0$$

It **always has the trivial solution** $x = 0$.

Example (Homogeneous):

$$x + y = 0, \quad 2x - y = 0$$

Here, the constant terms are zero.

- Trivial solution: $x = 0, y = 0$.
- Sometimes, there may also be **non-trivial solutions** if the equations are dependent (e.g., infinite solutions).

2. Non-Homogeneous System

A system of equations is called **non-homogeneous** if **at least one constant term is non-zero**.
General form:

$$Ax = b, \quad b \neq 0$$

Example (Non-Homogeneous):

$$x + y = 3, \quad 2x - y = 1$$

Here, constants are non-zero $(3, 1)$.

- Solving: From the first equation $y = 3 - x$. Substituting in the second:

$$2x - (3 - x) = 1 \implies 3x - 3 = 1 \implies 3x = 4 \implies x = \frac{4}{3}$$

Then $y = 3 - \frac{4}{3} = \frac{5}{3}$.
So, solution = $(x, y) = \left(\frac{4}{3}, \frac{5}{3}\right)$.

- Homogeneous → always has at least the trivial solution.
- Non-Homogeneous → may have **unique, infinite, or no solution** depending on consistency.

10.3 Solution Types

Type 1: Unique Solution

- The system has **exactly one solution**.
- Occurs when the **coefficient matrix is non-singular** ($\det(A) \neq 0$).

Example:

$$x + y = 3, \quad 2x - y = 1$$

- Coefficient matrix:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}, \quad \det(A) = (1)(-1) - (1)(2) = -1 - 2 = -3 \neq 0$$

- Unique solution: $x = \frac{4}{3}, y = \frac{5}{3}$.

Type 2: Infinitely Many Solutions

- The system has **infinite solutions**.
- Occurs when the **coefficient matrix is singular** ($\det(A) = 0$) but the system is consistent.

Example:

$$x + y = 2, \quad 2x + 2y = 4$$

- Coefficient matrix:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \quad \det(A) = 0$$

- The second equation is just a multiple of the first → **dependent equations**.
- Infinite solutions: $y = 2 - x$, x is free to choose.

Type 3: No Solution

- The system is **inconsistent**.
- No set of values satisfies all equations simultaneously.

Example:

$$x + y = 2, \quad x + y = 3$$

- Coefficient matrix:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \det(A) = 0$$

- Different constants (2 and 3) → **no solution** exists.

10.4 Consistency and Inconsistency

1. Consistent System

- A system of linear equations is **consistent** if it has **at least one solution**.
- The solution may be **unique** or **infinite**.

Example 1 (Unique solution):

$$x + y = 3, \quad 2x - y = 1$$

- Coefficient matrix:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}, \quad \det(A) = -3 \neq 0$$

- Unique solution: $x = \frac{4}{3}, y = \frac{5}{3}$.

Example 2 (Infinite solutions):

$$x + y = 2, \quad 2x + 2y = 4$$

- Second equation is a multiple of the first → dependent equations.
- Infinite solutions: $y = 2 - x$, x is free.

2. Inconsistent System

- A system is **inconsistent** if **no solution exists**.
- This happens when the equations contradict each other.

Example:

$$x + y = 2, \quad x + y = 3$$

- No pair (x, y) satisfies both equations simultaneously.
- Hence, the system has **no solution**.

3. Trivial Solution

- Occurs in **homogeneous systems** ($Ax = 0$).
- All variables are equal to **zero**.

Example:

$$x + y = 0, \quad 2x - y = 0$$

- Solution: $x = 0, y = 0$.

4. Non-Trivial Solution

- Occurs in **homogeneous systems** when at least **one variable is non-zero**.
- Appears when the coefficient matrix is **singular** ($\det(A) = 0$).

Example:

$$x + y + z = 0, \quad 2x + 2y + 2z = 0$$

- Non-trivial solution exists, e.g., $x = 1, y = -1, z = 0$.

SECTION 11: METHODS FOR SOLVING LINEAR SYSTEMS

11.1 Matrix Method

Matrix Representation

- A system of linear equations can be written in **matrix form** as:

$$Ax = b$$

where:

- A is the **coefficient matrix**
- x is the **column vector of variables**
- b is the **column vector of constants**

Solution Using Inverse

- If A^{-1} exists (i.e., $\det(A) \neq 0$), the solution can be found as:

$$x = A^{-1}b$$

Example

System of equations:

$$x + 2y = 5, \quad 3x + 4y = 11$$

Step 1: Write in matrix form

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad x = \begin{bmatrix} x \\ y \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

Step 2: Compute inverse of A

$$\det(A) = (1)(4) - (2)(3) = 4 - 6 = -2 \neq 0$$

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix}$$

Step 3: Multiply A^{-1} with b

$$x = A^{-1}b = \begin{bmatrix} -2 & 1 \\ 1.5 & -0.5 \end{bmatrix} \begin{bmatrix} 5 \\ 11 \end{bmatrix} = \begin{bmatrix} (-2)(5) + (1)(11) \\ (1.5)(5) + (-0.5)(11) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solution:

$$x = 1, \quad y = 1$$

This method is **efficient** for small systems where A^{-1} exists. For larger systems or singular matrices, other methods (like row reduction or Cramer's Rule) are preferred.

11.2 Cramer's Rule

Applicability

- Used for solving **n×n systems** of linear equations where the **coefficient matrix A is non-singular** ($\det(A) \neq 0$).

Formula

$$x_i = \frac{\det(A_i)}{\det(A)}$$

where:

- A_i is obtained by **replacing the i-th column of A with the constant vector** b.

Example

System of equations:

$$x + 2y = 5, \quad 3x + 4y = 11$$

Step 1: Coefficient matrix and constant vector

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

Step 2: Compute $\det(A)$

$$\det(A) = (1)(4) - (2)(3) = 4 - 6 = -2$$

Step 3: Replace columns to form A_1 and A_2

- Replace **1st column** with b to get A_1:

$$A_1 = \begin{bmatrix} 5 & 2 \\ 11 & 4 \end{bmatrix}$$

$$\det(A_1) = (5)(4) - (2)(11) = 20 - 22 = -2$$

- Replace **2nd column** with b to get A_2:

$$A_2 = \begin{bmatrix} 1 & 5 \\ 3 & 11 \end{bmatrix}$$

$$\det(A_2) = (1)(11) - (5)(3) = 11 - 15 = -4$$

Step 4: Compute solutions

$$x = \frac{\det(A_1)}{\det(A)} = \frac{-2}{-2} = 1$$

$$y = \frac{\det(A_2)}{\det(A)} = \frac{-4}{-2} = 2$$

Solution:

$$x = 1, \quad y = 2$$

Cramer's Rule provides a **direct formulaic solution** using determinants, but it is mostly practical for **small systems**, as determinant calculations for large matrices can become tedious.

11.3 Using Echelon and Reduced Echelon Forms

1. Row Echelon Form (REF)

Definition:

A matrix is in **row echelon form** if it satisfies these conditions:

1. All zero rows are at the bottom of the matrix.
2. The first non-zero element (pivot) in each row is to the right of the pivot in the previous row.
3. Entries below each pivot are zeros.

Purpose:

REF simplifies a system of linear equations so that the solution can be found using **back substitution**.

Example:

Consider the system of equations:

$$\begin{cases} x + y + z = 6 \\ 2x + 3y + 5z = 4 \\ 4x + y + 2z = 8 \end{cases}$$

Step 1: Write the augmented matrix

$$[A|b] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 2 & 3 & 5 & 4 \\ 4 & 1 & 2 & 8 \end{bmatrix}$$

Step 2: Apply elementary row operations to get REF

- Eliminate the first column below the pivot (row 1):

$$R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - 4R_1$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & -8 \\ 0 & -3 & -2 & -16 \end{bmatrix}$$

- Eliminate the second column below the pivot (row 2):

$$R_3 \to R_3 + 3R_2$$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & -8 \\ 0 & 0 & 7 & -40 \end{bmatrix}$$

This is now in **row echelon form (REF)**.

Step 3: Solve using back substitution

- From row 3: $7z = -40 \Rightarrow z = -40/7$
- From row 2: $y + 3(-40/7) = -8 \Rightarrow y = 16/7$
- From row 1: $x + 16/7 + (-40/7) = 6 \Rightarrow x = 26/7$

Solution:

$$x = 26/7, \quad y = 16/7, \quad z = -40/7$$

2. Reduced Row Echelon Form (RREF)

Definition:

A matrix is in **reduced row echelon form** (RREF) if it satisfies all conditions of **row echelon form (REF)**, plus:

1. All pivot elements are 1.
2. All entries above and below each pivot are 0.

Purpose:

RREF allows the solution of a linear system **directly**, without needing back substitution.

Example:

Consider the same system of equations:

$$\begin{cases} x + y + z = 6 \\ 2x + 3y + 5z = 4 \\ 4x + y + 2z = 8 \end{cases}$$

Step 1: Write the augmented matrix

$$[A|b] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 2 & 3 & 5 & 4 \\ 4 & 1 & 2 & 8 \end{bmatrix}$$

Step 2: Convert to REF first (as before)

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & -8 \\ 0 & 0 & 7 & -40 \end{bmatrix}$$

Step 3: Scale pivot rows so that each pivot is 1

- Row 3: $R_3 \to R_3/7$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & -8 \\ 0 & 0 & 1 & -40/7 \end{bmatrix}$$

Step 4: Eliminate entries above pivots

- Row 2: $R_2 \to R_2 - 3R_3$
- Row 1: $R_1 \to R_1 - R_3$

$$\begin{bmatrix} 1 & 1 & 0 & 62/7 \\ 0 & 1 & 0 & 16/7 \\ 0 & 0 & 1 & -40/7 \end{bmatrix}$$

- Row 1: $R_1 \to R_1 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 26/7 \\ 0 & 1 & 0 & 16/7 \\ 0 & 0 & 1 & -40/7 \end{bmatrix}$$

Step 5: Read the solution directly

$$x = 26/7, \quad y = 16/7, \quad z = -40/7$$

Observation:

- No back substitution is needed.
- Each pivot is 1 and all entries above/below pivots are zero.

3. Rank Analysis

- **Definition:** Rank = number of non-zero rows in REF
- **Usage:** Determine type of solution
  - If rank(A) = rank([A|b]) = n → Unique solution
  - If rank(A) = rank([A|b]) < n → Infinitely many solutions
  - If rank(A) < rank([A|b]) → No solution

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ 12 \\ 18 \end{bmatrix}$$

- REF of A:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{rank}(A) = 1, \quad \text{rank}([A|b]) = 1$$

- Number of variables $n = 3$
- **Conclusion:** Infinite solutions (since rank < n)

11.4 Gaussian Elimination Method

Process

- **Goal:** Solve a system of linear equations by transforming the augmented matrix into **row echelon form (REF)** using **elementary row operations**.
- **Steps:**
  1. Form the **augmented matrix** $[A|b]$ from the system $Ax = b$.
  2. Use **row operations** (swap, scale, add multiples) to create zeros below the pivot elements, forming **upper triangular form (REF)**.
  3. Use **back substitution** to solve for variables starting from the last row.

Advantages

- Works for **any system size**, whether small or large.
- Systematic and suitable for manual calculations or computer algorithms.

System of equations:

$$\begin{cases} x + y + z = 6 \\ 2x + 3y + 5z = 4 \\ 4x + y + 2z = 8 \end{cases}$$

Step 1: Form the augmented matrix

$$[A|b] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 2 & 3 & 5 & 4 \\ 4 & 1 & 2 & 8 \end{bmatrix}$$

Step 2: Eliminate entries below pivot in column 1

- $R_2 \to R_2 - 2R_1$
- $R_3 \to R_3 - 4R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & -8 \\ 0 & -3 & -2 & -16 \end{bmatrix}$$

Step 3: Eliminate entry below pivot in column 2

- Pivot in row 2: $R_2, 2 = 1$
- $R_3 \to R_3 + 3R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & -8 \\ 0 & 0 & 7 & -40 \end{bmatrix}$$

Step 4: Back substitution

- Row 3: $7z = -40 \implies z = -40/7$
- Row 2: $y + 3z = -8 \implies y + 3(-40/7) = -8 \implies y = 16/7$
- Row 1: $x + y + z = 6 \implies x + 16/7 - 40/7 = 6 \implies x = 26/7$

Solution:

$$x = \frac{26}{7}, \quad y = \frac{16}{7}, \quad z = -\frac{40}{7}$$

11.5 Gauss-Jordan Method

Process

- Transform the augmented matrix $[A|b]$ into **reduced row echelon form (RREF)** using **elementary row operations**.
- All pivot elements are 1, and all entries above and below pivots are 0.
- **Solution can be read directly** from the final matrix without back substitution.

Example

System of equations:

$$\begin{cases} x + y + z = 6 \\ 2x + 3y + 5z = 4 \\ 4x + y + 2z = 8 \end{cases}$$

Step 1: Form augmented matrix

$$[A|b] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 2 & 3 & 5 & 4 \\ 4 & 1 & 2 & 8 \end{bmatrix}$$

Step 2: Make pivot in row 1 column 1 = 1 (already 1), then eliminate other entries in column 1:

- $R_2 \to R_2 - 2R_1$
- $R_3 \to R_3 - 4R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & -8 \\ 0 & -3 & -2 & -16 \end{bmatrix}$$

Step 3: Pivot in row 2 column 2 = 1 (already 1), eliminate other entries in column 2:

- $R_1 \to R_1 - R_2$
- $R_3 \to R_3 + 3R_2$

$$\begin{bmatrix} 1 & 0 & -2 & 14 \\ 0 & 1 & 3 & -8 \\ 0 & 0 & 7 & -40 \end{bmatrix}$$

Step 4: Pivot in row 3 column 3 = 1

- $R_3 \to R_3/7$

$$\begin{bmatrix} 1 & 0 & -2 & 14 \\ 0 & 1 & 3 & -8 \\ 0 & 0 & 1 & -40/7 \end{bmatrix}$$

Step 5: Eliminate entries above pivot in column 3

- $R_1 \rightarrow R_1 + 2R_3$
- $R_2 \rightarrow R_2 - 3R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 26/7 \\ 0 & 1 & 0 & 16/7 \\ 0 & 0 & 1 & -40/7 \end{bmatrix}$$

Step 6: Read solution directly

$$x = \frac{26}{7}, \quad y = \frac{16}{7}, \quad z = -\frac{40}{7}$$

SECTION 12: ADVANCED MATRIX THEORY APPLICATIONS

12.1 Field Properties in Matrix Context

Matrices under **addition** and **scalar multiplication** form a **vector space**, meaning they follow certain rules similar to arithmetic in fields. Let's explain each property with examples.

1. Closure

- **Definition:** The sum of two matrices of the same order is also a matrix of the same order.
- **Example:**

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

- The result is also a 2×2 matrix → **closure holds**.

2. Commutativity

- **Definition:** The order of addition does not matter.
- **Example:**

$$A + B = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}, \quad B + A = \begin{bmatrix} 5+1 & 6+2 \\ 7+3 & 8+4 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

- $A + B = B + A$ → **commutative property verified**.

3. Associativity

- **Definition:** Grouping does not affect addition.
- **Example:**

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$(A + B) + C = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 11 & 12 \end{bmatrix}$$

$$A + (B + C) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 7 \\ 8 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 11 & 12 \end{bmatrix}$$

- Both are equal → **associativity holds**.

4. Existence of Identity

- **Definition:** There exists a zero matrix O such that $A + O = A$.
- **Example:**

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A + O = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

- Identity exists → **verified**.

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5. Existence of Inverse

- **Definition:** For every matrix A, there exists a **negative matrix** $-A$ such that $A + (-A) = O$.
- **Example:**

$$-A = \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}, \quad A + (-A) = \begin{bmatrix} 1-1 & 2-2 \\ 3-3 & 4-4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

6. Distributivity

- **Definition:** Scalar multiplication distributes over matrix addition.
- **Example:**

$$k = 2, \quad k(A+B) = 2\begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix} = \begin{bmatrix} 12 & 16 \\ 20 & 24 \end{bmatrix}$$

$$kA + kB = 2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + 2\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} + \begin{bmatrix} 10 & 12 \\ 14 & 16 \end{bmatrix} = \begin{bmatrix} 12 & 16 \\ 20 & 24 \end{bmatrix}$$

- Both are equal → **distributivity holds**.