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| Innopolis University | Practice Test #01<br>Model #01 | Fall 2022 | Name: ____________________<br>Surname: ____________________<br>Date: September $20^{th}$, 2022 |

- Each question has a value of 1 point.
- Each question is going to be binary checked (1 if it is correct, 0 otherwise; there are no decimal points).
- The practice has no grade.

Given the following IVP

$$\text{IVP}: \begin{cases} y'(x) = y^2 + x^2y^2 \\ y(0) = -1 \end{cases}$$

solve it using the **Euler** method

$$y_{k+1} = y_k + \Delta x \cdot f(x_k, y_k)$$

for the interval $x \in [0, 1]$ for $n = 10$ space subdomains. Use that information to

1. **(1 pt)** complete the table,

| Iteration $k$ | Space Step $x_k$ | Numerical Solution $y_k$ | Analytical Solution $f(x_k)$ | AVE $\lvert y_k - f(x_k)\rvert$ |
|---|---|---|---|---|
| 0 | 0 | -1.0 | -1.0 | 0.0 |
| 1 | | | | |
| 2 | | | | |
| 3 | | | | |
| 4 | | | | |
| 5 | | | | |
| 6 | | | | |
| 7 | | | | |
| 8 | | | | |
| 9 | | | | |
| 10 | | | | |

2. **(1 pt)** make a sketch of the numerical solution vs. the analytical solution, and
3. **(1 pt)** make a sketch of the error function.

**Solution:**

According to the Euler method we have

$$y_{k+1} = y_k + \Delta x \cdot f(x_k, y_k) \quad \rightarrow \quad y_{k+1} = y_k + \Delta x \cdot y_k^2(1 + x_k^2)$$

since $n = 10$ is the number of subdivisions of our space interval $[0, 1]$ this means

$$\Delta x = \frac{1-0}{10} = 0.1 \quad \rightarrow \quad x_k = k \cdot \Delta x = k \cdot 0.1$$

On the other hand, the differential equation is separable, so the solution can be readily obtained after standard integration

$$\begin{aligned}
\frac{dy}{dx} &= y^2(1+x^2) \\
\int y^{-2} dy &= \int (1+x^2) dx \\
\frac{1}{y} &= -\left(x + \frac{1}{3}x^3\right) + c \\
\frac{1}{-1} &= -\left(0 + \frac{1}{3}0^3\right) + c \\
\frac{1}{y} &= -\left(x + \frac{1}{3}x^3\right) - 1 \\
\frac{1}{y} &= -x - \frac{1}{3}x^3 - 1
\end{aligned}$$

which can be rewritten as

$$y(x) = -\frac{3}{x^3 + 3x + 3}$$

for the first computation $k = 1$ we have

$$y_1 = y_0 + \Delta x \cdot (y_0)^2(1 + (x_0)^2) \quad \rightarrow \quad y_1 = -1 + 0.1 \cdot (-1)^2(1 + (0)^2) = -0.9$$

for the second computation $k = 2$ we have

$$x_2 = y_1 + \Delta x \cdot (y_1)^2(1 + (x_1)^2) \quad \rightarrow \quad x_2 = -0.9 + 0.1 \cdot (-0.9)^2(1 + (0.1)^2) \approx -0.8$$

for the analytical solution we have

$$y(x) = -\frac{3}{x^3 + 3x + 3} \quad \rightarrow \quad f(x_k) = -\frac{3}{x_k^3 + 3x_k + 3}$$

which means, for $k = 1$

$$f(x_1) = -\frac{3}{(x_1)^3 + 3(x_1) + 3} \quad \rightarrow \quad f(0.1) = -\frac{3}{(0.1)^3 + 3(0.1) + 3} \approx -0.9088155$$

for $k = 2$

$$f(x_2) = -\frac{3}{(x_2)^3 + 3(x_2) + 3} \quad \rightarrow \quad f(0.2) = -\frac{3}{(0.2)^3 + 3(0.2) + 3} \approx -0.8314856$$

for the error function per step we have, for $k = 0$

$$\text{error}(y_k, f(x_k)) = |y_k - f(x_k)| \quad \rightarrow \quad \text{error}(y_0, f(x_0)) = |-1 - (-1)| =$$

for $k = 1$

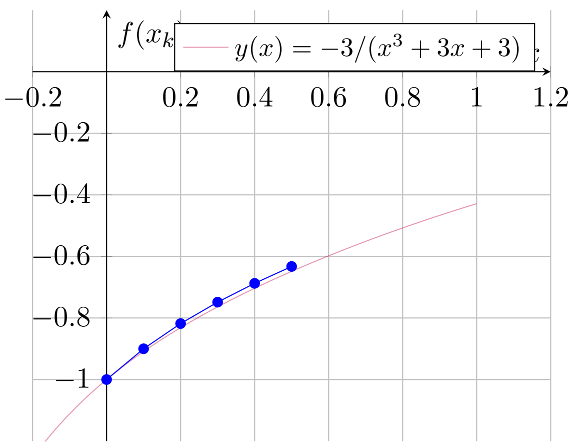
$$\text{error}(y_k, f(x_k)) = |y_k - f(x_k)| \quad \rightarrow \quad \text{error}(y_1, f(x_1)) = |-0.9 - (-0.9088155)| \approx 0.0088$$

continuing with this process till the fifth step we have the information

1. the table is

| Iteration<br>$k$ | Space Step<br>$x_k$ | Numerical Solution<br>$y_k$ | Analytical Solution<br>$f(x_k)$ | AVE<br>$\lvert y_k - f(x_k)\rvert$ |
|---|---|---|---|---|
| 0 | 0 | -1 | -1.0 | 0.0 |
| 1 | 0.1 | -0.9 | -0.9088155 | 0.0088155 |
| 2 | 0.2 | -0.81819 | -0.8314856 | 0.0132956 |
| 3 | 0.3 | -0.7485688 | -0.7639419 | 0.0153732 |
| 4 | 0.4 | -0.6874901 | -0.7035647 | 0.0160747 |
| 5 | 0.5 | -0.6326635 | -0.6486486 | 0.0159851 |

2. the sketch of the numerical solution is




3. the sketch of the error function

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