Differential equation

Numerical Methods

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Overview

1. Euler Method
2. Runge Kutta

Euler Method

Euler Method

Input, Output.

- Input: Initial value problem:
  - y' = f(x, y).
  - x $\in$ [a, b].
  - y(a) = c.
- Output:
  - Tabulated function $(x_m, u_m)|_{m=0}^{m=n}$.
  - A real value u_n approximating y(b).

Euler Method

Steps.

1. Choose an integer n ≥ 1 such that h = $\frac{b-a}{n}$ "What is h, can we choose h first?".
2. Let $x_0 = a, u_0 = y(a) = c$.
3. if $x_m (0 \leq m < n)$ is defined then:
   - $X_{m+1} = x_m + h$,
   - $d_m = h.f(x_m, u_m)$,
   - $u_{m+1} = u_m + d_m$.

Euler Method

Example

Find using Euler an approximation for the function $y = x^2$ for $x \in [1, 1.5]$ and y(1) = 1, The question could also be find the approximate value of y(1.5) using the Euler method if you know that y(1) = 1.

Euler Method

Solution

First we have to find the derivative:

$$y(x) = x^2 => y' = 2x$$

Then we have to calculate n or h depend on the information in the question:

if h = 0.1 then $n = \frac{b-a}{h} = \frac{1.5-1}{0.1} = 5.$

Euler Method

Solution cont...

Now we can start to calculate:

n	x_n	$y_n = y_{n-1} + d_n$	$f(x_n, y_n) = 2x$	$d_n = h.f(x_n, y_n)$
0	1	1	2	0.2
1	1.1	1.2	2.2	0.22
2	1.2	1.42	2.4	0.24
3	1.3	1.66	2.6	0.26
4	1.4	1.92	2.8	0.28
5	1.5	2.2	-	-

Euler Method

Example

Find using Euler an approximation for the solution of the equation $y' = x + y$ for $x \in [0, 1]$ and y(0) = 0 and (h = 0.2).

Euler Method

Solution

Now we can start to calculate:

n	x_n	$y_n = y_{n-1} + d_n$	$f(x_n, y_n) = 2x$	$d_n = h.f(x_n, y_n)$
0	0	0	0	0
1	0.2	0	0.2	0.04
2	0.4	0.04	0.44	0.088
3	0.6	0.128	0.728	0.1456
4	0.8	0.2736	1.0736	0.21472
5	1	0.48832	-	-

Euler Method

Example

Find using Euler an approximation for square root for real values $[1, \infty[$, use h = 0.1 and calculate $\sqrt{1.5}$.

Runge Kutta

Runge Kutta

Input, Output.

- Input: Initial value problem:
  - y' = f(x, y).
  - x $\in$ [a, b].
  - y(a) = c.
- Output:
  - Tabulated function $(x_m, u_m)|_{m=0}^{m=n}$.
  - A real value u_n approximating y(b).

Runge Kutta

Steps.

1. Choose an integer n ≥ 1 such that h = $\frac{b-a}{n}$ "What is h, can we choose h first?".
2. Let $x_0 = a, u_0 = y(a) = c$.
3. if $x_m (0 \leq m < n)$ is defined then:
   - $y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$,
   - $k_1 = h.y'(x_n, y_n)$,
   - $k_2 = h.y'(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$.
   - $k_3 = h.y'(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$.
   - $k_4 = h.y'(x_n + h, y_n + k_3)$.

Runge Kutta

Example

Using Runge Kutta method calculate an approximation for the solution of differential equation $y' = y - x$, where y(0) = 2 and h = 0.1, one step is enough.