EE243: Advanced Computer Vision
Assignment #7

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Problem 1

Given training data $\{x_i\}_{i=1}^{M}$, we want to learn the parameters $\theta = \{\lambda_j, \mu_j, \sum_j\}_{j=1}^{K}$ of a Mixture of Gaussians model using the EM algorithm. Derive the update rules for θ.

Answer: Assuming we have some data $\mathbf{x} = x^{(1)}, \ldots, x^{(m)}$, which some from K different Gaussian distributions (K mixtures). We will use the following notations:

- μ_k: the mean of the k^{th} Gaussian component.
- $\sum_k$: the covariance matrix of the k^{th} Gaussian component.
- λ_k: the multinomial parameter of a specific datapoint belonging to the component.
- $z^{(i)}$: the latent variable (multinomial) for each $x^{(i)}$.

We also assume that the dimension of each $x^{(i)}$ is n. The goal is: $max_{\mu,\Sigma,\lambda}\, ln\, p(\mathbf{x}; \mu, \Sigma, \lambda)$. Therefore lets follow exactly the EM framework.

E step

We set $w_j^{(i)} = q_i(z^{(i)} = j) = p(z^{(i)} = j|x^{(i)}; \mu, \Sigma, \lambda)$

M step

We will write down the lower bound and get derivatives for each of the three parameters.

$$\sum_i^m \sum_j^K q_i(z^{(i)} = j) ln \, \frac{p(x^{(i)}, z^{(i)} = j; \mu, \Sigma, \lambda)}{q_i(z^{(i)} = j)} \tag{1}$$

$$= \sum_i^m \sum_j^K q_i(z^{(i)} = j) ln \, \frac{p(x^{(i)} | z^{(i)} = j; \mu, \Sigma) p(z^{(i)} = j; \lambda)}{q_i(z^{(i)} = j)} \tag{2}$$

Note that:

- $x^{(i)} | z^{(i)} = j; \mu, \Sigma \sim \mathcal{N}(\mu_j, \Sigma_j)$
- $z^{(i)} = j; \lambda \sim Multi(\lambda)$

We can then leverage these probability distributions and continue

$$ll := \sum_i^m \sum_j^K w_j^{(i)} ln \, \frac{\frac{1}{\sqrt{(2\pi)^n |\Sigma_j|}} \, exp(- \frac{1}{2} (x^{(i)} - \mu_j)^T \Sigma_j^{-1} (x^{(i)} - \mu_j)) \, \lambda_j}{w_j^{(i)}} \tag{3}$$

Now, we need to maximize this lower bound for each of the three parameters.

Updating rules derivations

1- Derivation of μ_j

$$\nabla_{\mu_j} ll = \nabla_{\mu_j} \sum_i^m w_j^{(i)} ln \; \frac{\frac{1}{\sqrt{(2\pi)^n |\Sigma_j|}} \; exp(-\frac{1}{2}(x^{(i)} - \mu_j)^T \Sigma_j^{-1} (x^{(i)} - \mu_j)) \; \lambda_j}{w_j^{(i)}} \quad (4)$$

$$= \nabla_{\mu_j} \sum_i^m w_j^{(i)} [ln \; \frac{\frac{1}{\sqrt{(2\pi)^n |\Sigma_j|}} \; \lambda_j}{w_j^{(i)}} + ln \; exp(-\frac{1}{2}(x^{(i)} - \mu_j)^T \Sigma_j^{-1} (x^{(i)} - \mu_j))] \quad (5)$$

$$= \nabla_{\mu_j} \sum_i^m w_j^{(i)} [\frac{1}{2}(x^{(i)} - \mu_j)^T \Sigma_j^{-1} (x^{(i)} - \mu_j)] \quad (6)$$

$$= -\frac{1}{2} \sum_i^m w_j^{(i)} \nabla_{\mu_j} [(x^{(i)} - \mu_j)^T \Sigma_j^{-1} (x^{(i)} - \mu_j)] \quad (7)$$

[For $f(x) = x^T A x$: $\nabla_x f(x) = (A + A^T)x$] (8)

$$= \frac{1}{2} \sum_i^m w_j^{(i)} \nabla_{(x^i - \mu_j)} [(x^{(i)} - \mu_j)^T \Sigma_j^{-1} (x^{(i)} - \mu_j)] \quad (9)$$

$$= \frac{1}{2} \sum_i^m w_j^{(i)} [(\Sigma_j^{-1} + (\Sigma_j^{-1})^T)(x^{(i)} - \mu_j)] \quad (10)$$

[Note that Σ_j is symmetric so we have $(\Sigma_j^{-1})^T = (\Sigma_j^T)^{-1} = \Sigma_j^{-1}$] (11)

$$= \sum_i^m w_j^{(i)} [\Sigma_j^{-1} (x^{(i)} - \mu_j)] \quad (12)$$

Set the last term zero, we have:

$$\nabla_{\mu_j} ll = 0 \tag{13}$$

$$\sum_i^m w_j^{(i)} [\Sigma_j^{-1}(x^{(i)} - \mu_j)] = 0 \tag{14}$$

$$\sum_i^m w_j^{(i)} (x^{(i)} - \mu_j) = 0 \tag{15}$$

$$\sum_i^m w_j^{(i)} x^{(i)} = \sum_i^m w_j^{(i)} \mu_j \tag{16}$$

$$\mu_j = \frac{\sum_i^m w_j^{(i)} x^{(i)}}{\sum_i^m w_j^{(i)}} \tag{17}$$

2- Derivation of $\sum_j$

$$\nabla_{\Sigma_j} ll = \nabla_{\Sigma_j} \sum_i^m w_j^{(i)} ln \ \frac{\frac{1}{\sqrt{(2\pi)^n |\Sigma_j|}} \ exp(-\frac{1}{2}(x^{(i)} - \mu_j)^T \Sigma_j^{-1} (x^{(i)} - \mu_j)) \ \lambda_j}{w_j^{(i)}} \tag{18}$$

$$= \sum_i^m w_j^{(i)} \nabla_{\Sigma_j} [ln \ \frac{1}{\sqrt{|\Sigma_j|}} - \frac{1}{2}(x^{(i)} - \mu_j)^T \Sigma_j^{-1} (x^{(i)} - \mu_j)] \tag{19}$$

$$= -\frac{1}{2} \sum_i^m w_j^{(i)} [\frac{\partial \ ln \ |\Sigma_j|}{\partial \Sigma_j} + \frac{\partial}{\partial \Sigma_j}(x^{(i)} - \mu_j)^T \Sigma_j^{-1} (x^{(i)} - \mu_j)] \tag{20}$$

First, we consider the derivative of the first term in the square bracket:

$$\frac{\partial ln \ |\Sigma_j|}{\partial \Sigma_j} = \frac{1}{|\Sigma_j|} \frac{\partial |\Sigma_j|}{\partial \Sigma_j} \tag{21}$$

$$= \frac{1}{|\Sigma_j|} |\Sigma_j| (\Sigma_j^{-1})^T \tag{22}$$

$$= \Sigma_j^{-1} \tag{23}$$

Then, we do the second term:

$$\frac{\partial}{\partial \Sigma_j}(x^{(i)} - \mu_j)^T \Sigma_j^{-1} (x^{(i)} - \mu_j) = -\Sigma_j^{-1} (x^{(i)} - \mu_j)(x^{(i)} - \mu_j)^T \Sigma_j^{-1} \tag{24}$$

$$\tag{25}$$

Combined these results back and set it to zero, we have:

$$\nabla_{\Sigma_j} ll = -\frac{1}{2}\sum_i^m w_j^{(i)}[\Sigma_j^{-1} - \Sigma_j^{-1}(x^{(i)} - \mu_j)(x^{(i)} - \mu_j)^T \Sigma_j^{-1}] \tag{26}$$

$$= -\frac{1}{2}\sum_i^m w_j^{(i)}[I - \Sigma_j^{-1}(x^{(i)} - \mu_j)(x^{(i)} - \mu_j)^T]\Sigma_j^{-1} \overset{set}{=} 0 \tag{27}$$

$$\tag{28}$$

Rearrange the equation and we have:

$$\sum_i^m w_j^{(i)}[\Sigma_j - (x^{(i)} - \mu_j)(x^{(i)} - \mu_j)^T] = 0 \tag{29}$$

$$\sum_i^m w_j^{(i)}\Sigma_j = \sum_i^m w_j^{(i)}(x^{(i)} - \mu_j)(x^{(i)} - \mu_j)^T \tag{30}$$

$$\Sigma_j = \frac{\sum_i^m w_j^{(i)}(x^{(i)} - \mu_j)(x^{(i)} - \mu_j)^T}{\sum_i^m w_j^{(i)}} \tag{31}$$

3- Derivation of λ_j

This is relatively simpler but we need to apply Lagrange multipliers because $\sum_j \lambda_j = 1$.

$$ll = \sum_i^m \sum_l^k w_l^{(i)} ln \, \frac{\frac{1}{\sqrt{(2\pi)^n|\Sigma_l|}} \; exp(-\frac{1}{2}(x^{(i)} - \mu_l)^T \Sigma_l^{-1}(x^{(i)} - \mu_l)) \; \lambda_l}{w_l^{(i)}} \tag{32}$$

$$= \sum_i^m \sum_l^k w_l^{(i)} ln \; \lambda_l \tag{33}$$

$$\tag{34}$$

We need to construct Lagrangian, with α as the Lagrange multiplier:

$$\mathcal{L}(\lambda) = ll + \alpha(\sum_l^k \lambda_l - 1) \tag{35}$$

We will take derivative on $\mathcal{L}$ and set it to zero:

$$\frac{\partial \mathcal{L}(\lambda)}{\partial \lambda_j} = \frac{\partial}{\partial \lambda_j}[ll + \alpha(\sum_l^k \lambda_l - 1)] \tag{36}$$

$$= \sum_i w_j^{(i)} \frac{1}{\lambda_j} + \alpha \overset{set}{=} 0 \tag{37}$$

$$\tag{38}$$

Rearrange and we will have $\lambda_j = -\frac{\sum_i w_j^{(i)}}{\alpha}$. Recall that $\sum_j \lambda_j = 1$, we have:

$$\sum_j \lambda_j = \sum_j -\frac{\sum_i w_j^{(i)}}{\alpha} = 1 \tag{39}$$

$$\alpha = -\sum_j \sum_i w_j^{(i)} \tag{40}$$

$$= -\sum_j \sum_i p(z^{(i)} = j | x^{(i)}) \tag{41}$$

$$= -\sum_i 1 = -m \tag{42}$$

Finally, we have:

$$\lambda_j = \frac{\sum_i w_j^{(i)}}{m} \tag{43}$$

Problem 2

Consider the multi-class logistic regressions problem in Sec. 4.3.4 of Pattern Recognition and Machine Learning by Bishop. It provides an outline of an iterative algorithm for the multi-class classification problem using the softmax function. Complete the derivation of (4.110) by filling in the details in the outline.
Answer: In this question we are asked to prove equation 4.110 from the refrence book, which is shown as Equation 44 below:

$$\nabla_{w_k} \nabla_{w_j} E(w_1, ..., w_K) = -\sum_{n=1}^N y_{nk}(I_{kj} - y_{nj})\phi_n \phi_n^T \tag{44}$$

We have Equation 45:

$$p(C_k|\phi) = y_k(\phi) = \frac{e^{a_k}}{\sum_j e^{a_j}} \tag{45}$$

where the ‘activations’ a_k are given by Equation 46:

$$a_k = w_k^T \phi \tag{46}$$

$$\frac{\partial y_k}{\partial a_j} = y_k(I_{kj} - y_j) \tag{47}$$

In order to prove 4.110, we first have to prove how equation 4.106 (Equation 55) is derived from 4.104 (Equation 45).

0.1 Equation 55 derivation from Equation 45 proof

0.1.1 Assuming j=k

$$\begin{align}
\frac{\partial y_k}{\partial a_j}(Eq.45) &= \frac{e^{a_j} \sum_h e^{a_j} - (e^{a_j})^2}{(\sum_j e^{a_j})^2} \tag{48}\\
&= \frac{e^{a_j}}{\sum_j e^{a_j}} - (\frac{e^{a_j}}{\sum_j e^{a_j}})^2 \tag{49}\\
&= y_k - (y_k)^2 \tag{50}\\
&= y_k(1 - y_k) \tag{51}
\end{align}$$

0.1.2 Assuming j=k

$$\begin{align}
\frac{\partial y_k}{\partial a_j}(Eq.45) &= -\frac{e^{a_j} e^{a_k}}{(\sum_j e^{a_j})^2} \tag{52}\\
&= -\frac{e^{a_j}}{\sum_j e^{a_j}} \frac{e^{a_k}}{\sum_j e^{a_j}} \tag{53}\\
&= y_j y_k \tag{54}
\end{align}$$

0.1.3 Put them together

From Equation 51 and 54 we have the proof of derivation of Equation 55 from Equation 45, which is:

$$\frac{\partial y_k}{\partial a_j} = y_k(I_{kj} - y_j) \tag{55}$$

0.2 Derivation of Equation 44 (equation 4.110 in the reference book)

We have:

$$E(w_1, ..., w_k) = -ln(p(T|w_1, ..., w_k)) \tag{56}$$

$$= -\sum_{n=1}^{N}\sum_{k=1}^{K} t_{nk} ln(y_{nk}) \tag{57}$$

$$\xrightarrow{46\ in\ 45} y_{nk} = \frac{e^{w_k^T \phi_n}}{\sum_j e^{w_j^T \phi_n}} \tag{58}$$

Below we have some information to complete hessian derivayion:

$$\nabla E(w_1, ..., w_k) = \sum_{n=1}^{N}(y_{nj} - t_{nj})\phi_n \tag{59}$$

$$\frac{\partial y_{nj}}{\partial w_k} = \frac{\partial y_{nj}}{\partial a_{n_k}} . \cancel{\frac{\partial a_{nk}}{\partial w_k}} \phi_n = y_{nj}(I_{jk} - y_{nk})\phi_n \tag{60}$$

In order to derive hessian we have:

$$\nabla_{w_k}(\nabla E(w_1, ..., w_k)) = \tag{61}$$

$$\xrightarrow{59} = \nabla_{w_k}(\sum_{n=1}^{N}(y_{nj} - t_{nj})\phi_n) \tag{62}$$

$$= \sum_{n=1}^{N} \frac{\partial y_{nj}}{\partial w_k} \phi_n^T \tag{63}$$

$$\xrightarrow{60} = -\sum_{n=1}^{N} y_{nk}(I_{kj} - y_{nj})\phi_n \phi_n^T \tag{64}$$

Problem 3

Let $(\tilde{x}, \tilde{y})$ represent the image plane coordinates, whose corresponding representation in homogeneous coordinates is (x, y, z). Consider a camera model where $\tilde{x} = \tilde{x}$ and $\tilde{y} = \frac{y}{z}$. Prove that the projection of a $3D$ line on the image plane of this camera is a hyperbola.
Answer: Equation 65 shows the equation of a $3D$ line.

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \tag{65}$$

According to the question, the camera model is:

$$\tilde{x} = x \tag{66}$$

$$\tilde{y} = \frac{y}{z} \rightarrow y = z\tilde{y} \tag{67}$$

From the left most and the right most part of Equation 65 we have:

$$\frac{x - x_0}{a} = \frac{z - z_0}{c} \tag{68}$$

Which gives us:

$$z = \frac{c}{a}(x - x_0) + z_0 \tag{69}$$

By projecting the $3D$ line in $2D$ plane of camera using Equations 66 and 67, and replacing z in $3D$ line equation (Equation 65), we have:

$$\frac{\tilde{x} - x'_0}{a'} = \frac{z\tilde{y} - y'_0}{b'} \tag{70}$$

$$\frac{\tilde{x} - x'_0}{a'} = \frac{\frac{c'}{a'}(\tilde{x} - x'_0)\tilde{y} + z'_0\tilde{y} - y'_0}{b'} \tag{71}$$

$$\rightarrow \tilde{x} - x'_0 = \frac{c'}{b'}(\tilde{x} - x'_0)\tilde{y} + \frac{a'}{b'}z'_0\tilde{y} - \frac{a'}{b'}y'_0 \tag{72}$$

$$\tilde{x} - \frac{c'}{b'}\tilde{x}\tilde{y} - \frac{c'x'_0}{b'}\tilde{y} - \frac{a'z'_0}{b'}\tilde{y} = x'_0 - \frac{a'y'_0}{b'} \tag{73}$$

$$\tilde{x} - \frac{c'}{b'}\tilde{x}\tilde{y} - \tilde{y}(\frac{c'x'_0}{b'} - \frac{a'z'_0}{b'}) = x'_0 - \frac{a'y'_0}{b'} \tag{74}$$

$$\tilde{x} - \alpha\tilde{x}\tilde{y} - \beta\tilde{y} = \gamma \qquad \alpha,\ \beta,\ \gamma\ are\ constant \tag{75}$$

In Equations 70-74, constants with prime (') are the projection of their corresponding constants in $3D$ plane to the $2D$ plane which are also constants and are shown by a prime on them. We see that Equation 75 is a hyperbola equation.

Problem 4

Part a

In the notation we have followed for stereo reconstruction, prove that $F = e_r \times H$, where $p_r = Hp_l$.

Answer: Figure 1 shows an illustration of an epipolar geometry system. In the figure, all letters with prime are mapped to left side variables and they are used by l index in the following of equations. Similar to the left side variables, all the variables in the figure that are located on the right side are used by r index in the equations.

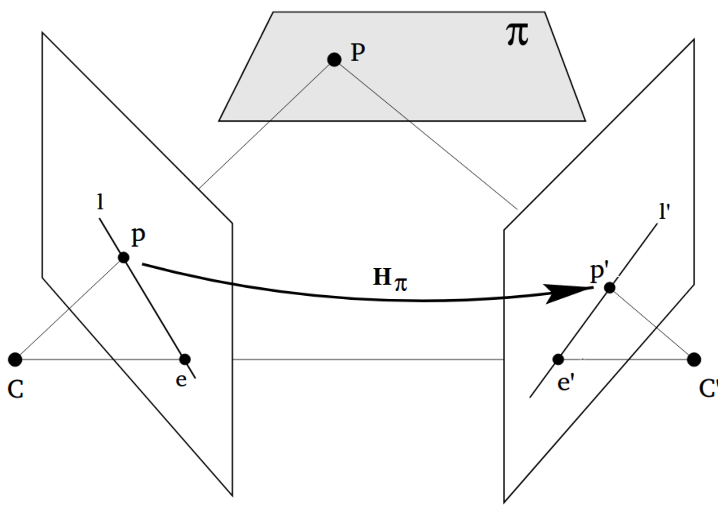

Figure 1: Epipolar Geometry.

From definition of epipolar line, we have:

$$l_r = e_r \times p_r \tag{76}$$

By combining Equation 76 and $p_r = Hp_l$, we have:

$$l_r = e_r \times Hp_l \tag{77}$$

We also have:

$$p_r^T F p_l = 0 \tag{78}$$

$$p_r^T l_r = 0 \quad \text{(because } p_r \text{ lies on a line } l_r\text{)} \tag{79}$$

$$\xrightarrow{78\ and\ 79} p_r^T l_r = p_r^T F p_l \tag{80}$$

$$\xrightarrow{80} l_r = F p_l \tag{81}$$

Using Equation 81 and 77, we have:

$$l_r = e_r \times H p_l = F p_l \tag{82}$$

$$\xrightarrow{82} F = e_r \times H = [e]_\times H \tag{83}$$

Part b

Consider the stereo system as defined in class, but with the world origin at the center of projection of the left camera and all distances represented with respect to this coordinate system. This implies $M_l = M_{l,int}[I_{3\times3}|0]$ and $M_r = M_{r,int}[R|t]$. Prove that the fundamental matrix $F = \hat{e}_r M_r M_l^+$. What is the center of projection of the right camera in this coordinate system?

Answer: In the epipolar geometry we know that for every point P in the $3D$ world:

$$p_l = M_l P \tag{84}$$

$$M_l^+ = M_l^T (M_l M_l^T)^{-1} \tag{85}$$

$$\xrightarrow{multiply\ M_l^+\ to\ 84} M_l^+ p_l = M_l^+ M_l P \tag{86}$$

$$M_l^+ \text{ is pseudo inverse of } M_l \tag{87}$$

$$\xrightarrow{86\ and\ 87} M_l^+ p_l = P \tag{88}$$

$$P = M_l^+ p_l \tag{89}$$

On the other camera we have:

$$p_r = M_r P \tag{90}$$

Using Equations 89 and 90, we have:

$$p_r = M_r P = M_r M_l^+ p_l \tag{91}$$

Epipolar line of p_r in M_r is:

$$l_r = e_r \times p_r \tag{92}$$

$$\xrightarrow{91} l_r = e_r \times M_r M_l^{+} p_l \tag{93}$$

We also have:

$$l_r = F p_l \tag{94}$$

Using Equations 93 and 94 we have:

$$l_r = e_r \times M_r M_l^{+} p_l = F p_l \tag{95}$$

$$\to F = e_r \times M_r M_l^{+} = \hat{e_r} M_r M_l^{+} \tag{96}$$

Since $t = C_r - C_l$, the center of projection of the right camera in this coordinate system is t, where t is the translation vector.

Problem 5

Consider a point which moves along the straight line AC with constant velocity $V = [V_X, 0, V_Z]^T$, i.e. the motion is constrained in the XZ plane. The starting point A is denoted as $[X_{ref}, Y_{ref}, Z_{ref}]$, and AC is at an angle θ to AB which is parallel to the image plane. The image plane is the XY plane of the coordinate system, the principal point is the center of the image and the focal length of the camera is f. Assume perspective projection which is shown in Figure 2.

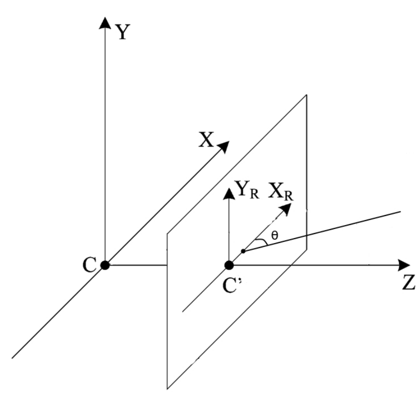

Figure 2: Perspective projection.

Part a

Let $[x, y]^T$ be the projection of a 3D point $[X, Y, Z]^T$, along AC, on the image plane. Write down the differential equations relating the velocity of the $[x, y]^T$ to the 3D velocity and depth, for the particular motion for this point.
Answer: We have:

$$R = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \tag{97}$$

$$T = \begin{vmatrix} t_x \\ t_y \\ t_z \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix} \tag{98}$$

$$P = KI = \begin{vmatrix} f & 0 & c_x \\ 0 & f & c_y \\ 0 & 0 & 1 \end{vmatrix} \tag{99}$$

$$x' = fX \rightarrow x = f\frac{X}{Z} \tag{100}$$

$$y' = fY \rightarrow y = f\frac{Y}{Z} \tag{101}$$

$$z' = Z \tag{102}$$

We have $[\frac{dX}{dt}, 0, \frac{dZ}{dt}]$ or $[V_X, 0, V_Z]$. We also know that:

$$\frac{V_Z}{V_X} = tan(\theta)\frac{V_X}{V_Z} = cot(\theta) \tag{103}$$

Thus, we can derive that:

$$v_x = \frac{dx}{dt} = \frac{dx}{dX}\frac{dX}{dt} + \frac{dx}{dZ}\frac{dZ}{dt} = f\frac{V_X Z - XV_Z}{Z^2} \tag{104}$$

$$v_y = \frac{dy}{dt} = \cancelto{0}{\frac{dy}{dY}\frac{dY}{dt}} + \frac{dy}{dZ}\frac{dZ}{dt} = -f\frac{YV_Z}{Z^2} \tag{105}$$

Part b

If $cot(\alpha) = \frac{\dot{x}}{\dot{y}}$, prove that

$$cot(\theta) = \frac{1}{f}(x_{ref} - y_{ref}cot(\alpha)) \tag{106}$$

where the lower case letters represent the perspective projections of the upper case letters. (Hint:Use the constant velocity equation of motion to express the position of $[X, Y, Z]^T$ in terms of $[X_{ref}, Y_{ref}, Z_{ref}]$ and V. Then follow the perspective projection equations.)

Answer: According to Equation 104 and 105, we have:

$$cot(\alpha) = \frac{v_x}{v_y} = \frac{V_X Z - V_Z X}{-V_Z Y} = -\frac{V_X}{V_Z}\frac{Z}{Y} + \frac{X}{Y} = -cot(\theta)\frac{Z}{Y} + \frac{X}{Y} \tag{107}$$

$$\rightarrow cot(\theta) = \frac{X}{Z} - cot(\alpha)\frac{Y}{Z} \tag{108}$$

$$\rightarrow cot(\theta) = \frac{1}{f}f\frac{X}{Z} - \frac{1}{f}f\frac{Y}{Z}cot(\alpha) \tag{109}$$

$$[\text{where (X, Y, Z)} \in \text{AC}] \tag{110}$$

$$\rightarrow cot(\theta) = \frac{1}{f}f\frac{X_{ref}}{Z_{ref}} - \frac{1}{f}f\frac{Y_{ref}}{Z_{ref}}cot(\alpha) \tag{111}$$

$$[\text{since } f\frac{X_{ref}}{Z_{ref}} = x_{ref} \text{ and } f\frac{Y_{ref}}{Z_{ref}} = y_{ref}] \tag{112}$$

$$\rightarrow cot(\theta) = \frac{1}{f}(x_{ref} - y_{ref}cot(\alpha)) \tag{113}$$

Part c

Write down the rotation matrix, in terms of θ, to rotate a point on AC to its corresponding location on AB.

Answer:

$$R_{AB}^{AC}(\theta) = R_Z R_Y R_x = I \begin{vmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & cos(\theta) \end{vmatrix} I = \begin{vmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & cos(\theta) \end{vmatrix} \tag{114}$$

Essentially, the answer is rotating the line around y-axis by $-\theta$.

Part d

If $[X_\theta, Y_\theta, Z_\theta]^T$ represents a point on AC, and $[X_0, Y_0, Z_0]^T$ represents the corresponding point on AB, write down the geometrical relationship between these corresponding pair of 3D point. Following some algebraic manipulation and using the equations of perspective projection and the assumption that $Z_\theta = Z_{ref}$ (this is actually an approximation), prove that

$$x_0 = f\frac{x_\theta cos(\theta) + x_{ref}(1 - cos(\theta))}{-sin(\theta)(x_\theta + x_{ref}) + f} \tag{115}$$

$$y_0 = \frac{y_\theta}{-sin(\theta)(x_\theta + x_{ref}) + f} \tag{116}$$

Answer: First we have perspective projection equations as below:

$$x_0 = f\frac{X_0}{Z_0}, x_\theta = f\frac{X_\theta}{Z_\theta} \tag{117}$$

$$y_0 = f\frac{Y_0}{Z_0}, y_\theta = f\frac{Y_\theta}{Z_\theta} \tag{118}$$

According to the provided information in the question and part c, we have:

$$\begin{vmatrix} X_0 \\ Y_0 \\ Z_0 \end{vmatrix} = R_{AB}^{AC} \begin{vmatrix} X_\theta \\ Y_\theta \\ Z_\theta \end{vmatrix} + \begin{vmatrix} X_{ref} \\ Y_{ref} \\ Z_{ref} \end{vmatrix} \tag{119}$$

$$= \begin{vmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & cos(\theta) \end{vmatrix} \begin{vmatrix} X_\theta \\ Y_\theta \\ Z_\theta \end{vmatrix} + \begin{vmatrix} X_{ref} \\ Y_{ref} \\ Z_{ref} \end{vmatrix} \tag{120}$$

$$= \begin{vmatrix} X_\theta cos(\theta) + Z_\theta sin(\theta) + X_{ref} \\ Y_\theta + Y_{ref} \\ -X_\theta sin(\theta) + Z_\theta cos(\theta) + Z_{ref} \end{vmatrix} \tag{121}$$

If we substitute the above equations in perspective projection equations, for X_{ref}, Y_{ref}, and Z_{ref} we have:

$$X_{ref} = \frac{x_{ref} Z_{ref}}{f} \tag{122}$$

$$Y_{ref} = \frac{y_{ref} Z_{ref}}{f} \tag{123}$$

$$Z_{ref} = Z_\theta \tag{124}$$

From Equation 119 and 121 we have:

$$\xrightarrow{117\ and\ 121} x_0 = f\frac{X_\theta cos(\theta) + Z_\theta sin(\theta) + X_{ref}}{Z_\theta} \tag{125}$$

$$\xrightarrow{118\ and\ 121} y_0 = f\frac{Y_\theta + Y_{ref}}{Z_\theta} \tag{126}$$

If we substitute the above equations in perspective projection equations, for x_0 we have:

$$x_0 = f\frac{\frac{x_\theta Z_\theta}{f}cos(\theta) + Z_\theta sin(\theta) + \frac{x_{ref}Z_{ref}}{f}}{-\frac{x_\theta Z_\theta}{f}sin(\theta) + Z_\theta cos(\theta) + Z_{ref}} \tag{127}$$

$$\xrightarrow{124} = f\frac{\frac{x_\theta}{f}cos(\theta) + sin(\theta) + \frac{x_{ref}}{f}}{-\frac{x_\theta}{f}sin(\theta) + cos(\theta) + 1} \tag{128}$$

$$\xrightarrow{\times\frac{\frac{f}{f}}{\frac{f}{f}}} = f\frac{x_\theta cos(\theta) + fsin(\theta) + x_{ref}}{-x_\theta sin(\theta) + fcos(\theta) + f} \tag{129}$$

For y_0 we have:

$$y_0 = f\frac{\frac{y_\theta Z_\theta}{f} + \frac{y_{ref}Z_{ref}}{f}}{-\frac{x_\theta Z_\theta}{f}sin(\theta) + Z_\theta cos(\theta) + Z_{ref}} \tag{130}$$

$$\xrightarrow{124} = f\frac{\frac{y_\theta}{f} + \frac{y_{ref}}{f}}{-\frac{x_\theta}{f}sin(\theta) + cos(\theta) + 1} \tag{131}$$

$$\xrightarrow{\times\frac{\frac{f}{f}}{\frac{f}{f}}} = f\frac{y_\theta + y_{ref}}{-x_\theta sin(\theta) + fcos(\theta) + f} \tag{132}$$

Problem 6

Consider the stereo system with the world origin at the center of projection of the left camera and all distances represented with respect to this coordinate system. This implies $M_l = K_l[I_{3\times3}|0]$ and $M_r = K_r[R|t]$, where K_l and K_r are the intrinsic parameters of the left and right camera, respectively. $(R, t) \in SE(3)$

represent the orientation and position of the right camera w.r.t. the left one. Recall that the fundamental matrix $F = \hat{e}_r M_r M_l^+$, where M_l^+ is the pseudo-inverse of M_l.

Part a

Prove that the left epipole $e_l = K_l R^T t$.
Answer: We know that for any $3D$ point P we have:

$$M_r = K_r[R|t] = [K_r R, K_r t] \tag{133}$$

$$M_l = K_l[I_{3\times 3}|0] = [K_l, 0] \tag{134}$$

$$M_l^+ = [K_l[I_{3\times 3}|0]]^{-1} = [[K_l, 0]]^{-1} = \begin{vmatrix} K_l^{-1} \\ 0 \end{vmatrix} \tag{135}$$

$$\rightarrow p_r = M_r \begin{vmatrix} P \\ 1 \end{vmatrix} = K_r R P + K_r t \tag{136}$$

$$\rightarrow p_l = M_l \begin{vmatrix} P \\ 1 \end{vmatrix} = K_l P \tag{137}$$

The right camera center coordination in the coordination system on the left camera is

$$C_r = \begin{vmatrix} R^T t \\ 1 \end{vmatrix} \tag{138}$$

Epipole is the image of the one camera's center of projection on the image plane of the other camera, and we know that the image of any point P on the left image plane is calculated based on Equation 137. Thus we have:

$$e_l = M_l C_r \tag{139}$$

$$\rightarrow e_l = M_l R^T t = K_l R^T t \tag{140}$$

Part b

Prove that the right epipole $e_r = K_r t$.
Answer: The left camera center coordination in the coordination system on itself is

$$C_l = \begin{vmatrix} 0 \\ 1 \end{vmatrix} \tag{141}$$

Same as previous part, we know that the image of any point P on the left image plane is calculated based on Equation 136. Thus for e_r, which is the image of the left camera center on the right camera image plane, we have:

$$e_r = M_r \begin{vmatrix} 0 \\ 1 \end{vmatrix} = K_r t \tag{142}$$

Part c

Prove that $M_r M_l^+ = K_r R K_l^{-1}$.
Answer: from Equation 133 and 135, we have:

$$M_r M_l^+ = \begin{vmatrix} K_r R & K_r t \end{vmatrix} \begin{vmatrix} K_l^{-1} \\ 0 \end{vmatrix} = K_r R K_l^{-1} \tag{143}$$

Part d

Using the result $\hat{x}S = S^{-T}[S^{-1}x]_\times$ ($[w]_\times$ is the same as $\hat{w} \in SO(3)$), and the above expression of the fundamental matrix, show that $F = K_r^{-T} R K_l^T [K_l R^T t]_\times$. (Hint: Express the fundamental matrix using the result in part (c). Apply the above result, three times, one matrix at a time, on the fundamental matrix.)
Answer: From Equation 143 we have:

$$F = \hat{e_r} M_r M_l^+ = \hat{e} K_r R K_l^{-1} \tag{144}$$
$$\xrightarrow{apply} F = K_r^{-T}[K_r^{-1} e_r]_\times R K_l^{-1} \tag{145}$$
$$\xrightarrow{part\ b} F = K_r^{-T}[K_r^{-1} K_r t]_\times R K_l^{-1} \tag{146}$$
$$= K_r^{-T}[t]_\times R K_l^{-1} = K_r^{-T} \hat{t} R K_l^{-1} \tag{147}$$
$$\xrightarrow{apply} F = K_r^{-T} R [R^T t]_\times K_l^{-1} \tag{148}$$
$$\xrightarrow{apply} F = K_r^{-T} R K_l^T [K_l R^{-1} t]_\times \tag{149}$$
$$\xrightarrow{part\ a} F = K_r^{-T} R K_l^T [e_l]_\times \tag{150}$$

Part e

If the right camera is only translated w.r.t. the left camera (no rotation) and the intrinsic parameters of the two cameras are the same, prove that $F = \hat{e}_r$. If the translation is parallel to the x-axis and the matrix of intrinsic parameters is identity (I), compute the fundamental matrix.

Answer: We have:

$$R = I \tag{151}$$

$$K_r = K_l \tag{152}$$

$$F = \hat{e_r} K_r R k_l^{-1} \tag{153}$$

$$\xrightarrow{151 \ and \ 152} F = \hat{e_r} \tag{154}$$

If the camera translation is parallel to the x-axis, then $e_r = (1, 0, 0)^T$, so

$$F = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} \tag{155}$$